7-1-2004

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Department of Mathematics
Research Report

2004 Series
#7

This research was partly supported by the Academy of Applied Science.

ASYMPTOTIC SOLUTIONS OF SEMILINEAR STOCHASTIC WAVE EQUATIONS

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Abstract

Large-time asymptotic properties of solutions to a class of semilinear stochastic wave equations with damping in a bounded domain are considered. First an energy inequality and the exponential bound for a linear stochastic equation are established. Under appropriate conditions, the existence theorem for a unique global solution is given. Next the questions of bounded solutions and the exponential stability of an equilibrium solution, in mean-square and the almost sure sense, are studied. Then, under some sufficient conditions, the existence of a unique invariant measure is proved. Two examples are presented to illustrate some applications of the theorems.

1 Introduction

Semilinear stochastic wave equations arise as mathematical models to describe the nonlinear vibration or wave propagation in a randomly excited continuous medium. To be specific the equation may take the form:

$$\partial_t^2 u(x,t) = c^2 \Delta u - 2\alpha \partial_t u(x,t) + f(u) + \sigma(u)\dot{W}(x,t), \tag{1.1}$$

in a domain $\mathcal{D}$ in $\mathcal{R}^d$, where $\partial_t = \frac{\partial}{\partial t}$, Δ is the Laplacian operator; c and 2α are some positive constants known as the wave speed and the damping coefficient, respectively. The nonlinear functions f and σ are given, and $\dot{W}(x,t) = \partial_t W(x,t)$ is a spatially dependent white noise, where $W(x,t)$ is a Wiener random field. In the papers [2, 3], we studied the local and global solutions of this type of equations without damping ($\alpha = 0$), where the nonlinear terms f and σ may admit a polynomial growth. As a sequel of our previous work, this paper is concerned with some qualitative behavior of asymptotic solutions to the equation

*Supported in part by a grant from the Academy of Applied Science.

AMS 2000 Subject classifications. Primary **60H15**; secondary **60H05**.

Key words and phrases: *Stochastic wave equation, semilinear, bounded solutions, exponential stability, invariant measure.*

(1.1) in a bounded domain $\mathcal{D}$ as $t \to \infty$. In addition to the global existence of solutions, we are interested in the questions of boundedness, asymptotic stabilities and the existence of a stationary solution or an invariant measure. For a solution of the wave equation to reach a statistical equilibrium, it is imperative to include the damping term in the equation (1.1) so that, in the physical term, the fluctuation-dissipation principle may hold. As a simple example, consider the randomly perturbed wave equation in one dimension:

$$\begin{cases} \partial_t^2 u = c^2 \partial_x^2 u - 2\alpha \partial_t u + \dot{W}(x,t), \;\; t > 0, \;\; x \in \mathcal{D} = (0, \pi), \\ u(x,0) = h(x), \;\; \partial_t(0,x) = 0; \;\; u(0,t) = u(\pi,t) = 0, \end{cases} \tag{1.2}$$

where h is a given continuous function and the Wiener field W is assumed to have the following Fourier series representation:

$$W(x,t) = \sum_{n=1}^{\infty} \sigma_n b_n(t) \phi_n(x),$$

where $\{b_n(t)\}$ is a sequence of independent copies of standard Brownian motions in one dimension, $\{\sigma_n\}$ is a sequence of reals such that $\sum_{n=1}^{\infty} \sigma_n^2 < \infty$, and $\phi_n = \sqrt{2/\pi} \sin nx, \quad n = 1, 2, \cdots$, are the normalized eigenfunctions associated with the problem (1.2). Then, by means of the eigenfunction expansion, it can be formally solved in the case $c > \alpha$ to give

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \phi_n(x), \tag{1.3}$$

where

$$u_n(t) = h_n e^{-\alpha t} \cos \omega_n t + \frac{\sigma_n}{\omega_n} \int_0^t e^{-\alpha(t-s)} \sin \omega_n (t-s) db_n(s). \tag{1.4}$$

with $h_n = \int_0^\pi h(x) \phi_n(x) dx$ and $\omega_n = \sqrt{(nc)^2 - \alpha^2}$ for $n = 1, 2, \cdots$. By some simple calculations, we obtain the mean

$$E u_n(t) = h_n e^{-\alpha t} \cos \omega_n t \to 0,$$

and the variance

$$Var.\{u_n(t)\} = \left(\frac{\sigma_n}{\omega_n}\right)^2 \int_0^t e^{-2\alpha s} \sin^2(\omega_n s) ds \to \frac{1}{4\alpha} \left(\frac{\sigma_n}{nc}\right)^2,$$

as $t \to \infty$. Then it follows from (1.4) that the solution $u(x,t)$ is a Gaussian random field with the mean $Eu(x,t) \to 0$, and the covariance function

$$Cov.\{u(x,t), u(y,t)\} = \sum_{n=1}^{\infty} \frac{1}{4\alpha} \left(\frac{\sigma_n}{nc}\right)^2 \phi_n(x) \phi_n(y).$$

In fact it can be shown that the solution $u(\cdot, t)$ converges in the mean-square to a Gaussian random field $\hat{u}(\cdot)$ with the above covariance function and its probability law is the invariant measure for the equation (1.2). On the other hand, without the damping ($\alpha = 0$), we would have $Eu_n(t) = h_n \cos nct$ and

$$Var.\{u_n(t)\} = (1/2)(\sigma_n/nc)^2[t - (1/2nc)\sin 2nct] \to \infty, \quad \text{as} \quad t \to \infty.$$

So the asymptotic solution will cease to exist. Clearly this can also happen in the nonlinear case. In fact it was shown that, with a cubic nonlinearity, the solution may explode in finite time [2], unless there exists a certain energy bound. As to be seen, the dissipation and the energy bound for a semilinear wave equation such as (1.1) are two major ingredients to ensure a proper asymptotic behavior of its solutions.

Our initial work on semilinear stochastic wave equations [2] was stimulated by two interesting papers by Mueller [14],[15] on the existence of large-time solutions to some nonlinear heat and wave equations with noise. If such a solution exists, it is natural to investigate its asymptotic behavior as $t \to \infty$. By a semigroup approach, asymptotic solutions to semilinear stochastic evolution equations have been studied by many authors. For the problems of boundedess and stability, see e.g. the papers [4], [9], [11], and [10], and, for the existence of invariant measures, we mention the articles [12], [4] and the book [8] for further references. In concrete terms, most of the above-mentioned results are applicable to the parabolic type of stochastic partial differential equations. The asymptotic solution of a stochastic hyperbolic or wave-like equation was studied in the paper [13] by the method of averaging. To our knowledge the asymptotic solutions of the semilinear wave equations under consideration have not been treated in the literature. For the deterministic case, the analysis of hyperbolic equations relies heavily on the so-called energy method [16], (Chap.4, [18]). Therefore the associated energy function plays an important role in the asymptotic analysis. Similarly we shall adopt the stochastic version of the energy method in the current study. In fact, in order to obtain the crucial exponential estimates, it is necessary to introduce a pseudo energy function, which can be interpreted physically as adding an artificial damping to the system.

The paper is organized as follows. In section 2 we present some technical lemmas concerning the energy equation and an essential exponential inequality for a linear stochastic wave equation. A pseudo energy function is introduced and shown to be equivalent to the usual energy function. For a class of semilinear stochastic wave equation, Theorem 3.1 given in section 3 ensures the existence of a unique global solution under suitable conditions. Subsequently the asymptotic properties of solutions are studied. In section 4 we give some sufficient conditions for a solution to be bounded or ultimately bounded in mean-square, as stated in Theorems 4.1 and 4.2, respectively. Then the questions of the mean-square and the almost sure exponential stabilities are considered in section 5. The results are summarized in Theorems 5.1 and 5.3. So far the stochastic wave equations in question admit nonlinear terms of a polynomial growth. For the existence of an invariant measure, this poses a challenging open problem. Instead, in section 6, we shall prove an existence theorem (Theorem 6.1) by assuming that the nonlinear terms are globally Lipschitzian and of a sublinear growth. Finally two examples are provided in section 7 to illustrate some applications of our theorems.

2 Energy Equation and Exponential Estimate

Let $\mathcal{D} \subset \mathcal{R}^d$ be a bounded domain with a smooth boundary $\partial\mathcal{D}$. We set $H := L^2(\mathcal{D})$ with the inner product and norm denoted by $(.,.)$ and $\|.\|$ respectively. Let $H^k = W^{k,2}(\mathcal{D})$ be the L^2-Sobolev space of order k with norm $\|.\|_k$, and denote by H_0^1 the closure in H^1 of the set of all C^1-functions with compact support in $\mathcal{D}$.[1].

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space for which a filtration $\mathcal{F}_t$ of sub $\sigma-$fields of $\mathcal{F}$ is given. Let $W(x,t)$, $x \in \mathcal{D}$, $t \geq 0$, be a continuous Wiener random field defined in this space with $W(x,0) = 0$. It has a zero mean, $E\ W(x,t) = 0$, and the covariance

$$EW(x,t)W(y,s) = (t \wedge s)r(x,y), \quad x, y \in \mathcal{D}, \tag{2.1}$$

where $(t \wedge s) = \min(t,s)$ for $0 \leq t, s \leq T$, and the covariance function $r(x,y)$ is bounded so that

$$\sup_{x \in \mathcal{D}} r(x,x) \leq r_0. \tag{2.2}$$

Let $\sigma(x,t) = \sigma(x,t,\omega)$ for $t \geq 0, x \in \mathcal{D}$ and $\omega \in \Omega$, be a continuous $\mathcal{F}_t-$predictable random field satisfying the condition:

$$E \int_0^T \|\sigma(\cdot,t)\|^p dt < \infty. \tag{2.3}$$

for $p \geq 2$. Then it can be shown that the stochastic integral

$$M(x,t) = \int_0^t \sigma(x,s)W(x,ds), \quad t > 0, x \in \mathcal{D}. \tag{2.4}$$

is well defined and $M_t = M(\cdot,t)$ is a continuous $H-$valued $\mathcal{F}_t-$martingale (see the Appendix). It has mean $E\ M(x,t) = 0$ and the covariation operator Q_t defined by

$$\langle (M_\cdot, g), (M_\cdot, h) \rangle_t = \int_0^t (Q_s g, h) ds,$$

for any $g, h \in H$, where the kernel function $q(x,y,t)$ of Q_t, defined by

$$(Q_t g)(x) = \int_{\mathcal{D}} q(x,y,t) g(y) dy,$$

is given by

$$q(x,y,t) = r(x,y)\sigma(x,t)\sigma(y,t).$$

In view of conditions (2.2) and (2.3), we have, by the B-D-G (Burkholder-Davis-Gundy) inequality (p.82, [7]),

$$E\,\|M_t\|^p \leq E\,C_p[\int_0^t Tr\,Q_s ds]^{p/2} = E\,C_p[\int_0^t \int_{\mathcal{D}} q(x,x,t) dx ds]^{p/2},$$

$$\leq C_p(T) E \int_0^T \|\sigma(\cdot,t)\|^p dt,$$

for some positive constants C_p and $C_p(T)$, where Tr denotes the trace operator in H. It is worth noting that the stochastic integration in (2.4) is taken with respect to a L^p-bounded integrand σ_t , instead of a Hilbert-Schmidt operator-valued process as usually done (see, Chap.4, [7]). This version of stochastic integral will be needed later on to deal with equations with a point-wise (in x) multiplicative noises (see Example 1). Since we have not been able to find a reference for this type of integral, it will be defined in the Appendix.

