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# A Topological Study Of Stochastic Dynamics On CW Complexes

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# DEDICATION

*To my wife, Jess*

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## CHAPTER 1 INTRODUCTION

While connections between topology and quantum field theory have been studied for several decades, bridges linking algebraic topology to statistical mechanics have gone relatively unnoticed and unexplored. One such link is empirical, or stochastic, current. This nexus allows for an abundance of ideas from algebraic topology, differential geometry, dynamical systems, and statistical mechanics to be mixed together in new and exciting ways. As such, the study of empirical currents permits a vast toolbox to be employed, and allows statements in one area to be translated into another. This dissertation is an investigation into what topological tools, ideas, and results we can discover by importing techniques from these other fields.

The mathematics behind empirical currents was first rigorously introduced for smooth manifolds in [11]. In this set-up, fix a smooth, Riemannian manifold  $(M, g)$  of dimension  $m$ , and a Morse function  $f : M \rightarrow \mathbb{R}$ . We shall often work in local coordinates  $(x^i)$ , and the Einstein summation convention is taken throughout. The stochastic nature of this work originates from a stochastic vector field  $\xi$  on  $M$ . By a stochastic vector field, we mean a time-dependent vector field  $\xi(x, t) = \xi^j(x, t) \frac{\partial}{\partial x^j}$ , satisfying Gaussian and Markovian statistics:

$$\langle \xi^j(x, t) \rangle = 0, \quad \langle \xi^j(x, t), \xi^k(x, t') \rangle = \beta^{-1} \delta(t - t') g^{jk}(x) \quad (1.1)$$

where  $\beta = \frac{1}{k_B T}$  is a positive real number given by the reciprocal of the Boltzmann constant  $k_B$  and temperature  $T$ . A particle on  $M$  will undergo motion governed by the Langevin equation

$$\frac{dx}{dt} = u(x) + \xi(x, t), \quad (1.2)$$

where the drift term  $u$  is linearly related to the driving force  $-\nabla f$ , expressed locally as

$$u^j(x) = -g^{jk}(x) \frac{\partial f_k(x)}{\partial x^k}.$$

Equivalently, there is a corresponding stochastic differential equation given by

$$dX_t = u(X_t)dt + \sqrt{\frac{g}{\beta}} dW_t, \quad (1.3)$$

where  $W_t$  denotes the standard  $m$ -dimensional Wiener process on  $\mathbb{R}^m \cong T_{X_t}M$ .

A stochastic trajectory, or solution to Equation (1.3), can be represented by a path  $\eta : [0, \tau] \rightarrow M$ . For long times  $\tau$ , one can assume the path is closed [11, p. 6], and so the trajectory can be represented by  $\eta : S^1 \rightarrow M$ . This gives rise to a class in the real-bordism homology of  $M$

$$Q_{\tau, \beta}(u) = \frac{1}{\tau}[\eta] \in H_1(M; \mathbb{R}),$$

known as the *average empirical current density associated to  $\eta$  and duration  $\tau$* . If we then fix an  $(m-1)$ -dimensional cycle, represented by a codimension 1 closed submanifold  $\alpha : K \rightarrow M$ , then the intersection product

$$Q^\alpha = \frac{1}{\tau}[\eta] \cdot [\alpha] \in H_m(M; \mathbb{R}) \cong \mathbb{R},$$

is the  $\alpha$ -*component of the empirical current*.

The prototypical example from which this terminology is motivated is that of an electron in an electrical wire  $M = S^1 \times D^2$ . Connecting the wire to a battery gives rise to the drift term  $u$ , and random collisions with phonons (quantized lattice vibrations) and impurities in the wire give rise to the stochastic vector field  $\xi$ . For an oriented cross-section  $\alpha : \{p\} \times D^2 \rightarrow S^1 \times D^2$  of manifolds with boundary, the intersection pairing  $[\eta] \cdot [\alpha]$  will be the flux, or signed number of times the electron passes through  $\{\alpha\} \times D^2$ . In this same sense,  $\frac{1}{\tau}[\eta] \cdot [\alpha]$  is the average current density at  $\alpha$  associated to the electron.



What we have discussed thus far can only be used to describe the motion of points on  $M$ . Our interest lies in the motion of extended objects on  $M$ , meaning submanifolds of dimension greater than zero. This problem requires more data than that given in Eq. (1.1). Specifically, we need to consider correlations of  $\xi$  at different points on  $M$ . To this end, we need a correlation function, or metric,  $G \in \Gamma(T^*M \boxtimes T^*M)$ , and replace Eq. (1.1) with

$$\langle \xi^j(x, t) \rangle = 0, \quad \langle \xi^j(x, t), \xi^k(y, t') \rangle = \beta^{-1} \delta(t - t') G^{jk}(x, y). \quad (1.4)$$

Imposing these conditions on  $\xi$  allows us to discuss the motion of extended objects in Eq. (1.2). We also impose the additional generic assumption that the pair  $(f, g)$  satisfies the Morse-Smale transversality condition. This means the stable and unstable manifolds of  $f$  intersect transversally. By [26], this implies the handlebody decomposition of  $M$  given by  $f$  is in fact a bona-fide CW decomposition.

The Markov process on smooth manifolds arising from Eq. (1.3) can be thought of as follows. Consider a  $(d-1)$ -cycle  $\eta_0$ , represented by a closed  $(d-1)$ -dimensional submanifold  $\eta_0 : N \rightarrow M$ . This should be thought of as an initial condition to the differential equation in Eq. (1.3). The process is best understood by a separation of time scales. Initially, the cycle  $\eta_0$  will evolve deterministically according to the gradient flow of  $f$ . The cycle will very quickly, independent of the noise, tend to the  $(d-1)$ -skeleton of  $M$  as determined by the Morse function. This is simply the decomposition determined by a Morse function: the open cells of the decomposition are given by the unstable manifolds of the (negative) gradient flow. In general, the cycle will continue to move along itself and perform fluctuations in some small neighborhood of the  $(d-1)$ -skeleton, since the Wiener process has mean zero.

On longer time scales, however, the stochastic vector field  $\xi$  can impact the dynamics

dramatically. That is, if we wait for a long enough time  $t > 0$ , the stochastic vector field  $\xi$  can push a segment of the evolved cycle  $\eta_t$  out of the  $(d - 1)$ -skeleton. If  $\xi$  is large enough, a segment of  $\eta_t$  can move against the gradient flow and up to a critical point of index  $d$ . With non-zero probability, the segment can cross the critical point. Once the segment crosses the critical point, no additional noise is needed: the gradient flow will push  $\eta_t$  back into a small neighborhood of the  $(d - 1)$ -skeleton. This scenario is known as a *rare event*, since the probability of  $\xi$  becoming large enough to temporarily push the cycle across a critical point of index  $d$  scales exponentially with  $-\beta$ . The random process consists of both the fast gradient flow relaxation and the slower, rare processes. In this scenario, the average empirical current associated to  $\eta_0$  is

$$Q_{\tau_D, \beta}(u) = \frac{1}{\tau}[\eta_\tau] \in H_d(M; \mathbb{R}).$$

In a generic situation, it will be useful to allow the Morse function  $f$ , and thus  $u$ , to vary periodically in time. The period of  $f$  is denoted  $\tau_D$ , so that  $f(x, t) = f(x, t + \tau_D)$  for all  $x \in M$ . For simplicity, we take the evolution time  $\tau = N\tau_D$ , and implicitly take  $N \rightarrow \infty$  throughout, so that we may observe many periods of  $f$ . For full generality in the continuous case, one should also allow families of generalized Morse functions, but this technicality is unimportant for our considerations.

Understanding solutions to Eq. (1.3), and Ito diffusions on Riemannian manifolds in general, is an entire subject unto itself. We will instead focus on a deterministic equation associated to Eq. (1.3), known as the Kolmogorov, or Fokker-Planck, equation. The Kolmogorov equation is a deterministic equation for distributions of processes, i.e., solutions of Eq. (1.3). This is in contrast to the Langevin equation, which is stochastic by its defi-

nition. Solutions to the Kolmogorov equation have deep connections to both the topology and geometry of the manifold. In terms of probability theory, the operator appearing in the Kolmogorov equation is known as the generator of the Markov process. In the case of a Riemannian manifold, it is well known that this operator is one-half of the Laplace-Beltrami operator [16].

The first complete and mathematically rigorous treatment of empirical currents was done for graphs, or one-dimensional CW complexes, in [14]. Here, the problem begins with a particle taking a random walk on a graph  $X$ , equivalent to a classical Markov chain with state space given by the vertices. The transitions occur through the edges, and the rate at which the particle jumps from vertex to vertex is determined by a set of real parameters  $\{E_i, W_\alpha\}$  for each vertex  $i$  and edge  $\alpha$ . The set of all such parameters is denoted  $\mathcal{M}_X = \{E_i, W_\alpha\}_{i,\alpha}$ . It is shown in [10] that the generator of this Markov process can also be written in terms of the Laplacian. To see this, define  $e^{\beta E} : C_0(X; \mathbb{R}) \rightarrow C_0(X; \mathbb{R})$  by extending  $i \mapsto e^{\beta E_i} \cdot i$  linearly, and similarly for  $e^{-\beta W} : C_1(X; \mathbb{R}) \rightarrow C_1(X; \mathbb{R})$ . The Fokker-Planck operator

$$\mathcal{H} = -\partial e^{-\beta W} \partial^* e^{\beta E} : C_0(X; \mathbb{R}) \rightarrow C_0(X; \mathbb{R}),$$

is the generator of the process. The Fokker-Planck operator is also known as the master operator on a graph, and governs the time evolution of distributions of the process. This is expressed in the Kolmogorov equation, also known as the master equation,

$$\frac{d\rho}{dt} = \mathcal{H}\rho,$$

where  $\rho = \rho(t) \in C_0(X; \mathbb{R})$  is a one parameter family of distributions.

Up to this point, we have only considered the case of time-independent  $E$  and  $W$ , and therefore time-independent rates of the process as well. This is a simplified situation for

a variety of reasons, most notably since the particle will tend to the vertex with lowest value of  $E$ , and no current will be generated. If we instead allow  $E$  and  $W$  to vary in time, we can attempt to force the particle to walk stochastically on the graph. We drive the system by taking a path of parameters, known as a *driving protocol*, and represented by  $\gamma : [0, \tau_D] \rightarrow \mathcal{M}_X$ . We restrict to the case of *periodic driving*, meaning  $\gamma$  can be represented as a smooth map  $\gamma : S^1 \rightarrow \mathcal{M}_X$ ; in particular,  $\gamma(0) = \gamma(\tau_D)$ . We can then associate a well-defined homology class

$$Q_{\tau_D, \beta}(\gamma) \in H_1(X; \mathbb{R}),$$

known as the *average empirical current density*.

Practical formulas for the average current density, as in [14], can be obtained under two limits. The first is the *adiabatic limit* of slow-driving, so that  $\tau_D \rightarrow \infty$ . It is remarkable that current is still generated in the adiabatic limit, since the parameters are changing infinitely slowly in time. Intuitively one would think this situation is very similar to that of time-independent parameters. The second limit is the *low-temperature limit*, in which  $\beta \rightarrow \infty$ . This is also known as the low noise limit, since the diffusion coefficient in Eq. (1.3) scales inversely with  $\beta$ . After taking this limit, the cycle will fluctuate entirely within the  $d$ -skeleton, and the process described above will take place on the CW complex as determined by  $f$ . The quantization result which we aim to generalize to arbitrary CW complexes is the following:

**Theorem 1.1** ([14, Theorem A]). *For a sufficiently generic driving protocol  $\gamma$ , the low noise, adiabatic limit of the average current tends to the integer lattice in real homology. That is,*

$$\lim_{\beta \rightarrow \infty} \lim_{\tau_D \rightarrow \infty} Q_{\tau_D, \beta}(\gamma) \in H_1(X; \mathbb{Z}) \subset H_1(X; \mathbb{R}).$$

This result was originally discovered physically by a number of experiments on molecular pumps [2, 10, 27] ratchets [23, 32], and other microscopic machines [3, 19, 34] (see the extensive references of [31]). The attempt to make these processes mathematically well-defined and this quantization precise resulted in the initial work on graphs [14]. The fact that the current lives in the integral lattice implies that it is a robust invariant of the physical processes under study, meaning it doesn't change under small perturbations of the system. The robustness of topological invariants is a key feature not shared by their geometric counterparts and has important physical implications for the system (see [12] and [13]). It is also interesting to note that the order of these limits is crucial, and in the case of reversed limits, it is hard to say much of anything. In at least one experiment, this was found to give zero current [3].

The physical process for extended objects given by Eq. (1.2) is motivated from statistical mechanics. Consider a dynamical system with state space given by  $M$ , a smooth manifold. We weakly couple the system to another dynamical system, known as the *bath*, represented by a large number of stochastic vector fields on  $M$ . From the perspective of statistical mechanics, the number of such vector fields is on the order of Avogadro's number, but mathematically, we only need enough to span the tangent space at all times. The bath serves as the origin of the noise, and correlations of the noise are governed by the metric  $\beta^{-1}G \in T^*M \boxtimes T^*M$ . This function  $G$  gives rise to a Riemannian metric on  $M$  by setting  $g(x) = G(x, x)$ , which in turn, governs the dissipative force  $u$  via  $u^j = g^{jk} \frac{\partial f}{\partial x^k}$ . The fact that these two metrics are proportional to one another is known as the fluctuation-dissipation relation. It is worth noting that these metrics equip  $M$  with a bi-Riemannian structure, and give physical meanings to the Riemannian metrics.

The low-temperature limit is the reason this dissertation is a study of stochastic topology and not of stochastic geometry. This limit, in which the main results of this work are stated, allows us to work completely with CW complexes instead of Riemannian manifolds. The main result of this dissertation is also stated in this limit.

**Theorem 1.2** (Theorem 5.17). *For a sufficiently generic periodic driving protocol  $\gamma$ , in the low-temperature, adiabatic limit, we have*

$$\lim_{\beta \rightarrow \infty} \lim_{\tau_D \rightarrow \infty} Q_{\tau_D, \beta}(\gamma) \in H_d(X; \mathbb{Z}[\frac{1}{D}]).$$

where  $D$  is determined by combinatorial data in terms of  $X$ .

Importantly, these limits also serve as both conceptual and algebraic simplifications. In the discrete setting, the only pertinent information about the Morse function one needs is its critical points, their open cells, and their critical values. This data is encoded in  $X$  and the space of parameters  $\mathcal{M}_X$ , respectively (see Definition 2.1).

The dissertation is organized as follows: The remainder of this chapter defines the spaces and notations we will use. Chapter 2 defines the aforementioned Markov process in a rigorous fashion. The Kirchhoff network problem is stated in Chapter 3, and a solution is constructed using the Higher Projection Formula in Theorem 3.19. An extremely general form of the higher matrix-tree theorem is given in Theorem 3.29. The Boltzmann distribution is defined in Chapter 4, and is shown to give an explicit solution to the combinatorial Hodge problem. In Chapter 5, the adiabatic and low-temperature limits are analyzed. It is shown in Proposition 5.9 that the current density can be expressed in terms of the Kirchhoff solution and Boltzmann distribution. All of this work culminates in the quantization result of Theorem 5.17. We believe this to be the main result of this work. Chapter 6 explores

applications of this work to physics, and includes various formulas for Reidemeister torsion in terms of spanning trees and spanning co-trees.

## 1.1 Conventions

Throughout this dissertation, let  $X$  denote a finite, connected CW complex of dimension  $d$ . Our results hold for any finite CW complex by truncation to the  $d$ -skeleton. We write  $X^{(k)}$  for its  $k$ -skeleton and  $X_k$  for its collection of  $k$ -cells. For a commutative ring  $A$ , the  $k$ -th chain group with coefficients in  $A$ , denoted  $C_k(X; A)$ , is the free  $A$ -module with basis  $X_k$ .

The structure of  $X$  is determined by inductively attaching cells of increasing dimension. The  $k$ -skeleton is formed from the  $(k - 1)$ -skeleton by means of attaching maps

$$S_\alpha^{k-1} \xrightarrow{\varphi_\alpha} X^{(k-1)},$$

where  $\alpha$  indexes the set of  $k$ -cells to be attached. Then

$$X^{(k)} = X^{(k-1)} \coprod_{\alpha} D_\alpha^n,$$

where the union is amalgamated along the attaching map  $S_\alpha^{k-1} \rightarrow X^{(k-1)}$ . Each  $k$ -cell  $\alpha$  has a boundary

$$\partial\alpha = \sum_{\substack{j \in X_{k-1} \\ \langle \partial\alpha, j \rangle \neq 0}} b_{\alpha, j} j$$

where  $b_{\alpha, j} := \langle \partial\alpha, j \rangle$  is the incidence number of  $\alpha$  and  $j$ . The incidence number can be explicitly described by means of the attaching maps. That is,  $b_{\alpha, j}$  is the degree of the composite

$$S_\alpha^{k-1} \xrightarrow{\varphi_\alpha} X^{(k-1)} \longrightarrow X^{(k-1)}/X^{(k-2)} \simeq \bigvee_i S_i^{k-1} \longrightarrow S_j^{k-1},$$

where the last map is given by projection onto the wedge summand corresponding to cell  $j$ .

With respect to the standard inner product, in which the set of cells form an orthonormal basis for  $C_i(X; A)$  for each  $i$ , the adjoint operator  $\partial^*$  on a  $(k-1)$ -cell  $j$  is given by

$$\partial^* j = \sum_{\substack{\alpha \in X_k \\ \langle \partial \alpha, j \rangle \neq 0}} b_{j,\alpha}^* \alpha$$

where  $b_{j,\alpha}^* := b_{\alpha,j}$ . We continue this convention of denoting  $(d-1)$ -cells with roman letters and  $d$ -cells with greek letters.

**Definition 1.3.** *A graph is a finite, one-dimensional CW complex.*

Every cell  $e_\alpha^n$  in a CW complex has a characteristic map

$$\Phi_\alpha : D_\alpha^n \longrightarrow X^{(k)},$$

extending the attaching map  $\phi_\alpha$  over the closed disk. By definition, the characteristic map is a homeomorphism from the interior of  $D_\alpha^n$  onto  $e_\alpha^n$ .

**Definition 1.4.** *A CW complex is regular if, for all  $\alpha$ , the characteristic map  $\Phi_\alpha$  is an embedding.*

Regular CW complexes are equipped with characteristic maps which are homeomorphisms on the *closed* cells, instead of just the open cells. This is equivalent to demanding that the closed cells of  $X$  be homeomorphic to closed euclidean cells [20].

**Definition 1.5.** *A CW complex is pseudo-regular if  $b_{\alpha,j} \in \{-1, 0, 1\}$  for every  $d$ -cell  $\alpha$  and  $(d-1)$ -cell  $j$ .*

If  $X$  is a regular CW complex, then it is pseudo-regular. In particular, any connected polyhedron or finite simplicial complex is pseudo-regular. Pseudo-regular complexes are very useful in homological calculations, as well as providing a simple cellular decomposition of  $X$



with fewer cells than that of a simplicial decomposition. We place the following assumption on the CW complexes we study.

**Assumption 1.6.** *X is a pseudo-regular CW complex.*

The boundary maps allow us to define the homology groups of  $X$ . Let  $Z_k(X; A)$  denote the kernel of  $\partial : C_k(X; A) \rightarrow C_{k-1}(X; A)$ , and let  $B_k(X; A)$  denote the image of  $\partial : C_{k+1}(X; A) \rightarrow C_k(X; A)$ . The  $k$ -th homology group with coefficients in  $A$  is the quotient  $H_k(X; A) = Z_k(X; A)/B_k(X; A)$ . The  $k$ -th Betti number of  $X$  is the dimension of  $H_k(X; \mathbb{Q})$ . For a finite, connected CW complex  $K$  of dimension  $d$ , let  $\theta_K$  be the order of the torsion subgroup of  $H_{d-1}(K; \mathbb{Z})$ .

In addition to the above structure, we also equip  $X$  with functions

$$E : X_{d-1} \rightarrow \mathbb{R} \qquad W : X_d \rightarrow \mathbb{R}. \tag{1.5}$$

Their values are denoted with subscripts:  $E_i := E(i)$ ,  $W_\alpha := W(\alpha)$ . We also fix a positive real number  $\beta$ . The number  $\beta$  should be thought of as a noise parameter. In fact, we are ultimately interested in the low-noise  $\beta \rightarrow \infty$  limit, so if the reader prefers a more deterministic viewpoint, this limit should be kept in mind. The same can be said for the energies  $E$  and  $W$ , which (topologically) can be taken to be identically zero.

We extend the operators of Eq. (1.5) to the chain complex

$$e^{\beta E} : C_{d-1}(X; \mathbb{R}) \rightarrow C_{d-1}(X; \mathbb{R}) \qquad e^{\beta W} : C_d(X; \mathbb{R}) \rightarrow C_d(X; \mathbb{R})$$

by

$$x \mapsto e^{\beta E_x} \cdot x \qquad \alpha \mapsto e^{\beta W_\alpha} \cdot \alpha. \tag{1.6}$$

This allows us to define modified inner products on  $C_d(X; \mathbb{R})$  and  $C_{d-1}(X; \mathbb{R})$

$$\langle x, y \rangle_E := e^{\beta E_x} \langle x, y \rangle \quad \langle \alpha, \gamma \rangle_W := e^{\beta W_\alpha} \langle \alpha, \gamma \rangle. \quad (1.7)$$

If we define the adjoint of  $\partial$  with respect to the modified inner product on both  $C_d(X; \mathbb{R})$  and  $C_{d-1}(X; \mathbb{R})$ , we obtain the *biased gradient operator*

$$\partial_{E,W}^* = e^{-\beta W} \partial^* e^{\beta E}. \quad (1.8)$$

Beginning in Chapter 5, we will take  $E$  and  $W$  to be time-dependent functions, so that the operators  $e^{\beta E(t)}, e^{\beta W(t)}$  and thus, the inner products  $\langle -, - \rangle_{E(t)}$  and  $\langle -, - \rangle_{W(t)}$  are all time-dependent.

