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Understanding Pre-Service Elementary Mathematics Teacher Learning In An Early Lesson Study Experience

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UNDERSTANDING PRE-SERVICE ELEMENTARY MATHEMATICS TEACHER LEARNING IN AN EARLY LESSON STUDY EXPERIENCE

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DEDICATION

I dedicate this work to my wife, Andrea. I know that you would never expect to be repaid for all of the selfless support that encouraged and enabled me to pursue this dream, but I will spend the rest of my life trying to do just that.

ACKNOWLEDGEMENTS

I am grateful to my advisor, Jennifer Lewis, for introducing me to new ideas, new opportunities, and new ambitions. Her boundless support, inspiring scholarship, and sincere kindness have provided an aspirational model for me as a teacher and as a person. In particular, working and learning with Jennifer within Project REALM: Realizing Equity and Achievement through Learning Mathematics was transformational for me as a teacher educator and foundational for this dissertation.

Steven Kahn is the reason I am a teacher. The programs that he has instituted at Wayne State University—the Emerging Scholars Program and the Math Corps, in particular—have given me the opportunity to work with students of all ages and backgrounds and alongside the best teachers I've ever known. Steve's philosophy of loving and believing in students has shown me that every person deserves the support of a caring community and excellent educational opportunities. All of these ideas informed this dissertation.

Asli Ozgun-Koca welcomed me into the countless programs that she leads; and, in all of them, she made me feel like a valued colleague rather than the novice that I was. I am thankful for her generosity in sharing her time and deep knowledge of mathematics education with me. Asli's guidance during my data analysis, her careful readings of early drafts, and her thoughtful suggestions were invaluable to me.

It has been an honor to collaborate with Deborah Zopf on improving the mathematics content courses at our institutions over the past five years. Deb's expertise in the mathematical knowledge needed for teaching pre-service teachers made our work so intriguing and rewarding that I chose to use one of these courses as the site for this study.

Her insights into the rich learning opportunities contained in these courses were instrumental in the ultimate design of this study.

Finally, I must acknowledge the generosity, openness, and hard work of the pre-service teachers involved in this study. I was excited by their deep thinking about teaching and learning mathematics and filled with hope for the elementary students who will soon be in their capable, caring hands. I thank them for their trust.

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CHAPTER 1 INTRODUCTION

Teachers matter. As the single most important in-school factor for improving student achievement (Darling-Hammond, Wei & Johnson, 2009), teachers need to be prepared to offer all of their students the high-quality mathematics experiences they deserve and require (National Council of Teachers of Mathematics, 2000; Conference Board of Mathematical Sciences, 2012). As the National Council of Teachers of Mathematics (2000) reminds us,

> Students learn mathematics through experiences that teachers provide. Thus students' understanding of mathematics, their ability to use it to solve problems, and their confidence in, and dispositions toward, mathematics are all shaped by the teaching they encounter in school.
> (p. 16)

Unfortunately, there is evidence that American students do not experience high-quality instruction (Ma, 1999; Schmidt, 2012; Stigler & Hiebert, 2004). The 2019 National Assessment of Educational Progress (NAEP), the "nation's report card", indicates that only 41% of fourth-grade and 34% of eighth-grade students were "proficient" at mathematics. Comparisons such as the Trends in International Mathematics and Science Study (National Research Council, 2010) and the Programme for International Student Achievement (OECD, 2016) show them lagging behind their international peers. The situation in large urban school districts is even more pressing. The NAEP report indicates that, in these districts, only 18% of fourth-grade African-American students were "proficient" at mathematics; and this percentage dwindles to 13% in the eighth grade. Sadly, these percentages have remained steady over the past decade (Higgins, 2018). Researchers posit a connection (see Figure 1) between the quality of classroom instruction and the opportunities to learn (OTLs) that teachers encounter during their pre-service preparation (Schmidt, Cogan, & Houang, 2011; Senk et al., 2012).

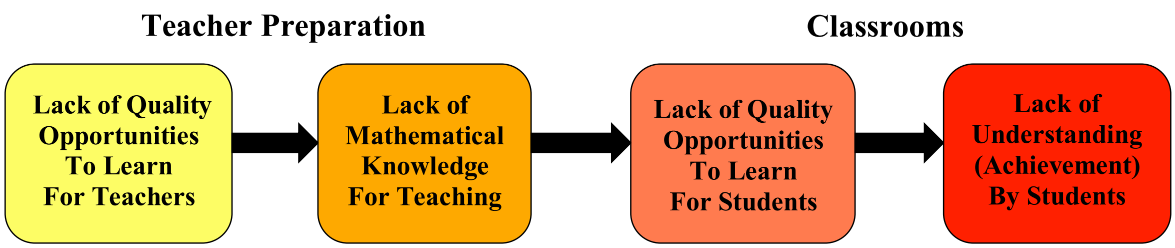


Figure 1. Hypothesized implications of a lack of quality opportunities to learn (OTLs) in teacher preparation on the mathematical knowledge for teaching, student OTLs, and student understanding of mathematics.

In this dissertation, I will concentrate on the first two links in this chain. In the following chapter, I will first summarize the literature regarding what teachers need to know (and to be able to do with that knowledge) in order to teach mathematics well. In the past, a search for the "right" teacher knowledge provided frustratingly inconsistent results (Begle, 1969; Monk, 1994). In the quest for a so-called "educational production function," researchers found that the same "input," such as the number of mathematics courses taken by a teacher, could produce a different "output" (student performance) depending on the study (Monk, 1989). More recent work in this area has significantly clarified our understanding of this knowledge base (Ball, Thames, & Phelps, 2008; National Research Council, 2010; Rowland, Huckstep, & Thwaites, 2005).

Secondly, I will consider the literature that provides guidance for the structure of quality teacher training programs. Among the many recommendations, there are common foci. I will explore four such points of convergence that I believe can be addressed in a mathematics content course during the earliest phases of elementary teacher preparation. They can be synthesized into a call for OTLs that allow pre-service teachers (PSTs) to *experience a new vision of mathematics*, *create new knowledge for action, engage in the complexity of instruction,* and *prepare to learn in and from practice.*

Lastly in Chapter 2, I will introduce lesson study, the professional development system for Japanese teachers at all stages of their training and practice, which has been proposed as a structure within which to address these calls (Hart, Alston, & Murata, 2011; Huang, Takahashi, & da Ponte, 2019). Lesson study activities in methods courses, field placements, and professional development in the United States are well documented in the literature (Fernandez, 2005; Hart et al., 2011; Larssen et al., 2018; Ponte, 2017). In order to contribute to the knowledge base regarding the effectiveness of lesson study in the development of PSTs in the United States, I situated such an activity within a mathematics content course *at the earliest possible stage* of a teacher preparation program.

In Chapter 3, I will describe the design of an interpretive case study (Merriam, 1988; Savin-Baden & Major, 2013) through which I sought to answer the following question:

> What types of learning are afforded or occluded by a modified lesson study activity within an early pre-service mathematics teacher content course for elementary school teachers?

In Chapter 4, I will present the data generated by the study as well as my initial analysis. Quantitative data collected at the beginning and end of the study will be presented and analyzed first. The presentation and analysis of the qualitative data collected throughout the study will help situate the reader in the activities and will proceed chronologically. That is, themes and categories arising from the data will be highlighted in the order in which they become visible.

Finally, in Chapters 5 and 6, I will synthesize broader themes and categories into a theory that will both explain the learning and will help fellow teacher educators understand, predict, and take advantage of the opportunities to learn that lesson study may offer (Charmaz, 2006; Glaser & Strauss, 1967).

CHAPTER 2 LITERATURE REVIEW

A Bit of Background

The way that teachers have been trained has always reflected what teacher educators thought they needed to know in order to carry out instruction. Available theories of learning would be used to conceptualize what good instruction should entail (Forzani, 2013). Although it would be incorrect to say that there was ever a single view of what constituted quality teaching (or, simply teaching, for that matter), trends are certainly discernable. For example, during the earlier part of the twentieth century, behaviorist learning theories led to instruction based on drill and external reinforcement (Skinner, 1954). Thus, it would be enough to assume (as many preparation programs and state licensing officials of that time did) that the mathematical knowledge that prospective teachers needed was simply the content that their students would learn. This, along with a few general pedagogical techniques would suffice (Fraser, 2007; Shulman, 1986).

As new theories of learning emerged, different models of instruction that operationalized them followed. Developmentalist, constructivist, and social theories sought to describe and explain the complex nature of learning (Dewey, 1938/1997; Lave & Wenger, 1991; Piaget, 1965; Vygotsky, 1978; Wenger, 1998). Figure 2 presents a view of instruction as interaction (Cohen, Raudenbush, & Ball, 2003), with the teacher facilitating the interactions between students, content, herself, and environments.

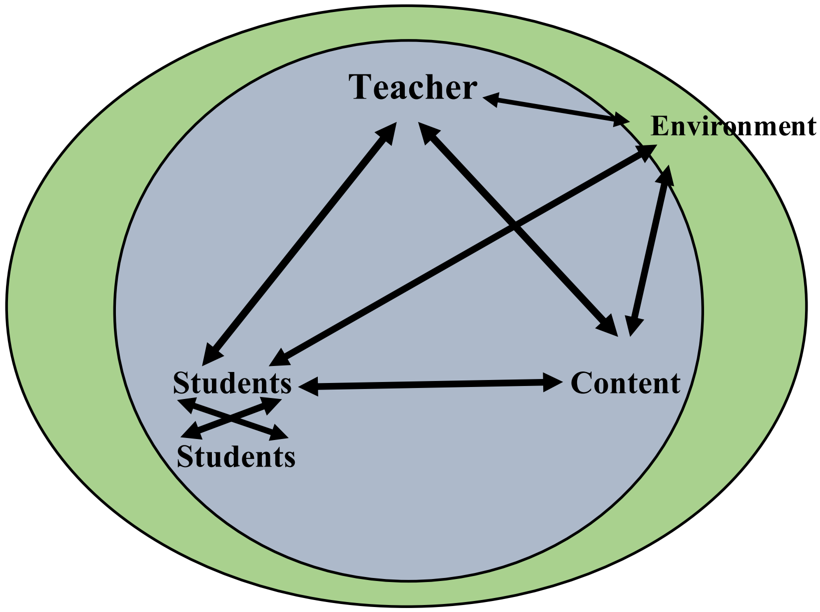


Figure 2. Instruction as Interaction. Adapted from D. Cohen, S. W. Raudenbush, & D. L. Ball (2003). Resources, instruction and research. *Educational Evaluation and Policy Analysis*, 25(2), p. 124. Copyright 2003 by Sage Publishing.

Clearly the knowledge demands on a teacher asked to facilitate the learning of students in this type of complex system are substantial (Lampert, 2001). Extending past subject-matter and generic pedagogical techniques, teachers must have knowledge of, for example, their students, what they know and what they will need to know, what they will find difficult, how to represent new mathematics that takes advantage of the students' and community's resources and address their challenges. We turn to look at the knowledge base that teachers draw on in practice, and how theorists have parsed this territory.

Shulman and Pedagogical Content Knowledge

In his Presidential Address to the American Educational Research Association, Shulman (1986), pointed to the lack of attention that was being paid (in research and teacher preparation) to the development of what he termed *pedagogical content knowledge*. This particular domain of teacher knowledge fell into a "blind spot" created by the field's focus on other, more traditional, domains: general pedagogical strategies (such as classroom management), content (such as the algorithms for multiplication and division of whole numbers), characteristics of learners (such as developmental stages), as well as educational philosophy and the structure of the profession. Researchers had concentrated on these important subject-independent domains through so-called “process-product” and “educational production function” investigations with some success, but inconsistent results confounded attempts to pinpoint the connection between mathematical knowledge and student performance (Begle, 1969; Berliner, 1986; Monk, 1989).

Shulman (1987) lamented that an emphasis on “discovering, explicating, and codifying *general* (emphasis added) teaching principles simplify the otherwise outrageously complex activity of teaching” (p. 11). The concept of pedagogical content knowledge captured this complexity because it represented “the blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organized, represented, and adapted to the diverse interests and abilities of learners, and presented for instruction” (p. 7). His call to elucidate further categorizations of this knowledge base, and in particular, his urging to look to practice in order to help with this work, was taken up by a new wave of researchers, including Ball and her colleagues in mathematics. We will turn to their conceptualization of the knowledge base: the Mathematical Knowledge for Teaching (MKT) (Ball et al., 2008).

Ball and the Mathematical Knowledge for Teaching

By examining the real work of teaching mathematics, Ball and her colleagues were able to create a practice-based model of the knowledge needed to carry out instruction. The tasks and problems that arose repeatedly, and the knowledge that teachers drew on to carry out and solve them—from planning for instruction, to interpreting and reacting to student thinking and modifying those plans—allowed the research team to refine and extend Shulman's domains of content and pedagogical knowledge.

Figure 3 displays these subdomains.

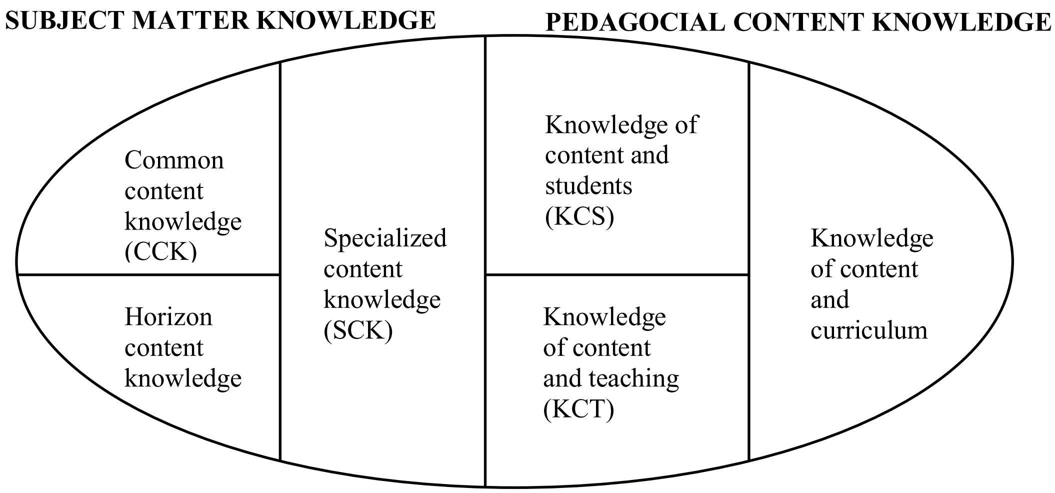


Figure 3. Domains of Mathematical Knowledge for Teaching. Adapted from "Content Knowledge for Teaching: What Makes it Special," by D. L. Ball, M. H. Thames, and G. Phelps, 2008, *Journal of Teacher Education,* 59(5), p. 403. Copyright 2008 by Sage Publishing.

Pedagogical content knowledge. Ball et al. (2008) were able to group Shulman's important categories into three subdomains of pedagogical content knowledge: knowledge of content and *students*, knowledge of content and *teaching*, and knowledge of content and *curriculum*.

The blended knowledge of content and students (KCS) is drawn on in when a teacher needs to anticipate student misconceptions (i.e. what they might find hard or confusing) of a particular topic, which questions she might ask students to help clarify their thinking, or what students might find intriguing and hence motivate them to persevere. That is, it uses the teacher's knowledge of students thinking around a particular topic rather than solely her own content knowledge (Hill, Ball, & Schilling, 2008).

When a teacher needs to think about how best to sequence topics (or examples within a single topic) within instruction, she is drawing on her knowledge of content and teaching (KCT). This knowledge is not only used during planning, but also when choosing (typically on-the-fly) from among all of the representations of, say, fractions that might best be used to address a student's question. The third subdomain, knowledge of content and curriculum, acts as a bit of a catch-all in this framework. It captures Shulman's idea of the curricular "tools of the trade:" the understanding of what instructional materials and programs are available to enrich instruction.

Subject matter knowledge. The biggest advancement of Shulman's framework came from the refinement and extension of the domain of subject matter knowledge. Content knowledge had been left unexamined--taken for granted as the information learned in previous mathematics courses, and then called upon in pedagogical action. For example, deciding if a student's claim that $e^{i\pi} + 1 = 0$ is correct or not draws on what Ball et al. (2008) classified as *common* content knowledge (CCK)—knowledge that other people might have and use. Of course, that knowledge is important; but it was surprising that simply increasing this type of knowledge (or its proxies) did not seem to translate into improved performance. For instance, Monk (1989, 1994) reported that there were correlations between one such proxy, the number of mathematics courses taken by a teacher, and his students' achievement, but only up to a certain point. After five classes, the benefits for instruction and students plateaued. Shulman's concept of pedagogical content knowledge was an important step at understanding this odd relationship: it was what teachers *did* with that knowledge, and *how* they understood it, that was important. These factors, as opposed to the more general measures of mathematics knowledge (such as number of classes taken) have been shown to be significant predictors of student performance, and are especially important at the lower grades (Hill, Rowan, Ball, 2005; Hill, Sleep, Lewis & Ball, 2007). I will discuss these measures of MKT more in Chapter 3.

The most important step was the recognition of a new distinction within content knowledge: *specialized* content knowledge (SCK) (Ball et al., 2008). Specialized content knowledge was detected in mathematics classrooms as the content knowledge particular to teaching. The hallmark of an action drawing on SCK is that it involves knowledge of mathematics *for others* rather than for oneself. Activities such as explaining "why" something works or is true, making connections between topics, designing representations that illuminate specific properties or ideas, and evaluating the generalizability of a student's inventive algorithm all draw on this particular knowledge base (Ball et al., 2008). Lastly, the subdomain of horizon content knowledge involves understanding what students will need in the future and tailoring current instruction so that it can be connected coherently to later content.

Why is a refined framework such as the Mathematical Knowledge for Teaching so important? In practice, tasks surely draw on multiple subdomains and their distinctions become unimportant. One benefit is particularly germane to my research question. The development of this framework offers the potential to

> …inform the design of support materials for teachers as well as teacher education and professional development. Indeed, it might clarify a curriculum for the content preparation of teachers that is professionally based—both distinctive, substantial and fundamentally tied to professional

> practice and to the knowledge and skill demanded by the work. (Ball et al., 2008, p. 405)

We will return to this shortly. Before we do, I will consider two other frameworks in order to provide a richer picture of teacher knowledge.

Other Conceptualizations Inspired by Shulman

Although the MKT framework has been the most influential work inspired by Shulman's initial call, there are others; and they allow us to see different aspects of or relationships between the various domains. One particularly compelling organization is the Knowledge Quartet (Rowland et al., 2005). Developed as an observational instrument to help teacher educators make visible the ways in which subject matter knowledge and pedagogical content knowledge arise in student teaching, the quartet helps focus attention on instances where pre-service teachers draw on particular domains of knowledge. Teaching for Mathematical Proficiency, a third framework, resonates with the quartet.

The Knowledge Quartet. The four components—foundation, transformation, connection, and contingency—reflect essential stages of what Shulman termed "pedagogical reasoning and action" (1987, p.15). I will quickly summarize the members of the quartet and connect them to the Mathematical Knowledge for Teaching.

Foundation refers to the knowledge that teachers *possess*. This includes knowledge *that* something is true as well as *why* it is true, beliefs about mathematics (what it *is*, what it *is for*, and *how* it should be learned), and pedagogical knowledge about content and students. It therefore incorporates all of the subject matter domains of MKT and introduces the additional aspect of pre-service teachers' beliefs into the construct. This is especially important, as a connection has been shown to exist between a teacher's beliefs about mathematics and the how he conceives of, learns about, and enacts instruction (Ball, 1990; Borko, 1992; Dunekacke, Jenssen, Eilerts, & Blomeke, 2016; Philipp, 2007).

Transformation is the action of converting foundational knowledge into action. (i.e. the operationalizing of subject matter and pedagogical content knowledge in instruction). Again, all subdomains of the MKT are drawn upon (e.g. choosing examples (KCT) or evaluating the quality of resources such as textbooks (knowledge of content and curriculum). Shulman was especially interested in understanding this transformation process.

Connection measures the coherence of the lesson and instruction. It involves the flow of the lesson as a whole, the sequencing of the topics and examples, and the demands placed on students. This component draws extensively on the middle three domains (as shown in Figure 3) of the Mathematical Knowledge for Teaching: SCK, KCS, and KCT.

The final component, contingency, reflects the unpredictable nature of teaching. Using and responding to student thinking as part of instruction introduces uncertainty and having to make in-the-moment instructional choices places intense demands on a teacher's knowledge (Lampert, 2001). These important practices, akin to what have been described as "high-leverage practices" (Ball, Sleep, Boerst, & Bass, 2009) for improving teaching and learning, show the clear relationship between knowledge and action.

Teaching for mathematical proficiency. The Knowledge Quartet aligns with the interwoven knowledge, actions, and beliefs needed to be able to carry out what the National Research Council (2001) calls "Teaching for Mathematical Proficiency" (p. 380). This involves possessing a conceptual understanding of the core knowledge of mathematics and a productive disposition (foundation), being competent in planning and fluent with instruction (transformation and connection), and being able to adapt one's practice to improve (contingency).

The Picture of Teacher Knowledge

What emerges from the discussion of these frameworks is an image of the necessary knowledge base for teaching mathematics that is substantial, interconnected, and, frankly, daunting. Looking at simply the *subject matter* knowledge of the MKT model, we can see that developing "only" part of this base is a considerable undertaking. The Conference Board of Mathematical Sciences recommends that "prospective elementary teachers should be required to complete at least 12 semester-hours on fundamental ideas of elementary mathematics, their early childhood precursors, and middle school successors" (2012, p. 18). How can preparation programs hope to develop, in such a short amount of time, all of the subject matter knowledge that teachers will need to draw on as they move into the classroom? They cannot.

This and other restrictions lead to the realization that the mathematical education of teachers must be seen as part of a long-term program of development that *begins* in the pre-service program and *continues* throughout a teacher's career. What would the experiences that prepare competent novice teachers and continue to support and develop them to teach well look like? I propose an answer to this question in the next section.

Preparing to Teach Mathematics

The Current Situation

The structure of teacher preparation programs offered by colleges and universities has remained amazingly consistent despite the vast recent changes in the demands placed on teachers stemming from the increase in the knowledge base for teaching, responsibilities within instruction, and in the diversity of their students. The familiar mix of content, methods, and social science foundational courses together with supervised practice teaching that was started in the early 1900s endures (Donoghue, 2006; Fraser, 2007). Unfortunately, the opinion that the formal training is disconnected from (or at best weakly related to) the work of teaching (with the exception, possibly, of the field experiences) has also been consistent (Feiman-Nemser, 1983, 2001).

The long-standing feeling that the theoretical underpinnings of the field are a form of “soft knowledge” where fundamental issues are always being reinterpreted (Labaree, 1998), leads to individualism and conservatism in new teachers. They rely on what they saw (or think that they saw) as the practice of teaching during their thirteen-year “apprenticeship of observation” (Lortie, 1975) as K-12 students. This apprenticeship tends to cement, in the prospective teacher’s mind, a vision of what good teaching entails and what “math class” should be (Seaman, Earles, Szydlik, & Beam, 2005). This vision is, of course, incomplete. Without being privy to all of the “underlying knowledge, skills, planning, and decision making” (Hammerness & Darling-Hammond, 2005, p. 367) it is understandable that pre-service teachers might not appreciate the value of this considerable professional knowledge needed for teaching. In addition to all of the “behind the scenes” work, much of the work of teaching is “invisible”—that is, even things visible to the eye of the apprentices may escape their notice (Lewis, 2007).

This resistance to change extends into in-service professional development as well. The Trends in International Mathematics and Science Study (TIMSS) allowed researchers to compare what happens in average U.S. math classrooms with what happens in other countries. The results were not promising. Stigler and Hiebert (2004) found that “…all available evidence suggests that classroom practice has changed little in the past 100 years" (p. 12). Despite an enormous professional development effort to help teachers improve their practice, U.S. classrooms remain mired in teacher-centered instruction and a focus on procedural rather than conceptual understanding (Stigler & Hiebert, 1999). Even when teachers believe that their instruction reflects current ideas, glimpses into their classrooms show otherwise.

For mathematical knowledge for teaching, resistance is particularly acute. Persistent myths that the elementary school mathematics curriculum is easy—and that pre-service teachers know enough because they can "*do* the math" leads to serious misconceptions about content knowledge (Ball, 1990; Ma, 1999). The math content knowledge of teachers is often fragile and needs to be reexamined and reinforced with emphasis on the specialized subdomain discussed above. We examine this in depth in the following section.

Despite these and other forces acting against them, it is clear that preparation programs and professional development can improve teaching and student learning (National Research Council, 2001; Hiebert, 2015). Earlier claims of the irrelevance of the knowledge and skills that come from formal training and certification have been challenged by more recent research (Darling-Hammond, Holzman, Gatlin, & Heilig, 2005); but it is the *quality* of the experiences of the program that matters, not simply the duration (Hammerness & Darling-Hammond, 2005). This maxim also holds for professional development (Desimone, Porter, Garet, Yoon, & Birman 2002; Kennedy, 1998; Loucks-Horsley, Stiles, Mundry, Love, & Hewson, 2010). We turn now to the types of quality experiences in pre-service programs that are needed to combat these forces and to help teachers develop the knowledge and skills that they need to teach mathematics well.

The Preparation of Pre-Service Teachers

In contrast to the disconnected, overly theoretical preparation that has been shown to have limited effect on advancing the knowledge and beliefs of PSTs beyond those formed during their apprenticeship of observation, the current literature offers an alternative. There is much to be done during the pre-service phase of training (Association of Mathematics Teacher Educators, 2017). Feiman-Nemser describes this as the period when PSTs must analyze their beliefs, increase their subject knowledge and knowledge of learners, build a beginning repertoire of teaching moves, and develop the tools to study teaching (2001). This view resonates with other literature for effective pre-service mathematics teacher preparation, and these four points of convergence can be addressed in the earliest phases of teacher preparation programs. For example, these ideas resonate with the three curriculum-based standards (out of five total) in the Association of Mathematics Teacher Educators' *Standards for Preparing Teachers of Mathematics* (2017). A quality mathematics teacher preparation program

- provides candidates with opportunities to learn mathematics and statistics that are purposefully focused on essential big ideas across content and processes that foster a coherent understanding of mathematics for teaching.