Now we consider the initial-boundary value problem for the linear damped hyperbolic equation with a random perturbation:

$$\begin{cases} [\partial_t^2 + 2\alpha\partial_t - A(x,D)]u(x,t) = f(x,t) + \partial_t M(x,t), \quad 0 < t < T, \\ u(x,0) = u_0(x), \quad \partial_t u(x,0) = v_0(x), \quad x \in \mathcal{D}, \\ u(\cdot,t)|_{\partial\mathcal{D}} = 0 \end{cases} \tag{2.5}$$

where α is a positive parameter, $D = \partial_x$ denotes the gradient operator, and $A(x,D)$ is a strongly elliptic operator of second order of the form:

$$A(x,D)\varphi(x) = \sum_{i,j=1}^{d} \partial_{x_i}[a^{ij}(x)\partial_{x_j}\varphi(x)] - b(x)\varphi(x). \tag{2.6}$$

In addition the coefficients $a^{ij} = a^{ji}$ and b are assumed to be smooth functions satisfying

$$a_0(1+|\xi|^2) \le \sum_{i,j=1}^{d} a^{ij}(x)\xi_i\xi_j \le a_1(1+|\xi|^2), \quad \xi, x \in \mathcal{D},$$

for some constants $a_1 \ge a_0 > 0$.

To consider (2.5) as an Itô equation in H, we set $u_t = u(\cdot,t), v_t = v(\cdot,t)$ and so on, and rewrite it as

$$\begin{cases} du_t = v_t dt, \\ dv_t = [Au_t - 2\alpha v_t + f_t]dt + dM_t, \quad 0 < t < T, \\ u_0 = g, \quad v_0 = h, \end{cases} \tag{2.7}$$

where the domain $\mathcal{D}(A) = H^2 \cap H_0^1$, $g \in H^1, h \in H$ and M_t is regarded as a H-valued Wiener martingale. The condition (2.6) implies that $(-A)$ is a self-adjoint, strictly positive linear operator in $H = L^2(D)$ and its square root $B = \sqrt{-A}$ is also a self-adjoint, strictly positive operator with domain $D(B)$, which is a Hilbert space under the inner product $(g,h)_B :=$ (Bg, Bh) (see Chap.1, [17]). Since $D(B) \cong H_1$, for convenience, we define $\|\cdot\|_1 = \|\cdot\|_B$ in the subsequent analysis. As usual, the Itô's differential equation (2.7) is interpreted as a stochastic integral equation:

$$\begin{cases} u_t = u_0 + \int_0^t v_s ds, \\ v_t = v_0 + \int_0^t Au_s ds - 2\alpha\int_0^t v_s ds + \int_0^t f_s ds + M_t. \end{cases} \tag{2.8}$$

Introduce the Hilbert space $\mathcal{H} = (H_1 \times H)$ equipped with the norm defined by

$$\|\phi\|_{\mathcal{H}} = \{\|u\|_1^2 + \|v\|^2\}^{1/2} = \{\|Bu\|^2 + \|v\|^2\}^{1/2},$$

for any $\phi = (u; v) \in \mathcal{H}$. Define the *energy function* $\mathbf{e}(\cdot) : \mathcal{H} \to \mathcal{R}^+ = [0, \infty)$ as follows:

$$\mathbf{e}(\phi) := \mathbf{e}(u; v) = \|Bu\|^2 + \|v\|^2, \text{ for } \phi = (u; v) \in H^1 \times H. \tag{2.9}$$

Notice that the norm $\|\phi\|_{\mathcal{H}} = \sqrt{\mathbf{e}(\phi)}$ is also called an energy norm. In what follows, we denote the $\mathcal{H}$-norm $\|\cdot\|_{\mathcal{H}}$ simply by $\|\cdot\|$ when there is no confusion.

Now regarding Eq.(2.8) as a stochastic evolution equation in $\mathcal{H}$ in a distributional sense, we have the following lemma:

Lemma 2.1 (Energy Equation) For $\phi_0 = (u_0, v_0) \in \mathcal{H}$, let f_t be a continuous predictable process in H, and let M_t be a continuous H-valued martingale with covariation operator Q_t such that

$$E\{\int_0^T \|f_t\|^2 dt + \int_0^T Tr\ Q_t\ dt\} < \infty. \tag{2.10}$$

Then the equation (2.8), or (2.7) has a unique solution $\phi_t = (u_t, v_t)$ which is a continuous predictable $\mathcal{H}$-valued process. Moreover it satisfies the energy equation:

$$\begin{cases} \mathbf{e}(\phi_t) = \mathbf{e}(\phi_0) - 4\alpha \int_0^t \|v_s\|^2 ds + 2\int_0^t (v_s, f_s) ds \\ +2\int_0^t (v_s, dM_s) + \int_0^t Tr\ Q_s\ ds \quad \text{a.s.,} \end{cases} \tag{2.11}$$

for $t \in [0, T]$. where the energy function $\mathbf{e}(\cdot)$ on $\mathcal{H}$ is defined by (2.9). Moreover, the following inequality holds:

$$E \sup_{t \leq T} \mathbf{e}(\phi_t) \leq C_1 + C_2 E \int_0^T \{\|f_s\|^2 + Tr\ Q_s\ \} ds, \tag{2.12}$$

where the constants C_1, C_2 depend on p, T and the initial conditions.

Notice that, due to the lack of required smoothness of solutions, the general Itô formula does not hold here. The energy equation (2.11) can be proved by a smoothing technique, such as the Yosida approximation (see, e.g. Chap.5, [7]), and then taking the limits properly. The energy inequality (2.12) follows from the energy equation and by invoking Burkholder type of a submartingale inequality and some other well-known inequalities. Since the proof is similar to the case when A is a Lapalcian as given in (pp.378-380, Chow [2]), it will be omitted.

Owing to the dissipation term in (2.11), in contrast with the energy inequality (2.12), it is possible to obtain an exponential estimate for the mean energy. To this end we introduce a *pseudo energy function* as follows:

$$\mathbf{e}^\lambda(\phi) := \mathbf{e}^\lambda(u; v) = \|Bu\|^2 + \|v + \lambda u\|^2, \text{ for } u \in H^1, v \in H, \tag{2.13}$$

where $\lambda > 0$ is a parameter. Let $v^\lambda = v + \lambda u$. Then we can write

$$\mathbf{e}^\lambda(u; v) = \mathbf{e}(u; v^\lambda) = \mathbf{e}(u; v) + 2\lambda(u, v) + \lambda^2\|v\|^2. \tag{2.14}$$

Since A is strongly elliptic and strictly positive, its smallest eigenvalue η_1 can be characterized as (p.38, [19])

$$\eta_1 = \sup_{\substack{g\in H^1\\ g\neq 0}} \frac{\|g\|_1^2}{\|g\|^2} > 0. \tag{2.15}$$

Lemma 2.2 (Exponential Estimate) Let the conditions for Lemma 2.1 be satisfied such that the inequality (2.10) holds for any $T > 0$. Then, if

$$\lambda \leq \lambda_0 := \min\{\frac{\alpha}{2}, \frac{\eta_1}{\alpha}\}, \tag{2.16}$$

there exists $\alpha_1 \in (0, \lambda)$ such that the following inequality holds

$$E\mathbf{e}^\lambda(\phi_t) \leq \mathbf{e}^\lambda(\phi_0)e^{-\alpha_1 t} + \int_0^t e^{-\alpha_1(t-s)}E\{\ \frac{2}{\alpha_1}\|f_s\|^2 + Tr\ Q_s\ \}ds. \tag{2.17}$$

Proof. It follows from Eq. (2.7) that $(u_t; v_t^\lambda)$ satisfies the perturbed system:

$$\begin{cases} du_t = [v_t^\lambda - \lambda u_t]dt, \\ dv_t^\lambda = [Au_t + \lambda(2\alpha - \lambda)u_t - (2\alpha - \lambda)v_t^\lambda + f_t]dt + dM_t, \\ u_0 = g, \quad v_0 = h, \quad 0 < t < T, \end{cases} \tag{2.18}$$

By applying Lemma 2.1 to the above system and noticing $\mathbf{e}^\lambda(u_t; v_t) = \mathbf{e}(u_t; v_t^\lambda)$, the pseudo energy function (2.13) satisfies

$$\begin{cases} d\mathbf{e}^\lambda(u_t; v_t) \;=\; 2[\lambda(2\alpha - \lambda)(u_t, v_t^\lambda) - \lambda\|u_t\|_1^2 - (2\alpha - \lambda)\|v_t^\lambda\|^2 \\ \qquad\qquad +(f_t, v_t^\lambda) + \frac{1}{2}Tr\ Q_t\ ]dt + 2(v_t^\lambda, dM_t), \end{cases} \tag{2.19}$$

with $\mathbf{e}^\lambda(u_0; v_0) = \mathbf{e}(u_0; v_0^\lambda)$. Now, in view of (2.15), we have, by using some simple inequalities,

$$\lambda(2\alpha - \lambda)(u, v^\lambda) - \lambda\|u\|_1^2 - (2\alpha - \lambda)\|v^\lambda\|^2$$

$$\leq \lambda(2\alpha - \lambda)\frac{\|u\|_1}{\sqrt{\eta_1}}\|v^\lambda\| - \lambda\|u\|_1^2 - (2\alpha - \lambda)\|v^\lambda\|^2$$

$$\leq \lambda(2\alpha - \lambda)[\lambda\frac{\|u\|_1^2}{\eta_1} + \frac{1}{4\lambda}\|v^\lambda\|^2] - \lambda\|u\|_1^2 - (2\alpha - \lambda)\|v^\lambda\|^2$$

$$\leq -\lambda(1-\frac{2\alpha\lambda}{\eta_1})\|u\|_1^2 - \frac{3\alpha}{4}\|v^\lambda\|^2 \leq -\frac{\lambda}{2}\|u\|_1^2 - \frac{3\lambda}{4}\|v^\lambda\|^2.$$

