An Extended Weighted Exponential Distribution

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A new class of weighted distributions is proposed by incorporating an extended exponential distribution in Azzalini's (1985) method. Several statistics and reliability properties of this new class of distribution are obtained. Maximum likelihood estimators of the unknown parameters cannot be obtained in explicit forms; they have to be obtained by solving some numerical methods. Two data sets are analyzed for illustrative purposes, and show that the proposed model can be used effectively in analyzing real data.

Keywords: Exponential distribution, extended exponential distribution, hazard rate function, maximum likelihood estimation, weighted exponential distribution

Introduction

Adding an extra parameter to an existing family of distribution functions is common in statistical distribution theory. Introducing an extra parameter often brings more flexibility to a class of distribution functions, and it can be very useful for data analysis purposes. Azzalini (1985) introduced the skew normal distribution by introducing an extra parameter to bring more flexibility to the normal distribution. Afterwards, extensive works on introducing shape parameters for other symmetric distributions have been defined, and several properties and their inference procedures have been discussed by several authors; see for example Balakrishnan and Ambagaspitiya (1994), Arnold and Beaver (2000), and Nadarajah (2009).

Recently, there has been an attempt to use Azzalini's method for non-symmetric distributions. Gupta and Kundu (2009) introduced a class of weighted exponential (WE) distribution that has a shape parameter. It is said that a random variable X follows the WE(α, λ) distribution if its density function is given by

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$$f(x;\alpha,\lambda) = \frac{\alpha+1}{\alpha}\lambda e^{-\lambda x}\left(1-e^{-\alpha\lambda x}\right) \tag{1}$$

where $x > 0$, $\alpha > 0$, and $\lambda > 0$.

Shakhatreh (2012) generalized the WE distribution to the two-parameter weighted exponential (TWE) distribution. A random variable X is said to have a TWE distribution with shape parameters $\alpha_1 > 0$, $\alpha_2 > 0$ and scale parameter $\lambda > 0$ if the PDF of X is given by

$$f(x;\lambda,\alpha_1,\alpha_2) = k(\alpha_1,\alpha_2)\lambda e^{-\lambda x}\left(1-e^{-\lambda\alpha_1 x}\right)\left(1-e^{-\lambda\alpha_2 x}\right), \quad x>0 \tag{2}$$

where $k(\alpha_1,\alpha_2) = \dfrac{(1+\alpha_1)(1+\alpha_2)(1+\alpha_1+\alpha_2)}{\alpha_1\alpha_2(2+\alpha_1+\alpha_2)}$.

It is observed that the WE and TWE distributions can provide a better fit for survival time data relative to other common distributions such as the gamma, Weibull, or generalized exponential distributions.

The aim of this study is to introduce an extended weighted exponential (EWE) distribution based on extended exponential (EE) distribution introduced by Gómez, Bolfarine, and Gómez (2014). A random variable X follows the EE distribution with parameters λ and β if its density function is given by

$$f(x;\lambda,\beta) = \frac{\lambda^2(1+\beta x)e^{-\lambda x}}{\lambda+\beta}, \quad x>0 \tag{3}$$

where $\lambda > 0$ and $\beta > 0$ with the notation $X \sim EE(\lambda, \beta)$.

One of the goals of the introduction of the EWE is that involves the WE as its sub-model. The EWE has three parameters, one scale parameter and two shape parameters, which makes it more flexible in describing different types of real data than its sub model.

It is observed that the EWE distribution has several desirable properties. The generation of random samples from the EWE is straight forward. The maximum likelihood estimators (MLEs) of unknown parameters can be obtained by solving three nonlinear equations. For illustrative purposes we have analyzed the two real data sets. After analyzed the data using the EWE model, observe that EWE provides a better fit than the WE model and TWE.

Definition, Interpretations, and Generation

Definition: A random variable X is said to have an extended weighted exponential distribution with shape parameters $\alpha > 0$, $\beta > 0$ and scale parameter $\lambda > 0$, denoted by EWE(α, β, λ), if the density function of X is given as below

$$f\left(x; \alpha, \beta, \lambda\right) = \frac{\left(1+\alpha\right)^2 \lambda}{\alpha\left(\lambda\left(1+\alpha\right)+\alpha\beta\right)} e^{-\lambda x}\left(\lambda + \beta - \left(\lambda + \beta + \alpha\beta\lambda x\right)e^{-\alpha\lambda x}\right) \quad (4)$$

for $x > 0$ and 0 otherwise.

