The Hatcher-Quinn Invariant And Differential Forms

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THE HATCHER-QUINN INVARIANT AND DIFFERENTIAL FORMS

by

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DISSERTATION

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Approved By:

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Advisor                   Date

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DEDICATION

To my wonderful daughter Ellie. You truly are the sun to me.
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Chapter 1 Introduction

1.1 Overview of the Program

In algebraic topology, intersection theory has been a fundamental tool with applications to fixed point theory and low dimensional topology. In the early 1970s, Hatcher and Quinn developed a bordism theoretic approach to intersection theory. In this context, we are given two submanifolds $P$ and $Q$ having a non-empty intersection in a third ambient manifold $M$. The basic problem was to find the obstructions to the existence of an isotopy of $P$ to a manifold whose image is disjoint from $Q$. Hatcher and Quinn defined an obstruction lying in a certain twisted normal bordism group. Specifically, assuming that $P$ and $Q$ are transverse in $M$, the obstruction is a twisted bordism class

$$
\chi^\text{HQ} \in \mathfrak{N}_{p+q-m}(E; \xi),
$$

where

- $E$ is the space of paths from a point of $P$ to a point of $Q$;
- $p, q$, and $m$ are the dimensions of $P, Q,$ and $M$, respectively;
- the virtual vector bundle $\xi$ models the stable normal bundle of the intersection manifold $D = P \cap Q$.

The obstruction $\chi^\text{HQ}$ may be defined as the bordism class of the map $D \to E$, together with its normal data. In what follows, we call $E$ the generalized path space. The main result of Hatcher and Quinn is:

**Theorem 1.1** (Hatcher-Quinn [6], Klein-Williams [7]). Assume $p + 2q, 2p + q \leq m - 3$. Then, the inclusion $P \subset M$ is isotopic to an embedding with image disjoint from $Q$ if and only if the Hatcher-Quinn invariant is trivial.
Remark 1.2. An important variant/generalization of the problem is the following: one is given a map \( f : P \to M \) and submanifold \( Q \subset M \). In this case, one wishes to find a homotopy of \( f \) to a map whose image misses \( Q \). Hatcher and Quinn show that their obstruction is both necessary and sufficient when \( p + 2q \leq m - 3 \). For this problem, the generalized path space \( E \) is given by the space of triples

\[(x, \lambda, y),\]

in which \( x \) is a point of \( P \), \( y \) is a point of \( Q \), and \( \lambda : [0, 1] \to M \) is a path from \( f(x) \) to \( y \).

Remark 1.3. Even more generally, one may also consider a pair of maps

\[ P \xrightarrow{f} M \xleftarrow{g} Q, \]

neither which are embeddings, and ask whether the product map \( f \times g : P \times Q \to M \times M \) can be deformed off of the diagonal inclusion \( M \subset M \times M \). If \( Q \) happens to be embedded, then the Hatcher-Quinn obstruction for the latter problem coincides with the one of Remark 1.2.

In this last case, the generalized path space \( E \) is defined as the space of triples

\[(x, \lambda, y)\]

in which \( x \) is a point of \( P \), \( y \) is a point of \( Q \), and \( \lambda : [0, 1] \to M \) is a path from \( f(x) \) to \( g(y) \).

For the rest of this dissertation, we do not assume that \( P \) or \( Q \) is embedded in \( M \).

Let \( \xi \mathbb{Z} \) denote the local coefficient system on \( E \) defined by the first Steifel-Whitney class of \( \xi \). Then one has a Hurewicz homomorphism

\[ H_{p+q-m}(E; \xi) \to H_{p+q-m}(E; \xi \mathbb{Z}), \]

which is an isomorphism if one disregards torsion. We now dispense with bordism altogether and work with (co-)homology instead.

Assume in what follows that \( \xi \) is oriented; for example, this will be the case if the
manifolds $P$, $Q$ and $M$ are oriented. Then modulo torsion, by means of the Kronecker pairing, we can think of the Hatcher-Quinn obstruction as a linear functional

$$\xi : H^{p+q-m}(E) \to \mathbb{R},$$

where $H^*(E)$ denotes the singular cohomology of $E$ with real coefficients.

If we redefine $E$ using piecewise smooth paths, then $E$ becomes a differentiable space in the sense of K. T. Chen [3]. In this setting, the functional $\xi$ is conjecturally induced by a co-closed linear functional

$$\chi : \Omega^{p+q-m}(E) \to \mathbb{R}$$

where $\Omega^*(E)$ denotes the complex of differential forms on $E$ in the sense of Chen.\footnote{We will instead be working with a slight variant of Chen's de Rham complex which reflects the fact that we are dealing with our notion of smooth space rather than Chen's notion of differentiable space. Every smooth space is canonically differentiable (see §Chapter 2).} The statement that $\chi$ is co-closed means that $\chi$ vanishes on exact forms, i.e., $\chi$ is a current.

1.2 Main results

This dissertation has three main results which we now outline. Our first result concerns finding a formula for $\chi$. Assume $M$ is oriented and comes equipped with a Riemannian metric. A tangential Thom form is a differential $m$-form $U$ on $M \times M$ such that

- $U$ is compactly supported on a sufficiently small tubular neighborhood of the diagonal $M \to M \times M$, and
- if $\pi : M \times M \to M$ is first factor projection, then $\pi_*(U) = 1$, where $\pi_*$ denotes integration along the fibers.

Equivalently, if we identify the tubular neighborhood with the total space of the tangent bundle $TM$, then $U$ represents the Thom class.
**Theorem A.** Given any finite family of Thom forms $U_1, \ldots, U_s$, there is an explicitly defined co-closed linear functional

$$\chi = \chi(U_1, \ldots, U_s) : \Omega^{p+q-m}(E) \to \mathbb{R}$$

that represents the Hatcher-Quinn invariant.

The relationship between the cohomology of the Chen complex $\Omega^*(E)$ and that of the singular cochain complex $S^*(E)$ remains obscure (cf. [8]). Rather than identifying the cohomology of the former, Chen produced a subcomplex of $\Omega^*(E)$ which computes the singular cohomology of $E$ when the manifold $M$ is 1-connected [4]. This subcomplex is quasi-isomorphic to the bar construction

$$(B^*(P; M; Q), d)$$

in which

$$B^k(P; M; Q) = \oplus_{r-s=k}\Omega^r(P \times M^s \times Q)$$

(actually, Chen preferred to work with a reduced version of the latter obtained by modding out by degeneracies).

Our second main result is an expression for the Hatcher-Quinn invariant on the bar complex.

**Theorem B.** Given any finite family of Thom forms $U_1, \ldots, U_s$, there is an explicitly defined co-closed linear functional

$$\bar{\chi} = \bar{\chi}(U_1, \ldots, U_s) : B^{p+q-m}(P; M; Q) \to \mathbb{R}$$

that represents the Hatcher-Quinn invariant.

One of the main tools in proving the above results is a model for the generalized path space $E$ that is inspired by Milnor’s model for the free path space of a Riemannian manifold [10].
Assume that $M$ has been equipped with a Riemannian metric and that the exponential map is a diffeomorphism onto its image for vectors of length less than or equal to a choice of positive real number $r$; in this case we say that $M$ is $r$-tempered. For a subset $T \subset (0, 1)$ of cardinality $s$ i.e., a subdivision $0 = t_0 < t_1 < \cdots < t_s < t_{s+1} = 1$ we define

$$E(T) \subset E$$

to be the space consisting of tuples

$$(x, \lambda, y)$$

such that

- $d(z_i, z_{i+1}) < r$, where $z_i = \lambda(t_i)$ and $d$ is the Riemannian distance;

- $f(x) = \lambda(0), g(y) = \lambda(1)$;

- $\lambda: [0, 1] \to M$ is a piecewise smooth path whose restriction to the subinterval $[t_i, t_{i+1}]$ is the unique minimal geodesic from $z_i$ to $z_{i+1}$.

If $T \subset T'$ is a refinement of subdivisions, then $E(T) \subset E(T')$. Hence,

$$T \mapsto E(T)$$

is a filtered diagram in the category of spaces.

