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Variational Analysis In Parametric Optimization

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VARIATIONAL ANALYSIS IN PARAMETRIC OPTIMIZATION

by

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DEDICATION

To my Grandmother

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Chapter 1

Introduction and Preliminaries

1.1 Introduction

The class of Asplund spaces, i.e., Banach spaces where every convex continuous function is generically Frétchet differentiable, is sufficiently rich and well-investigated in the geometric theory of Banach spaces and various applications; see [1, 13, 34, 43] and the references therein. In particular, it includes every Banach space with a Frétchet differentiable renorm (hence any reflexive space) and all spaces with separable duals. This dissertation is concerned with the family of parameterized minimization problems:

$$\text{Minimize } \psi_0(x,y)+\psi(x,y) \ \text{ over } \ x\in X, \tag{1.1}$$

where X, Y are Asplund spaces, and $\psi_0, \psi : X\times Y \longrightarrow \overline{I\!R}$ are lower semicontinuous, extended real-valued functions. In this model, the *cost functions* ψ_0 depend on the *parameter* x and the *decision variable* y, and the *constraint functions* ψ incorporate parameter-dependent constraints in the problem under consideration. In particular, this model covers parameterized problems of nonlinear programming, where the focus is on sensitivity analysis of *stationary point multifuctions* and *stationary point-multiplier-multifuntions* involving Karush-Kuhn-Tucker vectors associated with first-order necessary optimality conditions. A nonsmooth version of Fermat's rule [34, Proposition 1.114] gives a necessary optimality condition for problem (1.1) via the so-called *partial subdifferentials* considered as generalized partial derivatives. To be precise, if $\bar{x}$ is a solution to problem (1.1) with compatible parameter $\bar{y}$ then

$$0\in\widehat{\partial}_x(\psi_0+\psi)(\bar{x},\bar{y})\subset\partial_x(\psi_0+\psi)(\bar{x},\bar{y}),$$

where $\widehat{\partial}_x$ and ∂_x denote the Frétchet and basic partial subdifferentials in x, respectively. Under some suitable assumptions, in particular, when ψ_0 is strictly differentiable in x at $\bar{x}$, the *stationary point multifunction* is given by

$$S(y)=\big\{x\in X \mid 0\in\partial_x\psi_0(x,y)+\partial_x\psi(x,y)\big\}. \tag{1.2}$$

The set-valued mapping S in (1.2) is the *solution map* to the so-called *parameterized generalized equation* given in the form

$$0 \in \partial_x \psi_0(x, y) + \partial_x \psi(x, y), \tag{1.3}$$

which is a special case of

$$0 \in F(x, y) + Q(x, y), \tag{1.4}$$

where both *base* F and *field* Q are set-valued mappings depending on the parameter y.

Solution maps to parametric generalized equations of type (1.4) were recently studied by Mordukhovich and Nam [32] in Asplund space settings while their less general model with single-valued base $F(x, y) = f(x, y)$ was previously studied by Levy and Mordukhovich [21] in finite dimensions. In literature, the generalized equation (1.4) was introduced by Robinson [46] with single-valued base $F(x, y) := f(x, y)$ and parameter-independent field $Q(x, y) = Q(x)$, and has been well recognized to provide a convenient framework for the unified study of *optimal solutions* in many optimization-related areas including mathematical programming, complementarity, variational inequalities, optimal control, mathematical economics, equilibria, game theory, etc. We refer the readers to the recent texts [16, 34, 35, 42] and the bibliographies therein for various results, discussions, and applications regarding the parametric generalized equations of type (1.4) with $F(x, y) = f(x, y)$ and $Q(x, y) = Q(x)$ in both finite and infinite dimensions.

In this dissertation, we will focus our attention on the solution maps of type (1.2). Our primary goal is to study the dependence of S on the parameter y near the reference point. These questions are addressed to *local sensitivity analysis* of solution maps under parameter perturbations. One of the principal questions here is the solution *stability* under parameter perturbations, which is important not only for better understanding of the solution behavior with respect to perturbations, but also for constructing effective numerical algorithms to solve the problems. We particularly interested in *robust Lipschitzian stability* of multivalued solution maps (1.2) with respect to parameter perturbations. Note that both classical local Lipschitzian property and *Lipschitz-like* property, a natural extension of Lipschitz continuity to set-valued mappings introduced by Aubin [2], are *robust (stable)* with respect to small perturbations of initial data, which is of great significance for sensitivity analysis. Let us mention here some applications of coderivative analysis to various problems related to Lipschitzian stability of constraint and variational systems in finite-dimensional settings given in [14, 21, 22, 23, 27, 30, 41, 45], among other publications. However, not much attention has been paid to the study of Lipschitzian stability of solution maps (1.2) in infinite-dimensional settings.

To conduct such a local sensitivity analysis for solution maps (1.2), we estimate the coderivatives of S via the *second-order partial subdifferentials* of the cost and constraint functions, and employ the *pointbased coderivative characterization* of robust Lipchitzian behavior developed by Mordukhovich [26] to derive efficient *sufficient* (as well as *necessary and sufficient*) conditions for Lipschitzian stability of the solution maps (1.2) with evaluating the *exact Lipschitzian bound*. In order to do this, we need efficient calculus rules for second-order partial generalized differential constructions together with the corresponding sequential normal compactness. Nevertheless, while second-order ("full") subdifferential calculus (see [14, 24, 25, 34, 30, 38, 39, 44, 48]) and the SNC calculus (see [34, 36, 37]) have been developed by many authors, not much work has been done on second-order *partial* subdifferential calculus, especially in infinite dimensions. Thus, in this dissertation, we will develop as well the calculus rules for second-order partial subdifferentials of extended-real-valued functions in both Banach and Asplund space settings. This calculus is not only useful for the study of sensitivity analysis mentioned above but also of independent interest.

Along with the solution maps (1.2), we also pay our attention to the so-called *quasi-variational inequalities (QVIs)* of the following type: Given a parameter $y \in Y$, find a decision vector $x \in \Gamma(x,y) \subset X$ such that

$$\langle g(x,y), u-x \rangle \geq 0 \ \text{ for all } \ u \in \Gamma(x,y) \tag{1.5}$$

where X, Y are Asplund spaces, $g : X \times Y \to X^*$ is a single-valued continuously differentiable function, while $\Gamma : X \times Y \rightrightarrows X$ is a set-valued mapping.

QVIs were introduced by Bensoussa and Lions in a series of papers (see, e.g., [6]) in connection with *impulse optimal control* problems. They have been extensively studied in numerous publications, mainly from the viewpoints of existence of solutions and numerical methods; cf. [5, 11, 12, 16, 20, 40], among others. Much less attention has been paid to the study of *parameter-dependent* QVIs, especially those where *both* mappings g and Γ in (1.5) depend on parameters in infinite dimensions. The primary goal of our consideration is to undertake such a study concentrating mainly on the *sensitivity analysis* and conduct verifiable conditions ensuring *robust Lipschitzian stability* of the solution maps to QVIs (1.5) given by

$$S(y) = \big\{x \in X \big|\ \langle g(x,y), u - x\rangle \geq 0 \ \text{ whenever } \ u \in \Gamma(x,y)\big\}, \quad y \in Y, \tag{1.6}$$

which become

$$S(y) = \big\{x \in X \big|\ 0 \in g(x,y) + N\big(x; \Gamma(x,y)\big)\big\} \tag{1.7}$$

if Γ is of closed-graph and take convex values $\Gamma(x,y)$. It is obvious that solution maps (1.7) are special cases of solution maps to (1.4) with $F(x,y) := g(x,y)$ and $Q(x,y) := N\big(x; \Gamma(x,y)\big)$. However, the sensitivity analysis of solution maps to (1.4) does *not* implies the results derived in this dissertation. Our results on QVIs presented in what follows can be considered as generalization to Asplund space settings of the ones obtained by Mordukhovich and Outrata [31] in finite dimensions.

The outline of the dissertation is as follows. Chapter 1 provides preliminary material from variational analysis and generalized differentiation needed for the subsequent sections.

In Chapter 2, we study the first-order and second-order partial subdifferential calculi and obtain several sum rules and chain rules of rather general nonsmooth extended-real-valued functions in Banach space settings as well as more developed rules in Asplund space settings.

In Chapter 3, after establishing a coderivative estimate for the solution maps (1.2) in the case both ψ_0 and ψ are nonsmooth, we turn our attention to the case when cost function ψ_0 is $\mathcal{C}^2$ and study the parametric sensitivity of stationary points alone, as well as the stationary point-multiplier pairs asscociated with the parameterized optimization problems (1.1).

Finally, Chapter 4 is devoted to the study of coderivative analysis of parameter-dependent quasi-variational inequalities in Asplund spaces with application to stability.

1.2 Basic Definitions and Preliminaries

This part contains some material on generalized differentiation widely used in what follows. We refer the reader to the book by Mordukhovich [34] for more details, references, and discussions.

Unless otherwise stated, all spaces considered are *real Banach* whose norms are always denoted by $\|\cdot\|$. For any space X we consider its dual space X^* equipped with the weak-star topology, where $\langle\cdot,\cdot\rangle$ means the canonical pairing.

In contrast to the case of *single-valued mapping* $f : X \to Y$, the symbol $F : X \rightrightarrows Y$ stands for a *set-valued mapping (multifunction)* from X into Y with the *domain* and *kernel* denoted, respectively, by

$$\operatorname{dom} F := \big\{x \in X \big|\ F(x) \neq \emptyset\big\} \ \text{ and } \ \ker F := \big\{x \in X \big|\ 0 \in F(x)\big\}.$$

The *inverse* set-valued mapping $F^{-1} : Y \rightrightarrows X$ to F satisfies the relationships

$$x \in F^{-1}(y) \iff y \in F(x) \iff (x, y) \in \operatorname{gph} F,$$

and the *norm* of any positive homogeneous set-valued mapping is given by

$$\|F\| := \sup\big\{\|y\| \ \big|\ y \in F(x) \ \text{ and } \ \|x\| \le 1\big\}.$$

In particular, if $F : X \rightrightarrows X^*$, the expression

$$\operatorname*{Lim\,sup}_{x\to\bar{x}} F(x) := \big\{x^* \big|\ \exists x_k \to \bar{x} \text{ and } x_k^* \xrightarrow{w^*} x^* \text{ with } x_k^* \in F(x_k) \text{ for all } k \in I\!N\big\}$$

always means the *sequential Painlevé-Kuratowski upper/outer limit* with respect to the norm topology in X and the weak-star topology in X^*.

Given an extended-real-valued function $\varphi\colon X \to \overline{I\!R} := [-\infty, \infty]$ finite at $\bar{x}$, we start with the definition of its *ε-subgradients* $x^* \in \widehat{\partial}_\varepsilon\varphi(\bar{x})$, $\varepsilon \ge 0$, which form the enlargement of the *Fréchet subdifferential* $\widehat{\partial}\varphi(\bar{x}) := \widehat{\partial}_0\varphi(\bar{x})$ of φ at $\bar{x}$:

$$\widehat{\partial}_\varepsilon\varphi(\bar{x}) := \Big\{x^* \in X^* \Big|\ \liminf_{x\to\bar{x}} \frac{\varphi(x) - \varphi(\bar{x}) - \langle x^*, x - \bar{x}\rangle}{\|x - \bar{x}\|} \ge -\varepsilon\Big\}. \tag{1.8}$$

In (1.8), as usual, *subgradients* means actually *lower subgradients*, where the word "lower" is omitted by taking it for granted.

Based on (1.8), we introduce two *limiting subdifferentials* by using the sequential Painlevé-Kuratowski upper/outer limit. The first construction

$$\partial\varphi(\bar{x}) := \mathop{\mathrm{Lim\,sup}}_{\substack{x \xrightarrow{\varphi} \bar{x} \\ \varepsilon \downarrow 0}} \widehat{\partial}_\varepsilon \varphi(x) \tag{1.9}$$

is called the *basic subdifferential* of φ at $\bar{x}$ which reduces to the classical Fréchet derivative for strictly differentiable functions and to the subdifferential of convex analysis for convex functions. If X is an *Asplund space* and if φ is *lower semicontinuous* (l.s.c.) around $\bar{x}$, then we can equivalently put $\varepsilon = 0$ in (1.9). Moreover, we have $\partial\varphi(\bar{x}) \neq \emptyset$ for every locally Lipschitzian function on an Asplund space.

One obviously has $\widehat{\partial}\varphi(\bar{x}) \subset \partial\varphi(\bar{x})$ for any $\varphi\colon X \to \overline{I\!R}$ finite at $\bar{x}$. We say that φ is *lower regular* at $\bar{x}$ if the latter holds as equality. The collection of lower regular functions is sufficiently large including, besides convex and strictly differentiable functions, many other classes of functions important in variational analysis and optimization; see the books [34, 48] for more details, discussions, and applications.

The second limiting subdifferential construction defined by

$$\partial^\infty\varphi(\bar{x}) := \mathop{\mathrm{Lim\,sup}}_{\substack{x \xrightarrow{\varphi} \bar{x} \\ \varepsilon,\lambda \downarrow 0}} \lambda\widehat{\partial}_\varepsilon \varphi(x) \tag{1.10}$$

is called the *singular subdifferential* of φ at $\bar{x}$, where ε can be omitted for l.s.c. functions on Asplund spaces. Construction (1.10) carries nontrivial information only for non-Lipschitzian functions, since $\partial^\infty\varphi(\bar{x}) = \{0\}$ if φ is locally Lipschitzian around $\bar{x}$. Moreover, the latter condition is also *necessary* for φ to be locally Lipschitzian provided that it is l.s.c. around $\bar{x}$, that X is Asplund, and that φ has the so-called *sequential normal epi-compactness* property at $\bar{x}$ (see below), which is always the case when X is finite-dimensional ($X = I\!R^n$).

Given a set $\Omega \subset X$ and $\varepsilon \geq 0$, we define the *set of ε-normals* to Ω at $\bar{x} \in \Omega$ by

$$\widehat{N}_\varepsilon(\bar{x};\Omega) := \Big\{ x^* \in X^* \Big|\ \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle x^*, x - \bar{x}\rangle}{\|x - \bar{x}\|} \leq \varepsilon \Big\}, \tag{1.11}$$

where the symbol $x \xrightarrow{\Omega} \bar{x}$ signifies that $x \to \bar{x}$ with $x \in \Omega$. When $\varepsilon = 0$, this set is called the *Fréchet normal cone* to Ω at $\bar{x}$ and is denoted by $\widehat{N}(\bar{x};\Omega)$. The Fréchet normal cone looks like an adaption of the idea of Fréchet derivative to the case of sets. However, this construction does not have a number of natural properties expected from an appropriate notion of normals. In particular, we may have $\widehat{N}(\bar{x};\Omega) = \{0\}$ for boundary points of Ω even in finite dimensions, and calculus rules often fail. Letting for convenience $\widehat{N}(x;\Omega) = \emptyset$ if $x \notin \Omega$ and employing the outer limit to $\widehat{N}(\cdot;\Omega)$, we define its sequential regularization

$$N(\bar{x};\Omega) := \mathop{\mathrm{Lim\,sup}}_{\substack{x \to \bar{x} \\ \varepsilon \downarrow 0}} \widehat{N}_\varepsilon(x;\Omega), \tag{1.12}$$

where ε can be removed in Asplund space setting. This construction is known as the *basic/limiting/Mordukhovich normal cone* to Ω at $\bar{x}$. It is obvious that

$$\widehat{N}(\bar{x},\Omega) \subset N(\bar{x};\Omega) \ \text{ for any } \ \Omega \subset X \text{ and } \bar{x} \in \Omega.$$

The equality in this inclusion singles out a class of the so-called *normally regular* sets. An important example of set regularity is given by sets Ω locally convex around $\bar{x}$. In case the set $\operatorname{epi}\varphi := \big\{(x,\alpha) \in X \times I\!R \big|\ \alpha \geq \varphi(x)\big\}$ of an extended-real-valued function $\varphi : X \to \overline{I\!R}$ is normally regular at $\big(x,\varphi(x)\big)$ we say that φ is *epigraphically regular* at $\bar{x}$.

In contrast to (1.11), the basic normal cone (1.12) maybe *nonconvex* in very simple situations but enjoys some calculus rules in Asplund space setting; see [34]. Both Fréchet normal cone $\widehat{N}(\cdot;\Omega)$ and basic normal cone $N(\cdot;\Omega)$ reduce to the classical normal cone of convex analysis for convex sets Ω. These two constructions are *invariant* with respect to equivalent norms on X while the ε-normal sets $\widehat{N}_\varepsilon(\cdot;\Omega)$ depends on given norm $\|\cdot\|$ if $\varepsilon > 0$. Therefore, it is not hard to prove the following representations of Fréchet and basic normals to Cartesian product of sets:

$$\widehat{N}(\bar{x};\Omega_1 \times \Omega_2) = \widehat{N}(\bar{x}_1;\Omega_1) \times \widehat{N}(\bar{x}_2;\Omega_2),$$
$$N(\bar{x};\Omega_1 \times \Omega_2) = N(\bar{x}_1;\Omega_1) \times N(\bar{x}_2;\Omega_2),$$

for any point $\bar{x} = (\bar{x}_1,\bar{x}_2) \in \Omega_1 \times \Omega_2 \subset X_1 \times X_2$. On the other hand, let $\delta(\cdot;\Omega)$ be the *indicator function* of Ω, i.e., $\delta(x;\Omega) := 0$ if $x \in \Omega$ and $\delta(x;\Omega) = \infty$ otherwise, the Fréchet and basic normal cones to Ω at $\bar{x}$ are actually the corresponding subdifferentials of the indicator function. Indeed, we have

$$\widehat{N}(\bar{x};\Omega) = \widehat{\partial}\delta(\bar{x};\Omega) \ \text{ and } \ N(\bar{x};\Omega) = \partial\delta(\bar{x};\Omega) = \partial^{\infty}\delta(\bar{x};\Omega).$$

Given a set-valued mapping $F\colon X \rightrightarrows Y$ with the *graph*

$$\operatorname{gph} F := \big\{(x,y) \in X \times Y \big|\ y \in F(x)\big\},$$

we define its *Fréchet coderivative* at $(\bar{x},\bar{y}) \in \operatorname{gph} F$ and its *limiting coderivative* at $(\bar{x},\bar{y})$, respectively, by

$$\widehat{D}^*F(\bar{x},\bar{y})(y^*) := \big\{x^* \in X^* \big|\ (x^*,-y^*) \in \widehat{N}((\bar{x},\bar{y});\operatorname{gph} F)\big\}, \tag{1.13}$$

$$D^*_N F(\bar{x},\bar{y})(y^*) := \big\{x^* \in X^* \big|\ (x^*,-y^*) \in N((\bar{x},\bar{y});\operatorname{gph} F)\big\}. \tag{1.14}$$

Note that the coderivative construction (1.14) is called the *normal coderivative* to distinguish it from another limiting coderivative for mappings between infinite-dimensional spaces, which is called *mixed coderivative* and defined by

$$D^*_M F(\bar{x},\bar{y})(y^*) := \Big\{x^* \in X^* \Big|\ \exists\, \varepsilon_k \downarrow 0,\ (x_k,y_k) \xrightarrow{\operatorname{gph} F} (\bar{x},\bar{y}),\ x^*_k \xrightarrow{w^*} x^* \text{ and } y^*_k \xrightarrow{\|\cdot\|} y^* \ \text{ with } \ (x^*_k,-y^*_k) \in \widehat{N}_{\varepsilon_k}((x_k,y_k);\operatorname{gph} F)\Big\}, \tag{1.15}$$

where $\xrightarrow{w^*}$ signifies the weak* sequential convergence in X^*, while $\xrightarrow{\|\cdot\|}$ stands for the norm convergence in the dual space; we omit $\|\cdot\|$ in what follows. We can put $\varepsilon_k = 0$ in (1.15) if X and Y are Asplund and if the graph of F is closed around $(\bar{x},\bar{y})$. We also drop $\bar{y}$ in the coderivative notation (1.13), (1.14) and (1.15) if $F = f\colon X \to Y$ is single-valued. These coderivatives are extensions of the corresponding *adjoint* derivative operators, in the sense that

$$\widehat{D}^*f(\bar{x})(y^*) = \big\{\nabla f(\bar{x})^*y^*\big\} \ \text{ and } \ D^*f(\bar{x})(y^*) = \big\{\nabla f(\bar{x})^*y^*\big\} \tag{1.16}$$

for all $y^* \in Y^*$ if f is Fréchet differentiable and strictly differentiable at $\bar{x}$, respectively. In general, all of the three coderivatives defined above are positively homogeneous multifunctions from Y^* to X^* that satisfy the inclusions

$$\widehat{D}^*F(\bar{x},\bar{y})(y^*) \subset D^*_M F(\bar{x},\bar{y})(y^*) \subset D^*_N F(\bar{x},\bar{y})(y^*) \ \text{ for all } \ y^* \in Y^*.$$

If the equality holds in the former inclusion then F is said to be *M-regular* at $(\bar{x}, \bar{y})$. In the case that $D_N^* F(\bar{x}, \bar{y})(y^*) = \widehat{D}^* F(\bar{x}, \bar{y})(y^*)$ for all $y^* \in Y^*$, F is said to be *N-regular* at $(\bar{x}, \bar{y})$. Obviously, N-regularity always implies M-regularity but not vice versa; see [34] for examples. The N-regularity holds if, in particular, if $F = f$ is single-valued and *smooth* around $\bar{x}$ (or merely *strictly differentiable* at this point). Besides, this property holds for *convex-graph* mappings (i.e., mappings that have convex graphs) and other classes of set-valued mappings, while it may be violated in many important situations, e.g., for every locally Lipschitzian mapping $f\colon X \to Y$ that is not strictly differentiable at $\bar{x}$; see [34, Subsection 3.2.4] for exact results, proofs, and discussions.

Observe that there is a simple relationship between the Fréchet coderivative of a locally Lipschitzian mapping $f\colon X \to Y$ and the Fréchet subdifferential of its scalarization

$$\langle y^*, f\rangle(x) := \langle y^*, f(x)\rangle, \quad y^* \in Y^*.$$

This relationship called the *scalarization formula* is given by

$$\widehat{D}^* f(\bar{x})(y^*) = \widehat{\partial}\langle y^*, f\rangle(\bar{x}) \ \text{ for any } \ y^* \in Y^*. \tag{1.17}$$

A similar scalarization formula holds for the limiting constructions (1.9) and (1.14) but under an additional *strict Lipschitzian* assumption on f that reduces to the standard local Lipschitz continuity of f around $\bar{x}$ if Y is finite-dimensional; see [34, Subsection 3.1.3] and the references therein.

If a multifunction $F : X \times Y \rightrightarrows Z$ is of two variables $(x, y) \in X \times Y$, we denote by $D_x^* F(\bar{x}, \bar{y}, \bar{z})$ its *partial coderivative* (either normal or mixed) with respect to x at the point $(\bar{x}, \bar{y}, \bar{z}) \in \operatorname{gph} F$ which is defined as the corresponding coderivative of the multifunction $F(\cdot, \bar{y})$ at $(\bar{x}, \bar{z})$.

Let us mention also the relationships between subgradients and coderivatives. Given $\varphi : X \to \overline{I\!R}$, we associate with it the *epigraphical multifunction* $E_\varphi : X \to I\!R$ defined by

$$E_\varphi(x) := \big\{\alpha \in I\!R \big|\ \alpha \geq \varphi(x)\big\}.$$

Since E_φ takes values in $I\!R$, there is no difference between its normal and mixed coderivatives. Also note that $\operatorname{gph} E_\varphi = \operatorname{epi}\varphi$. We have

$$\partial\varphi(\bar{x}) = D^*E_\varphi\big(\bar{x},\varphi(\bar{x})\big)(1) = \big\{x^* \in X^* \big|\ (x^*,-1) \in N\big((\bar{x},\varphi(\bar{x}));\operatorname{epi}\varphi\big)\big\},$$
$$\partial^\infty\varphi(\bar{x}) = D^*E_\varphi\big(\bar{x},\varphi(\bar{x})\big)(0) = \big\{x^* \in X^* \big|\ (x^*,0) \in N\big((\bar{x},\varphi(\bar{x}));\operatorname{epi}\varphi\big)\big\}.$$

One of the most fundamental differences between variational analysis in finite and infinite dimensions is the necessity of imposing certain *normal compactness* requirements in infinite-dimensional settings that ensure certain *nontriviality* conclusions while passing to the limit in the weak* topology. We use the following normal compactness properties that are automatic in finite dimensions, hold for "reasonably good" sets and mappings, and are preserved under various operations.