## CHAPTER 2 EMPIRICAL CURRENTS

This chapter is devoted to making the ideas surrounding the Markov process described in Chapter 1 rigorous. We begin by discussing the analogous process on CW complexes. We shall quickly see that this differs greatly from the graph case, and some work must be done to define the process when  $X$  is not a graph. Fortunately, the study of current density and average current density allows for a reduction to the finite dimensional vector space  $C_{d-1}(X; \mathbb{R})$ .

The intuition which stands behind the process on general CW complexes is taken from a random walk on a graph. The process in higher dimensions can be formulated in the following fashion. Fix an integer cycle  $\hat{x} \in Z_{d-1}(X; \mathbb{Z})$ , thought of as the initial condition to the process, as well as the analogous object to a vertex in higher dimensions. This also fixes a class  $[\hat{x}] \in H_{d-1}(X; \mathbb{Z})$ . The cycle moves by stochastically ‘jumping across’  $d$ -cells, formally given by adding the boundaries of those  $d$ -cells to the cycle. We are interested in the long-time behavior of this process, and we allow the cycle to evolve in this way for an extended period of time. The evolved cycle may have substantially changed from the initial cycle  $\hat{x}$ , and may be supported on entirely distinct  $(d-1)$ -cells from  $\hat{x}$ , but by construction its homology class will remain fixed for all times. The rate at which the cycle evolves, as well as which  $d$ -cells it transitions over and which  $(d-1)$ -cells it is supported on, are in part determined by the energy functionals  $E$  and  $W$  of Eq. (1.5). The remainder of this chapter is devoted to rigorously defining this process.

## 2.1 The Process

**Definition 2.1.** *The space of parameters for  $X$  is the real vector space*

$$\mathcal{M}_X,$$

*consisting of pairs  $(E, W)$ , where  $E : X_{d-1} \rightarrow \mathbb{R}$  and  $W : X_d \rightarrow \mathbb{R}$ .*

**Definition 2.2.** *A periodic driving protocol is a smooth path*

$$\gamma : \mathbb{R} \rightarrow \mathcal{M}_X.$$

*such that  $\gamma(t) = \gamma(t + t_D)$  for all  $t \in \mathbb{R}$ . The real number  $\tau_D$  is the period of  $\gamma$ .*

It is convenient to represent a periodic driving protocol by a pair  $(\tau_D, \gamma)$ , where  $\gamma : S^1 \rightarrow \mathcal{M}_X$  is a smooth loop reparametrized to a total length of 1. The pair is also known as a smooth Moore loop. From the viewpoint of physics, this reparametrization is especially useful since  $\gamma$  is then specified by a dimensionless parameter.

A driving protocol gives a systematic way of changing the parameters of the process, and hence, a way of attempting to drive the system. By changing the parameters in a periodic fashion, we attempt to force the evolving  $(d - 1)$ -cycle to perform directed motion, even though the true motion is stochastic. If we were to take time-independent rates, so that  $E(t) \equiv E$ ,  $W(t) \equiv W$  for all  $t$ , then the cycle would tend to the  $(d - 1)$ -cells of lowest energy as in Lemma 5.5. In particular, no directed motion would occur (on average), and the current density, as defined in Definition 5.8 would be zero (see Proposition 5.10).

**Definition 2.3.** *Let  $X$  be a finite connected graph,  $(\tau_D, \gamma)$  a periodic driving protocol, and  $\beta > 0$ . The master operator for  $X$  is*

$$\mathcal{H}(t) := \mathcal{H}(\tau_D, \beta, \gamma)(t) = -\partial e^{-\beta W(t)} \partial^* e^{\beta E(t)} : C_0(X; \mathbb{R}) \rightarrow C_0(X; \mathbb{R}). \quad (2.1)$$

The master operator gives rise to a continuous time random walk, i.e., a Markov process on the graph  $X$ . The rate at which a particle sitting at a vertex  $j$  hops across an edge to an adjacent vertex  $i$  is governed by the  $(i, j)$ -entry of the master operator  $\mathcal{H}$ . Using the standard basis of  $C_0(X; \mathbb{R})$  given by the vertices, we can write the matrix elements for  $\mathcal{H}$  as

$$\mathcal{H}_{ij}(t) = \begin{cases} \sum_{\alpha \in d^{-1}\{i,j\}} e^{-\beta(W_\alpha(t) - E_i(t))} & i \neq j \\ - \sum_{i \subseteq d(\alpha)} e^{-\beta(W_\alpha(t) - E_i(t))} & i = j. \end{cases} \quad (2.2)$$

The master operator on a graph is also known as the *Fokker-Planck* operator in physics, since it governs the time evolution of probability distributions.

**Definition 2.4.** *Let  $(\tau_D, \gamma)$ , and  $\beta$  be as above. In addition, fix a vertex  $j \in X_0$ . The master equation for  $X$  is*

$$\frac{d\rho(t)}{dt} = \tau_D \mathcal{H}(t) \rho(t) \quad \rho(0) = \rho_0, \quad (2.3)$$

where  $\rho_0 \in C_0(X; \mathbb{R})$  is an initial 0-chain.

If  $\rho(t)$  is a normalized solution to Eq. (2.3), meaning  $\sum_{i \in X_0} \rho_i(t) = 1$ , then  $\rho_i(t)$  represents the probability density of observing state  $i$  at time  $t$ .

The behavior of solutions to Eq. (2.3) under the adiabatic and low-temperature limits, as well as the effect on the initial condition, was studied in [14]. We proceed to do the analogous analysis in higher dimensions. Using the boundary operator on the real chain complex of a general CW complex  $X$ , we form a higher dimensional version of the master operator.

**Definition 2.5.** *For  $(\tau_D, \gamma)$  and  $\beta > 0$  as above, the dynamical operator is defined to be*

$$\mathcal{H}(t) := \mathcal{H}(\tau_D, \beta, \gamma)(t) = -\partial e^{-\beta W(t)} \partial^* e^{\beta E(t)} : C_{d-1}(X; \mathbb{R}) \rightarrow C_{d-1}(X; \mathbb{R}). \quad (2.4)$$

As before, the matrix entries of  $\mathcal{H}$  can be written explicitly in terms of the basis of

$C_{d-1}(X; \mathbb{R})$  given by the set of  $(d-1)$ -cells:

$$\mathcal{H}_{ij}(t) = \begin{cases} -\sum_{\alpha} b_{\alpha,i} b_{\alpha,j} e^{-\beta(W_{\alpha}(t) - E_i(t))} & i \neq j \\ -\sum_{\alpha} b_{\alpha,i}^2 e^{-\beta(W_{\alpha}(t) - E_i(t))} & i = j, \end{cases} \quad (2.5)$$

where  $b_{\alpha,j}$  is the incidence number of  $\alpha$  and  $j$ . If  $\alpha$  is an edge connecting vertices  $i$  and  $j$  in a graph, then  $b_{\alpha,i} = -b_{\alpha,j} = 1$ , and Eq. (2.5) reduces to Eq. (2.2). However, from the viewpoint of probability theory, there is a tremendous difference between the master operator on a graph and the dynamical operator on a general CW complex. As defined in Eq. (2.4), the dynamical operator  $\mathcal{H}$  is *not* the generator of any random process on  $X$ . The easiest way to see this is to note that the generator of a Markov process must satisfy  $\mathcal{H}_{ij} > 0$  for  $i \neq j$ , and

$$\sum_j \mathcal{H}_{ij} = 0,$$

for all  $i$ . This need not be true for  $\mathcal{H}$  as in Definition 2.5, even under the pseudo-regularity condition on  $X$ . The reason for this failure lies in the fact that the domain of  $\mathcal{H}$  is  $C_{d-1}(X; \mathbb{R})$ , and this is *not* the state space of  $\mathcal{H}$ . Nevertheless, we still define the analogous dynamical equation.

**Definition 2.6.** Fix an initial cycle  $\hat{x} \in Z_{d-1}(X; \mathbb{Z})$ , a periodic driving protocol  $(\tau_D, \gamma)$ , and  $\beta > 0$ . The dynamical equation for  $\hat{x}$  is

$$\frac{d\rho(t)}{dt} = \tau_D \mathcal{H}(t) \rho(t) \quad \rho(0) = \hat{x}. \quad (2.6)$$

## 2.2 The Cycle-Incidence Graph

**Definition 2.7.** For any homology class  $[y] \in H_{d-1}(X; \mathbb{Z})$ , let

$$Z_{d-1}^{[y]}(X; \mathbb{Z})$$

denote the subspace of integral  $(d - 1)$ -cycles homologous to  $[y]$ .

Once we have fixed an initial condition  $\hat{x} \in Z_{d-1}(X; \mathbb{Z})$ , the state space of the process we are describing is given by  $Z_{d-1}^{[\hat{x}]}(X; \mathbb{Z})$ . This leads us to the following definition.

**Definition 2.8.** *For a finite, connected, pseudo-regular CW complex  $X$ , and choice of initial cycle  $\hat{x} \in Z_{d-1}(X; \mathbb{Z})$ , define an oriented graph as follows. The vertices are given by integer  $(d - 1)$ -cycles*

$$z \in Z_{d-1}^{[\hat{x}]}(X; \mathbb{Z}).$$

An oriented edge  $z_0 \rightarrow z_1$  is given by a triple

$$(\alpha, f, z_0),$$

with  $\alpha \in X_d$ ,  $f \in X_{d-1}$ , and satisfying the following:

- $\langle z_0, \alpha \rangle \neq 0$ ,
- $\langle \partial\alpha, f \rangle \neq 0$ , and
- $z_1 = z_0 - \langle z_0, f \rangle \langle \partial\alpha, f \rangle \partial\alpha$ .

The cycle incidence graph  $\Gamma_{X, \hat{x}}$  is the full subgraph generated by  $\hat{x}$ . That is, a vertex  $z$  lies in  $\Gamma_{X, \hat{x}}$  if there exists a finite sequence of oriented edges  $\hat{x} \rightarrow z_1 \rightarrow \cdots \rightarrow z_k \rightarrow z$ , i.e., there is a finite oriented path from  $\hat{x}$  to  $z$ . An edge belongs to the subgraph  $\Gamma_{X, \hat{x}}$  if it occurs in such a path.

The cycle-incidence graph gives a model for the state space of the process, and is motivated by the process described at the beginning of the chapter. To see this, fix a cycle  $z \in Z_{d-1}(X; \mathbb{Z})$  and a  $(d - 1)$ -cell  $i$  such that  $\langle z_0, i \rangle \neq 0$ . Furthermore, suppose there exists

$\alpha \in C_d(X; \mathbb{R})$  such that  $b_{\alpha, i} \neq 0$ . Then the cycle  $z$  can ‘jump off’ the cell  $i$ , ‘across’  $\alpha$ , forming  $z'$ . Specifically,

$$\begin{aligned} z' &= z - \sum_j \langle z, i \rangle \langle b_{\alpha, j}, i \rangle \left( \sum_k b_{\alpha, k} k \right) \\ &= z - \langle z, i \rangle b_{\alpha, i} \sum_k b_{\alpha, k} k. \end{aligned}$$

Equivalently, there is a path  $(\alpha, i, z) : z \rightarrow z'$  in  $\Gamma_{X, z_0}$ . Taking the inner product of  $z$  with  $i$  gives

$$\begin{aligned} \langle z_1, i \rangle &= \langle z_0, i \rangle - \langle z_0, i \rangle b_{\alpha, i}^2 \\ &= 0, \end{aligned}$$

where we have used the fact that  $b_{\alpha, i}^2 = 1$ , since  $X$  is pseudo-regular (Assumption 1.6). The cycle  $z$  is incident to  $i$  along  $\partial\alpha$ , whereas  $z'$  is not. However, the remainder of the cells of  $\partial\alpha$  are incident to  $z'$ , giving the ‘jump’ across  $\alpha$ . These types of jumps form the basis of the random process we study.

**Lemma 2.9.** *If  $X$  is a connected graph and  $i \in X_0$  is a vertex, then  $\Gamma_{X, i}$  is the double of  $X$ .*

*Proof.* Recall that the double of a graph  $X$  is an oriented graph  $\tilde{X}$  with the same set of vertices in which every edge is replaced with a pair of edges with opposite orientation.

Fix the canonical generator  $[i] \in H_0(X; \mathbb{Z}) \cong \mathbb{Z}$ . Since  $X$  is connected, every vertex is homologous to every other vertex, and  $X_0 \subset \Gamma_{X, i}$ . If  $\alpha \in X_1$  with  $\partial\alpha = \{j, k\}$  then the cycle-incidence graph has edges  $(\alpha, j, j) : j \rightarrow k$  and  $(\alpha, k, k) : k \rightarrow j$ . Edges in the cycle incidence graph are labelled by edges in the actual graph, and the edges in  $\Gamma_{X, i}$  must connect cycles represented by single vertices. Since  $\Gamma_{X, i}$  is generated by  $i$ , this completes the proof. □



To simplify notation, let  $\Gamma = \Gamma_{X,\hat{x}}$  for the remainder of this chapter. The boundary map on the cycle incidence graph  $d : \Gamma_1 \rightarrow \Gamma_0$  is given by

$$\begin{aligned} d(\alpha, f, z_0) &= z_1 - z_0 \\ &= -\langle z_0, \alpha \rangle \langle \partial \alpha, f \rangle \partial \alpha. \end{aligned}$$

We extend this to an operator on the chain complex in the usual manner. This allows us to rewrite the dynamical operator  $\mathcal{H}$  on the cycle incidence graph  $\Gamma$ . We weight the edges of the cycle-incidence graph with positive real numbers given by

$$k_{(\alpha, f, z)}(t) := e^{-\beta(W_\alpha(t) - E_f(t))}.$$

Define a homomorphism  $d_{E,W}^* : C_0(\Gamma; \mathbb{R}) \rightarrow C_1(\Gamma; \mathbb{R})$  by

$$d_{E,W}^*(z) = \sum_{(\alpha, f, z)} k_{(\alpha, f, z)}(t) (\alpha, f, z),$$

where the sum is indexed over all edges  $(\alpha, f, z)$  with initial vertex  $z$ .

**Definition 2.10.** *The master operator for  $\Gamma$  is defined to be*

$$H(t) := H(\tau_D, \beta, \gamma)(t) = dd_{E,W}^* : C_0(\Gamma; \mathbb{R}) \rightarrow C_0(\Gamma; \mathbb{R}). \quad (2.7)$$

**Remark 2.11.** *The map  $d_{E,W}^*$  is not the formal adjoint to the boundary operator  $d$  on  $\Gamma$ . The edges of  $\Gamma$  are oriented, since  $\Gamma$  is the state diagram of the process, and thus  $d_{E,W}^*$  is only ‘half’ the adjoint (see Lemma 2.9).*

For  $z_0 \neq z_1$ , let  $G(z_0, z_1) = d^{-1}(\{z_0, z_1\})$  be the set of all oriented edges in  $\Gamma$  from  $z_0$  to  $z_1$ . Define  $K(z) = \{(\alpha, f, z)\}$  be the set of all edges with initial vertex  $z$ . Equip  $C_0(\Gamma; \mathbb{R})$  with the natural basis given by the vertices of  $\Gamma$ , and thus, the integer  $(d-1)$ -cycles in  $X$

representing  $[\hat{x}]$ . In this basis, we compute the matrix elements of  $H$  to be

$$H_{z_i, z_j}(t) = \begin{cases} \sum_{(\alpha, f, z_j)} k_{(\alpha, f, z_j)}(t) & \text{if } i \neq j, \text{ summed over } (\alpha, f, z_j) \in G(z_j, z_i), \\ -\sum_{(\alpha, f, z_j)} k_{(\alpha, f, z_j)}(t) & \text{if } i = j, \text{ summed over } (\alpha, f, z_j) \in K(z_j). \end{cases} \quad (2.8)$$

The matrix  $H$  does form the generator of a Markov process on  $\Gamma$ . That is,  $H_{z_i, z_k} \geq 0$  and  $\sum_k H_{z_i, z_k} = 0$  for any  $i$ . The off-diagonal entry  $H_{z_k, z_i}$  measures the net flow of probability from state  $z_i$  to  $z_k$ . We will omit the explicit time dependence of the rates in what follows.

**Remark 2.12.** *The additional complexity of the possibly infinite cycle-incidence graph  $\Gamma$  as compared to the finite CW complex  $X$  is an artifact of the field-theory nature of this process. The state space is given by  $C_0(\Gamma; \mathbb{R})$ , the real Hilbert space with basis given by the collection of integer cycles homologous to a fixed cycle.*

**Definition 2.13.** *Let  $(\tau_D, \gamma)$  and  $\beta$  be as above. In addition, fix an integral cycle  $\hat{x} \in Z_{d-1}(X; \mathbb{Z})$ . The master equation for  $X$  is*

$$\frac{dp(t)}{dt} = H(t)p(t), \quad p(0) = \hat{x},$$

for  $p(t) \in C_0(\Gamma_{X, \hat{x}}; \mathbb{R})$ .

We can compare the situation on the CW complex to the cycle-incidence graph in a rigorous fashion. Define  $\pi_\bullet : C_\bullet(\Gamma; \mathbb{R}) \rightarrow C_{\bullet+d-1}(X; \mathbb{R})$  by

$$\pi_0(z) = z$$

to be the natural inclusion, and

$$\pi_1(\alpha, f, z) = -\langle z, f \rangle \langle \partial \alpha, f \rangle \alpha.$$

**Lemma 2.14.** *The following diagram*

$$\begin{array}{ccc} C_1(\Gamma; \mathbb{R}) & \xrightarrow{\pi_1} & C_d(X; \mathbb{R}) \\ d \downarrow \uparrow d_{E,W}^* & & \partial \downarrow \uparrow \partial_{E,W}^* \\ C_0(\Gamma; \mathbb{R}) & \xrightarrow{\pi_0} & C_{d-1}(X; \mathbb{R}) \end{array}$$

*commutes with respect to  $\{d, \partial\}$  and anti-commutes with respect to  $\{d_{E,W}^*, \partial_{E,W}^*\}$ .*

*Proof.* Since all the maps under consideration are linear, it suffices to check the statements on single elements.

For  $\{d, \partial\}$ , take an edge  $(\alpha, f, z) : z_0 \rightarrow z_1$  in  $C_1(\Gamma; \mathbb{R})$ . Then

$$\pi_0 d(\alpha, f, z_0) = \pi_0(z_1 - z_0) = z_1 - z_0,$$

whereas

$$\begin{aligned} \partial \pi_1(\alpha, f, z_0) &= -\partial \langle z_0, f \rangle \langle \partial \alpha, f \rangle \alpha \\ &= -\langle z_0, f \rangle \langle \partial \alpha, f \rangle \partial \alpha \\ &= z_1 - z_0. \end{aligned}$$

For  $\{d_{E,W}^*, \partial_{E,W}^*\}$ , take  $z \in C_0(\Gamma; \mathbb{R})$ . Then

$$\begin{aligned} \pi_1 d_{E,W}^* z &= \pi_1 \sum_{(\alpha, f, z)} k_{(\alpha, f, z)}(\alpha, f, z) \\ &= - \sum_{(\alpha, f, z)} k_{(\alpha, f, z)} \langle z, f \rangle \langle \partial \alpha, f \rangle \alpha. \end{aligned}$$

On the other hand, express  $z = \sum_f \langle z, f \rangle f$  as a sum of  $(d-1)$  cells. Then

$$\begin{aligned} \partial_{E,W}^* \pi_0 z &= \partial_{E,W}^* \sum_f \langle z, f \rangle f \\ &= \sum_{f, \alpha} \langle z, f \rangle \langle f, \partial \alpha \rangle e^{-\beta(W_\alpha - E_f)} \alpha. \end{aligned}$$

□

The dynamical equation on  $X$  has many advantages over the master equation on  $\Gamma$ , the most important of which is the finite-dimensional domain of its operator. While the dynamical equation is the obvious generalization one would guess from the graph case, it does not accurately describe the dynamics of the process we are considering. Nevertheless, we are interested in an observable of distributions of this process: the average current density. The average current density is the time average of the distribution on a cycle. There is no explicit formula for the current density on the cycle-incidence graph. However, using Lemma 2.14, we can push the current density, and thus average current density, into the CW complex  $X$ . Remarkably, the distributions of current densities satisfy the dynamical equation on  $X$ .

**Assumption 2.15.** *The current density  $\mathbf{J}_{\tau_D, \beta}(\gamma, \hat{x}) \in C_0(\Gamma_{X, \hat{x}}; \mathbb{R})$  is given by  $\partial_{E, W}^* \rho$  where  $\rho$  satisfies the dynamical equation (2.6).*

This assumption is the basis of our analysis. It has been shown to be valid at a physics level of rigor [33]. We believe it to be true at a mathematics level of rigor as well, but this is yet to be shown. In fact, all the finite moments of the process satisfy some closed equation on the finite CW complex.

## CHAPTER 3 THE KIRCHHOFF PROBLEM

In this chapter, we state and prove the Kirchhoff network problem for CW complexes. An explicit solution is constructed using a higher dimensional notion of a spanning tree. A useful characterization of the solution is given by splitting a certain short exact sequence, shown in Remark 3.3. The main tool used in this chapter is the higher projection formula given in Theorem 3.19. Based on this, we give a formal solution to the network theorem in Theorem 3.22, a general weighted matrix-tree theorem in Theorem 3.29, from which the classical matrix-tree theorem follows in Corollary 3.32. We note that this chapter significantly follows the work published in [9].