Plots of the EWE density function for fixed scale parameter $\lambda = 1$ and selected shape parameters are given in Figure 1. It is a unimodal density function for various values of the shape parameters. It is easy to show that if $\alpha \to 0$ then (4) converges to gamma(2, λ) and if $\alpha \to \infty$ then (4) converges to exp(λ). Note that, when $\beta = 0$, then EWE($\alpha, \beta = 1, \lambda$) = WE($\alpha, \lambda$).

Interpretation 1: EWE distribution can be obtained the same way that Azzalini obtained the skew-normal distribution. Suppose X_1 and X_2 are two independent variables and $X_1 \sim$ exp(λ), $X_2 \sim$ EE(λ, β). For any $\alpha > 0$, consider a new random variable $X = X_1$ given that $\alpha X_1 > X_2$. It can be easily observed that the density function of X is (4).

Interpretation 2: EWE distribution can be obtained by the hidden truncation model proposed by Arnold and Beaver (2000). Suppose Z and Y are two dependent random variables with the joint density function

$$f_{Z,Y}\left(z, y\right) = \frac{\lambda^3 z}{\lambda + \beta}\left(1 + \beta z y\right)e^{-\lambda z\left(y+1\right)}, \quad z > 0, y > 0 \quad (5)$$

It can be shown that the conditionally random variable $Z\,|\,Y \leq \alpha$ has the EWE distribution.
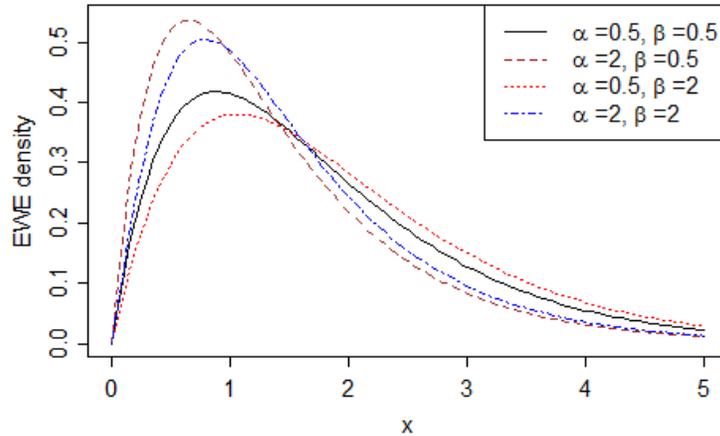
Figure 1. Plots of the EWE density function for fixed scale parameter $\lambda = 1$ and some selected shape parameters

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Interpretation 3:   Using the moment generating function (MGF) the stochastic representation of X can be easily obtained. Suppose U and V are two independent variables with distributions $\exp(\lambda)$ and $EE(\lambda(1 + \alpha), \alpha\beta)$, respectively. Then it can be observed that if

$$X = U + V \qquad (6)$$

then X has the density function (4).

Generation: All the above three interpretations can be used to generating random numbers from EWE distribution. Note that the simplest way to generate EWE random number is to use the stochastic representation (6).

Statistical and Reliability Properties

If $X \sim EWE(\alpha, \beta, \lambda)$, then the MGF of X for any $t < \lambda$ is given by

$$M_x(t) = \left(\frac{\lambda}{\lambda - t}\right)\left(\frac{\lambda(1+\alpha)}{\lambda(1+\alpha) - t}\right)^2\left(\frac{\lambda(1+\alpha) + \alpha\beta - t}{\lambda(1+\alpha) + \alpha\beta}\right) \qquad (7)$$

299

By straightforward integration, the row moments of X about the origin are found to be

$$E(X') = \frac{r!\left[(\lambda+\beta)(1+\alpha)^{r+2} - (\lambda+\beta)(1+\alpha) - \alpha\beta(r+1)\right]}{\lambda'\alpha(\lambda(1+\alpha)+\beta)(1+\alpha)'} \tag{8}$$