- provides candidates with multiple opportunities to learn to teach…in which mathematics, [and] practices for teaching mathematics…are integrated.

- includes clinical experiences that are guided on the basis of a shared vision of high-quality mathematics instruction and have sufficient support structures and personnel to provide coherent, developmentally appropriate opportunities for candidates to…learn from their own teaching and the teaching of others. (pp 29-37)

I believe these point to a set of goals that can be addressed in a mathematics content course at the early stage of pre-service preparation. PSTs should engage in activities that allow them to *construct a new vision of mathematics*, *create new knowledge for action, engage in the complexity of teaching,* and *prepare to learn in and from practice.* Figure 4 represents this framework. I will highlight, briefly, key points from the literature that speak to the importance of each of the four components.

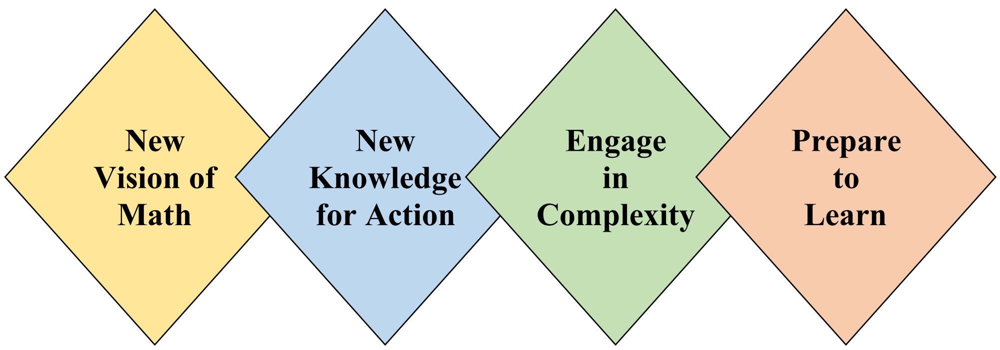


Figure 4. Framework of the Key Goals for Developing Mathematics Teachers: Early Stages of Pre-Service Preparation

Constructing a New Vision of Mathematics

When Ball et al. (2008) reminded readers that "…knowing a subject for teaching requires more than knowing its facts and concepts. Teachers must also understand the organizing principles and structures and the rules for establishing what is legitimate to do and say in a field" (p. 391), they were setting the stage for the introduction of their new construct: the mathematical knowledge for teaching; but they were also referring to the view of mathematics that teachers need to possess.

Both of these lines of thought can be traced back to Shulman's 1986 address. He added

> The teacher need not only understand *that* something is so; the teacher must further understand *why* it is so, on what grounds its warrant can be asserted, and under what circumstances our belief in its justification can be weakened or denied. Moreover, we expect the teacher to understand why a particular topic is particularly central to a discipline whereas another may be somewhat peripheral. (p. 9)

To organize this discussion, I will use their quotes, and see how the ideas that they contain, can guide the construction of a new vision of mathematics.

Idea 1: Knowing *what* mathematics is really about. Ball et al. (2008) stated that,

> Knowing a subject for teaching requires more than knowing facts and concepts. Teachers must also understand the organizing principles and structures and the rules for establishing what is legitimate to do and say in a field. (p. 391)

This idea appears in various forms throughout the literature. The National Research Council's Committee on the Study of Teacher Preparation points out that this goal for teacher knowledge was in place in the very first college teacher preparation programs at Teachers College and Michigan State University. Both of these pioneering institutions considered it essential to make sure that their candidates understood *the subject* itself, how that subject *worked*, as well as how to teach it (2010) (emphasis added). This focus on understanding the syntactic nature of mathematics (i.e. how it accepts or rejects new facts and concepts) means that a powerful alternative to the familiar structure of "math class" and "doing math" must be encountered (Schwab, 1978). Future teachers must experience mathematics as a dynamic field where they have opportunities " to construct viable arguments, to listen carefully to other people's reasoning, and to discuss and critique it"

(Conference Board of the Mathematical Sciences, 2012, p. 33), so that they will be able to facilitate experiences for their future students. For many pre-service teachers, fluency with procedures was what was valued in their previous content courses and "doing mathematics in ways consistent with mathematical practice is likely to be a new, and perhaps, alien experience for many teachers" (Conference Board of the Mathematical Sciences, 2012, p. 11). In fact, there is clear evidence that the activities of following rules and receiving validation of accuracy from the teacher remain a staple of American mathematics classrooms (Lortie, 1975; Ma, 1999; Stigler & Hiebert, 1999). In order to combat this common misconception of what "doing mathematics" entails, a new vision is needed. Schmittau (1991) proposed such a vision based on the work of Lakatos (1976). Schmittau (1991) argued that "allowing students to engage in a process of social interaction in 'proof and refutation'—as the new standards for school mathematics advocate—would do much to dislodge the notion of mathematical infallibility and broaden perspectives of mathematics methodology as well" (p. 129).

Research conducted as part of the Department of Education's Mathematics and Science Partnerships has shown that teachers can change their practice as a result of professional development that stresses this aspect of content knowledge. "This suggests that *doing* mathematics in ways consistent with the Common Core State Standards for Mathematical Practice is an important element in the mathematical education of teachers" (Conference Board of the Mathematical Sciences, 2012, p. 11).

Stigler and Hiebert, in their international comparisons, found that *how* a class interacts about mathematics can be more important than *what* mathematics they interact with (2004). Unfortunately, seeing and making meaning while doing mathematics has often been reserved for the elite while other students experienced only memorization and meaningless procedures (Moses & Cobb, 2001). This is particularly important for our discussion considering that most PSTs were students in classrooms that stressed an instrumental (i.e. tool-based) view of mathematics rather than a relational (sense-making) one (Skemp, 2006). Therefore, the preparation must engage teachers in a deep exploration, through sense-making activities, of the fundamental and connected ideas that they will teach. The PSTs must then unpack their own content knowledge of these key concepts to examine "why" things are so, transforming it into specialized content knowledge for teaching.

Idea 2: Knowing *why* mathematics is so. "The teacher need not only understand *that* something is so; the teacher must further understand *why* it is so" (Shulman, 1986, p. 9).

This idea hits squarely on a fundamental aspect of specialized content knowledge. The work of teaching involves knowing mathematics *for others*. Ball (2000) describes this as "the capacity to deconstruct one's own knowledge into a less polished and final form, where critical components are accessible and visible" (p. 245). In this single phrase, Ball captures the essence of why simply increasing the *common* content knowledge of teachers does not seem to translate consistently into more learning for their students (Begle, 1969; Berliner, 1986; Monk, 1989, 1994). Ball (2000) continues,

> This feature of teaching means that paradoxically, expert personal knowledge is often ironically inadequate for teaching. Because teachers must be able to work with content for students in its growing, unfinished state, they must be able to do something perverse: work backward from mature and compressed understanding of the content to unpack its constituent elements. (p. 245)

Therefore, the mathematics preparation of teachers must include opportunities for pre-service teachers to examine the elementary mathematics that they know, and to explore *why* they know it—unpacking that knowledge for themselves first, so that they will be able to unpack it for their future students. The elementary school mathematics curriculum is so deceptively deep that the challenge becomes deciding which topics to unpack. This leads us to our final idea.

Idea 3: Knowing *which* mathematics is important. “We expect the teacher to understand why a particular topic is particularly central to a discipline whereas another may be somewhat peripheral” (Shulman, 1986, p. 9).

Given an infinite amount of time, all of the compressed knowledge that pre-service teachers bring to their content courses could be unpacked and examined. This is, of course, not the case. Even the recommendation by the Conference Board of the Mathematical Sciences (2012) for *twelve* semester hours to be dedicated to this type of study exceeds what most programs are able to offer. Therefore, difficult choices must be made about which topics should be addressed. If one takes Idea 3 seriously, then it serves as a guide for such choices. Many groups offer particularly coherent guidelines for the crucial foundational topics of the elementary school curriculum (Conference Board of the Mathematical Sciences, 2012; National Council of Teachers of Mathematics, 2000; National Research Council, 2010).

Even more germane for our discussion are the recommendations for designing tasks that build mathematical knowledge for teaching (Suzuka et al., 2009). These tasks would, in part, uncover something *central* and allow the teachers to see *connections*. Synthesizing these recommendations, we come to the realization that the mathematics preparation should, therefore, seek to cover a small number of mathematical topics that are central to the elementary curriculum, and connected to each other in ways that present the coherent structure of mathematics. Rather than examining disconnected esoteric material, experiences can draw attention to the centrality of the topics covered by the level of attention lavished upon them and their repeated use throughout the program.

Creating New Knowledge for Action

The National Research Council (2001) emphasizes that "Effective programs of teacher preparation…cannot stop at simply engaging teachers in acquiring knowledge; they must challenge teachers to develop, *apply*, and analyze that knowledge in the context of their own classrooms so that knowledge and practice are integrated" (emphasis added) (p. 376). In other words, teachers must *do* things with their knowledge. The courses in mathematics, as well as the methods and social science foundations must be focused on how to "deploy" that knowledge in the teaching of others (Suzuka et al., 2009). This is what Ball and Forzani (2011) refer to as connecting knowledge and beliefs to judgment and practice. Rather than build knowledge that is separated from the practice of teaching, an understanding has emerged of the power in an opportunity to learn *immediately followed* by an opportunity to apply that knowledge (Darling-Hammond, Wei, & Johnson 2009).

Some researchers advocate for early field experiences where teachers must confront their pre-existing beliefs about mathematics teaching and learning. One advantage is the connection to practice that combats the feeling that teacher education is too theoretical. This experience can be especially useful when PSTs are able to observe highly proficient teachers engaging in the relational work of mathematics, as well as to see students persevering, interacting, and exercising collective authority to validate each other's discoveries (McDiarmid, 1990). This can highlight the relevance of the PST's own content learning.

This activity does contain certain pitfalls. An *unstructured* observation runs the risk of confirming pre-existing beliefs (Philipp et al., 2007), and as Lewis points out, "Learning a professional practice depends, in part, on *being able to notice* and attend to what practitioners do" (emphasis added) (2007, p. 6). This suggests a pre-observation meeting to discuss features to look for, as well as a post-lesson debriefing should be used to focus the learning on the important components of mathematics teaching. In addition, Putnam and Borko (2000) advise that early field experiences that allow PSTs to go beyond what they know and do might best be situated in unfamiliar contexts.

The next common theme recommends that the opportunity to apply that knowledge should represent the complexity of the real work of teaching.

Engaging in Complexity

As Lewis (2007) observes, teacher education often "disappears" teaching (p. 36). It does not allow the urgency of practice into the teacher education curriculum. Instead, it typically breaks practice down into fragments that can be practiced separately, and then leaves it to the teacher to reconstitute the complex practice once in the classroom. There are ways to conceptualize this complexity in order to train teachers to deal with it; one is the "Core Practices" model. Derived from the same analysis of the real work of teaching that led to the development of the Mathematical Knowledge for Teaching Framework (Ball et al., 2008), the core practices are those that are "most likely to equip beginners with capabilities for the fundamental elements of professional work and that are unlikely to be learned on one's own through experience" (Ball et al., 2009). These include practices such as eliciting student thinking and leading whole-class discussions or small group work. It conceptualizes teaching as a mix of routine and improvisational practices. The improvisational aspects stem from the interactional view of instruction (Figure 2), which, among other things, uses students' thinking as important classroom resources. Therefore, routines can be built into a PST's "instructional repertoire" in order to better manage the improvisational moments (Forzani, 2013). Theses routines, though, should be implemented within the complexity of live instruction. This answers the call made by Shulman (1987) for teacher preparation that would, "provide the students with the understandings and performance abilities they will need to reason their ways through and to enact a complete act of pedagogy" (p. 19). This brings up a question: what type of early field experience might be appropriate for PSTs at the *early* stages of preparation, that is, within the first year of a preparation program?

Both of the last two components--creating new knowledge for action and engaging in complexity—reflect an emphasis on the situated nature of learning that undergirds this structure for pre-service training. It is important to note that "situated" here does not necessarily mean in a classroom full of children; but rather within authentic activities that serve the goals of the preparation program (Putnam & Borko, 2000). Here, learning is viewed as the social identity-making process of becoming an active participant in this particular community of practice (Lave & Wenger, 2005; Wenger 1998). Thus, these "teaching activities" are viewed as loci for building the specialized knowledge for teaching, and present opportunities for PSTs to participate in a legitimate peripheral form. This offers newcomers what Lave and Wenger (1991) describe as "broad access to mature practice" (p. 100). Exposure to authentic sources such as student work and video recordings can begin to normalize the fourth component: learning in and from practice (Ball & Cohen, 1999).

Preparing to Learn In and From Practice

The acceptance of the reality that pre-service programs cannot possibly provide *all* of the training that teachers will need has motivated the structure of the experiences involved in this synthesis. The components described so far have targeted the knowledge and actions that proficient novice teachers will need to possess as they begin their in-service careers. But even that knowledge and action base will be fragile and will need to be solidified and deepened over the teacher's career. Therefore, the final component of our structure will engage PSTs in practices that build the skills and perspectives needed to learn in and from practice.

The literature is very clear on this point. Programs must prepare PSTs to be lifelong learners who draw from their practice, colleagues, curriculum, and research (Darling-Hammond et al., 2005; Feiman-Nemser, 2001; National Research Council 2001). Hiebert, Morris, Berk, and Jansen (2007) offer a term for this experience: the PSTs must learn to take a "research stance" in their practice. "Research" in this context refers to "intentional learning from carefully planned experiences as part of the daily routine of practice" (p. 50).

Taking a lesson as the unit of analysis, PSTs should be: 1) guided to attend to the decisions involved in planning, in particular the learning goals (what the students should learn), 2) instructed where to look for evidence of that learning, 3) asked to hypothesize about how the teaching decisions affected that learning, and 4) prompted to suggest improvements (Hiebert et al., 2007). This type of activity can (and should) be used within many of the traditional components of teacher preparation programs: content courses, methods courses, and field experiences.

The benefits are likely compounded when this type of work is woven through all of components, with the knowledge developed in a particular component being drawn on in others. Through skillful facilitation, this process of observing teaching from a research stance becomes a powerful learning experience. Facilitators that “render this experience educative” (Lewis, 2007, p. 49) by focusing attention on the preparation, deliberation, and reflection work involved offer a counterbalance to the PST’s uninformed “apprenticeship of observation” that they experienced during their own education.

A cohesive preparation program that emphasizes this type of training would require intense collaboration between teacher educators. This poses challenges as the formal mathematics education of teachers has been traditionally disconnected from the pedagogical training (Fraser, 2007; Heaton & Lewis, 2011). But, as Ball and Bass (2000) point out, “the gap between subject matter and pedagogy fragments teacher education by fragmenting teaching” (p. 85). By integrating the expertise of mathematicians, mathematics educators, and educational theorists, the program can model the type of professional learning community that we hope the PSTs will develop (Beers & Davidson, 2009; Hord, 2004), and represent the integrated and complex nature of teaching. Is there a particular format in which to frame such collaboration? I believe that there is, and we will examine it after a brief discussion on how in-service training can continue the work begun in the pre-service phase.

The Professional Development of Teachers

The structure of pre-service education presented above demands and relies upon an anticipated form of in-service development. Professional development should be seen as a way to deepen and extend the knowledge of content and students as well as to improve the study of teaching (i.e. learning in and from practice) (Feiman-Nemser, 2001). Again, the literature suggests the form of such development quite clearly. Professional development is most successful in improving teaching and learning when it involves sustained, teacher-led collaborative investigations of classroom experiences through observations and critical discussions of content and student thinking (Desimone et al., 2002; Kazemi & Franke, 2004; Kennedy, 1998; Lieberman & Pointer Mace, 2010). By honing the "professional vision" (i.e. the "noticing" that is part of the on-the-fly decision making that teachers do) initiated during the pre-service training, teachers can use it to extract more professional development "content" that they can engage in (Jacobs, Lamb, & Philipp, 2010; Sherin & Han, 2004; van Es & Sherin, 2008). That is, the curriculum of professional development springs from teachers' collective practice and is driven by their questions. This has been described as "practitioner inquiry" (Hammerness & Darling-Hammond, 2005):

> The process of practitioner inquiry includes all aspects of a research or inquiry process: identifying questions of compelling interest…pursuing those questions through the collection of data (which may include observations of children, class or other observational field notes, or interviews with children, parents, or other teachers); and reflecting upon the questions through written work…and oral discussion with peers… (p. 438)

The idea of practitioner inquiry aligns with the pre-service experiences suggested by Hiebert and his colleagues (2007) described above. There is a promising structure that could help PSTs adopt a beginning "research stance" and, if continued, could nurture it into full "practitioner inquiry". An exploration of this structure—lesson study—will complete our initial discussion.

Lesson Study

A Bit of Background

When the Meiji government of Japan sought to modernize its political and social structure in 1868, one of its goals was to institute compulsory schooling. It sought to emulate the newly formed common schools of the United States, and also looked to the normal schools for teacher preparation practices. One of the ideas that the Japanese reformers found promising was that of a "criticism lesson", where educators developed a lesson, observed the lesson's delivery, and then offered critique (Makinae, 2010). This practice evolved into what is now the main form of professional development in Japan: lesson study (Yoshida, 1999).

When American educational researchers looked to identify the factors that led to outstanding mathematics performance of Japanese students in the 1980s and 90s, they encountered this fully developed national system (Hiebert & Stigler, 2000; Lewis, 2000). Since that time, the research showing the potential for professional growth and improvement of student learning through lesson study have been trumpeted by many researchers and professional groups (Conference Board of the Mathematical Sciences, 2012; Gersten, Taylor, Keys, Rolfhus, & Newman-Gonchar, 2014; Lewis, Perry, Friedkin, & Roth, 2012; Lewis, Fischman, Riggs, & Wasserman, 2013, National Research Council, 2010). I will describe briefly the lesson study cycle, and how the earlier-proposed structure of pre-service programs resonates with it.

The Lesson Study Cycle

Figure 5 displays the four phases of lesson study. The cycle begins when a lesson study community (or "team") selects a particular question stemming from their practice. Participants then design a lesson that serves as a hypothesized answer to that question. A member of the lesson study team delivers that lesson to his students, and the other teachers observe. The observation roles are set out beforehand, with the intent to seek out evidence of the lesson's effectiveness in meeting the explicit learning goals. Following the observation, the team discusses the evidence that was collected, and considers the implications of the data.

All of the knowledge produced throughout the cycle is stored for later use by the team (and others). During the cycle, the lesson study team is often aided by an outside expert known as a "knowledgeable other", and often consults a wide range of instructional materials, to craft their lesson. The cycle typically involves weeks or months of study to create the lesson plan. Therefore, the emphasis is clearly on the process and not necessarily the final lesson plan product (Takahashi & McDougal, 2015). I will now summarize how the opportunities to learn within a lesson study cycle overlay nicely with the structure of pre-service education that came from the literature synthesis.

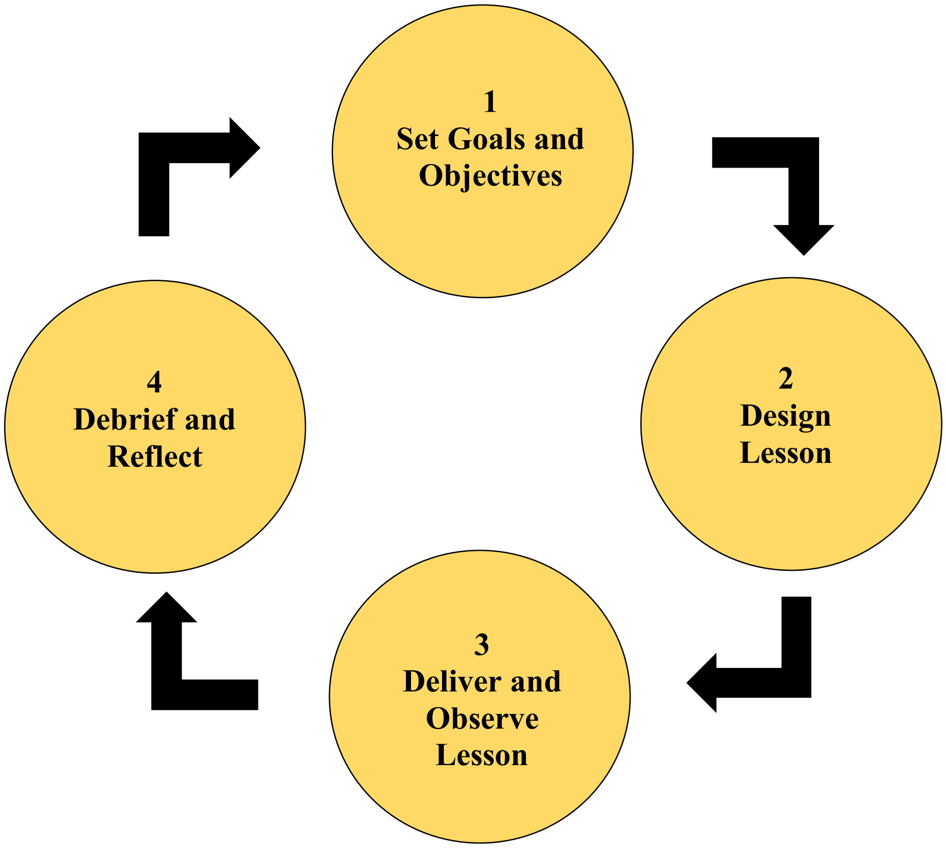


Figure 5. Phases of the lesson study cycle. Adapted from "Implementing Japanese Lesson Study in Foreign Countries: Misconceptions Revealed" by T. Fujii, 2014, *Mathematics Teacher Education and Development,* 16(1), p. 4. Copyright 2014 MERGA.

Constructing a New Vision of Mathematics within Lesson Study

There are many opportunities for pre-service teachers to construct a new view of mathematics within the lesson study cycle. Researchers have used the phrase "provoking a stumble" (Suzuka et al., 2009) to describe an activity which purposefully forces teachers to confront their misconceptions of a mathematical concept. Lesson study often provokes such a stumble (Fernandez, 2005). More importantly, lesson study can also provoke a *demand*. Through the deep study of other materials, discussions regarding student thinking, or reflections in the post-lesson discussion, the lesson study cycle often creates a demand for deeper learning (Lewis et al., 2013). If coordinated correctly, this can create a powerful connection between coursework and field experiences.

The examination of outside resources and observation phases of lesson study also present opportunities for PSTs to see mathematics differently. By integrating more student-centered activities, such as classroom discussion and student exploration, into the lesson plan, PSTs can witness the inventiveness and considerable mathematical talents that students naturally bring to the classroom. In addition, the "knowledgeable other" can introduce relevant research and curricular resources to help the group navigate the new territory (Watanabe, Takahashi, & Yoshida, 2008).

Creating New Knowledge for Action within Lesson Study

As mentioned above, the planning phase reveals gaps in content and pedagogical knowledge that can be addressed by deepening existing or creating new knowledge. The opportunity to deploy that new knowledge when the lesson is delivered offers an opportunity that is highly recommended in the literature. The "knowledgeable other" can help connect theory and practice as well as reflect on those connections in the post-lesson discussion (Takahashi, 2014). In a pre-service setting, the "knowledgeable other" might be made up of "knowledgeable *others*"—mathematicians, math educators, etc. Not only does the cycle allow PSTs to transform the knowledge learned in their coursework into usable knowledge in the classrooms, the cycle *creates* new knowledge. This is, by far, the most important point. Lesson study is a system within which teachers use their knowledge and skills to answer relevant problems, and, in doing so, create new information that can be drawn upon in other teaching situations.

Engaging in Complexity within Lesson Study

PSTs can experience the complexity of instruction in the planning and observation phases of the lesson study cycle. The planning phase allows "…the center of teaching expertise [to shift] from on-the-fly performance in the classroom to preparation and reflection outside the classroom" (Hiebert et al., 2007, p. 49). This allows the complexity to be pulled apart for study but then be reconstituted in the delivery of the lesson. By analyzing teaching for student learning (i.e. taking a research stance), the observers will also be exposed to specific teacher moves and routines in real time. Anticipating student responses during planning provides an opportunity to appreciate the complexity of instruction.

Preparing to Learn in and from Practice within Lesson Study

As lesson study is designed to create new knowledge, and has legitimate roles and levels of participation for pre-service, novice, and experienced teachers alike, it can serve as the system to *prepare* to learn from teaching during pre-service training, as well as to *continue* learning from teaching once in the classroom. Because collaboration and critical evaluation of teaching are so rare, the planning and post-lesson discussion phases of the cycle offer particularly rich environments to begin to engage in such work (Borko, 2004; Groth, 2011). Researchers often use the level of focus on student learning in lesson study discussions as a measure of lesson study's effectiveness (Murata, Bofferding, Pothen, Taylor, & Wischnia, 2012). That is, lesson study leads to teacher learning from practice that in turn leads to improved lesson study.

A Bit of Caution

I have presented a picture of lesson study as a system that is rich in practice-based opportunities to learn. Teachers in the United States are now more frequently encountering lesson study as an in-service professional development system (Takahashi & McDougal, 2015). Teacher educators have begun to use lesson study in teacher preparation programs in a variety of innovative ways (Hart et al., 2011; Huang et al., 2019; Larssen, Cajkler, Mosvold, & Bjuland, 2018; Ponte, 2017). If developed properly during pre-service, it is a practice that teachers can, with the proper support, bring with them into their first classrooms. Chokshi and Fernandez (2004) agree, but suggest a bit of caution, "Lesson study practitioners will encounter many opportunities to learn, but they will have to recognize them and develop productive strategies for doing so" (p. 521). To facilitate this recognition in preservice programs, teacher educators will have to collaborate and find the right mix of explicit and hidden scaffolding for this process.