The above result together with the fact:

$$(f_t, v^\lambda) \leq \frac{\lambda}{4}\|v^\lambda\|^2 + \frac{1}{\lambda}\|f_t\|^2$$

imply that

$$\lambda(2\alpha-\lambda)(u,v^\lambda) - \lambda\|u\|_1^2 - (2\alpha-\lambda)\|v^\lambda\|^2 + (f_t, v^\lambda) \leq -\frac{\lambda}{2}(\|u\|_1^2 + \|v^\lambda\|^2) + \frac{1}{\lambda}\|f_t\|^2. \quad (2.20)$$

In view of (2.20), the equation (2.19) yields

$$d\mathbf{e}^\lambda(u_t; v_t) \leq -\lambda \mathbf{e}^\lambda(u_t; v_t)dt + [\frac{2}{\lambda}\|f_t\|^2 + Tr\ Q_t\ ]dt + 2(v_t^\lambda, dM_t), \quad (2.21)$$

which can be integrated to yield the desired inequality (2.17), after taking the expectation, with any $\alpha_1 < \lambda$. □

It is easy to show that the energy norms induced by $\mathbf{e}$ and $\mathbf{e}^\lambda$ are equivalent. In fact the following lemma holds.

Lemma 2.3 For any $\lambda > 0$, the following inequality holds:

$$(\frac{\mu_1-\lambda}{\mu_1+\lambda})\ \mathbf{e}(u;v) \leq \mathbf{e}^\lambda(u;v) \leq (\frac{\mu_1+\lambda}{\mu_1-\lambda})\ \mathbf{e}(u;v), \quad (2.22)$$

where $\mu_1 = (\sqrt{4\eta_1 + \lambda^2})$. Moreover we have

$$E\mathbf{e}(\phi_t) \leq K(\lambda)\{\mathbf{e}(\phi_0)e^{-\alpha_1 t} + \int_0^t e^{-\alpha_1(t-s)}E(\ \frac{2}{\alpha_1}\|f_s\|^2 + Tr\ Q_s\ )ds\}, \quad (2.23)$$

where $K(\lambda) = \dfrac{\mu_1+\lambda}{\mu_1-\lambda}$.

Proof. By definition (2.14),

$$\mathbf{e}^\lambda(u;v) = \mathbf{e}(u;v) + 2\lambda(u,v) + \lambda^2\|u\|^2.$$

It follows that, for any $\beta > 0$,

$$\mathbf{e}^\lambda(u;v) \leq \mathbf{e}(u;v) + (1+\beta)(\lambda^2/\eta_1)\|u\|_1^2 + \frac{1}{\beta}\|v\|^2$$

$$\leq (1+\frac{1}{\beta})\ \mathbf{e}(u;v) = (\frac{\mu_1+\lambda}{\mu_1-\lambda})\ \mathbf{e}(u;v),$$

by choosing $\beta = \frac{1}{2}\{\sqrt{(4\eta_1/\lambda^2)+1} - 1\}$.

On the other hand, for any $\gamma > 0$,

$$\mathbf{e}^\lambda(u;v) \geq \mathbf{e}(u;v) - \{(1-\gamma)(\lambda^2/\eta_1)\|u\|_1^2 + \frac{1}{\beta}\|v\|^2\}$$

$$\geq (\frac{\mu_1 - \lambda}{\mu_1 + \lambda})\ \mathbf{e}(u;v),$$

by taking $\gamma = \frac{1}{2}\{\sqrt{(4\eta_1/\lambda^2)} + 1\}$.

Therefore we have verified the inequality (2.22), and the result (2.23) is now an direct consequence of (2.17) and (2.22). □

3 Semilinear Stochastic Hyperbolic Equations

Let us consider the initial-boundary value problem for the following hyperbolic equation:

$$\begin{cases} \partial_t^2 u(x,t) = [A(x,D) - 2\alpha\partial_t]u(x,t) \\ +f(u,Du,x,t) + \sigma(u,Du,x,t)\partial_t W(x,t), \quad t>0, \\ u(x,0) = u_0(x), \quad \partial_t u(x,0) = v_0(x), \quad x \in \mathcal{D} \subset \mathcal{R}^d, \\ u(\cdot,t)|_{\partial\mathcal{D}} = 0, \end{cases} \tag{3.1}$$

where, in contrast with the linear problem (2.1), $f(s,y,x,t)$ and $\sigma(s,y,x,t)$, for $x \in \mathcal{D}, t > 0, s \in \mathcal{R}, y \in \mathcal{R}^d$, are continuous functions, and $W_t = W(\cdot,t)$ is a continuous Wiener random field with covariance operator R of kernel $r(x,y)$, for $x,y \in \mathcal{D}$.

Similar to the linear case, we rewrite it as a system Itô equations in H:

$$\begin{cases} du_t = v_t dt, \\ dv_t = [Au_t - 2\alpha v_t + F_t(u_t)]dt + dM_t(u), \end{cases} \tag{3.2}$$

or

$$\begin{cases} u_t = u_0 + \int_0^t u_s ds, \\ v_t = v_0 + \int_0^t [Au_s - 2\alpha v_s + F_s(u_s)]ds + M_t(u), \end{cases} \tag{3.3}$$

where we set $F_t(u) := f(u, Du, \cdot, t)$,

$$M_t(u) = \int_0^t \Sigma_s(u_s)dW_s, \tag{3.4}$$

and $\Sigma_t(\cdot) : H^1 \to H$ is defined by $\Sigma_t(u)(x) := \sigma[u(x), Du(x), x, t]$ for any $u \in H^1$, $x \in \mathcal{D}$.

We are interested in the large-time solutions of Eq. (3.1) when the nonlinear terms allow a polynomial growth and are locally Lipschitz continuous. For the existence of solutions, we shall impose a set of sufficient conditions. In what follows, for $r, s \in \mathcal{R}$, let $b(r)$ and $k(r,s)$ be real-valued functions such that it is positive, locally bounded and monotonically increasing in each variable. Let us introduce a positive function $\Theta(\cdot;\cdot) : \mathcal{H} \to \mathcal{R}^+$ such that it is continuous, locally bounded and

$$\mathbf{e}(u;v) \leq \Theta(u;v) \leq \{\mathbf{e}(u;v) + c\|u\|_1^k\}, \tag{3.5}$$

for any $(u;v) \in \mathcal{H}$ and some constants $c \geq 0$ and $k \geq 2$. As a short-hand notation, we set

$$\|\Sigma_t(u)\|_R^2 = Tr\, Q_t(u) = \int_{\mathcal{D}} r(x,x)[\Sigma_t(u)(x)]^2 dx$$

and impose the following conditions which will be referred to later as Conditions A:

(A1) $A : H^2 \cap H_0^1 \to H$ is an elliptic operator as given in (2.6).

(A2) $F_t(\cdot) : H^1 \to H$ and $\Sigma_t(\cdot) : H_1 \to H$ are continuous in $t \geq 0$. There exists functions $b(r)$ and $k(r,s)$ as indicated above such that, for any $t \geq 0, u \in H^1$,

$$\|F_t(u)\|^2 + \frac{1}{2}\|\Sigma_t(u)\|_R^2 \leq b(\|u\|_1) + q(t),$$

for some $q \in L^1(\mathcal{R}^+)$, and

(A3)

$$\|F_t(u) - F_t(u')\|^2 + \frac{1}{2}\|\Sigma_t(u) - \Sigma_t(u')\|_R^2 \leq k(\|u\|_1, \|u'\|_1)\|u - u'\|_1^2,$$

for any $u, u' \in H_1, t \geq 0$.

(A4) There exists a positive function Θ depicted as above and constants $c_i > 0, i = 1,2,3$, and $\kappa < 1$ such that

$$\int_0^t \{(F_s(u_s), v_s) + \frac{1}{2}\|\Sigma_s(u_s)\|_R^2\} ds$$
$$\leq c_1 + c_2 \int_0^t \Theta(u_s; v_s) ds - c_3 \Theta(u_t; v_t) + \kappa \mathbf{e}(u_t; v_t),$$

for any $u_\cdot \in \mathcal{C}(\mathcal{R}^+; H^1) \cap \mathcal{C}^1(\mathcal{R}^+; H)$ with $v_t = \partial_t u_t$.

Theorem 3.1 Let Conditions A hold true. Then, for $u_0 \in H_1$ and $v_0 \in H$, the problem (3.1), or the system (3.2) has a unique continuous solution $u_\cdot \in \mathcal{C}([0,T]; H_1)$ with $\partial_t u_\cdot \in \mathcal{C}([0,T]; H)$, for any $T > 0$. Moreover the following energy equation holds

$$\begin{aligned} \mathbf{e}(u_t, v_t) &= \mathbf{e}(u_0, v_0) + 2\int_0^t [(v_s, F_s(u_s)) - 2\alpha\|v_s\|^2] ds \\ &+2\int_0^t (v_s, \Sigma_s(u_s) dW_s) + \int_0^t \|\Sigma_s(u_s)\|_R^2 ds \quad \text{a.s.}. \end{aligned} \tag{3.6}$$