In particular, mean and $E(X^2)$ are given, respectively, by

$$E(X) = \frac{(\lambda+\beta)(1+\alpha)^3 - (\lambda+\beta)(1+\alpha) - 2\alpha\beta}{\lambda\alpha(\lambda(1+\alpha)+\beta)(1+\alpha)} \tag{9}$$

$$E(X^2) = \frac{2\left[(\lambda+\beta)(1+\alpha)^4 - (\lambda+\beta)(1+\alpha) - 3\alpha\beta\right]}{\lambda^2\alpha(\lambda(1+\alpha)+\beta)(1+\alpha)^2} \tag{10}$$

The distribution function for the random variable X is given by

$$F_X(x;\alpha,\beta,\lambda) = 1 - C_0(\alpha,\beta,\lambda)e^{-\lambda x} + C_1(\alpha,\beta,\lambda)e^{-\lambda(\alpha+1)x} + C_2(\alpha,\beta,\lambda)xe^{-\lambda(\alpha+1)x} \tag{11}$$

where

$$C_0(\alpha,\beta,\lambda) = \frac{(1+\alpha)^2(\lambda+\beta)}{\alpha(\lambda(1+\alpha)+\alpha\beta)}, \quad C_1(\alpha,\beta,\lambda) = \frac{(1+\alpha)(\lambda+\beta)+\alpha\beta}{\alpha(\lambda(1+\alpha)+\alpha\beta)},$$

$$C_2(\alpha,\beta,\lambda) = \frac{\beta\lambda(1+\alpha)}{(\lambda(1+\alpha)+\alpha\beta)}$$

Also, the survival function and hazard rate function (HRF) of X can be placed in the following compact forms respectively:

$$\overline{F}_X(x;\alpha,\beta,\lambda) = C_0(\alpha,\beta,\lambda)e^{-\lambda x} - C_1(\alpha,\beta,\lambda)e^{-\lambda(\alpha+1)x} - C_2(\alpha,\beta,\lambda)xe^{-\lambda(\alpha+1)x} \tag{12}$$

$$h_X(x) = \frac{\dfrac{(1+\alpha)^2 \lambda}{\alpha(\lambda(1+\alpha)+\alpha\beta)} e^{-\lambda x} \left(\lambda + \beta - (\lambda + \beta + \alpha\beta\lambda x)^{-\alpha\lambda x}\right)}{C_0(\alpha,\beta,\lambda)e^{-\lambda x} - C_1(\alpha,\beta,\lambda)e^{-\lambda(\alpha+1)x} - C_2(\alpha,\beta,\lambda)xe^{-\lambda(\alpha+1)x}} \quad (13)$$
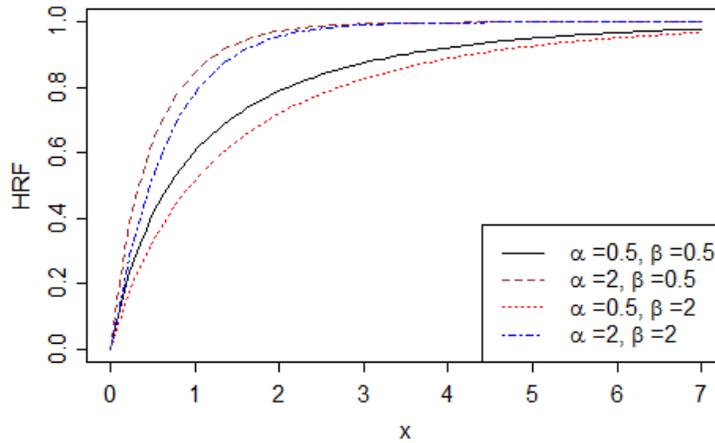

Figure 2. Plots of the EWE hazard rate function for fixed scale parameter $\lambda = 1$ and some selected shape parameters

In Figure 2, the HRF of the EWE distribution is plotted for selected values of the shape parameters and fixed scale parameter $\lambda = 1$. The HRF is an increasing function. The concept of an increasing failure rate is very attractive in an engineering context, where it has often been related to a mathematical representation of wear out (Marshall & Olkin, 2007).