This chapter is primarily focused on calculations with the  $d$ -cells of  $X$ . Recall the function  $W : X_d \rightarrow \mathbb{R}$ ,  $e^{\beta W}$ , and  $\langle -, - \rangle_W$  of Eqs. (1.5)-(1.7). To simplify the notation, set  $R = e^{\beta W}$  for the definition of the network problem, the discussion of its solution, and similarly for the higher network theorem. This notation is motivated from the corresponding problem in physics (see Remark 3.2).

### 3.1 Kirchhoff's Network Problem and Solution

**Definition 3.1.** *A network problem for  $X$  consists of a choice of  $(d-1)$ -boundary current  $p \in B_{d-1}(X; \mathbb{R})$  and  $d$ -cycle voltage  $q \in Z_d(X; \mathbb{R})$ . A solution consists of  $\mathbf{V}, \mathbf{J} \in C_d(X; \mathbb{R})$  such that*

$$\mathbf{V} = R\mathbf{J}, \quad (\text{Ohm's law}) \quad (3.1)$$

$$\partial\mathbf{J} = p, \quad (\text{current law}) \quad (3.2)$$

$$\langle \mathbf{V}, z \rangle = \langle q, z \rangle, \quad \text{for all } z \in Z_d(X). \quad (\text{voltage law}) \quad (3.3)$$

A solution to the network problem always exists and is unique. In the current notation, the modified inner product of Eq. (1.7) takes the form

$$\langle b, b' \rangle_R = \langle Rb, b' \rangle,$$

for  $b, b' \in X_d$ . Let

$$\partial_R^* : C_{d-1}(X; \mathbb{R}) \rightarrow C_d(X; \mathbb{R})$$

denote the formal adjoint to  $\partial$  using the standard inner product on  $C_{d-1}(X; \mathbb{R})$  and the modified inner product on  $C_d(X; \mathbb{R})$ . Let  $Z_d(X; \mathbb{R})^{\perp R}$  be the image of  $\partial_R^*$  and note that  $Z_d(X; \mathbb{R})^{\perp R}$  is the orthogonal complement to  $Z_d(X; \mathbb{R})$  in  $C_d(X; \mathbb{R})$  with respect to the modified inner product. By restricting the co-domain and factoring over the kernel, the map  $\partial : Z_d(X; \mathbb{R})^{\perp R} \rightarrow B_{d-1}(X; \mathbb{R})$  is an isomorphism. Hence, there is a unique  $\mathbf{J}_0 \in Z_d(X; \mathbb{R})^{\perp R}$  such that  $\partial \mathbf{J}_0 = p$ . Set  $\mathbf{V}_0 = R\mathbf{J}_0$ . Then  $\langle \mathbf{V}_0, z \rangle = \langle \mathbf{J}_0, z \rangle_R = 0$  for all  $z \in Z_d(X; \mathbb{R})$ . Let  $\mathbf{J}_1$  be the orthogonal projection of  $R^{-1}q$  onto  $Z_d(X; \mathbb{R})$  in the modified inner product, and set  $x = \mathbf{J}_1 - R^{-1}q$ . Then  $\langle R\mathbf{J}_1 - q, z \rangle = \langle x, z \rangle_R = 0$  for all  $z \in Z_d(X; \mathbb{R})$ . Set  $\mathbf{V}_1 = R\mathbf{J}_1$ . Then  $\mathbf{J} := \mathbf{J}_0 + \mathbf{J}_1$  and  $\mathbf{V} := \mathbf{V}_0 + \mathbf{V}_1$  solve the network problem.

If  $\mathbf{V}, \mathbf{J}$  and  $\mathbf{V}', \mathbf{J}'$  are two solutions to the network problem, then the current law implies that  $\mathbf{J} - \mathbf{J}'$  must be a cycle. Since the inner product on  $d$ -cycles is non-degenerate, the voltage law implies  $\mathbf{V} = \mathbf{V}'$ . Thus,  $\mathbf{J} = \mathbf{J}'$  since  $R$  is an invertible transformation, and solutions to the network problem are unique.

**Remark 3.2.** *The notation  $R = e^{\beta W}$  is suggestive. We may think of  $r_b = e^{\beta W_b}$  as the resistance across a  $d$ -cell  $b$ . In Proposition 5.9, we show the current  $\mathbf{J}$  can be written in the form  $\mathbf{J} = K(p)$ , where  $K : B_{d-1}(X; \mathbb{R}) \rightarrow C_d(X; \mathbb{R})$  is a linear operator equivalent to the orthogonal projection used above. Hence, we may think of  $K(p)$  as the current in  $X$  flowing*

through the  $d$ -cells of  $\alpha \in C_d(X; \mathbb{R})$  such that  $\partial\alpha = p$ . In this sense,  $r_\alpha K(p)_\alpha$  is the voltage across those same cells.

The classical case in which  $X$  is a graph gives the familiar picture from physics. Take  $p = j - i$  to be a difference of vertices and  $q = 0$ . Then  $\alpha$  is a sum of edges connecting  $i$  to  $j$  and  $K(j - i)$  is the current flowing through the edges (or wires) in  $\alpha$ . If we evaluate  $\partial K(j - i)$  at some intermediate vertex  $\ell$ , the voltage law implies

$$(\partial K(i - j))_\ell = \delta_{j\ell} - \delta_{i\ell}.$$

This is equivalent to the statement that at any node, the sum of current flowing into that node is equal to the sum of currents flowing out of that node. Since  $q = 0$ , the voltage law implies

$$\sum_{\gamma \in Z_1(X; \mathbb{R})} r_\gamma K(i - j)_\gamma = 0,$$

so that the sum of voltage differences (or drops) over any loop is zero.

The network problem for graphs has been studied in various contexts. For example, Roth [30] calls  $p$  a *node current* and  $q$  a *mesh voltage*, each arising from an external source. Bollobás [7, p. 41] only considers the case when  $q = 0$  and  $p$  is of the form  $p_i i + p_j j$  for a pair of distinct vertices  $i$  and  $j$ .

**Remark 3.3.** *The solution to Kirchhoff's network problem given above relies on the orthogonal projection  $C_d(X; \mathbb{R}) \rightarrow Z_d(X; \mathbb{R})$  under the modified inner product  $\langle -, - \rangle_W$  on  $C_d(X; \mathbb{R})$ .*

*This is equivalent to constructing a retract of  $i$ , i.e., splitting, the short exact sequence*

$$0 \longrightarrow Z_d(X; \mathbb{R}) \xrightarrow{i} C_d(X; \mathbb{R}) \xrightarrow{\partial} B_{d-1}(X; \mathbb{R}) \longrightarrow 0, \quad (3.4)$$

*with respect to the modified inner product. Explicitly, we mean a linear transformation*

*$A : C_d(X; \mathbb{R}) \rightarrow Z_d(X; \mathbb{R})$  such that  $A \circ i = \text{id}_{Z_d(X; \mathbb{R})}$  and  $\ker A \perp_W \text{im } i$ .*

In the case of a graph, Kirchhoff expressed the orthogonal projection as a weighted sum over spanning trees of  $X$  [17, 18]. We emulate this in higher dimensions.

## 3.2 Spanning Trees

**Definition 3.4.** A spanning tree for  $X$  (of dimension  $d$ ) is a subcomplex  $i_T : T \subset X$  provided

1.  $H_d(T; \mathbb{Z}) = 0$ ,
2.  $i_{T*} : H_{d-1}(T; \mathbb{Q}) \cong H_{d-1}(X; \mathbb{Q})$ , and
3.  $X^{(d-1)} \subset T$ .

A spanning tree is always of dimension  $d$ , unless otherwise stated.

**Remark 3.5.** Our definition of higher spanning tree is motivated by our ideas for current generation in higher dimensions. It should be noted that there is a variety of other requirements one could impose. The interested reader should refer to [25] for a detailed comparison of the different notions. Our definition agrees with that of [15]. In the case of a graph, our definition reduces to the classical notion of spanning tree.

Using these higher dimensional analogues of spanning trees, we shall construct the aforementioned orthogonal splitting. We first prove a sequence of lemmas on their existence and formal properties. We associate a linear transformation to each spanning tree, and after taking an appropriate linear combination, obtain the splitting.

**Definition 3.6.** A  $k$ -cell  $b \in X_k$  is essential if there exists a  $k$ -cycle  $z \in Z_k(X; \mathbb{R})$  such that  $\langle z, b \rangle \neq 0$ .

**Lemma 3.7.** Adding or removing an essential  $d$ -cell from  $X$  increases or decreases  $\beta_d(X)$  by one, respectively, and fixes  $\beta_{d-1}(X)$ .



*Proof.* Let  $Y \subset X$  be the result of removing a  $d$ -cell from  $X$ . Then we have an exact sequence in homology

$$0 \rightarrow H_d(Y) \rightarrow H_d(X) \xrightarrow{p} \mathbb{Z} \rightarrow H_{d-1}(Y) \rightarrow H_{d-1}(X) \rightarrow 0.$$

The above factors into two short exact sequences

$$0 \rightarrow H_d(Y) \rightarrow H_d(X) \rightarrow \text{im } p \rightarrow 0$$

$$0 \rightarrow \mathbb{Z}/\text{im } p \rightarrow H_{d-1}(Y) \rightarrow H_{d-1}(X) \rightarrow 0,$$

where  $\text{im } p$  is the image of  $p$ . If the attached cell is essential, then  $\text{im } p$  is a nontrivial subgroup of  $\mathbb{Z}$ . Therefore, the first sequence yields  $\beta_d(X) = \beta_d(Y) + 1$ , while the second implies  $\beta_{d-1}(Y) = \beta_{d-1}(X)$ .  $\square$

**Lemma 3.8.**  *$X$  has a spanning tree.*

*Proof.* If  $H_d(X; \mathbb{R}) = 0$ , then  $X$  is a spanning tree. If  $H_d(X; \mathbb{R}) \neq 0$ , then we can pick an essential  $d$ -cell and remove it, decreasing  $\beta_d(X)$  by one. Repeat this process until  $\beta_d$  is zero. Clearly the resulting subcomplex  $T$  contains  $X_{d-1}$ , and by Lemma 3.7, we have  $\beta_{d-1}(T) = \beta_{d-1}(X)$ . Hence,  $T$  is a spanning tree.  $\square$

**Corollary 3.9.** *Any spanning tree for  $X$  may be obtained by removing essential  $d$ -cells.*

*Furthermore, if  $T$  is a spanning tree of  $X$ , the number of essential  $d$ -cells withdrawn to construct  $T$  is equal to  $\beta_d(X)$ .*

*Proof.* Both statements follow immediately from Lemma 3.7 and from the fact that  $X^{(d-1)} \subset T$ , for any spanning tree  $T$ .  $\square$

**Lemma 3.10.** *Let  $T$  be a spanning tree of  $X$  and let  $\tilde{T} = T \cup b$ , where  $b$  is an essential cell in  $\tilde{T}$ . If  $b'$  is an essential  $d$ -cell of  $\tilde{T}$  different from  $b$ , then  $U := \tilde{T} \setminus b'$  is a spanning tree.*

*Proof.* Since  $b'$  is essential, Lemma 3.7 implies  $\beta_d(U) = 0$ . The same lemma also implies  $\beta_{d-1}(U) = \beta_{d-1}(\tilde{T}) = \beta_{d-1}(T)$ . Since our construction leaves the  $d - 1$  skeleton fixed, we conclude that  $U$  is a spanning tree.  $\square$

If  $b$  is a  $d$ -cell not in  $T$ , as in Lemma 3.10, then  $H_d(T \cup b; \mathbb{Z}) = Z_d(T \cup b; \mathbb{Z})$  is infinite cyclic. For any generator  $c$ , let  $t_b = \langle c, b \rangle$ . We use the inclusion  $Z_d(T \cup b; \mathbb{Z}) \subset C_d(X; \mathbb{Z})$  to define the inner product.

**Lemma 3.11.** *Let  $T$  be a spanning tree of  $X$  and let  $b \in X_d \setminus T_d$  be an essential  $d$ -cell. Then  $[\partial b] \in H_{d-1}(T; \mathbb{Z})$  is torsion of order  $|t_b|$ . In particular, there is a short exact sequence*

$$0 \rightarrow \mathbb{Z}/t_b\mathbb{Z} \rightarrow H_{d-1}(T; \mathbb{Z}) \rightarrow H_{d-1}(T \cup b; \mathbb{Z}) \rightarrow 0.$$

*Proof.* Since  $b$  is a  $d$ -cell not in  $T$ , the attaching map for  $b$  factors through  $T$ . Hence, the homology class  $[\partial b]$  lies in  $H_{d-1}(T; \mathbb{Z})$ . The isomorphism  $H_{d-1}(T; \mathbb{Q}) \cong H_{d-1}(X; \mathbb{Q})$ , along with the fact that  $\partial b$  bounds the cellular chain  $b$  in  $X$ , implies  $\partial b$  is torsion in  $H_{d-1}(T; \mathbb{Z})$ ; let  $t$  be its order.

By abuse of notation, we denote the cycle representing  $[\partial b]$  by  $\partial b$ . Then  $t\partial b$  is a cycle, which is also the boundary of a unique integral  $d$ -chain  $w \in C_d(T; \mathbb{Z})$ . It is straightforward to check that  $tb - w$  is a generator of  $H_d(T \cup b; \mathbb{Z}) = Z_d(T \cup b; \mathbb{Z})$ . Since  $\langle tb - w, b \rangle = t$ , we know  $t = \pm t_b$ , and the displayed sequence follows immediately.  $\square$

**Definition 3.12.** *For a spanning tree  $T$  of  $X$ , define a linear transformation*

$$\bar{T} : C_d(X; \mathbb{Q}) \rightarrow Z_d(X; \mathbb{Q}),$$

*on the  $d$ -cells of  $X$  and extend linearly. For a  $d$ -cell  $b$ , set*

$$\bar{T}(b) = \begin{cases} \frac{c}{t_b} & b \notin T, \text{ where } c \text{ is any generator for } H_d(T \cup b; \mathbb{Z}) \cong \mathbb{Z}, \\ 0 & b \in T. \end{cases}$$

The transformation of Definition 3.12 is well-defined since any two generators for  $H_d(T \cup b; \mathbb{Z})$  differ by a unit, and  $\langle c, b \rangle \neq 0$  for any spanning tree  $T$ .

**Lemma 3.13.** *Let  $T$  be a spanning tree of  $X$  and let  $b_i$  and  $b_j$  be essential  $d$ -cells such that  $b_i \in X_d \setminus T_d$  and  $b_j \in T_d$ . Let  $U := T \cup b_i \setminus b_j$ . Then*

$$\langle \overline{T}(b_i), b_j \rangle \langle b_i, \overline{U}(b_j) \rangle = 1.$$

*Proof.* We have  $T \cup b_i = U \cup b_j$ , so we may choose a common generator  $c$  for  $H_d(T \cup b_i) \cong H_d(U \cup b_j)$ . Let  $t_i = \langle c, b_i \rangle$  and  $t_j = \langle c, b_j \rangle$ , so that

$$\langle \overline{T}b_i, b_j \rangle \langle b_i, \overline{U}b_j \rangle = \frac{1}{t_i} \langle c, b_j \rangle \frac{1}{t_j} \langle b_i, c \rangle = 1. \quad \square$$

Recall that for a finite CW complex of  $N$  of dimension  $d$ ,  $\theta_N$  denotes the order of the torsion subgroup of  $H_{d-1}(N; \mathbb{Z})$ .

**Corollary 3.14.** *For  $T, U, b_i$ , and  $b_j$  as above,*

$$\theta_T^2 \langle \overline{T}(b_i), b_j \rangle = \theta_U^2 \langle b_i, \overline{U}(b_j) \rangle.$$

*Proof.* Set  $t_i := t_{b_i}$  and let  $Y = T \cup b_i = U \cup b_j$ . Then the exact sequence of Lemma 3.11

$$0 \rightarrow \mathbb{Z}/t_i\mathbb{Z} \rightarrow H_{d-1}(T; \mathbb{Z}) \rightarrow H_{d-1}(Y; \mathbb{Z}) \rightarrow 0$$

gives  $|t_i|\theta_Y = \theta_T$  and by symmetry  $|t_j|\theta_Y = \theta_U$ . Consequently,

$$\theta_T^2 \langle \overline{T}(b_i), b_j \rangle = \theta_Y^2 t_i t_j = \theta_U^2 \langle b_i, \overline{U}(b_j) \rangle. \quad \square$$

### 3.3 The Higher Network Theorem

Our proof is analogous to that of [22] and proceeds along the same steps as in [9]. The key difference from the classical network theorem lies in the fact that orders of torsion subgroups

must be introduced in higher dimensions. This is a natural generalization of the situation on graphs, where the relevant torsion subgroups are all trivial. For a spanning tree  $T$ , let  $\{e_1, e_2, \dots, e_k\}$  be elements of  $X_d \setminus T_d$ .

**Lemma 3.15.** *The collection  $\bar{T}(e_1), \dots, \bar{T}(e_k)$  forms a basis for  $Z_d(X; \mathbb{Q})$ .*

*Proof.* Since  $X$  is  $d$ -dimensional, we have  $Z_d(X; \mathbb{Q}) = H_d(X; \mathbb{Q})$ . If  $q : X \rightarrow X/T$  denotes the quotient map, then the homomorphism  $q_* : H_d(X; \mathbb{Q}) \rightarrow H_d(X/T; \mathbb{Q})$  is an isomorphism. The set  $\{e_1, e_2, \dots, e_k\}$  provides a basis for  $H_d(X/T; \mathbb{Q})$ . A simple computation shows that  $q_* \circ \bar{T} : C_d(X; \mathbb{Q}) \rightarrow H_d(X/T; \mathbb{Q})$  is given by

$$q_* \circ \bar{T}(e) = \begin{cases} e & \text{if } e \in X_d \setminus T_d, \\ 0 & \text{else.} \end{cases}$$

Hence,  $\bar{T}(e_1), \dots, \bar{T}(e_k)$  is a basis for  $H_d(X; \mathbb{Q})$ .  $\square$

**Corollary 3.16.** *For any  $z \in Z_d(X; \mathbb{R})$ , we have  $\bar{T}(z) = z$ .*

*Proof.* Lemma 3.15 permits us to write  $z = \sum_i s_i \bar{T}(e_i)$ . Then

$$\bar{T}(z) = \sum_i s_i \bar{T}(\bar{T}(e_i)) = \sum_i s_i \bar{T}(e_i) = z. \quad \square$$

**Definition 3.17.** *Recall the order of the torsion subgroup of  $H_{d-1}(T; \mathbb{Z})$  is denoted  $\theta_T$ .*

*Define the weight of  $T$  to be the positive real number*

$$w_T := \theta_T^2 \prod_{b \in T_d} e^{-\beta W_b}.$$

**Lemma 3.18.** *For distinct  $d$ -cells  $e_i, e_j \in X_d$ , let  $\mathcal{T}_{ij}$  be the set of all spanning trees such that  $\langle \bar{T}(b_i), b_j \rangle \neq 0$ . The operation that sends a tree  $T \in \mathcal{T}_{ij}$  to  $U := T \cup b_i \setminus b_j \in \mathcal{T}_{ji}$  is a bijection. Furthermore,*

$$\sum_{T \in \mathcal{T}_{ij}} w_T \langle \bar{T}(b_i), b_j \rangle_W = \sum_{U \in \mathcal{T}_{ji}} w_U \langle b_i, \bar{U}(b_j) \rangle_W.$$

*Proof.* The bijection claim is evident from Lemma 3.10. From the definition of the weights, we have  $e^{\beta W_j} w_T = e^{\beta W_i} w_U \frac{\theta_T^2}{\theta_U^2}$ . Note that  $\langle \bar{T}(b_i), b_j \rangle_W = e^{\beta W_j} \langle \bar{T}(b_i), b_j \rangle$ . Using Corollary 3.14, we infer

$$w_T \langle \bar{T}(b_i), b_j \rangle_W = w_U \langle b_i, \bar{U}(b_j) \rangle_W.$$

Now sum up over all  $T \in \mathcal{T}_{ij}$ . □

**Theorem 3.19** (Higher Projection Formula). *With respect to the modified inner product  $\langle -, - \rangle_W$ , the orthogonal projection  $C_d(X; \mathbb{R}) \rightarrow Z_d(X; \mathbb{R})$  is given by*

$$A = \frac{1}{\Delta} \sum_T w_T \bar{T}, \quad (3.5)$$

where the sum is over all spanning trees, and  $\Delta = \sum_T w_T$ .

*Proof.* Before commencing with the proof, recall the following fact from linear algebra. Suppose  $V$  is a real inner product space and  $U \subset V$  is a subspace. If  $G : V \rightarrow V$  is a self-adjoint operator such that  $G|_U = \text{id}_U$  and  $\text{im } G \subset U$ , then  $G$  is the orthogonal projection onto  $U$  [29].