Research Question

I have reviewed and synthesized literature that offers guidance on exploring two important areas. What teachers need to know in order to teach mathematics well, and the type of preparation that would best prepare them to do it, have been objects of an intense program of research. Long-standing assumptions about the nature of the mathematical knowledge needed for teaching, and the structure of pre-service teacher training have been challenged recently, and these challenges have opened up new possibilities to improve teaching and learning. The notions of specialized mathematical content knowledge and the social theories of learning have been particularly transformative. Connecting the knowledge and the training for teaching to the real work of the classroom offers new challenges to teacher educators. We must facilitate authentic experiences that allow PSTs to form a new vision of mathematics and its teaching, create new usable knowledge that can be deployed within the complexity of real teaching, and prepare to become lifelong learners from their own practice. Lesson study possesses great potential to serve as a broad structure for these experiences and serve teachers throughout all stages of their development.

Lesson study activities in methods courses and field placements, for both elementary and secondary PSTs, are well documented in the literature. For example, some researchers report on lesson study's ability to reveal gaps in the content knowledge of prospective secondary teachers and to address them (Fernandez, 2005). Recent large-scale literature reviews describe the use of lesson study in order to, for example, increase pre-service teachers' ability to attend to student thinking and to normalize collaboration (Hart et al., 2011; Ponte, 2017; Larssen et al., 2017). The study described in the next chapter was designed to extend the knowledge base regarding the effectiveness of lesson study in the development of PSTs in the United States. Its contribution is based on the placement of the lesson study activity at the *earliest possible stage of a teacher preparation program*—in the first mathematics content course designed for pre-service teachers. It was constructed in order to answer the question posed earlier:

> What types of learning are afforded or occluded by a modified lesson study activity within an early pre-service mathematics content course for elementary school teachers?

CHAPTER 3 THEORETICAL FRAMEWORK AND METHODOLOGY

Introduction

I undertook this study in order to understand the types of learning that are afforded or occluded by a modified lesson study activity within an early pre-service mathematics content course for elementary school teachers. It is hoped that this understanding will allow teacher educators more control over the opportunities to learn that PSTs encounter in their preparation programs (see Figure 1). Here I use "control" as Glaser (1978) intended: the ability to predict outcomes from new strategies and conditions, interpret what is happening, and to place unorganized specific incidents into a larger conceptual framework. A substantive theory offers this type of ideational organization; and it is this type of theory that I set out to induce from the data generated through this study.

Theoretical Framework

Learning

Let me first begin by defining what I mean by "learning." This definition is pivotal to the study because "our perspectives on learning matter: what we think about learning influences where we recognize learning, as well as what we do when we decide that we must do something about it" (Wenger, 1998, p. 9). Merriam-Webster (n.d.) defines learning as "knowledge or skill acquired by instruction or study, or modification of a behavioral tendency by experience (such as exposure to conditioning)." The models of teacher knowledge discussed earlier (e.g. MKT, the Knowledge Quartet) involve such facts and skills that could be acquired (i.e. learned) in this sense. But the models also involve notions of what they can *do* with the knowledge. That is, knowledge is recognized and categorized by how and what it allows people to participate in in the practice of teaching.

It is this engagement that will be at the heart of the definition of learning used in this study; and I turn to theories of situated learning and communities of practice as frameworks (Lave & Wenger, 1991; Wenger, 1998).

Learning within a Social Leaning System

A community of practice is a group of people mutually engaged in a joint enterprise through the creation and use of a shared set of resources. In a social learning system, "knowing" is defined as the ability to participate competently in a community of practice (Wenger, 1998). This competence is defined by the community of practice (and by other connected or surrounding communities) and may be at odds with a member's (or potential member's) own experience. This is an important state in a social learning system, because it is in the negotiation of this discord that learning occurs (Wenger, 2000).

Learning, from this perspective, is seen as "something that changes who we are by changing our ability to participate, to belong, to negotiate meaning" (Wenger, 1998, p. 226). A novice member would have little experience with the community's practice and would need to learn a great deal to be able to participate in its activities. Subsequent learning changes how the community views that person (as a more competent member) and how that person would view himself (as belonging to that community, with a better understanding of the meaning of the community's work). Conversely, the experiences (including new ideas and perspectives) of members—including novices—might force the community to adjust its notion of competence. That is, the negotiation/learning is not always one directional in a social learning system (see Figure 6).

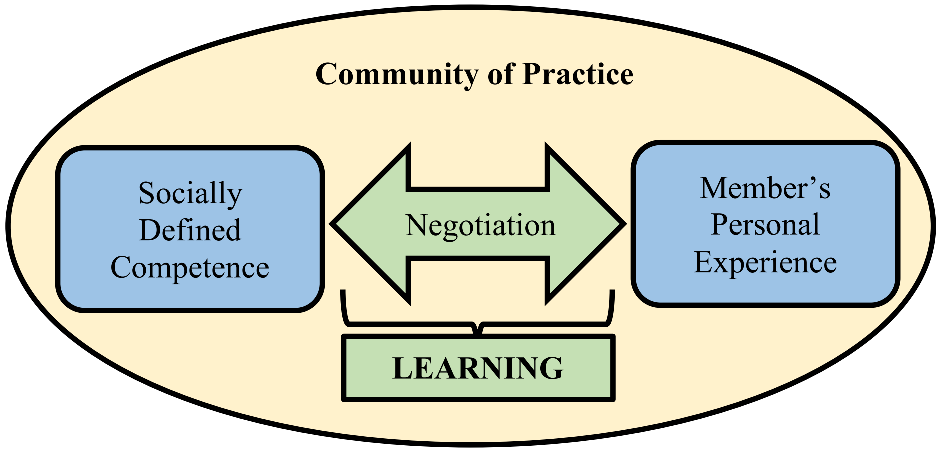


Figure 6. Learning in a Community of Practice

This social definition of learning incorporates more traditional notions of learning (e.g. acquisition of knowledge of facts and concepts, understanding of models and vocabulary, etc.). They are valued as learning insofar as they further a person's ability to engage in the practices of the community and deepen his sense of belonging to that community. In other words, there is no difference between knowing and doing (Wenger, 1998). This includes activities at the periphery of the community, as long as such participation involves legitimate, developmentally appropriate activities situated in that practice (Lave & Wenger, 1991).

Belonging to a community of practice involves members engaging in the practice, building an image of themselves and their community, and aligning their local work to broader communities. Engagement allows members to figure out what they are able to do, and what the effects of their actions can be. Again, this engagement can be at the periphery and often is, by design, for learners within the community. Imagining themselves and their community involves moving beyond direct engagement. It is an act of expanding their identities by means of images and models of what "could be." Adjusting practice, notions of competence, or visions of the possible due to pressures or influences from other communities, often communities that contain their own, are acts of alignment (Lave, 1998). These notions of engagement, imagination, and alignment harmonize nicely with the Framework of the Key Goals for Developing Mathematics Teachers: Early Stages, described in Chapter 1. Table 1 collects and connects their components. Together, these frameworks inform and support the study described in the following section.

Table 1

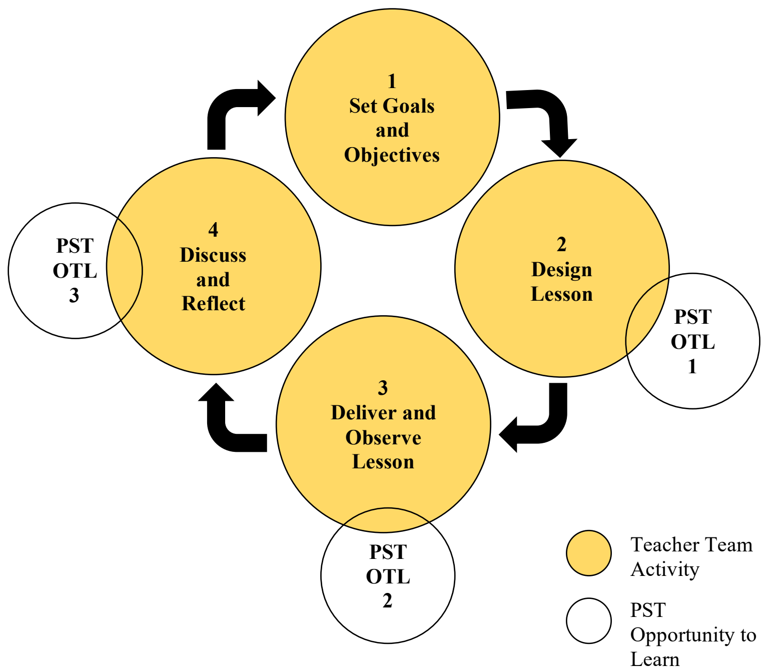
Key Goals: Pre-Service Preparation and Belonging to a Community of Practice

Key Goals for Developing Mathematics Teachers: Early Stages of Pre-Service Preparation	Belonging to a Community of Practice
Construct a New Vision of Mathematics	Imagination Building a new image of the subject and themselves as doers of mathematics. This new image should involve engagement in authentic mathematical activities.
Create New Knowledge for Action	Engagement Learning mathematics in a way that allows them to engage in the *teaching* of mathematics. Imagination Building a new image of themselves as teachers of mathematics.
Engage in the Complexity of Teaching	Engagement Participating peripherally in, and gaining an appreciation of, the complexity of live instruction. Imagination Forming an image of the possible in terms of mathematics instruction and student engagement.
Prepare to Learn in and from Practice	Engagement/Alignment Beginning to understand teaching practice as a social learning system where one continually negotiates the meaning of quality instruction.

Overview of the Study

PSTs from a university mathematics content course participated peripherally in a modified lesson study cycle initiated by an established lesson study team of veteran teachers (henceforth referred to as the teacher team) from a nearby elementary school. The particular content course from which the PSTs were selected is typically the first teacher education course in a PST's program, so, because of their inexperience with children, *peripheral* participation was appropriate (Lave & Wenger, 1991). Modifications to the traditional lesson study structure are common for pre-service teachers even in later stages of their preparation (Lewis, 2019), and these will be discussed shortly.

The PSTs were able to interact with veteran teachers engaging in the full complexity of delivering and reflecting on instruction, and to engage legitimately in this work. The PSTs participated in researching, planning and observing the lesson as well as a post-lesson discussion. The main charge of the PSTs was to conduct the content material study portion of the planning process known as *kyozaikenkyu* (Yoshida, 2012). After investigating and discussing the mathematics topic chosen for the research lesson, the PSTs incorporated that information into the teacher team's lesson plan. The PSTs were able to see their work's influence on the research lesson when it was delivered to students, and to participate in the post-lesson discussion. These activities, and the opportunities to learn, are represented in Figure 7.



PST OTL #

Opportunity To Learn #
Activity
(Potential PST Learning Focus Area)

PST OTL 1

Opportunity To Learn 1
Study Curriculum Materials: *Kyozaikenkyu*
(Constructing a New Vision of Mathematics, Creating New Knowledge for Action)

PST OTL 2

Opportunity to Learn 2
Observe Lesson and Focus on Student Thinking
(Engaging in Complexity of Instruction, Preparing to Learn In and From Practice)

PST OTL 3

Opportunity to Learn 3
Share Observation Data
Critique Lesson and Suggest Improvements
Observe Teacher Collaboration and Learning
(Constructing a New Vision of Mathematics, Preparing to Learn In and From Practice)

Figure 7. Phases of the lesson study cycle and potential opportunities to learn (OTLs) for pre-service teachers (PSTs) through peripheral participation, adapted from “Implementing Japanese Lesson Study in Foreign Countries: Misconceptions Revealed” by T. Fujii, 2014, *Mathematics Teacher Education and Development, 16*(1), p. 4. Copyright 2014 MERGA.

Throughout all of the activity's components, de-identified data—including prompted reflections, and discussion transcripts—were collected. All work products of the PSTs and the lesson study group were kept as reference points for analysis of the PSTs discussions and written reflections. The change in the PSTs' mathematical knowledge for teaching was measured using the Learning Mathematics for Teaching—Teacher Knowledge Assessment System (LMT-TKAS) (Hill et al., 2007), and the change in their beliefs regarding their own efficacy and their expectations of teaching outcomes were meaured using the Mathematics Teaching Efficacy Beliefs Instrument (MTEBI) (Enochs, Smith, & Huinker, 2000). This synthesis of the complementary qualitative and quantitative results helped create a more complete understanding of the PSTs' learning (Creswell & Plano Clark, 2011). In the following sections I will give a more detailed description of the structure of the study and its individual components.

Research Approach

I pursued the answer to my research question using the case study method. The case study approach has a long tradition in social science research and is identified as one of the primary approaches in the field of education (Merriam, 1988; Savin-Baden & Major, 2013; Yin, 1994). I investigated a bounded single unit of analysis in a holistic fashion (Savin-Baden & Major, 2013). The study was bounded because the number of participants, documents, and observation points of the study was finite. It considered a single unit of analysis: eleven PSTs enrolled in a mathematics content course designed for future elementary school teachers at a research university in a large city in the Midwestern United States, during the earliest stage of their professional preparation. For seven out of the eleven PSTs, the activity took place during the first semester of their teacher preparation program, but not necessarily their first semester at the university level.

The eleven PSTs making up the case represent the entire population of students enrolled in the first content course specifically designed for elementary school teachers. Despite the large total undergraduate enrollment (approximately 18,000 students), the number of students enrolled in these courses is small. Over the past three years, these courses have served, on average, approximately 15 PSTs per semester, 91% of whom were female and 58% were white. The percentage of white PSTs at the university is lower than the national average for elementary teachers (Snyder, de Brey, & Dillow, 2016), but reflects the diversity of the university's overall student body. Only students interested in entering the university's teacher preparation program enroll in these two content courses; so this nonprobability, purposive sample (Merriam, 1988) represented the earliest possible defined group of PSTs at this institution.

The study was holistic in the sense that it was used to understand the case as a whole but also considered the learning that occurs when individual participants interacted with particular opportunities to learn. The situated nature of the learning that this study sought to understand was, in the opinion of Yin (1994), a perfect candidate for a case study approach since, "a case study is an empirical inquiry that investigates a contemporary phenomenon within its real-life context, especially when the boundaries between phenomenon and context are not clearly evident" (p. 13).

This case study was conducted in order to understand the learning that takes place during an early lesson study activity. I sought to move beyond simply describing the learning to develop a theory of learning that other teacher educators could draw on in order to improve the preparation of their own PSTs. I wanted to construct a theory that would allow teacher educators to understand the relationships between the activity, the context, and the participants, and to help them anticipate particular outcomes. Merriam (1988) refers to this type of theory-building case study an *interpretive* case study, and offered guidance for the type of data collection best suited for this type of research:

> Naturalistic inquiry, which focuses on meaning in context, requires a data collection instrument sensitive to underlying meaning when gathering and interpreting data. Humans are best-suited for this task—and best when using methods that make use of human sensibilities such as interviewing, observing and analyzing. Nonprobability forms of sampling and inductive data analysis are consistent with the goals and assumptions of this paradigm, as are specific ways of ensuring for validity and reliability. (p. 3).

Heeding Merriam's advice, I used the data collection and analysis methods for this case study described in the following sections.

The Modified Lesson Study Activity and Data Collection

Lesson study activity. A modified lesson study activity was designed in order for the PSTs to participate in a legitimate peripheral manner, to foreground the study of mathematics content in the cycle, and to open up multiple data sources for analysis. Table 2 collects the Lesson Study Activities, the data collected, and the timing within the university's fifteen-week semester schedule. A more detailed description of the activities follows.

Lesson study introduction. I provided a brief (30-minute) introduction to lesson study and clarified the expectations of the PSTs during all of the modified lesson study activities.

Table 2

Overview of Lesson Study Activities, Data Collected, and Timing within the University Semester

	Data Collected	Timing
Lesson Study Introduction	Discussion Transcript & Written Reflections	Week 4
Kyozaikenkyu Discussion 1: Module Overview	Discussion Transcript	Week 8
Kyozaikenkyu Discussion 2: Terminology and Tools	Discussion Transcript	Week 9
Kyozaikenkyu Discussion 4: Three Steps to Mastering Multiplication Facts Article Discussion	Discussion Transcript	Week 10
Kyozaikenkyu Discussion 5: Lesson Plan – Revision and Anticipated Student Response Discussion	Discussion Transcript	Week 11
Review of Teacher Plan Discussion	Discussion Transcript	Week 13
Research Lesson and Post-Lesson Discussion	Video Recording & Discussion Transcript	Week 13
PST Discussion of Research Lesson and Post-Lesson Discussion	Discussion Transcript & Written Reflections	Week 14
Final Reflections	Written Reflections	Week 15

The introduction explained each component of the lesson study cycle, the typical use of lesson study in the professional development of in-service teachers, and the features of lesson study that run counter-cultural to teaching practice in the United States (e.g. intense collaboration, teacher agency in choosing which aspects of teaching and learning to study/improve, and the opening of classrooms to collegial observation and critique). The PSTs were prompted to reflect on any features of the lesson study cycle that stood out to them, in particular those features that challenged their notions of content knowledge demands and the amount of preparation and collaboration in teaching mathematics. Individual reflections were collected via Qualtrics, and the ensuing whole-class discussion was transcribed for analysis.

Kyozaikenkyu discussions. A few weeks after the lesson study introduction, the PSTs were informed of the foci of the teacher team's lesson study. These foci were based on the team's own learning goals, the timing of the study, and the school district's mandated curriculum: *Eureka Math* (Great Minds, 2015). The teacher team chose improving the understanding of multiplication for third-grade students as its content focus, and how to facilitate more effective whole-class discussions as its pedagogical focus. These goals meshed well with the content and format of the PST's college course, as the course included the study of whole-number multiplication (the definitions, properties, and models), and class sessions frequently involved small-group explorations and discussions in order to deepen understanding and promote equity of voice. The effect of this alignment will be discussed in subsequent chapters.

The work of the teacher team and the PSTs then proceeded in parallel. The PSTs began their own study of the instructional materials, focusing on the mathematical content of the lesson. The PSTs explored research-based materials including their own college class resources, the teacher team's district-mandated curriculum, and related research from practitioner journals (Kling & Bay-Williams, 2015). They used the following prompts provided by Watanabe and his colleagues (2008):

- What does this idea really mean mathematically?
- How does this idea relate to other ideas?
- What are common mistakes and how should teachers respond to those mistakes?
- What new ideas are students expected to build using this idea in the future? (p. 136).

In other words, they investigated the essential mathematics involved in the topic, the previous knowledge that students would draw on, and the future knowledge that would depend on this idea. The PSTs discussed their thinking in six full-class discussions that occupied one class period each week. These discussions were transcribed to allow for careful analysis and to create a chain of evidence. The teacher team met fortnightly throughout the semester, as part of a professional development series which focused on lesson study and equity.

Research lesson and post-lesson discussion. The learning from both groups was incorporated into a final lesson plan. I agreed to deliver the lesson to a class of third-grade students at the elementary school where the teacher team was based. The PSTs and the teacher team members observed and recorded data on the lesson's effectiveness. The PSTs were able to take note of how their *kyozaikenkyu* and discussions showed up in, and affected, the enacted lesson. In addition, the PSTs were instructed to observe how the students interacted around the content and to compare it to what they anticipated during their preparation. This allowed the PSTs to engage peripherally in the complexity of instruction and collaboration. The lesson was recorded to allow two of the PSTs who could not attend the live lesson to view the lesson being enacted. Immediately after the lesson's conclusion, one PST participated in the post-lesson discussion where observers shared the data that they collected during the lesson.

PST discussion of research lesson and post-lesson discussion. Following the observation of the live (or recorded) research lesson, the PSTs were asked to reflect, in writing, on the experience. In particular, they were asked to reflect on how the students interacted around the mathematical content (the explicit observation point from the lesson itself). These reflections were shared and expanded upon in a full-class discussion. In addition, the PST who participated in the post-lesson discussion described her experience with the class. The discussion was transcribed, and the written reflections were collected via Qualtrics.

Final reflections. PSTs were prompted to share their reflections of the entire process following the final activity. I asked them to write about what they learned, and which activity stood out to them the most.

Notable modifications of the lesson study cycle. It is important to acknowledge the modifications that were made to the lesson study cycle in order to make this activity accessible and meaningful to PSTs at such an early point in their preparation programs. Modifications are common when lesson study is used in teacher preparation programs, and most are introduced in an effort to accommodate the PSTs' fledgling subject-matter and pedagogical knowledge, inexperience with teaching children, or the lack of access to elementary students (Larssen et al. 2018; Lewis, 2019). Many of these modifications alter what many theorists and practitioners consider essential components of lesson study, often to the point of calling into question whether the activity can be considered "Lesson Study" (Ponte, 2017). However, even adjustments that seem to impinge on foundational ideas of lesson study (e.g. the teacher autonomy in constituting or joining a team, and the choice of focus for the study) can support PSTs in engaging with the powerful learning opportunities within lesson study (Lewis, 2019). But one must be careful to weigh the benefits and the costs of these modifications. The modifications for this study were based on the forementioned concerns as well as constraints of time and location.

Participation. Participation was, in some sense, involuntary. Most of the PSTs were in the first year of their preparation programs (for many it was the first semester/course), and all were enrolled in a mathematics content course for elementary school teachers. Their participation in the lesson study activities was a required part of this course, although they were given the opportunity to opt out of the lesson study activities and fulfil the requirement with an alternative assignment. Protocols for explaining and obtaining consent from the PSTs were approved by our university's Institutional Review Board. No one opted out.

Choice of focus. The PSTs had no experience teaching children, in general, and no experience with this particular population of children (context, norms, expectations, etc.). Therefore, they were not yet in a position to identify problems of practice or set overarching goals. The teacher team selected the foci of the lesson study, and the PSTs had no say in the matter.

Planning sessions. As mentioned earlier, the two groups studied and planned in parallel. The PSTs did not participate in the teacher team planning meetings which took place fortnightly on Saturdays as part of a school-university professional development partnership. The teacher team met for additional planning sessions after school in their school building.

The PSTs' university course sessions took place during the week, and the PSTs could not reasonably be required to attend extra sessions on weekends. Therefore, one university class session per week was dedicated to *kyonzaikenkyu* for the six consecutive weeks leading up to the research lesson. The contributions of both groups were then consolidated into a single lesson plan.

Delivering the lesson. A member of the teacher team did not teach the lesson; I did. This major modification stems from, among other concerns, the experience of the two constituent groups. Obviously, none of the PSTs was in a position to deliver a live lesson to students. Some members of the teacher team, although veteran teachers, were new to lesson study. Moreover, as part of the professional development series, they were exploring how mathematics instruction could address issues of equity. This was new to them, ambitious in scope, and, as one of the facilitators of the series, I felt that I could offer them an aspirational image of classroom instruction. I hypothesized that by delivering the lesson myself, it would create a sense of continuity for the PSTs between their university classroom and the third-grade classroom. There were other factors that led to this modification, not least of which was the unusual circumstances of the research lesson with nearly 20 observers (most of whom were from outside the school). These logistical issues and the learning goals for both the teacher team's own development and the PSTs led to the decision. I will discuss the rationale for and implications of this choice in the final chapters.

Post-lesson discussion. The final modification was based on the constraints of time and conflicts with the class schedules of the PSTs. Only one PST was able to participate in the post-lesson discussion with the teacher team. Important learning took place for that student, and this will be discussed in the next chapter. To mitigate the loss of this experience for the other PSTs, an additional post-lesson discussion without the teacher team was conducted in the university class session immediately following the research lesson.

Data Collection

Pre and post measures of Mathematical Knowledge for Teaching and beliefs. The Learning Mathematics for Teaching—Teacher Knowledge Assessment System (LMT-TKAS) (Hill et al., 2007) and the Mathematics Teaching Efficacy Beliefs Instrument (MTEBI) (Enochs et al., 2000) were administered online before and after the lesson study activities to measure changes in the PSTs' Mathematical Knowledge for Teaching and beliefs regarding teaching efficacy and outcomes.

Measuring the Mathematical Knowledge for Teaching. The Learning Mathematics for Teaching (LMT) measures were developed alongside the Mathematical Knowledge for Teaching framework (Ball et al., 2008) and contain items "intended to represent the mathematics problems encountered in teaching, rather than only 'common' content knowledge" (Hill et al., p.131). By turning to practice, i.e. looking at what teachers *do* in classrooms with their knowledge, Ball and her colleagues were able to ask important questions about content knowledge: what content do teachers need to know *for teaching* and *how* do they need to know it—for themselves or *for others*? The examination of practice that followed helped make visible an important new subdomain of subject matter knowledge discussed earlier: *specialized content knowledge* (2008).

Reliability estimates of between .72 and .81 have been reported (Hill et al., 2008) and higher scores on the LMT measures have been shown to be related to higher-quality instruction and increases in student learning (Hill et al., 2005). The LMT measures were administered via Qualtrics before and after the modified lesson study activities. This pre/post-test format to assess growth in the mathematical knowledge for teaching fit the intended use of the measures (Hill & Phelps, 2004).

Measuring teaching efficacy and outcome beliefs.

> Beliefs might be thought of as lenses through which one looks when interpreting the world, and affect might be thought of as a disposition or tendency one takes toward some aspect of his or her world; as such, the beliefs and affect one holds surely affect the way one interacts with his or her world. (Philipp, 2007, p. 258)

The literature supports a connection between teachers' beliefs about mathematics, their conceptions of math teaching and learning, and their instructional practice (Philipp, 2007). Researchers hypothesize that efforts to reform teaching practice may be hampered by a lack of attention paid to teachers' resilient beliefs (Hiebert & Stigler, 2000; Lortie, 1975; Philipp, 2007; Putnam & Borko, 2000). Therefore, it is warranted to investigate if the lesson study activity affected the beliefs of the PSTs.

The Mathematics Teaching Efficacy Beliefs Instrument (MTEBI) uses a set of Likert scale responses (Strongly Disagree to Strongly Agree) to measure the *personal mathematics teaching efficacy* (PMTE) and the *mathematics teaching outcome expectancy* (MTOE) of the respondents (Enochs et al., 2000). The PMTE is defined as "a belief in one's ability to teach effectively" (Enochs et al., 2000, p. 195), and the MTOE as "the belief that effective teaching will have a positive effect on student learning" (Enochs et al., 2000, p. 195). For example, the prompt "I understand mathematics concepts well enough to be effective in teaching elementary mathematics" reflects a belief in one's teaching efficacy; while the prompt "Students' achievement in mathematics is directly related to their teacher's effectiveness in mathematics teaching" seeks to measure the PST's belief in outcome expectancy. These two sub-scales have shown acceptable to good reliability levels (.77 for the MTOE and .88 for the PMTE), and strong construct validity (Enochs et al., 2000). The MTEBI were administered via Qualtrics in a pre/post-activity model.