Remark 1.4. In fact, $E(T)$ is an open manifold of dimension $p + ms + q$. One way to see this is to note that there is an open embedding $E(T) \to P \times M^{\times s} \times Q$ given by

$$(x, \lambda, y) \mapsto (x, z_1, \ldots, z_s, y).$$

Observe that this embedding only depends on the cardinality $s$ and is otherwise independent of $T$. Furthermore, by taking the image of $E(T)$ under this the embedding, we can conclude that $E(T)$ is diffeomorphic to the interior of a compact manifold with corners, whose corner
set is given by those points \((x, z_1, \ldots, z_s, y) \in P \times M^s \times Q\) such that \(d(z_i, z_{i+1}) = r\) for some \(i = 0, \ldots, s + 1\). It follows that the homology of \(E(T)\) is finitely generated.

Our third main result states that the spaces \(E(T)\) fit together to give a model for the homology of the generalized path space.

**Theorem C.** There is an isomorphism on integral homology

\[
\lim_{\overrightarrow{r}} H_*(E(T)) \cong H_*(E).
\]

In proving this last result it was useful to introduce a new concept associated with a fiber bundle \(\pi: E \to B\) with smooth base \(B\): a map \(U \to E\) is said fibered plot if \(U\) is a (possibly open) smooth manifold and the composition \(U \to E \to B\) is a smooth submersion. The category of fibered plots on \(\pi\) is a kind of “fibered diffeology.”
Chapter 2  Smooth Structure on the Path Space

2.1 Smooth spaces

We first introduce the notion of a smooth structure on a set by introducing the notion of plots. We then generalize this notion to the fibered setting. This is subsequently used as a tool to describe the singular homology $H_*(E)$ of the generalized path space $E$.

**Definition 2.1.** By a smooth structure on a set $E$ we mean a collection of functions

$$\phi : U \to E$$

called plots, in which $U$ is permitted to vary over all smooth manifolds without boundary, not necessarily compact. The plots are to satisfy the following axioms:

- If $f : U' \to U$ is a smooth map and $\phi : U \to E$ is a plot, then the composition $\phi \circ f : U' \to E$ is also a plot;

- If $\phi : U \to E$ is a function, and each $x \in U$ has an open neighborhood $U_x$ such that $\phi|_{U_x} : U_x \to E$ is a plot, then $\phi$ is a plot.

- Every function from a point to $E$ is a plot.

If a preferred smooth structure on $E$ is specified, then $E$ will be called a smooth space.

**Remark 2.2.** To avoid set theoretical difficulties, one may impose an additional condition on the domains of the plots: one may constrain them to be submanifolds of $\mathbb{R}^\infty$. We will not belabor this issue here, as it seems to be merely a matter of taste.

**Remark 2.3.** Our definition of smooth space may be regarded as a variant of the notion of diffeological space. The only difference is that a diffeological space requires the domains of the plots to be open subsets of some Euclidean space.
Let $E$ be a smooth space. The *smooth topology* declares a subset of $E$ to be open if and only if its preimage is open with respect to every plot of $E$. Note that any topological space has a natural smooth space structure: A function $\varphi : U \to E$ is a plot if and only if $U$ is a manifold and $\varphi$ is a continuous map. This gives us a pair of adjoint functors

$$L : \text{Smooth Spaces} \rightleftarrows \text{Topological Spaces} : R$$

in which $LRL = L$ and $RLR = R$. The adjoint functor condition shows that there are maps $M \to RL(M)$ and $LR(X) \to X$.

**Lemma 2.4.** If $X$ is a smooth manifold, then the map $LR(X) \to X$ is a homeomorphism.

**Proof.** The map $LR(X) \to X$ is the identity on underlying sets. Then a subset $O \subset LR(X)$ is open if and only if $\varphi^{-1}(O) \subset U$ is open for every continuous function $\varphi : U \to X$ with $U$ a manifold. Take $\varphi = \text{id} : X \to X$ to be the identity—this is a plot. Then, if $O \subset LR(X)$ is open, we see that $O$ is open in $X$. Hence, the identity function $X \to LR(X)$ is continuous. It follows that $LR(X) \to X$ is a homeomorphism.  

**Definition 2.5.** Suppose that a set $M$ is equipped with the structure of a topological space as well as the structure of a smooth space. To distinguish the two structures, let us write $M_t$ and $M_s$. If every plot $\varphi : U \to M_s$ is also a continuous function, we say that the structures are *compatible*. Equivalently, the structures are compatible if and only if the identity function $M_s \to RM_t$ defines a morphism of smooth spaces. By adjointness, this is the same as the condition that the identity $LM_s \to RM_t$ is a continuous map.

A *map* of smooth spaces $A \to B$ is a function which transfers plots in $A$ to plots in $B$. 
2.2 The de Rham complex of a smooth space

Given a smooth space $E$ we define a \textit{k-form} $\alpha$ to be, roughly speaking, a collection of compatible de Rham $k$-forms $\alpha_\phi \in \Omega^k(U)$, where $\phi: U \to E$ ranges over all plots. More precisely, the plots on $E$ form a filtered category $\text{plot}(E)$ and we set

$$\Omega^k(E) := \lim_{\leftarrow} \Omega^k(U).$$

where the inverse limit is taken in the category of vector spaces and is indexed over the opposite category $\text{plot}(E)^{\text{op}}$. Hence, the inverse limit is cofiltered.

The differential $d: \Omega^k(E) \to \Omega^{k+1}(E)$ is induced by the exterior derivative on the domains of the plots.

**Definition 2.6.** The \textit{de Rham complex} of the smooth space $E$ is the cochain complex $(\Omega^*(E), d)$.

**Remark 2.7.** The above gives rise to a contravariant functor on the category of smooth spaces and smooth maps. Furthermore, if $E$ is a smooth manifold with tautological smooth structure, then $\text{plot}(E)$ has $E$ as a preferred terminal object. It follows that the definition coincides with the usual de Rham complex.

**Remark 2.8.** The Chen de Rham complex is defined in a way similar to the above, but one only takes the inverse limit over plots whose domains are open convex subsets in Euclidean space. There is an obvious map from our de Rham complex to Chen’s, which we conjecture to be an isomorphism.

2.3 Fibered plots

In what follows we fix a map of smooth spaces

$$p : E \to X$$

such that
• $X$ is a smooth manifold with tautological smooth structure;

• $E$ is a topological space with smooth structure compatible with its topology;

• $p : E \to X$ is a fiber bundle.

**Definition 2.9.** A plot $\phi : U \to E$ is fibered if (and only if) the composition $p \circ \phi : U \to X$ is a smooth submersion. A morphism of fibered plots $\varphi : U \to E$ and $\psi : V \to E$ is a $C^\infty$ map $f : U \to V$ such that the following diagram commutes.

$$
\begin{array}{ccc}
U & \xrightarrow{f} & V \\
\phi \downarrow & & \downarrow \psi \\
E & & 
\end{array}
$$

The fibered plots form a category

$$
\text{fplot}(E).
$$

When there is no confusion, we will refer to a plot by its domain.

**Definition 2.10.** We say that a fibration $E \to X'$ has enough fibered plots if there is a functor $U_\bullet : I \to \text{fplot}(E)$ in which $I$ is a small, filtered indexing category such that the map

$$
\text{hocolim}_{\alpha \in I} : U_\alpha \to E
$$

is a homotopy equivalence, where the displayed homotopy colimit is taken in the category of spaces.

**Definition 2.11.** A system of enough fibered plots $U_\bullet$ is said to be hereditary if for every smooth map $X' \to X$ the base change $U'_\bullet$ defined by

$$
\begin{array}{rcl}
U_\alpha \times X' & \to & E \times X' \\
\downarrow & & \downarrow \\
\alpha \times X & \to & \alpha 
\end{array}
$$

is a system of enough fibered plots for the pullback fibration $E' \to X'$.
Lemma 2.12. If $M$ has enough fibered plots, then the homomorphism

$$\text{colim}_{\alpha \in I} : H_\bullet(U_\alpha) \to H_\bullet(M)$$

is an isomorphism.

Proof. This immediately follows from the fact that filtered homotopy colimits commute with homology (using e.g., [2, ch. XII§5.7] or [5, th. 18.3(a)] and the fact that the nerve of $I$ is contractible). \qed

Lemma 2.13. If $M$ is a smooth manifold, with its tautological smooth space structure, then there are enough plots.

Proof. Use the identity plot $M \to M$, where $I$ is the category with a single object and a single morphism. \qed

We now state the first main result of this thesis.

Theorem 2.14. The free path fibration $PM \to M$ is fiber bundle. Moreover, it admits a hereditary system of enough fibered plots $P_\bullet : I \to \text{fplot}(PM)$.

The first part of the Theorem 2.14 is due to Andrew Stacey and we reproduce his proof below. The second part is the result of several smaller results which appear below.