Consider the operator  $F : C_d(X; \mathbb{R}) \rightarrow Z_d(X; \mathbb{R})$  given by  $F := \sum_T w_T \bar{T}$ , where the sum is indexed over all spanning trees of  $X$ . For any pair of  $d$ -cells  $b_i$  and  $b_j$  of  $X$ , we have

$$\begin{aligned} \left\langle \sum_T w_T \bar{T}(b_i), b_j \right\rangle_W &= \sum_{T \in \mathcal{T}_{ij}} \langle \bar{T}(b_i), b_j \rangle_W, \\ &= \sum_{U \in \mathcal{T}_{ji}} w_U \langle b_i, \bar{U}(b_j) \rangle_W && \text{by Lemma 3.18,} \\ &= \langle b_i, \sum_U w_U \bar{U}(b_j) \rangle_W, \\ &= \langle b_i, \sum_T w_T \bar{T}(b_j) \rangle_W, \end{aligned}$$

where we have used the fact that  $\langle \overline{T}(b_i), b_j \rangle \neq 0$  if and only if  $\langle b_i, \overline{T}(b_j) \rangle \neq 0$ . Therefore  $F$  is self-adjoint with respect to the modified inner product. For  $z \in Z_d(X; \mathbb{R})$ , we have  $F(z) = (\sum_T w_T)z = \Delta z$ . Consequently,  $(1/\Delta)F$  restricts to the identity on  $Z_d(X; \mathbb{R})$ . Lemma 3.15 implies that  $\text{im } F \subset Z_d(X; \mathbb{R})$ , and hence  $(1/\Delta)F$  is the orthogonal projection in the modified inner product.  $\square$

As described in Remark 3.3, the orthogonal projection given in Theorem 3.19 can be interpreted as a retract of  $i$  in

$$0 \longrightarrow Z_d(X; \mathbb{R}) \xrightarrow{i} C_d(X; \mathbb{R}) \xrightarrow{-\partial} B_{d-1}(X; \mathbb{R}) \longrightarrow 0.$$

This gives rise to a section  $K : B_{d-1}(X; \mathbb{R}) \rightarrow C_d(X; \mathbb{R})$  of  $-\partial$  by

$$\begin{aligned} K(b) &= iA(\alpha) - \alpha, \\ &= \frac{1}{\Delta} \sum_T w_T \overline{T}\alpha - \alpha, \quad (\text{omitting } i) \\ &= \frac{1}{\Delta} \sum_T w_T (\overline{T}\alpha - \alpha), \end{aligned} \tag{3.6}$$

for any  $\alpha \in C_d(X; \mathbb{R})$  such that  $-\partial\alpha = b$ . Define  $K^T : B_{d-1}(X; \mathbb{Q}) \rightarrow C_d(T; \mathbb{Q})$  by

$$K_b^T := K^T(b) = \overline{T}\alpha - \alpha, \tag{3.7}$$

with  $\partial\alpha = b$ . The operator  $K^T$  is well-defined since the difference of any two  $d$ -chains with boundary  $b$  would give rise to a  $d$ -cycle on  $T$ , of which there are none. This also allows us to write

$$K = \frac{1}{\Delta} \sum_T w_T K^T, \tag{3.8}$$

and the preceding discussion shows this operator is well-defined.

**Proposition 3.20.** *For any  $b = -\partial\alpha \in B_{d-1}(X; \mathbb{Q})$ ,  $K_b^T$  is the unique  $d$ -chain in  $T$  so that  $-\partial K_b^T = b$ .*

*Proof.* Suppose that  $b$  lifts to a single cell  $\alpha \in X_d$ . The general case of  $b$  lifting to a superposition of cells follows from the linearity of  $\bar{T}$  and  $K$ . If  $\alpha \in T$ , so that  $\bar{T}\alpha = 0$ , then  $K_b^T = -\alpha$  and the statement is true. If  $\alpha \notin T$ , then

$$K_b^T = \frac{c}{\langle c, \alpha \rangle} - \alpha,$$

where  $c$  is any generator for  $H_d(T \cup \alpha; \mathbb{Z})$ . Corollary 3.16 implies

$$\bar{T} \left( \frac{c}{\langle c, \alpha \rangle} - \alpha \right) = \frac{\bar{T}c}{\langle c, \alpha \rangle} - \bar{T}\alpha = \frac{c}{\langle c, \alpha \rangle} - \frac{c}{\langle c, \alpha \rangle} = 0,$$

so that  $\frac{c}{\langle c, \alpha \rangle} - \alpha \in C_d(T; \mathbb{Q})$ . Since  $c$  is a cycle,  $\partial(\frac{c}{\langle c, \alpha \rangle} - \alpha) = b$ . The tree  $T$  has no integral cycles, and so  $K_b^T$  is the unique such  $d$ -chain in  $T$ .  $\square$

In particular, every rational  $(d-1)$ -boundary of  $X$  is contained in every spanning tree. This is analogous to the statement for graphs that, for any spanning tree and any two vertices, there is a unique path in the tree connecting them. For a finite, connected graph, the boundaries are spanned by the difference of vertices, and the analogy is exact.

**Remark 3.21.** *Proposition 3.20 is true under weaker hypotheses on the coefficients. This follows from Proposition 5.12.*

Although equivalent, the retract  $K$  rather than the section  $A$  of Theorem 3.19 is more useful in the calculation of average current density (see Proposition 5.9 and Lemma 5.16).

The higher projection formula gives rise to a simple proof of the higher network theorem. Recall the notation  $R = e^{\beta W}$ .

**Theorem 3.22** (Higher Network Theorem). *Given a vector  $\mathbf{V} \in C_d(X; \mathbb{R})$ , there is a unique*

$z \in Z_d(X; \mathbb{R})$  such that  $\mathbf{V} - Rz \in Z_d(X; \mathbb{R})^\perp$ . Furthermore, for each  $d$ -cell  $b$ , we have

$$\langle z, b \rangle = \frac{1}{\Delta} \sum_T \frac{w_T}{r_b} \langle \mathbf{V}, \bar{T}(b) \rangle.$$

*Proof.* Let  $z$  be the orthogonal projection of  $R^{-1}\mathbf{V}$  onto  $Z_d(X; \mathbb{R})$  with respect to the modified inner product. Then  $R^{-1}\mathbf{V} - z \in Z_d(X; \mathbb{R})^{\perp R}$ . For any  $w \in Z_d(X; \mathbb{R})$  we have

$$\langle R^{-1}\mathbf{V} - z, w \rangle_R = \langle \mathbf{V} - Rz, w \rangle = 0,$$

so that  $\mathbf{V} - Rz \in Z_d(X; \mathbb{R})^\perp$ . The cycle  $z$  is unique since the orthogonal decomposition is determined by an inner product.

For the displayed formula, we use the projection formula given in Theorem 3.19. For any  $d$ -cell  $b$ , we have

$$\begin{aligned} \langle z, b \rangle &= \frac{1}{r_b} \langle z, b \rangle_R, \\ &= \frac{1}{r_b} \left\langle \frac{1}{\Delta} \sum_T w_T \bar{T}(R^{-1}\mathbf{V}, b) \right\rangle_R, \\ &= \frac{1}{\Delta} \sum_T \frac{w_T}{r_b} \langle R^{-1}\mathbf{V}, \bar{T}(b) \rangle_R, \\ &= \frac{1}{\Delta} \sum_T \frac{w_T}{r_b} \langle \mathbf{V}, \bar{T}(b) \rangle. \end{aligned} \quad \square$$

### 3.4 The Higher Matrix-Tree Theorem

In this section, we derive a higher dimensional version of Kirchhoff's matrix-tree theorem. In the case of a graph, Kirchhoff proved that the determinant of the graph Laplacian can be used to count the number of spanning trees. In higher dimensions, we instead achieve a count of  $\sum_T \theta_T^2$ . We first prove the matrix-tree theorem up to a constant in Theorem 3.23, and later generalize it in a substantial way with Theorem 3.29.



Consider the operator

$$\partial e^{-\beta W} \partial^* : C_{d-1}(X; \mathbb{R}) \rightarrow C_{d-1}(X; \mathbb{R}).$$

It is clear that the image of this operator is contained in  $B_{d-1}(X; \mathbb{R})$ . If we restrict the domain to  $B_{d-1}(X; \mathbb{R})$ , then we obtain an isomorphism

$$\mathcal{L}(W) = \partial e^{-\beta W} \partial^*|_{B_{d-1}(X; \mathbb{R})} : B_{d-1}(X; \mathbb{R}) \xrightarrow{\cong} B_{d-1}(X; \mathbb{R}). \quad (3.9)$$

For notational convenience, we will often omit  $W$ , so that  $\mathcal{L} = \mathcal{L}(W)$ . The crux of proving the higher matrix tree theorem lies in the following theorem.

**Theorem 3.23.** *Theorem 3.19 implies the identity*

$$d \ln \det \mathcal{L} = d \ln \sum_T w_T.$$

*Proof.* First, note that the statement is equivalent to

$$\det \mathcal{L} = \gamma \sum_T w_T, \quad (3.10)$$

where  $\det \mathcal{L}$  and  $\sum_T w_T$  are functions of  $W$  (we omit the dependence), and  $\gamma$  is a constant independent of  $W$ , as yet to be determined. Second,  $\mathcal{L}$  is diagonalizable with positive eigenvalues, so  $\ln \mathcal{L}$  is defined.

We take the differential of the natural logarithm of  $\det \mathcal{L}$ :

$$\begin{aligned} d \ln \det \mathcal{L} &= d \operatorname{tr} \ln \mathcal{L} \\ &= \operatorname{tr} d(\ln \mathcal{L}) \\ &= \operatorname{tr}(\mathcal{L}^{-1} d\mathcal{L}), \end{aligned} \quad (3.11)$$

where  $d\mathcal{L} = \partial d e^{-W} \partial^* = -\partial d W e^{-W} \partial^*$ . The cyclic property of the trace implies

$$\operatorname{tr}(\mathcal{L}^{-1} d\mathcal{L}) = -\operatorname{tr}(\partial d W e^{-W} \partial^* \mathcal{L}^{-1}). \quad (3.12)$$

If we set  $\mathcal{K} := e^{-W} \partial^* \mathcal{L}^{-1} : B_{d-1}(X; \mathbb{R}) \rightarrow Z_d^{\perp R}(X; \mathbb{R})$ , then  $\text{tr}(\mathcal{L}^{-1} d\mathcal{L}) = -\text{tr}(\partial dW \mathcal{K}) = -\text{tr}(dW \mathcal{K} \partial)$ . Consequently,

$$\begin{aligned}
d\text{tr} \ln \mathcal{L} &= -\text{tr}(dW \mathcal{K} \partial) \\
&= - \sum_{b \in X_d} \langle b | dW \mathcal{K} \partial | b \rangle \\
&= - \sum_{b \in X_d} \langle b | dW \mathcal{K} | \partial b \rangle \\
&= - \sum_{b \in X_d} dW_b \langle b | \mathcal{K} | \partial b \rangle,
\end{aligned} \tag{3.13}$$

where  $dW_b$  denotes the  $b$ -coordinate function of  $dW$ , i.e.,  $dW_b(x) = dW(x)(b) = x_b$ , and  $\langle i | H | j \rangle$  stands for the inner product  $\langle i, H(j) \rangle$ .

By definition,  $\mathcal{K}$  is a left inverse to  $\partial : Z_d(X; \mathbb{R})^{\perp R} \rightarrow B_{d-1}(X; \mathbb{R})$ , so the expression  $\langle b | \mathcal{K} | \partial b \rangle$  is the same as  $\langle b, Pb \rangle$ , where  $P : C_d(X; \mathbb{R}) \rightarrow Z_d(X; \mathbb{R})^{\perp R}$  is the orthogonal projection in the modified inner product  $\langle -, - \rangle_W$ . By Theorem 3.19, we have

$$P = I - \frac{1}{\Delta} \sum_T w_T \bar{T}, \tag{3.14}$$

where  $I$  is the identity operator. By inserting this expression into  $\langle b, Pb \rangle$ , we obtain

$$\begin{aligned}
\langle b | \mathcal{K} | \partial b \rangle &= 1 - \frac{1}{\Delta} \langle b, \sum_T w_T \bar{T}(b) \rangle \\
&= 1 - \frac{1}{\Delta} \sum_{T, b \notin T} w_T \\
&= \frac{1}{\Delta} \sum_{T, b \in T} w_T
\end{aligned} \tag{3.15}$$

where  $\Delta = \sum_T w_T$  and the displayed sums run over trees  $T$  for which  $b$  does not, and does lie in  $T$ , respectively. This allows us to rewrite the expression appearing in the last line of

Eq. (3.13) as

$$\sum_{b \in X_d} dW_b \langle b | \mathcal{K} | \partial b \rangle = \frac{1}{\Delta} \sum_T \sum_{b \in T_d} w_T dW_b. \quad (3.16)$$

On the other hand, we have

$$d \ln \sum_T w_T = \frac{1}{\Delta} \sum_T dw_T, \quad (3.17)$$

where  $dw_T$  is given by

$$dw_T = \theta_T^2 d \prod_{b \in T_d} e^{-W_b} = - \sum_{b \in T_d} dW_b w_T. \quad (3.18)$$

Inserting Eq. (3.18) into Eq. (3.17) gives

$$d \ln \sum_T w_T = \frac{-1}{\Delta} \sum_T \sum_{b \in T_d} w_T dW_b. \quad (3.19)$$

Assembling equations (3.11), (3.13), (3.16), and (3.19), we conclude

$$d \ln \det \mathcal{L} = - \frac{1}{\Delta} \sum_T \sum_{b \in T_d} w_T dW_b = d \ln \sum_T w_T. \quad \square$$

### 3.5 Covolume

We now work to identify the prefactor  $\gamma$  appearing in Eq. (3.10). In fact, we generalize this greatly in Theorem 3.29, from which Corollary 3.32 follows directly. We first discuss the geometric notion of covolume, which we will use to state many of our results.

If  $A$  is a finitely generated abelian group, let

$$A_{\mathbb{R}} := A \otimes_{\mathbb{Z}} \mathbb{R}$$

denote its *realification*, and we let  $\beta(A) = \dim_{\mathbb{R}} A_{\mathbb{R}}$  denote the rank of  $A$ . Let  $t(A)$  be the order of the torsion subgroup of  $A$ . For a homomorphism  $\alpha : A \rightarrow B$  of abelian groups, we denote the induced homomorphism of real vector spaces by  $\alpha_{\mathbb{R}} : A_{\mathbb{R}} \rightarrow B_{\mathbb{R}}$ .

**Definition 3.24.** A homomorphism  $\alpha : A \rightarrow B$  of finitely generated abelian groups is called a real isomorphism if the induced homomorphism  $\alpha_{\mathbb{R}} : A_{\mathbb{R}} \rightarrow B_{\mathbb{R}}$  of real vector spaces is an isomorphism.

Clearly,  $\alpha$  is a real isomorphism if and only if both its kernel and its cokernel are finite. If  $\alpha$  is a real isomorphism, then  $\beta(A) = \beta(B)$ . For the remainder of this section, we will assume that  $A$  and  $B$  are free abelian. In this case,  $\alpha$  is a real isomorphism if and only if  $\alpha$  is a monomorphism with finite cokernel.

**Definition 3.25.** For  $\alpha : A \rightarrow B$  a real isomorphism with  $A$  and  $B$  free abelian, we let

$$t(\alpha) \in \mathbb{N}$$

denote the order of the cokernel of  $\alpha$ , i.e.,  $t(\alpha) := t(B/\alpha(A))$ .

**Proposition 3.26.** For a real isomorphism  $\alpha : A \rightarrow B$  of finitely generated free abelian groups we have  $|\det \alpha| = t(\alpha)$ .

*Proof.* By an appropriate choice of bases for  $A$  and  $B$ ,  $\alpha$  can be represented by a diagonal matrix. In this case, the claim is evident.  $\square$

An ordered basis for  $A$  determines an ordered basis for  $A_{\mathbb{R}}$ , and given any pair of ordered bases for  $A$ , the associated change of basis matrix for  $A_{\mathbb{R}}$  has determinant  $\pm 1$ . This defines an equivalence relation on ordered bases for  $A$  with exactly two distinct equivalence classes. A choice of equivalence class is referred to as an *orientation* of  $A$ . Consequently, when orientations for  $A$  and  $B$  are chosen, and  $\alpha : A \rightarrow B$  is a real isomorphism, then the determinant  $\det \alpha \in \mathbb{R}$  is defined and depends only on the choice of orientations. Furthermore, its absolute value  $|\det \alpha|$  is well defined and does not depend on the choice of orientations. The latter has the following interpretation: choose an ordered basis for  $B$ . This defines an inner

product on  $B_{\mathbb{R}}$  making the ordered basis for  $B$  into an orthonormal basis for  $B_{\mathbb{R}}$ . Then  $\alpha(A) \subset B_{\mathbb{R}}$  is a lattice and  $|\det \alpha|$  is its *covolume*, that is, the volume of the torus  $B_{\mathbb{R}}/\alpha(A)$  with respect to the induced Riemannian metric, or equivalently, the volume of a fundamental domain of the universal covering  $B_{\mathbb{R}} \rightarrow B_{\mathbb{R}}/\alpha(A)$ .

Recall the operator of Eq. (3.9)

$$\mathcal{L} = \mathcal{L}(W) = \partial e^{-W} \partial^* : B_{d-1}(X; \mathbb{R}) \xrightarrow{\cong} B_{d-1}(X; \mathbb{R}),$$

As we showed earlier in Proposition 3.23, we have the following representation:

$$\det \mathcal{L} = \gamma \sum_T w_T,$$

where the constant  $\gamma$  is still to be determined.

The main case of interest in the following definition is  $A = B_{d-1}(X; \mathbb{Z})$ . As pointed out in Remark 3.30, different choices of  $A$  give other versions of the higher matrix-tree theorem found in the literature [25].

**Definition 3.27.** *Let  $A \subset C_{d-1}(X; \mathbb{Z})$  be a subgroup. Define a natural number*

$$\mu(A) \in \mathbb{N}$$

*as follows: let  $\{e_i\}$  be a basis for  $A$ . Consider the matrix  $g$  whose  $(i, j)$ -entry is given by  $g_{ij} = \langle e_i, e_j \rangle$ , where we use the standard inner product on  $C_{d-1}(X; \mathbb{R})$ . Set  $\mu(A) := \det g$ .*

Since  $e_i$  expressed in the standard basis for  $C_{d-1}(X; \mathbb{R})$  has integer components, we infer that  $g_{ij} \in \mathbb{Z}$ , so  $\mu(A)$  is an integer. Alternatively, one can define  $\mu(A)$  as the *square* of the covolume of the lattice  $A \subset A_{\mathbb{R}}$  given by restricting the standard inner product of  $C_{d-1}(X; \mathbb{R})$  to  $A_{\mathbb{R}}$ . The equivalence of the two definitions can be seen as follows: let  $B$  be the matrix

whose rows are the vectors  $e_i$  expressed in an orthonormal basis for  $C_{d-1}$ . Then  $|\det B|$  is the covolume of  $A \subset A_{\mathbb{R}}$ . Furthermore,  $g = BB^*$ , so  $\mu(A) = \det g = (\det B)^2 \in \mathbb{N}$ .

For any abelian group  $U$ , set

$$B_{d-1}^U := B_{d-1}(X; U),$$

that is, the image of the boundary operator  $\partial : C_d(X; U) \rightarrow C_{d-1}(X; U)$  of the cellular chain complex of  $X$  with  $U$  coefficients. The following hypothesis will be assumed from now on, and holds for the main case of interest  $A = B_{d-1}(X; \mathbb{Z})$ .

**Assumption 3.28.** *The inclusion  $A \subset C_{d-1}(X; \mathbb{R})$  is such that the orthogonal projection  $P_A : B_{d-1}^{\mathbb{R}} \rightarrow A_{\mathbb{R}}$  is induced by a real isomorphism  $p_A : B_{d-1}^{\mathbb{Z}} \rightarrow A$ , i.e.,  $P_A = (p_A)_{\mathbb{R}}$ .*

Consider the composite operator

$$\mathcal{L}_A : A_{\mathbb{R}} \xrightarrow{\cong} A_{\mathbb{R}}$$

defined by  $\mathcal{L}_A = P_A \partial e^{-W} \partial^*|_{A_{\mathbb{R}}}$ .

**Theorem 3.29** (Generalized Higher Weighted Matrix-Tree Theorem). *We have*

$$\det \mathcal{L}_A = \gamma_A \sum_T w_T, \tag{3.20}$$

where the prefactor is given by

$$\gamma_A = \frac{\mu(A)t(p_A)^2}{\theta_X^2}. \tag{3.21}$$

**Remark 3.30.** *If  $A = A_S$  is the free abelian group generated by a judiciously chosen subset  $S \subset X_{d-1}$ , we will obtain  $\mu(A_S) = 1$ . Using this choice of  $A$  as well as  $W = 0$ , Theorem 3.29 gives a generalization of the main result of [25] to CW complexes.*

Before giving the proof, we recall a basic fact from linear algebra [29].

**Lemma 3.31.** *Let  $V$  and  $W$  be free abelian groups with inner products, and a choice of basis  $\{e_i\}$  and  $\{f_j\}$ , respectively. Write  $g^V$  for the matrix  $\langle e_i, e_j \rangle$  of inner products and similarly for  $g^W$ . If  $f : V \rightarrow W$  is a homomorphism and  $A$  is the matrix for  $f$  with respect to the bases  $\{e_i\}$  and  $\{f_j\}$ , then the matrix for  $f^*$  has the form  $(g^V)^{-1} A^T g^W$ .*

*Proof of Theorem 3.29.* As above, we have

$$\mathcal{L} := \partial \partial_W^* = \partial e^{-W} \partial^* : B_{d-1}(X; \mathbb{R}) \xrightarrow{\cong} B_{d-1}(X; \mathbb{R}).$$

Then

$$\mathcal{L}_A = P_A \partial e^{-W} \partial^* |_{A_{\mathbb{R}}} = P_A \mathcal{L} P_A^*,$$

where  $P_A^*$  denotes the adjoint of  $P_A$  with respect to the standard inner product on  $C_{d-1}(X; \mathbb{R})$ .