Proof. The proof is similar to that of Theorems 4.1 and 4.2 in [2] and will only be sketched. The main idea is to show that, by a smooth H_1-truncation, the conditions (A1)-(A3) reduce to the usual sublinear growth and the global Lipschitz conditions. By invoking a standard existence theorem (Theorem 7.4, Da Prato and Zabczyk [7]) for stochastic evolution equations in a Hilbert space, the truncated problem has a continuous solution $u^N_\cdot \in H_1$ for $t < (\tau_N \wedge T)$, where τ_N is a stopping time defined by

$$\tau_N = \inf\{t > 0 : \|u^N_t\|_1 > N\},$$

with N being a cut-off number. Hence, for $t < (\tau_N \wedge T)$, $u_\cdot = u^N_\cdot$ is the solution of (3.2) with $\partial_t u = v^N$. Noting that τ_N increases with N, let $\tau = \lim_{t\to\infty} \tau_N$. Define u_t for $t < (\tau_N \wedge T)$ by $u_t = u^N_t$ if $t < \tau_N \le T$. Then u_t thus defined is the unique local solution. By applying Lemma 2.1, the solution u^N_t satisfies the energy equation (3.6) for $t < (\tau_N \wedge T)$. Making use of condition (A4), this leads to an energy bound:

$$E\mathbf{e}(u_{\tau_N\wedge t}, v_{\tau_N\wedge t}) \le E\Theta(u_{\tau_N\wedge t}, v_{\tau_N\wedge t}) \le K(T),$$

for some constant $K(T) > 0$ and $t \le T$. This bound together with the fact

$$E\mathbf{e}(u_{\tau_N\wedge t}, v_{\tau_N\wedge t}) \ge N^2 P\{\tau_N \le T\}$$

imply that Prob $(\tau \le T) = 0$ for any $T > 0$.. Therefore the solution exists on any finite time interval $[0, T]$ as claimed. The energy equation (3.6) can be verified by taking the limit, as $N \to \infty$, in the $N - th$ approximating energy equation. □

Remark: In the above theorem, for simplicity, we assumed that $W(x,t)$ is a scalar Wiener random field. Under an obvious modification, Theorem 3.2 and the subsequent theorems still hold true when $W = (W^{(1)}, \cdots, W^{(k)})$ is a k-vector valued Wiener random field and $\sigma = (\sigma_1, \cdots, \sigma_k)$ is another k-vector valued, predictable random field such that the product, $\sigma(\cdot)W(\cdot) = \sum_{j=1}^{k} \sigma_j W^{(j)}(\cdot)$, is interpreted as a scalar product.

4 Bounded Solutions

Let us consider the hyperbolic system:

$$\begin{cases} du_t = v_t dt, \\ dv_t = [Au_t - 2\alpha v_t + F_t(u_t)]dt + \Sigma_t(u_t)dW_t, \quad t > 0, \end{cases} \tag{4.1}$$

with a given initial state $(u_0; v_0)$ which is a $\mathcal{F}_0-$ random vector in $\mathcal{H}$. For the existence of bounded solutions, we shall impose Conditions B as follows:

(B1) There exist $\Phi \in \mathcal{C}^1(H^1; \mathcal{R}^+)$, with Fréchet derivative $\Phi' \in \mathcal{C}(H^1; H)$, and $p_\cdot \in \mathcal{C}(\mathcal{R}^+ \times H^1; H)$ such that $F_t(u) = -\frac{1}{2}\Phi'(u) + p_t(u)$ for any $u \in H^1$ and

$$c_1 \le \Phi(u) \le c_2(1 + \|u\|_1^k)$$

for some constants c_1 and $c_2 > 0, k \ge 2$.

(B2) There exist constants $\beta_i \geq 0$ and $\gamma_i, \delta_1 \in \mathcal{R}$ with $i = 1, 2, 3$, and continuous functions θ and ρ on $\mathcal{R}^+$ such that

$$(\Phi'(u), u) \geq \beta_1 \Phi(u) - \gamma_1 \|u\|_1^2 - \delta_1,$$

$$\|p_t(u)\|^2 \leq \beta_2 \Phi(u) + \gamma_2 \|u\|_1^2 + \theta(t),$$

and

$$\|\Sigma_t(u)\|_R^2 \leq \beta_3 \Phi(u) + \gamma_3 \|u\|_1^2 + \rho(t),$$

for any $u \in H_1$ and $t > 0$.

(B3) The above constants satisfy

$$(\beta_1 - \frac{1}{2})\lambda^2 - \beta_3 \lambda - \beta_2 \geq 0,$$

$$(\gamma_1 - \frac{1}{2})\lambda^2 + \gamma_3 \lambda + \gamma_2 \leq 0.$$

Theorem 4.1 (Bounded in Mean-square) Suppose that Conditions A and B hold true, where the functions θ and ρ are bounded on $\mathcal{R}^+ := [0, \infty)$. Given $u_0 \in H^1$ and $v_0 \in H$ being $\mathcal{F}_0-$ random variables such that

$$E\{\mathbf{e}(u_0; v_0) + \Phi(u_0)\} < \infty,$$

then the solution of the problem (4.1) is bounded in mean-square. Moreover there exist positive constants K_1 and $\alpha_2 > 0$ such that

$$\begin{aligned} E\{\mathbf{e}(u_t; v_t) + \Phi(u_t)\} \leq K_1\{E[\mathbf{e}(u_0; v_0) + \Phi(u_0; v_0)]e^{-\alpha_2 t} \\ + \int_0^t e^{-\alpha_2(t-s)}[\frac{1}{\lambda}\theta(s) + \rho(s)]ds\} + |\delta_1|, \ \forall \ t > 0. \end{aligned} \tag{4.2}$$

Proof. By applying Lemma 2.1 to Equation (4.1), similar to (2.19), we obtain the perturbed energy equation:

$$\begin{aligned} d\mathbf{e}^\lambda(u_t; v_t) = 2[\lambda(2\alpha - \lambda)(u_t, v_t^\lambda) - \lambda\|u_t\|_1^2 - (2\alpha - \lambda)\|v_t^\lambda\|^2 \\ +(F_t(u_t), v_t^\lambda) + \frac{1}{2}\|\Sigma_t(u_t)\|_R^2]dt + 2\, dM_t^\lambda(u; v), \end{aligned} \tag{4.3}$$

where we set $dM_t^\lambda(u; v) = (v_t^\lambda, \Sigma(u_t)dW_t)$. Similar to (2.22) in Lemma 2.2, for $\lambda < \{\alpha \wedge \frac{\eta_1}{4\alpha}\}$, the above yields

$$\begin{aligned} d\mathbf{e}^\lambda(u_t; v_t) \leq -\lambda(\|u_t\|_1^2 + \frac{3}{2}\|v_t^\lambda\|^2)dt \\ +[2(F_t(u_t), v_t^\lambda) + \|\Sigma_t(u_t)\|_R^2]dt + 2\, dM_t^\lambda(u). \end{aligned} \tag{4.4}$$

By assumptions,

$$(F_t(u_t), v_t^\lambda) = -\frac{1}{2}(\Phi'(u_t), v_t) - \frac{1}{2}\lambda(\Phi'(u_t), u_t) + (p_t(u_t), v_t^\lambda)$$

$$\leq -\frac{1}{2}\frac{d}{dt}\Phi(u_t) - \frac{1}{2}\lambda(\Phi'(u_t), u_t) + \frac{\lambda}{2}\|v_t^\lambda\|^2 + \frac{1}{2\lambda}\|p_t(u_t)\|^2,$$

which, in view of condition (B2), implies that

$$\begin{aligned}(F_t(u_t), v_t^\lambda) \leq& -\frac{1}{2}\frac{d}{dt}\Phi(u_t) - \frac{1}{2}(\beta_1\lambda - \frac{\beta_2}{\lambda})\Phi(u_t)\\ &+\frac{\lambda}{2}\|v_t^\lambda\|^2 + \frac{1}{2}(\gamma_1\lambda - \frac{\gamma_2}{\lambda})\|u_t\|_1^2 + \frac{1}{\lambda}\theta(t) + \frac{\lambda}{2}\delta_1.\end{aligned} \tag{4.5}$$

Define a *super-energy function* $J : \mathcal{H} \to \mathcal{R}^+$ by

$$J(u; v) = \mathbf{e}(u; v) + \Phi(u), \tag{4.6}$$

with $J^\lambda = \mathbf{e}^\lambda + \Phi$. By applying (4.4), (4.5) and condition (B2) to the equation (4.3), we obtain

$$dJ^\lambda(u_t; v_t) \leq -\lambda\mathbf{e}^\lambda(u_t; v_t)dt - (\beta_1\lambda - \frac{\beta_2}{\lambda} - \beta_3)\Phi(u_t)dt$$

$$+\{\frac{\lambda}{2}\|v_t^\lambda\|^2 + (\gamma_1\lambda - \frac{\gamma_2}{\lambda} + \gamma_3)\|u_t\|_1^2 + \frac{1}{\lambda}\theta(t) + \rho(t) + \frac{\lambda}{2}\delta_1\}dt + 2\,dM_t^\lambda(u; v).$$

By invoking condition (B3), the above inequality gives

$$dJ^\lambda(u_t; v_t) \leq -\frac{\lambda}{2}J^\lambda(u_t; v_t)dt + \{\frac{1}{\lambda}\theta(t) + \rho(t) + \frac{\lambda}{2}\delta_1\}dt + 2\,dM_t^\lambda(u; v), \tag{4.7}$$

which implies that

$$\begin{aligned}EJ^\lambda(u_t; v_t) \leq& EJ^\lambda(u_0; v_0)e^{-\lambda t/2}\\ &+\int_0^t e^{-\lambda(t-s)/2}[\frac{1}{\lambda}\theta(s) + \rho(s)]ds\} + |\delta_1| < \infty,\end{aligned} \tag{4.8}$$

for all $t > 0$. Since, by assumption, θ and ρ are bounded, we have

$$EJ^\lambda(u_t; v_t) = E\{\mathbf{e}^\lambda(u_t; v_t) + \Phi(u_t; v_t)\} < \infty.$$

Now, by invoking Lemma 2.3, $J(u; v) \leq CJ^\lambda(u; v)$ for some $C > 0$. Therefore the result (4.2) holds with some constant $K_1 > 0$ and $\alpha_2 = \frac{\lambda}{2}$. □

In fact it is possible to show that the solution of the equation (4.1) is ultimately bounded in mean-square under somewhat stronger assumptions.

Theorem 4.2 Assume that Conditions A and B hold true with $\delta_1 = 0, \theta$ and $\rho \in L^1(\mathcal{R}^+)$. Then the solution $\phi_t = (u_t, v_t)$ is ultimately bounded in mean-square such that

$$E \sup_{0\le t\le T} \{\mathbf{e}(u_t; v_t) + \Phi(u_t)\} \le K_2 E\{\mathbf{e}(u_0; v_0) + \Phi(u_0)\} + K_3 \int_0^T [\theta(s) + \rho(s)]ds, \tag{4.9}$$

for some positive constants K_2 and K_3.