One of the well-known properties of the life time distribution is mean residual life time. For the EWE distribution it can be written as

301

$$m(t) = \mathrm{E}(X - t \mid X > t)$$

$$= \frac{1}{\lambda} \times \frac{\begin{bmatrix} C_0(\alpha, \beta, \lambda) \mathrm{e}^{-\lambda t} - C_3(\alpha, \beta, \lambda) \mathrm{e}^{-\lambda(\alpha+1)t} \\ -\dfrac{1}{1+\alpha} C_2(\alpha, \beta, \lambda) t \mathrm{e}^{-\lambda(\alpha+1)t} \end{bmatrix}}{\begin{bmatrix} C_0(\alpha, \beta, \lambda) \mathrm{e}^{-\lambda t} - C_1(\alpha, \beta, \lambda) \mathrm{e}^{-\lambda(\alpha+1)t} \\ -C_2(\alpha, \beta, \lambda) t \mathrm{e}^{-\lambda(\alpha+1)t} \end{bmatrix}} \qquad (14)$$

where $C_3(\alpha, \beta, \lambda) = \dfrac{(1+\alpha)(\lambda+\beta) + 2\alpha\beta}{\alpha(\alpha+1)(\lambda(1+\alpha) + \alpha\beta)}$.

Maximum Likelihood Estimation

The MLEs will be derived for the unknown parameters of the EWE distribution from complete samples only.

Let $X_1, \ldots, X_n$ be a random sample from GWE(α, β, λ). The log-likelihood function based on the observed sample $\{x_1, \ldots, x_n\}$ is

$$l(x_1, \ldots, x_n \mid \alpha, \beta, \lambda) = 2n \ln(1+\alpha) + n \ln \lambda - n \ln \alpha - n \ln\big(\lambda(1+\alpha) + \alpha\beta\big)$$

$$- \lambda \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \ln\Big(\lambda + \beta - (\lambda + \beta + \alpha\beta\lambda x_i) e^{-\alpha\lambda x_i}\Big) \qquad (15)$$

To find the MLE estimates for the EWE model parameters, differentiate the log-likelihood function and equating the resulting expressions to 0 as follows:

$$\frac{\partial l}{\partial \alpha} = \frac{2n}{1+\alpha} - \frac{n}{\alpha} - \frac{n(\lambda+\beta)}{\lambda(1+\alpha) + \alpha\beta} + \sum_{i=1}^{n} \frac{\alpha\beta\lambda^2 x_i^2 \mathrm{e}^{-\alpha\lambda x_i} - \beta\lambda x_i \mathrm{e}^{-\alpha\lambda x_i}}{\lambda + \beta - (\lambda + \beta + \alpha\beta\lambda x_i) \mathrm{e}^{-\alpha\lambda x_i}} = 0$$

$$\frac{\partial l}{\partial \beta} = \frac{-n\alpha}{\lambda(1+\alpha) + \alpha\beta} + \sum_{i=1}^{n} \frac{1 - (1+\alpha\lambda x_i) \mathrm{e}^{-\alpha\lambda x_i}}{\lambda + \beta - (\lambda + \beta + \alpha\beta\lambda x_i) \mathrm{e}^{-\alpha\lambda x_i}} = 0$$

$$\frac{\partial l}{\partial \lambda} = \frac{n}{\lambda} - \frac{n(1+\alpha)}{\lambda(1+\alpha) + \alpha\beta} + \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \frac{1 - (1+\alpha\beta x_i) \mathrm{e}^{-\alpha\lambda x_i} + \alpha\lambda x_i (1+\alpha\beta x_i) \mathrm{e}^{-\alpha\lambda x_i}}{\lambda + \beta - (\lambda + \beta + \alpha\beta\lambda x_i) \mathrm{e}^{-\alpha\lambda x_i}} = 0$$

The MLEs of the unknown parameters cannot be obtained explicitly. They have to be obtained by solving some numerical methods, like the Newton-

Raphson method, Gauss-Newton method, or their variants. In this paper we use the *optim* function from the statistical software R (R Core Team, 2013) to estimate the unknown parameters.