Lemma 2.15. Fibered plots are preserved under base change with respect to smooth maps.

Proof. Let $X' \to X$ be a smooth map of closed manifolds and let $E' = E \times_X X'$ and $U' = U \times_X X'$ be the fiber products. Then, $E' \to X'$ is a fibration.

First we will declare a smooth structure on $E'$. We say that a continuous map $V \to E'$ is a plot if and only if the composition $V \to E' \to X'$ is a plot.

Next, we need to show that $U' \to E'$ is a fibered plot. Let $F : X' \times U \to X \times X$ be the product map and let $\Delta : X \to X \times X$ be the diagonal. Since $U \to X$ is a submersion,
using a little linear algebra, we can show $F$ is transverse to $\Delta$. Hence, $U' = F^{-1}(\Delta(X))$ is a smooth manifold. Clearly, $U' \to E' \to X'$ is smooth. It remains to show that $U' \to X'$ is a submersion. To do this, recall the following: suppose $f : X \to Y$ is a map and $Z \subset Y$ is a submanifold transverse to $f$. Set $W = f^{-1}(Z)$. Then, for every $p \in W$ with $q = f(p)$ we have that the tangent space $T_pW$ is the kernel of the derivative $df_p : T_pX \to T_qY$. This gives us the following short exact sequence:

$$0 \to T_p W \to T_p X \to T_q Y / T_q Z \to 0.$$ 

We will apply this to the following situation: Consider the commutative pullback diagram

\[
\begin{array}{ccc}
U' & \longrightarrow & X \\
p \downarrow & & \downarrow \Delta \\
X' \times U & \overset{F}{\longrightarrow} & X \times X
\end{array}
\]

(2.2)

Note that $\Delta$ defines the diagonal submanifold and $F$ is transverse to it again because $U \to X$ is a submersion. It follows for $p = (u, y) \in U' = U \times X'$ there is a short exact sequence

$$0 \to T_p U \to T_u U \oplus T_y X' \to T_x X \to 0,$$

where $x$ is in the image of $y \in X'$ as well as in the image of $u \in U$ with respect to the maps $X' \to X$ and $U \to X$ (note: in the displayed diagram we have $T_x X$ identified with the quotient of $T_x X \oplus T_x X$ by the diagonal). It follows that the sequence

$$T_p U' \to T_y X' \to \text{cok}(T_u U \to T_x X) \to 0$$

is right exact. But the displayed cokernel is trivial since $U \to X$ is a submersion. Hence, $T_p U' \to T_y X'$ is onto, which is what we needed to prove. \qed

**Lemma 2.16.** If $U' \to E'$ is a base change of a fibered plot $U \to E$, then $U' \to U$ is proper.

**Proof.** Let $C \subset U$ be compact. Then, its preimage under $U' \to U$ is $C' := C \times_{X} X'$ which is a closed subspace of the compact Hausdorff space $C \times X'$. Hence it is compact. \qed
2.4 The free path space

For a closed smooth manifold $M$, the free path space $PM$ is the space of continuous, piecewise smooth paths $[0, 1] \to M$. This has a smooth structure in which a plot is a continuous map $\phi : U \to PM$ such that the adjoint $\hat{\phi} : U \times [0, 1] \to M$ is piecewise smooth in the sense that there is a subdivision $0 = a_0 < a_1 \cdots < a_k = 1$ such that the restriction $\hat{\phi} : U \times [a_j, a_{j+1}] \to M$ is a smooth map for $j = 0, \ldots, k - 1$.

Let $p : PM \to M \times M$ be the map which assigns to a path its endpoints. It is not hard to show that $p$ is a Hurewicz fibration. The following, which gives the first part of Theorem 2.14 was communicated to us by Andrew Stacey.

**Proposition 2.17** (Stacey [13]). The endpoint projection $p : PM \to M \times M$ is a fiber bundle.

**Proof.** For $A \subset M \times M$, let $P_A M$ be the set of all piecewise smooth paths $\gamma \in PM$ such that $p(\gamma) \in A$, i.e.

$$P_A M = A \times_{M} PM.$$ 

Let $(x, y) \in M \times M$. Then there are neighborhoods $U$ of $x$ and $V$ of $y$ along with families of compactly supported diffeomorphisms $\psi : \mathbb{R} \times U \to \text{Diff}(M)$ and $\phi : \mathbb{R} \times V \to \text{Diff}(M)$ such that $\psi_{0,p}$ is the identity on $M$ an $\psi_{1,p}(p) = x$, and similarly for $\phi$ and $y$. Now, choose smooth bump functions $\sigma : I \to \mathbb{R}$ and $\tau : I \to \mathbb{R}$, with disjoint supports, such that $\sigma(0) = 1$ and $\tau(1) = 1$. Let $W = U \times V \subset M \times M$. Then, $W$ is an open neighborhood of $(x, y)$. Define a map

$$\Psi : P_W M \to P_{(x,y)}(M) \times W$$

(where $P_{(x,y)}(M)$ is the fiber over $(x, y)$) by

$$\Psi(\gamma)(t) = (\psi_{\sigma(t),\gamma(0)} \circ \phi_{\tau(t),\gamma(1)}(\gamma(t)), (\gamma(0), \gamma(1)))$$.
Then we have

\[
\Psi(\gamma)(0) = \left( \psi_{\sigma(0), \gamma(0)} \circ \phi_{r(0), \gamma(1)}(\gamma(0)), (\gamma(0), \gamma(1)) \right)
\]

\[
= \left( \psi_{1, \gamma(0)} \circ \phi_{0, \gamma(1)}(\gamma(0)), (\gamma(0), \gamma(1)) \right)
\]

\[
= \left( \psi_{1, \gamma(0)} \circ \text{Id}(\gamma(0)), (\gamma(0), \gamma(1)) \right)
\]

\[
= \left( \psi_{1, \gamma(0)}(\gamma(0)), (\gamma(0), \gamma(1)) \right)
\]

\[
= (x, (\gamma(0), \gamma(1)))
\]

Similarly,

\[
\Psi(\gamma)(1) = (y, (\gamma(0), \gamma(1))).
\]

The inverse of this map \( P_{(x,y)}M \times W \) is:

\[
\Psi^{-1}(\gamma, (p, q))(t) = \left( \phi_{\tau(t),p}^{-1} \circ \psi_{\sigma(t),q}^{-1} \right)(\gamma(t)).
\]

This establishes local triviality. \( \square \)

### 2.5 Generalized paths again

Fix a smooth map \( X \to M \times M \), where \( X \) is a closed manifold.\(^2\) We set

\[
E = X \times_{M \times M} PM
\]

Then \( E \) is a piecewise smooth version of the generalized path space. By the proposition above the projection \( E \to X \) is a fiber bundle. Furthermore, \( E \) inherits a smooth structure from \( PM \).

\(^2\)In applications \( X \) will usually be the product \( P \times Q \).
2.5.1 Discussion

Suppose that there is a system of enough fibered plots $U_\bullet : \rightarrow fplot(E)$ such that each $U_n \rightarrow X$ is proper for $n \in I$. Then, by Ehresmann’s theorem, we infer that $U_n \rightarrow X$ is also a fiber bundle. As homotopy colimits commute with base changes of fibrations, it follows that $U_\bullet$ is hereditary. We show below that there is a system of enough fibered plots for the bundle $PM \rightarrow M \times M$. However, the hereditary system we are able to construct is such that the maps $U_n \rightarrow M \times M$ are not proper. Hence, in order to obtain a model for the homology of $E$, we will have to argue differently.

Fix a Riemannian metric on $M$. For $r > 0$, let $TM(r) \subset TM$ denote the open disk bundle of tangent vectors having length less than $r$.

**Definition 2.18.** By an exponential radius we mean an $r$ such that the exponential map $TM(r) \rightarrow M \times M$ is defined and is a diffeomorphism onto its image. We say that $M$ is $r$-tempered if an exponential radius $r$ has been fixed.

Henceforth, we assume that $M$ is $r$-tempered.

Let $J$ be the poset whose objects are finite subsets $T \subset (0,1)$ with order relation $T < T'$ if and only if $T \subset T'$. Note that $T \subset (0,1)$ inherits a total ordering from $(0,1)$ and defines a subdivision $0 < t_1 < \ldots < t_s < 1$ where $T = \{t_1, \ldots, t_s\}$ and by convention we set $t_0 := 0$ and $t_{s+1} := 1$. Let

$$P(T) \subset PM$$

be the subspace of piecewise smooth paths $\gamma : [0,1] \rightarrow M$ such that

- There is a unique minimal geodesic connecting the points $\gamma(t_{i-1})$ and $\gamma(t_i)$ and having length $< r$;
\* \( \gamma : [t_{i-1}, t_i] \to M \) is a unique minimal geodesic for \( i = 1, 2, \ldots, s \).