A set $\Omega \subset X$ is *sequentially normally compact (SNC)* at $\bar{x} \in \Omega$ if for any sequences $\varepsilon_k \downarrow 0$, $x_k \xrightarrow{\Omega} \bar{x}$, and $x_k^* \in \widehat{N}_{\varepsilon_k}(x_k;\Omega)$ such that $x_k^* \xrightarrow{w^*} 0$ one has

$$\|x_k^*\| \to 0 \ \text{ as } \ k \to \infty,$$

where ε_k can be omitted if X is Asplund and if Ω is locally closed around $\bar{x}$. The SNC property is always satisfied if Ω is *compactly epi-Lipschitzian (CEL)* at $\bar{x}$ in the sense of Borwein and Strójwas, that is, there are a compact set $C \subset \Omega$, a neighborhood U of $\bar{x}$, a neighborhood O of the origin in X, and a number $\gamma > 0$ such that

$$\Omega \cap U + tO \subset \Omega + tC \ \text{ for all } \ t \in (0,\gamma).$$

A set-valued mapping $F\colon X \rightrightarrows Y$ is *SNC* at $(\bar{x},\bar{y}) \in \operatorname{gph} F$ if its graph enjoys this property. In addition, more subtle *partial SNC* and *strongly partial SNC* (i.e., *PSNC* and *strongly PSNC*) properties can be defined. We say that F is *PSNC* at $(\bar{x},\bar{y}) \in \operatorname{gph} F$ if for any sequences $\varepsilon_k \downarrow 0$, $(x_k,y_k) \to (\bar{x},\bar{y})$ with $(x_k,y_k) \in \operatorname{gph} F$, $(x_k^*,y_k^*) \in \widehat{N}_{\varepsilon_k}((x_k,y_k);\operatorname{gph} F)$ one has

$$\big[x_k^* \xrightarrow{w^*} 0,\ \|y_k^*\| \to 0\big] \Longrightarrow \|x_k^*\| \to 0 \ \text{ as } \ k \to \infty, \tag{1.18}$$

where $\varepsilon_k = 0$ in the Asplund space and closed graph setting. F is said to be *strongly PSNC* at $(\bar{x},\bar{y})$ if (1.18) is replaced by

$$\big[x_k^* \xrightarrow{w^*} 0,\ y_k^* \xrightarrow{w^*} 0\big] \Longrightarrow \|x_k^*\| \to 0 \ \text{ as } \ k \to \infty.$$

The PSNC property always holds when F is *Lipschitz-like* around $(\bar{x}, \bar{y})$ in the following sense of Aubin [2]: there exist neighborhoods U of $\bar{x}$ and V of $\bar{y}$ as well as a modulus $\ell \geq 0$ such that

$$F(u) \cap V \subset F(v) + \ell \|u - v\| I\!B_Y \text{ whenever } u, v \in U, \tag{1.19}$$

where $I\!B_Y$, as usual, denotes the unit ball in Y. This reduces to the classical (Hausdorff) *local Lipschitz* behavior of F around $\bar{z}$ corresponding to $V = Y$ in (1.19). The infimum of all Lipschitzian moduli ℓ in (1.19) is called the *exact Lipschitzian bound* of F around $(\bar{x}, \bar{y})$ and is denoted by $\operatorname{lip} F(\bar{x}, \bar{y})$.

Note that there is no difference between the SNC and PSNC properties for extended-real-valued functions. We also consider an epigraphical version of the SNC property of $\varphi : X \to \overline{I\!R}$. We say that φ is *sequentially normally epi-compact (SNEC)* at $\bar{x} \in X$ if its epigraph is SNC at $(\bar{x}, \varphi(\bar{x}))$. This property always holds for locally Lipschitzian functions, their natural *directionally Lipschitzian* extensions (cf. Rockafellar [47]), etc.

We also use in this dissertation a less restrictive counterpart of Lipschitz-like property of set-valued mappings called *calmness*, where $u \in U$ that varies around $\bar{x}$ is replaced by $\bar{x}$ in (1.19): A set-valued mapping $F : X \rightrightarrows Y$ between Banach spaces X and Y is *calm* at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ with modulus $\ell \geq 0$ if there are neighborhood U of $\bar{x}$ and V of $\bar{y}$ such that

$$F(x) \cap V \subset F(\bar{x}) + \ell \|x - \bar{x}\| I\!B_Y \text{ for all } x \in U. \tag{1.20}$$

This calmness property was first recognized by Robinson [46] under the name *upper-Lipschitzian*, where $V = Y$ in (1.20). As observed by Henrion and Outrata [17], the calmness property of F at $(\bar{x}, \bar{y})$ is equivalent to the condition that there exist $\ell \geq 0$ and $\varepsilon > 0$ such that

$$d\big(y, F(\bar{x})\big) \leq \ell d(x, \bar{x}) \text{ for all } y \in F(x) \cap I\!B(\bar{y}, \varepsilon),\ x \in I\!B(\bar{x}, \varepsilon),$$

where $d\big(\cdot, F(\bar{x})\big)$ denotes the *distance function* to the set $F(\bar{x})$.

The following *pointbased coderivative characterization* of the Lipschitz-like property obtained by Mordukhovich [34, Theorem 4.10] is the basis for applications of the coderivative calculi to *robust Lipschitzian stability* of the extended generalized equations and their specifications. Note that this result provides not only *necessary and sufficient* conditions for the

Lipschitz-like property for generalized set-valued mappings but also establishes lower and upper *estimates* of the *exact Lipschitzian bound*, which thus give the *precise formula* when the coderivative norms $\|D^*_M F(\bar{x},\bar{y})\|$ and $\|D^*_N F(\bar{x},\bar{y})\|$ agree; see [34, Proposition 4.9] for efficient conditions ensuring this property in infinite-dimensional spaces.

Theorem 1.1 (pointbased coderivative characterization of Lipschitz-like property) *Let $F : X \rightrightarrows Y$ be closed-graph around a given point $(\bar{x},\bar{y}) \in \operatorname{gph} F$. Then F is Lipschitz-like around this point if and only if F is PSNC at $(\bar{x},\bar{y})$ and*

$$D^*_M F(\bar{x},\bar{y})(0) = \{0\}.$$

Furthermore, in this case one has the estimates

$$\|D^*_M F(\bar{x},\bar{y})\| \le \operatorname{lip} F(\bar{x},\bar{y}) \le \|D^*_N F(\bar{x},\bar{y})\|,$$

where the upper estimate holds if $\dim X < \infty$.

Our notation is basically standard; see the books by Rockafellar and Wets [48] and by Mordukhovich [34]. We assume that all of the operations on functions are well-defined.

Chapter 2

Partial Subdifferentials

There are numerous second-order generalized differential constructions introduced and applied in the framework of variational analysis and beyond, for example, in Aubin and Frankowska [3], Bonnans and Shapiro [8], Rockafellar and Wets [48], etc. We adopt the classical *dual derivative-of-derivative* approach developed by Mordukhovich who introduced in [29] the *second-order subdifferentials* of an extended-real-valued function as the *coderivatives* of the *(basic) first-order subgradient mapping.* The main motivation for introducing such a second-order subdifferential came from applications to sensitivity analysis for systems described via (first-order) subdifferentials or normal cones in Robinson's framework of generalized equations, which cover variational inequalities, complementarity conditions, etc. This approach follows from the classical development in view of the two facts: the basic subdifferential of an extended-real-valued function stands for a first-order generalized derivative; and the coderivatives of set-valued mappings play the role of adjoint derivative operators. In this approach, the *second-order partial subdifferentials* of an extended-real-valued function are defined as the coderivatives of the basic first-order partial subgradient mapping.

Unless otherwise stated, extended-real-valued functions under consideration are assumed to be proper and finite at reference points.

2.1 First-Order Partial Subdifferentials

2.1.1 Definitions and Properties

Definition 2.1 *Given $\varphi : X \times Y \to \overline{I\!R}$, the basic partial subdifferential and singular partial subdifferential in x at $(\bar{x}, \bar{y}) \in X \times Y$, denoted by $\partial_x \varphi(\bar{x}, \bar{y})$ and $\partial_x^\infty \varphi(\bar{x}, \bar{y})$, respectively, are the corresponding subdifferentials of the function $\varphi(\cdot, \bar{y})$ at $\bar{x}$.*

Example 2.2 *Consider $\varphi(x, y) := |x| + |y|$ for $(x, y) \in I\!R^2$. Using the above definitions,* *we get*

$$\partial_x\varphi(x,y) = \begin{cases} \{1\} & \text{if } x > 0, \\ [-1,1] & \text{if } x = 0, \\ \{-1\} & \text{if } x < 0, \end{cases}$$

and $\partial_x^\infty\varphi(x,y) = \{0\}$ *for all* $(x,y) \in I\!R^2$.

It is well known that $\partial_x\varphi(\bar{x},\bar{y}) = \{\nabla_x\varphi(\bar{x},\bar{y})\}$ if $\varphi(\cdot,\bar{y})$ is strictly differentiable at $\bar{x}$, and that $\partial_x^\infty\varphi(\bar{x},\bar{y}) = \{0\}$ if $\varphi(\cdot,\bar{y})$ is Lipschitz continuous around $\bar{x}$. Additionally, it happens that in the framework of Asplund spaces there are relations between "full" and "partial" subdifferentials as follows:

Proposition 2.3 *Let* X, Y *be Asplund spaces and* $\varphi : X \times Y \to \overline{I\!R}$ *be l.s.c around* $(\bar{x},\bar{y})$ *and SNEC at this point, and let the qualification condition*

$$\big[(0,y^*) \in \partial^\infty\varphi(\bar{x},\bar{y})\big] \Longrightarrow y^* = 0 \tag{2.1}$$

hold. Then one has the inclusions

$$\partial_x\varphi(\bar{x},\bar{y}) \subset \big\{x^* \in X\big|\, \exists y^* \in Y^* \text{ with } (x^*,y^*) \in \partial\varphi(\bar{x},\bar{y})\big\}, \tag{2.2}$$

$$\partial_x^\infty\varphi(\bar{x},\bar{y}) \subset \big\{x^* \in X\big|\, \exists y^* \in Y^* \text{ with } (x^*,y^*) \in \partial^\infty\varphi(\bar{x},\bar{y})\big\}. \tag{2.3}$$

Moreover, $\varphi(\cdot,\bar{y})$ *is lower regular at* $\bar{x}$ *and the equality holds in* (2.2) *if* φ *is lower regular at* $(\bar{x},\bar{y})$*. If in addition* φ *is epigraphically regular at* $(\bar{x},\bar{y})$ *then the equality holds also in* (2.3) *and* $\varphi(\cdot,\bar{y})$ *is epigraphically regular at* $\bar{x}$*.*

Proof. Letting $g : X \to Y$ be a smooth mapping given by $g(x) = \bar{y}$, we have $\varphi(\cdot,\bar{y}) = \varphi(\cdot,g(\cdot)) = (\varphi\circ g)(\cdot)$. The subdifferentiation of general compositions in Asplund spaces in

[34, Theorem 3.41] gives

$$\partial(\varphi \circ g)(\bar{x}) \subset \bigcup_{(x^*,y^*)\in\partial\varphi(\bar{x},\bar{y})} \big[x^* + \partial\varphi\langle y^*, g\rangle(\bar{x})\big], \tag{2.4}$$

$$\partial^\infty(\varphi \circ g)(\bar{x}) \subset \bigcup_{(x^*,y^*)\in\partial^\infty\varphi(\bar{x},\bar{y})} \big[x^* + \partial\varphi\langle y^*, g\rangle(\bar{x})\big]. \tag{2.5}$$

Since $\partial\langle y^*, g\rangle(\bar{x}) = \{0\}$, we obtain (2.2) and (2.3).

If φ is lower regular at $(\bar{x}, \bar{y})$ then by the equality statements in [34, Theorem 3.41], the composition $\varphi \circ g$, or $\varphi(\cdot, \bar{y})$, is lower regular at $\bar{x}$ and the equality holds in (2.4), hence in (2.2). If in addition φ is epigraphically regular at $(\bar{x}, \bar{y})$ then the composition $\varphi \circ g = \varphi(\cdot, \bar{y})$ is epigraphically regular at $\bar{x}$ and the equality holds in (2.5), hence in (2.3). △

Example 2.4 *For illustration let us reconsider the function $\varphi(x, y) = |x| + |y|$ for $(x, y) \in I\!R^2$. It is clear that φ is Lipschitz continuous on $I\!R^2$ and $\partial^\infty\varphi(\bar{x}, \bar{y}) = \{(0, 0)\}$, $\partial^\infty_x(\bar{x}, \bar{y}) = \{0\}$ for all $(\bar{x}, \bar{y}) \in I\!R^2$. Hence, (2.3) holds as equality for all $(\bar{x}, \bar{y})$. On the other hand, we can calculate via definition that the Fréchet subgradients of φ are given by*

$$\widehat{\partial}\varphi(\bar{x}, \bar{y}) = \begin{cases} \{(1,1)\} & \text{if } \bar{x} > 0, \bar{y} > 0, \\ \{(1,-1)\} & \text{if } \bar{x} > 0, \bar{y} < 0, \\ \{(-1,1)\} & \text{if } \bar{x} < 0, \bar{y} > 0, \\ \{(-1,-1)\} & \text{if } \bar{x} < 0, \bar{y} < 0, \\ [-1,1] \times \{1\} & \text{if } \bar{x} = 0, \bar{y} > 0, \\ [-1,1] \times \{-1\} & \text{if } \bar{x} = 0, \bar{y} < 0, \\ \{1\} \times [-1,1] & \text{if } \bar{x} > 0, \bar{y} = 0, \\ \{-1\} \times [-1,1] & \text{if } \bar{x} < 0, \bar{y} = 0, \\ [-1,1] \times [-1,1] & \text{if } \bar{x} = 0, \bar{y} = 0. \end{cases}$$

Using the representation $\partial\varphi(\bar{x},\bar{y}) = \operatorname*{Lim\,sup}_{(x,y)\xrightarrow{\varphi}(\bar{x},\bar{y})} \widehat{\partial}\varphi(x,y)$, *we obtain* $\partial\varphi(\bar{x},\bar{y}) = \widehat{\partial}\varphi(\bar{x},\bar{y})$ *for all* $(\bar{x},\bar{y}) \in I\!R^2$, *which means* φ *is lower regular at any* $(\bar{x},\bar{y}) \in I\!R^2$. *In fact, comparing the formulas for* $\partial_x\varphi(\bar{x},\bar{y})$ *and* $\partial\varphi(\bar{x},\bar{y})$ *calculated previously, we see that (2.2) holds as equality for all* $(x,y) \in I\!R^2$.

2.1.2 Partial Subdifferential Calculus in Banach spaces

In this subsection we present some partial subdifferential calculus rules for extended-real-valued functions valid in arbitrary Banach spaces. These results are easily proved based on the corresponding results for "full" subdifferentials. Let us begin with a partial subdifferential sum rule ensuring equalities with no regularity assumptions.

Proposition 2.5 (partial subdifferential sum rules with equalities in Banach spaces) *Let* $\psi : X \times Y \to \overline{I\!R}$ *be finite at* $(\bar{x},\bar{y})$. *The following assertions hold:*

(i) For any $\varphi : X \times Y \to \overline{I\!R}$, *if* $\varphi(\cdot,\bar{y})$ *is strictly differentiable at* $(\bar{x},\bar{y})$ *then one has*

$$\partial_x(\varphi+\psi)(\bar{x},\bar{y}) = \nabla_x\varphi(\bar{x},\bar{y}) + \partial_x\psi(\bar{x},\bar{y}).$$

(ii) If $\varphi(\cdot,\bar{y})$ *is Lipschitz continuous around* $(\bar{x},\bar{y})$ *one has*

$$\partial_x^\infty(\varphi+\psi)(\bar{x},\bar{y}) = \partial_x^\infty\psi(\bar{x},\bar{y}).$$

Proof. The results follow directly from the basic and singular subdifferential sum rules in [34, Proposition 1.107] applied to $\varphi(\cdot,\bar{y}) + \psi(\cdot,\bar{y})$. △

The next result gives a partial subdifferential chain rule for the standard compositions $(\varphi \circ g)(x,y) := \varphi(g(x,y))$ in a simple case.

Theorem 2.6 (partial subdifferential of composition with equality in Banach spac-es) *Let* $g : X \times Y \to Z$ *be Lipschitz continuous around* $(\bar{x},\bar{y})$, *and let* $\varphi : Z \to \overline{I\!R}$ *be* *finite at* $\bar{z} := g(\bar{x}, \bar{y})$*. If* φ *is strictly differentiable at* $\bar{z}$ *then*

$$\partial_x(\varphi \circ g)(\bar{x}, \bar{y}) = \partial_x \langle \nabla \varphi(\bar{z}), g \rangle (\bar{x}, \bar{y}).$$

Proof. Putting $h(\cdot) := (\varphi \circ g)(\cdot, \bar{y})$, and $\widetilde{g}(\cdot) := g(\cdot, \bar{y})$. Then $\partial_x(\varphi \circ g)(\bar{x}, \bar{y}) = \partial h(\bar{x})$ and $h(x) = (\varphi \circ \widetilde{g})(x)$. Since φ is strictly differentiable at $\bar{z}$ we have

$$\partial h(\bar{x}) = D^*_M \widetilde{g}(\bar{x})(\nabla \varphi(\bar{z}))$$

due to [34, Theorem 1.110]. On the other hand, $\widetilde{g}$ is Lipschitzian around $\bar{x}$ as g is Lipschitzian around $(\bar{x}, \bar{y})$, hence

$$D^*_M \widetilde{g}(\bar{x})(\nabla \varphi(\bar{z})) = \partial \langle \nabla \varphi(\bar{z}), \widetilde{g} \rangle (\bar{x}).$$

This implies

$$\partial h(\bar{x}) = \partial \langle \nabla \varphi(\bar{z}), \widetilde{g} \rangle (\bar{x}) = \partial_x \langle \nabla \varphi(\bar{z}), g \rangle (\bar{x}, \bar{y}).$$

Therefore, the result follows. △

Let us consider another important class of compositions

$$(\varphi \circ g)(x, y) := \varphi\big(x, y, g(x, y)\big), \tag{2.6}$$

which is the standard one $\varphi(g(x, y))$ when φ doesn't depend on (x, y). The next theorem contains *equality-type partial subdifferential chain rules* in the case of *surjective* partial derivatives of inner mappings.

Theorem 2.7 (partial subdifferentiation of compositions with surjective derivatives of inner mappings) *Let* $\varphi : X \times Y \times Z \to \overline{I\!R}$ *and* $g : X \times Y \to Z$*. Assume that* g *is strictly differentiable at* $(\bar{x}, \bar{y})$ *with the surjective partial derivative* $\nabla_x g(\bar{x}, \bar{y})$*, and that*

$$\varphi(x, y, z) = \varphi_1(x, y) + \varphi_2(z)$$

with $\varphi_2 : Z \to \overline{I\!R}$ *finite at* $\bar{z} := g(\bar{x}, \bar{y})$. *The following assertions hold:*

(i) If $\varphi_1 : X \times Y \to \overline{I\!R}$ *is strictly differentiable at* $(\bar{x}, \bar{y})$, *then*

$$\partial_x(\varphi \circ g)(\bar{x}, \bar{y}) = \nabla_x \varphi_1(\bar{x}, \bar{y}) + \nabla_x g(\bar{x}, \bar{y})^* \partial \varphi_2(\bar{z}).$$

(ii) If $\varphi_1 : X \times Y \to \overline{I\!R}$ *is Lipschitz continuous around* $(\bar{x}, \bar{y})$ *then*

$$\partial_x^\infty(\varphi \circ g)(\bar{x}, \bar{y}) = \nabla_x g(\bar{x}, \bar{y})^* \partial^\infty \varphi_2(\bar{z}).$$

Proof. We first prove assertion (i). Putting $\widetilde{\varphi}(x, z) := \varphi(x, \bar{y}, z)$, $\widetilde{\varphi}_1(x) := \varphi_1(x, \bar{y})$, $\widetilde{g}(x) = g(x, \bar{y})$, then

$$h(x) := (\varphi \circ g)(x, \bar{y}) = (\widetilde{\varphi} \circ \widetilde{g})(x).$$

Under the assumptions made in assertion (i), $\widetilde{g}$ is strictly differentiable at $\bar{x}$ with surjective coderivative $\nabla \widetilde{g}(\bar{x}) = \nabla_x g(\bar{x}, \bar{y})$, and $\widetilde{\varphi}(x, z) = \widetilde{\varphi}_1(x) + \varphi_2(z)$ with $\widetilde{\varphi}_1$ strictly differentiable at $\bar{x}$ and φ_2 finite at $\bar{z} = \widetilde{g}(\bar{x})$. Thus, [34, Proposition 1.112] applied to the composition $\widetilde{\varphi} \circ \widetilde{g}$ implies

$$\partial h(\bar{x}) = \nabla \widetilde{\varphi}_1(\bar{x}) + \nabla \widetilde{g}(\bar{x})^* \partial \varphi_2(\bar{z}).$$

This means

$$\partial_x(\varphi \circ g)(\bar{x}, \bar{y}) = \nabla_x \varphi_1(\bar{x}, \bar{y}) + \nabla_x g(\bar{x})^* \partial \varphi_2(\bar{z}).$$

Assertion (ii) is proved similarly by using the second part of the same [34, Proposition 1.112]. $\triangle$

2.1.3 Partial Subdifferential Calculus in Asplund spaces

This subsection is devoted to partial subdifferential calculus for extended-real-valued functions in Asplund spaces. We present here some principle calculus rules for basic and singular partial subgradients in fairly general settings.

Theorem 2.8 (sum rules for basic and singular subdifferentials in Asplund spaces)

Let X, Y *be Asplund spaces and* $\varphi_i : X \times Y \to \overline{I\!R}$, $i = 1, 2, \cdots, n \geq 2$ *be l.s.c around* $(\bar{x}, \bar{y})$

such that all but one of these functions are SNEC at $(\bar{x},\bar{y})$*. Assume that*

$$\big[x_1^*+\cdots+x_n^*=0, x_i^*\in\partial_x^\infty\varphi_i(\bar{x},\bar{y})\big]\Longrightarrow x_i^*=0\ \ \textit{for } i=1,\cdots,n.$$

Then one has the inclusions

$$\partial_x(\varphi_1+\cdots+\varphi_n)(\bar{x},\bar{y})\subset\partial_x\varphi_1(\bar{x},\bar{y})+\cdots+\partial_x\varphi_n(\bar{x},\bar{y}),\tag{2.7}$$

$$\partial_x^\infty(\varphi_1+\cdots+\varphi_n)(\bar{x},\bar{y})\subset\partial_x^\infty\varphi_1(\bar{x},\bar{y})+\cdots+\partial_x^\infty\varphi_n(\bar{x},\bar{y}).\tag{2.8}$$

If in addition each $\varphi_i(\cdot,\bar{y})$ *is lower regular at* $\bar{x}$ *then the sum* $(\varphi_1+\cdots+\varphi_n)(\cdot,\bar{y})$ *is lower regular at this point and* (2.7) *holds as equality. The equality also holds in* (2.8) *and* $(\varphi_1+\cdots+\varphi_n)(\cdot,\bar{y})$ *is epigraphically regular at* $\bar{x}$ *if each* $\varphi_i(\cdot,\bar{y})$ *is epigraphically regular at this point.*

Proof. We will apply [34, Theorem 3.36] to $\varphi_1(\cdot,\bar{y})+\cdots+\varphi_n(\cdot,\bar{y})$. It remains to verify that $\varphi_i(\cdot,\bar{y})$ is SNEC at $\bar{x}$ under the assumption of SNEC property for φ_i at $(\bar{x},\bar{y})$.