Therefore,

$$\det \mathcal{L}_A = \det(\mathcal{L}) \det(P_A P_A^*). \quad (3.22)$$

If we apply this to Eq. (3.10), we reproduce Eq. (3.20) with  $\gamma_A = \gamma \det(P_A P_A^*)$ . It suffices to identify the prefactor  $\gamma_A$ .

Consider the operator  $\mathcal{L}^T = \partial_T e^{-W} \partial_T^*$  for some spanning tree  $T$ . We have

$$\det \mathcal{L}^T = \det(\partial_T e^{-W} \partial_T^*) = \det(\partial_T \partial_T^* e^{-W}) = \frac{w_T}{\theta_T^2} \det(\partial_T \partial_T^*). \quad (3.23)$$

We can similarly restrict  $\mathcal{L}_A$  to a spanning tree  $T$ . The equation analogous to Eq. (3.22) for  $\mathcal{L}_A^T$ , combined with Eq. (3.23), gives

$$\det(\partial_T \partial_T^*) \det(P_A^T (P_A^T)^*) = \gamma_A \theta_T^2. \quad (3.24)$$

The real isomorphism  $p_A^T : B_{d-1}(T; \mathbb{Z}) \rightarrow A$  is obtained by composing the real isomorphism  $p_A : B_{d-1}(X; \mathbb{Z}) \rightarrow A$  with the inclusion  $B_{d-1}(T; \mathbb{Z}) \subset B_{d-1}(X; \mathbb{Z})$ . Lemma 3.31 gives

$$\det(P_A^T (P_A^T)^*) = \mu(A) (\mu(B_{d-1}(T; \mathbb{Z})))^{-1} (\det p_A^T)^2,$$

since  $P_A^T = (p_A^T)_{\mathbb{R}}$ . The free abelian group  $B_{d-1}(T; \mathbb{Z})$  has basis  $\{\partial e_1, \dots, \partial e_s\}$ , where  $e_1, \dots, e_s$  are the  $d$ -cells of  $T$ . The inner product matrix  $g$  has matrix elements  $g_{ij} = \langle \partial_T e_i, \partial_T e_j \rangle = \langle \partial_T^* \partial_T e_i, e_j \rangle$ , implying  $\mu(B_{d-1}(T; \mathbb{Z})) = \det(\partial_T^* \partial_T)$ . Then Eq. (3.24) assumes the form

$$\mu(A)(\det p_A^T)^2 = \gamma_A \theta_T^2.$$

Combining this with Proposition 3.26 results in

$$\gamma_A = \frac{\mu(A)t(p_A^T)^2}{\theta_T^2}. \quad (3.25)$$

The right side of Eq. (3.25) is written in terms of a particular spanning tree  $T$ , however, it does not actually depend on this choice. An invariant expression that does not contain  $T$  is obtained by using the following relations:

$$\frac{t(p_A^T)}{t(p_A)} = t(B_{d-1}(X; \mathbb{Z})/B_{d-1}(T; \mathbb{Z})) = \frac{\theta_T}{\theta_X}. \quad (3.26)$$

Substituting Eq. (3.26) into Eq. (3.25) results in an invariant expression for  $\gamma_A$ , given by Eq. (3.21). □

The choice  $A = B_{d-1}(X; \mathbb{Z})$  in Theorem 3.29 gives the following generalization of the classical matrix tree theorem. Set  $\mu_X = \mu(B_{d-1}(X; \mathbb{Z}))$  to be the square of the covolume of the lattice  $B_{d-1}(X; \mathbb{Z}) \subset B_{d-1}(X; \mathbb{R})$  with respect to the standard inner product on  $C_{d-1}(X; \mathbb{R})$ .

**Theorem 3.32** (Higher Weighted Matrix-Tree Theorem). *We have*

$$\det \mathcal{L} = \gamma_X \sum_T w_T,$$

where the sum is indexed over all spanning trees of  $X$ , and the prefactor is given by

$$\gamma_X = \frac{\mu_X}{\theta_X^2}.$$



The unweighted case  $W \equiv 0$  is worth highlighting, giving rise to the isomorphism

$$\mathcal{L} = \partial\partial^* : B_{d-1}(X; \mathbb{R}) \xrightarrow{\cong} B_{d-1}(X; \mathbb{R}).$$

**Corollary 3.33** (Higher Matrix-Tree Theorem). *For  $\mathcal{L}$  as above, we have*

$$\det \mathcal{L} = \gamma_X \sum_T \theta_T^2.$$

When  $X$  is a graph, we have  $\theta_T = 1 = \theta_X$ , and  $\mu_X$  is the number of vertices of  $X$ . This is the classical Kirchhoff matrix-tree theorem.

**Theorem 3.34.** *With  $A \subset C_{d-1}(X; \mathbb{Z})$  as in Assumption 3.28, we have*

$$\det \mathcal{L}_A = \sum_T \det \mathcal{L}_A^T.$$

*Proof.* Using Eq. (3.26) we infer that

$$\gamma_A = \frac{\mu(A)t(P_A)^2}{\theta_X^2} = \frac{\mu(A)t(P_A^T)^2}{\theta_T^2}$$

for any spanning tree  $T$ . Combining this with Theorem 3.29 in the case of a spanning tree  $T$  we obtain

$$\det \mathcal{L}_A^T = \gamma_A w_T.$$

The conclusion now follows by summing over all  $T$ . □

In the special case when  $A = B_{d-1}(X; \mathbb{Z})$ , Theorem 3.34 reduces to the generalization of the matrix-theorem in higher dimensions.

**Corollary 3.35.**  $\det \mathcal{L} = \sum_{\mathcal{T}} \det \mathcal{L}^{\mathcal{T}}.$

## CHAPTER 4 COMBINATORIAL HODGE THEORY

In this chapter, we define spanning co-trees and explore their combinatorial properties. Spanning co-trees should be thought of as generalizations of vertices of a graph. As its name suggests, it is homologically dual to spanning trees in the sense of rational homology. These objects arise naturally when attempting to solve the Hodge problem for CW complexes. Our method of proof is an application of the theory of generalized inverses. We note that this chapter significantly follows the published work in [8].

### 4.1 The Combinatorial Hodge problem

**Definition 4.1.** *Given  $x \in H_{d-1}(X; \mathbb{R})$ , the combinatorial Hodge problem for  $x$  is to find an explicit formula for the unique cycle  $\rho \in Z_{d-1}(X; \mathbb{R})$  such that*

- $\rho$  represents  $x$ , and
- $\rho$  is co-closed with respect to the modified inner product, i.e.,  $\partial_E^* \rho = 0$ .

The original formulation of the Hodge problem was for a compact, orientable Riemannian manifold. Given any real cohomology class, it asks to find a unique harmonic representative within that class. We state and prove the analogous result for the homology of a finite CW complex. Our explicit solution is given as a weighted sum over spanning co-trees, subcomplexes of dimension  $d - 1$ , similar to the solution of the Kirchhoff problem.

The original Hodge problem is with respect to the unmodified inner product. The general formulation given above reduces to the classical case by taking  $E \equiv 0$ . The condition that  $\rho$  be co-closed in the modified inner product is equivalent to the statement that  $\rho$  should be

orthogonal to any boundary with respect to the modified inner product:

$$\langle \alpha, \partial_E^* \rho \rangle = \langle \partial \alpha, \rho \rangle_E = 0.$$

Just as for the Kirchhoff problem, a solution to the combinatorial Hodge problem is equivalent to finding an orthogonal splitting, in this case of the quotient homomorphism  $p$  in the short exact sequence

$$0 \longrightarrow B_{d-1}(X; \mathbb{R}) \longrightarrow Z_{d-1}(X; \mathbb{R}) \xrightarrow{p} H_{d-1}(X; \mathbb{R}) \longrightarrow 0,$$

with respect to the modified inner product on  $Z_{d-1}(X; \mathbb{R}) \subset C_{d-1}(X; \mathbb{R})$ .

**Definition 4.2.** *A spanning co-tree for  $X$  (of dimension  $d-1$ ) is a subcomplex  $L \subset X$  such that*

1. *The inclusion  $j_L : L \subset X$  induces an isomorphism*

$$j_{L*} : H_{d-1}(L; \mathbb{Q}) \xrightarrow{\cong} H_{d-1}(X; \mathbb{Q});$$

2.  $\beta_{d-2}(L) = \beta_{d-2}(X)$ ;

3.  $X^{(d-2)} \subset L \subset X^{(d-1)}$ .

*Spanning co-trees will always be of dimension  $(d-1)$  unless otherwise stated.*

The long exact sequence of the pair  $(X, L)$  in homology implies that the first condition is equivalent to  $H_{d-1}(X, L; \mathbb{Q}) \cong 0$ .

**Lemma 4.3.**  *$X$  has a spanning co-tree.*

*Proof.* The homomorphism  $H_{d-1}(X^{(d-1)}; \mathbb{Q}) \rightarrow H_{d-1}(X; \mathbb{Q})$  is surjective with kernel  $K_1 := B_{d-1}(X; \mathbb{Q})$ . Set  $Y_1 := X^{(d-1)}$ . Suppose that  $c \in B_{d-1}(X; \mathbb{Q})$  is nontrivial. Let  $b$  be a

$(d-1)$ -cell of  $X$  such that  $\langle b, c \rangle \neq 0$ . Let  $Y_2$  be the result of removing  $b$  from  $X^{(d-1)}$ . The homomorphism  $H_{d-1}(Y_2; \mathbb{Q}) \rightarrow H_{d-1}(X; \mathbb{Q})$  is surjective; let  $K_2$  be its kernel. Then the rank of  $K_2$  is strictly less than that of  $K_1$  by Lemma 3.7. Furthermore,  $\beta_{d-2}(Y_2) = \beta_{d-2}(X)$ . By iterating (replacing  $Y_1$  with  $Y_2$ , and so forth) we obtain a subcomplex  $Y_k \subset X^{(d-1)}$  such that  $H_{d-1}(Y_k; \mathbb{Q}) \rightarrow H_{d-1}(X; \mathbb{Q})$  is an isomorphism. Since  $\beta_{d-2}(Y_k) = \beta_{d-2}(X)$ ,  $Y_k$  is a spanning co-tree.  $\square$

**Proposition 4.4.** *Let  $\mathbb{F}$  be a field of characteristic zero. Let  $L \subset X$  be a  $(d-1)$ -dimensional subcomplex that contains  $X^{(d-2)}$ . Then  $L$  is a spanning co-tree if and only if the composition*

$$C_{d-1}(L; \mathbb{F}) \rightarrow C_{d-1}(X; \mathbb{F}) \rightarrow C_{d-1}(X)/B_{d-1}(X; \mathbb{F}) \quad (4.1)$$

*is an isomorphism.*

*Proof.* We prove the proposition for  $\mathbb{F} = \mathbb{Q}$ , from which the general case follows by change of scalars. Suppose  $L$  is a subcomplex such that (4.1) is an isomorphism. Consider the following commutative diagram:

$$\begin{array}{ccccc} Z_{d-1}(L; \mathbb{Q}) & \xrightarrow{j_L} & Z_{d-1}(X; \mathbb{Q}) & \xrightarrow{p} & H_{d-1}(X; \mathbb{Q}) \\ \downarrow k & & \downarrow & & \downarrow b \\ C_{d-1}(L; \mathbb{Q}) & \xrightarrow{a} & C_{d-1}(X; \mathbb{Q}) & \xrightarrow{\pi} & C_{d-1}(X; \mathbb{Q})/B_{d-1}(X; \mathbb{Q}) \end{array}$$

By assumption, the bottom composite is Eq. (4.1) and an isomorphism. Since the left square is a pullback and the right square is a pushout, the top composite is also an isomorphism. Therefore,  $j_{L*} : H_{d-1}(L; \mathbb{Q}) \rightarrow H_{d-1}(X; \mathbb{Q})$  is an isomorphism. Since  $X^{(d-2)} \subset L$ , the remaining two conditions of Definition 4.2 follow trivially. Consequently,  $L$  is a spanning co-tree.

To prove the converse, we first show the composition is injective. Take  $x \in C_{d-1}(L; \mathbb{Q})$  be such that  $(\pi \circ a)(x) = 0$ . Then  $a(x) \in B_{d-1}(X; \mathbb{Q}) \subset Z_{d-1}(X; \mathbb{Q})$ . Since the left square

is a pullback, we infer that  $x \in Z_{d-1}(L; \mathbb{Q})$ . But  $p \circ i_L$  is an isomorphism, and  $b$  is injective, so  $x = 0$ . This establishes the injectivity of (4.1).

As for surjectivity, let  $z \in C_{d-1}(X; \mathbb{Q})/B_{d-1}(X; \mathbb{Q})$ . Lift this to an element  $y \in C_{d-1}(X; \mathbb{Q})$ . Then  $\partial(y) \in C_{d-2}(L; \mathbb{Q}) = C_{d-2}(X; \mathbb{Q})$  lies in  $Z_{d-2}(L; \mathbb{Q})$  since  $\partial^2 = 0$ . The pushforward of the homology class  $[\partial(y)] \in H_{d-2}(L; \mathbb{Q})$  in  $H_{d-2}(X; \mathbb{Q})$  is trivial, since  $H_{d-2}(L; \mathbb{Q}) \cong H_{d-2}(X; \mathbb{Q})$ . It follows that  $\partial(y)$  lies in  $B_{d-2}(X; \mathbb{Q}) = B_{d-2}(L; \mathbb{Q})$ . Hence,  $\partial(y) = \partial(x)$  for some  $x \in C_{d-1}(L; \mathbb{Q})$ . Then  $a(x) - y$  lies in  $Z_{d-1}(X; \mathbb{Q})$ , and since  $L$  is a spanning co-tree, there exists  $x' \in Z_{d-1}(L; \mathbb{Q})$  so that  $\pi(a(x) - y) = (b \circ p \circ j_L)(x')$ . But  $z = \pi(y)$ , so

$$z = \pi(y) = \pi(a(x)) - b(p(j_L(x'))) = \pi(a(x - k(x'))).$$

We conclude that (4.1) is surjective.  $\square$

**Lemma 4.5.** *Let  $\mathbb{F}$  be a field. A splitting of the quotient homomorphism  $C_{d-1}(X; \mathbb{F}) \rightarrow C_{d-1}(X; \mathbb{F})/B_{d-1}(X; \mathbb{F})$  restricts to a splitting of the quotient homomorphism  $Z_{d-1}(X; \mathbb{F}) \rightarrow H_{d-1}(X; \mathbb{F})$ .*

*Proof.* Consider the following commutative diagram, with exact rows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B_{d-1}(X; \mathbb{F}) & \longrightarrow & Z_{d-1}(X; \mathbb{F}) & \xrightarrow{p} & H_{d-1}(X; \mathbb{F}) & \longrightarrow & 0 \\ & & \parallel & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B_{d-1}(X; \mathbb{F}) & \longrightarrow & C_{d-1}(X; \mathbb{F}) & \xrightarrow{\pi} & C_{d-1}(X; \mathbb{F})/B_{d-1}(X; \mathbb{F}) & \longrightarrow & 0. \end{array}$$

Since  $H_{d-1}(X; \mathbb{F}) \subset C_{d-1}(X; \mathbb{F})/B_{d-1}(X; \mathbb{F})$ , we can restrict the given splitting to get a map  $H_{d-1}(X; \mathbb{F}) \rightarrow C_{d-1}(X; \mathbb{F})$ . A simple diagram chase shows that this map factors through  $Z_{d-1}(X; \mathbb{F})$ .  $\square$

## 4.2 The Boltzmann Distribution

Spanning co-trees are equipped with auxiliary data that will be used to obtain the desired splitting. Observe that the projection  $Z_{d-1}(L; \mathbb{Z}) \rightarrow H_{d-1}(L; \mathbb{Z})$  is an isomorphism since  $L$  has no  $d$ -cells. Let  $\phi_L$  be the composite

$$\phi_L : Z_{d-1}(L; \mathbb{Z}) \xrightarrow{=} H_{d-1}(L; \mathbb{Z}) \xrightarrow{j_{L*}} H_{d-1}(X; \mathbb{Z}).$$

Then  $\phi_L$  becomes an isomorphism after tensoring with the rational numbers by the defining properties of  $L$ . Hence, its cokernel  $\text{cok } \phi_L$  is finite; let  $c_L := |\text{cok } \phi_L|$  be its cardinality. We define the *weight* of  $L$  to be the real number

$$\tau_L = c_L^2 \prod_{b \in L_{d-1}} e^{-\beta E_b}.$$

We tensor with  $\mathbb{Q}$  to invert  $\phi_L$  and obtain

$$\psi_L : H_{d-1}(X; \mathbb{Q}) \xrightarrow{(\phi_L \otimes \mathbb{Q})^{-1}} Z_{d-1}(L; \mathbb{Q}) \xrightarrow{j_L} Z_{d-1}(X; \mathbb{Q}),$$

where by slight notational abuse, we have used  $i_L$  to denote the homomorphism induced by the map with the same name. Using these data, we can state the solution to the combinatorial Hodge problem.

**Theorem 4.6.** *The solution to the combinatorial Hodge problem is given by  $\rho = \Psi(x)$ , in which  $\Psi : H_{d-1}(X; \mathbb{R}) \rightarrow Z_{d-1}(X; \mathbb{R})$  is the homomorphism*

$$\Psi = \frac{1}{\Lambda} \sum_L \tau_L \psi_L,$$

where the sum runs over all spanning co-trees  $L$  and  $\Lambda = \sum_L \tau_L$ .

**Definition 4.7.** Let  $x \in H_{d-1}(X; \mathbb{Z})$  be an integer homology class. The higher Boltzmann distribution at  $x$  is the real  $(d-1)$ -cycle

$$\rho^B := \frac{1}{\Lambda} \sum_L \tau_L \psi_L(\bar{x}) \in Z_{d-1}(X; \mathbb{R}), \quad (4.2)$$

where  $\bar{x} \in H_{d-1}(X; \mathbb{Q})$  is the image of  $x$  with respect to the homomorphism  $H_{d-1}(X; \mathbb{Z}) \rightarrow H_{d-1}(X; \mathbb{Q})$ .

**Remark 4.8.** Consider the special case when  $X$  is a simple graph, i.e. a graph with no self loops. The spanning co-trees of  $X$  are given by the vertices. Choosing a vertex of  $X$  gives rise to a canonical generator for  $H_0(X; \mathbb{Z}) \cong \mathbb{Z}$ . If  $L = j$  is a vertex, then  $\phi_L$  is an integral isomorphism so that  $|c_L| = 1$ . The weight of  $L$  is  $\tau_L = e^{-\beta E_j}$ . Then  $\psi_L(\bar{x}) = j$  and

$$\rho^B = \frac{\sum_j e^{-\beta E_j} j}{\sum_j e^{-\beta E_j}}$$

coincides with the classical Boltzmann distribution. This is the sense in which Definition 4.7 generalizes the Boltzmann distribution and acquires its name [28].

### 4.3 Generalized Inverses

Our proof of Theorem 4.6 relies on the theory of generalized inverses. Generalized inverses were developed to study linear systems  $Ax = b$  for which  $A^{-1}$  does not exist.

Let  $A$  be an  $m \times n$  matrix over  $\mathbb{R}$ , and let  $b \in \mathbb{R}^m$  be given. Consider the linear system  $Ax = b$ . In general, such systems need not have a (unique) solution. One way to study the system is to attempt to minimize the norm of the residual vector  $Ax - b$ . Among all such  $x$  for which the norm of  $Ax - b$  is minimizing, we impose the additional constraint that the norm of  $x$  is minimizing. This is called a *least squares problem*. Note that this description is slightly more general than the usual formulation. The classical least squares problem

assumes that  $A$  is injective. We will be primarily concerned here with the case when  $A$  is surjective.

**Remark 4.9.** *When  $A$  is surjective, the residual vector having minimum norm is the zero vector. In this case the least squares problem reduces to the problem of finding a solution of  $Ax = b$  such that the norm of  $x$  is minimized.*

The Moore-Penrose pseudoinverse  $A^+$  gives a preferred solution to the least squares problem. If  $b \in \text{im}(A)$ , then a solution to  $Ax = b$  exists and the Moore-Penrose solution  $x = A^+b$  will be a solution with the smallest norm. Furthermore, the matrix  $A^+$  exists and is unique [24], [4, p. 109].

The operation of sending a transformation to its Moore-Penrose pseudoinverse,  $A \mapsto A^+$ , satisfies the identities

$$A^+ = A^t(AA^t)^+ = (A^tA)^+A^t, \quad (4.3)$$

where  $A^t$  is the transpose of  $A$  (cf. [4, chap. 1.6, ex. 18(d)]). In particular, when  $A$  is surjective, we obtain the formula

$$A^+ = A^t(AA^t)^{-1}. \quad (4.4)$$

**Remark 4.10.** *If  $A$  is surjective, then one may drop the requirement that the target of  $A$  is based. That is, suppose more generally that  $A : \mathbb{R}^n \rightarrow W$  is a surjective linear transformation where  $W$  is not necessarily based. Then the least squares problem as well as the formula (4.4) make sense if we use the formal adjoint  $A^* : W^* \rightarrow (\mathbb{R}^n)^* = \mathbb{R}^n$  in place of the transpose.*

We will need a weighted version of the least squares problem. For this, we weight the standard basis elements  $\{e_i\}_{i=1}^n$  of  $\mathbb{R}^n$  by means of a positive functional  $\mu : \{e_i\}_{i=1}^n \rightarrow \mathbb{R}_+$ . Then  $\mu$  defines a modified inner product  $\langle -, - \rangle_\mu$  on  $\mathbb{R}^n$ , determined by  $\langle e_i, e_j \rangle_\mu := \mu(e_i)\delta_{ij}$



(compare  $\mu$  and  $\langle -, - \rangle_\mu$  with  $E$  and  $\langle -, - \rangle_E$  from Eq. (1.7)). The *weighted least squares problem* is to minimize  $|Ax - b|$  such that  $|x|_\mu$  is also minimized. Again, the solution  $x = A^+b$  exists and is unique, where now  $A^+$  is the weighted version of the Moore-Penrose pseudoinverse.

