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Jerome Kaminker

Claude Schochet
Wayne State University, clsmath@gmail.com

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STEENROD HOMOLOGY AND OPERATOR ALGEBRAS
BY JEROME KAMINKER AND CLAUDE SCHOCHET
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The recent work of Larry Brown, R. G. Douglas, and Peter Fillmore (referred to as BDF) [2], [3], and [4] on operator algebras has created a new bridge between functional analysis and algebraic topology. This note and a subsequent paper [5] constitute an effort to make that bridge more concrete.

We first briefly describe the BDF framework. This requires the following C*-algebras: \( C(X) \), the continuous complex-valued functions on a compact metric space \( X \); \( L \), the bounded operators on an infinite dimensional separable Hilbert space; \( K \subseteq L \), the compact operators; and \( L/K \), the Calkin algebra. (Let \( \pi: L \rightarrow L/K \) be the projection.) An extension is a short exact sequence of C*-algebras and C* algebra morphisms of the form \( 0 \rightarrow K \rightarrow E \rightarrow C(X) \rightarrow 0 \) where \( E \) is a C*-algebra containing \( K \) and \( I \) (the identity operator) and contained in \( L \). Unitary equivalence classes of extensions form an abelian group, denoted \( \text{Ext}(X) \).

\( \text{Ext}(X) \) was invented by BDF in order to study essentially normal operators, that is, operators \( T \in L \) with \( \pi T \) normal. Let \( E_T \) denote the C*-algebra generated by \( I \), \( T \), and \( K \), and let \( X = \sigma(\pi T) \), the spectrum of \( \pi T \). Then the exact sequence \( 0 \rightarrow K \rightarrow E_T \rightarrow C(X) \rightarrow 0 \) represents an element of \( \text{Ext}(X) \). This element is zero if and only if \( T \) is a compact perturbation of a normal operator. For \( X \subseteq C \), BDF prove that

\[ \text{Ext}(X) \simeq \tilde{H}^0(C - X). \]

This isomorphism assigns to \( E_T \) a sequence of integers corresponding to the Fredholm index of \( T - \lambda J \) on the various bounded components of \( C - X \).

The isomorphism (1) was subsequently generalized [3]. Let \( E_{2n+1}(X) = \text{Ext}(X) \) and \( E_{2n}(X) = \text{Ext}(SX) \), where \( SX \) is the suspension of \( X \). Then BDF show that \( E_* \) satisfies (on compact metric pairs) all of the Eilenberg-Steenrod


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axioms for an homology theory, except the dimension axiom. For finite CW complexes, $E_*(X) = \tilde{K}_*(X)$, where $\tilde{K}_*$ is the reduced homology theory corresponding to complex $K$-theory [1] and [10].

The homology theory $E_*$ satisfies two additional axioms:

(RH) Let $f: (X, A) \to (Y, B)$ be a relative homeomorphism (i.e., $f|X - A$ is a homeomorphism onto $Y - B$). Then $f_*: E_*(X, A) \to E_*(Y, B)$ is an isomorphism.

(Wedge) Let $\bigvee_j X_j$ be the strong wedge of a sequence of pointed compact metric spaces. Then $E_*(\bigvee_j X_j) = \prod_j E_*(X_j)$. (The strong wedge of a family of pointed spaces is the subspace of the product consisting of all points with at most one coordinate not a basepoint.)

In 1941, Steenrod introduced a homology theory on compact metric spaces via “regular cycles” [9] and [8]. This theory, which we denote by $^sH_*$, satisfies all seven of the usual axioms as well as (RH) and (Wedge).