Analysis

As with the collection of data, the case study strategy does not prescribe a specific form of analysis. Because I sought to build a theory that will contribute to the understanding of PST learning in early stages of their preparation, grounded theory methods guided my analysis. Analysis grounded in the data fits Yin's (1994) criteria for high-quality analysis because it relies on all relevant evidence, considers "rival" interpretations, is driven by the most important emergent aspects of the case, and allows me to use my own expertise regarding the context of the study.

The constant comparative method of data analysis, described by Merriam (1988) as dynamic and recursive, involved joint coding and analysis of data (Charmaz, 2006; Glaser & Strauss, 1967). The analysis was dynamic in that incoming data narrowed the focus to key emerging ideas as well suggested new data collection opportunities. The analysis was recursive because it involved iterations of explicit coding and analytic procedures based on previous categories and properties.

Repeated themes that emerged from the initial data analysis were recorded as analytic categories. These categories had clear inclusion criteria and had to be relevant to the study, mutually exclusive, independent, and exhaustive (Merriam, 1988). The inclusion criteria as well as the properties of the categories were recorded using analytic memos to increase the rigor and clarity of this first level of analysis and abstraction.

Hypotheses, in the form of links between categories and their properties, were formed via constant comparison of new categories and properties (Charmaz, 2006; Glaser & Strauss, 1967; Yin, 1994). This increased level of abstraction allowed a theory to be generated.

Validity and Reliability

The question of validity—how closely the results match reality—can be viewed as problematic for the qualitative aspects of this study. As the primary research instrument, I understand that my findings will be judged by how well they represent and explain the experiences of the participants. In order for interested readers to feel comfortable using the (hopefully practical and actionable) results, they must trust them. To help ensure the *credibility* and *transferability* of this explanation, Glaser and Strauss (1967) urge a presentation of the methods, analysis, and results that make the reader feel vicariously immersed in the field. This resonates with Merriam's (1988) belief that, in educational research, concerns over validity and reliability can be addressed by the level of attention given to issues of why and how the study was conceived, how the data is collected and analyzed, and how theoretical connections were constructed.

The methodological coherence of the study, the convergence of multiple sources of evidence (i.e. triangulation), and the maintenance of a chain of evidence, mentioned previously, are strategies that I used to ensure the quality of this study. Follow-up discussions allowed me to verify my interpretations of their experiences (so-called "member checking") and I engaged experts in peer audits of my analysis. The dense description of the context, analysis, and findings will enable other researchers to judge the transferability and dependability (in quantitative contexts referred to as external validity and reliability) of the results of the study (Merriam, 1988; Savin-Baden & Major, 2013).

Conclusion

To understand the learning that occurs in a modified lesson study activity, I created an interpretive case study of pre-service teachers enrolled in a mathematics content course at a large research university in the Midwestern United States. Based on the recommendations of qualitative researchers, I chose data collection and analysis methods that best align with the goal of constructing a grounded theory to help understand and interpret the learning that the activity affords or occludes. My goal was to contribute to the knowledge base of teacher education and to offer mathematics educators more control over a specific set of learning outcomes in their preparation programs.

I acknowledge that it was challenging to remain open to the emerging data. As a passionate participant, I needed to heed the advice of Glaser (1978): "Generating good ideas also requires the analyst to be a non-citizen for the moment so he can come closer to objectivity and to letting the data speak for itself" (p. 8). The data spoke volumes. I share what I heard from the data, i.e. the results of the study, in the next Chapter.

CHAPTER 4 RESULTS

Introduction

In this chapter, I will organize the presentation and the analysis of the data generated by the study into two parts: analysis of the quantitative data and analysis of the qualitative data. The quantitative data will allow me to identify any significant change in the PSTs' Mathematical Teaching Efficacy Beliefs and Mathematical Knowledge for Teaching over the course of the study. The complementary qualitative data will allow me to better understand that change as well detect learning viewed from the perspective of Wenger's Community of Practice framework (1998). In addition, the analysis will be further supported and organized by my framework of Key Goals for Developing Mathematics Teachers: Early Stages of Pre-Service Preparation (see p. 17). I begin with the quantitative results.

A Bit of Context

Early in the design phase of this study, I realized that it would be difficult to distinguish between the effects of the lesson study activities and the university course activities that the PSTs engaged in during the semester. This attribution problem is common in educational research (Everson, 2017). However, when the *course* activities are viewed as a source of new information (both in terms of content and pedagogy) for the lesson study activities, then they can be viewed as resources that support a particularly deep exploration of curriculum materials: *kyozaikenkyu* (Watanabe et al., 2008). In short, what and how the PSTs experience mathematics in the course become the "raw materials" that can be drawn upon as they engage in the lesson study cycle.

In effective lesson study cycles, this mathematical knowledge for teaching is exactly the type of information that is explored by the teacher team and introduced by the Knowledgeable Other (Takahashi & McDougal, 2015). Philipp (2007) reminds us, “While students are learning mathematics, they are also learning lessons about what mathematics is, what value it has, how it is learned, who should learn it, and what engagement in mathematical reasoning entails” (p. 257). Thus, the university course work and the values it stems from inform and support the activities of the study. This is important to keep in mind since Fujii (2014) sees lesson study as an “organic system” where the *educational values* of the team “is always tied to, influenced by, and reflected in, the key features of lesson study” (p. 13). In order to understand what the PSTs learned during the lesson study activities, it is important to understand the values that were embedded in what and how students engaged in mathematics during their university course.

The structure of the university course was designed to offer PSTs opportunities to learn within authentic mathematical practices. Bass (2015) describes this type of mathematical work as “progress[ing] through a trajectory that [he] describes as *exploration, discovery, conjecture, proof, and certification*” (p. 631). Therefore, each week, the PSTs were asked to analyze, present, and explain key mathematical ideas embedded problems that involve key topics from the elementary mathematics curriculum. This collection of analysis, presentation, and explanation has come to be known as the APEX Cycle (Ozgun-Koca, Zopf, & Nazelli, 2020). During the analysis phase, PSTs work on problem sets that force them to confront their often fragile and compressed knowledge of elementary mathematics concepts. They struggle productively to understand the concept and to connect different representations and models. They are then asked to present their understandings in a whole-class discussion where the PSTs build a coherent, mutually accepted solution. Each is asked to defend his or her claims against the collegial scrutiny and critique of others. These rich discussions produce frequent stumbles for the presenter and opportunities for the classroom community to notice value (either potential or clouded) in another's thinking. Bass (2015) highlights this ability as a crucial aspect of high-quality teaching:

> The kind of mathematical knowledge required includes not only a robust understanding of the mathematical terrain of the work and corresponding learning goals but also an ability to hear, in incipient and undeveloped form, significant, though often not entirely correct, mathematical ideas in student thinking. I emphasize that the latter entails a special kind of knowledge of mathematics, not just psychology. Teaching requires the skills to not only hear and validate and give space to these ideas, but also to help students reshape them in mathematically productive ways. (p. 636)

The notions of *being heard* and *ideas being validated* in the APEX Cycle were mentioned frequently by the PSTs in their reflections on the lesson study activities. I will explore this connection between our course and lesson study activities more deeply in the sections that follow. Lastly, the PSTs were asked to explain, in their own terms, the solution to the problems. These journal entries involve thorough explanations of the conceptual underpinnings of and connections between the problems, including all algorithms and representations.

Quantitative Analysis

Beliefs

Phillip (2007) acknowledged that there is not agreement in the field about the definition of *belief*. He offered the following working definition of beliefs that I will adopt. Beliefs are

> [p]sychologically held understandings, premises, or propositions about the world that are thought to be true. Beliefs are more cognitive, are felt less intensely, and are harder to change than attitudes. Beliefs might be thought of as lenses that affect one's view of some aspect of the world or as dispositions toward action. Beliefs, unlike knowledge, may be held with varying degrees of conviction and are not consensual. Beliefs are more cognitive than emotions and attitudes. (p. 259)

In this study I choose to view learning as defined within the Community of Practice Framework (Wenger, 1998). Therefore, it is important to measure change in the PSTs' beliefs about the efficacy of their actions and the efficacy of teaching in general, as they affect their dispositions to actively engage in the practice of teaching.

Mathematics Teaching Efficacy Beliefs Inventory. The MTEBI (Enochs et al., 2000) consists of 21 prompts with 5-point Likert scale responses (Strongly Agree = 5, Agree = 4, Uncertain = 3, Disagree = 2, and Strongly Disagree = 1). Therefore, the range of possible total scores on the MTEBI is 21 to 105. The PMTE subscale is made up of 13 prompts, and the remaining 8 prompts constitute the MTOE subscale, with ranges of 13 to 65 and 8 to 40, respectively. Recall that the PMTE, the Personal Mathematics Teaching Efficacy subscale, measures the "belief in one's ability to teach effectively," while the MTOE, the Mathematics Teaching Outcome Expectancy subscale mesures the "belief that effective teaching will have a positive effect on student learning" (Enochs et al., 2000, p. 195). "I know how to teach mathematics concepts effectively" is an example form the

PMTE subset, while "Students' achievement in mathematics is directly related to their teacher's effectiveness in mathematics teaching" (Enochs et al., 2000, p. 200) is an example from the MTOE subset. The boxplots shown in Figure 8 give an overview of the data for the total MTEBI, the PMTE, and the MTOE pretest and post-test scores. I conducted paired-samples *t*-tests on each pre/post score pair and report the results separately.

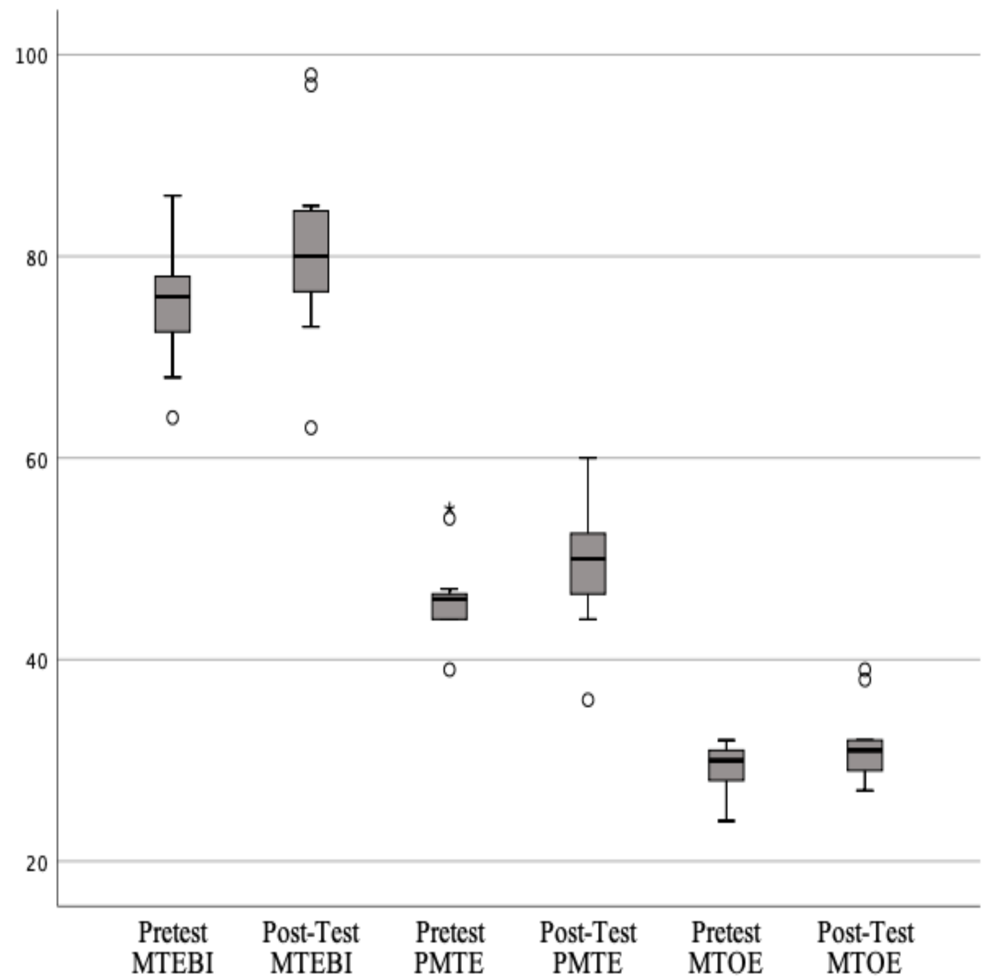


Figure 8: Boxplot displays for pretest and post-test scores for the Mathematics Teacher Efficacy Beliefs Inventory (MTEBI), the Personal Mathematics Teaching Efficacy (PMTE) Subscale, and the Mathematics Teaching Outcome Expectancy (MTOE) Subscale.

MTEBI total score. Table 3 displays the means, standard deviations, and sample sizes (*n*) for the MTEBI pretest and post-test. On average, MTEBI scores increased by 5.64 points. This difference was not significant t (10) = 2.138, p =. 058, but represented a large-sized effect, with Cohen's $d = (81.09 - 75.45)/6.44 = .87522$ (Cohen, 1992). Due to concerns over questionable normality of the data and small sample size a (non-parametric) Wilcoxon Signed Rank Test was performed. This analysis showed that the increase in student MTEBI scores *was significant*, T (10) = 55.5, p =. 045.

Table 3

Descriptive Statistics for the MTEBI Pretest and Post-Test Scores

MTEBI Scores	Mean	Standard Deviation	*n*
Pretest	75.45	6.44	11
Post-test	81.09	10.08	11

This significant increase in the total MTEBI score can be interpreted as a change in the PSTs' identities, as it shows that they believe that they can engage more effectively in the practice of mathematics teaching. This positive change seems to align with other studies that suggest activities involving observations, discussion, and reflection on teaching have strong potential to change participant beliefs. This leads to natural questions: What led to this growth? Is it possible to track this change in beliefs through the class discussions and reflections? We will turn to the qualitative data to help answer these questions. To better understand this increase in the PSTs beliefs about their efficacy, I first examined the change in scores on its two subscales.

PMTE subscale. Table 4 displays the means, standard deviations, and sample sizes for the PMTE subscale scores. On average, students PMTE scores increased by 3.18 points. This increase was *not significant*, t (10) = 1.606, p = .139, but represented a medium-sized effect, d = .7026. A Wilcoxon Signed Rank Test affirmed the result, without the assumption of normality, T (10) = 40, p = .201.

Table 4

Descriptive Statistics for the PMTE Pretest and Post-Test Scores

PMTE Scores	Mean	Standard Deviation	*n*
Pretest	46.36	4.54	11
Post-test	49.55	6.61	11

This indicates that the PSTs' believe that they are now able to teach more effectively, but the effect of the lesson study activities on this particular aspect of their beliefs about teaching was not as strong as the overall effect. This is understandable. The lesson study and university course activities foregrounded issues of *content*. The PSTs might not yet fully appreciate the impact of their increased content knowledge on their ability to teach effectively. Evidence from classroom discussions and PST reflections throughout the lesson study cycle indicate substantial increases in mathematical knowledge for teaching—more substantial, possibly, then the PSTs might recognize themselves. For example, I will present examples of discussions where the PSTs accessed an expanding repertoire of representations of multiplication of whole numbers and explored the benefits and limitations of each. The PSTs might not appreciate this as mathematical knowledge *for teaching*; but it surely is. I will delve into that qualitative evidence shortly.

MTOE subscale. Table 5 displays the means, standard deviations, and sample sizes for the MTOE subscale scores. On average, students MTOE scores increased by 2.45 points. This difference was *significant*, t (10) = 2.540, p = .029, and represented a large-sized effect,

d = .8978. A Wilcoxon Signed Rank Test affirmed this significance, without the assumption of normality, T (10) = 48.500, p = .032.

Table 5

Descriptive Statistics for the MTOE Pretest and Post-Test Scores

MTOE Scores	Mean	Standard Deviation	*n*
Pretest	29.09	2.74	11
Post-test	31.55	3.90	11

Thus, there was a large, significant effect on the PSTs' beliefs about the connection between effective teaching and student learning. This increase was in the beliefs about the outcomes of effective *teaching* as opposed to beliefs about themselves as *teachers*. Again, the lesson study activity was situated in the first content course of their university preparation programs, and there was relatively little time dedicated to opportunities where the PSTs took on the explicit role of teacher. In contrast, there was a large amount of time dedicated to negotiating the practices of, and the concerns surrounding, teaching. These included working through the mathematics problems that students will encounter, discussing challenges that affect teaching inside and outside of the classroom, exploring curriculum materials, anticipating student responses and planning responses, etc. The PSTs were able to experience the effect of a new type of mathematics class themselves through their university classroom (especially the APEX cycles) and observe this effect in elementary school students during the research lesson. In addition, lesson study purposely takes the attention away from the teacher and places it on teaching (Hiebert & Morris, 2012; Lewis, Perry, Friedkin, & Roth, 2012). This is a subtle change of focus away from the more visible performance and stylistic aspects of teaching—teacher traits—to the methods used in teaching mentioned above. Again, I will theorize about this growth with the help of the discussion transcripts and student reflections during the analysis of the qualitative data.

Mathematical Knowledge for Teaching

The Learning Mathematics for Teaching—Teacher Knowledge Assessment System. The Learning Mathematics for Teaching—Teacher Knowledge Assessment System (LMT-TKAS) (Hill et al., 2007) was used to assess any change in the PSTs' Mathematical Knowledge for Teaching (MKT) during this study (Ball et al., 2008). The PSTs responded to prompts that required them to draw upon the different domains of of this knowledge. An example is given in Figure 9. In order to respond to the prompt, the PSTs need to do something that is not commonly (if ever) done outside of teaching: to look at a computation and decide if the method used is generalizable to other numbers or specific to this particular computation. The thinking is wholly based in mathematical content, and specific to teaching (Hill et. al, 2007).

Imagine that you are working with your class on multiplying large numbers. Among your students' papers, you notice that some have displayed their work in the following ways:

Student A	Student B	Student C
35	35	35
× 25	× 25	× 25
125	175	25
+75	+700	150
875	875	100
		+600
		875

Which of these students would you judge to be using a method that could be used to multiply any two whole numbers?

	Method would work for all whole numbers	Method would *not* work for all whole numbers.	I'm not sure
Student A	1	2	3
Student B	1	2	3
Student C	1	2	3

Figure 9: Example of an LMT-TKAS item adapted from "Assessing teachers' mathematical knowledge: What knowledge matters and what evidence counts?" by H. Hill et al., 2007, *Second handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics*, 1, p. 132. Copyright 2017 Information Age Publisher.

The LMT-TKAS produced scores for each PST, given in standard-deviation units. These standard deviation units were created using the large data sets collected by the system's developers, and to which my data was added. Table 6 displays the means, standard deviations, and sample sizes for the LMT-TKAS scores.

Table 6

Descriptive Statistics for the LMT-TKAS Pretest and Post-Test Scores

LMT-TKAS Scores	Mean	Standard Deviation	*n*
Pretest	- .1127	.628	11
Post-test	- .1068	.615	11

On average, students LMT scores increased by only .00592 points. This increase was not significant t (10) = .029, p = .978, and represents almost no effect: d = .01. A Wilcoxon Signed Rank Test affirmed the result, without the assumption of normality, T (10) = 30, p = .790. I attribute this lack of any noticeable change in the PST's LMT scores to a few important extenuating circumstances.

I selected the Elementary Number Concepts and Operations – Content Knowledge as the specific LMT for this study. The pretest contained 28 questions covering topics that include whole numbers, integers, fractions, and decimals, and the post-test contained 29 items. This learning measure has the largest intersection with the mathematical concepts covered in the PSTs' content course and the topics explored in the lesson study, making it the best fit of all of the available measures. The intersection still represented only 75% (21 out of 28 questions) of the LMT pretest and 79% (23 out of 29 questions) of the LMT post-test. The intersection was decreased further by a limitation of the study that I will discuss in greater detail in the next chapter; but I mention the relevant portion here. The lesson study activities took a portion of the already limited instructional time away from the standard course plan. One result of this was that the planned unit on Rational Numbers (i.e. fractions) had to be postponed until the second course of the university sequence (and, hence, after the measures were taken by the PSTs). This lowered the intersection of topics to only 36% (10 out of 28 questions) and 52% (15 out of 29 questions) of the pre- and post-test, respectively.

In addition to the lack of alignment with the LMTs and the content of both the lesson study and the university course, the relatively short length of the study's activities must be taken into account. The university course, in its entirely, represented roughly 40 hours of instruction time; and the lesson study activities occupied approximately 10 hours of time both in and outside of class. Lesson study is built on the idea of steady, incremental improvement (Stigler & Hiebert, 1999), and the current study's activities are conceived as an initial step in the PSTs' multi-year preparation programs and careers. The amount of change that I expected to see over such a short time period of time was small to begin with, and these circumstances made change even more difficult to discern.

The developers of the LMT have acknowledged that both of the limitations mentioned above could make it difficult for the measures to detect the effects of educational and professional development activities (Hill & Phelps, 2004). Therefore, it is understandable, but still concerning, that the LMTs did not show growth in the PSTs' Mathematical Knowledge for Teaching (Ball et al., 2008). A teacher's Mathematical Knowledge for Teaching is essential to teaching well (Ball et al., 2008; Hill et al., 2005; Hill et al, 2008); and the lack of change on this assessment is important to note. Thankfully, the design of the study offered other data sources that help us to better understand change in the PSTs knowledge base and ability to participate in the practice of teaching. I will now turn to the qualitative data they produced.

Qualitative Analysis

The initial open coding of six classroom discussion transcripts and three written reflections of the PSTs yielded 56 individual codes. These codes were subsumed within ten categories that brought the data back together. These ten categories led to creation of three broader themes: *Teacher Challenges, Past-Present-Future: Math and Math Class, and Preparing for, Enacting,* and *Learning from Engaging and Validating Instruction*. The progression from codes to categories and themes is presented in Table 7.

For reference, I include my initial codebook in the Appendix A. During the coding process, I engaged expert qualitative researchers in peer audits of my analysis. During these peer audits, we first coded selections of randomly selected discussion transcripts independently and then compared our results. At first, the early initial coding checks resulted in approximately 70% agreement of code assignments. This showed that I needed to refine my code descriptions and to include more context to help my coding partner understand the codes. Through these efforts and additional discussion, most differences in coding were able to be reconciled, and later coding audits of other documents resulted in consistent agreement rates of over 80%.

Table 7

Coding Progression: Initial Codes to Categories to Broader Themes

Initial Codes	Categories	Broader Themes
Teacher Challenges (Time) Teacher Challenges (Diversity of Learners) Teacher Challenges (MKT) Teacher Challenges (Competency) Teacher Challenges (Management) Teacher Challenges (Different Curricula) Math Phobia/Animosity	Negotiating Teacher Challenges: Classroom	Teacher/Teaching Challenges
Teacher Challenges (Pay) Teacher Challenges (Status) Teacher Challenges (Funding) Teacher Challenges (Autonomy) Teacher Challenges (Structural) Teaching and Learning US Culture	Negotiating Teacher Challenges: Broader Community	
Comparing/Contrasting Own Experience Past Teachers Memorization Comparing/Contrasting Other Observations	Reflecting on Past Experience	Math and Math Class, Past-Present-Future
New View of Mathematics Content Knowledge Changing View of Math Self Connections Multiple Solution Paths Representations Connections to Our Course	Reflecting on Current Experience	
My Future Classroom Validating Students Student Engagement Presenting Content Enthusiasm Classroom Discourse	New Vision of Math Class	
Resources Curriculum Guidance Curriculum Progression Curriculum Applications Building Community (Context) Building Community (Language) Building Community with Inservice Teachers	Understanding Resources and Context	Engaging and Validating Instruction: Preparing, Enacting, and Learning
New Knowledge for Action Complexity of Live Instruction Proposing Teacher Move Live Lesson Depth of Understanding Instructional Decisions Behind the Scenes	Preparing for Instruction	
Teacher Student Relationships Lesson-Teaching Effect on Students Teacher Learning and Student Performance	Teacher-Student Interaction	
Conjecturing about Student Thinking Understanding Student Thinking Assessing Student Learning Describing Student Actions	Student Thinking and Actions	
Research Stance Continual Learning Opening Up Classrooms Collaboration Early Stage of Preparation	Preparing to Learn in and from Practice	

I will begin the analysis of the qualitative data chronologically, according to the opportunities to learn (OTLs) identified in Chapter 3 (see Figure 7 on page 41). I will then present an analysis of themes that emerge when one steps back to view the activities as a whole. Each proposed opportunity to learn did, in fact, open up space for the PSTs to struggle productively with issues of mathematical content, pedagogy, and their visions of mathematics class as a site of learning for their students and themselves; but there was also a surprise. This unexpected opportunity to learn brought up issues with which the PSTs continued to negotiate across the entirety of the lesson study activities. I will begin with this unforeseen opportunity: the presentation that introduced lesson study to the PSTs.

Please note, all names used in the following transcriptions and reflections, except my own, are pseudonyms. In addition, filler words (e.g. "um" and "like") have been removed from discussion transcripts in the service of clarity.

Opportunities to Learn

An unanticipated opportunity to learn: The introduction to lesson study. At the end of the first month of the university class, I delivered a brief introductory presentation on lesson study (Figure 10). This presentation described the lesson study cycle, offered a bit of history of lesson study in Japan and US contexts, and highlighted the features of lesson study that make it a model of high-quality professional learning (Smith, 2001). This presentation was intended to be informational, that is, to provide the PSTs with a broad overview of lesson study as context for the activities that they would be engaging in throughout the semester. Unexpectedly, it opened up a passionate conversation about the PSTs' understanding of teaching and learning in the United States, and the concerns that they were bringing into our class and to the lesson study activity. In addition, it created an opportunity for the PSTs to share their perspectives on both teaching in general and their initial understanding of the affordances and challenges of lesson study. The majority of the post-presentation discussion (54% of the codes) centered on what the PSTs understood as challenges of teaching.