Proof. In view of (4.7), it is clear that

$$J^\lambda(u_t; v_t) + \frac{\lambda}{2}\int_0^t J^\lambda(u_s; v_s)ds \le J^\lambda(u_0; v_0) + \int_0^t [\frac{1}{\lambda}\theta(s) + \rho(s)]ds + 2M_t^\lambda(u; v),$$

Hence

$$EJ^\lambda(u_t; v_t) + \frac{\lambda}{2}\int_0^t EJ^\lambda(u_s; v_s)ds \le E\, J^\lambda(u_0; v_0) + \int_0^t [\frac{1}{\lambda}\theta(s) + \rho(s)]ds, \tag{4.10}$$

and

$$E \sup_{0\le t\le T} J^\lambda(u_t; v_t) \le E\, J^\lambda(u_0; v_0) + \int_0^t [\frac{1}{\lambda}\theta(s) + \rho(s)]ds + 2E \sup_{0\le t\le T} |M_t^\lambda(u; v)|. \tag{4.11}$$

By means of the B-D-G inequality for a submartingale, we can deduce that

$$\begin{aligned} E \sup_{0\le t\le T} |M_t^\lambda(u; v)| &= E \sup_{0\le t\le T} |\int_0^t (v_s^\lambda, \Sigma_s(u_s)dW_s)| \\ &\le 3E\{\textstyle\int_0^T (R\Sigma_s(u_s)v_s^\lambda, \Sigma_s(u_s)v_s^\lambda)ds\}^{1/2} \\ &\le 3E\{\sup_{0\le t\le T} \|v_t^\lambda\|\}\{\int_0^T \|\Sigma_s(u_s)\|_R^2 ds\}^{1/2} \\ &\le \frac{1}{4}E \sup_{0\le t\le T} \|v_t^\lambda\|^2 + 9E\int_0^T \|\Sigma_s(u_s)\|_R^2 ds. \end{aligned} \tag{4.12}$$

The above results (4.8), (4.12) and condition (B2) imply that

$$\begin{aligned} E \sup_{0\le t\le T} J^\lambda(u_t; v_t) &\le EJ^\lambda(u_0; v_0) + \frac{1}{2}E \sup_{0\le t\le T} \|v_t^\lambda\|^2 \\ &\quad + 18E\int_0^T \|\Sigma_s(u_s)\|_R^2]ds + \int_0^t [\frac{1}{\lambda}\theta(s) + \rho(s)]ds \\ &\le EJ^\lambda(u_0; v_0) + \frac{1}{2}E \sup_{0\le t\le T} \|v_t^\lambda\|^2 + 18E\int_0^T [\beta_3\Phi(u_s) + \gamma_3\|u_s\|_1^2]ds \end{aligned}$$

$$+\int_0^t [\frac{1}{\lambda}\theta(s) + 19\rho(s)]ds$$

Therefore there exist positive constants $c_i, i = 1, 2, 3$, such that

$$E \sup_{0\le t\le T} J^\lambda(u_t; v_t) \le c_1 E J^\lambda(u_0; v_0) + c_2 \int_0^T E \sup_{0\le\tau\le s} J^\lambda(u_\tau; v_\tau)ds$$

$$+c_3 \int_0^T [\theta(s) + \rho(s)]ds.$$

From this together with the bound (4.10) and the Gronwall lemma, we can infer that there exists a pair of positive constants k_2, k_3 such that

$$E \sup_{0\le t\le T} J^\lambda(u_t; v_t) \le k_2 E J^\lambda(u_0; v_0) + k_3 \int_0^T [\theta(s) + \rho(s)]ds,$$

which, by Lemma 2.3, leads to the desired inequality (4.9). □

5 Asymptotic Stabilities of Solutions

Suppose that the hyperbolic system (4.1) has an equilibrium solution $\hat{\phi} = (\hat{u}; \hat{v})$. Without loss of generality, assume that $(\hat{u}; \hat{v}) \equiv (0; 0)$. We are interested in the asymptotic stabilities of the null solution in the following sense.

Definitions:

(1) The null solution $\phi = (u; v) \equiv (0; 0)$ of Equation (4.1) is said to be *asymptotically stable in mean-square* in $\mathcal{H}$ if $\exists\ \delta > 0$ such that, for $\|\phi_0\| < \delta$,

$$\lim_{t\to\infty} E\|\phi_t\|^2 = 0,$$

and it is *exponentially stable in mean-square* if there exist positive constants $K(\delta)$ and ν such that

$$E\,\|\phi_t\|^2 \le K(\delta)e^{-\nu t}, \quad \forall\, t > 0,$$

where $\|\phi\|^2 = \|u\|_1^2 + \|v\|^2$.

(2) The null solution is said to be *a.s. (almost surely) asymptotically stable* if

$$P\{\lim_{t\to\infty} \|\phi_t\| = 0\} = 1,$$

and it is *a.s. exponentially stable* if there exist positive constants $K_2(\delta), \nu_2$ and a random time $T(\omega) > 0$ such that

$$\|\phi_t\| \le K_2(\delta)e^{-\nu_2 t}, \quad \forall\, t > T,\ a.s..$$

Remark: In view of the the above definitions, it is clear that the exponential stability implies the asymptotic stability.

To proceed we assume that $F_t(0) = 0$ and $\Sigma_t(0) = 0$ for any $t > 0$ so that $\phi = (u, v) \equiv (0, 0)$ is an equilibrium solution of the equation (4.1). In addition we suppose, under suitable conditions such as Conditions A, the equation has a unique global solution.

Theorem 5.1 (Stability in Mean-square) Suppose that Conditions B hold true with the following provisions:

(1) $\Phi(0) = 0$ and $\Phi(u) > 0$ if $u \neq 0$.

(2) In condition (B2), $\delta_1 = 0$ and there exists $\alpha_0 > 0$ such that

$$\int_0^\infty e^{\alpha_0 t}\theta(t)dt + \int_0^\infty e^{\alpha_0 t}\rho(t)dt < \infty.$$

Then the null solution of the equation (4.1) is exponentially stable in mean-square. Moreover, if $\phi_0 = (u_0, v_0)$ be a $\mathcal{F}$-measurable random variable in $\mathcal{H}$ satisfying

$$E\{\mathsf{e}(u_0; v_0) + \Phi(u_0; v_0)\} < \infty,$$

there exists positive constants α_2 and C such that

$$E\{\mathsf{e}(u_t; v_t) + \Phi(u_t)\} \leq E\{\mathsf{e}(u_0; v_0) + \Phi(u_0) + C\}e^{-\alpha_2 t}, \tag{5.1}$$

for any $t > 0$, where $C = 0$ if $\theta = \rho \equiv 0$. □

The theorem follows immediately from the inequality (4.2) in Theorem 4.1 by choosing $\alpha_2 \leq \alpha_0$. In fact it is possible to show that the null solution is a.s. exponentially stable. Before stating the next theorem, we need a lemma which is a simple consequence of Theorem 4.2.

Lemma 5.2 Under the conditions for Theorem 5.1, the solution $\phi_t = (u_t, v_t)$ is ultimately bounded in mean-square such that

$$\begin{aligned} E \sup_{0\leq t\leq T} \{\mathsf{e}^\lambda(u_t; v_t) + \Phi(u_t)\} \leq K_1 E\{\mathsf{e}^\lambda(u_0; v_0) + \Phi(u_0)\} \\ + K_2 \int_0^T [\theta(t) + \rho(t)]dt, \end{aligned} \tag{5.2}$$

for some constants $K_1, K_2 > 0$. □

With the aid of Theorem 5.1 and Lemma 5.2, we can prove the following theorem.

Theorem 5.3 (Almost Sure Stability) Assume that all of the conditions for Theorem 5.1 hold true with $\theta = \rho \equiv 0$. Then the null solution of the equation (4.1) is exponentially stable almost surely. Moreover there exists positive constants C, ν and a random variable $T(\omega) > 0$ such that

$$\mathbf{e}(u_t;v_t)+\Phi(u_t)\le C\{\mathbf{e}(u_0;v_0)+\Phi(u_0)\}e^{-\nu t}\text{ a.s.,} \tag{5.3}$$

for any $t>T$.

Proof. Owing to Lemma 2.3, instead of (5.3), it suffices to show that

$$\{\mathbf{e}^\lambda(u_t;v_t)+\Phi(u_t)\}\le C_0E\{\mathbf{e}^\lambda(u_0;v_0)+\Phi(u_0)\}e^{-\nu t},\ t>T,\quad \text{a.s.,} \tag{5.4}$$

for some constant $C_0>0$ and for λ satisfying (2.16). To this end, set $\mathcal{R}^+=\cup_{n=0}^\infty[n,n+1]$ and consider the solution $(u_t;v_t)$ for $n\le t<n+1$. Similar to the steps leading to the inequality (4.8) in the proof of Theorem 4.1, it can be shown that

$$E\sup_{n\le t\le n+1}J^\lambda(u_t;v_t)\le EJ^\lambda(u_n;v_n)+2E\sup_{0\le t\le T}|M_t^\lambda(u)|, \tag{5.5}$$

where we recall that $J^\lambda(u;v)=\mathbf{e}^\lambda(u;v)+\Phi(u)$. Similar to (4.12), we have

$$\begin{aligned}E\sup_{n\le t\le n+1}|M_t^\lambda(u;v)|&\le 3E\{\sup_{n\le t\le n+1}\|v_t^\lambda\|\}\{\int_n^{n+1}\|\Sigma_s(u_s)\|_R^2ds\}^{1/2}\\&\le 3\{E\sup_{n\le t\le n+1}\|v_t^\lambda\|^2\}^{1/2}\{E\int_n^{n+1}\|\Sigma_s(u_s)\|_R^2ds\}^{1/2}\\&\le 3\{E\sup_{n\le t\le n+1}\|v_t^\lambda\|^2\}^{1/2}\{(\beta_3\vee\gamma_3)\int_n^{n+1}EJ^\lambda(u_s;v_s)ds\}^{1/2}.\end{aligned}$$

By making use of the inequalities (5.1) and (5.2), the above gives rise to the upper bound:

$$\begin{aligned}&E\sup_{n\le t\le n+1}|M_t^\lambda(u;v)|\\&\le 3\{EJ^\lambda(u_0;v_0)\}\{CK(\beta_3\vee\gamma_3)\int_n^{n+1}e^{-\alpha_2 s}ds\}^{1/2}\\&\le 3\{EJ^\lambda(u_0;v_0)\}\{\frac{CK}{\alpha_2}(\beta_3\vee\gamma_3)\}^{1/2}e^{-n\alpha_2/2}.\end{aligned} \tag{5.6}$$

By taking (5.1), (5.5) and (5.6) into account, we get

$$E\{\sup_{n\le t\le n+1}J^\lambda(u_t;v_t)\}\le C_0\ \{EJ^\lambda(u_0;v_0)\}e^{-n\alpha_2/2}, \tag{5.7}$$

for constant $C_0>0$ and $\alpha_2=\dfrac{\lambda}{2}$.