Putting $F(x,y):=\{\alpha|\ \alpha\ge\varphi(x,y)\}$, and $\widetilde{F}:=F(\cdot,\bar{y})$. Taking any sequences $\varepsilon_k\downarrow 0$, $(x_k,\alpha_k)\xrightarrow{\text{gph}\widetilde{F}}(\bar{x},\widetilde{\varphi}(\bar{x}))$, and $x_k^*\in\widehat{D}^*_{\varepsilon_k}\widetilde{F}(x_k,\alpha_k)(\alpha_k^*)$ with $(x_k^*,\alpha_k^*)\xrightarrow{w^*}(0,0)$. Then

$$\limsup_{(x,\alpha)\xrightarrow{\text{gph}\widetilde{F}}(x_k,\alpha_k)}\frac{\langle x_k^*,x-x_k\rangle-\alpha_k^*(\alpha-\alpha_k)}{\|x-x_k\|+|\alpha-\alpha_k|}\le\varepsilon_k.$$

This implies

$$\limsup_{(x,y,\alpha)\xrightarrow{\text{gph}F}(x_k,\bar{y},\alpha_k)}\frac{\langle(x_k^*,0),(x-x_k,y-\bar{y})\rangle-\alpha_k^*(\alpha-\alpha_k)}{\|x-x_k\|+\|y-\bar{y}\|+|\alpha-\alpha_k|}\le\varepsilon_k.$$

Hence

$$(x_k^*,0,-\alpha_k^*)\in\widehat{N}_{\varepsilon_k}\big((x_k,\bar{y},\alpha_k);\text{gph}\,F\big),$$

which means

$$(x_k^*,0)\in\widehat{D}^*_{\varepsilon_k}F(x_k,\bar{y},\alpha_k)(\alpha_k^*).$$

Using the SNC property of F, we obtain

$$\|(x_k^*, 0, \alpha_k^*)\| \to 0.$$

Therefore, $\widetilde{F}$ is SNC at $(\bar{x}, \varphi(\bar{x}, \bar{y}))$, which means $\varphi(\cdot, \bar{y})$ is SNEC at $\bar{x}$. △

Theorem 2.9 (partial subdifferentiation of general compositions in Asplund spaces) *Let X, Y, Z be Asplund spaces, $g : X \times Y \to Z$ be Lipschitz continuous around $(\bar{x}, \bar{y})$, and $\varphi : X \times Y \times Z \to I\!R$ be l.s.c around $(\bar{x}, \bar{y}, \bar{z})$ with $\bar{z} := g(\bar{x}, \bar{y})$. Then one has the following assertions:*

(i) Assume that either $\varphi(\cdot, \bar{y}, \cdot)$ is SNEC at $(\bar{x}, \bar{z})$ or $g(\cdot, \bar{y})$ is SNC at $(\bar{x}, \bar{z})$, and that the qualification

$$\partial^\infty_{(x,z)} \varphi(\bar{x}, \bar{y}, \bar{z}) \cap \Big[-N\big((\bar{x}, \bar{z}); \operatorname{gph} g(\cdot, \bar{y})\big)\Big] = \{0\}$$

is satisfied. Then the basic and singular partial subdifferentials in x of the composition $\varphi \circ g := \varphi\big(x, y, g(x, y)\big)$ satisfy the inclusions

$$\partial_x(\varphi \circ g)(\bar{x}, \bar{y}) \subset \bigcup_{(x^*, z^*) \in \partial_{(x,z)} \varphi(\bar{x}, \bar{y}, \bar{z})} \big[x^* + D^*_{N,x} g(\bar{x}, \bar{y}, \bar{z})(z^*)\big], \tag{2.9}$$

$$\partial^\infty_x(\varphi \circ g)(\bar{x}, \bar{y}) \subset \bigcup_{(x^*, z^*) \in \partial^\infty_{(x,z)} \varphi(\bar{x}, \bar{y}, \bar{z})} \big[x^* + D^*_{N,x} g(\bar{x}, \bar{y}, \bar{z})(z^*)\big]. \tag{2.10}$$

(ii) Assume in addition to (i) that $\varphi(\cdot, \bar{y}, \cdot)$ is lower regular at $(\bar{x}, \bar{y}, \bar{z})$, and that either $g(\cdot, \bar{y})$ is strictly differentiable at $\bar{x}$ or it is N-regular at this point with $\dim Z < \infty$. *Then* (2.9) *holds as equality. Furthermore, if $\varphi(\cdot, \bar{y}, \cdot)$ is epigraphically regular at $(\bar{x}, \bar{z})$, then* (2.10) *holds as equality.*

(iii) Let $\varphi = \varphi(z)$ and assume that φ is SNEC at $\bar{z}$ or $\widetilde{g}^{-1}$ is PSNC at $(\bar{z}, \bar{x})$ for $\widetilde{g}(x) := g(x, \bar{y})$, and that the qualification condition

$$\partial^\infty \varphi(\bar{z}) \cap \big(-D^*_M \widetilde{g}^{-1}(\bar{z}, \bar{x})(0)\big) = \{0\}$$

holds. Then one has the inclusions

$$\partial_x(\varphi \circ g)(\bar{x}, \bar{y}) \subset \bigcup_{z^* \in \partial\varphi(\bar{z})} D^*_{N,x} g(\bar{x}, \bar{y})(z^*),$$

$$\partial^\infty_x(\varphi \circ g)(\bar{x}, \bar{y}) \subset \bigcup_{z^* \in \partial^\infty\varphi(\bar{z})} D^*_{N,x} g(\bar{x}, \bar{y})(z^*),$$

where the equalities hold under the additional assumptions of (ii).

Proof. Putting $\widetilde{\varphi}(x, z) := \varphi(x, \bar{y}, z)$, we have

$$(\varphi \circ g)(x, \bar{y}) = \widetilde{\varphi}\big(x, \widetilde{g}(x)\big).$$

The results in part (i), (ii) and (iii) follow directly from the corresponding part of [34, Theorem 3.41] applied to the composition $\widetilde{\varphi} \circ \widetilde{g}$. △

Remark 2.10

(i) If $g(\cdot, \bar{y})$ is strictly Lipschitzian around $\bar{x}$, that is, there is a neighborhood V of the origin in X such that the sequence

$$y_k := \frac{g(x_k + t_k v, \bar{y}) - g(x_k, \bar{y})}{t_k}, \quad k \in I\!N,$$

contains a norm convergent subsequence whenever $v \in V$, $x_k \to \bar{x}$, and $t_k \downarrow 0$, then due to the characterization of normal coderivative in [34, Theorem 3.28] we have

$$D^*_N \widetilde{g}(\bar{x}, \bar{y}, \bar{z})(z^*) = \partial\langle z^*, g(\cdot, \bar{y})\rangle(\bar{x}).$$

Therefore, (2.9) and (2.10) reduce to

$$\partial_x(\varphi \circ g)(\bar{x}, \bar{y}) \subset \bigcup_{(x^*, z^*) \in \partial_{(x,z)}\varphi(\bar{x}, \bar{y}, \bar{z})} \big[x^* + \partial\langle z^*, g(\cdot, \bar{y})\rangle(\bar{x})\big], \tag{2.11}$$

$$\partial^\infty_x(\varphi \circ g)(\bar{x}, \bar{y}) \subset \bigcup_{(x^*, z^*) \in \partial^\infty_{(x,z)}\varphi(\bar{x}, \bar{y}, \bar{z})} \big[x^* + \partial\langle z^*, g(\cdot, \bar{y})\rangle(\bar{x})\big]. \tag{2.12}$$

(ii) Taking into account [34, Corollary 3.17], we have the following estimate

$$D^*_{N,x}g(\bar{x},\bar{y},\bar{z})(z^*) \subset \text{proj}_{\,x}D^*_N g(\bar{x},\bar{y},\bar{z})(z^*),$$

where $\text{proj}_{\,x}D^*_N g(\bar{x},\bar{y},\bar{z})(z^*)$ *denotes the* projection *of the set* $D^*_N g(\bar{x},\bar{y},\bar{z})(z^*) \subset X^* \times Y^*$ *on the space* X^**. This inclusion holds as equality if* g *is* N*-regular at* $(\bar{x},\bar{y},\bar{z})$*. Moreover, in the latter case,* $g(\cdot,\bar{y})$ *is also* N*-regular at* $(\bar{x},\bar{z})$*.*

(iii) Observe also that the qualification condition of Theorem 2.9(iii) is implied by the qualification condition

$$\partial^\infty\varphi(\bar{z}) \cap \ker\big(\text{proj}_{\,x}D^*_N g(\bar{x},\bar{y})\big) = \{0\}.$$

2.2 Second-Order Partial Subdifferentials

2.2.1 Definittions and Properties

Definition 2.11 *Given* $\varphi : X\times Y \to \overline{I\!R}$ *with* $(\bar{x},\bar{y}) \in X\times Y$ *and* $\bar{u} \in \partial_x\varphi(\bar{x},\bar{y})$*. The normal (or mixed) second-order partial subdifferential in* x *at* $(\bar{x},\bar{y},\bar{u})$*, denoted by* $\partial^2_{N,x}\varphi(\bar{x},\bar{y},\bar{u})$ *(or* $\partial^2_{M,x}\varphi(\bar{x},\bar{y},\bar{u})$*, respectively), is the normal (or mixed, respectively) coderivative of* $\partial_x\varphi$ *at* $(\bar{x},\bar{y},\bar{u})$*. That means*

$$\partial^2_{N,x}\varphi(\bar{x},\bar{y},\bar{u})(w) := D^*_N\partial_x\varphi(\bar{x},\bar{y},\bar{u})(w),$$

$$\partial^2_{M,x}\varphi(\bar{x},\bar{y},\bar{u})(w) := D^*_M\partial_x\varphi(\bar{x},\bar{y},\bar{u})(w),$$

for all $w \in X^{**}$*.*

There is no difference between $\partial^2_{N,x}\varphi(\bar{x},\bar{y},\bar{u})$ and $\partial^2_{M,x}\varphi(\bar{x},\bar{y},\bar{u})$ if the normal and mixed coderivatives agree for $\partial_x\varphi$ at $(\bar{x},\bar{y},\bar{u})$; then we use the symbol $\partial^2_x\varphi(\bar{x},\bar{y},\bar{u})$ in Definition

2.11. It happens, in particular, if X is finite-dimensional and also if $\partial_x\varphi$ is N-regular at $(\bar{x},\bar{y},\bar{u})$. The latter always holds for $\mathcal{C}^2$ (and for a slightly more general) functions when, moreover, the values of the second-order partial subdifferential mappings are singletons and coincide with the image of the adjoint operator to the classical second-order partial derivative.

Proposition 2.12 *Let $\varphi\in\mathcal{C}^1$ around $(\bar{x},\bar{y})$, and let its partial derivative operator in x, $\nabla_x\varphi: X\times Y\to X^*$, be strictly differentiable at $(\bar{x},\bar{y})$ with the strict derivative denoted by $\big(\nabla^2_{xx}\varphi(\bar{x},\bar{y}),\nabla^2_{xy}\varphi(\bar{x},\bar{y})\big)$. Then for all $w\in X^{**}$ one has*

$$\partial^2_{N,x}\varphi(\bar{x},\bar{y})(w)=\partial^2_{M,x}\varphi(\bar{x},\bar{y})(w)=\Big\{\big(\nabla^2_{xx}\varphi(\bar{x},\bar{y})^*w,\nabla^2_{xy}\varphi(\bar{x},\bar{y})^*w\big)\Big\}.$$

Proof. Since φ is $\mathcal{C}^1$ around $(\bar{x},\bar{y})$ there exists a neighborhood U of $(\bar{x},\bar{y})$ such that

$$\partial_x\varphi(x,y)=\nabla_x\varphi(x,y)$$

for all $(x,y)\in U$. Applying [34, Theorem 1.38] for the strictly differentiable mapping $\nabla_x\varphi$, we get

$$\partial^2_x\varphi(\bar{x},\bar{y})(w)=D^*\nabla_x\varphi(\bar{x},\bar{y})(w)=\Big\{\big(\nabla^2_{xx}\varphi(\bar{x},\bar{y})^*w,\nabla^2_{xy}\varphi(\bar{x},\bar{y})^*w\big)\Big\}$$

for all $w\in X^{**}$, where D^* stands for either normal coderivative ($D^*=D^*_N$) or mixed coderivative ($D^*=D^*_M$). $\triangle$

In general, both $\partial^2_{N,x}\varphi(\bar{x},\bar{y},\bar{u})$ and $\partial^2_{M,x}\varphi(\bar{x},\bar{y},\bar{u})$ are positively homogeneous mappings from X^{**} to $X^*\times Y^*$ whose calculation involves evaluations of generalized normals to $\operatorname{gph}\partial_x\varphi$. In finite dimensions it is convenient to use the following representations of basic normals

$$N(\bar{x},\Omega)=\operatorname*{Lim\,sup}_{x\to\bar{x}}\widehat{N}(x;\Omega)=\operatorname*{Lim\,sup}_{x\to\bar{x}}\big[\operatorname{cone}(x-\Pi(x;\Omega))\big]$$

for any set Ω locally closed around $\bar{x}\in\Omega$.

Example 2.13 *We consider again the function $\varphi(x,y):=|x|+|y|$ on $I\!R^2$ and compute*

$\partial_x^2\varphi(0,0,1)$. *We have*

$$\partial_x\varphi(x,y) = \begin{cases} 1 & \textit{if } x > 0, \\ [-1,1] & \textit{if } x = 0, \\ -1 & \textit{if } x < 0. \end{cases}$$

For $(\bar{x},\bar{y},\bar{u})$ *near* $(0,0,1)$ *we have*

$$\widehat{N}\big((x,y,u);\mathrm{gph}\,(\partial_x\varphi)\big) = \begin{cases} \big\{(0,0,0)\big\} & \textit{if } x < 0 \textit{ or } x > 0, u \neq 1, \\ \big\{(0,0)\big\} \times I\!R & \textit{if } x > 0, u = 1 \\ (-\infty,0] \times \{0\} \times [0,\infty) & \textit{if } x = 0, u = 1 \\ I\!R \times \big\{(0,0)\big\} & \textit{if } x = 0, u < 1. \end{cases}$$

Using representation $N\big((0,0,1);\mathrm{gph}\,(\partial_x\varphi)\big) = \mathop{\mathrm{Lim\,sup}}_{(x,y,u)\to(0,0,1)} \widehat{N}\big((x,y,u);\mathrm{gph}\,(\partial_x\varphi)\big)$, *we obtain the following representation for* $N\big((0,0,1);\mathrm{gph}\,(\partial_x\varphi)\big)$

$$\big[(-\infty,0] \times \{0\} \times [0,\infty)\big] \cup \big[\{(0,0)\} \times (-\infty,0)\big] \cup \big[(0,\infty) \times \{(0,0)\}\big]$$

Since $\partial_x^2\varphi(0,0,1)(w) = \Big\{(\nu_1,\nu_2) \in I\!R^2 \big|\ (\nu_1,\nu_2,-w) \in N\big((0,0,1);\mathrm{gph}\,(\partial_x\varphi)\big)\Big\}$, *we get*

$$\partial_x^2\varphi(0,0,1)(w) = \begin{cases} \big\{(0,0)\big\} & \textit{if } w > 0 \\ (-\infty,\infty) \times \{0\} & \textit{if } w = 0 \\ (-\infty,0] \times \{0\} & \textit{if } w < 0. \end{cases}$$

Next, we consider a class of functions consisting of functions φ that are continuously differentiable around $(\bar{x},\bar{y})$ with the partial gradient in x locally Lipschitzian around this point. The calculation of the mixed second-order partial subdifferential for such functions can be essentially simplified due to the following representation:

Proposition 2.14 *Let $\varphi : X \times Y \to Z$ be $\mathcal{C}^1$ around $(\bar{x}, \bar{y})$ with $\nabla_x \varphi$ Lipschitz continuous around $(\bar{x}, \bar{y})$. Then*

$$\partial^2_{M,x}\varphi(\bar{x}, \bar{y})(w) = \partial\langle w, \nabla_x \varphi\rangle(\bar{x}, \bar{y})$$

for all $w \in X^{**}$.

Proof. Since φ is $\mathcal{C}^1$ around $(\bar{x}, \bar{y})$ there exists a neighborhood U of $(\bar{x}, \bar{y})$ such that

$$\partial_x \varphi(x, y) = \nabla_x \varphi(x, y)$$

for all $(x, y) \in U$. Taking into account that $\nabla_x \varphi$ is Lipschitz continuous around $(\bar{x}, \bar{y})$, we have

$$D^*_M \nabla_x \varphi(\bar{x}, \bar{y})(w) = \partial\langle w, \nabla_x \varphi\rangle(\bar{x}, \bar{y})$$

for all $w \in X^{**}$ due to [34, Theorem 1.90]. This implies the result. $\triangle$

2.2.2 Second-Order Partial Subdifferential Calculus in Banach Spaces

Our primary goal in the second-order theory is to develop principle calculus (sum and chain) rules for the second-order partial subdifferentials defined above. In this subsection we present results obtained in general Banach spaces. To derive second-order partial sum and chain rules for $\partial^2_{N,x}$ and $\partial^2_{M,x}$, we proceed via Definition 2.11 applying calculus rules for the normal and mixed coderivatives to set-valued mapping generated by the basic first-order partial subdifferential. In this way we have to restrict ourselves to favorable classes of functions for which the corresponding first-order partial subdifferential calculus rules hold as *equalities*, since neither normal nor mixed coderivative enjoys monotonicity properties that may allow one to use an inclusion-type partial subdifferential calculus. We begin with a simple sum rule for the second-order partial subdifferentials.

Proposition 2.15 (equality sum rules for second-order partial subdifferentials) *Let $\bar{u} \in \partial_x(\varphi_1 + \varphi_2)(\bar{x}, \bar{y})$, where φ_1 is $\mathcal{C}^1$ around $(\bar{x}, \bar{y})$ with $\nabla_x \varphi_1$ strictly differentiable at $(\bar{x}, \bar{y})$ while φ_2 is finite at $(\bar{x}, \bar{y})$ with $\bar{u}_2 := \bar{u} - \nabla_x \varphi_1(\bar{x}, \bar{y}) \in \partial_x \varphi_2(\bar{x}, \bar{y})$. Then one has*

$$\partial^2_x(\varphi_1 + \varphi_2)(\bar{x}, \bar{y}, \bar{u})(w) = \big(\nabla^2_{xx}\varphi_1(\bar{x}, \bar{y})^* w, \nabla^2_{xy}\varphi_1(\bar{x}, \bar{y}, \bar{u})^* w\big) + \partial^2_x \varphi_2(\bar{x}, \bar{y}, \bar{u}_2)(w)$$

for all $w \in X^{**}$, for both normal ($\partial_x^2 = \partial_{N,x}^2$) and mixed ($\partial_x^2 = \partial_{M,x}^2$) second-order partial subdifferentials in x.

Proof. There exists a neighborhood U of $(\bar{x}, \bar{y})$ such that φ_1 is $\mathcal{C}^1$ at every $(x, y) \in U$. Using Proposition 2.5, we have

$$\partial_x(\varphi_1 + \varphi_2)(x, y) = \nabla_x \varphi_1(x, y) + \partial_x \varphi_2(x, y)$$

for all $(x, y) \in U$. Since $\nabla_x \varphi_1$ is strictly differentiable at $(\bar{x}, \bar{y})$, it follows that

$$\partial_x^2(\varphi_1 + \varphi_2)(\bar{x}, \bar{y}, \bar{u})(w) = \big(\nabla_{xx}^2 \varphi_1(\bar{x}, \bar{y})^* w, \nabla_{xy}^2 \varphi_1(\bar{x}, \bar{y}, \bar{u})^* w\big) + \partial_x^2 \varphi_2(\bar{x}, \bar{y}, \bar{u}_2)(w)$$

for all $w \in X^{**}$ due to the coderivative sum rule in [34, Therem 1.62(ii)]. $\triangle$

Let us now present the central result of the second-order partial subdifferential calculus in general Banach spaces.

Theorem 2.16 (second-order partial chain rules with surjective derivatives of inner mappings) *Let $\bar{u} \in \partial_x(\varphi \circ g)(\bar{x}, \bar{y})$ with $g : X \times Y \to Z$ and $\varphi : Z \to I\!R$. Assume that g is $\mathcal{C}^1$ around $(\bar{x}, \bar{y})$ with the surjective partial derivative $\nabla_x g(\bar{x}, \bar{y}) : X \to Z$, and that the mapping $\nabla_x g : X \times Y \to \mathcal{L}(X, Z)$ is strictly differentiable at $(\bar{x}, \bar{y})$. Let $\bar{p} \in Z^*$ be a unique functional satisfying*

$$\bar{u} = \nabla_x g(\bar{x}, \bar{y})^* \bar{p} \;\; \text{and} \;\; \bar{p} \in \partial \varphi(\bar{z}) \;\; \textit{with} \;\; \bar{z} := g(\bar{x}, \bar{y}).$$

Then for all $w \in X^{**}$ one has

$$\partial_{M,x}^2(\varphi \circ g)(\bar{x}, \bar{y}, \bar{u})(w) = \big(\nabla_{xx}^2 \langle \bar{p}, g \rangle (\bar{x}, \bar{y})^* w, \nabla_{xy}^2 \langle \bar{p}, g \rangle (\bar{x}, \bar{y})^* w\big)$$
$$+ \nabla g(\bar{x}, \bar{y})^* \partial_M^2 \varphi(\bar{z}, \bar{p})\big(\nabla_x g(\bar{x}, \bar{y})^{**} w\big),$$

$$\partial^2_{N,x}(\varphi \circ g)(\bar{x},\bar{y},\bar{u})(w) \subset \big(\nabla^2_{xx}\langle \bar{p}, g\rangle(\bar{x},\bar{y})^* w, \nabla^2_{xy}\langle \bar{p}, g\rangle(\bar{x},\bar{y})^* w\big)$$
$$+\nabla g(\bar{x},\bar{y})^*\partial^2_N\varphi(\bar{z},\bar{p})\big(\nabla_x g(\bar{x},\bar{y})^{**}w\big).$$

Proof. It follows from Theorem 2.7 that

$$\partial_x(\varphi \circ g)(x,y) = \nabla_x g(x,y)^*\partial\varphi(g(x,y))$$

for all (x,y) around $(\bar{x},\bar{y})$. We then can represent $\partial_x(\varphi \circ g)$ as the composition $f \circ G$ where $(f \circ G)(x,y) := f(x,y,G(x,y))$ with

$$f(x,y,p) := \nabla_x g(x,y)^* p,$$
$$G(x,y) := (\partial\varphi) \circ g.$$

Let us check that the assumptions in [34, Lemma 1.126] hold under the assumptions made in this therem. Actually, the only assumption needs to be checked is the injectivity of the operator $\nabla_x g(\bar{x},\bar{y})^* : Z^* \to X^*$, which follows from the assumed surjectivity of $\nabla_x g(\bar{x},\bar{y})$ due to [34, Lemma 1.18]. Thus, the special chain rule for coderivative in [34, Lemma 1.126] applied to the composition $f \circ G$ gives

$$D^*_M(f \circ G)(\bar{x},\bar{y},\bar{u})(w) = (\nabla_x f(\bar{x},\bar{y},\bar{p})^* w, \nabla_y f(\bar{x},\bar{y},\bar{p})^* w) + D^*_M G(\bar{x},\bar{y},\bar{p})\big(f(\bar{x},\bar{y},\cdot)^* w\big),$$

$$D^*_N(f \circ G)(\bar{x},\bar{y},\bar{u})(w) \subset (\nabla_x f(\bar{x},\bar{y},\bar{p})^* w, \nabla_y f(\bar{x},\bar{y},\bar{p})^* w) + D^*_N G(\bar{x},\bar{y},\bar{p})\big(f(\bar{x},\bar{y},\cdot)^* w\big).$$

On the other hand, it follows from the constructions of f and G that

$$(\nabla_x f(\bar{x},\bar{y},\bar{p})^* w, \nabla_y f(\bar{x},\bar{y},\bar{p})^* w) = \big(\nabla^2_{xx}\langle \bar{p}, g\rangle(\bar{x},\bar{y})^* w, \nabla^2_{xy}\langle \bar{p}, g\rangle(\bar{x},\bar{y})^* w\big)$$

and

$$D^* G(\bar{x},\bar{y},\bar{p})\big(f(\bar{x},\bar{y},\cdot)^* w\big) = D^*(\partial\varphi \circ g)(\bar{x},\bar{y},\bar{p})\big(\nabla_x g(\bar{x},\bar{y})^{**}w\big)$$
$$= \nabla g(\bar{x},\bar{y})^*\partial^2_N\varphi(\bar{z},\bar{p})\big(\nabla_x g(\bar{x},\bar{y})^{**}w\big),$$

where the latter equality is obtained due to the coderivative chain rule in [34, Theorem 1.66] for the case that the inner mapping g is strictly differentiable at $(\bar{x},\bar{y})$ with $\nabla g(\bar{x},\bar{y})$ is surjective. The proof is complete. △

The last result of this subsection provides equalities for both second-order partial subdifferentials of compositions $\varphi \circ g$ in general Banach spaces, where φ but not g is assumed to be twice partial differentiable. Before stating the result, we define the so-called *second-order partial coderivative sets*, which will be used in formulations of the next theorem and related results in the next subsection.

Definition 2.17 (second-order partial coderivative set) *Given a Lipschitz continuous mapping $g : X \times Y \to Z$ between Banach spaces, the second-order partial coderivative sets in x for g at $(\bar{x}, \bar{y}, \bar{p}, \bar{u}) \in X \times Y \times Z^* \times X^*$ with $\bar{u} \in \partial_x \langle \bar{p}, g\rangle(\bar{x}, \bar{y})$ is defined by*

$$D_x^2 g(\bar{x}, \bar{y}, \bar{p}, \bar{u})(w) := D^*(\partial_x \langle \cdot, g\rangle)(\bar{x}, \bar{y}, \bar{p}, \bar{u})(w)$$

for all $w \in X^{**}$, where D^* stands for both normal ($D^* = D_N^*$) and mixed ($D^* = D_M^*$) coderivatives of the mapping $(x, y, p) \rightrightarrows \partial\langle p, g\rangle(x, y)$. If g is strictly differentiable at $(\bar{x}, \bar{y})$ then

$$\partial_x \langle \bar{p}, g\rangle(\bar{x}, \bar{y}) = \nabla_x g(\bar{x}, \bar{y})^* \bar{p}$$

and we omit $\bar{u}$ in the arguments for $D_x^2 g$.