For what follows, assume that  $A$  has rank  $m$ , i.e.,  $A$  is surjective. Let  $A_S$  be the submatrix whose rows correspond to indices in the set  $S \subset \{1, 2, \dots, m\}$ :

$$[A_S]_{ij} := [A]_{ij}, \quad \text{for } i = 1, \dots, m, j \in S.$$

We restrict our attention to those  $S$  such that  $A_S$  is invertible. Let  $i_S : \mathbb{R}^m \rightarrow \mathbb{R}^n$  denote the inclusion given by the rows corresponding to  $S$ . Set

$$t_S := \det(A_S)^2 \prod_{i \in S} \frac{1}{\mu(e_i)}$$

and set  $\Lambda := \sum_S t_S$ . We can now state the summation formula for  $A^+$  in the case of surjective  $A$ .

**Theorem 4.11** (cf. [6, Theorem 1], [5, Theorem 2.1]). *Let  $A$  be an  $m \times n$  matrix of rank  $m$  defined over  $\mathbb{R}$ . Then the weighted Moore-Penrose pseudoinverse of  $A$  is given by*

$$A^+ = \frac{1}{\Lambda} \sum_S t_S i_S (A_S)^{-1},$$

where the sum is taken over all indices  $S \subset \{1, 2, \dots, n\}$  such that  $A_S$  is invertible.

**Remark 4.12.** *The splitting only uses the weighted basis for  $\mathbb{R}^n$  and not the basis for  $\mathbb{R}^m$ . Hence, Theorem 4.11 holds whenever  $A : \mathbb{R}^n \rightarrow W$  is a surjective linear transformation in which  $W$  is not necessarily based (cf. Remark 4.10).*

*Proof of Theorem 4.6.* By Lemma 4.5, it suffices to produce a splitting of the quotient homomorphism  $\pi : C_{d-1}(X; \mathbb{R}) \rightarrow C_{d-1}(X; \mathbb{R})/B_{d-1}(X; \mathbb{R})$ . Here we use the weighted basis of

$C_{d-1}(X; \mathbb{R})$  defined by the cells and the weighting given by  $b \mapsto e^{\beta E_b}$ . Applying Theorem 4.11 and Remark 4.12 to  $\pi$  gives a splitting, written as a sum over subsets  $S$  of the basis elements of  $C_{d-1}(X; \mathbb{R})$ . By Proposition 4.4, the collection of these subsets are in bijection with the set of spanning co-trees. The inclusion  $i_S$  corresponds to the inclusion  $C_{d-1}(L; \mathbb{R}) \rightarrow C_{d-1}(X; \mathbb{R})$  and  $\phi_L$  corresponds to  $A_S$ . Since  $\phi_L$  is a real isomorphism, it is straightforward to verify that  $\det(\phi_L) = |\text{cok}(\phi_L)|$ , and the result follows.  $\square$

**Remark 4.13.** *If we fix a weighting  $X_d \rightarrow \mathbb{R}$ , we may instead apply [6, Theorem 1] to the inclusion map  $i : Z_d(X; \mathbb{R}) \rightarrow C_d(X; \mathbb{R})$ . This produces an orthogonal splitting  $C_d(X; \mathbb{R}) \rightarrow Z_d(X; \mathbb{R})$  to  $i$  in the modified inner product on  $C_d(X; \mathbb{R})$ . The splitting is written as a sum indexed over the set of spanning trees as in Theorem 3.19. In fact, this gives quick alternative proofs to Theorem 3.19 and Theorem 3.22.*

## CHAPTER 5 QUANTIZATION

We now turn to the actual observable of the process we are interested in. We first prove the adiabatic theorem, and then analyze the operators of the previous chapters in the low-temperature limit. All of this is combined to prove the main quantization result of Theorem 5.17.

### 5.1 The Adiabatic Theorem

The adiabatic theorem states that, for slow enough driving, a periodic solution to the dynamical equation exists and is unique. It should be noted that this adiabatic theorem is for the current density and average current density on the finite CW complex, and not for the actual process on the cycle-incidence graph. There is no stationary distribution for the actual process on the cycle-incidence graph, since the process will blow up in finite time (compare to the appendix of [14, Appendix]). Our proof is similar to that of [14], but modified appropriately to the higher dimensional setting.

**Theorem 5.1.** *Let  $(\tau_D, \gamma)$  be a periodic driving protocol and fix  $\hat{x} \in Z_{d-1}(X; \mathbb{Z})$ . There exists  $\tau_0 = \tau_0(\beta, \gamma)$  such that for all  $\tau_D > \tau_0$ , a periodic solution  $\rho = \rho(\beta, \tau_D, \gamma)(t)$  of the dynamical equation for  $\hat{x}$  exists and is unique. Furthermore,*

$$\lim_{\tau_D \rightarrow \infty} \rho(t) = \rho^B(\gamma(t)),$$

where  $\rho^B$  is the Boltzmann distribution at  $\hat{x}$ .

**Remark 5.2.** *It should be noted that for the remainder of this dissertation, we have fixed once and for all  $\hat{x} \in Z_{d-1}(X; \mathbb{Z})$ , and therefore a class  $[\hat{x}] \in H_{d-1}(X; \mathbb{Z})$  as well.*

The dynamical equation is a first order differential equation, and by specifying the initial

condition  $\hat{x}$ , we have the existence of a unique solution [1]. We introduce the *time-ordered exponential*  $U(t, t_0)$  for  $0 \leq t_0 \leq t \leq 1$ , which gives the unique solution to the initial value problem

$$\frac{d}{dt}U(t, t_0) = \tau_D \mathcal{H}(t)U(t, t_0) \quad U(t_0, t_0) = I.$$

Explicitly,

$$U(t, t_0) = \lim_{N \rightarrow \infty} e^{\varepsilon \tau_D \mathcal{H}(t_N)} e^{\varepsilon \tau_D \mathcal{H}(t_{N-1})} \dots e^{\varepsilon \tau_D \mathcal{H}(t_0)},$$

where  $\varepsilon = t/N$  and  $t_j = j\varepsilon$ . It is easy to verify that the formal solution

$$\rho(t) = U(t, 0)\rho(0) = \left( \lim_{N \rightarrow \infty} e^{\varepsilon \tau_D \mathcal{H}(t_N)} e^{\varepsilon \tau_D \mathcal{H}(t_{N-1})} \dots e^{\varepsilon \tau_D \mathcal{H}(t_0)} \right) \rho(0)$$

solves the dynamical equation Eq. (2.6) for  $\rho(0) = \hat{x}$ . The time-ordered exponential is often denoted

$$\hat{T} \exp \left( \tau_D \int_{t_0}^t \mathcal{H}(\tau) d\tau, \right)$$

in analogy with the solution to a one-dimensional differential equation.

If  $A$  is an operator on a real inner product space  $V$ , define

$$\|A\| = \sup_{v \neq 0} \frac{|Av|}{|v|} = \sup_{|v|=1} |Av|$$

to be the standard operator norm.

**Proposition 5.3.** *Let  $(\tau_D, \gamma)$  be a periodic driving protocol. There exist positive constants  $\lambda$  and  $c$  so that for all  $t < t_0 \in [0, 1]$ ,*

$$\|U(t, t_0)\| < ce^{-\lambda \tau_D(t-t_0)}.$$

*Proof.* Let  $\tilde{\mathcal{H}}(t)$  denote the restriction of  $\mathcal{H}(\gamma(t))$  to  $B_{d-1}(X; \mathbb{R})$ . The operator  $\tilde{\mathcal{H}}(t)$  is negative definite and self-adjoint with respect to the inner product  $\langle -, - \rangle_{E(t)}$  on  $B_{d-1}(X; \mathbb{R}) \subset$

$C_{d-1}(X; \mathbb{R})$ . Define  $\lambda = -\sup_{t \in [0,1]} \sigma(\tilde{\mathcal{H}}(t))$ , where  $\sigma(A)$  denotes the spectrum of an operator  $A$ . Then  $\lambda > 0$  and  $\tilde{\mathcal{H}}_0(t) := \tilde{\mathcal{H}}(t) + \lambda I$  is negative semi-definite. Let  $\tilde{U}(t, t_0)$  and  $\tilde{U}_0(t, t_0)$  denote the time-evolution operators for  $\tilde{\mathcal{H}}$  and  $\tilde{\mathcal{H}}_0$ , respectively. Then  $U(t, t_0) = e^{-\lambda \tau_D(t-t_0)} U_0(t, t_0)$ , and so

$$\|U(t, t_0)\| = e^{-\lambda \tau_D(t-t_0)} \|U_0(t, t_0)\|.$$

It remains to show that  $\|U_0(t, t_0)\|$  is uniformly bound.

Let  $\nu(t)$  be the formal solution to the dynamical equation  $\dot{\nu}(t) = \tau_D \tilde{\mathcal{H}}_0(t) \nu(t)$ . Then

$$\begin{aligned} \frac{d}{dt} \langle \nu(t), \nu(t) \rangle_{E(t)} &= \langle \nu(t), \nu(t) \rangle_{\dot{E}(t)} + 2\tau_D \langle \tilde{\mathcal{H}}_0(t) \nu(t), \nu(t) \rangle_{E(t)} \\ &\leq \langle \nu(t), \nu(t) \rangle_{\dot{E}(t)} \end{aligned}$$

since  $\tilde{\mathcal{H}}_0(t)$  is negative semi-definite. Here  $\langle -, - \rangle_{\dot{E}(t)}$  denotes the time-derivative of the time-dependent inner product. Compactness of  $[0, 1]$  implies both  $\langle \nu(t), \nu(t) \rangle_{E(t)}$  is bounded below and  $\langle \nu(t), \nu(t) \rangle_{\dot{E}(t)}$  is bounded above. Hence, there exists a positive constant  $A$  so that

$$\frac{\langle \nu(t), \nu(t) \rangle_{\dot{E}(t)}}{\langle \nu(t), \nu(t) \rangle_{E(t)}} < A.$$

Together with the previous inequality, we have the bound  $\frac{d}{dt} \ln(\langle \nu(t), \nu(t) \rangle_{E(t)}) < A$ . Integrating this quantity and plugging in the formal solution yields

$$\begin{aligned} \frac{\langle U_0(t, t_0) \nu(t_0), U(t, t_0) \nu(t_0) \rangle}{\langle \nu(t_0), \nu(t_0) \rangle} &\leq \frac{\langle U_0(t, t_0) \nu(t_0), U(t, t_0) \nu(t_0) \rangle_{E(t)}}{\langle \nu(t_0), \nu(t_0) \rangle_{E(t_0)}}, \\ &< e^{A(t-t_0)}, \\ &< c^2, \end{aligned}$$

for some  $c$ , since  $t < t_0$  and both lie in  $[0, 1]$ . Since the initial condition  $\nu(t_0)$  was arbitrary, we have  $\|U(t, t_0)\| < c$ . □

*Proof of Thm 5.1.* To simplify notation, write  $\rho^B(\gamma(t)) = \rho^B(t)$ . Let  $\rho(t)$  denote any solution to the dynamical equation Eq. (2.6). We can uniquely express  $\rho(t) = \rho^B(t) + \xi(t)$ , where  $\xi : [0, 1] \rightarrow B_{d-1}(X; \mathbb{R})$  is a path in the  $(d-1)$ -boundaries of  $X$ . The solution  $\rho(t)$  is periodic precisely when  $\xi(t)$  is periodic. Applying the dynamical operator to this, we can re-write the dynamical equation as

$$\dot{\xi}(t) = \tau_D \mathcal{H}(t) \xi(t) - \dot{\rho}^B(t). \quad (5.1)$$

The solution to the dynamical equation in Eq. (5.1) is given by

$$\xi(t) = U(t, 0) \xi(0) - \int_0^t U(t, t') \dot{\rho}^B(t') dt'. \quad (5.2)$$

Evaluating at  $t = 1$ , the requirement that  $\rho$  be periodic reads

$$(I - U(1, 0)) \xi(0) = - \int_0^1 U(1, t') \dot{\rho}^B dt',$$

where  $I$  is the identity operator. Take  $\tau_0 = \lambda^{-1} \ln(2c)$  as in Proposition 5.3, so that  $\|U(1, 0)\| < 1/2$  and thus  $I - U(1, 0)$  is invertible. Then

$$\xi(0) = - (I - U(1, 0))^{-1} \int_0^1 U(1, t') \dot{\rho}^B(t') dt', \quad (5.3)$$

implying that the solution  $\rho(t)$  exists and is unique.

As for the adiabatic limit, it suffices to show that  $\xi(t) \rightarrow 0$  as  $\tau_D \rightarrow \infty$ . From Eq. (5.2), we have

$$\begin{aligned} |\xi(t)| &\leq \|U(t, 0)\| |\xi(0)| + \int_0^t \|U(t, t')\| |\dot{\rho}^B(t')| dt', \\ &\leq \frac{1}{2} |\xi(0)| + cr \int_0^1 e^{-\lambda \tau_D (1-t)} dt, \\ &\leq \frac{1}{2} |\xi(0)| + \frac{cr}{\lambda \tau_D}, \end{aligned}$$

where  $r = \sup_{t \in [0,1]} |\dot{\rho}^B(t)|$ . To bound  $\xi(0)$ , we use similar bounds applied to Eq. (5.3):

$$\begin{aligned} |\xi(0)| &\leq r \|I - U(1,0)\| \int_0^1 \|U(1,t')\| dt' \\ &\leq \frac{2rc}{\lambda\tau_D}. \end{aligned}$$

We have used the fact that  $\|U(1,0)\| < 1/2$  implies  $\|I - U(1,0)\| < 2$ . Therefore, for any  $t \in [0, 1]$ , we have

$$|\xi(t)| \leq \frac{2rc}{\lambda\tau_D} \rightarrow 0 \text{ as } \tau_D \rightarrow \infty. \quad \square$$

## 5.2 The Low-Temperature Limit

**Definition 5.4.** *Let  $V$  be a real vector space with basis set  $B$ . A functional  $f : V \rightarrow \mathbb{R}$  is very non-degenerate if for all subsets  $S, T \subset B$  of the same cardinality, we have*

$$\sum_{s \in S} f(s) \neq \sum_{t \in T} f(t).$$

The very non-degenerate condition is clearly generic. The *energy* of a co-tree is given by the value of the functional

$$L \mapsto \sum_{a \in L_{d-1}} E_a.$$

If  $E$  is very non-degenerate, then this functional has a unique minimum for some co-tree  $L^m$ .

In this case, we say  $L^m$  is the *minimal co-tree*.

We remind the reader that we have fixed a class  $\hat{x} \in Z_{d-1}(X; \mathbb{Z})$ . The operator  $\rho^B$  will be taken to be the Boltzmann distribution at  $\hat{x}$ .

**Lemma 5.5.** *Let  $E(t)$  be a very non-degenerate function for all  $t$ . Then the low-temperature,  $\beta \rightarrow \infty$ , limit of  $\rho^B$  is supported on the minimal co-tree  $L^m$ :*

$$\lim_{\beta \rightarrow \infty} \rho^B(t) = \psi_{L^m},$$

and the convergence is uniform.

To ease the notation, we shall write  $b \in K$  to mean  $b \in K_{d-1}$  for the remainder of this section. The letter  $K$  will refer to a general spanning co-tree, so  $\sum_K$  refers to a sum over all spanning co-trees. The explicit time dependence of the rates is omitted.

*Proof.* Since the domain of  $\rho^B(t)$  is  $S^1$ , uniform convergence follows from pointwise convergence and Proposition 5.7. We proceed by studying the components of  $\rho^B$  individually.

Multiply the numerator and denominator of  $\rho_L^B$  by  $\exp(-\beta \sum_{a \in L^m} E_a)$  to obtain

$$\rho_L^B = \frac{c_L^2 \exp \left\{ -\beta \left( \sum_{b \in L} E_b - \sum_{a \in L^m} E_a \right) \right\} \psi_L}{\sum_K c_K^2 \exp \left\{ -\beta \left( \sum_{e \in K} E_e - \sum_{a \in L^m} E_a \right) \right\}}. \quad (5.4)$$

Since  $L^m$  is minimal, the numerator tends to zero for all  $L \neq L^m$ . When  $L = L^m$ , the sum vanishes and the numerator tends to  $c_{L^m}^2 \psi_{L^m}$ . The same argument is true for the sum in the denominator, in which case we have

$$\lim_{\beta \rightarrow \infty} \rho^B = \frac{c_{L^m}^2 \psi_{L^m}}{c_{L^m}^2} = \psi_{L^m}. \quad \square$$

If  $W$  is very non-degenerate, then just as for co-trees, the functional on the set of trees given by

$$T \mapsto \sum_{\alpha \in T} W_\alpha$$

has a unique minimum  $T^\mu$ .

**Lemma 5.6.** *Let  $W(t)$  be a very non-degenerate function for all  $t$ . Then the low-temperature,  $\beta \rightarrow \infty$  limit of  $A$  is supported on a single tree:*

$$\lim_{\beta \rightarrow \infty} A = \bar{T}^\mu,$$

where  $T^\mu$  is minimal tree and the convergence is uniform.



*Proof.* The proof is exactly the same as the proof of Lemma 5.5.  $\square$

**Proposition 5.7.** *Let  $L$  be a spanning-co-tree and let  $E(t)$  be very non-degenerate for all  $t$ . The  $L$ -component of the time-derivative of the Boltzmann distribution tends to 0 uniformly in the low-temperature limit.*

*Proof.* A straightforward computation of the time-derivative of Eq. (4.2) gives

$$\dot{\rho}_L^B = \frac{\beta c_L^2 \exp(-\beta \sum_{b \in L} E_b) \sum_K \left[ c_K^2 \exp(-\beta \sum_{a \in K} E_a) \left( \sum_{a \in K} \dot{E}_a - \sum_{b \in L} \dot{E}_b \right) \right]}{\left[ \sum_K c_K^2 \exp\left(-\beta \sum_{a \in K} E_a\right) \right]^2} \psi_L. \quad (5.5)$$

For convergence in the low-temperature limit, we only need to verify the statement pointwise since  $S^1$  is compact, and it suffices check the statement for each component  $\dot{\rho}_L^B$ . First, multiply the numerator and denominator of Eq. (5.5) by  $\exp\{-2 \sum_{b \in L} E_b\}$  to get

$$\dot{\rho}_L^B = \frac{\beta c_L^2 \left[ \sum_K c_K^2 \exp\left\{-\beta \left( \sum_{a \in K} E_a - \sum_{b \in L} E_b \right)\right\} \left( \sum_{a \in K} \dot{E}_a - \sum_{b \in L} \dot{E}_b \right) \right]}{\left[ \sum_K c_K^2 \exp\left\{-\beta \left( \sum_{a \in K} E_a - \sum_{b \in L} E_b \right)\right\} \right]^2} \quad (5.6)$$

There are two cases to consider: either  $L$  is the minimal co-tree or it is not.

If  $L$  is the minimal co-tree, so that  $\sum_{b \in L} E_b < \sum_{a \in K} E_a$  for every other co-tree  $K$ , then the denominator of Eq. (5.6) is given by

$$\left( c_L^2 + \sum_{K \neq L} c_K^2 \exp\left(-\beta \left( \sum_{a \in K} E_a - \sum_{b \in L} E_b \right)\right) \right)^2,$$

which tends to  $c_L^4 < \infty$  as  $\beta \rightarrow \infty$ . As for the numerator of Eq. (5.6), when  $L = K$ , we have  $\sum_{a \in K} \dot{E}_a = \sum_{b \in L} \dot{E}_b$  and the numerator is exactly zero. If  $L \neq K$ , then the exponential factor is negative and tends to zero as  $\beta \rightarrow \infty$ .

If  $L$  is not the minimal co-tree, then some other co-tree will be minimal. Therefore, at least one of the exponents  $-\beta(\sum E_a - \sum E_b)$  will be positive. Since the denominator is

squared, Eq. (5.5) is dominated by  $A\beta/e^{B\beta}$  for some constants  $A$  and  $B$  with  $B > 0$  for large  $\beta$ . It is easy to see this expression tends to zero as  $\beta \rightarrow \infty$ .  $\square$

### 5.3 Current Quantization

For the remainder of this chapter, take  $\tau_D$  large enough so that a unique periodic solution to Eq. (2.6) exists by Theorem 5.1. Furthermore, fix a cycle  $\hat{x} \in Z_{d-1}(X; \mathbb{Z})$  as in Chapter 2.

**Definition 5.8.** *For a periodic driving protocol  $(\tau_D, \gamma)$  and  $\beta > 0$ , the current density at  $t \in [0, 1]$  is defined to be*

$$\mathbf{J}(t) = \mathbf{J}(\beta, \tau_D, \gamma)(t) := \tau_D e^{-\beta W(t)} \partial^* e^{\beta E(t)} \rho(t) = \partial_{E,W}^* \rho(t) \in C_d(X; \mathbb{R}) \quad (5.7)$$

where  $\rho(t)$  is the unique periodic solution to the dynamical equation given by Theorem 5.1.

The time-averaged current density is

$$Q_{\tau_D, \beta}(\gamma) := Q(\beta, \tau_D, \gamma) := \int_0^1 \mathbf{J}(t) dt. \quad (5.8)$$

These definitions are motivated by traditional current in electrical networks. We can think of  $\partial^*$  as a discrete analog of the gradient operator. From this perspective, the current density is given by taking the biased gradient of the formal solution to the dynamical equation. The boundary operator  $\partial$  can then be thought of as the divergence operator. Then the continuity equation for the current density

$$\partial \mathbf{J}(t) = -\dot{\rho}$$

holds due to Eq. (2.6).