Therefore, by using the Markov inequality and (5.7),

$$\begin{aligned}&P\{\sup_{n\le t\le n+1}J^\lambda(u_t;v_t)>C_0\ EJ^\lambda(u_0;v_0)e^{-n\alpha_2/4}\}\\&\le\frac{E\{\sup_{n\le t\le n+1}J^\lambda(u_t;v_t)\}}{C_0\ E\{J^\lambda(u_0;v_0)\}e^{-n\alpha_2/4}}\le e^{-n\alpha_2/4}.\end{aligned}$$

Since $\sum_{n=1}^{\infty} e^{-n\alpha_2/4} < \infty$, it follows from the Borel-Cantelli lemma that there exists a random number $N(\omega) > 0$ such that, for $n > N$,

$$\sup_{n\leq t\leq n+1} J^\lambda(u_t; v_t) \leq C_0 \; \{EJ^\lambda(u_0; v_0)\} e^{-n\alpha_2/4}, \quad \text{a.s.},$$

which, by definition (4.6), implies the inequality (5.4) with $\nu = \alpha_2/4$. □

6 Invariant Measures

Let us consider the autonomous version of the system (4.1):

$$\begin{cases} du_t = v_t dt, \\ dv_t = [Au_t - 2\alpha v_t + F(u_t)]dt + \Sigma(u_t)dW_t, \quad t > 0, \end{cases} \tag{6.1}$$

with a given initial state $(u_0; v_0)$, where F and Σ do not depend on t explicitly.

Let $\phi_t = (u_t; v_t)$ and rewrite the system (6.1) as an evolution equation in the differential form:

$$d\phi_t = \mathcal{A}\phi_t dt + \mathcal{F}(\phi_t)dt + d\mathcal{M}_t(\phi), \tag{6.2}$$

with $\phi_0 = (u_0; v_0)$, where we set

$$\phi_t = \begin{bmatrix} u_t \\ v_t \end{bmatrix}, \quad \mathcal{F}(\phi) = \begin{bmatrix} 0 \\ F(u) \end{bmatrix}, \quad \mathcal{M}_t(\phi) = \begin{bmatrix} 0 \\ M_t(u) \end{bmatrix},$$

and

$$\mathcal{A} = \begin{bmatrix} 0 & I \\ A & -2\alpha I \end{bmatrix}$$

with I being the identity operator on H.

Under Conditions C given below, the solution ϕ_t is a diffusion process in $\mathcal{H}$. The transition probability function is given by

$$P_t(\xi; \mathcal{B}) = P\{\phi_t \in \mathcal{B} | \phi_0 = \xi\}, \;\; \xi \in \mathcal{H}, \;\; \mathcal{B} \in \sigma(\mathcal{H}).$$

Suppose there exists an invariant measure μ on $(\mathcal{H}, \sigma(\mathcal{H}))$, where $\sigma(\mathcal{H})$ denotes the Borel $\sigma-$ field of $\mathcal{H}$. Then it satisfies (p.12, [8]):

$$\mu(\mathcal{B}) = \int_{\mathcal{H}} P_t(\xi; \mathcal{B})\mu(d\xi), \;\; \forall \; \mathcal{B} \in \sigma(\mathcal{H}).$$

To show the existence of an invariant measure, we shall specialize Conditions A by assuming that the nonlinear terms satisfy a uniform Lipschitz-continuity condition. To be precise, assume the following Conditions C:

(C1) Let $F(\cdot) : H^1 \to H$ and $\Sigma(\cdot) : H^1 \to H$, and let there exist positive constants b_i, c_i for $i = 1, 2$, such that

$$\|F(u)\|^2 \leq b_1 \|u\|_1^2 + c_1$$

and

$$\|\Sigma_t(u)\|_R^2 \leq b_2 \|u\|_1^2 + c_2,$$

for any $u \in H^1$.

(C2) There exist positive constants k_1, k_2 such that

$$\|F(u) - F(u')\|^2 \leq k_1 \|u - u'\|_1^2$$

and

$$\|\Sigma(u) - \Sigma(u')\|_R^2 \leq k_2 \|u - u'\|_1^2,$$

for any $u, u' \in H^1$.

(C3) The constants b_i and k_i satisfy

$$(b_1 + b_2\lambda) \wedge (k_1 + k_2\lambda) \leq \frac{\lambda^2}{2}.$$

To show the existence of an invariant measure, we shall follow an approach by Da Prato and Zabcyzk (Theorem 6.3.2, [8]) for some stochastic dissipative systems. Though not directly applicable to the present problem, it can be adapted to proving the following theorem.

Theorem 6.1 (Invariant Measures) Suppose that the system (6.1) satisfies Conditions C. Then there exists a unique invariant measure μ on $(\mathcal{H}, \sigma(\mathcal{H}))$. Moreover, given any bounded Lipschitz continuous function G on $\mathcal{H}$ there are positive constants C and α_2 such that

$$|\int_{\mathcal{H}} G(\eta) P_t(\xi; d\eta) - \int_{\mathcal{H}} G(\eta)\mu(d\eta)| \leq C(1 + \|\xi\|)e^{-\alpha_2 t}, \tag{6.3}$$

for any $t > 0$ and $\xi \in \mathcal{H}$.

Proof. To extend the time domain for the system (6.1) to the whole real line $\mathcal{R}$, introduce an independent copy V_t of the Wiener process W_t for $t \geq 0$. Define $\hat{W}_t$ by

$$\hat{W}_t = \begin{cases} W_t & \text{for } t \geq 0, \\ V_{-t} & \text{for } t \leq 0, \end{cases} \tag{6.4}$$

and let $\hat{\mathcal{F}}_t = \sigma\{\hat{W}_s : s \leq t\}$, for $t \in \mathcal{R}$. Now let $\phi_t(\tau; \xi) = (u_t; v_t)(\tau; \xi) = (u_t(\tau; \xi); v_t(\tau; \xi))$ be the solution of the extended system:

$$
\begin{cases}
du_t = v_t dt, \\
dv_t = [Au_t - 2\alpha v_t + F(u_t)]dt + \Sigma(u_t)d\hat{W}_t, \quad t > \tau, \\
u_\tau = \xi_1, \quad v_\tau = \xi_2,
\end{cases}
\tag{6.5}
$$

where $\xi = (\xi_1; \xi_2) \in \mathcal{H}$.

Similar to the derivation of the inequality (4.4) in the proof of Theorem 4.1, it can be shown that, for $\lambda < \{\alpha \wedge \frac{\eta_1}{4\alpha}\}$,

$$
\begin{aligned}
d\mathbf{e}^\lambda(u_t; v_t) \le -\lambda \mathbf{e}^\lambda(u_t; v_t)dt + [\frac{\lambda}{2}\|v_t^\lambda\|^2 + \frac{1}{\lambda}\|F(u_t)\|^2 \\
+\|\Sigma(u_t)\|_R^2]dt + 2(v_t^\lambda, \Sigma(u_t)d\hat{W}_t),
\end{aligned}
\tag{6.6}
$$

where $d\hat{M}_t^\lambda(u;v) = (v_t^\lambda, \Sigma(u_t)d\hat{W}_t)$. By making use of conditions (C1) and (C2), the above yields

$$
\begin{aligned}
d\mathbf{e}^\lambda(u_t; v_t) &\le -\lambda \mathbf{e}^\lambda(u_t; v_t)dt + [\frac{\lambda}{2}\|v_t^\lambda\|^2 + (\frac{b_1}{\lambda} + b_2)\|u_t\|_1^2 \\
&\quad +(\frac{c_1}{\lambda} + c_2)]dt + 2\, d\hat{M}_t^\lambda(u;v), \\
&\le [-\frac{\lambda}{2}\mathbf{e}^\lambda(u_t; v_t) + (\frac{c_1}{\lambda} + c_2)]dt + 2\, d\hat{M}_t^\lambda(u;v),
\end{aligned}
$$

so that

$$
E\mathbf{e}^\lambda(\phi_t) \le \mathbf{e}^\lambda(\xi)e^{-\lambda(t-s)/2} + (\frac{c_1}{\lambda} + c_2)\int_s^t e^{-\lambda(t-r)/2}dr.
$$

Therefore there exists a constant $K_1 > 0$ such that

$$
E\mathbf{e}^\lambda(\phi_t) \le K_1\{1 + e^\lambda(\xi)\}, \quad \text{for any}\, t \ge s. \tag{6.7}
$$

For $\tau_1 > \tau_2 > 0$, let

$$
u_t^i = u_t(-\tau_i; \xi), \quad v_t^i = v_t(-\tau_i; \xi), \quad \text{for} \quad t > -\tau_2,
$$

with $i = 1, 2$, and

$$
\tilde{u}_t = u_t^1 - u_t^2, \quad \tilde{v}_t = v_t^1 - v_t^2.
$$

Then it follows from (6.5) that they satisfy

$$\begin{cases} d\tilde{u}_t = \tilde{v}_t dt, \\ d\tilde{v}_t = [A\tilde{v}_t - 2\alpha\tilde{v}_t + \delta F(u_t^1; u_t^2)]dt \\ \qquad +\delta\Sigma(u_t^1; u_t^2)d\hat{W}_t, \quad t > -\tau_2, \\ \tilde{u}_{-\tau_2} = (u_{-\tau_2}^1 - \xi_1), \quad \tilde{v}_{-\tau_2} = (v_{-\tau_2}^1 - \xi_2), \end{cases} \tag{6.8}$$

where

$$\delta F(u^1; u^2) = F(u^1) - F(u^2), \quad \delta\Sigma(u^1; u^2) = \Sigma(u^1) - \Sigma(u^2). \tag{6.9}$$

Let $\tilde{v}^\lambda = \tilde{v} + \lambda\tilde{u}$ and $\tilde{\mathbf{e}}_t^\lambda = \mathbf{e}(\tilde{u}_t; \tilde{v}_t^\lambda)$. Similar to (6.6), we can obtain the energy inequality:

$$\begin{aligned} d\tilde{\mathbf{e}}_t^\lambda \leq & -\lambda\tilde{\mathbf{e}}_t^\lambda dt + [\frac{\lambda}{2}\|\tilde{v}_t^\lambda\|^2 + \frac{1}{\lambda}\|\delta F(u_t^1; u_t^2)\|^2 \\ & +\|\delta\Sigma(u_t^1; u_t^2)\|_R^2]dt + 2(\tilde{v}_t^\lambda, \delta\Sigma(u_t)d\hat{W}_t), \end{aligned} \tag{6.10}$$