Note that for \( T \subseteq T' \) we have \( P(T) \subseteq P(T') \). Consider the base change to \( X \), i.e.,

\[
E(T) := X \times_{M \times M} P(T).
\]

Then a point of \( E(T) \) is a pair \((z, \lambda) \in X \times P(T)\) such that

\* \( z \in X \);

\* \( d(\gamma(t_i, \gamma(t_{i+1}))) < r \) for \( i = 0, \ldots, s \);

\* \( \gamma : [0, 1] \to M \) is a piecewise smooth path in which \( \gamma|[t_{i-1}, t_i] \) is a unique minimal geodesic;

\* \( x = \gamma(0), y = \gamma(1), \) where \((x, y) \in M \times M\) is the image of \( z \).

Let \( f : PM \to [0, \infty) \) be the energy functional defined by

\[
f(\gamma) = \frac{1}{2} \int_0^1 \|\gamma'(t)\|^2 dt.
\]

For \( c \in [0, \infty) \), let

\[
P^c M := f^{-1}[0, c)
\]

be the subspace of piecewise smooth paths whose energy is less than \( c \). Similarly, we define

\[
E^c(T) = P^c M \cap E(T).
\]

Lastly, for points \( x, y \in M \), let \( E^c_{x,y}(T) \) and \( P^c_{x,y} M \) denote the subspaces of such paths from \( x \) to \( y \).

**Theorem 2.19** (Milnor [10, §16]). For \( T \) sufficiently fine, we have that the inclusion

\[
P^c_{x,y}(T) \to P^c_{x,y} M
\]
is a deformation retract. Furthermore, $P_{x,y}^c(T)$ has the structure of a smooth finite dimensional manifold without boundary.

Proof. A retraction is defined by mapping a path $\gamma$ to a piecewise smooth path obtained by gluing together the unique geodesic that joins each pair $(\gamma(t_i), \gamma(t_{i+1}))$. If $T$ is sufficiently fine, then the energy bound guarantees that there is a unique such geodesic (here we use the compactness of $M$ and the Hopf-Rinow Theorem.) Milnor defines an elementary homotopy showing the retraction to be a deformation retraction. □

Next, consider the base changes

$$E^c(T) := X \times_{M \times M} P^c T$$

and

$$E := X \times_{M \times M} P^c M.$$

Corollary 2.20. If $T$ is sufficiently fine, then the inclusion $E^c(T) \to E^c$ is a deformation retract. Furthermore, $E^c(T)$ is a finite dimensional manifold.

Proof. Milnor’s deformation retraction $P_{x,y}^c M \to P_{x,y}^c(T)$ is continuous with respect to $(x, y)$. But this is identified with the map of fibers $E_z^c \to E_z^c(T)$, where the image of $z \in X$ in $M \times M$ is $(x, y)$. We can see this as a fiberwise deformation retract $E^c \to E^c(T)$. Note: for our applications, we are using the compactness of $X$ in order to get the required fineness for the subdivision of $T$. The manifold structure comes from the fact that $E^c(T)$ can be regarded as an open subset of $X \times M^s$ where $|T| = s$. □

From now on, we will only consider the case where $c$ is a non-negative integer. Define a filtered poset $I$ whose objects are pairs $(T, i)$ for $T \in I$ and $i \in \mathbb{N}$. The order relation is defined as follows: $(T, i) \leq (S, j)$ if and only if $T \subset S$ and $i \leq j$. This gives rise to a functor

$$(T, i) \mapsto E^i(T).$$

Lemma 2.21. The map

$$\text{hocolim}_{(T, i) \in I} : E^u(T) \to E$$
is a weak homotopy equivalence.

Proof. We have that \( I = J \times \mathbb{N} \). So, it is enough to show that the map

\[
\text{hocolim}_{i \in J} \text{hocolim}_{T \in I} : E^n(T) \to E
\]

is a weak homotopy equivalence. By Corollary 2.20, the map \( E^n(T) \to E^n \) is a homotopy equivalence for \( T \) sufficiently fine. By homotopy invariance of the homotopy colimit, we infer that

\[
\text{hocolim}_{T \in I} : E^n(T) \to E^n
\]

is a homotopy equivalence. Applying the homotopy colimit over all \( n \) and using the homotopy invariance once again gives the result. \( \square \)

**Proposition 2.22.** The bundle \( E \to X \) has enough fibered plots.

Proof. It will suffice to show that the maps \( E(T) \to E \) as \( T \in J \) varies equips \( E \) with enough fibered plots. To do this, we first note that the map \( E(T) \to X \) is a submersion since it factors as a projection

\[
E(T) \subset X \times M^s \to X,
\]

where the first map is the inclusion and the second map is a projection. Since the second map is a submersion, the map itself is also one. To finish the proof, we will show that the map

\[
\text{hocolim}_{T \in I} : E(T) \to E
\]

is a weak homotopy equivalence. First note that the map

\[
\text{hocolim}_{n} E^n(T) \to \text{colim}_{n} E^n(T) := E(T)
\]

is a weak equivalence, since for each \( T \) the system \( E^n(T) \to E^{n+1}(T) \) is a directed system
of closed cofibrations. By the previous lemma, the map
\[
\operatorname{hocolim}_{T \in I} E(T) \simeq \operatorname{hocolim}_{T \in I} E^n(T) \to E
\]
is a weak equivalence. \qed

The following result completes the proof of Theorem 2.14:

**Corollary 2.23.** The functor \( T \mapsto P(T) \) is a hereditary system of enough fibered plots for \( PM \to M \times M \).

**Proof.** This follows from the definition of \( E(T) \) as the base change
\[
E(T) := X \times_{M \times M} P(T)
\]
and the fact that \( T \mapsto E(T) \) was shown to define a system of enough fibered plots. \qed
Chapter 3  Hatcher-Quinn Forms

3.1  Diagonal forms

Let $\Delta : M \to M \times M$ denote the diagonal map. Recall that $M$ is a closed Riemannian manifold equipped with volume form. We also assume that $M$ is $r$-tempered.

**Definition 3.1.** A diagonal form is a differential $m$-form $\delta_M \in \Omega^m(M \times M)$ such that

$$\int_M \Delta^* \alpha = \int_{M \times M} \alpha \wedge \delta_M$$

for all closed $m$-forms $\alpha$ on $M \times M$. In other words $\delta_M$ is any form that represents the Poincaré dual to the diagonal.

**Remark 3.2.** Clearly, a diagonal form exists and any two diagonal forms differ by an exact form. Hence, the affine space of diagonal forms is a torsor over the exact $m$-forms.

3.2  Diagonal forms from Thom forms

Recall the fibered plots $P(T)$ for $T \subset (0, 1)$. In particular, $P(\emptyset)$ is a tubular neighborhood of the diagonal $\Delta : M \to M \times M$.

**Definition 3.3.** A (tangential) Thom form is a differential form $U \in \Omega^m(P(\emptyset))$ such that

- $U$ is closed;
- $U$ is compactly supported;
- if $\pi : P(\emptyset) \to M$ is first factor projection, then

$$\pi_*(U) = 1,$$

where $\pi_*$ denotes integration along the fibers.
Remark 3.4. Compare [1, rem. 6.17.1]. Note that the evident embedding $e: M \to P(\emptyset)$ is Poincaré dual to $U$ in the sense that

$$\int_M e^*\alpha = \int_{M(\emptyset)} \alpha \wedge U$$

where $\alpha$ ranges over all $m$-forms on $P(\emptyset)$.

Definition 3.5. Let $\iota: P(\emptyset) \to M \times M$ be the inclusion. Given a Thom form $U$, its associated diagonal form is

$$\iota_!(U) \in \Omega^m(M \times M),$$

where $\iota_!$ is extension by zero. (It is straightforward to check that $\iota_!(U)$ is a diagonal form.)

3.3 An ‘explicit’ Thom form

As above, let $M$ be a closed, oriented Riemannian manifold $M$ of dimension $m$. Then Mathai and Quillen [9] provide an explicit construction of a “Gaussian shaped” closed differential form $\omega$ representing the Thom class of the tangent bundle $TM$. This differential form has the following properties:

1. $\omega$ is rapidly decaying along the fibers of $TM$;

2. $p_*(\omega) = 1$, where $p: TM \to M$ is the bundle projection, and $p_*$ denotes integration along the fibers.