Theorem 2.18 (second-order partial chain rules with twice differentiable outer mappings) *Let g be strictly differentiable at $(\bar{x}, \bar{y})$, let $\varphi \in \mathcal{C}^1$ around $\bar{z} := g(\bar{x}, \bar{y})$ with $\nabla\varphi$ strictly differentiable at this point, and let $\bar{p} := \nabla\varphi(\bar{z})$. Assume that the operator $\nabla^2\varphi(\bar{z})\nabla g(\bar{x}, \bar{y}) : X \times Y \to Z^*$ is surjective. Then*

$$\partial_x^2(\varphi \circ g)(\bar{x}, \bar{y})(w) = \bigcup_{(x^*, y^*, q) \in D_x^2 g(\bar{x}, \bar{y}, \bar{p})(w)} \left[(x^*, y^*) + \nabla g(\bar{x}, \bar{y})^* \nabla^2\varphi(\bar{z})^* q\right]$$

for all $w \in X^{**}$, where ∂_x^2 and D_x^2 stand for the corresponding normal and mixed partial second-order constructions.

These chain rules hold without the above surjectivity assumption if $\nabla_x g$ is strictly subdifferentiable at $(\bar{x}, \bar{y})$. In the latter case one has

$$D_x^2 g(\bar{x}, \bar{y}, \bar{p})(w) = \big(\nabla^2_{xx}\langle \bar{p}, g\rangle(\bar{x}, \bar{y})^* w, \nabla^2_{xy}\langle \bar{p}, g\rangle(\bar{x}, \bar{y})^* w, \nabla_x g(\bar{x}, \bar{y})^{**} w\big),$$

for both $D_x^2 = D_{N,x}^2$ and $D_x^2 = D_{M,x}^2$.

Proof. Since φ is C^1 around $\bar{z}$ and g is Lipschitz continuous around $(\bar{x}, \bar{y})$ due to its strict subdifferentiability at this point, it implies by Theorem 2.6 that there is a neighborhood U of $(\bar{x}, \bar{y})$ such that for all $(x, y) \in U$ we have

$$\partial_x(\varphi \circ g)(x, y) = \partial_x\big\langle \nabla\varphi\big(g(x, y)\big), g\big\rangle(x, y) =: (F \circ h)(x, y)$$

for $F : X \times Y \times Z^* \rightrightarrows X^*$ and $h : X \times Y \longrightarrow X \times Y \times Z^*$ defined by

$$\begin{aligned} F(x, y, p) &:= \partial_x\langle p, g\rangle(x, y), \\ h(x, y) &:= \Big(x, y, \nabla\varphi\big(g(x, y)\big)\Big). \end{aligned}$$

Note that since $\nabla^2\varphi(\bar{z})\nabla g(\bar{x}, \bar{y})$ is surjective, so is $\nabla h(\bar{x}, \bar{y})$. We thus have, due to [34, Theorem 1.66], for both mixed and normal coderivatives that

$$D^*(F \circ h)(\bar{x}, \bar{y}, \bar{p})(w) = \nabla h(\bar{x}, \bar{y})^* D^* F(\bar{x}, \bar{y}, \bar{p}, \bar{u})(w)$$

for $\bar{u} \in \partial_x\langle \bar{p}, g\rangle(\bar{x}, \bar{y}) = \{\nabla_x g(\bar{x}, \bar{y})^* \bar{p}\}$ and any $w \in X^{**}$.

On the other hand, we have

$$D^* F(\bar{x}, \bar{y}, \bar{p})(w) = D^*\partial_x\langle \cdot, g\rangle(\bar{x}, \bar{y}, \bar{p})(w) = D_x^2 g(\bar{x}, \bar{y}, \bar{p})(w).$$

Therefore, taking into account the construction of h, we arrive at

$$D^*(F \circ h)(\bar{x}, \bar{y}, \bar{p})(w) = \bigcup_{(x^*, y^*, q) \in D_x^2 g(\bar{x}, \bar{y}, \bar{p})(w)} \big[(x^*, y^*) + \nabla g(\bar{x}, \bar{y})^* \nabla^2\varphi(\bar{z})^* q\big]$$

for all $w \in X^{**}$. This proves the Theorem in the case of surjectivity.

The last claim in the theorem easily follows from the above procedure due to [34, Theorem 1.65]. $\triangle$

2.2.3 Second-Order Partial Subdifferential Calculus in Asplund Spaces

In this subsection we continue developing the second-order partial subdifferential calculus started in the preceding subsection in the framework of general Banach spaces. Here we follow the same scheme that leads us to second-order partial subdifferential sum and chain rules by using coderivative calculus applied to *equality-type* sum and chain rule for first-order partial subgradients. In contrast to the previous consideration, we assume in this subsection that some of the spaces in question are Asplund. This allows us to employ extended first-order partial calculus rules obtained in the framework of Asplund spaces.

We start as usual with sum rules and obtain the following three versions for extended-real-valued functions defined on spaces that are Asplund together with their duals. The major source of such spaces are *reflexive* Banach spaces. On the other hand, there are interesting examples of even *separable* spaces X, which are nonreflexive but Asplund together with X^*. Let us mention the famous *long James* space whose natural embedding in the second dual is of *codimension one* but which is nevertheless isometrically isomorphic to its second dual. Other examples, discussion, and references can be found, e.g., in the book of Bourgin [10].

Recall also that all the functions under consideration below are assumed to be proper and finite at reference points.

Theorem 2.19 (second-order partial subdifferential sum rules in Asplund spaces) *Let X, Y, X^* and Y^* be Asplund spaces. Given $\varphi_i : X \times Y \to \overline{I\!R}$, $i = 1, 2$ with $\bar{u} \in \partial_x(\varphi_1 + \varphi_2)(\bar{x}, \bar{y})$. The following assertions hold for both normal ($\partial_x^2 = \partial_{N,x}^2$) and mixed ($\partial_x^2 = \partial_{M,x}^2$) second-order partial subdifferentials in x:*

(i) Assume that $\varphi_1 \in \mathcal{C}^1$ around $(\bar{x}, \bar{y})$ with $\bar{u}_1 := \nabla_x \varphi_1(\bar{x}, \bar{y})$ and that the graph of $\partial_x \varphi_2$ is norm-closed around $(\bar{x}, \bar{y}, \bar{u}_2)$ with $\bar{u}_2 := \bar{u} - \bar{u}_1$. Assume also that either $\nabla_x \varphi_1$ is Lipschitzian around $(\bar{x}, \bar{y})$ or $\partial_x \varphi_2$ is PSNC at $(\bar{x}, \bar{y}, \bar{u}_2)$ and

$$\partial_{M,x}^2 \varphi_1(\bar{x}, \bar{y}, \bar{u}_1)(0) \cap \big(-\partial_{M,x}^2 \varphi_2(\bar{x}, \bar{y}, \bar{u}_2)(0)\big) = \{0\}. \tag{2.13}$$

Then for all $w \in X^{**}$ one has

$$\partial_x^2(\varphi_1+\varphi_2)(\bar{x},\bar{y},\bar{u})(w) \subset \partial_x^2\varphi_1(\bar{x},\bar{y},\bar{u}_1)(w) + \partial_x^2\varphi_2(\bar{x},\bar{y},\bar{u}_2)(w). \tag{2.14}$$

(ii) Let both φ_i be l.s.c around $(\bar{x},\bar{y})$ and let $S : X \times Y \times X^* \rightrightarrows X^* \times X^*$, with

$$S(x,y,u) := \big\{(u_1,u_2) \in X^* \times X^* \big| \; u_i \in \partial_x\varphi_i(x,y), i = 1,2, \text{ and } u_1+u_2 = u\big\},$$

be inner semicontinuous at $(\bar{x},\bar{y},\bar{u},\bar{u}_1,\bar{u}_2)$ for a given $(\bar{u}_1,\bar{u}_2) \in S(\bar{x},\bar{y},\bar{u})$. Assume that the graph of each $\partial_x\varphi_i$ is norm-closed around $(\bar{x},\bar{y},\bar{u}_i)$ for $i = 1,2$, that one of $\partial_x\varphi_i$ is PSNC at the corresponding $(\bar{x},\bar{y},\bar{u}_i)$, and that the qualification condition (2.13) is fulfilled. Suppose also that there is a neighborhood U of $(\bar{x},\bar{y})$ such that

$$\partial_x^\infty\varphi_1(x,y) \cap \big(-\partial_x^\infty\varphi_2(x,y)\big) = \{0\}$$

for all $(x,y) \in U$, that one of φ_i SNEC at every $(x,y) \in U$ (both assumptions are fulfilled when one of φ_i is Lipschitz continuous around $(\bar{x},\bar{y})$), and that each φ_i are partially lower regular in x at every $(x,y) \in U$. Then the sum rule (2.14) holds for all $w \in X^{**}$.

(iii) Assume that the above set-valued mapping S be inner semicompact at $(\bar{x},\bar{y},\bar{u})$, that the graph of each $\partial_x\varphi_i$ is norm-closed whenever (x,y) is near $(\bar{x},\bar{y})$, and that other assumptions in (ii) are fulfilled for any $(\bar{u}_1,\bar{u}_2) \in S(\bar{x},\bar{y},\bar{u})$. Then for all $w \in X^{**}$ one has

$$\partial_x^2(\varphi_1+\varphi_2)(\bar{x},\bar{y},\bar{u})(w) \subset \bigcup_{(\bar{u}_1,\bar{u}_2)\in S(\bar{x},\bar{y},\bar{u})} \partial_x^2\varphi_1(\bar{x},\bar{y},\bar{u}_1)(w) + \partial_x^2\varphi_2(\bar{x},\bar{y},\bar{u}_2)(w).$$

Proof.

Since φ_1 is $\mathcal{C}^1$ around $(\bar{x},\bar{y})$, Proposition 2.5 assures that there is a neighborhood U of $(\bar{x},\bar{y})$ such that

$$\partial_x(\varphi_1+\varphi_2)(\bar{x},\bar{y},\bar{u})(w) = \nabla_x\varphi_1(x,y) + \partial_x\varphi_2(x,y)$$

for all $(x, y) \in U$. We then apply to this equality the coderivative sum rule in Asplund spaces from [34, Theorem 3.10(i)] with $F_1 := \nabla_x \varphi_1$ and $F_2 := \partial_x \varphi_2$. It yields

$$D^*\partial_x(\varphi_1 + \varphi_2)(\bar{x}, \bar{y}, \bar{u})(w) \subset D^*(\nabla_x \varphi_1)(\bar{x}, \bar{y}, \bar{u}_1) + D^*(\partial_x \varphi_2)(\bar{x}, \bar{y}, \bar{u}_2).$$

This proves (i).

In the same way we justify the second-order partial sum rules in (ii) and (iii) by respectively applying [34, Theorem 3.10(i) and (ii)] to the equality

$$\partial_x(\varphi_1 + \varphi_2)(x, y) = \partial_x \varphi_1(x, y) + \partial_x \varphi_2(x, y), \text{for } (x, y) \in U,$$

which follows from Theorem 2.8. △

Next, we derive second-order partial subdifferential chain rules for composition $(\varphi \circ g)(x, y) := \varphi\big(g(x)\big)$ of a function $\varphi : Z \to \overline{I\!R}$ and a mapping $g : X \times Y \to Z$, where the spaces X, Y, Z and Z^* are Asplund. In contrast to Theorem 2.16, the following theorem doesn't require the surjectivity of $\nabla g(\bar{x}, \bar{y})$ while imposing mor assumptions on the outer function φ under first-order and second-order qualification conditions.

Theorem 2.20 (second-order partial subdifferential chain rules with smooth inner mappings in Asplund spaces) *Assume that $g \in \mathcal{C}^1$ around some $(\bar{x}, \bar{y})$ with the partial derivative $\nabla_x g$ strictly differentiable at this point, that φ is l.s.c and lower regular around $\bar{z} := g(\bar{x}, \bar{y})$, and that the inverse mapping g^{-1} is PSNC at $(\bar{z}, \bar{x}, \bar{y})$. Suppose also that φ is SNEC around $\bar{z}$ and that the first-order qualification condition*

$$\partial^\infty \varphi\big(g(x, y)\big) \cap \ker \nabla_x g(x, y)^* = \{0\} \tag{2.15}$$

is satisfied around $(\bar{x}, \bar{y})$ (the last two assumptions are automatic when φ is locally Lipschitzian around $(\bar{x}, \bar{y})$). Then the following assertions hold for both normal $\partial_x^2 = \partial_{N,x}^2$ and mixed $\partial_x^2 = \partial_{M,x}^2$ second-order partial subdifferentials:

(i) Given $\bar{u} \in \partial_x(\varphi \circ g)(\bar{x}, \bar{y})$, we assume that the mapping $S : X \times Y \times X^* \rightrightarrows Z^*$ with the values

$$S(x, y, u) = \big\{p \in Z^* \big|\ p \in \partial\varphi\big(g(x, y)\big), \nabla_x g(x, y)^* p = u\big\}$$

is inner semicontinuous at $(\bar{x}, \bar{y}, \bar{u}, \bar{p})$ for some fixed $\bar{p} \in S(\bar{x}, \bar{y}, \bar{u})$, that the graph of the subdifferential mapping $\partial\varphi$ is norm-closed around $(\bar{z}, \bar{p})$, and that the mixed second-order qualification condition

$$\partial^2_M \varphi(\bar{z}, \bar{p})(0) \cap \ker \nabla g(\bar{x}, \bar{y})^* = \{0\}$$

is satisfied. Then for all $w \in X^{**}$ we have

$$\partial^2_{N,x}(\varphi \circ g)(\bar{x}, \bar{y}, \bar{u})(w) \subset \big(\nabla^2_{xx}\langle \bar{p}, g\rangle(\bar{x}, \bar{y})^* w, \nabla^2_{xy}\langle \bar{p}, g\rangle(\bar{x}, \bar{y})^* w\big)$$
$$+\nabla g(\bar{x}, \bar{y})^* \partial^2_N \varphi(\bar{z}, \bar{p})\big(\nabla_x g(\bar{x})^{**} w\big).$$

(ii) Given $\bar{u} \in \partial_x(\varphi \circ g)(\bar{x}, \bar{y})$, we assume that the above mapping S is inner semicompact at $(\bar{x}, \bar{y}, \bar{u})$, that the graph of $\partial\varphi$ is norm-closed whenever z is near $\bar{z}$, and that the mixed second-order qualification condition in (i) is satisfied for every $\bar{p} \in S(\bar{x}, \bar{y}, \bar{u})$. Then for all $w \in X^{**}$ we have

$$\partial^2_{N,x}(\varphi \circ g)(\bar{x}, \bar{y}, \bar{u})(w) \subset \bigcup_{\bar{p} \in S(\bar{x}, \bar{y}, \bar{u})} \Big[\big(\nabla^2_{xx}\langle \bar{p}, g\rangle(\bar{x}, \bar{y})^* w, \nabla^2_{xy}\langle \bar{p}, g\rangle(\bar{x}, \bar{y})^* w\big)$$
$$+\nabla g(\bar{x}, \bar{y})^* \partial^2_N \varphi(\bar{z}, \bar{p})\big(\nabla_x g(\bar{x})^{**} w\big)\Big].$$

Proof. Since g is $\mathcal{C}^1$ around $(\bar{x}, \bar{y})$ we have

$$D^*_{N,x} g^{-1}(z, x, y)(0) = -\ker \nabla_x g(x, y)^*$$

for (x, y) near $(\bar{x}, \bar{y})$ and $z := g(x, y)$. Thus, the first-order qualification implies

$$\partial^\infty \varphi\big(g(x, y)\big) \cap \big(- D^*_{N,x} g^{-1}(z, x, y)(0)\big) = \{0\}$$

for (x, y) near $(\bar{x}, \bar{y})$. Theorem 2.9(iii) then assures that there is a neighborhood U of $(\bar{x}, \bar{y})$ such that for all $(x, y) \in U$ we have

$$\partial_x(\varphi \circ g)(x, y) = \bigcup_{p \in \partial\varphi(g(x,y))} \nabla_x g(x, y)^* p.$$

Let us denote $\partial_x(\varphi \circ g)(x, y) = (f \circ G)(x, y)$ in U, with

$$G(x, y) = \Big(x, y, \partial\varphi\big(g(x, y)\big)\Big),$$
$$f(x, y, p) = \nabla_x g(x, y)^* p, \ p \in Z^*.$$

We notice that f is smooth and $G \circ f^{-1}$ is inner semicontinuous at $(\bar{x}, \bar{y}, \bar{u})$ under the assumptions made. Also, we have for both $D^* = D^*_N$ and $D^* = D^*_M$ that

$$D^*G\big((\bar{x}, \bar{y}), (\bar{x}, \bar{y}, \bar{p})\big)(x^*, y^*, q) = (x^*, y^*) + D^*(\partial\varphi \circ g)\big((\bar{x}, \bar{y}), (\bar{x}, \bar{y}, \bar{p})\big)(q)$$

for $(x^*, y^*, q) \in X^* \times Y^* \times Z^{**}$. On the other hand, we conclude by [34, Theorem 1.65(i)] that, for all $w \in X^{**}$,

$$\partial_x^2(\varphi \circ g)(\bar{x}, \bar{y}, \bar{p})(w) \subset \Big(D^*_N G\big((\bar{x}, \bar{y}), (\bar{x}, \bar{y}, \bar{p})\big) \circ \nabla f\big((\bar{x}, \bar{y}), (\bar{x}, \bar{y}, \bar{p})\big)^*\Big)(w).$$

Therefore

$$\partial_x^2(\varphi \circ g)(\bar{x}, \bar{y}, \bar{p})(w) \subset \Big\{\big(\nabla^2_{xx}\langle \bar{p}, g\rangle(\bar{x}, \bar{y})^* w, \nabla^2_{xy}\langle \bar{p}, g\rangle(\bar{x}, \bar{y})^* w\big) + D^*_N(\partial\varphi \circ g)(\bar{x}, \bar{y}, \bar{p})\big(\nabla_x g(\bar{x}, \bar{y})^{**} w\big)\Big\}.$$

It remains to compute $D^*_N(\partial\varphi \circ g)$. To furnish this, we use [34, Theorem 3.13(i)] that provides the coderivative chain rule

$$D^*_N(\partial\varphi \circ g)(\bar{x}, \bar{y}, \bar{p})(q) \subset \big(D^*_N g(\bar{x}, \bar{y}) \circ D^*_N(\partial\varphi)(\bar{z}, \bar{p})\big)(q)$$
$$= \nabla g(\bar{x}, \bar{y})^* \partial^2_N \varphi(\bar{z}, \bar{p})(q)$$

for all $q \in Z^{**}$ under the PSNC assumption on g^{-1} and the mixed qualification condition

$$(D^*_M \partial\varphi)(\bar{z}, \bar{p})(0) \cap \ker \nabla g(\bar{x}, \bar{y})^* = \{0\},$$

which reduces to the second-order qualification condition of the theorem. We complete the proof for (i).

The same arguments work for (ii), in which we use part (ii) instead of part (i) of [34, Theorem 1.65]. $\triangle$

When Z is finite-dimensional (X and Y may be not), some of the assumptions of Theorem 2.20 either are satisfied automatically or can be essentially simplified. In this way we get the following result, where $\partial_x^2\varphi$ stands for the common second-order partial subdifferential of $\varphi : I\!R^m \to \overline{I\!R}$ while $\partial_x^2(\varphi \circ g)$ is the same as in the above theorem.

Corollary 2.21 (second-order partial subdifferential chain rules for compositions with finite-dimensional intermediate space) *Let $\bar{u} \in \partial_x(\varphi \circ g)(\bar{x}, \bar{y})$, where $\varphi : I\!R^m \to \overline{I\!R}$ and $g : X \times Y \to I\!R^m$ with Asplund spaces X, Y. Assume that $g \in \mathcal{C}^1$ around some $(\bar{x}, \bar{y})$ with the partial derivative $\nabla_x g$ strictly differentiable at this point, that φ is l.s.c and lower regular around $\bar{z} := g(\bar{x}, \bar{y})$ with closed graphs of $\partial\varphi$ and $\partial^\infty\varphi$ near $\bar{z}$. Suppose also that the first-order qualification condition*

$$\partial^\infty\varphi\big(g(\bar{x}, \bar{y})\big) \cap \ker \nabla_x g(\bar{x}, \bar{y})^* = \{0\}$$

is satisfied and one has the second-order qualification condition in the form

$$\partial^2\varphi(\bar{z}, \bar{p})(0) \cap \ker \nabla g(\bar{x}, \bar{y})^* = \{0\} \ \textit{if}\ \bar{p} \in \partial\varphi(\bar{z}) \ \textit{with}\ \nabla g(\bar{x})^*\bar{p} = \bar{u}.$$

Then the second-order partial chain rule of Theorem 2.20(ii) holds for all $w \in X^{**}$.

Proof. The SNEC property of $\varphi : X \times Y \to \overline{I\!R}$ and the PSNC property of g^{-1} are automatic when $dim Z < \infty$.

We next check that in the case Z is finte-dimensional, the first-order QC is satisfied in a neighborhood of $\bar{x}$ if it is satisfied at $\bar{x}$. In fact, assuming the contrary and taking into account that $\partial^\infty\varphi(.)$ is a cone, there exist sequences $(x_k, y_k) \to (\bar{x}, \bar{y})$ and $v_k \in \partial^\infty\varphi(g(x_k, y_k))$

with $\nabla_x g(x_k, y_k)^* v_k = 0$ and $\|v_k\| = 1$. We then extract a convergent subsequence of v_k, which is denoted by the same notation, such that v_k converges to some $v \in \partial^\infty \varphi(\bar{z})$, due to the closedness property of the graph of $\partial^\infty \varphi$ near $\bar{z}$, with $\|v\| = 1$ and $\nabla g(\bar{x}, \bar{y})^* v = 0$, which contradicts (2.15).

The result follows from Theorem 2.20(ii). The only assumption needs to be checked is that the mapping S is inner semicompact at $(\bar{x}, \bar{y})$. Taking sequences $\{(x_k, y_k, u_k)\} \to (\bar{x}, \bar{y}, \bar{u})$ and $\{p_k\}$ with $p_k \in S(x_k, y_k, u_k)$, we will show that $\{p_k\}$ is bounded, and thus contains a convergent subsequence. This can also be proved by contradiction. Were $\{p_k\}$ not bounded, we would find a subsequence of $\{\frac{p_k}{\|p_k\|}\}$, which is denoted by $\{p'_k\}$, such that $p'_k \to p'$ with $\|p'\| = 1$, $p' \in \partial^\infty \varphi(\bar{z})$ and $\nabla_x g(\bar{x}, \bar{y})^* p' = 0$. This contradicts (2.15). $\triangle$

The next corollary justifies the second-order partial subdifferential chain rule for an important class of functions that automatically satisfy all the first-order assumptions in Corollary 2.21. Recall that a function $\psi : X \times Y \to \overline{I\!R}$ is said to be *strongly amenable* in x at $\bar{x}$ with compatible parameterization in y at $\bar{y}$ if there is a neighborhood U of $(\bar{x}, \bar{y})$ such that $\psi = \varphi \circ g$ with a $\mathcal{C}^2$ mapping $g : U \to I\!R^m$ and a proper l.s.c. convex function $\varphi : I\!R^m \to \overline{I\!R}$ that satisfy

$$\partial^\infty \varphi(g(\bar{x}, \bar{y})) \cap \ker \nabla_x g(\bar{x}, \bar{y})^* = \{0\}. \tag{2.16}$$

Moreover, according to [34, Proposition 1.112] and Theorem 2.9, if ψ is finite around $(\bar{x}, \bar{y})$ then the basic subgradient mapping, the singular subgradient mapping and the first-order basic partial subgradient mapping of ψ are given by

$$\partial \psi(x, y) = \nabla g(x, y)^* \partial \varphi(g(x, y)), \tag{2.17}$$

$$\partial^\infty \psi(x, y) = \nabla g(x, y)^* \partial^\infty \varphi(g(x, y)), \tag{2.18}$$

$$\partial_x \psi(x, y) = \nabla_x g(x, y)^* \partial \varphi(g(x, y)). \tag{2.19}$$

for all (x, y) near $(\bar{x}, \bar{y})$.