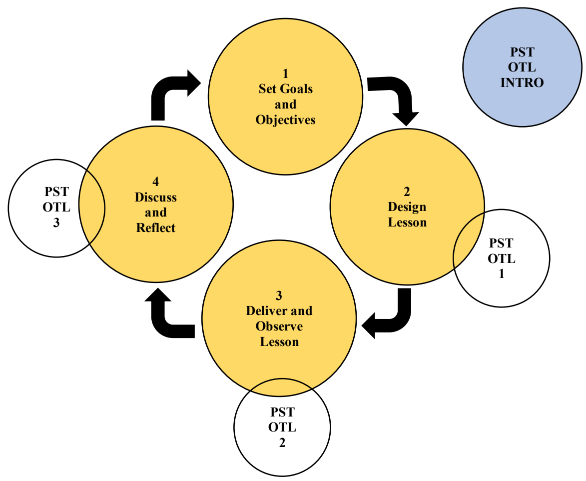


Figure 10. Opportunity to Learn (OTL): Introduction to Lesson Study Presentation.

Time. Several PSTs voiced trepidation over the time investment required for lesson study in light of already high demands on teachers' time. This concern is well founded, and the issue of time demands within lesson study has been described by Lewis (2016) as "profoundly countercultural…in the US context" (p. 539). However, their statements were not wholly about lesson study but rather dealt with broader concerns of teaching and teachers.

Abbey, a PST who described herself as "not extremely confident" in mathematics and as someone who would "often struggle with it," captured the feelings of other PSTs with respect to time in her post-presentation reflection. Her response indicated that she was also concerned with the work ethic of her soon-to-be colleagues as they undertake the important work of teaching. She wrote,

> My first impression of the lesson study was it seemed time consuming. I figured that is a main reason it usually is not applied in the United States because Americans have a stigma of being lazy. They want to do the work that is required for them and only that work. Most are not dedicated to constantly perfecting their craft, and that goes for even [*sic*] profession, not just teaching. However, it is especially crucial teachers are constantly trying to improve their skills, since their performance directly effects the future of our country. (Abbey, Post-Introduction Reflection)

Abbey's concern was echoed by Sadie who added,

> I think that it is a very good idea, however, the amount of time concerns me. Teachers have a bunch of stuff already on their plate that this doesn't seem extremely realistic. The time aspect is extremely challenging, and I feel like getting teachers on board may also be challenging. (Sadie, Post-Introduction Reflection)

It is interesting to note that Abbey, Sadie and other PSTs, at such an early point in their preparation programs, were already grappling with issues of teaching culture and negotiating the expectations they had of themselves and of other teachers. The introductory presentation on lesson study served as a sound board for the PSTs' concerns with which they entered our course. One challenge in particular—the diversity of learners that the PSTs will be working with in their future classrooms—featured prominently in what the PSTs said and wrote after being introduced to lesson study.

Diversity of learners. A small detail from the introductory presentation initiated a discussion that would continue throughout the entirety of the lesson study activity for the PSTs. The detail that prompted this discussion was that, in Japan, lesson study is used as a mechanism to introduce and test new topics that are being considered for their national curriculum. I mentioned that, because the curriculum was shared by every school in the nation, on any given day one could be fairly confident that every Japanese third-grade student would be going over the same lesson. Two PSTs had strong reactions to this idea. For Abbey the uniformity of a common curriculum suggested a rigid system that could stunt individual thinking and teacher-student relationships.

> I don't get how that's productive, because not all kids are going to learn the exact same way. I always feel in that sense of such stringent structure and everything, you're not giving kids the opportunity to think about things in a different way, and develop their own sense of learning, and their own understanding of concepts. In that sense, then *why even have teachers?* (emphasis added) Because, why not just play a video for them? Have them learn…like the teacher's supposed to adapt to the class…and teach the kids how they're learning and how they're understanding the concepts. So, if it's so centralized then it just…it takes the whole relationship and purpose away from the teacher. (Abbey, Post-Introduction Discussion Transcript)

Prudence, a PST who had spent time in elementary classrooms before as part of an early field experience, added that she was concerned that a common curriculum and the prescribed pacing that would necessarily accompany it could have implications on equity. She followed up on Abbey's comment by sharing,

> I just think with the whole lesson thing you were saying and how you brought up, Abbey, how every kid is learning the same thing every day…it kind of makes me think of what about the kids that can't keep up with that? So, if they don't take time to go at a slower pace, it makes me feel, in a way, they're weeding out the kids that can't keep up to their standards and their structure. (Prudence, Post-Introduction Discussion Transcript)

Although motivated by the discussion about the Japanese system, it is clear that issues of how teachers reconcile a common curriculum with a classroom full of students who bring different competencies, needs, and ways of thinking were very much on the minds of these PSTs at the onset of our activity. Their comments indicate a more than nascent sense of the complexity of teaching, especially as it pertains to the understanding and inclusion of all students. As I report on the analysis of the data springing from the other opportunities to learn, I will track how these issues of managing the diversity of students evolve for Abbey and Prudence, and how other PSTs are drawn into this negotiation.

A pause in the lesson study activities for the pre-service teachers. The next lesson study activity, the curriculum study, occurred three weeks after the introductory presentation. During this three-week period, the teacher team from the local elementary school finalized their choices for the foci of their lesson study. They chose to focus on the commutative property of multiplication as a tool for supporting third-grade students as they learn their multiplication facts. Additionally, the teacher team decided to study how to engage students in productive whole-class discussions.

These choices aligned perfectly with the content and pedagogical aims of the PSTs' university course; that is, they would soon be studying the properties of multiplication of whole numbers and do so through collaborative exploration and discussion. As mentioned earlier, the PSTs would engage in the APEX cycle (Analysis, Presentation, and Explanation) while studying the topics of elementary school curriculum—in particular multiplication of whole numbers. Thus, the PSTs began to study multiplication in their university course at the same time as they began the curriculum materials study component of the lesson study cycle. Again, the alignment of the university course and the lesson study activities, in terms of content, pedagogical goals, and now *timing*, allowed the activities to feel like a natural part of the course.

Opportunity to learn 1: Curriculum material study (*kyozaikenkyu*). In this section, I will analyze the data that was collected during the PSTs' study of curriculum materials (or *kyozaikenkyu)*. Figure 11 shows this opportunity to learn (PST OTL 1) and where it falls within the lesson study cycle. I will divide my analysis of the data from this opportunity into five parts as this data was generated by the five separate, whole-class discussions during the curriculum material study. The main resource for the study was the *Eureka Math Grade 3 Module 3 Teacher Guide* (Great Minds, 2015), the adopted curriculum of the teacher team's school district. In addition, as mentioned above, our university course's curriculum also represented a valuable source of information to help understand the mathematics involved in, and surrounding, the lesson.

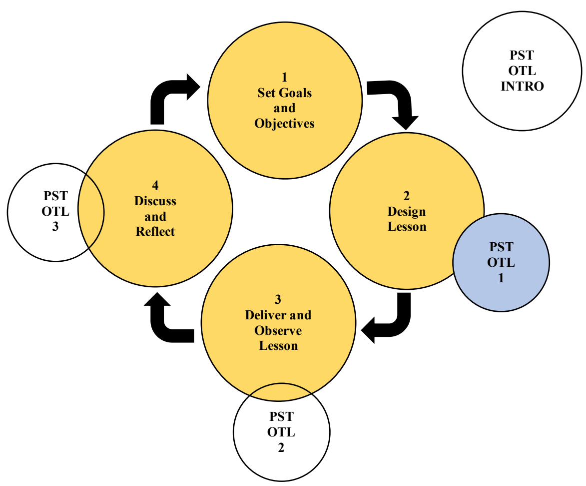


Figure 11. Opportunity to Learn (OTL):
Curriculum Material Study (*Kyozaikenkyu*).

The PSTs studied, and then discussed, the *Eureka Math* Teacher Guide's Math Grade 3 Module 3 Overview, Terminology and Suggested Tools information sheet, and the Module 3 Lesson 1 Plan (Great Minds, 2015). The lesson plan was adapted by the teacher team, and the PSTs discussed the team's edits. The Module 3 Lesson 1 Plan is included in

Appendix B for reference. In addition to the *Eureka Math* (Great Minds, 2015) materials, the PSTs read and discussed the article *Three Steps to Mastering Multiplication Facts* (Kling & Bay-Williams, 2015). This article was chosen taken from a practitioner journal as an example of a professional resource readily available to practicing teachers.

Grade 3 Module 3 Overview discussion. The *Math Grade 3 Module 3 Overview* (Great Minds, 2015) seeks to provide a broad outline of the collection of 21 lessons that make up the module entitled "Multiplication and Division with Units of 0, 1, 6-9, and Multiples of 10" (p. 2) The nine-page document lays out the learning objectives for the module, how individual lessons progress within the module, how the lessons are connected to and build upon previous learning, and how address individual state standards. It is a dense collection of information designed to help in-service teachers understand how the individual lessons of the module are connected to each other and how this module connects to previous and future modules. The lesson chosen by the teacher team was the first lesson of this module. This lesson, together with the two following constitute "Topic A", which the *Overview* explains

> …begins by revisiting the commutative property. Students study familiar facts from Module 1 to identify known facts using units of 6, 7, 8, and 9 (**3.OA.5, 3.OA.7**). They realize that they already know more than half of their facts by recognizing, for example, that if they know 2×8, they also know 8×2 through commutativity. This begins a study of arithmetic patterns that becomes an increasingly prominent theme in the module (**3.OA.9**). The subsequent lesson carries this study a step further; students apply the commutative property to relate 5×8 and 8×5 and then add one more group of 8 to solve 6×8 and, by extension, 8×6. The final lesson in this topic builds fluency with familiar multiplication and division facts, preparing students for the work ahead by introducing the use of a letter to represent the unknown in various positions (**3.OA.3**, **3.OA.4**). (Great Minds, 2015, p. 2)

As the above excerpt shows, the *Overview* provides a thorough explanation of the mathematical progression and highlights the key mathematical ideas that teachers and students will interact with in the lessons. It also identifies the state standards (the **3.OA.#** tags indicate that these are third-grade standards involving operations and algebra thinking) that the lessons are designed to address. For example, 3.OA.5 states:

> Apply properties of operations as strategies to multiply and divide. (Students need not use formal terms for these properties.). *Examples: If* $6 \times 4 = 24$ *is known, then* $4 \times 6 = 24$ *is also known. (Commutative property of multiplication.)* $3 \times 5 \times 2$ *can be found by* $3 \times 5 = 15$*, then* $15 \times 2 = 30$*, or by* $5 \times 2 = 10$*, then* $3 \times 10 = 30$*. (Associative property of multiplication.) Knowing that* $8 \times 5 = 40$ *and* $8 \times 2 = 16$*, one can find* 8×7*as* $8 \times (5 + 2) = (8 \times 5) + (8 \times 2) = 40 + 16 = 56$*. (Distributive property.).* (Great Minds, 2015, p. 5)

The document thus served as an authentic, content-rich resource for the PSTs to study.

The study of this document and the discussion it sparked occurred at an important time, and the focus and depth of the discussion reflects this timing. Recall that the PSTs had been immersed in the university course's activities for nearly two months before beginning the *kyozaikenkyu*. This gave them opportunities to participate in multiple iterations of the APEX cycle (Ozgun-Koca et al., 2020), allowing them to engage in what Bass (2015) characterized as authentic mathematical: "exploration, discovery, conjecture, proof, and certification" (p. 631).

Their experiences in the university course allowed them to engage with the curriculum resources differently, as they now had two different frames of reference with which to explore the document: their view of math and math class *before* our course and the new vision of mathematics and math class that had been forming over the past two months. Analysis of the coding of the discussion transcript shows that the majority (51%) of the PSTs' comments focused, understandably, on the clear *Curriculum Progression* found in the materials, the *Depth of Understanding* needed for teaching, and the *Curriculum Guidance* (Table 7) provided by the Module Overview (Great Minds, 2015). That is, the PSTs noticed and commented frequently on the quality of this resource in terms of understanding how the topics fit together and the knowledge needed to understand and convey the mathematics the topics involved. For example, Rita noticed that the curriculum did not cover the multiplication tables in numerical order. For her, the module was a thoughtful progression which left the tables for 0 and 1 until the end of the unit.

> Also, the fact that they have to build a base. They have to start, if you start with the first day…they have to understand what's happening on the first day otherwise it won't translate into the next day's. I think that is really important for the whole 0 1 thing, because if you don't understand what all the other numbers are going to do, nothing is going to be a lot harder, so actually going and doing the other numbers first is going to be a lot easier. (Rita, Module Overview Discussion Transcript)

However, further analysis revealed that 20 out of the 65 coded statements (31%) dealt with the broad code category *Math and Math Class, Past-Present-Future.* Thus, this first component of the *kyozaikenkyu* not only introduced them to a rich teaching resource, it provided an opportunity to negotiate collaboratively their new vision of mathematics and mathematics instruction.

The curriculum study, and the discussion of their observations, created a space where the PSTs could connect what they had been experiencing in the university course with what they were seeing in the school curriculum. In particular, the PSTs frequently referred to the flexible thinking and multiple solution paths that had come to represent competence in our course, how these were also valued in the *Eureka Math Module Overview* (Great Minds, 2015), and how it differed from their own school experience. For example, Julia noted how the curriculum guidance on learning the multiplication facts for seven included a flexible method for computing 3×7,

> …what I wanted to highlight was how they did the sevens. So, if you're trying to get 21, they use the $14 + 6$ method, plus 1. When I was younger, they just told us to memorize the sevens…have us write it down. Just kept writing it down versus giving us different methods or different ways of solving the problem. (Julia, Module Overview Discussion Transcript)

Lucy saw this flexibility and sense-making approach as *empowering* for students.

> So, I find that interesting because, like Julia said, we just learned to memorize it. It wasn't, "Oh, this is a pattern. Now you can figure out the next one." (Lucy, Module Overview Discussion Transcript)

Additionally, these alternate methods also seemed to present new options for the *PSTs*. Maxwell, a PST whose preparation program concentration was mathematics, commented on how the curriculum guidance suggested that students could draw upon previous knowledge and the associative property to figure out new facts. He shared,

> So, at the beginning of page 3, where it's $8 \times 5 = 4 \times 2 \times 5$. That way of showing the [associative] property is very interesting I guess you can say. Because they do break it down into where 2×5 is actually 10…which [4×10] is the same as 8×5. Which is just showing the [associative] property, but that is not at all the way I would have learned it or…it's very new, but it makes sense. (Maxwell, Module Overview Discussion Transcript)

The PSTs had commented on the difference between the sense-making activities of the university course and their previous experiences in their own courses, but this first *kyozaikenkyu* discussion gave a sharper focus to the difference and allowed the PSTs to situate and discuss this distinction as learners *and as future teachers*. Later in the discussion, Jude shared that Maxwell's comment resonated with him and made him think about the mathematics in terms of *teaching*.

> …I think Maxwell talked about how they split up the $8 \times 5 = 4 \times 10$, or whatever. I kind of agree that's not how we learned it I feel like; but I can see how it would actually help us be able to *show them* (emphasis added) how to understand the similarities instead of just memorization. (Jude, Module Overview Discussion Transcript)

I consider Maxwell's statement to be a record of a shift that occurred during the lesson study activities: from learning mathematics for himself to learning mathematics for others. Ball (2000) characterized this as "a transcendence of the tacit understanding that characterizes and is sufficient for personal knowledge and performance" (p. 245).

Prudence's comments during the discussion also indicate a shift. For Prudence, though, the exploration of the *Module Overview* (Great Minds, 2015) allowed her to address an earlier, more abstract, question in a more concrete way. Recall that Prudence, during the discussion of the introductory presentation on lesson study, expressed concerns with addressing the needs of all learners, and how a common curriculum and pacing might create difficulties when differentiation is needed. This concern persisted, and the exploration of the curriculum guidance allowed Prudence to move this negotiation within *authentic teaching activities* (e.g. the study of teacher guides) rather than within the more general and cultural terms of the earlier discussion.

> Prudence: I kind of like what Jude said because even though it's the first sentence and you just keep reading through everything. They only have 25 days to really get everyone on the same page and understand it. So, it's almost like a process so they can't go a certain pace even if the teacher wants to.
>
> Chris: Hmm.
>
> Prudence: But I definitely like all the different methods they're using because I think it makes it easier to understand all this in 25 days.
>
> Chris: Are you seeing the 25 days as kind of constraining? A bit?

Prudence: A little bit, yes. But at the same time, I get how they have to get all the third graders to this level, on the same page, if they want to continue.

Chris: Mm Hmm, okay.

Prudence: I think they should make it more flexible, not just saying that the module has to be…you have to get each day done, everyone on the same pace.

Chris: Okay.

Prudence: We talked about that earlier. I think I just worry a little bit if they leave the hardest [concept], 0, at the end. And if the kid doesn't fully understand and goes to the fourth grade? That was the end of it. Because it doesn't really make sense when you're that age…thinking of nothing as a number.

A fellow PST, Pam, who tended to read the curriculum materials with great care, pointed out that the *Eureka Math Overview* (Great Minds, 2015) did attempt to address Prudence's concern, albeit implicitly.

> …Lucy is talking about how 9 is introduced later. But it does clarify that they give three days to learn the units of 9. So, maybe is it a signal that it will be a bit more difficult, or that it needs to be focused on…it needs to be emphasized? Because it specifically says it gets three days, and none of the other topics get that many days. (Pam, Module Overview Transcript)

In summary, the comments of PSTs during the first discussion provoked by the curriculum material study seem to indicate that the PSTs were engaging in a negotiation of how mathematical competence was defined. I claim that this initial *kyozaikenkyu* activity helped bring into clearer focus the similarities between their university course experience and the classroom experience envisioned by the *Eureka Math* curriculum designers. The lesson study activity can be viewed as the lens that allowed this alignment of sense-making approaches to be more visible and served as a bridge between the university and the elementary classrooms. The activities that the PSTs engaged in as part of the university course generated new knowledge; but this lesson study activity allowed the PSTs to see the competence defined in our course reflected in the *Eureka Math* curriculum. At the same time, this focal lens also helped the PSTs juxtapose their own elementary school math class experience and the sense-making approach that they had been engaging in as university students and, now, as future teachers.

Terminology and Suggested Tools discussion. The *Module 3 Overview* (Great Minds, 2015) concludes with a description of the terminology and suggested tools that the students encounter throughout the module. Because the university course emphasized multiple representations of mathematical ideas, I asked the PSTs to study this section separately and to share what they learned in a whole-class discussion.

The *Terminology and Suggested Tools* section is a listing of terms that often includes a clarifying example. For instance, one of the "Familiar Terms and Symbols" is "Distribute" and it reads, "Distribute (with reference to the distributive property; e.g., in $12 \times 3 = (10 \times 3) + (2 \times 3)$, the 3 is the multiplier for each part of the decomposition)" (Great Minds, 2015; p. 9). There are 21 total terms defined in this section, including "Array," "Number bond," and "Tape diagram" (Great Minds, 2015; p. 9). These three particular terms, along with "Place value disk," are also included in the Suggested Tools collection. Here, the word "tools" refers to the representations that students will be expected to use in order to make sense of the mathematics in the module. During the discussion sparked by the study of this document, the statements of the PSTs can be interpreted as a continuation of the construction of a new view of mathematics and math class, especially as it pertains to the valuing of multiple solution paths. In addition, the

PSTs built new knowledge for teaching that complemented and deepened the knowledge constructed in the university course.

During this second discussion of the curriculum materials, the PSTs more frequently connected what they noticed in the materials with the ideas and experiences from their university course. A bit of context might help to understand some of these connections. Prior to studying and discussing the terminology listing, the PSTs engaged in an activity that explored alternative mental-math procedures for addition and subtraction. This activity required the PSTs to leverage their understanding of place value to, say, think of 23 as 2 tens and 3 ones or 1 ten and 13 ones. This activity was on Julia's mind as she read through the terminology list, and it brought her back to the notion of multiple solution paths.

> Julia: I don't know about you guys…but it kind of interests me how, say if they we're taking a test, and they didn't specify what kind of properties or something like that…do you think that they would…how would they grade that? You know, would they mark them off for doing a certain property? I don't know…it's kind of weird. What I was thinking about…you know all the different properties that they have that they can use at their disposal. Would they…how would a teacher grade it? If it's the right answer, how would they use a different property? Is it kind of weird? Because how I was brought up, we were taught to do it a certain way; and if it wasn't that certain way you got it wrong or something. So, it's kind of intriguing to see more than one way to do something. Then they might not get it right…or I mean they might not get it wrong for doing it…

> Chris: So, a different…when you say "property" are you really meaning a different "solution path"? Like different "techniques"?

> Julia: Well, say for an example we…the child…had to do what we did…23 minus 7, and instead of using…instead of just doing 20…whatever I just said…they got the answer using the distributive property, commutative property, or the short-cut

way to get the answer. For me, at that age, I would have probably gotten it wrong doing it a different way. But now, they probably wouldn't because it's a…I don't know…it's kind of a different way of getting the answer.

Chris: Okay.

Julia: It kind of intrigues me to know there's more than one way to do something and they won't get it wrong because it's different.

Lucy, an older student with a son who attends school in the same district and, hence, uses the same *Eureka Math* curriculum, remained skeptical because of the contrast with her own experience.

Lucy: So, if the teacher teaches it one way, but because I'm old school and I teach my son to do it another way…

Chris: [Chuckles]

Leslie: …and he does it my way…will he get it wrong?

Julia: It's kind of interesting to know…there's like twelve different ways to do one problem and like they get it wrong if it's…even though it's the right answer…but they get it wrong doing it a different…a certain way.

Chris: How many other people were kind of…remember it as there was one way--and the teacher was expecting it to be that way. And if you tried something else…even if you got the right answer…that's what you're saying Julia…that you might get marked down? Like, this isn't the way we did it in class. Anybody else like that? **[All PSTs raise their hands]**. Wow! Everybody? Whoa…okay…so this is quite a change, huh? So, you're noticing already that they're valuing different solution paths?

Julia: Yes. (Terminology and Tools Discussion Transcript)

Jude also connected the flexible thinking addressed in the document with our course work, but just as he had in the previous discussion, Jude seems to be viewing this new knowledge as new knowledge *for action*.

> Bringing up Julia's distributive property stuff…It looks like how they broke it up was like in the ones place and the tens place. Kind of like how we practiced our [alternate algorithms for] addition. So that kind of made it easier for me to understand why they would reinforce that so much because *that helped us understand the actual process of addition* (emphasis added). (Jude, Terminology and Tools Discussion Transcript)

Jude has taken the knowledge that he constructed from our course activity, noticed it embedded in the curriculum guidance, and assessed its value in terms of teaching. Maxwell's comments indicate a similar shift to thinking about mathematics for others. Again, it was Julia's comment that prompted his comment.

> I like how Julia brought up the distributive property. I was going to say that actually. Because, same boat as her, that is not…I wouldn't have ever thought or done it that way…breaking up the 12 into 10 and 2. So, I think it's kind of important to learn how they do it instead of how we're used to doing it *so we can teach it better. We know what they're doing* (emphasis added). (Maxwell, Terminology and Tools Discussion Transcript)

The PSTs made numerous connections between the curriculum guidance, their university course, and their previous experience. There were also opportunities to learn when the materials under study made visible the gaps in the PSTs' mathematical knowledge for teaching. The lesson study activity offered a chance to extend and deepen the understanding they had begun to build in their college course. In other words, the PSTs were able to add to their teaching repertoire. Many of the common representations in *Eureka Math* (Great Minds, 2015), such as a number bond, were new to the PSTs.

> Jude: For me on the terminology page, page 9, the term "number bond"…I don't think that sounds [inaudible] I know anything

about that. I was trying to read it, and I was trying to understand it, and for a number bond, my first thought was maybe like multiplication and equal signs. How the numbers are "bonded" through those expressions. So, that was just my first takeaway.

Chris: And you saw that in…it's kind of in the middle of our, what they're calling "Familiar Terms and Symbols". That "Familiar" is really intended for the students and the instructor. So, a child who had been coming up through pre-K, K, Grade 1, Grade 2…would have developed all these terms and tools on the next page. In fact, they do have that as the suggested tool and representation: the number bond, on page 10. They don't talk about it…or describe it…but they say, "These are the things we'd like you to use." So, it's something that's sort of in the kids' toolbox. Gotcha. Okay, so that was new to you? Never studied that yourself?

Jude: No. not the way that they said it anyway. (Terminology and Tools Discussion Transcript)

Pam added to the list of representations that were new to her.

Pam: …when I was doing this, there were two things that I did not know…actually one, two, three: the array, the number bond, and the tape diagram. So, I didn't recall any of those. And…array… I kind of felt stupid after I looked it up…because I was like, oh, it was just a picture. So, then I did look them all up, and I was, like, oh, okay, that's what it is. But when I was reading this, I had no idea. That kind of goes up with what Lucy said…she didn't know what the tape diagram was either.

Chris: So, you said that the array, the number bond…Jude brought that up initially…and this tape diagram that Lucy brought up.

Pam: Yes…and I had heard of them…I totally remember the word, but I was like I don't know what that means in this context anymore. (Terminology and Tools Discussion Transcript)

The *kyozaikenkyu* had motivated Pam to research these mathematical models in order to understand the tools that the students would be accustomed to using. This move shows her willingness to explore multiple solution paths in anticipation of her future students' thinking and responses—a most productive habit of teaching in general, and a crucial component of lesson study.

The PSTs continued to comment on the valuing and emphasis of multiple solution path; but now this idea was discussed in terms of *teaching*. Recall that during the discussion of the lesson study introduction presentation, Abbey indicated that she was concerned with how to reconcile a common curriculum and a student's individual thinking. Her work during the intervening weeks and her examination of the curriculum materials had allowed her to reassess her earlier thinking.