We also see from this definition that in the limit of slow-driving (though not necessarily adiabatic),  $Q_{\tau_D, \beta}(\gamma)$  defines a real  $d$ -dimensional homology class of  $X$ . Since  $\tau_D$  is large

enough, applying  $\partial$  to Eq. (5.8) gives

$$\begin{aligned}\partial Q_{\tau_D, \beta}(\gamma) &= \tau_D \int_0^1 \partial \partial_{E, W}^* \rho(t) dt \\ &= -\tau_D \int_0^1 \dot{\rho}(t) dt \\ &= 0,\end{aligned}$$

since  $\rho(t)$  is the periodic solution to the dynamical equation.

**Proposition 5.9.** *The current density  $\mathbf{J}(t)$  is equivalent to*

$$\mathbf{J}(t) = K(\gamma, \dot{\rho}(\gamma)), \quad (5.9)$$

where  $K$  is the operator of Equation (3.8) and  $\rho(t)$  is the unique periodic solution to Eq. (2.6).

*Proof.* The method of proof relies on identifying properties which uniquely characterize  $\mathbf{J}(t)$ .

Consider the set of all  $w(t) \in C_d(X; \mathbb{R}) \times [0, 1]$  for which

- (i)  $\partial w(t) = -\dot{\rho}(t)$ , and
- (ii)  $\langle w(t), z \rangle_{W(t)} = 0$  for all  $z \in Z_d(X; \mathbb{R})$ ,

for all  $t \in [0, 1]$ . First, it is easy to see that any  $w \in C_d(X; \mathbb{R})$  satisfying Conditions (i) and (ii) must be unique. Condition (i) implies that the difference of any two solutions must be a cycle, whereas Condition (ii) shows their difference is zero. We omit the explicit time dependence in what follows.

For the formula of  $\mathbf{J}$  given in Definition 5.8, Condition (i) is verified directly by Equa-

tion (2.6). Condition (ii) is a direct consequence of the definition:

$$\begin{aligned} \langle e^{-\beta W} \partial^* e^{\beta E} \rho, z \rangle_W &= \langle \partial^* e^{\beta E} \rho, z \rangle \\ &= \langle e^{\beta E} \rho, \partial z \rangle \\ &= 0. \end{aligned}$$

As for Eq. (5.9), we have

$$\begin{aligned} \partial K(\dot{\rho}) &= \partial \left( \frac{1}{\Delta} \sum_T w_T (\alpha - \bar{T}\alpha) \right), \quad \text{for any } \alpha \text{ so that } \partial\alpha = \dot{\rho} \\ &= \frac{1}{\Delta} \sum_T w_T \partial\alpha - \frac{1}{\Delta} \sum_T w_T \partial \bar{T}\alpha \\ &= \frac{1}{\Delta} \sum_T w_T \partial\alpha, \\ &= \dot{\rho}, \end{aligned}$$

where we have used the fact that  $\bar{T}\alpha$  is a cycle. For Condition (ii), we have

$$\begin{aligned} \langle K(\dot{\rho}), z \rangle &= \langle \frac{1}{\Delta} \sum_T w_T \alpha, z \rangle_W - \langle \frac{1}{\Delta} \sum_T w_T \bar{T}\alpha, z \rangle_W \\ &= \frac{1}{\Delta} \sum_T w_T (\langle \alpha, z \rangle_W - \langle \alpha, \bar{T}z \rangle_W) \\ &= 0. \end{aligned}$$

Here we have used the fact that  $\frac{1}{\Delta} \sum_T w_T \bar{T}$  is self-adjoint in the modified inner product and  $\bar{T}z = z$  for any cycle  $z$  (by Theorem 3.19 and Corollary 3.16, respectively).  $\square$

**Lemma 5.10.** *If  $\gamma$  is a constant driving protocol, so that  $\gamma(t) = \gamma(0)$  for all  $t$ , then*

$$Q_{\tau_D, \beta}(\gamma) = 0.$$

*Proof.* Recall that  $\tau_D$  is large enough to guarantee the existence of unique solution to

Eq. (2.6). Since  $\gamma$  is constant, the weights appearing in  $K$  are time-independent. By Proposition 5.9,

$$\begin{aligned} Q_{\tau_D, \beta}(\gamma) &= \int_0^1 K(\dot{\rho}(t)) dt \\ &= K \int_0^1 \dot{\rho}(t) dt \\ &= 0, \end{aligned}$$

since  $\rho(t)$  is a periodic solution. □

As pointed out above, the average current density  $Q_{\tau_D, \beta}(\gamma) \in H_d(X; \mathbb{R})$  has real coefficients in general. We would like to have greater control over the coefficients. In particular, we will show the values the current acquires in the low-temperature, adiabatic limit is restricted to certain rational numbers. This leads us to a refinement of the coefficients appearing in Theorem 3.19 and Theorem 4.6.

**Lemma 5.11.** *Let  $R$  be an integral domain, and let  $f : A \rightarrow B$  be a rational isomorphism between finitely generated  $R$ -modules, with  $A$  free. Let  $B_{\text{tors}}$  denote the torsion subgroup of  $B$  and  $\pi : B \rightarrow B/B_{\text{tors}} =: B_0$  denote the projection onto the torsion-free summand. Then  $|\text{cok } f| = |B_{\text{tors}}| |\det f_0|$ , where  $f_0 = \pi \circ f$ .*

*Proof.* Since  $f$  is a rational isomorphism, we know  $\text{cok } f$  must be a finite group. With this

in mind, we apply the snake lemma to obtain the following diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & B_{\text{tors}} & \xrightarrow{\cong} & \ker j & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & A & \xrightarrow{f} & B & \longrightarrow & \text{cok } f & \longrightarrow & 0 \\
& & \downarrow \text{id} & & \downarrow \pi & & \downarrow j & & \\
0 & \longrightarrow & A & \xrightarrow{\pi f} & B_0 & \longrightarrow & \text{cok } (\pi f) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
& & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

The ‘snake’ morphism implies  $B_{\text{tors}} \cong \ker j$ , and therefore the right-most vertical exact sequence gives the desired equation. By Proposition 3.26, we identify  $|\text{cok } f_0|$  with  $\det f_0$ .  $\square$

Let  $H_{d-1}(T; \mathbb{Z})_0$  denote the image of the coefficient homomorphism  $H_{d-1}(T; \mathbb{Z}) \rightarrow H_{d-1}(T; \mathbb{R})$  and similarly denote  $H_{d-1}(X; \mathbb{Z})_0$ . Each of these has a preferred isomorphism to the torsion-free part of their respective integral homology groups. Let

$$\nu_T = \det i_{*0} : H_{d-1}(T; \mathbb{Z})_0 \rightarrow H_{d-1}(X; \mathbb{Z})_0,$$

where  $i_{*0}$  denotes the projection of the map  $i_*$  onto the torsion-free summands. Similarly, let

$$\mu_L = \det j_{*0} : H_{d-1}(L; \mathbb{Z})_0 \rightarrow H_{d-1}(X; \mathbb{Z})_0.$$

**Proposition 5.12.** *The transformation  $K^T : B_{d-1}(X; B) \rightarrow C_d(T; B)$  of Eq. (3.8) can be defined for any abelian group  $B$  such that  $\theta_T$ ,  $\theta_X$ , and  $\nu_T$  are invertible. Similarly the map  $\psi_L : H_d(X; C) \rightarrow Z_d(X; C)$  can be defined for any abelian group  $C$  such that  $\theta_X$  and  $\mu_L$  are invertible. In particular,*

$$K^T(\psi_L) \in C_d \left( X; \mathbb{Z} \left[ \frac{1}{\theta_X \theta_T \nu_T \mu_L} \right] \right).$$

*Proof.* Consider the long exact sequence in homology of the pair  $(X, L)$  with coefficients in  $B$ :

$$0 \rightarrow H_d(X; B) \rightarrow H_d(X, T; B) \xrightarrow{\delta} H_{d-1}(T; B) \xrightarrow{i_*} H_{d-1}(X; B) \rightarrow H_{d-1}(X, T; B) \rightarrow 0.$$

We want to identify the initial or universal group  $\tilde{B}$  for which  $i_*$  is an isomorphism. By Lemma 3.11, the image of  $\delta$  is torsion. Tensoring the above sequence with  $\mathbb{Z}[\frac{1}{\theta_T}]$  kills the torsion subgroup, including  $\ker i_*$ , and yields the short exact sequence

$$0 \rightarrow H_{d-1}(T; \mathbb{Z}[\frac{1}{\theta_T}]) \xrightarrow{i_*} H_{d-1}(X; \mathbb{Z}[\frac{1}{\theta_T}]) \rightarrow H_{d-1}(X, T; \mathbb{Z}[\frac{1}{\theta_T}]) \rightarrow 0.$$

This change of scalars implies  $H_{d-1}(T; \mathbb{Z}[\frac{1}{\theta_T}])$  is free abelian. We now apply Lemma 5.11 to find  $|\operatorname{cok} i_*| = \theta_X \cdot \nu_T$ , and tensoring with  $\mathbb{Z}[\frac{1}{\theta_X \nu_T}]$  forces  $i_*$  to be an isomorphism.

Set  $n = \theta_X \theta_T \nu_T$  and take  $b \in B_{d-1}(X; \mathbb{Z}[\frac{1}{n}])$ , so that  $b = \partial\alpha$  for some  $\alpha \in C_d(X; \mathbb{Z}[\frac{1}{n}])$ . If  $\alpha \in C_d(T; \mathbb{Z}[\frac{1}{n}])$ , then  $\bar{T}\alpha = 0$  and hence  $K_b^T \in C_d(T; \mathbb{Z}[\frac{1}{n}])$ . If  $\alpha \notin C_d(T; \mathbb{Z}[\frac{1}{n}])$ , then  $0 \neq [b] \in H_{d-1}(T; \mathbb{Z}[\frac{1}{n}])$ , but  $i_*[b] = 0 \in H_{d-1}(X; \mathbb{Z}[\frac{1}{n}])$ , which is a contradiction.

The statement for  $\psi_L$  is proven similarly. Since a co-tree has no  $d$ -cells, we have  $Z_{d-1}(L; \mathbb{Z}) = H_{d-1}(L; \mathbb{Z})$ , and so inverting  $\phi_L$ , as required in the definition of  $\psi_L$ , is equivalent to inverting the map induced by the inclusion  $j_* : H_{d-1}(L; \mathbb{Z}) \rightarrow H_{d-1}(X; \mathbb{Z})$ . The domain of  $j_*$  is free abelian, and so we may directly apply Lemma 5.11. We invert  $|\operatorname{cok} \phi_L| = \theta_X \mu_L$  by tensoring with  $\mathbb{Z}[\frac{1}{\theta_X \mu_L}]$ .

In both cases, we have only used the invertibility of certain numbers. This perspective implies we can tensor with

$$\mathbb{Z} \left[ \frac{1}{\theta_T \theta_X \nu_T \mu_L} \right]$$

from the outset in both cases, and the maps  $i_*$  and  $j_*$  will both remain isomorphisms.

Since we haven't changed the cell structure, the other two properties of Definition 3.4 and Definition 4.2 both still hold.  $\square$

**Definition 5.13.** *The space of good parameters*

$$\mathcal{M}_X^g \subset \mathcal{M}_X$$

*consists of those  $(E, W) \in \mathcal{M}_X$  such that either  $E$  or  $W$  is very non-degenerate.*

**Remark 5.14.** *We restrict our attention to the space of good parameters for the remainder of this chapter. The space of good parameters does not depend on the CW structure of  $X$ , only on the number of  $d$  and  $(d - 1)$  cells. When  $X$  is a graph, it is possible to extend the space of good parameters to a space of robust parameters, as in [14]. This space does depend on the structure of the graph, and all of the results utilizing good parameters also hold for the robust parameters. We believe such a space of robust parameters exists for general CW complexes, but we do not include it here.*

The space of good parameters admits a simple decomposition. Let  $U$  be the set of  $(E, W)$  such that  $E$  is very non-degenerate, and  $V$  be the set of  $(E, W)$  for which  $W$  is very non-degenerate. Then we can write

$$\mathcal{M}_X^g = U \cup V,$$

where  $U$  and  $V$  are both open sets.

A periodic driving protocol  $(\tau_D, \gamma)$  whose image is completely contained in  $\mathcal{M}_X^g$  is known as a *loop of good parameters*. It is convenient to represent such maps by  $\gamma : C \rightarrow \mathcal{M}_X^g$ , where  $C$  is a circle of radius  $1/(2\pi)$ . For a closed arc  $I \subset C$ , the contribution along  $I$  to the



average current density is given by

$$Q_{\tau_D, \beta}(\gamma) = \int_I \mathbf{J}(s) ds,$$

where we parametrize by arc length.

We take a simplicial decomposition of  $C$  in terms of alternating segments in  $U$  and  $V$ .

Let  $I_1, I_2, \dots, I_n$  be a simplicial decomposition of  $C$  into closed arcs such that

(i)  $\gamma(I_i) \subset U$ , or

(ii)  $\gamma(I_i) \subset V$  and  $\gamma(\partial I_i) \subset U$ ,

for every  $i$ . The segments satisfying (i) are said to be of *type U* and those satisfying (ii) are of *type V*. The decomposition implies

$$Q_{\tau_D, \beta}(\gamma) = \sum_{k=1}^n \int_{I_k} \mathbf{J}(\gamma) ds.$$

Theorem 5.1 implies that, in the adiabatic limit,  $\lim_{\tau_D \rightarrow \infty} \mathbf{J}(t) = K(\gamma, \dot{\rho}^B)$ . Therefore, we set

$$Q_\beta(\gamma) := \lim_{\tau_D \rightarrow \infty} Q_{\tau_D, \beta}(\gamma) = \int_0^1 K(\gamma, \dot{\rho}^B).$$

**Lemma 5.15.** *Suppose that  $I$  is of type U. In the low-temperature, adiabatic limit, the contribution to  $Q_\beta(\gamma)$  along  $I$  is trivial.*

*Proof.* By Proposition 5.9, the average current density along  $I$  is given by

$$Q_\beta(\gamma) = \int_I K(\gamma, \dot{\rho}^B(\gamma)) ds.$$

Since  $E$  is very non-degenerate on segments of type  $U$ , Proposition 5.7 implies that  $\dot{\rho}^B \rightarrow 0$  uniformly in the low temperature limit. Therefore,  $K$ , and both  $\mathbf{J}$  and  $Q_\beta$  tend to zero as well. □

**Lemma 5.16.** *Suppose that  $I = [a, b]$  is of type  $V$ . In the low-temperature limit, the contribution to  $Q_\beta(\gamma)$  along  $I$  lies in*

$$C_d \left( X; \mathbb{Z} \left[ \frac{1}{\theta_X \mu_{L(a)} \mu_{L(b)} \theta_T \nu_T} \right] \right),$$

where  $T$  is the unique tree on  $I$ ,  $L(a)$  is the unique co-tree at  $\gamma(a)$ , and  $L(b)$  is the unique co-tree at  $\gamma(b)$ .

*Proof.* Note that the argument for segments of type  $U$  does not apply, since the function  $E$  need not be very non-degenerate (except at the endpoints of  $I$ ), and so  $\dot{\rho}^B \not\rightarrow 0$ . Equation 5.9 gives

$$Q_\beta(\gamma) = \int_I K(\gamma, \dot{\rho}^B),$$

Lemma 5.6 implies that  $K \rightarrow K^T$  uniformly as  $\beta \rightarrow \infty$ . Therefore,

$$\begin{aligned} \lim_{\beta \rightarrow \infty} Q_\beta(\gamma) &= \lim_{\beta \rightarrow \infty} \int_a^b K(\gamma, \dot{\rho}^B(\gamma)) ds \\ &= K^T \left( \lim_{\beta \rightarrow \infty} \int_a^b \dot{\rho}^B(\gamma) ds \right) \\ &= K^T(\psi_{L(b)} - \psi_{L(a)}). \end{aligned}$$

Set  $n = \theta_X \mu_{L(a)} \mu_{L(b)} \theta_T \nu_T$ . By Proposition 5.12,  $K^T : B_{d-1}(X; \mathbb{Z}[\frac{1}{n}]) \rightarrow C_d(X; \mathbb{Z}[\frac{1}{n}])$ . The same result implies the difference  $\psi_{L(b)} - \psi_{L(a)} \in B_{d-1}(X; \mathbb{Z}[\frac{1}{\theta_X \mu_{L(a)} \mu_{L(b)}}])$ , and the statement follows.  $\square$

Define

$$D = \theta_X \prod_L \mu_L \prod_T \theta_T \nu_T,$$

where the products are taken over all spanning co-trees  $L$  and spanning trees  $T$ . The following is a straightforward consequence of the previous two lemmas.

**Theorem 5.17.** *If  $\gamma : S^1 \rightarrow \mathcal{M}_X^g$  is a loop of good parameters, then in the low-temperature, adiabatic limit, we have*

$$\lim_{\beta \rightarrow \infty} \lim_{\tau_D \rightarrow \infty} Q_{\tau_D, \beta}(\gamma) \in H_d(X; \mathbb{Z}[\frac{1}{D}]) \subset H_d(X; \mathbb{R}).$$

This theorem gives the desired generalization of the integer quantization theorem on graphs. If  $X$  is a graph, then every factor appearing in  $D$  equals 1, and we recover the result of [14, Theorem A].

It is interesting to note the factors appearing in  $D$ , and whether they have geometric significance. The numbers  $\mu_L$  and  $\nu_T$  are determinants, and as such, should have some geometric interpretation, certainly up to some additional data. Ideas like this will be used in Chapter 6 when discussing Reidemeister torsion. The other point of interest lies in the torsion terms  $\theta_X$  and  $\theta_T$ . If an experimental study of extended empirical currents could be performed, one would be able to see the effects of algebraic torsion. This is unexpected, since physics tends to be done over a characteristic zero field like  $\mathbb{C}$ , where all torsion vanishes.

## CHAPTER 6 APPLICATIONS

This chapter is intended to provide computations using ideas from the previous chapters.

This follows a computation from [9].

### 6.1 Reidemeister Torsion

In the mid twentieth century, Franz, Reidemeister and De Rham classified three dimensional lens spaces using a combinatorial invariant of triangulated spaces. This invariant was subsequently called Reidemeister torsion, or R-torsion, and has found many applications. This is a subtle, secondary invariant, defined up to a choice of representation of the fundamental group. Intuitively, R-torsion is a generalized determinant, describing how the cells of the universal cover  $\tilde{X}$  fit together with the action of  $\pi_1(X)$  to form  $X$ . The torsion is computed using the simplicial chain complex twisted by the representation. The most important and well-known case of this occurs when the twisted chain complex is acyclic. Milnor [21] extended the notion of Reidemeister torsion to a not necessarily acyclic finite chain complex  $C_*$  over a field in which a preferred basis is chosen for  $C_*$  as well as its homology. We point out that Milnor's invariant is not preserved under chain homotopy equivalence. In this setting, the torsion *a priori* depends not only on the chain complex, but also the equivalence class of the preferred bases. We restrict to the case of chain complexes defined over the real numbers.

Consider the case of a chain complex  $C_*$  of finite dimensional vector spaces over  $\mathbb{R}$  having non-trivial terms in degrees  $0 \leq * \leq d$ . Let  $\partial : C_k \rightarrow C_{k-1}$  be the boundary operator. Let  $Z_k \subset C_k$  be the subspace of  $k$ -cycles and let  $B_k \subset Z_k$  the subspace of  $k$ -boundaries. We also

set  $H_k = Z_k/B_k$ . We then have short exact sequences

$$0 \rightarrow Z_k \rightarrow C_k \rightarrow B_{k-1} \rightarrow 0 \quad \text{and} \quad 0 \rightarrow B_k \rightarrow Z_k \rightarrow H_k \rightarrow 0.$$

If we choose splittings  $s_{k-1} : B_{k-1} \rightarrow C_k$  and  $t_k : H_k \rightarrow Z_k$ , we are entitled to write  $C_k \cong Z_k \oplus B_{k-1} \cong B_k \oplus H_k \oplus B_{k-1}$ .

Pick bases  $\mathfrak{b}_k := \{b_k^i\}$ ,  $\mathfrak{c}_k := \{c_k^i\}$ ,  $\mathfrak{h}_k := \{h_k^i\}$  for  $B_k$ ,  $C_k$ , and  $H_k$ , respectively. It follows that  $\{b_k^i, t_k(h_k^i), s_{k-1}(b_{k-1}^i)\}_i$  forms another basis for  $C_k$ . Let  $\{\mathfrak{b}_k \mathfrak{h}_k \mathfrak{b}_{k-1}\}$  denote this basis and let

$$[\mathfrak{b}_k \mathfrak{h}_k \mathfrak{b}_{k-1} / \mathfrak{c}_k]$$

denote the change of basis matrix that expresses the basis  $\mathfrak{b}_k \mathfrak{h}_k \mathfrak{b}_{k-1}$  in terms of the basis  $\mathfrak{c}_k$ .

Let  $\mathfrak{c} = \{\mathfrak{c}_k\}$  and  $\mathfrak{h} = \{\mathfrak{h}_k\}$ .

**Definition 6.1** (Milnor [21, p. 365]). *The torsion of the pair  $(C_*, \mathfrak{h})$  is defined by*

$$\tau(C_*, \mathfrak{h}) = \prod_{k \geq 0} \det[\mathfrak{b}_k \mathfrak{h}_k \mathfrak{b}_{k-1} / \mathfrak{c}_k]^{(-1)^k},$$

which is consistent with Milnor's definition with respect to the identification of  $K_1(\mathbb{R}) \cong \mathbb{R}^\times$  given by the determinant function.