Corollary 2.22 (second-order partial chain rule for amenable functions) *Let* $\psi : X \times Y \to \overline{I\!R}$ *be strongly amenable in* x *at* $\bar{x}$ *with compatible parameterization in* y *at* $\bar{y}$, $\bar{u} \in \partial_x \psi(\bar{x}, \bar{y})$, *and* $\varphi : I\!R^m \to \overline{I\!R}$, $g : U \to I\!R^m$ *be mappings from its composite representation. Assume that* X *and* Y *are Asplund spaces and the second-order qualification condition*

$$\partial^2 \varphi(\bar{z}, \bar{p})(0) \cap \ker \nabla g(\bar{x}, \bar{y})^* = \{0\} \text{ if } \bar{p} \in \partial\varphi(\bar{z}) \text{ with } \nabla g(\bar{x}, \bar{y})^* \bar{p} = \bar{u} \tag{2.20}$$

holds with $\bar{z} = g(\bar{x}, \bar{y})$. *Then for all* $w \in X^{**}$ *one has the inclusion*

$$\partial_x^2 \psi(\bar{x}, \bar{y}, \bar{u})(w) \subset \bigcup_{\bar{p} \in S(\bar{x}, \bar{y}, \bar{u})} \Big[\nabla^2 \langle \bar{p}, g \rangle (\bar{x}, \bar{y})^*(w, 0) + \nabla g(\bar{x}, \bar{y})^* \partial_x^2 \varphi(\bar{z}, \bar{p})(\nabla_x g(\bar{x}, \bar{y})^{**} w) \Big]$$

where $\partial_x^2 \psi$ *stands for either* $\partial_{N,x}^2 \psi$ *or* $\partial_{M,x}^2 \psi$ *and*

$$S(x, y, u) = \big\{ p \in Z^* \big| \; p \in \partial\varphi(g(x, y)), \nabla_x g(x, y)^* p = u \big\}.$$

Proof. The lower regularity of φ, and the closedness of the graphs of $\partial\varphi$ and $\partial^\infty \varphi$ are implied by the convexity of φ. Hence the result follows from Corollary 2.21. △

Finally, let us consider a second-order chain rule for compositions $\varphi \circ g$ involving $\mathcal{C}_x^{1,1}$ functions φ and Lipschitzian mappings g. In the next theorem we use the second-order partial coderivatives (normal and mixed) of Lipschitzian mappings defined in Definition 2.17.

Theorem 2.23 (second-order partial chain rule with Lipschitzian inner mappings in Asplund spaces) *Let* X, Y, Z, X^*, Y^* *and* Z^* *are Asplund spaces. Given* $\bar{u} \in \nabla_x(\varphi \circ g)(\bar{x}, \bar{y})$, *where* $g : X \times Y \to Z$ *is Lipschitz continuous around* $(\bar{x}, \bar{y})$, *and* $\varphi : Z \to \overline{I\!R}$ *is* $\mathcal{C}^{1,1}$ *around* $\bar{z} := g(\bar{x}, \bar{y})$ *with* $\bar{p} := \nabla\varphi(\bar{z})$. *Assume that the graph of the set-valued mapping* $(x, y, p) \longmapsto \partial_x \langle p, g \rangle (x, y)$ *is norm-closed in* $X \times Y \times Z^* \times X^*$ *whenever* (x, y, p) *are* *near $(\bar{x}, \bar{y}, \bar{p})$. Then one has the second-order partial chain rule*

$$\partial_x^2(\varphi \circ g)(\bar{x}, \bar{y}, \bar{u})(w) \subset \bigcup_{(u,v,q) \in D_x^2 g(\bar{x},\bar{y},\bar{p},\bar{u})(w)} \big[(u, v) + D_N^* g(\bar{x}, \bar{y}) \circ \partial_N^2 \varphi(\bar{z})(q)\big]$$

for all $w \in X^{**}$, where ∂^2 and D^2 stand for the corresponding normal and mixed second-order partial constructions.

Proof. Using the same arguments as in the proof of Theorem 2.16 we have the representation

$$\partial_x(\varphi \circ g)(x, y) = (F \circ h)(x, y)$$

for all (x, y) in a neighborhood U of $(\bar{x}, \bar{y})$, where the mappings $F : X \times Y \times Z^* \rightrightarrows X^*$ and $h : X \times Y \to X \times Y \times Z^*$ are defined by

$$\begin{aligned} F(x, y, t) &:= \partial_x \langle t, g\rangle(x, y), \\ h(x, y) &:= \Big(x, y, \nabla\varphi\big(g(x, y)\big)\Big), (x, y) \in U. \end{aligned}$$

Let us apply to this composition the coderivative chain rule from [34, Theorem 3.13(i)]. This gives

$$D^*(F \circ h)(\bar{x}, \bar{y}, \bar{p}, \bar{u})(w) \subset D_N^* h(\bar{x}, \bar{y}) \circ D^* F(\bar{x}, \bar{y}, \bar{p}, \bar{u})(w), w \in X^{**}$$

for both normal and mixed coderivatives under the assumptions made. We also have the conclusion

$$D^*(\nabla\varphi \circ g)(\bar{x}, \bar{y})(q) \subset D_N^* g(\bar{x}, \bar{y}) \circ \partial_N^2 \varphi(\bar{z})(q), \; q \in Z^{**}$$

from the same [34, Theorem 3.13(i)] with the assumption that $\nabla\varphi$ is Lipschitz around $\bar{z}$. Combining the two inclusions and taking into account the fact that $D^* F(\bar{x}, \bar{y}, \bar{p}, \bar{u}) = D_x^2 g(\bar{x}, \bar{y}, \bar{p}, \bar{u})$ by notation, we arrive at the result. △

Chapter 3

Coderivatives in Parametric Optimization in Asplund Spaces

In this chapter we mostly focus on the solution map to the parameter-dependent generalized equation given by

$$S(y) = \big\{x \in X \mid 0 \in \partial_x \psi_0(x, y) + \partial_x \psi(x, y)\big\}, \tag{3.1}$$

where X, Y are arbitrary Asplund spaces and $\psi_0, \psi : X \times Y \to \overline{I\!R}$ are extended-real-valued functions. Using coderivatives to analyze solution map (3.1) results in an estimate for the coderivative of S in terms of the normal second-order partial subdifferentials in x of ψ_0 and ψ. Thus, to carry the analysis further, besides the calculus developed in chapter 2, we also develop in this chapter an estimate for second-order *partial* subdifferentials in terms of the normal *full* one.

In particular, when the cost function ψ_0 is $\mathcal{C}^2$, the solution map S in (3.1) become

$$S(y) = \big\{x \in X \mid 0 \in \nabla_x \psi_0(x, y) + \partial_x \psi(x, y)\big\}, \tag{3.2}$$

and is the *stationary point multifunction* of the parameterized minimization problem

$$\text{Minimize } \psi_0(x, y) + \psi(x, y) \quad \text{over} \quad x \in X. \tag{3.3}$$

We focus particular attention on the case when the constraint function is *strongly-amenable*. These covers many interesting examples including nonlinear programs, and our final estimates are given entirely in terms of standard derivative conditions on the original data of the problem.

In the penultimate section of this chapter, we study *stationary point-multiplier* pairs and make connections to our results from earlier sections on stationary points. The final section is devoted to the special case when *canonical perturbations* are present in the parameterization of the optimization model. In this case, the constraint qualifications that restrict the more general work are all satisfied automatically.

3.1 General Coderivative Analysis

Let us introduce a new sequential normal compact property of sets, which is essential in establishing an upper estimate for the second-order partial subdifferentials.

Definition 3.1 *Let X be a Banach space. A set $\Omega \subset X$ is basically sequentially normally compact (BSNC) at $\bar{x} \in \Omega$ if for any sequence $(x_k, x_k^*) \in \Omega \times X^*$ satisfying*

$$x_k \to \bar{x},\ x_k^* \in N(x_k; \Omega), \quad \text{and} \quad x_k^* \xrightarrow{w^*} 0,$$

one has $\|x_k^*\| \to 0$.

Remark 3.2 *Since ε_k can be equivalently removed from the definition of SNC property in the Asplund space setting, it is not hard to see that if a closed subset Ω of an Asplund space is BSNC at $\bar{x}$, then it is SNC at this point.*

The following lemma sets up the relationship between the compactly epi-Lipschitzian (CEL) property and the basic sequential normal compactness.

Lemma 3.3 *Let $\Omega \subset X$ be compactly epi-Lipschitzian around $\bar{x}$. Then it is basically sequentially normally compact at this point.*

Proof: Assumming that Ω is CEL at $\bar{x}$, we find a compact set $C \subset X$ and positive numbers γ and η such that

$$\Omega \cap (\bar{x} + \eta I\!B) + t\eta I\!B \subset \Omega + tC \ \text{ for all } \ t \in (0, \gamma).$$

First, let us show that this implies the existence of a constant $\alpha > 0$ for which

$$\widehat{N}_\varepsilon(x; \Omega) \subset \left\{x^* \in X^* |\ \eta\|x^*\| \le \varepsilon(\alpha + \eta) + \max_{c \in C}\langle x^*, c\rangle\right\} \tag{3.4}$$

whenever $x \in \Omega \cap (\bar{x} + \eta B)$. Indeed, fixing $x \in \Omega + (\bar{x} + \eta I\!B)$ and employing the CEL property of Ω, for any $e \in I\!B$ and $t \in (0, \gamma)$ we pick a point $c_t \in C$ such that $x + t(\eta e - c_t) \in \Omega$. Due to the compactness of C, a subsequence of c_t converges to some point $\tilde{c} \in C$ as $t \downarrow 0$. This implies, by the definition of $\widehat{N}_\varepsilon(x;\Omega)$, that

$$\langle x^*, \eta e - \tilde{c}\rangle - \varepsilon\|\eta e - \tilde{c}\| \leq 0 \quad \text{for all} \quad x^* \in \widehat{N}_\varepsilon(x;\Omega).$$

Since $e \in I\!B$ was chosen arbitrarily, the latter gives inclusion (3.4) with $\alpha := \max_{c\in C}\|c\|$.

Next, let us show that

$$N(x;\Omega) \subset \big\{x^* \in X^* \big|\ \eta\|x^*\| \leq \max_{c\in C}\langle x^*, c\rangle\big\} \tag{3.5}$$

whenever $x \in \Omega \cap (\bar{x} + \frac{\eta}{2} I\!B)$. Indeed, fixing any $x \in \Omega \cap (\bar{x} + \frac{\eta}{2} I\!B)$ and $x^* \in N(x;\Omega)$, we find sequences $\varepsilon_k \downarrow 0$, $u_k \xrightarrow{\Omega} x$, and $u_k^* \in \widehat{N}_{\varepsilon_k}(u_k;\Omega)$ such that $u_k^* \xrightarrow{w^*} x^*$. It follows from (3.4) that

$$\eta\|u_k^*\| \leq \varepsilon_k(\alpha+\eta) + \max_{c\in C}\langle u_k^*, c\rangle$$

when k is sufficiently large. Hence

$$\eta\|u_k^*\| \leq \varepsilon_k(\alpha+\eta) + \max_{c\in C}\langle u_k^* - x^*, c\rangle + \max_{c\in C}\langle x^*, c\rangle. \tag{3.6}$$

The compactness of C together with $u^* \xrightarrow{w^*} x^*$ implies that $\langle u_k^* - x^*, c\rangle \to 0$ *uniformly* in $c \in C$. Therefore, (3.6) ensures (3.5) as $k \to \infty$.

Now we are ready to prove that Ω is BSNC at $\bar{x}$. Indeed, taking any sequences $x_k \xrightarrow{\Omega} \bar{x}$ and $x_k^* \in N(x_k;\Omega)$ such that $x_k^* \xrightarrow{w^*} 0$, we have

$$\eta\|x_k^*\| \leq \max_{c\in C}\langle x_k^*, c\rangle,$$

when k is sufficiently large. By a similar argument we can see that $\|x_k^*\| \to 0$ as $k \to \infty$. The proof is complete. $\triangle$

Remark 3.4 *As proved in Fabian and Mordukhovich [15], the SNC and CEL properties agree for closed subsets of WCG spaces, and hence the CEL, BSNC and SNC properties all agree. This implies, in particular, that the BSNC (SNC) property of closed sets in such spaces is actually around $\bar{x} \in \Omega$. Moreover, in this case, the mapping $x \rightrightarrows N(x,\Omega)$ is closed-graph in the norm×weak* topology of $X \times X^*$ around $\bar{x}$ due to [33, Proposition 3.4].*

Next, let us provide another sufficient condition for the basic sequential normal compactness.

Proposition 3.5

(i) Let $F : X \rightrightarrows Y$ be a closed-graph set-valued mapping which is Lipschitz-like around $(\bar{x}, \bar{y})$, where X is an arbitrary Banach space while Y is finite dimensional. Then $\operatorname{gph} F$ *is BSNC around $(\bar{x}, \bar{y})$.*

(ii) Let $\varphi : X \to \overline{I\!R}$ be locally Lipschitzian around $\bar{x}$, then $\operatorname{epi} \varphi$ *is BSNC at $(\bar{x}, \varphi(\bar{x}))$.*

Proof. Let ℓ be the Lipschitz modulus of F around $(\bar{x}, \bar{y})$. First, we prove that under the assumptions made in (i) there is $\eta > 0$ for which

$$\|x^*\| \leq \ell \|y^*\| \tag{3.7}$$

whenever $(x^*, y^*) \in N\big((x, y); \operatorname{gph} F\big)$ with $x \in \bar{x} + \eta I\!B$ and $y \in F(x) \cap (\bar{y} + \eta I\!B)$. Indeed, since F is Lipschiz-like around $(\bar{x}, \bar{y})$, for any $\varepsilon \geq 0$ there exists, by [34, Theorem 1.43], a positive number η such that

$$\sup \Big\{ \|x^*\| \ \Big| \ (x^*, y^*) \in \widehat{N}_\varepsilon\big((x, y); \operatorname{gph} F\big) \Big\} \leq \|y^*\| + \varepsilon(1 + \ell) \tag{3.8}$$

whenever $x \in \bar{x} + \eta I\!B$, and $y \in F(x) \cap (\bar{y} + \eta I\!B)$. Fix any $x \in \bar{x} + \eta I\!B$, $y \in F(x) \cap (\bar{y} + \eta I\!B)$, and $(x^*, y^*) \in N\big((x, y); \operatorname{gph} F\big)$. Using the definition of the basic normals, we find sequences $\varepsilon_k \downarrow 0$, $(x_k, y_k) \xrightarrow{\text{gphF}} (x, y)$, $(x_k^*, y_k^*) \in \widehat{N}_{\varepsilon_k}\big((x_k, y_k); \operatorname{gph} F\big)$ such that $(x_k^*, y_k^*) \xrightarrow{w^*} (x^*, y^*)$ for all $k \in I\!N$. Due to (3.8) we have

$$\|x_k^*\| \leq \ell \|y_k^*\| + \varepsilon_k(1 + \ell)$$

for all k sufficiently large. Note that $\|y_k^*\| \to \|y^*\|$ since Y is finite-dimensional, and that the norm function is weak* lower semicontinuous on X^*. Passing to the limit in the latter inequality, we get (3.7).

Next, we show that $\operatorname{gph} F$ is BSNC at any point (x, y) such that $x \in \bar{x} + \eta I\!B$, $y \in F(x) \cap (\bar{y} + \eta I\!B)$. Indeed, taking any $(x_k, y_k) \xrightarrow{\text{gphF}} (x, y)$ and $(x_k^*, y_k^*) \in N\big((x_k, y_k); \operatorname{gph} F\big)$

such that $(x_k, y_k) \xrightarrow{w^*} (0,0)$, we have $\|y^*\| \to 0$ due to the finite dimension of Y, and hence $\|x_k^*\| \to 0$ by (3.7). Therefore, (i) is proved.

To prove (ii), we observe easily that if φ is locally Lipschitz around $\bar{x}$, then its epigraphical mapping from X to $I\!R$ defined by $x \rightrightarrows E_\varphi(x)$ is Lipschitz-like around $(\bar{x}, \varphi(\bar{x}))$. Since $\operatorname{gph} E_\varphi = \operatorname{epi}\varphi$, (ii) follows directly from (i). $\triangle$

The next theorem establishes a sufficient condition guaranteeing that if the qualification condition holds at a point then that condition holds *around* that point. This enables us to develop the calculi for second-order subdifferentials in a more convenient way.

Theorem 3.6 *Let Ω_1 and Ω_2 be two closed subsets of an Asplund space and let $\bar{x} \in \Omega_1 \cap \Omega_2$. Assume that the graphs of the set-valued mappings $x \rightrightarrows N(x;\Omega_i)$ are closed around $\bar{x}$ in norm$\times$weak* topology of $X \times X^*$, for both $i = 1, 2$, and that either Ω_1 or Ω_2 is BSNC at $\bar{x}$. Then the qualification condition*

$$N(\bar{x};\Omega_1) \cap \big(-N(\bar{x};\Omega_2)\big) = \{0\}$$

guarantees that there exists a neighborhood V of $\bar{x}$ such that

$$N(x;\Omega_1) \cap \big(-N(x;\Omega_2)\big) = \{0\} \tag{3.9}$$

for all $x \in \Omega_1 \cap \Omega_2 \cap V$.

Proof. Let us assume that (3.9) does not hold. Then there exists a sequence $x_k \xrightarrow{\Omega_1\cap\Omega_2} \bar{x}$ such that

$$N(x_k;\Omega_1) \cap \big(-N(x_k;\Omega_2)\big) \neq \{0\}.$$

Fixing $x_k^* \in N(x_k;\Omega_1) \cap \big(-N(x_k;\Omega_2)\big)$ satisfying $x_k^* \neq 0$, we have

$$w_k^* := \frac{x_k^*}{\|x_k^*\|} \in N(x_k;\Omega_1) \cap \big(-N(x_k;\Omega_2)\big).$$

Since $\{w_k^*\}$ is bounded in the Asplund space X, we can extract a convergent subsequence (without relabeling) such that $w_k^* \xrightarrow{w^*} w^*$. Using the closed property of the mappings $x \rightrightarrows N(x;\Omega_i)$ we get

$$w^* \in N(\bar{x};\Omega) \cap (-N(\bar{x};\Omega_2).$$

Let us show that $w \neq 0$, which leads to a contradiction. On the contrary, assume that $w^* = 0$. Since either Ω_1 or Ω_2 is BSNC at $\bar{x}$ we have $\|w_k^*\| \to 0$ as $k \to \infty$. This is a contradiction since $\|w_k^*\| = 1$. This completes the proof. △

The intersection rule for basic normal cones plays a central role in the calculi of variational analysis. We will establish in what follows the intersection rules involving points around a reference point.

Corollary 3.7 *In addition to the assumptions made in Theorem 3.6, assume that one of Ω_i is SNC around $\bar{x}$. Then there exists a neighborhood V of $\bar{x}$ such that*

$$N(x;\Omega_1 \cap \Omega_2) \subset N(x;\Omega_1) + N(x;\Omega_2) \tag{3.10}$$

for all $x \in \Omega_1 \cap \Omega_2 \cap V$.

Proof. This result follows directly from Theorem 3.6 and [34, Corollary 3.5]. △

Taking into account Lemma 3.3, we get another corollary of Theorem 3.6.

Corollary 3.8 *Let Ω_1 and Ω_2 be two closed subsets of a WCG Asplund space and let $\bar{x} \in \Omega_1 \cap \Omega_2$. Assume that both of Ω_i are SNC at $\bar{x}$, and that the following qualification condition holds*

$$N(\bar{x};\Omega_1) \cap \big(-N(\bar{x};\Omega_2)\big) = \{0\}.$$

Then there exists a neighborhood V of $\bar{x}$ such that

$$N(x;\Omega_1 \cap \Omega_2) \subset N(x;\Omega_1) + N(x;\Omega_2) \tag{3.11}$$

for all $x \in \Omega_1 \cap \Omega_2 \cap V$.

Let us present a lemma that gives a sufficient condition for the constraint qualification condition

$$\big[(0, y^*) \in \partial^\infty \varphi(x, y)\big] \Longrightarrow y^* = 0 \tag{3.12}$$

to hold around a reference point $(\bar{x}, \bar{y})$. This is essential to express the partial subgradient mapping as a projection of the full subgradient mapping.

Lemma 3.9 *Let* X *and* Y *be Asplund spaces and* $(\bar{x}, \bar{y}) \in X \times Y$. *Suppose that* $\varphi : X \times Y \to \overline{I\!R}$ *is continuous at* $(\bar{x}, \bar{y}) \in X \times Y$ *and that the constraint qualification (3.12) holds at* $(\bar{x}, \bar{y})$. *Then (3.12) also holds for all* (x, y) *in a neighborhood of* $(\bar{x}, \bar{y})$ *in each of the following cases:*

(i) $\operatorname{epi}\varphi$ *is BSNC at* $\big(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y})\big)$, *and* $(x, y, \alpha) \rightrightarrows N\big((x, y, \alpha); \operatorname{epi}\varphi\big)$ *is closed-graph in the norm×weak** *topology of* $(X \times Y \times I\!R) \times (X^* \times Y^* \times I\!R)$ *at that point.*

(ii) X *and* Y *are WCG Asplund spaces and* φ *is SNEC at* $(\bar{x}, \bar{y})$.

Proof. Taking Remark 3.4 into account, we easily see that (ii) follows from (i). It remains to prove (i). Let us assume the contrary. There exists a sequence $(x_k, y_k) \to (\bar{x}, \bar{y})$ and $y_k^* \neq 0$ such that $(0, y_k^*) \in \partial^\infty \varphi(x_k, y_k)$, which implies

$$(0, y_k^*, 0) \in N\big((x_k, y_k, \varphi(x_k, y_k)); \operatorname{epi}\varphi\big).$$

Putting $w_k^* = \dfrac{y_k^*}{\|y_k^*\|}$, we have

$$(0, w_k^*, 0) \in N\big((x_k, y_k, \varphi(x_k, y_k)); \operatorname{epi}\varphi\big).$$

Due to the weak-star sequentially compactness of the dual unit ball in Asplund space, we can extract (without relabeling) a convergent subsequence of $\{w_k^*\}$, which is also denoted by the same notation, such that $w_k^* \xrightarrow{w^*} w^*$. Using the closed-graph property of the multifunction $(x, y, \alpha) \rightrightarrows N\big((x, y, \alpha); \operatorname{epi}\varphi\big)$ around $\big(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y})\big)$, we get

$$(0, w^*, 0) \in N\big((\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y})); \operatorname{epi}\varphi\big).$$

This means

$$(0, w^*) \in \partial^\infty \varphi(\bar{x}, \bar{y}).$$

Thus, (3.12) implies $w^* = 0$, which in turn yields $\|w_k\| \to 0$ by the BSNC property of epi φ at $\big(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y})\big)$. This is a contradiction since $\|w_k^*\| = 1$ for all k. △

Theorem 3.10 *Let X and Y be Asplund spaces. Assume that $\varphi : X \times Y \to \overline{I\!R}$ is continuous at $(\bar{x}, \bar{y})$ and l.s.c around this point. Assume also that φ is lower regular around $(\bar{x}, \bar{y})$, and that the constraint qualification (3.12) holds at $(\bar{x}, \bar{y})$. Then there exists a neighborhood of $(\bar{x}, \bar{y})$ such that*

$$\partial_x \varphi(x, y) = \big\{x^* \in X^* \big| \, \exists y^* \in Y^* \text{ with } (x^*, y^*) \in \partial \varphi(x, y)\big\}$$

for any (x, y) in that neighborhood in each of the cases (i) and (ii) in Lemma 3.9.

Proof. The result follows from Lemma 3.9 and Proposition 2.3. △

Putting Theorem 3.10 together with the chain rule for normal coderivative in [34, Theorem 3.13] yields the following result which gives an upper estimate for the second-order partial subdifferentials.

Corollary 3.11 *Let $\bar{u} \in \partial_x \varphi(\bar{x}, \bar{y})$. Under the same assumptions as in Theorem 3.10, the following inclusion holds for both normal ($\partial_x^2 = \partial_{N,x}^2$) and mixed ($\partial_x^2 = \partial_{M,x}^2$) second-order partial subdifferentials in x:*

$$\partial_x^2 \varphi(\bar{x}, \bar{y}, \bar{u})(w) \subset \bigcup_{(\bar{u}, v) \in \partial \varphi(\bar{x}, \bar{y})} \partial_N^2 \varphi(\bar{x}, \bar{y}, \bar{u}, v)(w, 0)$$

for all $w \in X^{**}$, provided that the graph of the multifunction $\partial \varphi$ is norm-closed around $(\bar{x}, \bar{y})$, and that the multifunction $P(x, y, u) := \big\{v \in Y^* \big| \, (u, v) \in \partial \varphi(x, y)\big\}$ is inner semicompact at $(\bar{x}, \bar{y}, \bar{u})$.

Proof. Putting $G(x,y) := \partial\varphi(x,y)$ and $F := \mathrm{proj}_1(X^*,Y^*)$, we find, by Theorem 3.10, a neighborhood of $(\bar{x},\bar{y})$ such that $\partial_x\varphi = F\circ G$ on that neighborhood. Observe that $D^*_M F(u,v,u)(0) = 0$, the qualification condition in [34, Theorem 3.13] holds for any $(\bar{u},v)\in P(\bar{x},\bar{y},\bar{u})$, so the result follows from the chain rule for normal coderivative in that theorem. △

Next, we will employ the coderivative calculus in [32, Corollary 4.3], which enables us to estimate coderivative of the stationary point multifunction stationary point multifunction in terms of the initial data. This result on coderivatives of solution maps for the so called generalized equations can be treated as an extended implicit mapping theorem (see [32] for details).

To proceed, we need to define a notion used in what follows.