> I know that it seems like it would be more challenging for the teacher [inaudible] but I feel like it kind of makes it less daunting…because…*you don't have to make sure that everyone is on the exact same page.* In one of the sentences it says, "Apply properties of operations as strategies to multiply and divide." So, you present all this information, and the child chooses the one that connects with them and the way they conceptualize it…rather than telling them which way they conceptualize it with. So, I feel like it makes it almost easier for the teacher because you present them with the information and *they can understand it in their own way rather than saying, "You have to understand it this way."*(emphasis added) (Abbey, Terminology and Tools Discussion Transcript)

In Abbey's comment about what exploring multiple solution paths offers to students and teachers, I see a sizeable shift in her understanding of how a common curriculum can coexist with, and even support, students' individual thinking. I theorize that this shift was made possible by experiencing the diversity of thinking—and the valuing of that thinking—in our college course, and by being able to connect that experience to teaching practice through the lesson study activity. Julia began to understand the implications of this new vision of math class, where the teacher would have to anticipate and react to student thinking, and how she would now need to learn mathematics.

> I know that for sure when we're looking at this we're like, "Wait, hold up…I have to learn this new way of doing math, you know, and I have to teach this to our students." So, for me, it's a high expectation that we have to have. Because we have to teach this, we have to know this in general. So, it's pretty interesting to just see that we have to really…I don't want to say we learn math…we *learn math a certain way*. (Julia, Terminology and Tools Discussion Transcript)

The analysis of the transcript of the discussion of the *Terminology and Suggested Tools* portion of the *Eureka Math Module 3 Overview* (Great Minds, 2015) allows a glimpse into the increasing ability to participate in teaching activities (i.e. learning) of the PSTs. I propose Julia's negotiation of a new vision of mathematical competence for both students and teachers, Maxwell's and Jude's development of new knowledge for teaching, Pam's independent exploration of mathematical terminology provoked by her study of the curriculum resource, and Abbey's growing understanding of how teachers can use a common curriculum while still incorporating and valuing individual thinking, as evidence of this learning. This learning continued as we moved into the third component of the curriculum material study: the review of a practitioner research article dealing with children's learning of multiplication.

Research article discussion. The PSTs were asked to read and reflect on the article *Three Steps to Mastering Multiplication Facts*, by Kling and Bay-Williams (2015). I selected this particular article because the authors presented a conceptualization of *fluency* with multiplication facts that aligned with our university course and the *Eureka Math* curriculum materials we had been studying. Kling and Bay-Williams present fluency in terms of "noticing relationships and using strategies" (2015, p. 550), as a way to help students achieve the Common Core State Standards' definition of fluency: "skill in carrying out procedures flexibly, accurately, efficiently, and appropriately" (National

Governors Association Center for Best Practices & Council of Chief State School Officers, 2010, p. 6). The authors also emphasize multiple representations of multiplication, and a variety of strategies to support this relational understanding. Again, this mirrored the work that the PSTs were doing themselves as learners and what they were immersed in, as future teachers, with the *Eureka Math* materials. An excerpt from the article is shown in Figure 12.

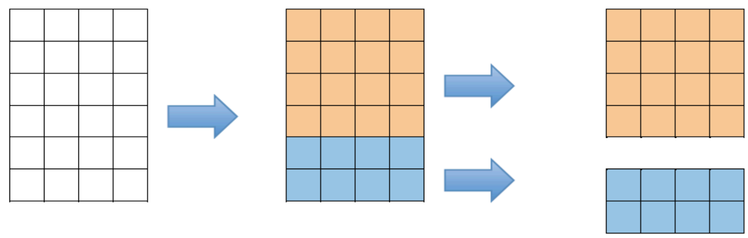
FIGURE 4

Array and area models and equal-groups interpretations work well for the early stages of learning the decomposing strategy, when using a representation is a crucial part of a student's process.

(a) Using an equal-groups interpretation to decompose the fact 7 × 8

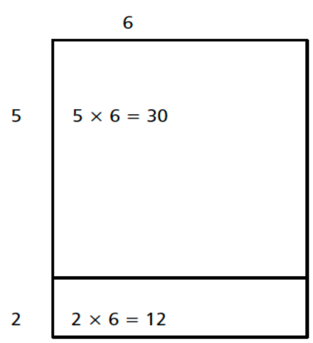
I think of 7 × 8 as 7 groups of 8 things. I don't know what that is, so I start with 5 groups of 8 things, which is 40.	5 × 8 = 40
I have to have 7 groups in the end, so I need to add 2 more groups of 8 things. I know that 2 groups of 8 things is 16.	2 × 8 = 16
So, to find 7 groups of 8 things, I add 40 + 16, which is 56.	7 × 8 = 5 × 8 + 2 × 8 = 40 + 16 = 56

(b) Using an array representation to decompose the fact 6 × 4



I can split my 6 × 4 array into two smaller arrays, one that is 4 × 4 and one that is 2 × 4. I know that 4 × 4 =16 and 2 × 4 = 8. I then add the smaller products of 16 and 8 and get 24 for my answer.

(c) Using an area representation to decompose the fact 7 × 6



7 × 6 = 5 × 6 + 2 × 6
= 30 + 12
= 42

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Figure 12. Strategies and representations in learning multiplication facts, from "Three Steps to Mastering Multiplication Facts" by G. Kling & J. Bay-Williams, 2015, *Teaching Children Mathematics*, 21(9), p. 554. Copyright 2015 NCTM.

One idea in particular stood out to the PSTs and sparked a discussion over the notion of competence. Pam was the first to comment on the idea:

Pam: The…second paragraph on page 551. It says, second note…that the phrase "know from memory" is used not the term "memorization." So, I looked back in our Eureka and it still says, "know from memory", it doesn't use "memorization." I also think about Educational Psychology classes…how they talk about "retrieval" a lot. I know that first-hand from learning Spanish and using flashcards, where you actually need to know the material on the opposite side of the flashcard before you just turn it over and look at it. Because you can memorize that really quick and pass your test, but then when you have to know what's going on, there's no retrieval time in your brain that actually goes to get that material. Loose terms, you know…

Chris: Yes.

Pam: But it's the same thing with math. Using those math flashcards…sure they can memorize it for their times tables test, but when it comes down to it, are they actually going to know [it]? (Article Discussion Transcript)

This distinction between *know from memory* and *memorization* seemed to touch upon the many PSTs' developing new vision of mathematics and what competence meant within that vision. The emphasis on memorization in the PSTs' previous experiences has been noted (see, for example, Lucy's and Julia's comments in the previous section), but now the PSTs were even more reflective of what the consequences of this emphasis on their own mathematical paths.

Abbey: Kind of piggy-backing on what Pam said…in the last page it talks about this negative disposition against math. I feel like I've always kind of thought that because I wasn't good at memorization. And I just…surface-level third-grade me…was like, "Oh, I can't remember these. I'm just not good at math." And that kind of carried over and continued to decrease my interest in anything in math. I just didn't like it. So…I think if I was taught more of the retrieval rather than the memorization, I would have had a more open mind.

Chris: And think how great you are…I mean at *this*…at this sort of visualization and explaining this stuff…how everything is fitting together. That's the skill. You're really really good at that.

Abbey: Thank you. (Article Discussion Transcript)

Rita shared her experience that showed that, even when students are successful at memorizing facts and procedures, there are consequences to the this strictly instrumental understanding.

> It was a little bit different for me because I…through elementary and middle school, math was the easiest possible thing for me. And then as soon as I got to the harder stuff that actually needed like the build-up of the easier stuff, I was, like, wait…no no no no no, I don't know how to do *any* of this. Like Calculus threw me for a loop because I had been focusing just on…not necessarily just on memorization…but I didn't really go through the process of how I needed to go through the process. It really screwed me over when I got to higher levels of math. *That's* when it discouraged me. Rather than in third grade. The teacher was like, "Memorize this," and I was like, "No problem! I can do that!" But this is *so* much better. If I learned math this way, I'd probably still be in calculus classes and still be going into that path. (Rita, Article Discussion Transcript)

I view these exchanges as records of a change going on inside the minds of these PSTs. They are learning a new way to view mathematics, themselves as doers of mathematics, and how they want to teach it. They have experienced this different version of math class as students in their university course, they have become aware of the difference and its implications, and they have seen it proposed as high-quality instruction by both the *Eureka Math* curriculum materials and this article. This activity has helped these PSTs move this reconceptualization of mathematical competence one step closer to their future elementary classrooms.

Lesson plan discussion. The final component of the *kyozaikenkyu* was the study of the specific lesson plan and the tweaks offered by the lesson study teacher team. As mentioned earlier, the teacher team chose *Eureka Math Grade 3 Module 3 Lesson 1* for their research lesson, which had the stated objective of having students "study commutativity to find known facts of 6, 7, 8, and 9" (Great Minds, 2015, p. 7). That is, the goal was to have students realize that, because they know $3 \times 9 = 27$, they also know the value of 9×3. In addition, the teacher team sought to increase the amount and quality of the classroom discourse throughout the lesson. For the research lesson, the teacher team decided to use a single problem, the lesson's *Application Problem* (Great Minds, 2015), to serve as the anchor for a whole-class discussion in which students could discover the commutativity of multiplication. The *Eureka Math* curriculum developers allotted 20 minutes for the students to do and discuss the problem, but the teacher team decided to triple that amount of time. Here was the problem:

> ***Application Problem***
> Geri brings 3 water jugs to her soccer game to share with teammates. Each jug contains 6 liters of water. How many liters of water does Geri bring? (Great Minds, 2015, p. 14).

One of the fundamental practices of lesson study is having all members of the team work through the problems in many different ways in order to better understand the mathematics involved and to anticipate student thinking and responses. We began with this practice, and it showed that the PSTs had developed a substantial body of *actionable* mathematical knowledge. Here is one particularly rich exchange showing this.

Chris: Let's just brainstorm. We'll dump them all up on the board. This is the reason we're doing this. If we're going to open this up and invite all kinds of different ways of thinking about this, we should really take a moment to plan how we will react. If we're going to go through all the different ones, or if we want to pick particular ones that kind of help us move the lesson along….we should think about not just randomly saying, "What'd you get?" and then, "What'd you get?" or "What'd you get?" But looking for particular…particularly rich representations that we can draw the whole class' attention to. We might not see everybody's way. We should know what we are getting into when we open it up. So, I don't know if this is something maybe later for your methods classes…but discussions are rarely, if done well, unplanned. They might seem like just bopping around and asking people what they're thinking; but a good discussion usually has a focus and has picked strategically which things they want you to talk about and compare, to contrast, so…that's sort of what we need to do here. Any suggestions for different models? Just yell them out and if I know what they are I'll draw them. If not, I'll ask you to explain.

Rita: An array.

Chris: An array. So how are you seeing it, Rita, for the array? What do you expect them to write down?

Rita: I did three of six horizontally.

Chris: Like this? [Draws three rows of six circles]

Rita: Yes.

Chris: So, three like that?

Rita: Yes.

Chris: So, three rows of six. Okay, so did you use…

Rita: Circles.

Chris: All right, what else?

Pam: Repeated addition.

Chris: Repeated addition. Visually or numerically?

Pam: Numerically.

Chris: Numerically, okay. [writes on board] 6 plus 6 plus 6. Like that? Good one. Any others? That would be a nice, cool connection, right? Six plus six plus six. Jude?

Jude: I just drew like three circles and then six circles in each. To kind of represent the liters in the jugs.

Chris: So, three like this?

Jude: Yeah.

Chris: Like that?

Jude: Yep.

Rita: Would that be called a grouping?

Julia: Is that a number bond?

Chris: Great that actually has a name. They call these number bonds, and they're built exclusively for addition like this. So, what Jude's really doing, an abstracted version of this, could be this [Points to 6+6+6=18 on board] adds up to 18. That's really what's going on here, right? 6 here 6 here 6 here. So [inaudible] you can sort of see the richness. Jude has them grouped this way. Rita's had them listed out that way; and Pam just went to the numbers to represent it. Six plus six more plus six more…there's like this thing underneath here… is 18. So, number bonds…did you think of it as a number bond or was it more visual?

Jude: I was just trying to represent three groups of six.

Chris: Good. Yes, so this this is…they have a name for that, and it actually starts way early on. Where they just have…so Jude's version of it is kind of like the earlier version. We actually see the things in there; and then in the grades to follow they have this more abstracted version…6 plus 6 plus 6…and what they what they do, I just say it for a second they value like being able to break this up differently. Sort of more flexible. How can we write 18? Someone could say well I've got…something like that [points to Jude's]. And someone else says oh I can do 18 as 9 plus 9. Something like that…sort of used to break these up in ways that they use for different purposes. Okay, pretty cool. Any other ones?

Rita: An area model.

Chris: An area model? Say more about how you are thinking of area. How could I draw it?

Rita: Um, I used boxes.

Chris: Would it help for you to draw it? It might be easier for you...

Rita: Yes, sure.

Chris: Which, by the way, we'll see in the class. That's probably going to happen. Kids are probably going be a little more adept at drawing themselves than explaining it.

Rita: So, it’s basically my array with boxes.

Chris: Gotcha. Okay. Thank you. Yes, nice. We use this in our own discussion, right? A three by six, so the area is 18 squares…related to that. Awesome. Any other ones? I'm curious, did you not do the tape diagram because it's sitting here in front of us, because you don't like it, or you don't want to use it? (Lesson Plan Review Discussion Transcript)

In this single, brief (roughly eight-minute-long) exchange, the PSTs were able to suggest multiple representations—some that they had used in their university class, some that they had been introduced to in the *Eureka Math* materials—and to state connections between them. The curriculum guide suggested that a tape diagram be used for the representation of the problem, a representation that was unfamiliar to the PSTs but was supposed to be in the elementary students’ repertoire of tools and representations. The representations and the scripted interaction surrounding this is presented in Figure 13.

Part 1: Explore commutativity as it relates to multiplication.

Draw or project the tape diagrams shown to the right.

T: Talk to your partner. Which tape diagram represents the Application Problem? How do you know? (Allow time for discussion.)

T: Draw both tape diagrams on your personal white board. Write a multiplication sentence for each. (Allow time for students to work and finish.)

T: How are the multiplication sentences related?

S: They use the same numbers. → Both have a product of 18. → They use the same factors but in a different order. The product is the same.

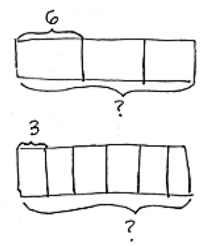


Figure 13. Tape Diagrams and Sample Teacher-Student Interaction, from "*Eureka Math Grade 3 Module 3 Teacher Guide*", 2015, p. 14. Copyright 2015: Great Minds.

The PSTs felt comfortable offering critique of the suggested representation, and to consider alternative models that they thought might help the elementary students' understanding.

Pam: Well what I was going to say earlier was I don't like how the numbers in the tape diagram aren't written in the boxes. They're written on the outside. And here [in another example of a tape diagram] it has them written in. I get that they want it to be more efficient probably, where you see a box and just know that it represents 3. So, you're not saying like 3 + 3 + 3 + 3 + 3 +3, you're saying, "Oh there's six boxes that are three."

Chris: Gotcha. So, if you're saying if we went with a tape diagram *with the numbers in here...*like that...

Pam: That makes more sense to me. I don't know. Maybe I'm just thinking of this too hard.

Chris: Maybe the kids would have...no, this is a great idea. Maybe they would have a better chance of saying, "Oh, that's six threes; and the other one is three sixes." That's a great point. I mean...as written…I think that's not as compelling, and you're saying you agree...but the one [with the numbers written in] actually is really compelling. If we did those two…like [that]. That might be a little bit more compelling.

Pam: If you go from the array of seeing like a figure of six dots. And then you move to the tape diagram where you're seeing just the digit, six, written in there, and then you can go to the regular tape diagram. But just looking at that tape diagram; seeing six boxes, that's kind of like…okay.

Chris: Yes, yes. I agree. Well here's something really important in lesson study. If you're not sure, well what do we do? Do we do that with the numbers in there or the arrays? We can make a choice and test it out. That's what these lessons are for. (Lesson Plan Review Discussion Transcript)

In this negotiation, Pam was engaging in the complexity of instruction as she considered the sequencing of models to best develop student understanding. This is an excellent example of a teaching decision that can completely reshape the impact of a mathematics lesson. She was also forming an image of the possible in terms of student engagement. In the end, the PSTs decided to replace the tape diagram with an array as the representation that would ultimately undergird the classroom discussion. We will see the consequences of that decision when we analyze the data generated by the research lesson and the post-lesson discussion.

The PSTs continued to explore the question that Abbey first expressed during the discussion that followed the introductory presentation on lesson study: how can teachers reconcile a standard curriculum and a classroom of diverse learners? As we have seen, throughout the lesson study activities, Abbey had begun to see how individual thinking and multiple solution paths can be compatible with, and enrich, a standard curriculum. She and the other PSTs noticed the additional inclusive questioning and attention to equity of voice incorporated into the lesson play by the teacher team. For example, one of the team's stated goals was to explore alternate views of questioning. They wrote

> We believe that the questions that teachers ask their students are powerful, intentional decisions that have been made to increase understanding through student discourse. We will be utilizing a variety of techniques including starter questions, questions to stimulate mathematical thinking, assessment questions, and final discussion questions. These will be non-judgmental

> questions that promote equity in the classroom. (Teacher Team Lesson Plan)

In addition, the teacher team emphasized having the elementary students *explain their thinking* throughout the lesson. Martha, one of the most reserved of the PSTs, noted this and her comment opened up space for others to express their excitement at this prospect. This discussion also introduced a new term, *validating*, with which the PSTs began to describe instruction. They would continue to use this term for the remainder of the study.

> Martha: I liked that students are going to be able...the students are going to *explain their thought process* (emphasis added) while solving the content.
>
> Chris: They want to hear about what the students are thinking. Good that you picked up on that. Abbey?
>
> Abbey: Yes, I was just going to mention about how I think the language of explaining their thought process is really crucial. I feel like when it's more individual it decreases the animosity a lot of kids feel towards math. Because it can be so stringently taught. So, I really like that they have the opportunity to explain how they came to their answer. And I think whether it was the correct one or not, it's still worth *validating* (emphasis added) and explaining in a different way.
>
> Chris: Can you say a couple more sentences about what you mean by "stringently taught"? You're sort of setting up this difference between what you see here and this "stringently taught." Can you tell us what you mean by that? Just so we're all on the same page.
>
> Abbey: Mm Hmm. In my experience in elementary education, I always liked English because I could write my way through answers. It wasn't just like the answer is 24, and then if you don't have 24 you're right...or you're wrong. But I think, with this, although there's a right answer, there's *different methods to get to that answer* that could also be correct and *could also help a different key aspect* of the lesson (emphasis added).
>
> Chris: Okay.

Abbey: So, I liked different things [about the lesson plan additions]

Chris: Very good...

Abbey: And I feel a lot of the struggle that kids feel with math is knowing if I didn't get this answer then I'm wrong completely. Like there's no right part of what they did, for all the work, to show...

Chris: Gotcha. Or even taking it one step further like if I didn't do it *the* way...

Abbey: It's the wrong way.

Chris: Right. Even though you saw it and got the 24, you went a different way...right. I think that's, that's also something that they're going to spend a lot of attention on.

Jude: Yes. Kind of going along with what Pam and Rita said: the long-term goals for Number 2. They're not really focusing on the content, they're trying to—I guess what Abbey said too—focus on how they get there, and how they can explain their processes. Which in the long term will help them solve more problems, I guess.

Chris: Yes, I actually have been really amazed at this lesson as we've dug more deeply. This could be a two-second lesson, right? Teaching it the way you're saying "the stringent way"…one way. 5 times 3 is 3 times 5. How long does that take to just tell somebody that? How long did it just take me to tell you that? Two seconds? But…we've built this entire...this whole process—we've been talking about it, reading articles about it. There's so much to learn about it, and then how you get kids talking about it...as opposed to the teacher just saying, "You guys know that 3 time 5 is 5 times 3? Cool. Let's move on." It lets them discover it, lets them represent it in lots of different ways.

Abbey: Yes, I was also just going say...because they promote us to be critical thinkers...well, my education system did, in my experience...but they didn't give you the opportunity to be a critical thinker. Because it was just, they were telling you how to think. You know what I'm saying?

Chris: Yes.

> Abbey: It was just...I think with this, it allows them to actually critically think for themselves and problem solve, which I think is more useful than knowing an answer to a problem.
>
> Jude: And that's probably more *validating* (emphasis added) for the students…to feel more competent. (Lesson Plan Review Discussion Transcript)

In this discussion, I propose that Abbey continued to refine the answer to her initial question of how to incorporate, and, using her term, to *validate* the different thinking of students. I view this as further evidence of the PSTs reimagining of mathematics as a sense-making activity; one where thinking is valued.

Maxwell noted that, despite all of the studying, planning and discussions that both we and the teacher team had engaged in, the moment of truth would come when the lesson was delivered to a classroom of students. He recognized that our hypotheses about teaching and learning would be tested and that

> I think it will be interesting, first-hand, to see the questions that we've come up with and they've come up with…to see if we get the actual answers we want. (Maxwell, Lesson Plan Review Discussion Transcript)

It is exactly this affordance of lesson study, the ability to walk questions of both content and pedagogy into a classroom, that we turn to next. The opportunities to learn offered by the research lesson observation and the post-lesson discussion will be analyzed together in the following section.

Opportunities to learn 2 and 3: Research lesson and post-lesson discussion. In order to understand the discussions and reflections of the PSTs that followed and were sparked by the lesson observation, I offer a quick summary of, and a few important vignettes from, the research lesson. The lesson took place on Tuesday, November 26, 2019 in Mrs. Cuoco's third-grade classroom. On that day, the day before the Thanksgiving vacation was to begin, there were 14 students present. In addition to the students, there were five members of the teacher team (including Mrs. Cuoco), nine PSTs, and the school district's Executive Director of Educator Excellence present in the classroom as observers. Thus, there were 15 observers. As mentioned earlier, I agreed to deliver the lesson.

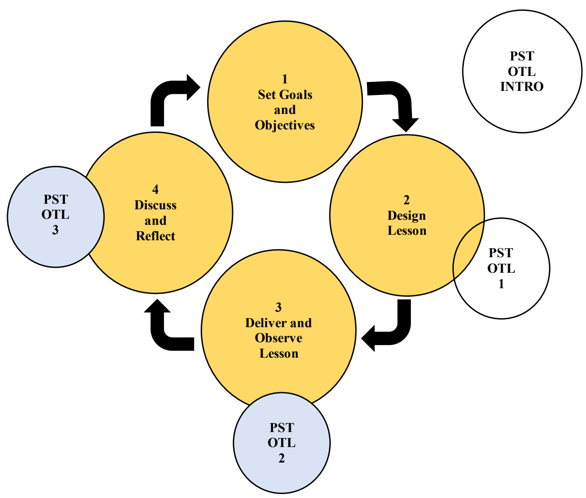


Figure 14. Opportunity to Learn (OTL):
Research Lesson and Post-Lesson Discussion

It is helpful to recall that the stated lesson objective was to have students, "Study commutativity to find known facts of 6, 7, 8, and 9" (Great Minds, 2015; p. 13), and that the teacher team chose to focus the research lesson on the motivating application problem shown here:

Application Problem
Geri brings 3 water jugs to her soccer game to share with teammates. Each jug contains 6 liters of water. How many liters of water does Geri bring? (Great Minds, 2015, p. 14).

The teacher team's hypothesis was that, by creating additional opportunities for students to hear each other's thinking, that the lesson could offer more equity of voice and a clearer focus on student ideas. The students were asked to work independently and then share their thinking with their tablemates. After this, a whole-class discussion was orchestrated that allowed students to share their representations or build upon the ideas of others. The students were able to see and hear the ideas of others, and a variety of representations were able to be compared and evaluated. I present two moments of the classroom discussion that served as focal points for the PSTs' discussions and reflections. The first occurred at the beginning of the whole-class discussion:

Chris: I'm going to have a few people come up and put their different versions on the board…I saw some really cool stuff, but there are a few that I want to make sure that everybody sees. Michelle, would you be willing to share what you had; and I'll draw it?

Michelle: I drew a number bond.

Chris: How many people used number bonds? [Many hands go up]. Ooh, a very common way. Michelle, you tapped into something really important. It looks like a lot of people did that. Can you describe what your number bond looked like?

Michelle described the number bond with six legs, each containing 3, with a mysterious total of 47. Her number bond is reproduced in Figure 15.

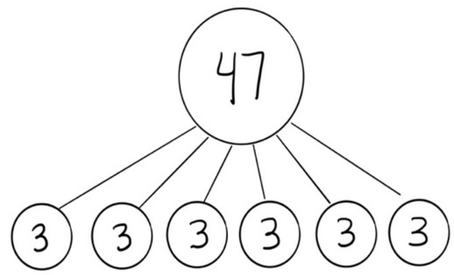


Figure 15. Michelle's Number Bond

Chris: How many people have a number bond that looked like that? [many hands go up]. Same numbers? How many people had *exactly the same numbers*? [no hands go up] Does anyone have anything to say about this number bond set up? If you have a different one. Mary? Can you add to this?

Mary: My esteemed colleagues, eighteen goes at the top. [other students give the agreement signal[1]]

Chris: Mary has a different theory on this. She thinks eighteen would go at the top. That's a little bit different than the 47. Pepper, I see you shaking your head…are you in the 18 group? You think there should be an 18 here? Can you get inside Mary's head and tell us why there should be eighteen?

Pepper: Because I added.

Chris: So, you had a number bond that looks like this [points to Michelle's]? Is this the way yours looked?

Pepper: No.

Chris: Oh! Something different? Can you say how yours looks?

Pepper: I have two legs. [Audible gasps from other students]

Chris: Two legs? Okay…and what's in each of these?

[1] I visited Mrs. Cuoco's class the previous day in order to get to the know the students before the research lesson. During that class, I introduced non-verbal hand signals that we would use during class: "I agree," "I disagree,", and "I support you." These signals allow students to silently, yet publicly, express their thinking. The students used these signals enthusiastically throughout the lesson. I also encouraged them to address each other as "my esteemed colleagues."

Pepper: Three and six.

Chris: Interesting. Okay, and in the top you had…

Pepper: Nine.