Simulation

Some simulation results are presented to see how the maximum likelihood estimators behave for different sample sizes and for different parameter values. The sample sizes, namely $n = 20$, 40, 60, and 80 and two different sets of parameter values: Set 1: $\alpha = 0.5$, $\lambda = \beta = 1$, and Set 2: $\beta = 0.5$, $\alpha = \lambda = 1$. In each case, the maximum likelihood estimators of the unknown parameters are computed by maximizing the log-likelihood function (15). The average estimates and mean squared errors were computed over 1000 replications and the results are reported in Table 1. In all the cases the performances of the maximum likelihood estimates are quite satisfactory. As sample size increases the average estimates and the mean squared error decrease for all the parameters, as expected. It verifies the consistency properties of the MLEs.

Table 1. The average MLEs and the associated square root of the mean squared errors (within brackets)

	Set 1			Set 2		
n	*α*	*β*	*λ*	*α*	*β*	*λ*
20	0.6143	1.1257	1.1014	1.1316	0.6013	1.1286
	(0.0726)	(0.0793)	(0.0563)	(0.0811)	(0.0701)	(0.0599)
40	0.5825	1.1094	1.0614	1.1105	0.5784	1.0742
	(0.0592)	(0.0696)	(0.0352)	(0.0713)	(0.0501)	(0.0431)
60	0.5675	1.0835	1.0452	1.0922	0.0553	1.0562
	(0.0411)	(0.0536)	(0.0261)	(0.0658)	(0.0398)	(0.0371)
80	0.5595	1.0658	1.0352	1.0715	0.5462	1.0402
	(0.0388)	(0.0414)	(0.0201)	(0.0456)	(0.0321)	(0.0245)

Data Analysis

Two real data sets are considered to demonstrate the performance of the proposed distribution in practice. For each data set, the results of the fitted proposed model are compared with the WE, TWE, and EE models. To see which one of these models is more appropriate to fit the data set, the MLEs of unknown parameters and Akaike information criterion (AIC) were computed. The Kolmogorov-

Smirnov (K-S) distance between the empirical cumulative distribution function and the fitted distribution function was obtained in each case, as well as the associated p-value.

Data Set 1: Bjerkedal (1960) provided a data set consisting of survival times of 72 Guinea pigs injected with different amount of tubercle. We consider only the study in which animals in a single cage are under the same regimen. The data represents the survival times of Guinea pigs in days. The data are given below:

12 15 22 24 24 32 32 33 34 38 38 43 44 48 52 53 54 54 55 56 57 58 58 59 60 60 60 60 61 62 63 65 65 67 68 70 70 72 73 75 76 76 81 83 84 85 87 91 95 96 98 99 109 110 121 127 129 131 143 146 146 175 175 211 233 258 258 263 297 341 341 376

Table 2. The MLEs of parameters, AIC, and K-S statistics for the Guinea pigs data

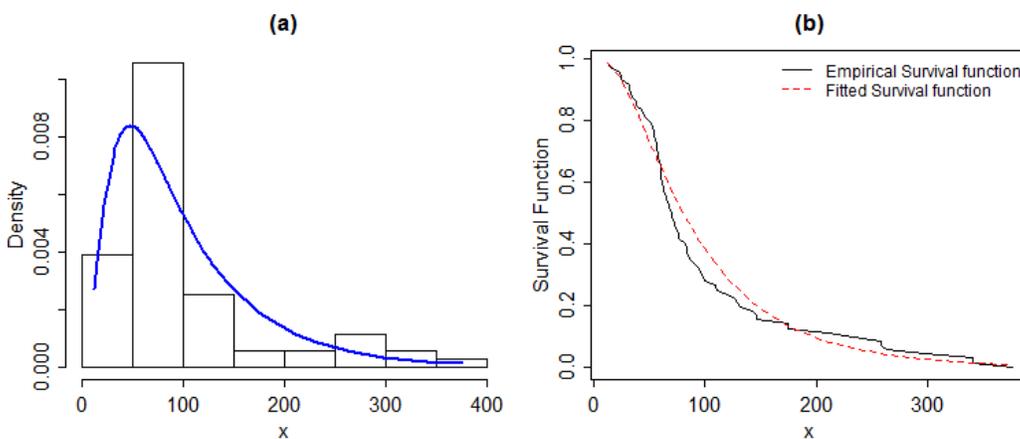| Model	MLE of the parameters	AIC	K-S statistics	*p*-value
EE(β, λ)	10.1738, 0.0200	792.6086	0.1334	0.1544
WE(α, λ)	1.6312, 0.0138	791.1381	0.1153	0.2939
TWE(α_1, α_2, λ)	2.8013, 2.8013, 0.0142	789.0153	0.1132	0.3147
EWE(α, β, λ)	3.9035, 3.0313, 0.0141	**788.7657**	**0.1129**	**0.3174**