In view of (6.9) and the conditions (C2) and (C3), the equation (6.10) yields

$$\begin{aligned} d\tilde{\mathbf{e}}_t^\lambda &\leq -\lambda\tilde{\mathbf{e}}_t^\lambda dt + [\frac{\lambda}{2}\|\tilde{v}_t^\lambda\|^2 + (\frac{k_1}{\lambda} + k_2)\|\tilde{u}_t\|_1^2]dt + 2(\tilde{v}_t^\lambda, \delta\Sigma(u_t)d\hat{W}_t) \\ &\leq -\frac{\lambda}{2}\tilde{\mathbf{e}}_t^\lambda dt + 2(\tilde{v}_t^\lambda, \delta\Sigma(u_t)d\hat{W}_t), \end{aligned}$$

which implies that

$$E\tilde{\mathbf{e}}_t^\lambda \leq E\tilde{e}_{-\tau_2}^\lambda e^{-\lambda(t+\tau_2)/2}.$$

In view of the bound (6.7) and the initial conditions in (6.8), we can show that

$$E\tilde{\mathbf{e}}_{-\tau_2}^\lambda \leq 2K_1\{1 + \mathbf{e}^\lambda(\xi)\},$$

so that

$$E\tilde{\mathbf{e}}_t^\lambda = E\mathbf{e}^\lambda(u_t^1 - u_t^2; v_t^1 - v_t^2) \leq 2K_1\{1 + \mathbf{e}\lambda(\xi)\}e^{-\lambda(t+\tau_2)/2}. \tag{6.11}$$

Let

$$\psi_\tau = (u_0(-\tau;\xi); v_0(-\tau;\xi)).$$

By setting $t = 0$ in (6.11), we obtain, for any $\xi \in \mathcal{H}$,

$$E\mathbf{e}^\lambda(\psi_{\tau_2} - \psi_{\tau_1}) \leq 2K_1\{1 + \mathbf{e}^\lambda(\xi)\}e^{-\lambda\tau_2/2}, \tag{6.12}$$

which goes to zero as $\tau_2 \to \infty$. By Lemma 2.3 with $\lambda > 0$, the energy functions $\mathbf{e}^\lambda$ and $\mathbf{e}$ are equivalent so that $\mathbf{e}(\cdot) \leq C\mathbf{e}^\lambda(\cdot)$ for some constant $C > 0$. Since $\mathbf{e}$ defines the energy norm $\|\cdot\|$ on $\mathcal{H}$ by $\|\phi\|^2 = \mathbf{e}(\phi)$, $\phi \in \mathcal{H}$, the set $\{\psi_\tau : \tau \geq 0\}$ is a Cauchy family of random variables in $L^2(\Omega;\mathcal{H})$. Therefore, there exists a unique random variable $\psi_\infty \in L^2(\Omega;\mathcal{H})$ such that, for any $\xi \in \mathcal{H}$,

$$\lim_{\tau\to\infty} E\,\|\psi_\tau - \psi_\infty\|^2 = 0.$$

Since the random variables $\psi_\tau = \phi_\tau(0;\xi)$ in distribution, $\phi_\tau(0;\xi)$ converges weakly to ψ_∞ as $\tau \to \infty$. It follows that $P_t(\xi;\cdot)$ converges weakly to the probability measure μ for ψ_∞ as $t \to \infty$, and μ is the desired invariant measure for the system (6.1).

To verify the inequality (6.3), let $G : \mathcal{H} \to \mathcal{R}$ be bounded and Lipschitz continuous on $\mathcal{H}$ such that

$$|G(\xi) - G(\eta)| \le \gamma\|\xi - \eta\|.$$

Then, for any $t > s > 0$, we have

$$\begin{aligned}|\int_{\mathcal{H}} G(\eta)P_t(\xi;d\eta) - \int_{\mathcal{H}} G(\eta)P_s(\xi,d\eta)| &= |EG(\psi_t) - EG(\psi_s)| \\ &\le \gamma\{E\|\psi_t - \psi_s\|^2\}^{1/2} \le K\{1 + \mathrm{e}^{\lambda}(\xi)\}^{1/2}e^{-\lambda s/4},\end{aligned}$$

due to the bound (6.12), for some constant $K > 0$. Therefore, by letting $s \to \infty$ and invoking Lemma 2.3, the inequality (6.3) follows. □

7 Examples

To illustrate the application of the stated theorems, let us specialize $A = (\Delta - 1)$ in the equation (3.1) to get

$$\begin{cases} \partial_t^2 u(x,t) = (\Delta - 1)u - 2\alpha\partial_t u + f(u,x,t) \\ +\sigma(u,Du,x,t)\partial_t W(x,t), \quad 0 < t < T, \quad x \in \mathcal{D} \subset \mathcal{R}^d,\ d \le 3. \\ u(x,0) = u_0(x) \in H^1 \cap L^{2n}, \quad \partial_t u(x,0) = v_0(x) \in H, \\ u(\cdot,t)|_{\partial D} = 0, \end{cases} \tag{7.1}$$

where $n \ge 1$. We shall present two examples with different kinds of nonlinear terms.

(Example 1) Let

$$\begin{aligned} f(u,x,t) &= -\kappa u^{2n-1} + \beta(x,t)u^m, \\ \sigma(u,Du,x,t) &= \zeta(x,t)(1 + |Du|^2)^\delta u^k, \end{aligned} \tag{7.2}$$

where $\kappa > 0$ is a constant, β and ζ are some functions on $\mathcal{D} \times \mathcal{R}^+$ as yet to be specified. The positive integers n, m and r are given such that $2 \le m < n$, where n is any natural number for $d \leq 2$ but, for $d = 3$, $n \leq 2$, (for reason, see [2]). Also we assume $\delta \in (0, \frac{1}{2})$ and $0 < k < m(1 - 2\delta)$.

Similar to the results in [2], we can show that Conditions A are satisfied so that the equation has a unique global solution. In view of (7.2) and condition (A2), we set $\Phi'(u) = 2\kappa u^{2n-1}$ so that $\Phi(u) = \frac{\kappa}{n}\|u^n\|^2$, and $p_t(u) = \beta(\cdot, t)u^m$. Therefore we have

$$(\Phi'(u), u) = 2\kappa\|u^n\|^2. \quad \text{and } \|p_t(u)\|^2 = \|b_2 u^m\|^2. \tag{7.3}$$

By means of an elementary Young's inequality (p.61, [6]), it can be shown that, for any $\varepsilon > 0$, we have

$$(\beta u^m)^2 \leq \varepsilon u^{2n} + \frac{\beta^{2q}}{q(q'\varepsilon)^q},$$

where $q = n/(n-m)$ and $q' = n/m$, so that

$$\|p_t(u)\|^2 = \int_{\mathcal{D}} \beta^2 u^{2m} dx \leq \varepsilon\|u^n\|^2 + C_1 \int_{\mathcal{D}} \beta^{2q} dx, \tag{7.4}$$

for some constant $C_1 > 0$. Next consider the term

$$\|\Sigma_t(u)\|_R^2 = \int_{\mathcal{D}} r(x,x)\zeta^2(x,t)(1 + |Du|^2)^{2\delta} u^{2k} dx.$$

By a repeated application of the Young's inequality, we can deduce that, for any $\varepsilon', \varepsilon" > 0$, there exists $C_2 > 0$, such that

$$\|\Sigma_t(u)\|_R^2 \leq \int_{\mathcal{D}} r(x,x)\zeta^2(x,t)\{\varepsilon' u^{2n} + \varepsilon" |Du|^2 + C_2\} dx. \tag{7.5}$$

Suppose that β, η and r are bounded and continuous such that

$$|\beta(x,t)| \leq \beta_0, \ |\eta(x,t)| \leq \eta_0, \text{ and } \ |r(x,x)| \leq r_0, \tag{7.6}$$

for any $x \in \mathcal{D}, t \geq 0$. In view of (7.6), the inequalities (7.4) and (7.5) yield

$$\begin{aligned} &\|p_t(u)\|^2 \leq \frac{\varepsilon\kappa}{n}\Phi(u) + C_1 \int_{\mathcal{D}} \beta^{2q}(x,t) dx, \\ &\|\Sigma_t(u)\|_R^2 \leq r_0\zeta_0^2\{\frac{\varepsilon'\kappa}{n}\Phi(u) + \varepsilon" \|u\|_1^2\} + C_2 r_0 \int_{\mathcal{D}} \zeta^2(x,t) dx. \end{aligned} \tag{7.7}$$

From (7.3) and (7.4), using the notation in condition (B2), we see that

$$\begin{aligned} &\beta_1 = 2\kappa, \ \ \gamma_1 = \delta_1 = 0; \ \ \beta_2 = \frac{\varepsilon\kappa}{n}, \ \ \gamma_2 = 0; \\ &\beta_3 = r_0\zeta_0^2\frac{\varepsilon'\kappa}{n}, \ \ \gamma_3 = r_0\zeta_0^2\varepsilon", \end{aligned} \tag{7.8}$$

and

$$\theta(t) = C_1 \int_{\mathcal{D}} \beta^{2q}(x,t) dx, \ \ \rho(t) = C_2 r_0 \int_{\mathcal{D}} \zeta^2(x,t) dx. \tag{7.9}$$

Therefore the condition (B3) takes the form:

$$\begin{aligned} &2\kappa\lambda^2 - r_0\zeta_0^2\frac{\varepsilon'\kappa}{n}\lambda - \frac{\varepsilon\kappa}{n} > \frac{\lambda^2}{2}, \\ &r_0\zeta_0^2\varepsilon''\lambda < \frac{\lambda^2}{2}. \end{aligned} \tag{7.10}$$

Since $\varepsilon, \varepsilon'$ and ε'' are arbitrary, they can be chosen so small that condition (B3) holds simply for $\kappa > \frac{1}{4}$. Assume this is the case. Then, by applying Theorems 4.1, 4.2 and 5.1, 5.2, depending on the properties of θ and ρ as defined in (7.9), we can draw the following conclusions:

(1) By the conditions in (7.6), it is clear that both θ and ρ are bounded on $\mathcal{R}^+$ so that, by invoking Theorem 5.1, we can conclude that the solution of the problem (7.1) is bounded in mean-square and there exists a constant $K_1 > 0$ such that

$$\sup_{t>0} E\{\|u_t\|_1^2 + \|\partial_t u_t\|^2 + \|u^n\|^2\} \leq K_1.$$

(2) Suppose that the functions θ and ρ defined by (7.9) belong to $L^1(\mathcal{R}^+)$ so that

$$\int_0^\infty \int_{\mathcal{D}} \beta^{2q}(x,t)dxdt < \infty, \quad \int_0^\infty \int_{\mathcal{D}} \zeta^2(x,t)dxdt < \infty.$$

Then, by Theorem 4.2, the solution is ultimately bounded in mean-square if $\kappa > \frac{1}{4}$, and furthermore there is $K_2 > 0$ such that

$$E\sup_{t>0}\{\|u_t\|_1^2 + \|\partial_t u_t\|^2 + \|u^n\|^2\} \leq K_2.$$

(3) Note that the equation (7.1) has a null solution $(u; v) = (0; 0)$. Assume there exists $\alpha_0 > 0$ such that $(e^{\alpha_0 t}\theta)$ and $(e^{\alpha_0 t}\rho)$ belong to $L^1(\mathcal{R}^+)$, or

$$\int_0^\infty \int_{\mathcal{D}} e^{\alpha_0 t}\beta^{2q}(x,t)dxdt < \infty, \quad \int_0^\infty \int_{\mathcal{D}} e^{\alpha_0 t}\zeta^2(x,t)dxdt < \infty.$$

Then Theorem 5.1 shows that the null solution is exponentially stable in mean-square, and moreover, according to Theorem 5.3, the solution is in fact a.s. exponentially stable.

(Example 2) Consider a mildly nonlinear equation of the form:

$$\begin{cases} \partial_t^2 u(x,t) = (\Delta - 1)u - 2\alpha\partial_t u + f(u) \\ +\sigma(Du)\partial_t W(x,t), \quad 0 < t < T, \quad x \in \mathcal{D} \subset \mathcal{R}^d, \\ u(\cdot,t)|_{\partial D} = 0, \end{cases} \tag{7.11}$$

subject to the initial conditions: $u(x,0) = u_0(x) \in H^1$ and $\partial_t u(x,0) = v_0(x) \in H$, where

$$\begin{aligned} f(u) &= -\kappa u \tan^{-1}(1+u^2), \\ \sigma(Du)\partial_t W &= \sigma_1\{1+|Du|^2\}^{1/2}\partial_t W^{(1)} + \sigma_2 \partial_t W^{(2)}. \end{aligned} \tag{7.12}$$

In the above equations $\kappa > 0, \sigma_1$ and σ_2 are some constants, and $W^{(1)}(\cdot,t), W^{(2)}(\cdot,t)$ are independent Wiener random fields with bounded, continuous covariant functions r_1, r_2, respectively. Rewritting the equation (7.11) in the system form (6.1) and noting (7.12), it is easy to verify that

$$\|F(u)\|^2 \le (\frac{\kappa\pi}{2\eta_1})^2 \|u\|^2, \tag{7.13}$$

and

$$\begin{aligned} \|\Sigma(u)\|_R^2 &= \sigma_1^2 \int_{\mathcal{D}} r_1(x,x)(1+|Du|^2)dx + \sigma_2^2 \int_{\mathcal{D}} r_2(x,x)dx \\ &\le \sigma_1^2 r_0 \|u\|_1^2 + c_2, \end{aligned} \tag{7.14}$$

for some $c_2 > 0$, where η_1 is the smallest eigenvalue of $A = (-\Delta + 1)$ and $r_0 = \sup_{x\in\mathcal{D}} |r_1(x,x)|$. Similarly we can obtain the following bounds:

$$\begin{aligned} &\|F(u) - F(u')\|^2 \le \frac{\kappa^2}{\eta_1}(1+\frac{\pi}{2})^2 \|u-u'\|_1^2, \\ &\|\Sigma(Du) - \Sigma(Du')\|_R^2 \le \sigma_1^2 r_0 \|u-u'\|_1^2, \end{aligned} \tag{7.15}$$

for any $u, u' \in H^1$. In the notation of Conditions C, we can read off from (7.13) to (7.15) and find $c_1 = 0$ and

$$b_1 = (\frac{\kappa\pi}{2\eta_1})^2, \; b_2 = \sigma_1^2 r_0,$$

$$k_1 = \frac{\kappa^2}{\eta_1}(1+\frac{\pi}{2})^2, \; k_2 = \sigma_1^2 r_0.$$

To satisfiy the last condition (C4), we require that

$$(\frac{\kappa\pi}{2\eta_1})^2 + \sigma_1^2 r_0 \lambda \le \frac{\lambda^2}{2}$$

and

$$\frac{\kappa^2}{\eta_1}(1+\frac{\pi}{2})^2 + \sigma_1^2 r_0 \lambda \le \frac{\lambda^2}{2}.$$

Then Theorem 6.1 (see the remark following Theorem 3.1) ensures the existence of a unique invariant measure μ in the state space $\mathcal{H}$ for the equation (7.11) and the the corresponding transition probability converges weakly to μ at an exponential rate.

8 Appendix

Let $W(x,t)$ be a continuous Wiener random field as given in section 2. Then it may be regarded as a $H-$valued Wiener process with a finite-trace covariance operator R. We first define the stochastic integral with an a.s. bounded integrand. To this end, let $\sigma(x,t)$ be an a.s. bounded, continuous predictable random field such that

$$E\int_0^T \|\sigma_t\|^2 dt < \infty. \tag{8.1}$$

We may consider σ_t as a linear operator in H such that $[\sigma_t h](x) = \sigma(x,t)h(x)$ for any $h \in H$. Then it is easy to check that $\sigma_t : H \to H$ is Hilbert-Schmidt a.s. and, noting (8.1),

$$E\int_0^T Tr\,(\sigma_t R \sigma_t^\star)dt = E\int_0^T \int_{\mathcal{D}} r(x,x)\sigma^2(x,t)dxdt$$

$$\leq r_0 E\int_0^T \|\sigma_t^2\|^2 dt < \infty,$$

where $\star$ denotes the conjugation. Therefore the stochastic integral

$$M(x,t) = \int_0^t \sigma(x,s)W(x,ds) \tag{8.2}$$

or

$$M_t = \int_0^t \sigma_s dW_s$$

is well defined as a continuous $H-$valued martingale with mean zero and the covariation operator Q_t defined as (see p.90, [7])

$$\langle (M_t,g),(M_s,h)\rangle = \int_0^{t\wedge s} (Q_\tau g,h)d\tau, \tag{8.3}$$

where Q_s has the kernel $q(x,y,s) = r(x,y)\sigma(x,s)\sigma(y,s)$.

Now we shall define a stochastic integral with a L^p- bounded integrand as shown in the proof of the following theorem.

Theorem A.1 Let $W(.,t)$ be a continuous Wiener random field with a bounded covariance function $r(x,y)$ such that

$$\sup_{x\in\mathcal{D}} r(x,x) \leq r_0. \tag{8.4}$$

Suppose that $\sigma_t = \sigma(\cdot,t)$ is a predictable, continuous $H-$valued process satisfying the condition

$$E\int_0^T \|\sigma(\cdot,t)\|^p dt = E\int_0^T \int_{\mathcal{D}} |\sigma(x,t)|^p dxdt < \infty, \tag{8.5}$$

for an integer $p \geq 2$. Then the stochastic integral M_t (8.2) is well defined as a continuous $H-$valued, L^p-martingale with mean zero and the covariation operator Q_t for $t \in [0,T]$, as given by (8.3).

Proof. Since the set $\mathcal{C}_b$ of bounded continuous functions on $\mathcal{D}$ is dense in $L^p(\mathcal{D})$ (p.28, [1]), by smoothing, there exists a sequence $\{\sigma_t^n\}$ of predictable continuous random fields converging to σ_t such that it satisfies condition (8.1) and

$$\lim_{n\to\infty} E\int_0^T \|\sigma_t^n - \sigma_t\|^p dt = 0. \tag{8.6}$$

Therefore, as in (8.2), the stochastic integral

$$M_t^n = \int_0^t \sigma_s^n dW_s$$

exists as a continuous $H-$valued martingale for each n. Let $\mathcal{M}_T^p$ denote the Banach space of continuous L^p-martingales $N_t \in H$ with norm (p.79, [7])

$$\|N\|_T = \{E \sup_{0\le t\le T} \|N_t\|^p\}^{1/p}.$$

Then the sequence $\{M_t^n\}$ belongs to $\mathcal{M}_T^p$, since, by the B-D-G inequality,

$$\|M^n\|_T^p = E\sup_{0\le t\le T}\|M_t^n\|^p \le C_pE\{\int_0^T\int_{\mathcal{D}} r(x,x)|\sigma(x.t)|^2dxdt\}^{p/2}$$
$$\le C_p r_0^{p/2} E\{\int_0^T \|\sigma_t\|^2dt\}^{p/2} \le C_p(T)E\int_0^T \|\sigma_t\|^p dt,$$

which is finite by (8.5), where $C_p, C_p(T)$ are some positive constants.

Now, for $n > m$,

$$\|M^n - M^m\|_T^p = E\sup_{0\le t\le T}\|M_t^n - M_t^m\|^p$$
$$\le C_pE\{\int_0^T Tr\, Q_s^{mn}ds\}^{p/2},$$

where

$$Tr\, Q_s^{mn} = \int_D q^{mn}(x,x,s)dx = \int_D r(x,x)[\sigma^n(x,s) - \sigma^m(x,s)]^2dx \le r_0\|\sigma_s^n - \sigma_s^m\|^2.$$

It follows from (8.4) and (8.6) that

$$\|M^n - M^m\|_T^p \le C_p(T)E\int_0^T \|\sigma_t^n - \sigma_t^m\|^p dt,$$

which goes to zero as $n > m \to \infty$ due to (8.6). Therefore the sequence $\{M_t^n\}$ converges to the limit denoted by M_t, which is defined as a stochastic integral given by (8.2). We can check that it preserves the properties of M_t^n as stated in the theorem. □

Acknowledgment

The author wishes to thank Professor Jerzy Zabczyk of the Polish Academy of Science for a helpful discussion about the problem of invariant measures given in Theorem 6.1.

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