Definition 3.12 *The pair $\{F_1,F_2\}$ satisfies the limiting qualification condition at $(\bar{x},\bar{y})$ if for any sequences $(x_{ik},y_{ik})\xrightarrow{\mathrm{gph}F_i}(\bar{x},\bar{y})$ and $(x^*_{ik},y^*_{ik})\xrightarrow{w^*}(x^*_i,y^*_i)$ with $x^*_{ik}\in\widehat{D}^*F_i(x_{ik},y_{ik})(-y^*_{ik})$ for $i=1,2$ one has*

$$\Big[\big|\big|(x^*_{1k}+x^*_{2k},y^*_{1k}+y^*_{2k})\big|\big|\to 0\Big]\Longrightarrow(x^*_1,y^*_1)=(x^*_2,y^*_2)=(0,0).$$

The aforementioned [32, Corollary 4.3] applied to the stationary point multifunction S in (3.1) gives the following result:

Proposition 3.13 *Let $(\bar{y},\bar{x})\in\mathrm{gph}\,S$ for S given in (3.1). We have the following estimates of the normal coderivative of S at $(\bar{y},\bar{x})$:*

(i) Suppose that there is $\bar{u}\in\partial_x\psi_0(\bar{x},\bar{y})\cap\big(-\partial_x\psi(\bar{x},\bar{y})\big)$ such that the graphs of $\partial_x\psi_0$ and $\partial_x\psi$ are locally norm-closed around $(\bar{x},\bar{y},\bar{u})$ and $(\bar{x},\bar{y},-\bar{u})$, respectively, and that the intersection mapping $\partial_x\psi_0\cap\big(-\partial_x\psi\big)$ is inner semicontinuous at $(\bar{x},\bar{y},\bar{u})$. Then for all

$x^* \in X^*$ we have the inclusion

$$D_N^* S(\bar{y}, \bar{x})(x^*) \subset \Big\{ y^* \in Y^* \Big| \ \exists w \in X^{**} \text{ such that } (-x^*, y^*) \in \partial^2_{N,x} \psi_0(\bar{x}, \bar{y}, \bar{u})(w) + \partial^2_{N,x} \psi(\bar{x}, \bar{y}, -\bar{u})(w) \Big\} \qquad (3.13)$$

in each of the following cases:

a) The pair $\{\partial_x \psi_0, -\partial_x \psi\}$ satisfies the limiting qualification condition at $(\bar{x}, \bar{y}, \bar{u})$, and either $\partial_x \psi_0$ is PSNC at $(\bar{x}, \bar{y}, \bar{u})$ and $(\partial_x \psi)^{-1}$ is strongly PSNC at $(-\bar{u}, \bar{x}, \bar{y})$, or $(\partial_x \psi_0)^{-1}$ is PSNC at $(\bar{u}, \bar{x}, \bar{y})$ and $\partial_x \psi$ is strongly PSNC at $(\bar{x}, \bar{y}, -\bar{u})$, or the similar PSNC/SNC conditions hold with changing places of ψ_0 and ψ.

b)The pair $\{\partial_x \psi_0, -\partial_x \psi\}$ satisfies the limiting qualification condition at $(\bar{x}, \bar{y}, \bar{u})$, and one of the mappings $\partial_x \psi_0$ and $-\partial_x \psi$ is SNC at $(\bar{x}, \bar{y}, \bar{u})$.

(ii) Suppose that $\partial_x \psi_0 \cap \big(-\partial_x \psi \big)$ is inner semicompact at $(\bar{x}, \bar{y})$ and that the assumptions of (i) are fulfilled whenever $\bar{u} \in \partial_x \psi_0(\bar{x}, \bar{y}) \cap \big(-\partial_x \psi(\bar{x}, \bar{y}) \big)$. Then for all $x^* \in X^*$ we have

$$D_N^* S(\bar{y}, \bar{x})(x^*) \subset \Big\{ y^* \in Y^* \Big| \ \exists \bar{u} \in \partial_x \psi_0(\bar{x}, \bar{y}) \cap \big(-\partial_x \psi(\bar{x}, \bar{y}) \big), \ w \in X^{**} \text{ with } (-x^*, y^*) \in \partial^2_{N,x} \psi_0(\bar{x}, \bar{y}, \bar{u})(w) + \partial^2_{N,x} \psi(\bar{x}, \bar{y}, -\bar{u})(w) \Big\}. \qquad (3.14)$$

Corollary 3.14 *If in addition to the assumptions made in Proposition 3.13, suppose that ψ_0 and ψ satisfy the assumptions given in Corollary 3.11. Then we can respectively replace (3.13) and (3.14) with*

$$D_N^* S(\bar{y}, \bar{x})(x^*) \subset \Big\{ y^* \in Y^* \Big| \ (-x^*, y^*) \in \bigcup_{(\bar{u}, v) \in \partial \psi_0(\bar{x}, \bar{y})} \partial^2_N \psi_0(\bar{x}, \bar{y}, \bar{u}, v)(w, 0) + \bigcup_{(-\bar{u}, v') \in \partial \psi(\bar{x}, \bar{y})} \partial^2_N \psi(\bar{x}, \bar{y}, -\bar{u}, v')(w, 0), \ w \in X^{**} \Big\}, \qquad (3.15)$$

and

$$D_N^* S(\bar{y},\bar{x})(x^*) \subset \Big\{ y^* \in Y^* \Big| \; \exists \bar{u} \in \partial_x \psi_0(\bar{x},\bar{y}) \cap \big(-\partial_x \psi(\bar{x},\bar{y})\big), \; w \in X^{**} \text{ with}$$

$$(-x^*, y^*) \in \bigcup_{(\bar{u},v)\in\partial\psi_0(\bar{x},\bar{y})} \partial_N^2 \psi_0(\bar{x},\bar{y},\bar{u},v)(w,0) + \bigcup_{(-\bar{u},v')\in\partial\psi(\bar{x},\bar{y})} \partial_N^2 \psi(\bar{x},\bar{y},-\bar{u},v')(w,0) \Big\}. \tag{3.16}$$

Remark 3.15 *It is easy to see that the limiting QC for the pair* $\{\partial_x\psi_0, -\partial_x\psi\}$ *at* $(\bar{x},\bar{y},\bar{u})$ *is implied by the normal qualification condition*

$$N\big((\bar{x},\bar{y},\bar{u}); \mathrm{gph}\,(\partial_x\psi_0)\big) \cap \Big[-N\big((\bar{x},\bar{y},\bar{u}); \mathrm{gph}\,(-\partial_x\psi)\big)\Big] = \{0\},$$

which is equivalent to the qualification condition: $(x^*, y^*, w) = (0,0,0)$ *is the only triple satisfying the inclusion*

$$(x^*, y^*) \in \partial_{N,x}^2 \psi_0(\bar{x},\bar{y},\bar{u})(w) \bigcap \Big[-\partial_{N,x}^2 \psi(\bar{x},\bar{y},-\bar{u})(w)\Big]. \tag{3.17}$$

Furthermore, if ψ_0 *is* $\mathcal{C}^2$ *around* $(\bar{x},\bar{y})$ *then the latter is simplified to:*

$$\Big[0 \in \nabla^2\psi_0(\bar{x},\bar{y})^*(w,0) + \partial_{N,x}^2 \psi\big(\bar{x},\bar{y},-\nabla_x\psi_0(\bar{x},\bar{y})\big)(w)\Big] \Longrightarrow w = 0. \tag{3.18}$$

Now we focus our attention to the case when X, Y are arbitrary Asplund spaces and the cost function $\psi_0 \colon X \times Y \to \overline{I\!R}$ is $\mathcal{C}^2$. The stationary point multifunction to the parameterized minimization problem (3.3) becomes (3.2), which has the following coderivative estimate:

Corollary 3.16 *Let the cost function* $\psi_0 \colon X \times Y \to \overline{I\!R}$ *be* $\mathcal{C}^2$ *and* $\bar{x} \in S(\bar{y})$ *for the stationary point multifunction* S *given in (3.2). Assume that the graph of* $\partial_x\psi$ *is norm-closed around* $(\bar{x},\bar{y},-\nabla_x\psi_0(\bar{x},\bar{y}))$*, and that either* $\dim X < \infty$ *or* $(\partial_x\psi)^{-1}$ *is strongly PSNC at*

$\big(-\nabla_x\psi_0(\bar{x},\bar{y}),\bar{x},\bar{y}\big)$. *Assume also that the constraint qualification condition (3.18) is satisfied. Then for all* $x^*\in X^*$ *the normal coderivative of* S *has each image* $D^*_N S(\bar{y},\bar{x})(x^*)$ *contained in the set of all* $y^*\in Y^*$ *for which there exists* $w\in X^{**}$ *such that*

$$(-x^*,y^*)-\nabla^2\psi_0(\bar{x},\bar{y})^*(w,0)\in\partial^2_{N,x}\psi\big(\bar{x},\bar{y},-\nabla_x\psi_0(\bar{x},\bar{y})\big)(w). \tag{3.19}$$

Proof. Taking Remark 3.15 and Proposition 2.12 into account, the result follows from Proposition 3.13. △

We can combine the pointbased characterization of Lipschitz-like property for set-valued mappings in Apslund space settings given in [34, Theorem 4.10] with the coderivative estimate in Corollary 3.16 to deduce the following sufficient condition for the Lipschitz-like property of the general stationary point multifuction (3.2).

Corollary 3.17 *Let* S *be the stationary point multifunction (3.2) with cost function* ψ_0 *is* $\mathcal{C}^2$, *and let* $\bar{x}\in S(\bar{y})$. *Assume that the graph of* $\partial_x\psi$ *is norm-closed around* $\big(\bar{x},\bar{y},-\nabla_x\psi_0(\bar{x},\bar{y})\big)$ *and SNC at this point. If the following qualification conditions*

$$\Big[(0,y^*)-\nabla^2\psi_0(\bar{x},\bar{y})^*(w,0)\in\partial^2_{N,x}\psi\big(\bar{x},\bar{y},-\nabla_x\psi_0(\bar{x},\bar{y})\big)(w)\Big]\Longrightarrow y^*=0, \tag{3.20}$$

$$\Big[(0,0)\in\nabla^2\psi_0(\bar{x},\bar{y})^*(w,0)+\partial^2_{N,x}\psi\big(\bar{x},\bar{y},-\nabla_x\psi_0(\bar{x},\bar{y})\big)(w)\Big]\Longrightarrow w=0 \tag{3.21}$$

are satisfied then S *is Lipschitz-like around* $(\bar{y},\bar{x})$.

Proof. It is clear that all assumptions in Corollary 3.16 are satisfied. Thus, the normal coderivative of S is estimated by (3.19). Using the QC (3.20), we obtain

$$D^*_N S(\bar{y},\bar{x})(0)=\big\{0\big\},$$

which implies

$$D^*_M S(\bar{y},\bar{x})(0)=\big\{0\big\}.$$

To show that S is Lipschitz-like around $(\bar{y}, \bar{x})$, it remains to verify that S is PSNC at this point. Indeed, we will show that S is SNC under the assumptions made. Observe that

$$(y, x) \in \operatorname{gph} S \iff (x, y) \in \ker\left(\nabla_x \psi_0 + \partial_x \psi\right) = \operatorname{dom}\big(\nabla_x \psi_0 \cap (-\partial_x \psi)\big).$$

Thus, due to [32, Proposition 4.1], the set $\operatorname{gph} S$ is SNC at $(\bar{y}, \bar{x})$ if the intersection mapping $\nabla_x \psi_0 \cap (-\partial_x \psi)$ is PSNC at $(\bar{x}, \bar{y})$. The latter means that the intersection set

$$\operatorname{gph}\big(\nabla_x \psi_0 \cap (-\partial_x \psi)\big) = \operatorname{gph}\left(\nabla_x \psi\right) \cap \operatorname{gph}\left(-\partial_x \psi\right) \subset X \times Y \times X^*$$

is PSNC at $\big(\bar{x}, \bar{y}, \nabla_x \psi_0(\bar{x}, \bar{y})\big)$ with respect to $X \times Y$. Employing [34, Corollary 3.80] on PSNC property of set intersections in product spaces, the PSNC property of $\nabla_x \psi_0 \cap (-\partial_x \psi)$ is ensured if $\partial_x \psi$ is SNC around $\big(\bar{x}, \bar{y}, -\nabla \psi_0(\bar{x}, \bar{y})\big)$, and the pair $\big\{\operatorname{gph} \nabla_x \psi_0, \operatorname{gph}\left(-\partial_x \psi\right)\big\}$ satisfies the mixed qualification condition at $\big(\bar{x}, \bar{y}, \nabla_x \psi_0(\bar{x}, \bar{y})\big)$ relative to X^*. It is easy to derive directly from definitions that the mixed qualification condition for the set systems $\nabla_x \psi_0 \cap (-\partial_x \psi)$ is implied by the qualification condition (3.21). This completes the proof of the corollary. △

Example 3.18 *Consider the problem*

$$\textit{Minimize } \frac{1}{2} \sum_{k=1}^{\infty} x_k^2 - \sum_{k=1}^{\infty} \frac{y^{k-1}}{(k-1)!} x_k$$

$$\textit{over } x = \{x_k\} \in \ell^2 \textit{ satisfying } x_1 \leq 0,$$

which conforms to our model by defining

$$\psi_0(x, y) := \frac{1}{2} \sum_{k=1}^{\infty} x_k^2 - \sum_{k=1}^{\infty} \frac{y^{k-1}}{(k-1)!} x_k,$$

$$\psi(x, y) := \begin{cases} 0 & \textit{if } \; x_1 \leq 0, \\ \infty & \textit{otherwise.} \end{cases}$$

Then ψ_0 *is* $\mathcal{C}^2$, *and*

$$\nabla_x\psi_0(x,y) = (x_1 - 1, x_2 - y, x_3 - \frac{y^2}{2!}, x_4 - \frac{y^3}{3!}, \cdots),$$
$$\nabla^2\psi_0(x,y)^*(w,0) = \Big(w, -\sum_{k=2}^{\infty}\frac{y^{k-2}}{(k-2)!}w_k\Big), \text{ for any } w \in \ell^2.$$

Also,

$$\partial\psi(x,y) = \partial^\infty\psi(x,y) = N(x_1; I\!R_-) \times \big\{(0,0,0,\cdots)\big\}.$$

The stationary point multifunction is given by

$$\begin{aligned} S(y) &= \Big\{x = (x_1, x_2, x_3, \cdots) \in \ell^2 \Big| \; 0 \in \nabla_x\psi_0(x,y) + \partial_x\psi(x,y)\Big\} \\ &= \Big\{x = (x_1, x_2, x_3, \cdots) \in \ell^2 \Big| \; x_1 \in 1 - N(x_1; I\!R_-), \; x_k = \frac{y^{k-1}}{(k-1)!} \text{ for } k \geq 2\Big\}. \end{aligned}$$

It is easy to see that

$$N(x_1, I\!R_-) = \begin{cases} \emptyset & \text{if } x_1 > 0, \\ [0,\infty) & \text{if } x_1 = 0, \\ 0 & \text{if } x_1 < 0. \end{cases}$$

Thus, $x_1 \in 1 - N(x_1; I\!R_-)$ *implies* $x_1 = 0$ *(and hence* $\partial\psi(x,y) = \partial^\infty\psi(x,y)$ *is the singleton* $\{(1,0,0,\cdots) \in \ell^2\}$*). The mapping* $S : I\!R \to \ell^2$ *is single-valued, which is given by*

$$S(y) = (0, y, \frac{y^2}{2!}, \frac{y^3}{3!}, \cdots).$$

Take any $\bar{y} \in I\!R$ *and* $\bar{x} = (0, \bar{y}, \frac{\bar{y}^2}{2!}, \frac{\bar{y}^3}{3!}, \cdots) \in S(\bar{y})$, *it follows directly from formula of* S *that*

$$D_N^*S(\bar{y},\bar{x})(x^*) = \Big\{\sum_{k=2}^{\infty}\frac{\bar{y}^{k-2}}{(k-2)!}x_k^*\Big\}, \text{ for any } x^* \in \ell^2.$$

On the other hand, it is easy to see that $\operatorname{gph}(\partial_x\psi)$ *is normed-closed in* $\ell^2 \times I\!R \times \ell^2$ *around* $\big(\bar{x}, \bar{y}, (1,0,0,0,\cdots)\big)$ *and* $\partial_x\psi$ *is strongly PSNC at this point. We can also check that*

$(\nabla_x\psi_0)^{-1}$ is PSNC at $\big(\nabla_x\psi_0(\bar{x},\bar{y}),\bar{x},\bar{y}\big)$. It remains to justify that the qualification in Corollary 3.16 is satisfied. In fact, for $\bar{x}\in S(\bar{y})$ we have

$$-\nabla_x\psi_0(\bar{x},\bar{y})=(1,0,0,\cdots),$$

and

$$\partial^2_{N,x}\psi\big(\bar{x},\bar{y},-\nabla_x\psi_0(\bar{x},\bar{y})\big)(w)=D^*_N N(\cdot;I\!R_-)(0,1)(w_1)\times\{(0,0,0,\cdots)\}$$

for $w=\{w_k\}\in\ell^2$. Thus, the inclusion

$$0\in\nabla^2\psi_0(\bar{x},\bar{y})^*(w,0)+\partial^2_{N,x}\psi\big(\bar{x},\bar{y},-\nabla_x\psi(\bar{x},\bar{y})\big)(w)$$

implies $w_k=0$ for all $k\geq 2$, and $-w_1\in D^*_N N(\cdot,I\!R_-)(0,1)(w_1)$. The latter inclusion results in $w_1=0$, since

$$D^*_N N(\cdot,I\!R_-)(0,1)(w_1)=\begin{cases} I\!R & \text{if } \; w_1=0,\\ \emptyset & \text{if } \; w_1\neq 0.\end{cases}$$

Therefore, Corollary 3.16 claims that $D^*_N S(\bar{y},\bar{x})(x^*)$ is contained in the set

$$\Big\{y^*\in Y^*\Big|\ \exists w\in\ell^2 \text{ such that } \big(-x^*-\nabla^2_{xx}\psi_o(\bar{x},\bar{y})^*(w),\ y^*-\nabla^2_{xy}\psi_o(\bar{x},\bar{y})^*(w)\big)\in \partial^2_{N,x}\psi\big(\bar{x},\bar{y},-\nabla_x\psi_0(\bar{x},\bar{y})\big)(w)\Big\}.$$

In fact, the latter set also reduces to single point $\Big\{\sum_{k=2}^{\infty}\dfrac{\bar{y}^{k-2}}{(k-2)!}x^*_k\Big\}$. Thus, in this case we actually have the exact formula for the coderivative of stationary point multifunction.

3.2 Coderivative Analysis of Composite Constraint Functions

In this section, we consider further the case when the constraint function ψ is strongly amenable in x at $\bar{x}$ with compatible parameterization in y at $\bar{y}$. The following result holds:

Corollary 3.19 *For the stationary point multifunction (3.2) with* $(\bar{y},\bar{x}) \in \mathrm{gph}\, S$*, assume that the constraint function* ψ *is strongly amenable in* x *at* $\bar{x}$ *with compatible parameterization in* y *at* $\bar{y}$*, and that the second-order qualification condition (2.20) holds with mappings* $\varphi : I\!R^m \to \overline{I\!R}$*,* $g : X \times Y \to I\!R^m$ *from the composite representation of* ψ *and* $\bar{u} = -\nabla_x \psi_0(\bar{x},\bar{y})$*. If* $w = 0$ *is the only solution to the inclusion*

$$0 \in \nabla^2 \psi_0(\bar{x},\bar{y})^*(w,0) + \bigcup_{\substack{\bar{p} \in \partial\varphi(g(\bar{x},\bar{y})) \\ \nabla_x g(\bar{x},\bar{y})^*\bar{p} = -\nabla_x \psi_0(\bar{x},\bar{y})}} \Big[\nabla^2 \langle \bar{p}, g\rangle(\bar{x},\bar{y})^*(w,0) + \nabla g(\bar{x},\bar{y})^* \partial^2 \varphi(g(\bar{x},\bar{y}),\bar{p})(\nabla_x g(\bar{x},\bar{y})^{**} w)\Big],$$

then the normal coderivative of S *has each image* $D_N^* S(\bar{y},\bar{x})(x^*)$ *contained in the set of* $y^* \in Y^*$ *for which there exists* $w \in X^{**}$ *such that*

$$(-x^*, y^*) - \nabla^2 \psi_0(\bar{x},\bar{y})^*(w,0) \in \bigcup_{\substack{\bar{p} \in \partial\varphi(g(\bar{x},\bar{y})) \\ \nabla_x g(\bar{x},\bar{y})^*\bar{p} = -\nabla_x \psi_0(\bar{x},\bar{y})}} \Big[\nabla^2 \langle \bar{p}, g\rangle(\bar{x},\bar{y})^*(w,0) + \nabla g(\bar{x},\bar{y})^* \partial^2 \varphi(g(\bar{x},\bar{y}),\bar{p})(\nabla_x g(\bar{x},\bar{y})^{**} w)\Big], \tag{3.22}$$

provided that either $\nabla\psi_0$ *is SNC at* $\big(\bar{x},\bar{y},\nabla\psi_0(\bar{x},\bar{y})\big)$ *(which holds, in particular, when* $\dim X < \infty$*) or* $(\partial_x \psi)^{-1}$ *is strongly PSNC at* $\big(-\nabla_x \psi_0(\bar{x},\bar{y}),\bar{x},\bar{y}\big)$*.*

Proof. We check that the graph of $\partial_x \psi$ is norm-closed around $\big(\bar{x},\bar{y},-\nabla_x\psi_0(\bar{x},\bar{y})\big)$. Indeed, since ψ is strongly amenable in x at $\bar{x}$ we have the following representation

$$\partial_x \psi(x,y) = \nabla_x g(x,y)^* \partial\varphi\big(g(x,y)\big)$$

for all (x,y) near $(\bar{x},\bar{y})$. Therefore, the closedness of $\mathrm{gph}\,\partial_x\psi$ around $\big(\bar{x},\bar{y},-\nabla_x\psi_0(\bar{x},\bar{y})\big)$ follows from the twice differentiability of g and the closedness of $\mathrm{gph}\,\partial\varphi$, which is implied by the convexity of φ. Employing Proposition 3.13 together with Theorem 2.22 under the assumptions made, we obtain the result. △

On the other hand, we can incorporate multipliers into our analysis of stationary points in the following way: A multiplier associated with the stationary point $x \in X$ is any vector $p \in \partial\varphi(g(x,y))$ for which $0 = \nabla_x\psi_0(x,y) + \nabla_x g(x,y)^* p$. The sets of stationary point-multiplier pairs associated with each parameter define the *stationary point-multipler multifunction*

$$SM(y) := \Big\{(x,p) \in X \times I\!R^m \Big| \, p \in \partial\varphi(g(x,y)) \text{ and } 0 = \nabla_x\psi_0(x,y) + \nabla_x g(x,y)^* p\Big\}.$$

This multifunction can be rewritten in the form

$$SM(y) := \Big\{(x,p) \in X \times I\!R^m \Big| \, 0 \in \Big(\nabla_x\psi_0(x,y) + \nabla_x g(x,y)^* p, -g(x,y)\Big) + \{0\} \times (\partial\varphi)^{-1}(p)\Big\}, \tag{3.23}$$

which is exactly the kind of multifunction covered by [32, Corollary 4.3] with

$$F(x,p,y) := \big(\nabla_x\psi_0(x,y) + \nabla_x g(x,y)^* p, -g(x,y)\big) \tag{3.24}$$

and

$$Q(x,p,y) := \{0\} \times (\partial\varphi)^{-1}(p). \tag{3.25}$$

As a result, we have the following corollary.

Corollary 3.20 *For the stationary point-multiplier mapping (3.23) with* $(\bar{x},\bar{p}) \in SM(\bar{y})$*, if the constraint qualification holds that* $(w,v) = (0,0)$ *is the only solution to*

$$\begin{cases} 0 = \nabla^2\psi_0(\bar{x},\bar{y})^*(w,0) + \nabla^2\langle\bar{p},g\rangle(\bar{x},\bar{y})^*(w,0) + \nabla g(\bar{x},\bar{y})^* v \\ v \in \partial^2\varphi\big(g(\bar{x},\bar{y}),\bar{p}\big)\big(\nabla_x g(\bar{x},\bar{y})^{**} w\big) \end{cases}$$

then the coderivative has each image $D^*SM(\bar{y},\bar{x},\bar{p})(x^*,q)$ *contained in the set of all* $y^* \in Y^*$ *for which there exists* $w \in X^{**}$ *with*

$$\begin{aligned} (-x^*,y^*) - \nabla^2\psi_0(\bar{x},\bar{y})^*(w,0) &\in \nabla^2\langle\bar{p},g\rangle(\bar{x},\bar{y})^*(w,0) \\ &+ \nabla g(\bar{x},\bar{y})^*\partial^2\varphi\big(g(\bar{x},\bar{y}),\bar{p}\big)\big(\nabla_x g(\bar{x},\bar{y})^{**} w + q\big). \end{aligned}$$

provided that $(\partial\varphi)^{-1}$ is closed graph around $\bar{p}$ and the set-valued mapping F defined by (3.24) is SNC at $(\bar{x}, \bar{p}, \bar{y})$.