Remark 3.6. The Thom form $\omega$ is universal in the sense that $\omega = \bar{\omega}(\tilde{\omega})$, where $\tilde{\omega} \in \Omega^m_{SO(n)}(\mathbb{R}^n)$ is an equivariant Gaussian-shaped differential form and $\bar{\omega}: \Omega^m_{SO(n)}(\mathbb{R}^n) \to \Omega^m(TM)$ is the Chern-Weil homomorphism. For an explicit expression for $\bar{\omega}$, see [9, eqn. 6.2].

The form $\omega$ is not a Thom form in the above sense. We will show how an elementary modification of $\omega$ produces a form $U$ satisfying Definition 3.3.
Assume $M$ is $r$-tempered and let $\epsilon = r/2$. Then we have a fiberwise map

$$h: TM \to TM, \quad h(y) = \frac{\epsilon y}{(1 + |y|^2)^{1/2}},$$

which is a fiberwise diffeomorphism of $TM$ onto $TM(\epsilon)$, where the latter is the open $\epsilon$-disk bundle of $TM$. Let $\Omega^m_\epsilon(TM)$ be the space of $m$-forms on $TM$ which are supported in the closed $\epsilon$-disk bundle, and note that $\Omega^m_\epsilon(TM) = \Omega^m_\epsilon(TM(r))$. Then $h$ induces an isomorphism

$$h^*: \Omega^m_\epsilon(TM) \xrightarrow{\cong} \Omega^m_{rd}(TM),$$

where the target consists of rapidly decaying $m$-forms. Let $\bar{\omega} \in \Omega^m_\epsilon(TM(r))$ be the unique form satisfying $h^*(\bar{\omega}) = \omega$. Then $\bar{\omega}$ is closed and compactly supported. Moreover, as integration is invariant with respect to orientation preserving diffeomorphisms, we infer that $p_*(\bar{\omega}) = 1$.

Lastly, let $\exp: TM(r) \xrightarrow{\cong} P(\emptyset)$ be the diffeomorphism defined by the exponential map. Set

$$U := (\exp^{-1})^*\bar{\omega} \in \Omega^m_{cs}(P(\emptyset)).$$

Then $U$ is a Thom form in the sense of Definition 3.3.

### 3.4 Construction of Hatcher-Quinn forms

Consider the Cartesian product $M^{s+2}$. Then, for $0 \leq i < j \leq s + 1$, let

$$p_{i,j}: M^{s+2} \to M \times M$$

be the projection to the $i$ and $j$ factors. Let $U_0, \ldots, U_s \in \Omega^m(M \times M)$ be a collection of Thom forms with associated diagonal forms

$$\delta_i = \iota_i(U_i) \quad i = 0, \ldots, s.$$

---

3 As Quillen and Mathai remark, one may use this isomorphism as a definition of the space of rapidly decaying $m$-forms.
Consider the closed form
\[ \lambda := \lambda(\delta_0, \ldots, \delta_s) := p_{0,1}^* \delta_0 \wedge \ldots \wedge p_{s,s+1}^* \delta_s. \]

The pullback of \( \lambda \) along the map \( f \times \text{id} \times g : P \times M^s \times Q \to M^{s+2} \) defines an \( m(s+1) \)-form
\[ \omega_s := \omega(\delta_0, \ldots, \delta_s) := (f \times \text{id} \times g)^* \lambda(\delta_0, \ldots, \delta_s) \in \Omega^{m(s+1)}(P \times M^s \times Q). \]

**Definition 3.7.** The form \( \omega_s \) is said to be the *Hatcher-Quinn form* associated with the Thom forms \( U_1, \ldots, U_s \).

**Remark 3.8.** Set \( X = P \times Q \). Let \( T \subset (0,1) \) be a subdivision of size \( s \), and recall that \( E(T) = X \times_{M \times M} P(T) \). Then there is a canonical open embedding \( E(T) \to P \times M^s \times Q \) whose image is independent of the choice of \( T \). Identifying \( E(T) \) with its image in \( P \times M^s \times Q \), it is easy to see that \( \omega_s \) is compactly supported inside \( E(T) \). For this reason, we sometimes abuse notation and write \( \omega_s \in \Omega^{m(s+1)}_{cs}(E(T)) \).

The following result justifies the terminology.

**Proposition 3.9.** The form \( \omega_s \) is Poincaré dual to the transversal intersection of the map \( f \times \text{id} \times g : P \times M^s \times Q \to M^{s+2} \) with the diagonal \( M \to M^{s+2} \).

**Proof.** We will only establish the case when \( s = 0 \) as the other cases are similar. Set \( \omega := \omega_0 \). We can assume that \( f \) and \( g \) are transverse; this is equivalent to the assumption that \( f \times g \) is transverse to the diagonal. Consider first the case when \( f \) and \( g \) are embeddings making \( P \times Q \) into a submanifold of \( M \times M \). Let \( \mathcal{D} \) be the intersection of \( P \) and \( Q \) in \( M \). Identify De Rham cohomology with singular cohomology using the Stokes map. We need to show that the Poincaré Duality isomorphism
\[ \cap[P \times Q] : H^m(P \times Q) \to H_{p+q-m}(P \times Q) \]
maps \( \omega \) to \( \iota_*(\mathcal{D}) \). Let \( \xi \) denote the normal bundle of \( P \times Q \) in \( M \times M \). Let \( (D(\xi), S(\xi)) \) be the disk bundle and sphere bundle of \( \xi \), respectively. This is a compact, oriented manifold
with boundary of dimension $2m$. Let its fundamental class be denoted by $\mu_\xi$ and let the fundamental class of $M \times M$ be denoted by $\mu_{M \times M}$. Let the inclusion $D(\xi) \subset M \times M$ be denoted by $j$. Note that $H_*(D(\xi), S(\xi)) \cong \tilde{H}_*((P \times Q)^\xi)$, where $(P \times Q)^\xi = D(\xi)/S(\xi)$ is the Thom space and $\tilde{H}_*$ refers to reduced homology. Then one has a commutative diagram

$$
\begin{array}{ccc}
H^m(M \times M) & \xrightarrow{j^*} & H^m(D(\xi)) \\
\cap_{\mu_{M \times M}} \cong & \cap_{\mu_\xi} \cong & \cong \cap_{\mu_{P \times Q}} \\
H_m(M \times M) & \xrightarrow{\Delta_*} & \tilde{H}_m((P \times Q)^\xi) \cong H_{p+q-m}(P \times Q)
\end{array}
$$

where the vertical arrows are the isomorphisms given by Poincare duality. The composite along the top row is $(f \times g)^*$, and the right map on the top row is induced by the zero section of $\xi$. The left bottom map is induced by the Thom construction. The second bottom map is given by the Thom isomorphism (given by $\cap U_\xi$, where $U_\xi \in \tilde{H}_*((P \times Q)^\xi)$, is the Thom class). The left square commutes because the Thom collapse $M \times M \to (P \times Q)^\xi$ is of degree one on $2m$-dimensional homology. The right square commutes for the following reason: the diagonal map

$$
P \times Q \to (P \times Q) \times (P \times Q)
$$

has the property that the pullback of the bundle $1 \times \xi$ coincides with $\xi$; the Thomification of this will give the diagonal map

$$
(P \times Q)^\xi \to (P \times Q)_+ \wedge (P \times Q)^\xi.
$$

The result will then follow from this observation together with the fact that the Thom isomorphism $H_{2m}(D(\xi), S(\xi)) \cong H_{p+q}(P \times Q)$ equates $\mu_\xi$ with $\mu_{P \times Q}$. The image $\Delta_* (\mu_M)$ with respect to the left bottom map of the diagram is given by the cycle determined by the inclusion $i^\xi : D^\xi \to (P \times Q)^\xi$ (note that $H_m(D^\xi) = H_{p+q-m}(D)$). This follows from the
commutativity of the square

\[
\begin{array}{ccc}
M & \longrightarrow & D^\xi \\
\Delta \downarrow & & \downarrow \xi \\
M \times M & \longrightarrow & (P \times Q)^\xi
\end{array}
\] (3.1)

where the horizontal maps are the Thom constructions. Let us denote this class by \((i^\xi)_*(\mu_D^\xi)\).