Steenrod showed that one may obtain $^sH_n(X)$ as follows. Write $X = \lim X_j$, where $\{X_j, p_j\}$ is an inverse system of finite simplicial complexes obtained as the nerves of open covers which successively refine each other and whose mesh goes to zero. Assume also that $X_0 = \text{point}$. Let $FX$ be the infinite mapping cylinder—that is, $FX = (\bigcup_j X_j \times [j, j + 1])/\sim$, where $\sim$ is the equivalence relation corresponding to pasting the cylinders $X_j \times [j, j + 1]$ together at their ends via the maps $\{p_j\}$. Then $FX$ admits the structure of a countable, locally finite CW-complex. Steenrod proved that $^sH_n(X)$ is isomorphic to the $(n + 1)$st homology group of $FX$ based on infinite chains. We thus obtain a useful characterization of the groups $^sH_*(X)$.

Steenrod [9] and Milnor [7] also proved that $^sH_n(X)$ is related to the more common Čech homology group $\check{H}_n(X) = \lim_{\leftarrow j} H_n(X_j)$ by a split exact sequence

$$0 \to \lim_{\leftarrow j} H_{n+1}(X_j) \to ^sH_n(X) \to \check{H}_n(X) \to 0.$$ (2)

Milnor also showed that $^sH_*$ is the dual theory to Čech cohomology theory on compact metric spaces. Since $E_*$ bears the same relationship to cohomology $K$-theory on compact metric spaces, we were led to make precise the relation between $E_*$ and $^sH_*$.

An important tool is the spectral sequence provided by the following theorem.

**Theorem 1.** Let $X$ be compact metric of dimension $d < \infty$. Then there is a spectral sequence $\{E^p_{p,q}\}$ which converges to $E_*(X)$, is natural in $X$,
has $E^{d+1} = E^\infty$ and $E^2_{p,q} = \delta H_p(X; E_q(\text{point}))$.

For finite CW-complexes this spectral sequence is equivalent to the Atiyah-Hirzebruch spectral sequence.

If $X \subset R^3$ then $E_\ast(X)$ is determined by Steenrod homology. Precisely, $\text{Ext}(X) = \delta H_1(X)$ and there is an exact sequence $0 \to \delta H_0(X) \to E_0(X) \to \delta H_2(X) \to 0$. This is useful in studying the following question. Let $A_1$ and $A_2$ be essentially normal operators such that $\pi A_1$ and $\pi A_2$ commute. When do there exist compact perturbations $A_j = B_j + K_j$, $j = 1, 2$, with $B_1$ and $B_2$ commuting normals? If $A_2$ is selfadjoint then the obstruction to perturbation is an element of $\text{Ext}(X) = \delta H_1(X)$, where $X = \text{joint}(\pi A_1, \pi A_2) \subset R^3$. So, for example, if $\delta H_1(X) = 0$ then the $B_j$ exist. If $A_2$ is just normal then $X \subset R^4$ and the obstruction group $\text{Ext}(X)$ is an extension of $\delta H_1(X)$ by a certain subgroup of $\delta H_3(X)$. The applicability of higher dimensional computations to operator theory was first observed by BDF [4, p. 119].

In analogy to $K$-theory there is a Chern character useful in comparing $E_\ast$ with homology. This yields $\text{ch} \otimes Q: E_\ast(X) \otimes Q \to \delta H_\ast(X; Q)$ which is not always an isomorphism, in contrast to the cohomology $K$-theory situation.

**Theorem 2.** The following are equivalent:

(a) The differentials in $\{E^r_{p,q}\}$ are torsion-valued and $\text{ch} \otimes Q$ is an isomorphism.

(b) $\text{hom}(\delta H_\ast(X), Q/Z) \otimes Q = 0$.

Finally, an analog of (2) holds for $E_\ast$. If $X$ is the inverse limit of finite CW-complexes $X_j$, then there is a split exact sequence

$$0 \to \lim_{\leftarrow} K_0(X_j) \to \text{Ext}(X) \to \lim_{\leftarrow} K_1(X_j) \to 0$$

and thus $\text{Ext}(X)$ is completely determined by $K$-theory on finite complexes. Also, if $X$ and $Y$ have the same shape [6] then $E_\ast(X) \simeq E_\ast(Y)$.

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**REFERENCES**