Proof. Employing [32, Corollary 4.3], the coderivative of the stationary point-multiplier mapping (3.23) has each image $D^*SM(\bar{y}, \bar{x}, \bar{p})(x^*, q)$ contained in the set of all $y^* \in Y^*$ for which there exists $(w, v) \in X^{**} \times I\!R^m$ with

$$(-x^*, -q, y^*) \in D^*_N F(\bar{x}, \bar{p}, \bar{y})(w, v) + D^*_N Q\big(\bar{x}, \bar{p}, \bar{y}, 0, g(\bar{x}, \bar{y})\big)(w, v).$$

Since $F : X \times I\!R^m \times Y \to X^* \times I\!R^m$ is $\mathcal{C}^1$, and thus strictly differentiable at $(\bar{x}, \bar{p}, \bar{y})$, its normal coderivative has each image given by

$$\begin{aligned} D^*_N F(\bar{x}, \bar{p}, \bar{y})(w, v) &= \big\{\nabla F(\bar{x}, \bar{p}, \bar{y})^*(w, v)\big\} \\ &= \left\{ \begin{bmatrix} \nabla^2_{xx}\psi_0(\bar{x}, \bar{y})^* w + \nabla^2_{xx}\langle \bar{p}, g\rangle(\bar{x}, \bar{y})^*(w) - \nabla_x g(\bar{x}, \bar{y})^* v \\ \nabla_x g(\bar{x}, \bar{y})^{**} w \\ \nabla^2_{xy}\psi_0(\bar{x}, \bar{y})^* w + \nabla^2_{xy}\langle \bar{p}, g\rangle(\bar{x}, \bar{y})^*(w) - \nabla_y g(\bar{x}, \bar{y})^* v \end{bmatrix} \right\}. \end{aligned}$$

Also, the coderivative of the multifunction Q in this case satisfies

$$D^*_N Q(\bar{x}, \bar{p}, \bar{y}, 0, g(\bar{x}, \bar{y}))(w, v) = \begin{bmatrix} 0 \\ D^*_N(\partial\varphi)^{-1}\big(\bar{p}, g(\bar{x}, \bar{y})\big)(v) \\ 0 \end{bmatrix}.$$

Taking into account the fact that

$$\begin{aligned} -q - \nabla_x g(\bar{x}, \bar{y})^{**} w \in D^*_N(\partial\varphi)^{-1}\big(\bar{p}, g(\bar{x}, \bar{y})\big)(v) \Longleftrightarrow \\ -v \in \partial^2\varphi\big(g(\bar{x}, \bar{y}), \bar{p}\big)\big(q + \nabla_x g(\bar{x}, \bar{y}) w\big), \end{aligned}$$

the result follows. $\triangle$

3.3 Sensitivity Analysis under Canonical Perturbations

Our sensitivity analysis applies only when certain important constraint qualifications are satisfied. In this section, we focus our attention on a broad and important model when the constraint qualifications are automatic. Everything revolves around the following result that covers implicit mappings defined by canonical perturbations $\tilde{z}$.

Corollary 3.21 *Let $S : Y \times Z \to X$ be an implicit multifunction of the form*

$$S(y, \tilde{z}) := \big\{x \in X \,\big|\, \tilde{z} \in f(x, y) + Q(x, y)\big\} \tag{3.26}$$

with $f : X \times Y \to Z$ a $\mathcal{C}^1$ mapping and $Q : X \times Y \rightrightarrows Z$ a multifuction with closed graph, and let $(\bar{y}, \bar{z}, \bar{x}) \in \operatorname{gph} S$. Suppose that either f is SNC at $(\bar{x}, \bar{y}, \bar{z})$ (which holds, in particular, when $\dim Z < \infty$) or Q^{-1} is strongly PSNC at $\big(\bar{z} - f(\bar{x}, \bar{y}), \bar{x}, \bar{y}\big)$. Then the coderivative of S at $(\bar{y}, \bar{z}, \bar{x})$ has each image $D^*_N S(\bar{y}, \bar{z}, \bar{x})(x^*)$ contained in the set

$$\big\{(y^*, -z^*) \in Y^* \times Z^* \big|\, (-x^*, y^*) \in \nabla f(\bar{x}, \bar{y})^* z^* + D^*_N Q(\bar{x}, \bar{y}, \bar{z} - f(\bar{x}, \bar{y}))(z^*)\big\}.$$

Proof. Putting $\tilde{y} := (y, \tilde{z})$, and

$$\begin{aligned} \tilde{f}(x, \tilde{y}) &:= f(x, y) - \tilde{z}, \\ \widetilde{Q}(x, \tilde{y}) &:= Q(x, y), \end{aligned}$$

the result follows directly from [32, Corollary 4.3] applied to $\tilde{f}$ and $\tilde{Q}$ since the coderivatives of these mappings satisfy

$$\begin{aligned} \nabla \tilde{f}(\bar{x}, \bar{y}, \bar{z})^* p &= \big(\nabla f(\bar{x}, \bar{y})^* p, -p\big), \\ D^*_N \widetilde{Q}(\bar{x}, \bar{y}, \bar{z}, \bar{z} - f(\bar{x}, \bar{y}))(p) &= \Big(D^*_N Q\big(\bar{x}, \bar{y}, \bar{z} - f(\bar{x}, \bar{y})\big)(p), 0\Big), \end{aligned}$$

for any $p \in Z^*$. △

Canonical perturbations arise in our optimization model as a linear "tilt" term (parameterized by $u \in X^*$) in the object function and a constant shift ($v \in I\!R$) in the constraint function as follows:

$$\text{Minimize } \psi_0(x, y) - \langle u, x \rangle + \varphi(g(x, y) + v) \text{ over } x \in X, \tag{3.27}$$

where ψ_0 is $\mathcal{C}^2$ as before, and $\varphi \circ g$ is a corresponding composite representation of the constraint function ψ, which is strongly amenable in x at $\bar{x}$ with compatible parameterization in y at $\bar{y}$. With the parameterization thus enriched by the canonical perturbations, the corresponding stationary point-multiplier multifunction becomes

$$SM(y,u,v) := \Big\{(x,p) \in X \times I\!R^m \Big|\ p \in \partial\varphi\big(g(x,y)+v\big),\\ u = \nabla_x\psi_0(x,y) + \nabla_x g(x,y)^* p\Big\},$$

which can be rewritten in the form

$$SM(y,u,v) := \Big\{(x,p) \in X \times I\!R^m \Big|\ (u,v) \in \big(\nabla_x\psi_0(x,y) + \nabla_x g(x,y)^* p, -g(x,y)\big)\\ + \{0\} \times (\partial\varphi)^{-1}(p)\Big\}. \tag{3.28}$$

Thus, the following results are implied from Corollary 3.21.

Corollary 3.22 *Let $(\bar{x},\bar{p}) \in SM(\bar{y},\bar{u},\bar{v})$ with the stationary point-multiplier mapping SM (3.28) associated with the canonically perturbed optimization problem (3.27). Assume that* $\dim X < \infty$ *and that $(\partial\varphi)^{-1}$ is closed-graph near $\big(\bar{p}, g(\bar{x},\bar{y})+\bar{v}\big)$. The normal coderivative of SM has each image $D^*_N SM\big((\bar{y},\bar{u},\bar{v}),(\bar{x},\bar{p})\big)(u^*,v')$ contained in the set of all $(y^*,-u^*,v') \in Y^* \times X^{**} \times I\!R^m$ such that*

$$v' \in \partial^2\varphi\big(\bar{v}+g(\bar{x},\bar{y}),\bar{p}\big)\big(p' + \nabla_x g(\bar{x},\bar{y})^* u^*\big),\ p' \in I\!R^m,$$

and for which

$$(-x^*,y^*) - \nabla^2\psi_0(\bar{x},\bar{y})^*(u^*,0) = \nabla^2\langle\bar{p},g\rangle(\bar{x},\bar{y})^*(u^*,0) + \nabla g(\bar{x},\bar{y})^* v'.$$

Proof. The result follows directly from Corollary 3.21 with

$$f(x,p,y) = \Big(\nabla_x\psi_0(x,y) + \nabla_x g(x,y)^* p,\ -g(x,y)\Big),$$
$$Q(x,p,y) = \{0\} \times (\partial\varphi)^{-1}(p),$$

and hence,

$$\nabla f(\bar{x},\bar{p},\bar{y})^*(u^*,-v') = \begin{pmatrix} \nabla^2_{xx}\psi_0(\bar{x},\bar{y})^*u^* + \nabla^2_{xx}\langle\bar{p},g\rangle(\bar{x},\bar{y})^*u^* + \nabla_x g(\bar{x},\bar{y})^*v' \\ \nabla_x g(\bar{x},\bar{y})^*u^* \\ \nabla^2_{xy}\psi_0(\bar{x},\bar{y})^*u^* + \nabla^2_{xy}\langle\bar{p},g\rangle(\bar{x},\bar{y})^*u^* + \nabla_y g(\bar{x},\bar{y})^*v' \end{pmatrix},$$

$$D^*_N Q\big((\bar{x},\bar{p},\bar{y}),(\bar{u},\bar{v})\big)(u^*,-v') = \begin{pmatrix} 0 \\ D^*_N(\partial\varphi)^{-1}\big(\bar{p},\bar{v}+g(\bar{x},\bar{y})\big)(-v') \\ 0 \end{pmatrix},$$

where the coderivative of $(\partial g)^{-1}$ is related to the second-order subdifferential of g by

$$\begin{aligned} -p' - \nabla_x g(\bar{x},\bar{y})^*u^* \in & D^*_N(\partial\varphi)^{-1}\big(\bar{p},\bar{v}+g(\bar{x},\bar{y})\big)(-v') \\ & \Longleftrightarrow v' \in \partial^2\varphi\big(\bar{v}+g(\bar{x},\bar{y}),\bar{p}\big)\big(p' + \nabla_x g(\bar{x},\bar{y})^*u^*\big). \end{aligned}$$

$\triangle$

Chapter 4

Coderivative Analysis of Quasi-Variational Inequalities in Asplund Spaces

In this chapter, we consider the so-called parameterized *quasi-variational inequalities (QVIs)* of the following type: Given a parameter $y \in Y$, find a decision vector $x \in \Gamma(x,y) \subset X$ such that

$$\langle g(x,y), u-x\rangle \geq 0 \text{ for all } u \in \Gamma(x,y) \tag{4.1}$$

where $g : X \times Y \to X^*$ is a single-valued continuously differentiable function, while $\Gamma : X \times Y \rightrightarrows X$ is a set-valued mapping. We always assume that the spaces under consideration are Asplund spaces, and that the mapping Γ is of closed graph and take convex values $\Gamma(x,y)$. The solution map to (4.1) is defined by

$$S(y) := \big\{x \in X \big|\ \langle g(x,y), u-x\rangle \geq 0 \text{ whenever } u \in \Gamma(x,y)\big\},\ \ y \in Y.$$

Using the standard definition of the normal cone to convex sets, we can rewrite the QVI (4.1) in Robinson's form of the generalized equation (GE):

$$0 \in g(x,y) + N_{\Gamma(x,y)}(x), \quad x \in \Gamma(x,y). \tag{4.2}$$

The solution map to the QVI (4.1) written in the GE form (4.2) is given by

$$S(y) = \big\{x \in X \big|\ 0 \in g(x,y) + N_{\Gamma(x,y)}(x)\big\}. \tag{4.3}$$

4.1 New Rules of Coderivative Calculus

Throughout this section, we assume that the set-valued mapping Γ generating the QVI in (4.1) admits the representation

$$\Gamma(x,y) := \big\{u \in X \big|\ q(x,y,u) \in \Theta\big\} \tag{4.4}$$

where $q : X \times Y \times X \to Z$ is *twice continuously differentiable* around the points in question, and where Θ is a *closed convex* subset of Z such that $\text{int}\Theta \neq \emptyset$. In addition, q and Θ have to satisfy certain requirements ensuring that Γ in (4.4) is convex-valued, which is essential to ensure the strong amenable structure in the representation of $N_{\Gamma(x,y)}(z)$. The convex-valuedness property of $\Gamma(x,y)$ holds, e.g., if Θ is a convex cone with vertex at 0, and if $q(x,y,\cdot)$ is Θ-convex for all $(x,y) \in X \times Y$. Furthermore, we impose the basic constraint qualification (CQ) condition

$$\Big[\nabla q(\bar{x},\bar{y},\bar{x})^* z^* = 0 \text{ and } z^* \in N_\Theta\big(q(\bar{x},\bar{y},\bar{x})\big)\Big] \Longrightarrow z^* = 0. \tag{4.5}$$

We will study the coderivative of the set-valued mapping $(x,y) \rightrightarrows N_{\Gamma(x,y)}(x)$ in the generalized equation form (4.2) of the QVI under consideration.

It follows from the structure of Γ in (4.4) that $N_{\Gamma(x,y)}$ admits the composite subdifferential representation

$$N_{\Gamma(x,y)}(u) = \partial_u \psi(x,y,u) \text{ with } \psi := \delta_\Theta \circ q.$$

Since Θ is convex and q is smooth, the multifunction $(x,y,u) \rightrightarrows N_\Theta\big(q(x,y,u)\big)$ is closed-graph in norm$\times$weak* topology of $(X \times Y \times X) \times Z^*$. Thus, the basic CQ (4.5) is persistent in a neighborhood of $(\bar{x},\bar{y},\bar{x})$ and the composite function ψ is *strongly amenable* around $(\bar{x},\bar{y},\bar{x})$. It then follows from Theorem 2.9 that

$$N_{\Gamma(x,y)}(u) = \partial_u \psi(x,y,u) = \nabla_u q(x,y,u)^* N_\Theta\big(q(x,y,u)\big).$$

Therefore, we can replace the GE (4.2) by

$$0 \in g(x,y) + \nabla_3 q(x,y,x)^* N_\Theta\big(q(x,y,x)\big) \tag{4.6}$$

considered for all (x,y,u) around $(\bar{x},\bar{y},\bar{x})$.

We will focus our attention on the multivalued term in (4.6) denoted by

$$Q(x,y) := \nabla_3 q(x,y,x)^* N_\Theta\big(q(x,y,x)\big). \tag{4.7}$$

This set-valued mapping $Q : X \times Y \rightrightarrows X^*$ is closed-graph in norm$\times$weak* topology of $(X \times Y) \times X^*$ due to the robustness of the normal cone N_Θ when Θ is convex and the continuity of q.

Let us present an upper estimate for the basic normal cones of a special form of sets, which is essential to establish a coderivative calculus for set-valued mappings of type (4.7).

Theorem 4.1 *Let $M : Y \rightrightarrows X$ be the set-valued mapping given by*

$$M(y) = \big\{x \in C \big|\ g(x) + y \in D\big\}$$

where C and D are closed subsets of X and Y, respectively, and $g : X \to Y$ is strictly differentiable at $\bar{x}$. Assume that either $\dim Y < \infty$ *or D is SNC at $g(\bar{x})$, and that M is calm at $(0, \bar{x})$. Then we have the following inclusion*

$$N_{M(0)}(\bar{x}) \subset \bigcup_{y^* \in N_D(g(\bar{x}))} \nabla g(\bar{x})^*(y^*) + N_C(\bar{x}). \tag{4.8}$$

Proof. Letting $f : Y \times X \to Y$ be the function given by $f(y, x) := g(x) + y$, then f is PSNC at $(0, \bar{x})$. In addition, f is SNC at this point if $\dim Y < \infty$. We rewrite M as

$$M(y) = \big\{x \in X \big|\ f(y, x) \in D, (y, x) \in Y \times C\big\}.$$

It is easy to see that for function f defined above we have $\ker D_N^* f(0, \bar{x}) = \{0\}$. Thus, the coderivative estimate for constraint systems in [34, Theorem 4.32] gives

$$\begin{aligned} D_N^* M(0, \bar{x})(x^*) &\subset \big\{y^* \in Y^* \big|\ (y^*, -x^*) \in D_N^* f(0, \bar{x}) \circ N_D(g(\bar{x})) + N_{Y \times C}(0, \bar{x})\big\} \\ &= \big\{y^* \in N_D\big(g(\bar{x})\big) \big| -x^* \in \nabla g(\bar{x})^* y^* + N_C(\bar{x})\big\}. \end{aligned} \tag{4.9}$$

Take any $u^* \in N_{M(0)}(\bar{x})$. Since $M(0) = C \cap g^{-1}(D)$ is closed, there exist, by [34, Theoem 1.97], $\lambda > 0$ and $x^* \in \partial d_{M(0)}(\bar{x})$ such that $u^* = \lambda x^*$. Then, for $\varepsilon_k \downarrow 0$, there are sequences $x_k \in M(0)$, $x_k^* \in \widehat{\partial} d_{M(0)}(x_k)$ and $r_k \downarrow 0$ satisfying $x_k \to \bar{x}$, $x_n^* \to x^*$, and

$$d_{M(0)}(x) - \langle x_k^*, x - x_k\rangle \ge -\varepsilon_k \|x - x_k\|, \quad \text{for all } x \in B(x_k, r_k).$$

Let L be the modulus of calmness of M at $(0, \bar{x})$. We find $r > 0$ and $\eta > 0$ satisfying

$$d_{M(0)}(x) \le L\|y\| \quad \text{whenever } x \in B(\bar{x}, r) \cap M(y), \text{ for } y \in B(0, \eta).$$

Thus, for k sufficiently large we have

$$L\|y\| + \varepsilon_k \|x - x_k\| - \langle x_k^*, x - x_k\rangle \ge \|x - x_k\| \text{ for all } x \in B(x_k, r_k) \cap M(y), y \in B(0, \eta).$$

This yields

$$L\|y\| + \varepsilon_k\|x - x_k\| - \langle x_k^*, x - x_k\rangle + \delta\big((y,x); \operatorname{gph} M\big) \geq 0 \text{ for all } x \in B(x_k, r_k), y \in B(0,\eta).$$

Therefore, $(0, x_k)$ is a local minimum of the function

$$\varphi(y,x) := L\|y\| + \varepsilon_k\|x - x_k\| - \langle x_k^*, x - x_k\rangle + \delta\big((y,x); \operatorname{gph} M\big).$$

Hence, by the non-smooth version of Fermat's rule in [34, Proposition 1.114], we get

$$0 \in \widehat{\partial}\varphi(0, x_k).$$

Evoking the semi-Lipschitzian sum rule for Fréchet subdifferentials in [34, Theorem 2.33] we find for each k a point $(y_k, u_k) \in \operatorname{gph} M$ such that $\|y_k\| + \|u_k - x_k\| < \frac{1}{k}$ and

$$\widehat{\partial}\varphi(0, x_k) \subset L\mathbb{B}_{Y^*} \times \{0\} + \{0\} \times \varepsilon_k\mathbb{B}_{X^*} - (0, x_k^*) + \widehat{N}_{\operatorname{gph} M}(y_k, u_k) + \frac{2}{k}\big(\mathbb{B}_{Y^*} \times \mathbb{B}_{X^*}\big).$$

It follows that

$$\big(Ly_k^* - \frac{2}{k}v_k^*, u_k^* - \varepsilon_n x_k^* - \frac{2}{k}w_k^*\big) \in \widehat{N}_{\operatorname{gph} M}(y_k, u_k),$$

for some $y_k^*, v_k^* \in \mathbb{B}_{Y^*}$ and some $x_k^*, w_k^* \in \mathbb{B}_{X^*}$. Since B_{Y^*} is sequentially weak*-compact, we can assume (by taking a subsequence if necessary) that $Ly^* \overset{w^*}{\to} y^* \in L\mathbb{B}_Y^*$ as $k \to \infty$. Letting $k \to \infty$ in the latest inclusion we get

$$(y^*, x^*) \in N_{\operatorname{gph} M}(0, \bar{x}),$$

which means $y^* \in D_N^* M(0, \bar{x})(-x^*)$. Taking (4.9) into account we obtain

$$y \in N_D\big(g(\bar{x})\big) \text{ and } x^* \in \nabla g(\bar{x})^* y^* + N_C(\bar{x}).$$

Thus, $u^* = \lambda x^* \in \nabla g(\bar{x})^*(\lambda y^*) + N_C(\bar{x})$ for $\lambda y^* \in N_D\big(g(\bar{x})\big)$. Inclusion (4.8) follows. $\triangle$

We are now ready to establish coderivative calculus for multifunction of the special type (4.7). For simplicity, we use the notations

$$r(x,y) := q(x,y,x) \text{ and } p(x,y) := \nabla_3 q(x,y,x).$$

Theorem 4.2 *Under the standing assumptions above, suppose that the basic CQ (4.5) is strengthened as*

$$\Big[p(\bar{x},\bar{y})^*z^* = 0 \text{ and } z^* \in N_\Theta\big(r(\bar{x},\bar{y})\big)\Big] \Longrightarrow z^* = 0 \tag{4.10}$$

and let $\bar{x}^* \in Q(\bar{x},\bar{y})$. Then the following assertions hold:

(i) For all $x^{**} \in X^{**}$ we have the coderivative upper estimate

$$D_N^*Q(\bar{x},\bar{y},\bar{x}^*) \subset \bigcup_{\substack{z^*\in N_\Theta(r(\bar{x},\bar{y}))\\ p(\bar{x},\bar{y})^*z^*=\bar{x}^*}} \Big[\big(\nabla_{(x,y)}p(\bar{x},\bar{y})^*z^*\big)^*x^{**} + D_N^*(N_\Theta\circ r)(\bar{x},\bar{y},z^*)\big(p(\bar{x},\bar{y})^{**}x^{**}\big)\Big].$$

(ii) Define the set-valued mapping $M: Z\times Z^* \rightrightarrows X\times Y\times Z^*$ by

$$M(\nu,\nu^*) := \big\{(x,y,z^*)\in X\times Y\times Z^* \big|\, \big(r(x,y)+\nu, z^*+\nu^*\big)\in \operatorname{gph} N_\Theta\big\}, \tag{4.11}$$

and assume that it is calm at the points $(0,0,\bar{x},\bar{y},z^*)$ satisfying

$$z^* \in N_\Theta\big(r(\bar{x},\bar{y})\big) \text{ and } p(\bar{x},\bar{y})^*z^* = \bar{x}^*. \tag{4.12}$$

Assume further that either $\dim Z < \infty$ *or N_Θ is SNC at all points $\big(r(\bar{x},\bar{y}),z^*\big)$ satisfying (4.12). Then for all $x^{**}\in X^{**}$ we have the following inclusion*

$$D_N^*Q(\bar{x},\bar{y},\bar{x}^*)(x^{**}) \subset \bigcup_{\substack{z^*\in N_\Theta(r(\bar{x},\bar{y}))\\ p(\bar{x},\bar{y})^*z^*=\bar{x}^*}} \Big[\big(\nabla_{(x,y)}p(\bar{x},\bar{y})^*z^*\big)^*x^{**} + \nabla r(\bar{x},\bar{y})^* D_N^*N_\Theta(r(\bar{x},\bar{y}),z^*)\big(p(\bar{x},\bar{y})^{**}x^{**}\big)\Big]. \tag{4.13}$$

Proof. (i) We represent the multifunction Q under consideration in (4.7) as the composition

$$Q(x,y) = (f\circ F)(x,y) \tag{4.14}$$

of a single-valued mapping $f : X \times Y \times Z^* \to X^*$ defined by

$$f(x, y, z^*) = p(x, y)^* z^* \tag{4.15}$$

and a set-valued mapping $F : X \times Y \to X \times Y \times Z^*$ defined by

$$F(x, y) = \big(x, y, N_\Theta(r(x, y))\big). \tag{4.16}$$

Put

$$\begin{aligned} G(x, y, x^*) : &= F(x, y) \cap f^{-1}(x^*) \\ &= \Big\{(x, y, z^*) \in X \times Y \times Z^* \Big|\ z^* \in N_\Theta\big(r(x, y)\big),\ \text{and } p(x, y)^* z^* = x^*\Big\}. \end{aligned}$$

In order to employ the coderivative chain rule in [34, Theorem 1.65] for the composition $f \circ F$, the only assumption that needs checking is that G is inner semicompact at $(\bar{x}, \bar{y}, \bar{x}^*)$. In fact, suppose the contrary, there exists a sequence $(x_k, y_k, x_k^*) \to (\bar{x}, \bar{y}, \bar{x}^*)$ such that for each k there is $z_k^* \in N_\Theta\big(r(x_k, y_k)\big)$ satisfying $p(x_k, y_k)^* z_k^* = x_k^*$ and $\|z_k^*\| \geq k$. Put $d_k^* = \frac{z_k^*}{\|z_k^*\|}$, then $\|d_k^*\| = 1$. By passing to a subsequence if necessary, we find $d^* \in Z^*$ such that $d_k^* \stackrel{w^*}{\to} d^*$. The continuity of r and p and the robustness of the normal cone $N_\Theta(\cdot)$ yield

$$d^* \in N_\Theta\big(r(\bar{x}, \bar{y})\big) \quad \text{and} \quad p(\bar{x}, \bar{y})^* d^* = 0.$$

This implies, by the constraint qualification condition (4.10), that $d^* = 0$, which means $d_k^* \stackrel{w^*}{\to} 0$. On the other hand, since Θ is convex with $\text{int}\Theta \neq \emptyset$ it is BSNC at $r(\bar{x}, \bar{y})$. Hence, $\|d_k^*\| \to 0$. This is a contradiction as $\|d_k^*\| = 1$.