If we apply the second map of the bottom row of diagram (3.1) to this class, we obtain the class \(i_*(\mu_D) \in H_{p+q-m}(P \times Q)\). This is because the Thomification of the cycle \(D \to P \times Q\) gives the cycle \(D^\xi \to (P \times Q)^\xi\). According to Milnor and Stasheff [11], \(\delta_M/\mu_M = 1\); This is the same thing as writing

\[
\Delta_* (\mu_M) \cap \mu_{P \times Q} = i_*(\mu_D).
\]

This handles the case when \(f\) and \(g\) are embeddings. In the general case, we can choose an embedding \(e : P \times Q \to \mathbb{R}^k\) for some large \(k\) and construct the embedding

\[
(f \times g, e) : P \times Q \to M \times M \times \mathbb{R}^k.
\]

Then we can use a similar argument as above but keeping in mind that \(\Delta : M \to M \times M\) should be replaced by \(\Delta \times \text{id} : M \times \mathbb{R}^k \to M \times M \times \mathbb{R}^k\). \qed
Chapter 4  The Cobar Construction

4.1 Cosimplicial Spaces

Let $\Delta$ denote the category of finite ordered sets and order preserving maps. Up to unique isomorphism, an object of $\Delta$ is of the form $[k] = \{0 < 1 < \cdots < k\}$ for some $k \geq 0$.

**Definition 4.1.** A cosimplicial space is a covariant functor

$$Y : \Delta \to \text{Top}$$

A map of cosimplicial spaces is a natural transformation of such functors.

A standard reference for cosimplicial spaces is [2].

**Example 4.2.** The assignment $[n] \mapsto \Delta^n$, where $\Delta^n$ is the standard topological $n$-simplex, defines a cosimplicial space. We denote it by $\Delta^\ast$.

**Definition 4.3.** The totalization of a cosimplicial space $Y$, is the topological space

$$\text{tot}(Y) := \text{hom}(\Delta^\ast, Y)$$

which is topologized as a subspace of the Cartesian product $\prod_n \text{hom}(\Delta^n, Y_n)$, where the mapping space $\text{hom}(\Delta^n, Y_n)$ is equipped with the compact open topology.

**Example 4.4.** If $X$ is a topological space, then $X \times \Delta^\ast$ is a cosimplicial space, with $X \times \Delta^n$ in cosimplicial degree $n$. There is a canonical adjunction

$$\text{hom}(X \times \Delta^\ast, Y) = \text{hom}(X, \text{tot}(Y)) .$$

**Remark 4.5.** A cosimplicial space $Y$, is equivalently described as a collection of spaces $Y_n$ together with co-face maps $d_i : Y_{n-1} \to Y_n$ for $i = 0, \ldots, n$ and co-degeneracy maps $s_i : Y_n \to Y_{n-1}$ for $i = 0, \ldots, n - 1$ satisfying the standard identities [12, 14.5.3].
4.2 Geometric cobar construction

Recall that a simplicial set is a contravariant functor $X : \Delta \to \text{Sets}$. The simplicial standard $k$-simplex is the simplicial set $\Delta[k]$, defined by $[n] \mapsto \text{hom}([n], [k])$. Given a simplicial set $X$, and a topological space $Z$, one has a cosimplicial space $\text{hom}(X, Z)$ which is given by

$$[n] \mapsto \text{hom}(X_n, Z),$$

where $\text{hom}(X_n, Z)$ is the function space of maps $X_n \to Z$.

With respect to the diagram of smooth maps

$$P \to M \leftarrow Q,$$

the geometric cobar construction is the cosimplicial space $E := E(P; M; Q)$ given by

$$[n] \mapsto P \times M^n \times Q.$$

More precisely, if we think $M \times M$ and $P \times Q$ as cosimplicial spaces which are constant in each degree, then there is a diagram of cosimplicial spaces

$$\text{hom}(\Delta[1], M) \to M \times M \leftarrow P \times Q$$

in which the arrow on the left is given by restriction along $\Delta[0] \times \Delta[0] \subset \Delta[1]$.

We then define $E$, as the pullback of this diagram (note that in degree $n$, one has $\text{hom}(\Delta[1], M) = M^{n+2}$ and the pullback in this degree is identified with $P \times M^n \times Q$). As taking pullbacks preserves functoriality, it follows that the construction describes a cosimplicial space.

Remark 4.6. In fact, $E$ possesses additional structure: it is a cosimplicial object in the category of manifolds and smooth maps.

Remark 4.7. The co-degeneracy maps of $E$ are given by various projections and the co-face maps are given by various diagonals.
4.3 The cobar evaluation map

Let \( \Delta^r \) denote the standard geometric \( r \)-simplex consisting of sequences of real numbers \( 0 = t_0 \leq t_1 \leq \cdots \leq t_r \leq t_{r+1} = 1 \). In what follows, such a sequence will be denoted as \( t_\bullet \).

For all \( r \geq 0 \) there is a canonical evaluation map

\[
e : E \times \Delta^r \to P \times M^{\times r} \times Q
\]

defined by \( e((x, \lambda, y), t_\bullet) = (x, \lambda(t_1), \ldots, \lambda(t_r), y) \). If we vary \( r \), these maps fit together to give a map of cosimplicial spaces

\[
e : E \times \Delta^\cdot \to E_.
\]  

(4.2)

as well as its associated adjoint

\[
\hat{e} : E \to \text{tot}(E.).
\]  

(4.3)

The next result shows that \( E_\cdot \) is a cosimplicial model for the generalized path space \( E \).

**Proposition 4.8.** The map of cosimplicial spaces \( e \) (cf. (4.2)) induces a homotopy equivalence on totalizations. Furthermore, the map of spaces \( \hat{e} \) (cf. (4.3)) is a homotopy equivalence.

**Proof.** The map \( e \) factors as

\[
E \times \Delta^\cdot \xrightarrow{\text{id} \times \hat{e}} \text{hom}(\Delta^\cdot, E_\cdot) \times \Delta^\cdot, \to E,
\]

where the first displayed map is the product of the adjoint to \( e \) with the identity map of \( \Delta^\cdot \), and the second displayed map is given by evaluation. The second of these maps is a degreewise homotopy equivalence and so it is one after totalizing. Moreover \( \Delta^\cdot \) has contractible totalization. Consequently, the first assertion will follow from the second one.

To prove the second assertion, first note that general case reduces to the special case when \( P = M = Q \) by taking a base change along \( P \times Q \to M \times M \). In the special case \( E_\cdot \) is the cosimplicial space which is \( \text{hom}(\Delta[1], M) \), and \( E \) is identified with the free path space \( PM \).
Both are identified up to homotopy with $M$: in the first case a map $\text{hom}(\Delta[1], M) \to M$ is defined using restriction to $\Delta[0] \subset \Delta[1]$, and in the second case the identification $PM \to M$ is given by evaluating at the initial point of a path. The assertion follows as both of these identifications are compatible. \hfill \square

### 4.4 The Hatcher-Quinn functional

Recall that for a given subdivision $T \subset (0, 1)$ of size $s$, the map

$$E(T) \to P \times M^s \times Q$$

defined by evaluating paths at the points of $T$ is an open embedding. In what follows, we identify $E(T)$ with its image inside $P \times M^s \times Q$.

As mentioned above, every Hatcher-Quinn form $\omega_s \in \Omega^{m(s+1)}(P \times M^s \times Q)$ is compactly supported inside the image of $E(T)$ in $P \times M^s \times Q$ (cf. Remark 3.8). Hence, by slight notational abuse, we may regard $\omega_s$ as an element of $\Omega^{m(s+1)}(E(T))$. Also recall that $\omega_s$ depends on a preferred selection of Thom forms $U_1, \ldots, U_s$.

**Definition 4.9.** The *Hatcher-Quinn functional* for the generalized path space $E$ is the composition

$$\chi : \Omega^{p+q-m}(E) \to \Omega^{p+q-m}(E(T)) \overset{\wedge \omega_s}{\longrightarrow} \Omega^{p+q+ms}_{cs}(E(T)) \overset{\int_{E(T)}}{\longrightarrow} \mathbb{R}$$

The definition of $\chi$ depends on the subdivision $T$ of size $s$ and Thom forms and $U_1, \ldots, U_s$. By the integration-by-parts formula, $\chi$ is a co-closed linear functional, in the sense that $\chi(d\alpha) = 0$ for all $\alpha \in \Omega^{p+q-m-1}(E)$.