Chris: You had nine. [Draws the number bond shown in Figure 16]. Okay guys, here's something really super important. I love that you're trying to use these number bonds; that's a really powerful representation. But I think our number bonds have a real specific meaning here. What's inside these things…we should be able to see them in this problem. And what's up here, should have something to do with the problem…So, I love the format…John?...Can you share what you did? Did you also have a number bond?...Does it look like this [points to Michelle's]?

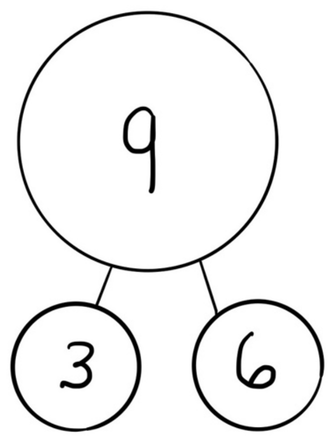


Figure 16. Pepper's Number Bond

John: I had three legs.

Chris: Three legs. Wait a minute…Michelle and Pepper, you were so great to get these on the board, I love it. But let's listen how John is using these number bonds to help him make sense of *this* problem. We can see if you want to change how you're thinking about your number bonds. John, why three legs?

John: Because there are three jugs of water. (Research Lesson Transcript)

John went on to describe a number bond, the concept introduced to the discussion by Michelle, that incorporated Pepper's idea of the addition of the lower numbers and represented the application problem (see Figure 17). This collaborative building on of student ideas, even when some parts of the ideas were considered and discarded, seemed to capture the attention of the PSTs. Before I turn to their observations, I offer one more vignette from the research lesson.

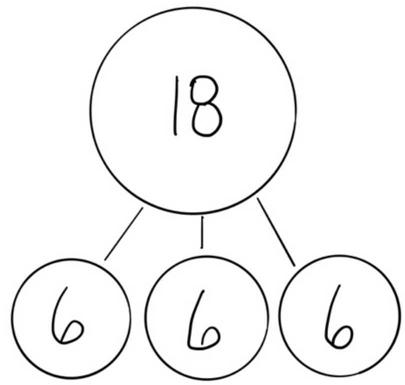


Figure 17. John's Number Bond

During the curriculum study phase, the PSTs decided to build the discussion towards an array representation of 3×6. Their hypothesis was that this particular visual representation would most easily allow the students to see the connection between 3×6 and 6×3, that is, to discover the commutative property of multiplication. A few students had used arrays to solve the application problem, and after developing the number bond representation to answer the original question, it was now time to turn to the commutative property. Anna's thinking helped the class get started:

Chris: Okay. [Anna] has a bunch of dots…quite a few of them. Did anyone else use dots or circles to represent this? [A few hands go up]. Okay, so guys, again we tapped into something that a lot of people are using. Penny? If you don't mind, can you tell us about that one? [Points to her paper]

Penny: My esteemed colleagues, I did three of six.

Chris: Three rows of six? Like that? [Writes the array shown on the left of Figure 18]…Penny, what does this *mean* to you?

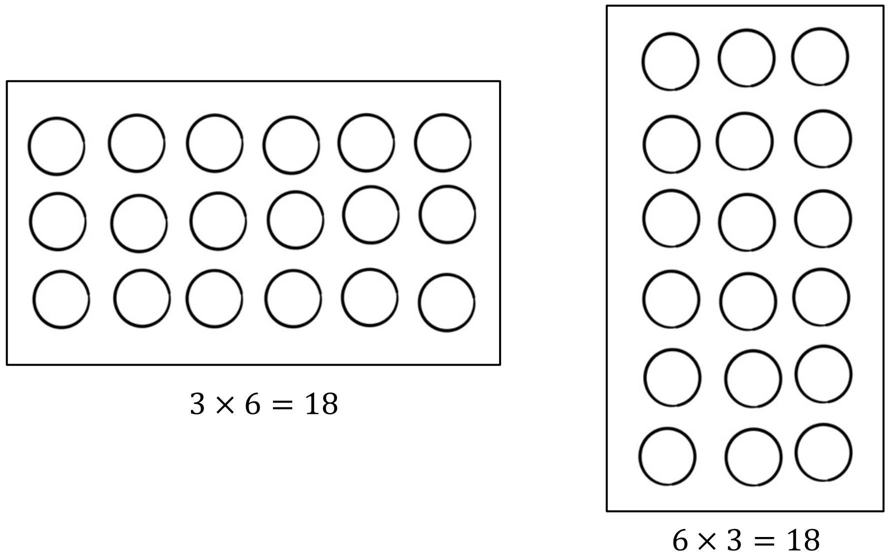


Figure 18. Penny's array (left) and the rotated array exemplifying the commutative property of multiplication.

Penny: I…I…because there's three…three…three jugs of water and six liters, so I…so I did the circles like that.

Chris: I should credit Anna on this. She had a similar idea. She had all of the dots written out. She had exactly the same circles, the same *number* of circles didn't you, Anna? Sort of written out in long form. (Research Lesson Transcript)

After Penny's array was made public and accepted as a representation of $3x6$, the students were able to rotate their personal whiteboards 90° to see the array for 6×3 (See the array shown on the right in Figure 18). When the commutative property was brought up by one of her colleagues, Anna recalled the previous day's lesson.

Anna: We talked about this yesterday.

Chris: You know what? Anna remembers that we talked about this yesterday, but it wasn't multiplication. Do you remember what the commutative property was about yesterday, Michelle?

Michelle: It was adding.

Chris: It was adding! So yesterday we could *add* them in any order, but today it turns out that, when you *multiply*, the order doesn't matter either. The *product* stays the same. [Points to Anna] That's a great connection!

Anna: A lot of people did $3 \times 6 = 18$. A lot of people did that. Was that actually a lesson today…or? Were you planning this?

Chris: I was *hoping* that would happen. You guys are so great… (Research Lesson Transcript)

The next meeting of the university course was devoted to a discussion of the research lesson. In between, I asked the PSTs to reflect on the lesson. Anna was not the only person to detect that the successful discovery of the commutative property of multiplication was not accidental, but the product of planning. Jude, in our post-lesson discussion, expressed his surprise at how the curriculum guidance together with our adjustments so closely matched what he observed play out during the lesson:

Jude: I guess after watching the lesson, it was actually…it lined up a lot, more than I thought, with the dialogue that Eureka provided. And how the students would understand the problem and say, "Oh, 18 is the same…or 6×3 is the same as 3×6…Well, I think a lot about how you asked them to represent the problem in any way they wanted to. Most of mine, that I observed, were doing the number bonds and they didn't really use like the tape diagrams [the suggested representation in the Eureka materials]…and it still illustrated the same issues that they [Eureka] thought they [the students] would have. Some of them drew like six lines with three [circles], some of them drew three lines with six. (Post-Lesson Discussion Transcript)

Eleanor, who was not able to attend the live lesson but watched a recording, agreed with Jude and noted that there was additional knowledge that was incorporated into the teacher team's plan:

Eleanor: I think it just means a lot more preparing. They give the Eureka to the teachers and say, "Okay, here...do this." But even that, you know, just knowing what's in it…you can know it top to bottom, but even that

> isn't enough to prepare you for what you're going to experience in the classroom. So, I think for the teacher that you all went to see, she had to think about not only the Eureka, but then how it will go with her specific kids. So, she knows her kids more so than the people that made the lesson plan. She has to add the Eureka on with what she knows about her kids and that's like double planning on top of what you have to know from the lesson plan that they give you teach. So, that's something interesting...(Eleanor, Post-Lesson Discussion Transcript)

I view Eleanor's insight as evidence of an emerging awareness of Knowledge of Content and Students (Ball, et al., 2008) (see Figure 3). That is, Eleanor understands that even deep knowledge of content alone is not enough. In addition, a teacher must be aware of what his or her students bring to a lesson—for example, what they know, what their understanding will allow them to see, what they will find difficult, and what scaffolds will allow them to overcome those difficulties. In addition, her short statement offers evidence that Eleanor has made progress towards all four of the key goals for developing mathematics teachers (see Figure 4 and Table 1). She discusses instruction in light of a new vision of mathematics where the students interact with the ideas of others. She understands that enacting this instruction, with all its complexity, depends on a deep knowledge of content and students; and she views this instruction as being informed by a teacher's own practice—specifically the learning that comes from knowing one's students and their thinking.

One other aspect of planning that was raised by many PSTs in their written reflections. The PSTs seemed acutely aware of the benefits of their deep study of the curriculum materials during planning. In particular, they noted frequently how this planning allowed them to notice and understand student thinking. I offer comments from Rita, Pam, and Jude as examples.

> Our work studying the lesson in depth helped to look for *what was important in the students' work* (emphasis added). Being able to study all the visual representations helped the most because that is the main method of concept development in this lesson. (Rita, Research Lesson Reflection)

> I think having background study done before the lesson over the content *made the misconceptions more noticeable* (emphasis added). In class, we went over all the different ideas that the students may have when asked to represent the story problem. Many of them clung to the number bond, which genuinely surprised me. We had all assumed the majority of representations would be of arrays or tape diagrams. When the number bonds or arrays were incorrect*, it was easier to see where the student went wrong because I myself had already worked through it and I can remember what my classmates or I had gotten tripped up on* (emphasis added). (Pam, Research Lesson Reflection)

> *My work on the curriculum had me looking for the thought process of students understanding the first story problem* (emphasis added). It was interesting to see how many used the number bond and after looking around the class, I assume that is something they previously learned and seem comfortable with. It was super interesting to see how the teacher used guided questions and select student examples to bring them all to the correct answer. It honestly shocked me how smoothly the commutative property fell into the lesson toward the end. Students exclaimed that 6x3 was the same as 3x6 even before the array was introduced. I was shocked at how accurate the dialogue provided by Eureka played out into the actual lesson by the teacher. (Jude, Research Lesson Reflection)

Within the brief reflections of Pam and Jude, I see a nascent "research stance" (Hiebert et al., 2007) which allowed them to make predictions about student thinking that were tested within teaching practice. This stance is what allowed Pam to be "surprised" and Jude to be "shocked" by what they saw.

Coding of the post-lesson discussion revealed that student engagement, classroom discourse, and the instructional decisions that fostered them were the issues that most captured the PSTs attention. Roughly 67% of the coded comments dealt with student engagement and this was echoed in their individual reflections.

> The first thing I noticed was how engaged the kids were. A part of me was skeptical about how effective the lesson study could be because I thought it would be too distracting to have all those adults in the room observing. However, it didn't seem to be a problem for the students at all. I also thought that the discussions were the most important part of the learning process. Apart from the instruction, the students seemed to really enjoy discussing their answers with one another. Not only did they enjoy it, but it also made them really try to have a deeper understanding of the material because they had to explain their answers to their classmates. (Eleanor, Research Lesson Reflection)

> What stood out to me was that the students were actually very engaged and enthusiastic during and after the lesson. In Eureka's plan, it said students would leave enthusiastic and I had doubts about that, but after seeing the actual lesson I see how they were right. (Maxwell, Research Lesson Reflection)

The teacher team also found the amount of student engagement fostered by the lesson remarkable. The lone PST who was able to participate in the teacher team's post-lesson discussion immediately following the lesson, Pam, reported to her fellow PSTs:

> Pam: …they were saying that [Anna] never speaks. Her teacher said that was the most she's heard her speak all year. And I was, like, "Okay!" I was kind of shocked by that.

> Chris: Me too.

> Pam: And then [Mrs. Cuoco] said that they've never seen that much interaction with the kids. They loved the way that [Chris] was asking questions and he was like saying, "If this is this and...what do you think?" and "How can we add onto this?" And I...they were like so in love with the idea. And I was, like, "That's how our math class is all the time." That is our class every day. We go back and forth all day, bouncing off each other and put different ideas up. But when they talked about how they, the kids, didn't talk that much [normally]…especially when they said Anna was silent; I was like "What?!" (Post-Lesson Discussion Transcript)

The interactive instruction that had, at this point, become what "math class is all the time" to Pam, and the engagement it generated throughout the classroom was noticed by Abbey as well.

> Abbey: …well there were three boys at the back table that I was watching. Two of them were really engaged, but they had different representations of it. And, there wasn't conflict...I wouldn't call it. But just basically [they asked], "Well, which one should we do?" "Well, what about yours?" "What about mine?" Kind of like a back-and-forth. Then they kind of had to merge their ideas together because you said come up with one idea per table. So, I think that working together aspect of it, and the discussion part of it, is really really useful for any career, job or anything. (Post-Lesson Discussion Transcript)

Pam and Jude noted, however, that instructors wishing to engage students with this type of interactive instruction need to be aware of and anticipate more than just the mathematics. They need to consider social aspects such as their students' self-confidence. They pointed to Penny, the student whose arrays ultimately made possible the powerful visual representation of the commutative property (see Figure 18). Note the level of detail in their observational data, another affordance of lesson study:

> Pam: For the tables that I watched, I felt like it was a lot of them following each other's lead. There wasn't that discussion...more or less...Penny watched a lot and she, like I had mentioned before, she had arrays drawn...both of them...and then she drew the number bond. Then she saw that someone had suggested number bonds to be drawn on the board; and she erased all her stuff. And [she] left the number bond. Then she rewrote the arrays again; both of them again. And then somebody else put a number bond on the board and she erased them again! She kept going back and forth with it, but everybody at her table only had number bonds. She's also very shy.
>
> Jude: Yes, I was going to say...she looked at Mary's a lot, but Mary wasn't right. Penny was right, but she kept erasing her stuff. (Post-Lesson Discussion Transcript)

How could this lack of confidence be addressed? Jude hypothesized that it had something to do with *validation*, an idea first verbalized by Abbey during the review of the lesson plan.

> Jude: Yes, and I don't think that they were ever told that their answer was wrong or was not correct. Because I feel like I was a lot like Penny when I was a kid. I probably knew the answer, but I was too scared to be wrong to vocalize it. But I feel like she was even more comfortable sharing her arrays after she got that *validation* (emphasis added). (Post-Lesson Discussion Transcript)

Jude reflected about how this validation could also engage struggling students. He wrote, "The teacher offered validation for participation and ideas even when they were not correct. This fostered more participation and trust from the students." Abbey added that the hand signals, which students were encouraged to use during the lesson to indicate "support", helped students feel heard:

> Abbey: I loved the hand signals. When you said engagement, it reminded me of it. I even told my family about it at Thanksgiving, and we were laughing so hard. Because, every time someone was talking, we would be like [shows the "support" signal]. I just thought it was the cutest way to keep everyone interactive and like...I don't know...I think it makes the classroom feel a lot smaller when you know people are like...other children...are listening to you. (Post-Lesson Discussion Transcript)

The observation of the live lesson was fruitful because, in addition to providing answers to some of the PSTs questions, it also generated new questions. For example, Pam revisited a common theme for our discussions: how to engage equitably diverse learners in a whole-class discussion. This time, however, she was concerned with a high-achieving student, John, who quickly found the answer to the application problem.

> Pam: I guess I wonder...do kids, like John, in these specific classes get that scaffolding? Eureka gave the suggestion for it, and we decided not to differentiate in this lesson and keep it all on the same thing. But those would be the students that would get the, you know...

> Chris: Oh, the higher numbers [for the application problem]?

Pam: Right. How do you incorporate that into this type of lesson where you're bouncing back and forth? How do you deal with the three kids who are way over here with their answers and then you everybody else with like the basic ones? (Post-Lesson Discussion Transcript)

I interpret the comments of Jude, Abbey, and Pam to reflect learning about the complexity of live instruction, in particular the efficacy of particular teaching strategies. That is, instructional decisions regarding classroom discussion norms or differentiation have an effect on student engagement and learning.

Three PSTs, however, seemed to be more concerned with challenges that might limit their ability to implement the type of instruction that they had just planned and observed. Maxwell, Lucy, and Prudence were enrolled in a course that involved an early field experience. Each was assigned to observe classes in a local elementary school. When asked if they thought they would want to engage their future students in discussions, their reactions were mixed.

Maxwell: So, I think that the discussion part in the classroom was super beneficial, because at least for me I always learn better from group discussions. One-on-one, it's, "Yeah I get it," but you learn more and the concept is built at a deeper level of understanding. So, being able to do that would be something I really want to do…I think, well I'm...at least with the teacher whose classroom I'm in now, pacing is a big issue for her. And she's like trying this in kindergarten right now. And it's an issue...so...

Chris: Okay, she's trying Eureka?

Maxwell: Yes. (Post-Lesson Discussion Transcript)

In addition to class pacing, Lucy added the concerns of larger, more diverse classrooms.

Lucy: So yes, I think discussion will be part of what I would like to do in my class. My only concern is if you have a larger class. I'm sure everybody wasn't there because it was the day before the holiday.

Chris: No.

Leslie: But with a larger class, it would be a little difficult to have a discussion and then trying to bring everybody back down. I'm at [the same local elementary school as Maxwell] as well. Third grade class, and they're just [whistles].

[students laugh]

Chris: So, this is the same grade level, right?

Leslie: Same grade level.

Chris: So, night and day kind of difference? Between this and there?

Leslie: Yes. It's like with the larger class and then everybody *not* being on the same level mathematically. So, yes, that would be my only concern. But, so yes, I would definitely try to have some discussion. (Post-Lesson Discussion Transcript)

Prudence seemed to attribute the differences between what she had experienced in her field placements and the research lesson to the difference in students and teachers.

This made her even more skeptical about the likelihood of success of whole-class discussions in all classrooms.

Prudence: I'm noticing now especially when we talked about the discussion, it's really helpful in the classroom. But in some of the classes that I've job-shadowed in, the teacher can't really control the class if she lets discussion happen during math or science. So, she worries about if she lets them start talking none of them will pay attention. So, she makes sure like it's dead silent, like, during the whole lesson. And these kids, I think in my reflections I said this, this is the most well-behaved class I've seen. They were really, really well-behaved.

Chris: So, in the classrooms you've been in, and you've actually seen a lot of the stuff that we've been working on, right, in the classrooms? You mentioned that before. So, when those teachers try to get a discussion going, a whole-class discussion, what...so, total silence, you said? That's sort of an attempt to control...

Prudence: Yes, and they have to raise their hand, there's no, like, hand motions. So, I feel like it makes the kid more scared that they can be wrong. If it's a dead-silent class and they have to raise their hand. Maybe just the lesson plan would have to be shifted a little depending on the school, where the school is. Because also, in the school I'm in, if one kid gets a question wrong, he'll flip his desk. There's a lot of anger.

Jude: Do you think that he would have, if he was sitting in that classroom, would he have flipped his desk? The way the conversation went.

Prudence: I think he would have...I sit with two kids in the back because they can get very aggressive. It's just because once they make one little mistake, they just go from one to ten. They can't control their emotion…So, I'm just worried, but I think this is more of a positive, safe, environment…everyone is supporting everybody. (Post-Lesson Discussion Transcript)

In contrast, Pam shared that she was eager to include discussions in her future classroom.

Pam: I like the idea of doing discussions...not every day…but incorporating them into different lessons. And I want all my future students to feel *validated* (emphasis added). All the Annas, all the Johns who are excelling, and all the Pennys who are shy…I want them all to feel like they're a crucial part of the classroom. Discussions can help that. (Post-Lessson Discussion Transcript)

The analysis of the quantitative and qualitative data suggests that the PSTs learned much throughout the components of the lesson study activities. The significant change in their beliefs regarding the positive effect that teaching can have on student learning is better understood after looking at the data produced during each component of the lesson study. The PSTs were first able to experience this effect themselves as students in their university course. Moving into the periphery of teaching practice, they began their *kyozaikenkyu*: the intense—both in terms of effort and time—study of curriculum materials. This study allowed the PSTs to juxtapose their previous experience with mathematics, their current experience in the university course, and the new elementary curriculum which allowed them to envision and value a new type of mathematics class. The research lesson and post-lesson discussions were powerful focusing activities where the PSTs could see their new mathematical knowledge for teaching put into action through their enhanced ability to notice and interpret student thinking.

Broader Themes: Lesson Study as Lens

So far, the analysis and discussion has been presented chronologically, and this view of the data was helpful to see the substantial learning that took place within the different components of the lesson study activities. In the final sections of this chapter, I will take a step back, and view the learning from a higher vantage point. This "bird's eye view" will provide an opportunity to present, explain, and connect three broader themes that emerged from my analysis of the data: *Teacher/Teaching Challenges*, *Math and Math Class: Past-Present-Future*, and *Engaging and Validating Instruction: Preparing, Enacting, and Learning*. All three share a common bond. In each, the lesson study activities acted as a powerful focal lens through which the PSTs could see their own experiences, their new understandings and questions, their developing identities as teachers come together with and within authentic elements of teaching practice—live students, in a real classroom, engaged in an exploration of mathematics.

Teacher/Teaching Challenges

The unexpectedly rich discussion that was sparked by the presentation that introduced the PSTs to lesson study made it clear that the PSTs were coming into the university course aware of many of the perennial challenges of teaching. The description of the highly centralized educational system of Japan that so effectively uses lesson study to improve teaching and learning, provoked many PSTs to share their reservations about the prospect of engaging in lesson study. This is not an uncommon reaction, even with inservice teachers, as many of the principles of lesson study (e.g. collaboration and slow, incremental improvement) are profoundly countercultural to the educational system in the United States (Lewis, 2019). The fact that the PSTs had reservations is not the point. What is important, is that their written and oral comments made visible and audible the issues and practices of teaching that they were already wrestling with as they began their preparation programs. Abbey shared one of her concerns:

> I feel like the end product is too idealistic of a concept. Because, not criticizing, obviously what Japan is doing is working. They're cranking out these insanely smart children. But…I don't think you can structure…you can't assume someone's education level. I guess if the system worked right, you could. But you can't come into a class saying, "Okay everyone in this class is going to learn the exact same thing." It's just not feasible. I feel like that can never work. Every third grader in Japan is learning the exact same thing…they're doing the exact same questions*? I don't get how that's productive, because not all kids are going to learn the exact same way* (emphasis added). (Introduction to Lesson Study Discussion Transcript)

I interpret Abbey's concern as one of the most fundamental for teachers: how does a teacher engage all students who each enter the classroom with their own unique skills, needs, and ideas? Abbey points to the uniform curriculum of the Japanese system as exacerbating this concern. In addition, I categorize her comment as a view from the periphery of teaching practice. Abbey, in her first semester at the university, had not yet had an opportunity to engage with this issue within the practice of teaching. The lesson study activities would allow her to do just that.

I have noted Abbey's learning at different points in the lesson study activities. Abbey's engagement with this issue is an exemplar of how the lesson study activities allowed the PSTs to see this common challenge negotiated within the curriculum materials, in the planning of the teacher team, and in a live lesson (see Figure 19). I will retrace Abbey's progression through this process to highlight the key moments in this negotiation.

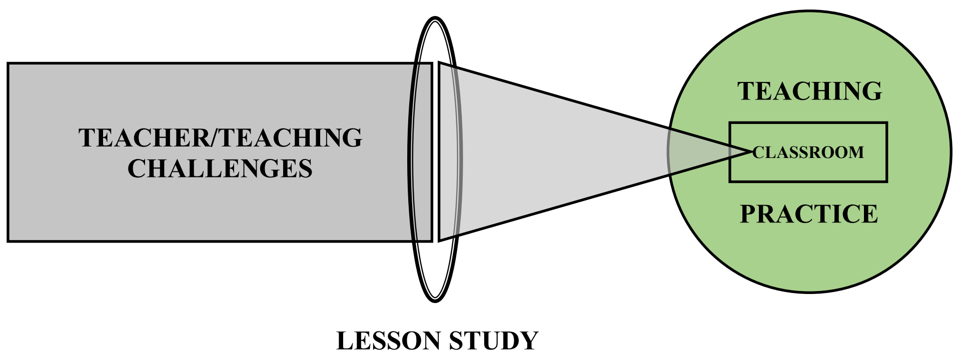


Figure 19: Lesson Study as Lens: Allowing the PSTs to address challenges within practice.

Within our university course, Abbey was immersed in mathematical discourse that centered around different solution paths suggested by the PSTs. That is, different ways of thinking were honored and compared on a daily basis. Because of her own experience with this instructional model, Abbey was able to notice how it served as the foundation for the *Eureka Math* curriculum (Great Minds, 2015).

> *…you don't have to make sure that everyone is on the exact same page.* In one of the sentences it says, "Apply properties of operations as strategies to multiply and divide." So, you present all this information, and the child chooses the one that connects with them and the way they conceptualize it…rather than telling them which way they conceptualize it with. So, I feel like it makes it almost easier for the teacher because you present them with the information and *they can understand it in their own way rather than saying, "You have to understand it this way."*(emphasis added) (Abbey, Terminology and Tools Discussion Transcript)

As the PSTs explored the teacher team's plan, the team's emphasis on having students explain their ideas during the lesson helped Abbey see how her concern was being addressed by practicing teachers.

> Yes, I was just going to mention about how I think the language of explaining their thought process is really crucial. *I feel like when it's more individual it decreases the animosity a lot of kids feel towards math*. Because it can be so stringently taught. So, I really like that they have the opportunity to explain how they came to their answer. (Abbey, Lesson Plan Review Discussion)

This statement captures an important shift. Diverse thinking is viewed as an asset— as the raw materials that are used to fuel a productive classroom discussion—rather than something to be corralled to "fit" a uniform curriculum. This difference is what initially concerned Abbey, and now she has been able to see how a uniform curriculum and diverse thinking can not only coexist, but can complement and enrich each other.

The research lesson enabled Abbey and the other PSTs to see this play out in the classroom.

> But just basically [they asked], "Well, which one should we do?" "Well, what about yours?" "What about mine?" Kind of like a back-and-forth. Then they kind of had to merge their ideas together… (Abbey, Post-Lesson Discussion Transcript)

The fact that Abbey participated in the intense curriculum study that *culminated* in the research lesson observation allowed her to gain an appreciation of the amount of preparation required to facilitate a classroom where the diverse thinking of students can be harnessed for learning that she had come to value.