Figure 3. The fitted EWE distribution and the relative histogram for the Guinea pigs data (a); Empirical and fitted survival functions for the Guinea pigs data (b)

Table 3. The MLEs of parameters, AIC, and K-S statistics for the melanoma data

Model	MLE of the parameters	AIC	K-S statistics	p-value
EE(β, λ)	4.1321, 0.0019	913.2957	0.1551	0.1158
WE(α, λ)	1.6197 ,0.0010	912.5643	0.0767	0.8651
TWE(α_1, α_2, λ)	0.0136 0.2099 0.0022	913.3586	0.0696	0.9271
EWE(α, β, λ)	0.0375, 0.1435, 0.0021	**912.0859**	**0.0620**	**0.9710**

It is clear from Table 2 that, based on the AIC value and also based on the K-S statistic, the proposed EWE model provides a better fit than the WE, TWE and EE models for this specific data set. The relative histogram and the fitted EWE distribution are plotted in Figure 3. In order to assess if the model is appropriate, the plots of the fitted EWE survival function and empirical survival function are displayed in Figure 3.

Data Set 2:   This data set relates to survival time for 57 patients in Denmark with malignant melanoma (Andersen, Borgan, Gill, & Keiding, 1993). The data are given below:

185 204 210 232 279 295 386 426 469 529 621 629 659 667 718 752 779 793
817 833 858 869 872 967 977 982 1041 1055 1062 1075 1156 1228 1252 1271
1312 1435 1506 1516 1548 1560 1584 1621 1667 1690 1726 1933 2061 2062
2103 2108 2256 2388 2467 2565 2782 3042 3338

The results are given in Table 3. The lowest values of the AIC and K-S test statistics are obtained for the EWE distribution. Based on these measures, the EWE is the best distribution among all those used here to fit the data set. In order to assess if the model is appropriate, the histogram of the data and the plot of the fitted EWE model are displayed in Figure 4.
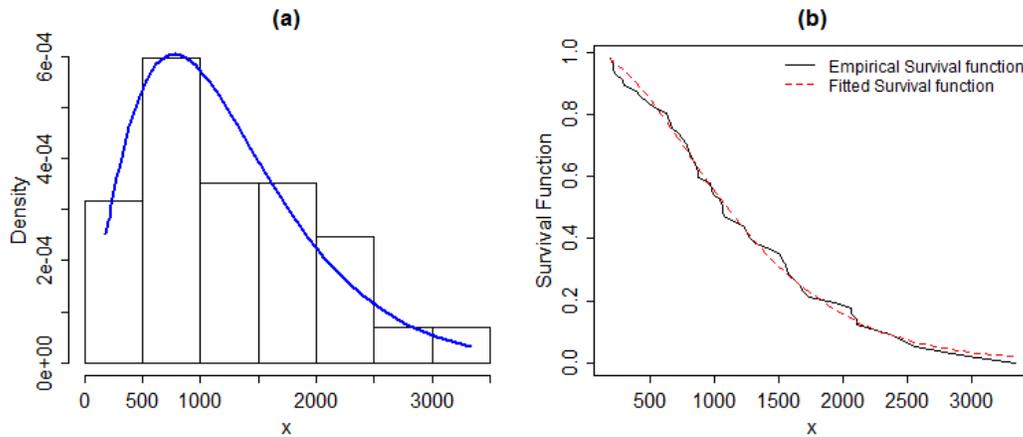
Figure 4. The fitted EWE distribution and the relative histogram for the melanoma data (a); Empirical and fitted survival functions for melanoma data (b)

Conclusion

A new class of weighted distributions based on the extended exponential distribution were introduced. The proposed model contains the WE model as its submodel. It is shown that the distribution function, hazard function, and moment generating function can be obtained in closed form. The MLEs can be computed using numerical algorithms. The failure rate function of proposed distributions is an increasing function. The flexibility of the proposed distribution and increased range of skewness was able to fit and capture features in two real data sets much better than the WE and other popular distributions.

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