### 4.5 The linear functional $\bar{\chi}$

Using the evaluation map

$$E \times \Delta^r \to P \times M^x \times Q$$

we may form the composition
\[
\bar{\chi} : \Omega^{p+q-m+r}(P \times M^{\times r} \times Q) \xrightarrow{\epsilon^*_T} \Omega^{p+q-m+r}(E \times \Delta^r) \xrightarrow{\int_{\Delta^r}} \Omega^{p+q-m}(E) \xrightarrow{\chi} \mathbb{R}
\]
which is given by the explicit formula
\[
\bar{\chi}(\alpha) := \int_{E(T)} \int_{\Delta^r} e^*_T \alpha \wedge \omega_n.
\] (4.4)

4.5.1 An alternative description of $\bar{\chi}$

Here we provide an alternative description of $\bar{\chi}$ that arises from writing the standard simplex as a union of prisms. We include this description for the sake of completeness. It will not be used in this dissertation, but it will be used in future applications.

Let $S \subset (0,1)$ be a subset of cardinality $r$, say $S = s_1 < \ldots < s_r$. Then, in barycentric coordinates, $S$ is associated with a point in the interior of the $r$-simplex $\Delta^r$. Assume further that $S$ is disjoint from $T$. Then, we have a map of totally ordered sets:
\[
\sigma : S \rightarrow T_+,
\]
where $T_+ = T \sqcup 0 = T \sqcup t_0$ which is given by $\sigma(x) = t_i$ when $t_i < x < t_{i+1}$. This, in turn, canonically determines an $(r-1)$-simplex
\[
[\sigma] : [r-1] \rightarrow [n]
\]
of the simplicial set $\Delta[n]$. Conversely, such a simplex, together with choices of $S$ and $T$ determines $\sigma$ uniquely. In addition, by way of $\sigma$, we have a decomposition of $S$ into totally ordered sets
\[
S = S_0 \sqcup S_1 \sqcup S_2 \sqcup \ldots \sqcup S_n,
\]
where $S_i = \sigma^i(i)$. Set $r_i = |S_i|$.

If $J$ is a closed interval, we denote $\Delta^s(J)$ as the simplex of order preserving functions
\[ [s-1] \to J; \] (so when \( J = [0,1] \) we get the standard simplex \( \Delta^s \)). Then \( \sigma \) gives rise to a map

\[
\psi_\sigma : \Delta_\sigma := \Delta^r_0(I_0) \times \Delta^r_1(I_1) \times \ldots \times \Delta^r_n(I_n) \to \Delta^r,
\]

where \( I_j = (t_j, t_{j+1}) \) and \( r = \sum r_i \). We then, consider the restricted map:

\[
e_\sigma : E(T) \times \Delta_\sigma \xrightarrow{\psi_\sigma} E(T) \times \Delta^r \hookrightarrow E \times \Delta^r \xrightarrow{e_s} P \times M^{\times r} \times Q.
\]

Then

\[
\bar{\chi}(\alpha) = \sum_{\sigma : S \to T_+} \int_{E(T)} \int_{\Delta_\sigma} e_\sigma^* \alpha \wedge \omega_n ,
\]

where \( T \) is fixed, \( S \subset (0,1) \) varies over all subsets of cardinality \( r \) and \( \sigma : S \to T_+ \) is an order preserving function.

The above construction may be simplified as follows. For an \((r-1)\)-simplex \( u : [r-1] \to [n] \) we write

\[
\Delta_u := \Delta^r_0 \times \ldots \times \Delta^r_n
\]

which is a product of standard simplices and \( r_i \) is the cardinality of \( u^{-1}(i) \). Then there is an evident concatenation map

\[
\Delta_u \to \Delta^r.
\]

Additionally, given \( S \) and \( T \) as above, we have a linear change of variables

\[
\Delta_{i[\sigma]} \xrightarrow{\cong} \Delta_\sigma
\]

where the domain is a product of standard simplices and the codomain is a product of non-standard simplices. Then

\[
\bar{\chi}(\alpha) = \sum_u \int_{E(T)} \int_{\Delta_u} e_u^* \alpha \wedge \omega_n \tag{4.5}
\]

where we sum over all simplices \( u \) of \( \Delta[n] \), and where for the \((r-1)\)-simplex \( u : [r-1] \to [n] \),
$e_u$ is the map

$$e_u : E(T) \times \Delta_u \to P \times \mathcal{M}^r \times Q$$

defined by the composition

$$E(T) \times \Delta_u \to E(T) \times \Delta^r \xleftarrow{c} E \times \Delta^r \xrightarrow{e} P \times \mathcal{M}^r \times Q$$

Note that in the expression (4.5), $T$ is held fixed, but we no longer have specified choices of $S$. 
Chapter 5 The Bar Construction

Recall the cosimplicial space \( E = E(P; M; Q) \) defined by

\[ [s] \mapsto P \times M^s \times Q . \]

Applying the de Rham functor to the latter, we obtain a simplicial object in the category of cochain complexes:

\[ [s] \mapsto \Omega^*(P \times M^s \times Q) . \]

In bi-degree \((r, s)\) we have a commutative square given by:

\[
\begin{array}{ccc}
\Omega^r(P \times M^s \times Q) & \xrightarrow{d_H} & \Omega^{r+1}(P \times M^s \times Q) \\
\downarrow d_V & & \downarrow d_V \\
\Omega^r(P \times M^{s-1} \times Q) & \xrightarrow{d_H} & \Omega^{r+1}(P \times M^{s-1} \times Q)
\end{array}
\]

which is part of a double complex. The horizontal differential is given by the exterior derivative inherited from the de Rham functor, The left vertical differential arises from alternating sum of face operators \( d^i : \Omega^r(P \times M^s \times Q) \to \Omega^r(P \times M^{s-1} \times Q) \) arising from the geometric cobar construction. The right vertical differential is defined similarly.

**Definition 5.1.** The (unreduced) bar construction \( B^* \) is the total complex of the above double complex, i.e.,

\[
B^k := \bigoplus_{r-s=k} \Omega^r(P \times M^s \times Q)
\]

with differential \( d : B^k \to B^{k+1} \) defined by \( d := d_H + (-1)^s d_V \). The sign is given by the vertical degree, which is \((-1)^s\) if the form lies in \( \Omega^r(P \times M^s \times Q) \).

**Remark 5.2.** Chen showed that the cohomology of this complex coincides with the singular cohomology of generalized path space \( E \) under the assumption that \( M \) is 1-connected (cf. [4]). Actually, Chen only considered a reduced version of the above complex which is obtained by quotienting out by the degeneracies (it is well-known that the cohomology is invariant under
this operation).

5.1 The Hatcher-Quinn functional on the bar complex

The cosimplicial evaluation map

\[ e : E \times \Delta^r \to E. \]

induces, upon application of the de Rham functor, a map of simplicial cochain complexes which in bi-degree \((r, s)\) is given by the homomorphism

\[ e^* : \Omega(r(P \times M^s \times Q)) \to \Omega(r(E \times \Delta^s)) \]

The latter induces a homomorphism of total complexes

\[ B^k \to \bigoplus_{r-s=k} \Omega^r(E \times \Delta^s). \]

Composing the latter with integration along the simplex factor results in a cochain map

\[ B^k \to \bigoplus_{r-s=k} \Omega^r(E \times \Delta^s) \xrightarrow{\oplus_s f_{\Delta^s}} \Omega^k(E) \]

which is an unreduced version of the cochain map considered by Chen.

Remark 5.3. Although the cohomology of the bar construction \(B^*\) coincides with the singular cohomology of \(E\) when \(M\) is 1-connected, the relationship between the cohomology of \(\Omega^*(E)\) and the singular cohomology of \(E\) remains obscure (cf. [8]).

Definition 5.4. The Hatcher-Quinn functional on the bar complex is the homomorphism

\[ \bar{\chi} : B^{p+q-m} \to \mathbb{R} \]

which is given on the summand \(\Omega^{p+q-m+r}(P \times M^s \times Q) \subset B^{p+q-m}\) by (4.4).

Proposition 5.5. The linear functional \(\bar{\chi}\) is co-closed.