Milnor showed that the definition is independent of the choice of  $\mathfrak{b}$  as well as the splittings.

Thus the torsion is really an invariant of the triple  $(C_*, \mathfrak{c}, \mathfrak{h})$ .

In what follows,  $C_* = C_*(X; \mathbb{R})$  is the cellular chain complex of a finite, connected CW complex  $X$ , which has a preferred basis consisting of the set of cells. In this case, we think of the torsion as an invariant of the pair  $(X, \mathfrak{h})$  and set

$$\tau(X; \mathfrak{h}) := \tau(C_*(X; \mathbb{R})),$$

where we have indicated the dependence on the choice of homology basis. It will be useful to single out a specific kind of homology basis. Let  $H_*(X; \mathbb{Z})_0 \subset H_*(X; \mathbb{R})$  be the lattice given by taking the image of the evident homomorphism  $H_*(X; \mathbb{Z}) \rightarrow H_*(X; \mathbb{R})$ . Note that  $H_*(X; \mathbb{Z})_0$  has a preferred isomorphism to the torsion free part of  $H_*(X; \mathbb{Z})$ .

**Definition 6.2.** *A combinatorial basis for  $H_*(X; \mathbb{R})$  consists of a basis for  $H_k(X; \mathbb{Z})_0$  for  $k \geq 0$ .*

Fix a combinatorial basis  $\mathfrak{h}$  and a positive-valued function  $W = \prod_k W_k : \prod_k X_k \rightarrow \mathbb{R}$  on the set of cells of  $X$ . This gives rise to an operator

$$\mathcal{L}_k(W) = \partial \partial_{W_{k,k+1}}^* := \partial e^{-W_{k+1}} \partial^* e^{W_k} : B_k(X; \mathbb{R}) \rightarrow B_k(X; \mathbb{R}),$$

where the adjoint is defined in the modified inner product on both source and target. This operator gives a version of the dynamical operator of Eq. (2.4) for every degree of the chain complex. We define  $B_k^{\perp W}(X; \mathbb{R})$  to be the orthogonal complement of  $B_k(X; \mathbb{R})$  in  $Z_k(X; \mathbb{R})$  with respect to modified inner product on  $C_k(X; \mathbb{R})$ . There is a preferred identification  $B_k^{\perp W}(X; \mathbb{R}) \cong H_k(X; \mathbb{R})$  given by sending a cycle to its homology class. As in Section 3.5, let  $\eta_k$  be the square of the covolume of  $H_k(X; \mathbb{Z})_0 \subset B_k^{\perp W}(X; \mathbb{R})$ , with respect to the combinatorial basis  $\mathfrak{h}_k$  for  $H_k(X; \mathbb{Z})_0$  and the inner product on  $B_k^{\perp W}(X; \mathbb{R})$  obtained by restricting the modified inner product on  $C_k(X; \mathbb{R})$ .

**Theorem 6.3.** *Let  $X$  be a finite, connected CW complex. Then*

$$\tau^2(X; \mathfrak{h}) = \frac{\prod_k \text{even} \det \mathcal{L}_k(W)}{\prod_k \text{odd} \det \mathcal{L}_k(W)} \cdot \frac{\prod_k \text{odd}, b \in X_k e^{W_{kb}}}{\prod_k \text{even}, b \in X_k e^{W_{kb}}} \cdot \frac{\prod_k \text{even} \eta_k}{\prod_k \text{odd} \eta_k}.$$

**Remark 6.4.** *If we take  $W = 0$ , then Theorem 6.3 immediately implies that  $\tau^2(X; \mathfrak{h})$  is an invariant of the lattice  $H_*(X; \mathbb{Z})_0 \subset H_*(X; \mathbb{R})$  rather than just an invariant of the specific choice of combinatorial basis  $\mathfrak{h}$ . Since this lattice doesn't depend on any choices, we infer*

that  $\tau^2(X; \mathfrak{h})$  depends only on the CW structure of  $X$ . In fact, the method of proof of [21, Thm. 7.2] shows that  $\tau^2(X; \mathfrak{h})$  is invariant under subdivision.

*Proof of Theorem 6.3.* For the purpose of this proof, we suppress  $W$  and write  $\mathcal{L} = \mathcal{L}(W)$ .

We also set  $C_* := C_*(X; \mathbb{R})$ . Define the splitting maps  $s_{k-1} : B_{k-1} \rightarrow C_k$  by

$$s_{k-1}(b^i) = e^{-W_k} \partial^* e^{W_{k-1}} \mathcal{L}_{k-1}^{-1}(b^i) = \partial_{W_{k,k+1}}^* \mathcal{L}_{k-1}^{-1}(b^i).$$

Let  $Z_k^{\perp W}(X; \mathbb{R})$  denote the image of  $s_{k-1}$ , and similarly we define  $Z_k^{\perp W}(X; \mathbb{Z})$  to be  $s_{k-1}(B_k(X; \mathbb{Z}))$ .

Note that  $Z_k^{\perp W}(X; \mathbb{R})$  is the orthogonal complement to  $Z_k$  in the modified inner product on  $C_k$ .

Let  $\gamma^k$  denote the square of the covolume of  $Z_k^{\perp W}(X; \mathbb{Z}) \subset Z_k^{\perp W}(X; \mathbb{R})$ , using the inner product induced by the modified inner product on  $C_k$ . Similarly, let  $\gamma_{k-1}$  denote the square of the covolume of  $B_{k-1}(X; \mathbb{Z}) \subset B_{k-1}(X; \mathbb{R})$ , where  $B_{k-1}(X; \mathbb{R})$  is given the inner product by restricting the modified inner product on  $C_{k-1}$ . Using the isomorphism determined by the splitting  $B_k \oplus B_k^{\perp W} \oplus Z_k^{\perp W} \xrightarrow{\cong} C_k$ , we infer

$$\det[\mathfrak{b}_k \mathfrak{h}_k \mathfrak{b}_{k-1} / \mathfrak{c}_k]^2 = \frac{\gamma_k \eta_k \gamma^k}{\prod_{b \in X_k} e^{W_{kb}}}, \quad (6.1)$$

so the square of the Reidemeister torsion is

$$\tau^2(X; \mathfrak{h}) = \frac{\prod_{k \text{ even}} \gamma_k \eta_k \gamma^k \prod_{k \text{ odd}, b \in X_k} e^{W_{kb}}}{\prod_{k \text{ odd}} \gamma_k \eta_k \gamma^k \prod_{k \text{ even}, b \in X_k} e^{W_{kb}}}. \quad (6.2)$$

Since  $s_k = \partial_{W_{k+1, k+2}}^* \mathcal{L}_k^{-1}$ , its adjoint is given by  $s_k^* = \mathcal{L}_k^{-1} \partial$  (since  $\mathcal{L}$  is self-adjoint). Therefore,

$$s_k^* s_k = \mathcal{L}_k^{-1} \partial \partial_{W_{k+1, k+2}}^* \mathcal{L}_k^{-1} = \mathcal{L}_k^{-1}.$$

Since  $\gamma^k$  is given as the determinant of an inner product matrix from Section 3.5, we

make the following computation:

$$\begin{aligned} \langle s_{k-1}(b^i), s_{k-1}(b^j) \rangle_W &= \langle s_{k-1}^* s_{k-1}(b^i), (b^j) \rangle_W \\ &= \langle \mathcal{L}_{k-1}^{-1}(b^i), (b^j) \rangle_W. \end{aligned}$$

If  $U$  denotes the change of basis matrix expressing  $\mathbf{b}_{k-1}$  in terms of an orthonormal basis for  $B_{k-1}(X; \mathbb{R})$  in the modified inner product, then the determinant of the matrix with entries given by the previous display is  $(\det U)^2 \det \mathcal{L}_{k-1}$ , by definition. A similar observation shows that the determinant of the matrix whose entries are  $\langle b_{k-1}^i, b_{k-1}^j \rangle_W$  is  $(\det U)^2$ , and this is just  $\gamma_{k-1}$ . Consequently, the quotient of these determinants is

$$\frac{\gamma^k}{\gamma_{k-1}} = \frac{1}{\det \mathcal{L}_{k-1}}. \quad (6.3)$$

Inserting Eqn. (6.3) into Eqn. (6.2) and performing the evident cancellations, we conclude

$$\tau^2(X; \mathfrak{h}) = \frac{\prod_{k \text{ even}} \det \mathcal{L}_k(W)}{\prod_{k \text{ odd}} \det \mathcal{L}_k(W)} \cdot \frac{\prod_{k \text{ odd}, b \in X_k} e^{W_{kb}}}{\prod_{k \text{ even}, b \in X_k} e^{W_{kb}}} \cdot \frac{\prod_{k \text{ even}} \eta_k}{\prod_{k \text{ odd}} \eta_k}. \quad \square$$

In the special case of  $W = 0$ , we can combine Theorem 6.3 with Corollary 3.32 to give the following.

**Corollary 6.5** (Torsion-Tree Theorem). *For a finite, connected CW complex  $X$ , we have*

$$\tau^2(X; \mathfrak{h}) = \prod_{k \geq 0} (\delta_k \sum_{T \in \mathcal{T}_{k+1}} \theta_T^2)^{(-1)^k},$$

where  $\mathcal{T}_k$  denotes the spanning trees of  $X^{(k)}$ , and

$$\delta_k = \frac{\eta_k \mu_k}{\theta_k^2},$$

where

- $\eta_k$  is the square of the covolume of  $H_k(X; \mathbb{Z})_0 \subset B_k^{\perp W}(X; \mathbb{R})$ ,



- $\mu_k$  is the square of the covolume of  $B_k(X; \mathbb{Z}) \subset B_k(X; \mathbb{R})$ , and
- $\theta_k$  is the order of the torsion subgroup of  $H_k(X; \mathbb{Z})$ .

### 6.1.1 An Alternative Formula

For each  $k \geq 1$ , fix a spanning tree  $T^k$  and spanning co-tree  $L^k$  for  $X^{(k)}$ . Our convention is to set  $T^0 = L^0 = \emptyset$ . Then we have an increasing filtration

$$T^0 \subset L^0 \subset \dots \subset X^{(k-1)} \subset T^k \subset L^k \subset X^{(k)} \subset \dots$$

Define a basis for  $Z_k^\perp(X; \mathbb{Z})$ ,  $\mathfrak{b}^k = \{b_i^k\}$ , as given by  $k$ -the cells of  $T^k$ , denoted  $T_k^k$ . Here we are using the preferred isomorphism  $Z_k^\perp(X; \mathbb{Z}) \cong C_k(T^k; \mathbb{Z})$ . For a basis  $\mathfrak{b}_{k-1}$  of  $B_{k-1}(X; \mathbb{R})$ , we take the image of the standard basis for  $C_k(T^k; \mathbb{R})$  under the composition

$$C_k(T^k; \mathbb{R}) \xrightarrow{\partial} B_{k-1}(T^k; \mathbb{R}) \rightarrow B_{k-1}(X; \mathbb{R}).$$

The basis for homology in degree  $k$  is the combinatorial basis  $\mathfrak{h}_k$  given as an input to the torsion. As always, the basis for  $C_k(X; \mathbb{R})$  is given by the set of  $k$ -cells.

Before explicitly identifying the torsion, note that in each dimension  $k$  there are essentially three types of cells:

$$X_k = (T_k^k) \cup (L_k^k \setminus T_k^k) \cup (X_k \setminus L_k^k).$$

Roughly speaking, the first set of cells contributes to  $Z_k^\perp$ , the second set contributes to  $B_k^\perp$  and the last set contributes to  $B_k$ . This gives us a decomposition of the  $k$ -chains

$$C_k(X; \mathbb{R}) = C_k(T^k; \mathbb{R}) \oplus C_k(L^k/T^k; \mathbb{R}) \oplus C_k(X/L^k; \mathbb{R}). \quad (6.4)$$

(when  $k = 0$ , we replace  $C_0(V^0/T^0; \mathbb{R})$  with  $C_0(V^0, T^0; \mathbb{R}) = \mathbb{R}$ , etc.) Furthermore, the cell decomposition above implies the change-of-basis matrix  $[\mathbf{b}_k \mathbf{h}_k \mathbf{b}_{k-1}/\mathbf{c}]$  has the following form.

$$\begin{bmatrix} * & * & * \\ * & * & 0 \\ * & 0 & 0 \end{bmatrix}$$

Therefore, the determinant decomposes as the product of three sub-determinants.

We first identify the contribution of  $\mathbf{h}_k$  to the torsion. With respect to the splitting Eq. (6.4), the combinatorial basis  $\mathbf{h}_k$  has image contained in the direct sum

$$C_k(T^k; \mathbb{R}) \oplus C_k(L^k/T^k; \mathbb{R}) = C_k(L^k; \mathbb{R}).$$

Hence, its contribution to the torsion is left invariant if we project these elements onto  $C_k(L^k/T^k; \mathbb{R}) = H_k(L_k/T_k; \mathbb{R}) = H_k(X; \mathbb{R})$  (since the other summand  $C_k(T^k; \mathbb{R}) = Z_k^\perp(T^k; \mathbb{R})$  maps to  $Z_k^\perp(X; \mathbb{R})$  and the relevant determinant remains unchanged if we project away from  $Z_k^\perp(X; \mathbb{R})$ ). Consequently, the homological contribution to the torsion in degree  $k$  is given by the determinant of the composite

$$H_k(X; \mathbb{R}) \xrightarrow{i_*^{-1}} \cong H_k(L^k; \mathbb{R}) \xrightarrow{p_*} \cong H_k(L^k/T^k; \mathbb{R}),$$

where  $p : L^k \rightarrow L^k/T^k$  is the quotient map. So we wish to identify  $\det p_*/\det i_*$ .

**Definition 6.6.** *Let*

$$\chi_k \in \mathbb{N}$$

*denote the square of the determinant of  $i_* : H_k(L^k; \mathbb{R}) \rightarrow H_k(X; \mathbb{R})$ , i.e., the square of the covolume of the lattice  $i_*(H_k(L^k; \mathbb{Z})) \subset H_k(X; \mathbb{R})$ .*

Applying Proposition 3.26 to the real isomorphism  $H_k(L^k; \mathbb{Z}) \rightarrow H_k(L^k/T^k; \mathbb{Z})$ , we infer the following result.

**Lemma 6.7.** *The determinant of  $p_*$  is the ratio  $\pm\theta_{T^k}/\theta_{L^k}$ .*

Consequently, up to sign, the contribution of  $\mathfrak{h}_k$  to the determinant defining the Reidemeister torsion is

$$\frac{\theta_{T^k}}{\theta_{L^k}\sqrt{\chi_k}}. \quad (6.5)$$

We next identify the contribution in degree  $k$  to the torsion provided by the basis  $\mathfrak{b}_k$ . As defined above this basis is given by the boundaries of the cells of  $T_{k+1}$ . This leads us to consider the composite

$$C_{k+1}(T^{k+1}; \mathbb{Z}) \xrightarrow{\partial} B_k(T^{k+1}; \mathbb{Z}) \xrightarrow{q_k} C_k(X/L^k; \mathbb{Z}), \quad (6.6)$$

where  $q_k$  is induced by the quotient map  $T^{k+1} \rightarrow X/L^k$ . The homomorphism  $\partial$  is an isomorphism and so it has determinant  $\pm 1$ . The second homomorphism  $q_k$  is a real isomorphism and therefore the determinant of its realification,  $\det((q_k)_{\mathbb{R}})$ , has value  $\pm t(q_k)$  by Proposition 3.26. Note that  $(q_k)_{\mathbb{R}}$  is the restriction of the orthogonal projection  $C_k(X; \mathbb{R}) \rightarrow C_k(X/L^k; \mathbb{R})$  to the subspace  $B_k(T_{k+1}; \mathbb{R}) \subset C_k(X; \mathbb{R})$ . and the projection of  $\mathfrak{b}_k$  onto this summand gives its contribution to the torsion. Hence, the determinant of the composition  $(q_k)_{\mathbb{R}} \circ \partial$  is  $\pm t(q_k)$ . So the contribution in degree  $k$  of  $\mathfrak{b}_k$  to the torsion is  $\pm t(q_k)$ .

Lastly, the contribution to the torsion in degree  $k$  provided by the basis  $\mathfrak{b}_{k-1}$  is given by the standard basis of  $C_k(T_k; \mathbb{R})$  via the splitting Eq. (6.4). It is then evident that the contribution in degree  $k$  of  $\mathfrak{b}_{k-1}$  to the torsion is 1.

Assembling, we obtain

$$\det[\mathfrak{b}_k \mathfrak{h}_k \mathfrak{b}_{k-1} / \mathfrak{c}] = \pm t(q_k) \cdot \frac{\theta_{T^k}}{\theta_{L^k}\sqrt{\chi_k}} \cdot 1. \quad (6.7)$$

**Theorem 6.8.** *For a connected, finite CW complex  $X$  with combinatorial homology basis  $\mathfrak{h}$ , spanning tree data  $\{T^k\}$  and spanning co-tree data  $\{L^k\}$ , we have*

$$\tau^2(X; \mathfrak{h}) = \prod_{k \geq 0} \left( \frac{\theta_{T^k}^2 t(q_k)^2}{\theta_{L^k}^2 \chi_k} \right)^{(-1)^k},$$

where  $q_k : B_k(T^{k+1}; \mathbb{Z}) \rightarrow C_k(X/L^k; \mathbb{Z})$  and  $\chi_k \in \mathbb{N}$  are as above.

**Example 6.9.** *If  $X$  has dimension one, then all terms appearing in Theorem 6.8 are equal to one. Hence,  $\tau^2(X; \mathfrak{h}) = 1$  whenever  $X$  is a connected finite graph.*

**Example 6.10.** *Let  $A$  be a finitely generated torsion abelian group and let  $n$  be a positive integer. Up to isomorphism  $A$  can be expressed as the cokernel of a real isomorphism  $h\mathbb{Z}^k \rightarrow \mathbb{Z}^k$ . Choose a self-map of a  $k$ -fold wedge of  $n$ -spheres  $f : \vee_k S^n \rightarrow \vee_k S^n$  which induces  $h$  on homology in degree  $n$ . There is only one such map up to homotopy. Let  $M(A, n)$  be the mapping cone of  $f$ . Then  $M(A, n)$  is a Moore space of type  $(A, n)$ .*

Set  $T^i = * = L^i$  for  $0 < i < n$  and  $T^n = M(A, n) = L^n$ . Then  $T^i$  is a spanning tree for the  $i$ -skeleton of  $M(A, n)$  and  $L^i$  is a spanning co-tree of  $M(A, n)$  in degree  $i$  with respect to  $T^i$ . In this instance, the only non-trivial term appearing in Theorem 6.8 is  $t(q_n)$  and in this case  $q_n = h$ . Consequently,

$$\tau^2(M(A, n); \mathfrak{h}) = (t(h))^{2(-1)^n} = |A|^{2(-1)^n}.$$

For example, if  $A = \mathbb{Z}/2$  and  $n = 1$ , then  $M(A, n) = \mathbb{R}P^2$ . We conclude that  $\tau^2(\mathbb{R}P^2; \mathfrak{h}) = \frac{1}{4}$ .

The results of Corollary 6.5 and Theorem 6.8 are both novel and surprising as statements in mathematical physics. While there are a variety of definitions one could give of spanning trees in higher dimensions [25], Definition 3.4 was motivated from physical reasoning related to empirical currents on CW complexes. The same can be said for spanning co-trees, which That this same definition can be interpreted in terms of Reidemeister torsion

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**ABSTRACT****A TOPOLOGICAL STUDY OF STOCHASTIC DYNAMICS  
ON CW COMPLEXES**

by

**MICHAEL J. CATANZARO****MAY 2016**

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**Major:** Mathematics  
**Degree:** Doctor of Philosophy

In this dissertation, we consider stochastic motion of subcomplexes of a CW complex, and explore the implications on the underlying space. The random process on the complex is motivated from Ito diffusions on smooth manifolds and Langevin processes in physics. We associate a Kolmogorov equation to this process, whose solutions can be interpreted in terms of generalizations of electrical, as well as stochastic, current to higher dimensions. These currents also serve a key function in relating the random process to the topology of the complex. We show the average current generated by such a process can be written in a physically familiar form, consisting of the solution to Kirchhoff's network problem and the Boltzmann distribution, suitably generalized to arbitrary dimensions. We analyze these two components in detail, and discover they reveal an unexpected amount of information about the topology of the CW complex. The main result is a quantization result for the average current in the low temperature, adiabatic limit. As an application, we express the Reidemeister torsion of the complex, a topological invariant, in terms of these quantities.

## AUTOBIOGRAPHICAL STATEMENT

Michael Joseph Catanzaro was born in Detroit, Michigan in 1987. He grew up in Clinton Township, Michigan, and began attending Wayne State University in the fall of 2005. He earned a B.S. in Mathematics and a B.S. in Physics in 2010, an M.A. in Mathematics in 2011, and a Ph.D. in Mathematics in 2016, all from Wayne State. He has been involved in a variety of research projects in the fields of algebraic topology, statistical mechanics, and physical chemistry. His work has been published in mathematics, physics, and chemistry journals. He was awarded a summer research position at Los Alamos National Laboratory in 2012, and was a recipient of the Rumble fellowship in 2015. In 2016, he accepted a post-doctoral position at the University of Florida.

He married his beautiful wife Jessica in October 2014. They currently reside in the Detroit area with their adorable puppy Louie. Outside of academia, he enjoys many sports and hobbies including golf, basketball, baseball, and playing guitar.