Employing the aforementioned chain rule in [34, Theorem 1.65], we obtain

$$D_N^*(f \circ F)(\bar{x}, \bar{y}, \bar{x}^*)(x^{**}) \subset \bigcup_{\substack{z^* \in N_\Theta(r(\bar{x}, \bar{y})) \\ p(\bar{x}, \bar{y})^* z^* = \bar{x}^*}} D_N^* F\big((\bar{x}, \bar{y}), (\bar{x}, \bar{y}, z^*)\big) \circ \nabla f(\bar{x}, \bar{y}, \bar{x}^*)^*(x^{**}).$$

Furthermore, since p is smooth around $(\bar{x}, \bar{y})$ due to the twice continuously differentiability of q, we get

$$\nabla f(\bar{x}, \bar{y}, z^*)^*(x^{**}) = \Big(\nabla_{x,y}\big(p(\bar{x}, \bar{y})^* z^*\big)^* x^{**}, p(\bar{x}, \bar{y})^{**} x^{**}\Big).$$

To justify the coderivative upper estimate in (i), it remains to observe that for any (u^*, v^*, z^{**}) in $X^* \times Y^* \times Z^{**}$ we have the equality

$$D_N^* F\big((\bar{x}, \bar{y}), (\bar{x}, \bar{y}, z^*)\big)(u^*, v^*, z^{**}) = (u^*, v^*) + D_N^*(N_\Theta \circ r)(\bar{x}, \bar{y}, z^*)(z^{**})$$

due to the coderivative sum rule from [34, Theorem 1.62].

(ii) Next, we show that the coderivative upper estimate in (ii) follows from the one in (i) under additional calmness assumption made. The difference between these two estimates is that instead of the normal coderivative of the composition $(N_\Theta \circ f)$ in (i) we obtain the estimate in (ii) via the gradient of r and the coderivative of N_Θ separately, which is much more convenient for further applications. To proceed, consider the set-valued mapping $N_\Theta \circ r$ and observe that

$$\text{gph}\,(N_\Theta \circ r) = \big\{(x, y, z^*) \in X \times Y \times Z^* \big|\, \big(r(\bar{x}, \bar{y}), z^*\big) \in \text{gph}\, N_\Theta\big\}.$$

Since $\text{gph}\,(N_\Theta \circ r) = M(0, 0)$ for the mapping M in (4.11), employing Theorem 4.1 under the calmness assumption for M and the SNC assumption for $\text{gph}\, N_\Theta$, we get

$$N_{\text{gph}(N_\Theta \circ r)}(\bar{x}, \bar{y}, z^*) \subset \begin{bmatrix} \nabla r(\bar{x}, \bar{y})^* & 0 \\ 0 & E \end{bmatrix} \circ N_{\text{gph} N_\Theta}\big(r(\bar{x}, \bar{y}), z^*\big),$$

which is equivalent to the inclusion

$$D_N^*(N_\Theta \circ r)(\bar{x}, \bar{y}, z^*) \subset \nabla r(\bar{x}, \bar{y})^* D_N^* N_\Theta\big(r(\bar{x}, \bar{y}), z^*\big)(z^{**})$$

for all $z^{**} \in Z^{**}$. Substituting this in to the coderivative upper estimate for Q in (i) we arrive at the one in (ii) and complete the proof of the theorem. △

Observing that $D_N^* N_\Theta = D_N^*(\partial \delta_\Theta)$ is the second-order subdifferential of the indicator function of Θ, the final upper estimate of Theorem (4.2) contains second-order information on the data involved. Moreover, the calmness assumption in the second part of that theorem automatically holds under the surjectivity of $\nabla r(\bar{x}, \bar{y})^*$ imposed in the following corollary:

Corollary 4.3 *In addition to the first-order CQ (4.5), assume that* N_Θ *is SNC at all points* $\big(r(\bar{x}, \bar{y}), z^*\big)$ *for* z^* *satisfying (4.12), and that* $\nabla r(\bar{x}, \bar{y})$ *is surjective. Then the coderivative upper estimate (4.13) is satisfied.*

Proof. We need to check that the surjectivity of $\nabla r(\bar{x},\bar{y})$ implies the calmness requirement of Theorem 4.2.

As mentioned previously, the calmness property of a set-valued mapping at a reference point is automatic when the mapping is Lipschitz-like around the point. The latter property is characterized via the coderivative criterion in [34, Theorem 4.10]. Employing that criterion, we need to check that for all z^* satisfying (4.12) the set-valued mapping M in (4.11) is PSNC at the points $(0,0,\bar{x},\bar{y},z^*)$ and

$$D^*_M(0,\bar{x},\bar{y},z^*)(0)=\{0\}.$$

Putting

$$h(\nu,\nu^*,x,y,z^*):=\big(r(x,y)+\nu,z^*+\nu^*\big)$$

for $(\nu,\nu^*,x,y,z^*)\in Z\times Z^*\times X\times Y\times Z^*$, we have the representation

$$\operatorname{gph}M=h^{-1}(\operatorname{gph}N_\Theta).$$

For any z^* satisfying (4.12), $\operatorname{gph}N_\Theta$ is SNC at $\big(r(\bar{x},\bar{y}),z^*\big)$ and h is C^1 around $(0,0,\bar{x},\bar{y},z^*)$ with surjective derivative $\nabla h(0,0,\bar{x},\bar{y},z^*)$ due to the surjectivity of $\nabla r(\bar{x},\bar{y})$, which implies

$$\ker D^*_N h(0,0,\bar{x},\bar{y},z^*)=\ker\nabla h(0,0,\bar{x},\bar{y},z^*)^*=\{(0,0)\}.$$

Thus, the SNC property of inverse images in [34, Theorem 3.84] implies the SNC property of $\operatorname{gph}M$ at $(0,0,\bar{x},\bar{y},z^*)$ for all z^* satisfying (4.12).

It remains to show that $D^*_M(0,0,\bar{x},\bar{y},z^*)(0,0,0)=\big\{(0,0)\big\}$. Indeed, the construction of M in (4.11) is covered in [34, Theorem 4.31] on computing coderivatives of constraint systems. Employing the result of this theorem, we obtain

$$\begin{aligned}D^*_N M(0,0,\bar{x},\bar{y},z^*)(0,0,0)=\Big\{(s_1,s_2)\in Z^*\times Z^{**}\Big|\ \exists(t_1,t_2)\in N_{\operatorname{gph}N_\Theta}\big(r(\bar{x},\bar{y}),z^*\big)\\ \text{such that }\ (s_1,s_2,0,0,0)\in\nabla h(0,0,\bar{x},\bar{y},z^*)^*(t_1,t_2)\Big\},\end{aligned}$$

which reduces to $\big\{(0,0)\big\}$. We complete the proof of the corollary. △

4.2 Coderivatives of Solution Maps to QVIs

The main goal of this section is to derive upper estimates for the coderivative of the solution map (4.3) to the initial QVI (4.1) with $\Gamma(x, y)$ given in (4.4). The results obtained in what follows are largely based on the coderivative estimates for the multivalued term (4.7) of this QVI established in previous section via coderivative calculus.

To begin, we consider the parameter-dependent GE

$$0 \in g(x, y) + Q(x, y) \tag{4.17}$$

with both single-valued term $g : X \times Y \to X^*$ and set-valued term $Q : X \times Y \rightrightarrows X^*$ depending on the parameter $y \in Y$, where g is smooth as in section one, while Q is an arbitrarily set-valued mapping, which may not be in the special form (4.7). The following proposition provides a coderivative upper estimate for the solution map to the parameter-dependent GE (4.17) via the adjoint Jacobian of g and the coderivative of Q under the appropriate calmness assumption.

Proposition 4.4 (coderivative of solution maps to parameter-dependent GEs) *Let $(\bar{x}, \bar{y})$ satisfy the GE (4.17), and let*

$$S(y) = \big\{x \in X \big|\, 0 \in g(x, y) + Q(x, y)\big\}$$

be the solution map to this GE. Assume that g is continuously differentiable around $(\bar{x}, \bar{y})$, that Q is locally closed-graph around $(\bar{x}, \bar{y})$ and SNC at this point, and that the set-valued mapping $\Xi : X \times Y \times X^* \rightrightarrows X \times Y$ defined by

$$\Xi(u, v, u^*) = \Big\{(x, y) \in X \times Y \Big|\, \big(x + u, y + v, -g(x, y) + u^*\big) \in \operatorname{gph} Q\Big\} \tag{4.18}$$

is calm at $(0, 0, 0, \bar{x}, \bar{y})$. Then for all $x^* \in X^*$ one has the estimate

$$\begin{aligned} D^*_N S(\bar{y}, \bar{x})(x^*) \subset \Big\{y^* \in Y^* \Big|\, \exists x^{**} \in X^{**} \text{ with } (-x^*, y^*) \in \nabla g(\bar{x}, \bar{y})^* x^{**} \\ + D^*_N Q\big(\bar{x}, \bar{y}, -g(\bar{x}, \bar{y})\big)(x^{**})\Big\} \end{aligned} \tag{4.19}$$

Proof. Under the calmness assumption, it holds by Theorem 4.1 that

$$N_{\Xi(0,0,0)}(\bar{x},\bar{y}) \subset \begin{bmatrix} E & 0 & -\nabla_x g(\bar{x},\bar{y})^* \\ 0 & E & -\nabla_y g(\bar{x},\bar{y})^* \end{bmatrix} \circ N_{\mathrm{gph}Q}\big(\bar{x},\bar{y},-g(\bar{x},\bar{y})\big).$$

Observing that for the solution map S to the GE (4.17) we have

$$(y,x) \in \mathrm{gph}\, S \iff (x,y) \in \Xi(0,0,0),$$

which implies

$$y^* \in D_N^* S(\bar{y},\bar{x})(x^*) \iff (-x^*,y^*) \in N_{\Xi(0,0,0)}(\bar{x},\bar{y})$$

for any $x^* \in X^*$. Hence, there is $(u^*,v^*,-x^{**}) \in N_{\mathrm{gph}Q}\big(\bar{x},\bar{y},-g(\bar{x},\bar{y})\big)$ such that

$$(-x^*,y^*) = \big(u^* + \nabla_x g(\bar{x},\bar{y})^* x^{**}, v^* + \nabla_y g(\bar{x},\bar{y})^* x^{**}\big) = (u^*,v^*) + \nabla g(\bar{x},\bar{y})^* x^{**}.$$

Taking into account that $(u^*,v^*) \in D_N^* Q\big(\bar{x},\bar{y},-g(\bar{x},\bar{y})\big)(x^{**})$, we arrive at the coderivative estimate for S in the proposition. △

Now we proceed by studying the *solution map* (4.3) to the *QVI* (4.1) generated by the parameterized set Γ from (4.4). To simplify formulas of type (4.19) in what follows, we introduce the Lagrangian mapping $\mathcal{L} : X \times Y \times Z^* \to X^*$ defined by

$$\mathcal{L}(x,y,z^*) := g(x,y) + p(x,y)^* z^*. \tag{4.20}$$

Therefore, the adjoint Lagrange partial derivative is represented as

$$\nabla_{x,y}\mathcal{L}(x,y,z^*)^* = \nabla g(x,y)^* + \big(\nabla_{x,y} p(x,y)^* z^*\big)^*.$$

To formulate the main result of this section, we define the mapping $\Lambda : X \times Y \rightrightarrows Z^*$ by

$$\Lambda(x,y) := \big\{z^* \in Z^* \big|\ \mathcal{L}(x,y,z^*) = 0\big\}.$$

Theorem 4.5 (coderivative estimate for solution maps to QVIs) *Let $S : Y \rightrightarrows X$ be the solution map (4.3) to the original QVI represented by (4.6) around the reference point $(\bar{y},\bar{x}) \in \mathrm{gph}\, S$ with the Lagrangian $\mathcal{L}$ defined by (4.20). Assume that the CQ (4.10)* *holds and the multifunction M given by (4.11) is calm at all the points $(0,0,\bar{x},\bar{y},z^*)$ with $z^*\in\Lambda(\bar{x},\bar{y})$. We suppose also that N_Θ is SNC at all points $\big(r(\bar{x},\bar{y}),z^*\big)$ with $z^*\in\Lambda(\bar{x},\bar{y})$, and that the mapping Q in (4.7) is SNC at $\big(\bar{x},\bar{y},-g(\bar{x},\bar{y})\big)$. If in addition the multifunction $P: X^*\times Z\times Z^*\rightrightarrows X\times Y\times Z^*$ defined by*

$$P(x^*,\nu,\nu^*)=\big\{(x,y,z^*)\in X\times Y\times Z^*\big|\ \mathcal{L}(x,y,z^*)+x^*=0\big\}\cap M(\nu,\nu^*)$$

is calm at the points $(0,0,0,\bar{x},\bar{y},z^*)$ with $z^*\in\Lambda(\bar{x},\bar{y})$ then for all $x^*\in X^*$ we have

$$\begin{aligned} D_N^*S(\bar{y},\bar{x})(x^*)\subset \bigcup_{z^*\in\Lambda(\bar{x},\bar{y})} \Big\{\nabla_y\mathcal{L}(\bar{x},\bar{y},z^*)^*x^{**}+\nabla_y r(\bar{x},\bar{y})^*z^{**}\Big|\ \exists x^{**}\in X^{**} \\ \textit{such that } z^{**}\in D_N^*N_\Theta\big(r(\bar{x},\bar{y}),z^*\big)\big(p(\bar{x},\bar{y})^{**}x^{**}\big) \\ \textit{and }\ 0=x^*+\nabla_x\mathcal{L}(\bar{x},\bar{y},z^*)^*x^{**}+\nabla_x r(\bar{x},\bar{y})^*z^{**}\Big\}. \end{aligned} \tag{4.21}$$

Proof. We can easily see that the coderivative estimate for S follows from assertion (ii) of Theorem 4.2 and Proposition 4.4 provided that the multifunction Ξ from (4.18) with Q given by (4.7) is calm at $(0,0,0,\bar{x},\bar{y})$. We will show that the calmness property of P at the points $(0,0,0,\bar{x},\bar{y},z^*)$ with $z^*\in\Lambda(\bar{x},\bar{y})$ implies the calmness property of Ξ at $(0,0,0,\bar{x},\bar{y},z^*)$. Indeed, since the set-valued mapping

$$\begin{aligned} P(x^*,\nu,\nu^*) &= \big\{(x,y,z^*)\in X\times Y\times Z^*\big|\ \mathcal{L}(x,y,z^*)+x^*=0\big\}\cap M(\nu,\nu^*) \\ &= \Big\{(x,y,z^*)\in X\times Y\times Z^*\Big|\ g(x,y)+p(x,y)^*z^*+x^*=0\ \text{ and} \\ &\qquad \big(r(x,y)+\nu,z^*+\nu^*\big)\in \operatorname{gph} N_\Theta\Big\} \end{aligned}$$

is calm at the points $(0,0,0,\bar{x},\bar{y},z^*)$ with $z^*\in\Lambda(\bar{x},\bar{y})$, it follows by the result of [17, Lemma 1] that the (only canonically perturbed) mapping $\bar{P}: X\times Y\times X^*\rightrightarrows X\times Y\times Z^*$ defined by

$$\begin{aligned} \bar{P}(u,v,u^*):=\big\{(x,y,z^*)\in X\times Y\times Z^*\big|\ -g(x,y)+u^*=p(x+u,y+v)^*z^*, \\ z^*\in N_\Theta\big(r(x+u,y+v)\big)\big\} \end{aligned}$$

is calm at all the points $(0, 0, 0, \bar{x}, \bar{y}, z^*)$ with $z^* \in \Lambda(\bar{x}, \bar{y})$.

On the other hand, in the case under consideration the following representation holds:

$$\begin{aligned}
\Xi(u, v, u^*) &= \big\{(x, y) \in X \times Y \big|\, (x + u, y + v, -g(x, y) + u^*) \in \operatorname{gph} Q\big\} \\
&= \big\{(x, y) \in X \times Y \big| - g(x, y) + u^* \in p(x + u, y + v)^* N_\Theta\big(r(x + u, y + v)\big)\big\}, \\
&= \big\{(x, y) \in X \times Y \big|\, \exists z^* \in N_\Theta\big(r(x + u, y + v)\big) \text{ such that} \\
&\qquad\qquad -g(x, y) + u^* \in p(x + u, y + v)^* z^*\big\},
\end{aligned}$$

which implies the relationship

$$\Xi(u, v, u^*) = \operatorname{proj}_{u,v} \bar{P}(u, v, u^*).$$

Therefore, Ξ is calm at all points $(0, 0, 0, \bar{x}, \bar{y})$. This completes the proof of the theorem. $\triangle$

Remark 4.6 *The assumption on SNC property of Q in Theorem 4.5 is satisfied if $p(\bar{x}, \bar{y})^*$ is surjective and the second-order QC*

$$D_N^* N_\Theta\big(r(\bar{x}, \bar{y}), z^*\big)(0) \cap \ker\big(\nabla r(\bar{x}, \bar{y})^*\big) = \{0\} \tag{4.22}$$

is satisfied for all $z^* \in \Lambda(\bar{x}, \bar{y}) \cap N_\Theta\big(r(\bar{x}, \bar{y})\big)$.

Indeed, we have the representation $Q(x, y) = (f \circ F)(x, y)$ with f and F are given in (4.15) and (4.16), respectively, and recall that the set-valued mapping $f^{-1} \cap F$ is inner semi-compact at $\big(\bar{x}, \bar{y}, -g(\bar{x}, \bar{y})\big)$ when the basic CQ (4.10) is satisfied. Due to the result on SNC property of compositions in [34, Theorem 3.98], the SNC assumption on N_Θ combined with the second-order CQ (4.22) implies the SNC property of $N_\Theta \circ r$ at all points $(\bar{x}, \bar{y}, z^*)$, and hence the SNC property of F at $(\bar{x}, \bar{y}, \bar{x}, \bar{y}, z^*)$ whenever $z^* \in \Lambda(\bar{x}, \bar{y}) \cap N_\Theta\big(r(\bar{x}, \bar{y})\big)$. Also note that under the assumptions made, f^{-1} is PSNC at $\big(-g(\bar{x}, \bar{y}), \bar{x}, \bar{y}, z^*\big)$ if $\nabla f(\bar{x}, \bar{y}, z^*)$ is surjective, which is implied by the surjectivity of $p(\bar{x}, \bar{y})^*$. Therefore, Q is SNC at $\big(\bar{x}, \bar{y}, -g(\bar{x}, \bar{y})\big)$ by the aforementioned [34, Theorem 3.98].

Taking Remark 4.6 into account, we arrive at a corollary of Theorem 4.5:

Corollary 4.7 *For the solution map S considered in Theorem 4.5 and $\bar{x} \in S(\bar{y})$, assume that $p(\bar{x}, \bar{y})^*$ is surjective and the first-order CQ (4.10) holds. Assume also that for all $z^* \in \Lambda(\bar{x}, \bar{y}) \cap N_\Theta\big(r(\bar{x}, \bar{y})\big)$ the second-order CQ (4.22) holds and $\operatorname{gph} N_\Theta$ is SNC at $\big(r(\bar{x}, \bar{y}), z^*\big)$. If in addition the set-valued mapping M is calm at $(0, 0, \bar{x}, \bar{y}, z^*)$ and the set valued mapping P is calm at $(0, 0, 0, \bar{x}, \bar{y}, z^*)$ then we have the coderivative estimate (4.21).*

4.3 Robust Lipschitzian Stability of QVIs

By *robust Lipschitzian stability* of QVIs we understand in this section the fulfillment of the Lipschitz-like property of the solution map (4.3) to the QVI (4.2) with the generating sets $\Gamma(x, y)$ given by (4.4) around the reference point $(\bar{y}, \bar{x})$. This type of Lipschitz behavior has been recognized as an appropriate stability property of local sensitivity analysis, which is *robust* (i.e., *preserved*) under small parameter perturbations.

To derive efficient conditions for robust Lipschitzian stability of the QVIs under consideration, we utilize in what follows the pointbased characterization of Lipschitz-like property in [34, Theorem 4.10] combined with the constructive derivative estimate for solution map (4.3) established in the previous section. Furthermore, the coderivative results developed above allows us to conduct not only qualitative but also quantitative analysis of robust Lipschitzian stability for QVIs by providing an estimate of the *exact Lipschitzian bound*.

Theorem 4.8 *Let $S : Y \rightrightarrows X$ be the solution map (4.3) to the original QVI represented by (4.6) around the reference point $(\bar{y}, \bar{x}) \in \operatorname{gph} S$ with the Lagrangian $\mathcal{L}$ defined by (4.20). Suppose that all the assumptions of Corollary 4.7 are satisfied. Then the solution map S is Lipschitz-like around $(\bar{y}, \bar{x})$ if and only if the following condition holds:*

$$\begin{cases} 0 = \nabla_x \mathcal{L}(\bar{x}, \bar{y}, z^*)^* x^{**} + \nabla_x r(\bar{x}, \bar{y})^* z^{**} \\ z^* \in \Lambda(\bar{x}, \bar{y}) \\ z^{**} \in D_N^* N_\Theta\big(r(\bar{x}, \bar{y}), z^*\big)\big(p(\bar{x}, \bar{y})^{**} x^{**}\big) \end{cases} \Longrightarrow \nabla_y \mathcal{L}(\bar{x}, \bar{y}, z^*)^* x^{**} + \nabla_y r(\bar{x}, \bar{y})^* z^{**}.$$

Furthermore, if $\dim Y < \infty$ *we have the upper estimate*

$$\begin{aligned} \operatorname{lip} S(\bar{y}, \bar{x}) \le \sup \Big\{ \big\| \nabla_y \mathcal{L}(\bar{x}, \bar{y}, z^*)^* x^{**} + \nabla_y r(\bar{x}, \bar{y})^* z^{**} \big\| \;\Big|\; z^* \in \Lambda(\bar{x}, \bar{y}) \in X^{**} \\ 0 = x^* + \nabla_x \mathcal{L}(\bar{x}, \bar{y}, z^*)^* x^{**} + \nabla_x r(\bar{x}, \bar{y})^* z^{**} \\ z^{**} \in D_N^* N_\Theta\big(r(\bar{x}, \bar{y}), z^*\big)\big(p(\bar{x}, \bar{y})^{**} x^{**}\big),\ \|x\| \le 1 \Big\}. \end{aligned}$$

for the exact Lipschitzian bound of S *around* $(\bar{y}, \bar{x})$*.*

Proof. Under the assumptions of Corollary 4.7 we have the coderivative estimate (4.21). The new imposed condition in this theorem implies

$$D_N^* S(\bar{y}, \bar{x})(0) = \{0\},$$

and thus $D_M^* S(\bar{y}, \bar{x})(0) = \{0\}$. Due to the coderivative criterion in [34, Theorem 4.10], it remains to verify that S is PSNC at that point.

Let us show that S is actually SNC at $(\bar{y}, \bar{x})$ under the assumptions made. Introducing the mapping $h(x, y) = \big(x, y, -g(x, y)\big)$, we have

$$(y, x) \in \operatorname{gph} S \iff (x, y) \in h^{-1}(\operatorname{gph} Q).$$

Thus, $\operatorname{gph} S$ is SNC at $(\bar{x}, \bar{y})$ if and only if $h^{-1}(\operatorname{gph} Q)$ is SNC at $(\bar{x}, \bar{y})$. The latter follows from the SNC property of inverse images in [34, Theorem 3.84].

The estimate for the exact Lipschitz bound of S follows from the aforementioned [34, Theorem 4.10] as we have

$$\operatorname{lip} S(\bar{y}, \bar{x}) \le \|D_N^* S(\bar{y}, \bar{x})\|$$

when $\dim Y < \infty$. △

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ABSTRACT

VARIATIONAL ANALYSIS IN PARAMETRIC OPTIMIZATION

by

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Advisor: Boris S. Mordukhovich

Major: Mathematics

Degree: Doctor of Philosophy

The dissertation is devoted to the development of variational analysis and generalized differentiation in infinite dimensions. We derive new calculus rules for both first-order partial subdifferentials and second-order partial subdifferentials in the framework of general Banach spaces as well as more developed rules in the framework of Asplund spaces. This calculus is applied in the study of sensitivity analysis for solution maps to the parameterized generalized equations in Asplund spaces, where both bases and fields are parameter-dependent set-valued mappings. We analyze the parametric sensitivity of either stationary points or stationary point-multiplier multifunctions associated with parameterized optimization problems under consideration. The dissertation also focus on a family of parameterized quasi-variational inequalities and conduct a sensitivity analysis for their solution maps.

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