The proof of the proposition requires some preparation. We first review integration along the fibers.
5.1.1 Integration along the fibers

Let $B$ be a closed $n$-manifold and $F$ be a compact, oriented $r$-manifold with boundary $\partial F$. We will use a negative orientation on the boundary. A basis for $T_x \partial F$ is said to be negatively oriented if adding an outward pointing normal vector to that basis results in a positively oriented basis for the ambient tangent space $T_x F$. The vector space of $n$-forms $\Omega^n(B \times F)$ splits as a direct sum over $k$ of subspaces $\Omega^{k,n-k}(B \times F)$ where the latter are known as forms of type $(k,n-k)$. A form of this type is a rule, which assigns to a point $(x,y) \in B \times F$ a linear functional $\varphi_{x,y} : \Lambda^k(T_x B) \otimes \Lambda^{n-k}(T_y F) \to \mathbb{R}$, where, for a vector space $V$, $\Lambda^j(V)$ denotes the $j$-th exterior power. By taking the adjoint, we get a map:

$$\hat{\varphi}_{x,y} : \Lambda^k(T_x B) \to (\Lambda^{n-k}(T_y F))^*$$

where $V^* = \text{hom}(V, \mathbb{R})$ is the linear dual. Now, let $\Lambda^k_BTB$ be the total space of the fiberwise $k$-th exterior power of the tangent bundle of $B$. We can now reformulate $\hat{\varphi}_{x,y}$ as a map

$$\hat{\varphi} : \Lambda^k_BTB \to \Omega^{n-k}(F).$$

Taking $n = k + r$ gives a fiberwise linear functional:

$$\Lambda^k_BTB \xrightarrow{\hat{\varphi}} \Omega^r(F) \xrightarrow{\int_F} \mathbb{R}$$

i.e., an element of $\Omega^k(B)$.

**Definition 5.6.** Integration along the fibers is the homomorphism

$$\int_{\text{fib}} : \Omega^{k+r}(B \times F) \rightarrow \Omega^{k}(B)$$

which is supported entirely on $\Omega^{k+r}(B \times F)$ and which is given by the above construction.

Now, consider the vertical differential $d_v : \Omega^k(B \times F) \to \Omega^k(B \times \partial F)$ given by restriction, and the horizontal differential $d_h : \Omega^k(B \times F) \to \Omega^{k+1}(B \times F)$ defined by the exterior derivative on the de Rham complex. Now set $\delta = d_v + (-1)^{|v|}d_h$, where $|v|$ denotes the
Lemma 5.7. With respect to the negative orientation, the following diagram is commutative.

\[
\begin{array}{ccc}
\Omega^{k+r}(B \times F) & \xrightarrow{f_{\text{fib}}} & \Omega^{k}(B) \\
\downarrow_{\delta} & & \downarrow_{(-1)^k d} \\
\Omega^{k+r}(B \times \partial F) \bigoplus \Omega^{k+r+1}(B \times \partial F) & \xrightarrow{f_{\text{fib}}} & \Omega^{k+1}(B)
\end{array}
\]

Proof. Inspection shows that there are only two cases to consider: forms on \( B \times F \) of type \((k, r)\) and those of type \((k + 1, r - 1)\) (in all other cases, going around the diagram in each direction gives the zero homomorphism). In the first case, \( d_v \) acts trivially since there are no \( s \)-forms on the boundary. Also, exterior differentiation has two terms according to the Leibniz rule:

\[
d : \Omega^{k,r}(B \times F) \to \Omega^{k+1,r}(B \times F) \oplus \Omega^{k,r+1}(B \times F).
\]

There are no forms of type \((k, r + 1)\) for dimensional reasons. Moreover, on the summand \( \Omega^{k,r+1}(B \times F) \), the displayed map \( d \) is induced by the exterior derivative on \( B \). In this case we need to show that the following diagram commutes:

\[
\begin{array}{ccc}
\Omega^{k,r}(B \times F) & \xrightarrow{f_{\text{fib}}} & \Omega^{k}(B) \\
\downarrow_{(-1)^k d \times 1} & & \downarrow_{(-1)^k d} \\
\Omega^{k+1,r}(B \times F) & \xrightarrow{f_{\text{fib}}} & \Omega^{k+1}(B)
\end{array}
\]

But this is clear since exterior differentiation on \( B \) commutes with integration along the fibers. Next, we look at the forms of type \((k + 1, r - 1)\). A similar analysis reduces to establishing that the following diagram is commutative

\[
\begin{array}{ccc}
\Omega^{k+1,r-1}(B \times F) & \xrightarrow{0} & \Omega^{k}(B) \\
\downarrow_{d_v + (d \times \text{id})} & & \downarrow_{(-1)^k d} \\
\Omega^{k+1,r-1}(B \times \partial F) \bigoplus \Omega^{k+1,r}(B \times F) & \xrightarrow{f_{\text{fib}}} & \Omega^{k+1}(B)
\end{array}
\]

where the upper horizontal map is the trivial homomorphism. Note that we have used the
Leibniz rule here and removed the sign in \((-1)^{d_h}\), where \(d_h\) is identified with \((-1)^{k+1}(\operatorname{id} \times d)\). In this case, we are asking whether 

\[
\int_F (1 \times d)\phi = -\int_{\partial F} d_v \phi,
\]

where both sides are given by integration along the fibers. But this follows by Stoke’s Theorem (note: this is why we chose the orientation on \(\partial F\) the way we did).

**Proof of Proposition (5.5).** The proof is a direct consequence of Lemma (5.7) by setting \(B = P \times M^{x_r} \times Q\), \(F = \Delta^r\), where we have used the fact that \(\Delta^r\) is a smooth manifold with boundary (by rounding corners). In this instance, due to orientation conventions, the vertical differential \(d_v : \Omega^k(B \times F) \to \Omega^k(B \times \partial F)\) coincides with the alternating sum \(\sum (-1)^i d^i\).
REFERENCES


ABSTRACT

THE HATCHER-QUINN INVARIANT AND DIFFERENTIAL FORMS

by

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An intersection problem consists of submanifolds \( P, Q \subset M \) having non-empty intersection. In 1974, Hatcher and Quinn introduced a bordism-theoretic obstruction to finding a deformation of \( P \) off of \( Q \) by an isotopy. This dissertation studies the problem of finding an analytical expression for the Hatcher-Quinn obstruction—one which involves the language of differential forms.

We first introduce the notion of a smooth structure on a set by introducing a system of mappings called plots. By generalizing this to the fibered setting, we use the concept to give a model for the homology of the generalized path space \( E \) i.e., the space of paths in \( M \) which start at a point of \( P \) and end at a point of \( Q \); this is the content of chapter 2. In chapter 3, we introduce diagonal forms and Thom forms. We then use these to construct Hatcher-Quinn forms. Chapter 4 introduces cosimplicial spaces and the geometric cobar construction to give a combinatorial model for the generalized path space. The end of chapter 4 gives a construction of the Hatcher-Quinn functional on the generalized path space. In chapter 5, we develop the bar complex, develop the Hatcher-Quinn functional on the bar complex and show that the functional is co-closed.
AUTOBIOGRAPHICAL STATEMENT

I was born in the conservative city of Kokomo, Indiana. In my formative years, I spent my time in nature camping and in the alleyways of Kokomo making large oil paintings from stolen canvas stapled to low profile box springs. In my youth, I was entranced by art so much that I hated the idea of doing a single math problem. As a result, I failed eighth-grade pre-algebra twice. During my time in high school, I discovered the guitar; this would end up giving me my first taste of self-confidence. After graduating high school, I played in local bands. We would mostly play for friends but, unlike most musicians, I was lucky. I got to tour in a band with my best friends. We played shows around the Midwest for college communities and nineties vegans. We pretended we were rock stars. We pretended we were cool. For money, I did construction work. I laid brick with my dad, but eventually, I wanted more. I had a wife and I wanted a family. So, I enrolled in the local university, “IUK”. During my time at IUK, I traversed several majors, but eventually saw the beauty in math; this would shape the rest of my life. The endeavor was fruitful. I received the “Mathematics Student of the Year Award” twice, once in 2012 and again in 2013. This led me to pursue a Master’s degree at Ball State. Afterwards, I began my PhD work at Wayne State, settling on a degree in Differential Topology. I worked with two gifted topologists - John R. Klein and Vladimir Chernyak. Most importantly, at this time, I had come to appreciate the greatest joy in my life, my daughter, Elle Akira Turner. Ellie was born in 2014 and proved to be my greatest accomplishment. My magnum opus. In 2019, I experienced two life changing events – a messy divorce from Ellie’s mother and a worldwide pandemic resulting from the SARS-Cov-2 virus. My hopes of a life in academia were dashed as a hiring freeze took away all of my job prospects. I am currently a data scientist, building algorithms with neural networks and statistical models. So far, the job seems enjoyable and hopeful. Although I am a mathematician at heart, I remain optimistic about the future of this field and hope it will afford Ellie and I a wonderful life.