> Being able to observe and participate, and not just simply being taught about it made me desire this method of instruction in every classroom…We don't live in a Utopian Society where every child will ask the perfect question that connects one idea to the next, even if it's what the script says. Every

> classroom dynamic is different and valuable. Thus, in practice, this process could get messy. However, that does not make it any less important and crucial to incorporate. Education is unarguably the best product of living in a free country. Education is predicated on curiosity and uniqueness. How can we promote education if we are not enabling these qualities amongst young people? Therefore, lesson study is the best option we have to allow positive discourse to flow in the classroom. (Abbey, Final Reflection)

Thus, Abbey came to understand that the diversity of thinking that students bring to their classrooms *is* challenging, but a standard curriculum does not necessarily need to exacerbate that challenge. She came to view the amount of preparation needed to engage with student thinking as daunting, yet worthwhile. Other PSTs showed similar growth along these lines, but, in some instances, the learning was impeded. I offer the case of Prudence as an example.

Recall that early on in the lesson study activities, Prudence seemed to agree with Abbey's initial concern and shared

> It kind of makes me think of what about the kids that can't keep up with that? So, if they don't take time to go at a slower pace, it makes me feel, in a way, they're weeding out the kids that can't keep up to their standards and their structure. (Prudence, Post-Introduction Discussion Transcript)

Like Abbey, Prudence was concerned with the important ideas of equity and the individual needs of students. Through the lesson study, Prudence came to see that a common lesson could engage all students, and that every student could contribute their thinking.

> In class I learned that everyone can see a problem a different way and its good for the rest of the class to see the different solving methods. Also, it's important for the students to explain how they got their answers. This allows the teacher to see their way of thinking and show the rest of the class that their way is just as good. (Prudence, Post-Lesson Reflection)

Her enthusiasm, however, seemed tempered by her other field experience. As noted earlier, Prudence shared her experiences in other classrooms where classroom management issues were given as the reason her cooperating teacher structured the class quite differently than what she observed during the research lesson. Conflicting approaches to teaching (teacher-centered versus student centered) also created dissonance between the two field experiences. After the research lesson, Prudence shared her concerns over the ability to engage *all* students in the type of lesson that she had just observed.

> Prudence: I think with Eureka, including the discussion, that's how they planned it timewise. But I think some teachers might think that, again, like I said, they can't control [the class] as well so they cut out the discussion. So that way *they can go over more material in that time* (emphasis added). Because you gave a lot of time to the discussion, which really helped make sure that everybody and every table understood it.

Pam responded by pointing out how the teacher team had managed their limited time.

> Pam: That's the opposite of what the teachers you said. They were just talking about they're struggling with the pacing and they couldn't keep up with the curriculum because they want to have these discussions. But they have so much material to go over. They just can't keep up with it. So, they were saying they were cutting out...they were like trimming the lessons to take out material to allow for discussion time. So, it's interesting that it's flip-flopped there. Because they have to take into account the behavioral issues.
>
> Prudence: That makes sense. The teacher I've been with, she tells me she doesn't have any time for more than two questions because *she has to get through the lesson* (emphasis added). (Post-Lesson Discussion Transcript)

This brings to light an important feature of this study that has been mentioned earlier: the close alignment of pedagogical approaches of the university course, the *Eureka Math* curriculum materials, and the research lesson. For Abbey, the lesson study activities constituted the entirety of her field experiences. I theorize that the consistency of the approaches and the opportunity to see the approach in action with students provided enough evidence for Abbey to conclude that this type of instruction was possible, and valuable, for all students. Prudence's additional field experiences, which contrasted markedly with what she experienced within the lesson study activities—including the research lesson observation—seemed to refract the image that was so clearly visible to Abbey and other PSTs.

This difference in learning echoes Jacobson's (2017) statement that "prior experience shapes how people interpret, and hence, what they learn in learning situations" (p. 157). Jacobson's work suggests that the "applicative knowledge" (i.e. knowledge for action) that Abbey, Prudence, and the other PSTs were able to create during the lesson study activities was a function of "interpretive knowledge" (i.e. knowledge that allows them to notice and make sense of new experiences) that may have been developed in previous field experiences (Jacobson, 2017). The limiting potential of negative (or simply misaligned) earlier or concurrent negative field experiences was evident in the differences between Abbey's and Prudence's learning and should be considered when implementing similar pre-service learning opportunities.

There is evidence that the inverse of Prudence's situation may also hold true. That is, alignment of experiences can solidify beliefs and learning. For example, Eleanor's concurrent field experience aligned with the feedback system of our lesson study activities. The classroom teacher that she was paired with possessed a research stance that Eleanor noticed and valued.

> Eleanor: I go to [a local elementary school] once a week for [another] class, and the teacher does discussions all the time...and it wasn't

smooth. So, every week I would come, the desks would be different, she would have handouts that they could base their discussions on. She would be trying different activities and stuff. And I [think] that's so cool that she doesn't NOT do it because it's difficult. But she is constantly working trial and error to try to figure stuff out.

Chris: Very nice.

Eleanor: And I was like, "That's something that I want to implement in my classroom." Just so that discussions can take place...like everybody has said, it is super important. Even though it may not always be easy to draw people back in. Just trying different stuff to *make* it work. (Post-Lesson Discussion Transcript)

Math and Math Class: Past-Present-Future

Each component of the lesson study activities provided opportunities for the PSTs to create a new vision of mathematics and math class; and these were analyzed separately in the previous chapter. Recall, for example, that the study of the Module Overview (Great Minds, 2015) prompted Julia to share:

> Because *how I was brought up, we were taught to do it a certain way* (emphasis added); and if it wasn't that certain way you got it wrong or something. So, it's kind of intriguing to see more than one way to do something. (Julia, Module Overview Discussion Transcript)

During the discussion of the research article on multiplication facts, Rita expressed how the difference in seeing mathematics as a sense-making activity rather than a rule-memorizing/following activity might have made a significant difference in her educational trajectory. She shared, "If I learned math this way, I'd probably still be in calculus classes and still be going into that path."

When viewed as a whole, what emerges is another instance of lesson study acting as a focusing device—collecting the PSTs' views of past and current experiences with mathematics and juxtaposing them with views of elementary students engaging with mathematics (see Figure 20). This juxtaposition of images helped the PSTs imagine their future classrooms.

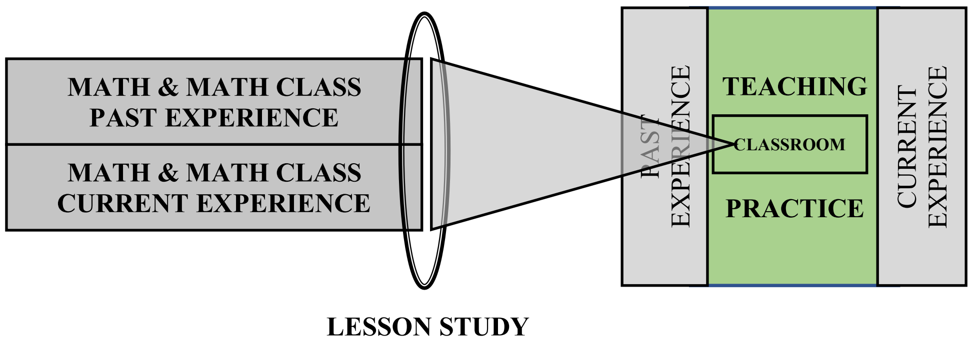


Figure 20: Lesson study as lens: Allowing the PSTs to juxtapose past and current experiences with classroom observation.

> This lesson study definitely *changed the way I think about math content and the way it can be presented to students*. Our math course alone did this solely from the amount of discussion and discourse we have. I really hadn't thought about how effectively a math lesson of that nature could be implemented into an elementary class; but it worked beautifully in my opinion…It really did change my perspective on how math content can become lessons with such rich development for the students. *Had I not seen it first-hand, I think I would have had an idea of the possibility, but now I know it to be true*. (Pam, Final Reflection)

For Pam, the research lesson observation helped her connect what she was experiencing in the university classroom with what she saw the elementary students experiencing. In her view, this shifted what she was feeling and thinking from theory to practice. This connection was made by Jude as well.

> What stood out to me in the lesson study activity was how consistent [Chris] treated his lessons. *His college course and the third grade classroom held the same rhythm* (emphasis added). By that I mean that both lessons had an established respect and communication system. Both classes were

> encouraged to share there [*sic*] thought process, and that was the part [Chris] cared more about. Taking the importance of the answer out of the room takes away the scary pressure of being wrong in front of your peers. I think that is what made the lesson and semester with Nazelli a success. I largely looked at this because the math content stayed the same. The commutative property did not change. How it was taught is what was changed. The lesson emphasized student discourse and I found this to be a valuable way of teaching a scary subject like math…(Jude, Final Reflection)

I claim that the inclusion of the research lesson observation was a key step in the learning that was experienced by the PSTs. Although, throughout the study, many PSTs commented on how different they felt our university course was from their past experiences, and how different of a picture of instruction the *Eureka Math* curriculum materials presented, it was the research lesson that showed them that this type of math class is possible with elementary students.

> In our class, working within groups and having class discussions, everyone has a different way in solving problems. *As I observed the students, they were the same way* (emphasis added). (Lucy, Research Lesson Reflection)

> What stood out to me was that the students were actually very engaged and enthusiastic during and after the lesson. In Eureka's plan, it said students would leave enthusiastic and I had doubts about that, *but after seeing the actual lesson I see how they were right* (emphasis added). My work during the whole thing made me realize that teaching is much more effective in group discussions. *I never had that in elementary school really* (emphasis added), it was all just us being told things to remember. So, I enjoyed *seeing* (emphasis added) a new type of teaching. (Maxwell, Research Lesson Reflection)

Although some researchers have found that PSTs were able to increase their understanding of reform-oriented teaching (and their ability to implement it) through modified lesson study activities *without* observing a research lesson involving school-aged students (Fernandez, 2005), the impact of the research lesson on these particular PSTs was substantial.

Engaging and Validating Instruction: Preparing, Enacting, and Learning

The third theme that emerges from the data is compelling in terms of clarity and the intensity with which it was felt by the PSTs. It is one that encompasses all of the major goals of preservice preparation laid out previously. Additionally, it brings together the university course, the curriculum study, and the research lesson. Here, I consider lesson study as a lens that allowed the PSTs to see how their new vision of mathematics and math class, new knowledge for action, and preparation to observe and learn from instruction within an elementary classroom (see Figure 21).

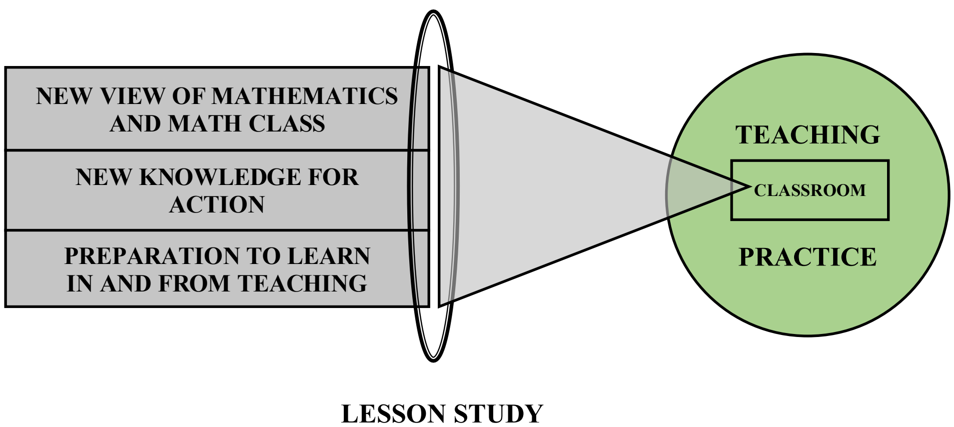


Figure 21: Lesson study as lens: Allowing the PSTs to see their new vision, new knowledge, and preparation to learn from practice within an elementary classroom.

As part of the university course, the PSTs engaged with mathematics in a way that differed markedly from their previous experiences. The focus on ideas and the attention lavished on flexible thinking and multiple solutions throughout multiple APEX cycles (Ozgun-Koca et al., 2020) had created an atmosphere where the PSTs felt that their ideas were being heard, valued, and used to help others understand mathematics. This feeling was first vocalized by Abbey during the review of the teacher team's lesson plan.

> So, I really like that they have the opportunity to explain how they came to their answer. And I think whether it was the correct one or not, it's still worth *validating* (emphasis added) and explaining in a different way. (Abbey, Lesson Plan Review Transcript)

I posit that, despite the fact that Abbey was referring to the validation of elementary students' ideas, this originated from feelings of validation that she felt during our course—something that she relayed was absent from her own elementary school experience. The atmosphere of mathematical discussion, considering different ways of thinking, and honoring of ideas—what came to be known as *validating* instruction—that Abbey and her colleagues were immersed in was also advocated for in the *Eureka Math* curriculum materials (Great Minds, 2015). I propose that this alignment, made visible through lesson study, is what helped Abbey notice and name this type of instruction.

> Previous to participating in lesson study, the content of math seemed very abstract. There was little reasoning/considering in my solving of math problems. I assumed that's because there just wasn't any to be considered. However, throughout the course of this class, I learned that all those operations have extremely rich and dense explanation that goes beyond surface level. I felt genuinely discouraged and intimidated when I entered my first college math class (this one), because I had never been taught any explanation. I grasped onto the assumption that some people were just naturally better at math. However, this is only true to an extent. Some people are just naturally better at memorization, which is how math was presented to me in school. Thus, I was apprehensive towards learning how to teach a topic I 'wasn't good at'. Lesson study taught me that all these feelings and assumptions were incorrect. I thought I hated math, but it turns out, I just hate the way math was taught to me. If I was given logical and critical answers to my questions, I would have been able to process that information in a proactive way. Yet, in a traditional classroom, these questions are discouraged, regardless of whether they are valid or not. This is my favorite part of lesson study. There's ambiguity in the answers, which allows for unique and wholesome interpretations. This doesn't just allow for a rich conceptualization; it diminished all the animosity that grows between most students and math. I believe this animosity is rooted in the heart of traditional [teaching], where they are not afraid to tell kids, "Memorize this because I said so, it will help you in the long run."...I am very extremely grateful *that lesson study broke down these stereotypes about math that I had and gave me a new perspective that I will carry with me into my*

> *development as a teacher* (emphasis added) and an adult. (Abbey, Final Reflection)

The PSTs also gained an appreciation of how much preparation is involved in order to enact engaging instruction—instruction that uses student thinking as a critical component of the lesson—that allows students to feel validated. Multiple PSTs commented on this important mathematics-based facet of teaching.

> The students were so inclined to help each other. I also think that it made the students understand the material better because explaining it to each other helps the students understand why the concepts work. The lesson study also made me take into consideration just how much work goes into preparing to teach a lesson. The commutative property is simple. However, when preparing to teach it, *it is necessary to take into consideration how my students will react, what questions they will have, how much prior knowledge they have, etc. All these things make preparing for a lesson time consuming, rightfully so* (emphasis added). (Eleanor, Final Reflection)

> I never knew how much was behind the scenes when it comes to math. I learned to appreciate math more because of this newfound fact of math. Being able to see the "why" behind a problem opens up a teacher's eyes so much to how to explain it to their students. I plan on using the skills I learned in this lesson study when I become a teacher. Also, *I learned that a lot of preparation needs to go into lesson plans because each class can react to a lesson plan differently* (emphasis added). (Prudence, Final Reflection)

Prudence's comments assume an interactive model of instruction (Cohen et al., 2003) (see Figure 2), one that, like Abbey, she plans on incorporating into her own practice.

This preparation allowed the PSTs to both notice and understand student thinking during the research lesson. Recall that Rita noted, "Our work studying the lesson in depth helped to look for *what was important in the students' work* (emphasis added)." The PSTs were able to make sense of the student thinking and better understand the curating of those ideas during the discussion. As mentioned in the previous section, this indicates that the PSTs were beginning to understand what it means to be able to learn in and from practice.

Their deep understanding of the mathematics allowed them to attend to issues of content and to issues of validation simultaneously during the observation.

> The students seemed engaged and it was obvious that they all were looking for some validation within their efforts, and this lesson gave them just that. We have discussed this before in class but I, along with most of my colleagues, want all students to feel validated in my classroom. *After doing this lesson study, I can see that an engaging lesson such as this makes that wish a reality. It was truly fascinating to me that a validating atmosphere was created during a math lesson* (emphasis added). The teachers from the school were shocked at the amount of participation but I see it as a direct result [of] the way the lesson was taught to the kids and the way they were all made to feel like an essential part of the lesson. All in all, I think discussion-based lessons are now a future goal of mine to incorporate into future classes of my own. (Pam, Final Reflection)

As the PSTs' comments show, the lesson study activity enabled the PSTs to see engaging, validating lessons as possible and powerful means of helping elementary students understand mathematics and feel competent and valued. Lesson study opened up a space for the PSTs to appreciate the amount of mathematical knowledge for teaching (in particular, common and specialized content knowledge, and knowledge of content and students) (Ball et al., 2008) required to implement this work and to be able to use their practice as a source of their own learning.

CHAPTER 5 DISCUSSION

Introduction

My study was designed to investigate the following research question: What types of learning are afforded or occluded by a modified lesson study activity within an early pre-service mathematics teacher content course for elementary school teachers? The chronological and "bird's eye view" analyses of the data generated by this study show that the PSTs involved in this study learned a great deal. I will now move to an even higher vantage point in order to look at the implications for other PSTs and for teacher educators in whose university classrooms the PSTs learn about mathematics and teaching.

The Learning Afforded by the Modified Lesson Study Activity

The lesson study activities provided opportunities for the PSTs in this study to negotiate important challenges of teaching, including how teachers can capitalize on diverse thinking about mathematics and how teachers manage limited time—both inside the classroom and while preparing for instruction. Furthermore, this negotiation was situated within teaching practice. In addition, the curriculum study components helped the PSTs build Mathematical Knowledge for Teaching (Ball et al, 2008) that they were able to call upon during preparation for, the observation of, and the discussion following the research lesson. That is, the PSTs were able to better engage in key elements of teaching practice connected to mathematics content, such as considering the most compelling mathematical representation, anticipating student responses to mathematical questions, and interpreting the matheamtical thinking of students during a live lesson. The lesson study activities allowed the PSTs to imagine their own future classroom as a place where students with diverse, flexible mathematical thinking will be heard and validated. Their experiences within our university course helped the PSTs to belief that math class could be different, the study of the curriculum materials offered evidence that this type of elementary math class could be designed, but it was the research lesson that showed that it could be a reality. This is what happened with this particular group of PSTs.

In Chapter 2, I introduced my conceptualization of the Key Goals for Developing Mathematics Teachers in the Early Stages of Pre-Service Preparation (See Figure 2 on page 18). The four goals were to *construct a new vision of mathematics*, *create new knowledge for action, engage in the complexity of teaching,* and *prepare to learn in and from practice*. In Chapter 3, the notion of a belonging to a Community of Practice (Wenger, 1998) was aligned with the four goals (see Table 1 on page 40). Belonging involved *engagement* in the practice, *imagination* of what the member's role could be, and the *alignment* of one's work with the community and communities that surround it. Using these frameworks, I will now present the implications for other PSTs. That is, I will now make the shift from discussing what happened to the *specific participant PSTs* in my study to making claims that I feel apply to *PSTs in general*.

A New Vision of Mathematics and Math Class

If I learned math this way, I'd probably still be in calculus classes and still be going into that path. (Rita, Article Discussion Transcript)

A thorough curriculum study can allow PSTs to experience a coherent progression and development of mathematical ideas. A connected, sense-making, coherent elementary curriculum can provide a powerful counterimage to their own memories of math class. For example, *Eureka Math* (Great Minds, 2015) valued flexible thinking and multiple solution paths. For PSTs who are used to a more instrumental or procedural view of mathematics, a curriculum such as the one the PSTs in this study examined can help form a new image of mathematical competence. The construction of this image can be aided by a university classroom experience that parallels the curriculum. In this study, the university course heeded the call of Schmittau (1991) to include opportunities for PSTs to "engage in a process of social interaction…[in order to] broaden perspectives of mathematics methodology" (p. 129), and this emphasis was mirrored in the curriculum materials.

The introduction of research literature that again shared emphasis brings the new vision of mathematics and math class into even clearer focus. For example, the particular research article examined during this study emphasized "knowing from memory" and deemphasized "memorizing"—an idea also found in the *Eureka Math* materials. This notion served as a soundboard for the PSTs' new vision of mathematics and triggered an outpouring of reflections of previous mathematics classes that they collectively agreed were for them, at best, ineffective, and, at worst, demoralizing. Thus, engaging in lesson study within a content course, even at the earliest stages of a preparation program, can provide a compelling alternative to the conservative pull of PSTs' "apprenticeship[s] of observation" (Lortie, 1975).

Through the research lesson, PSTs are able to glimpse the possible in an elementary classroom. For the PSTs in this study, they saw it as a space where student ideas could be heard, understood, built upon, and validated. Fujii (2014) noted that "the consideration of educational values is always tied to, influenced by, and reflect in, the key features of lesson study" (p. 13). When the educational values of the university course, the school curriculum, and the research materials that constitute the curriculum study materials align, lesson study can *focus* these values into a new coherent image of teaching and learning.

The significant increase in the PSTs belief in the efficacy of this new vision of mathematics teaching was striking considering the relatively short length of the lesson study activities. I posit that the alignment was responsible for much of this important change.

New Knowledge for Action

> …I think Maxwell talked about how they split up the $8 \times 5 = 4 \times 10$, or whatever. I kind of agree that's not how we learned it I feel like; but I can see how it would actually help us be able to *show them* (emphasis added) how to understand the similarities instead of just memorization. (Jude, Module Overview Discussion Transcript)

The curriculum study is situated within, and grounded by, a lesson plan that PSTs know will be *enacted*. As they study and discuss the curriculum materials (including the university course materials), the upcoming live lesson can focus the PSTs own learning on learning *for others*. Concepts that were new to PSTs may be considered in terms of how they might be help elementary school students learn. The discussion transcripts from the Terminology and Suggested Tools (Great Minds, 2015) and review of the lesson plan show that the PSTs had created substantial new knowledge. This knowledge may have remained inert, but the research lesson provided the opportunity to immediately apply it. Thus, the lesson study activity seems to respond to the National Research Council's (2001) call for preparation programs to emphasize the *application* of new knowledge. The lesson study activity does this with the urgency suggested by Darling-Hammond and Wei (2009): the application occurs *immediately* after the knowledge is created. This is a more common structure in later methods courses; and it is important to keep in mind that this can occur productively within a content course at the earliest stages of pre-service preparation. Even though PSTs at this early stage of their preparation are not ready to teach a live lesson, they have opportunities to draw upon their new knowledge as they, for example, work the mathematics problems in the lesson to anticipate student responses, discover and consider the affordances of different representations, or explore research literature. In short, they are able to apply their new knowledge as they engage in the authentic work of teaching.

Engage in Complexity

> How do you incorporate [differentiation] into this type of lesson where you're bouncing back and forth? How do you deal with the three kids who are way over here with their answers and then you everybody else with like the basic ones? (Pam, Post-Lesson Discussion Transcript)

From the beginning of the study, it was clear that the PSTs had some understanding of the complexity of instruction. The discussion transcripts show that important challenges such as time and the diversity of student thinking were very much on the minds of the PSTs. Therefore, it reasonable to conclude that other PSTs will also already be thinking about such perpetual challenges of teaching as they enter their preparation programs. The study shows that this type of lesson study activity can open up space for PSTs to study how the field confronts these challenges through the examination of curriculum materials, research literature, and the thinking of the teacher team recorded in their lesson plan. For example, during the study of the teacher team's modified lesson plan, the PSTs were able to experience how issues of student voice and equity were worked out within a single mathematics lesson. The research lesson observation enables PSTs to both appreciate the complexity of teaching and understand the amount of preparation needed to harness it. By this, I refer to issues of subject-matter knowledge. The deep study of the mathematics underlying, surrounding, and inside the lesson can counteract the common misconception that one need only know how to *do* the mathematics for oneself (Ball, 1990, Ma, 1999). The study shows that the urgency of practice, so often "disappeared" by teacher education (Lewis, 2007), can be experienced and appreciated by PSTs despite their peripheral participation in teaching practice. The research lesson observation presents an opportunity for PSTs to make on-the-fly interpretations of student thinking and to watch an experienced teacher react to unanticipated responses. As the quote at the beginning of this section makes clear, the research lesson is also an opportunity for PSTs to form new questions about teaching and learning. In short, the lesson study activities can act as a vehicle for moving PSTs from the extreme periphery of teaching practice towards fuller participation by allowing them to experience and appreciate the complexity of instruction (Lave & Wenger, 1991).

Prepare to Learn in and from Practice

> I want to continue to be a part of lesson studies throughout my career because I see now how beneficial they are. I think that there is so much to learn from other instructors and I never want to get complacent in my teaching and think there is no room for improvement. This shows that even the best instructors can always improve. (Eleanor, Final Reflections)

This study shows that the activities in this study can enable PSTs to better anticipate, notice, and understand student thinking. This, in turn, allows PSTs to better learn in and from teaching. In short, the analysis indicates that PSTs can begin to develop a research stance (Hiebert et al., 2007). This feature of lesson study has been noted since its introduction in the United States (Stigler & Hiebert, 1999), but the study indicates that the development of this research stance can begin at the earliest stages of a preparation program. Thus, PSTs can begin to view their practice and their colleagues as resources for continual learning and see this as a natural part of teaching practice. Thus, they will have a built-in mechanism for aligning their practice with best practices as new information becomes available via their own teaching, their colleagues' teaching, and the broader field of education. The PSTs in this study expressed that they valued the lesson study activities and felt strongly about implementing it in their future practice.

Generalization Considerations

The Benefits and Challenges of Alignment

One of the most important factors involved in the learning of PSTs described above is the *alignment* of four major facets of the lesson study activities: the pedagogical approach of the university course, the curriculum materials, the goal of the teacher team, and the delivery of the classroom lesson. If all of these facets consistently and explicitly incorporate the same (or at least similar) values, as they did during this study, PSTs will have multiple opportunities to feel the effects as learners and to study the curricular and teacher moves that reflect those values as future teachers. I will zoom back to a close-up view of my particular study to help clarify the type of alignment I am referring to. Figure 22 represents this alignment within the conceptualization of instruction as interaction (Cohen et al., 2003).

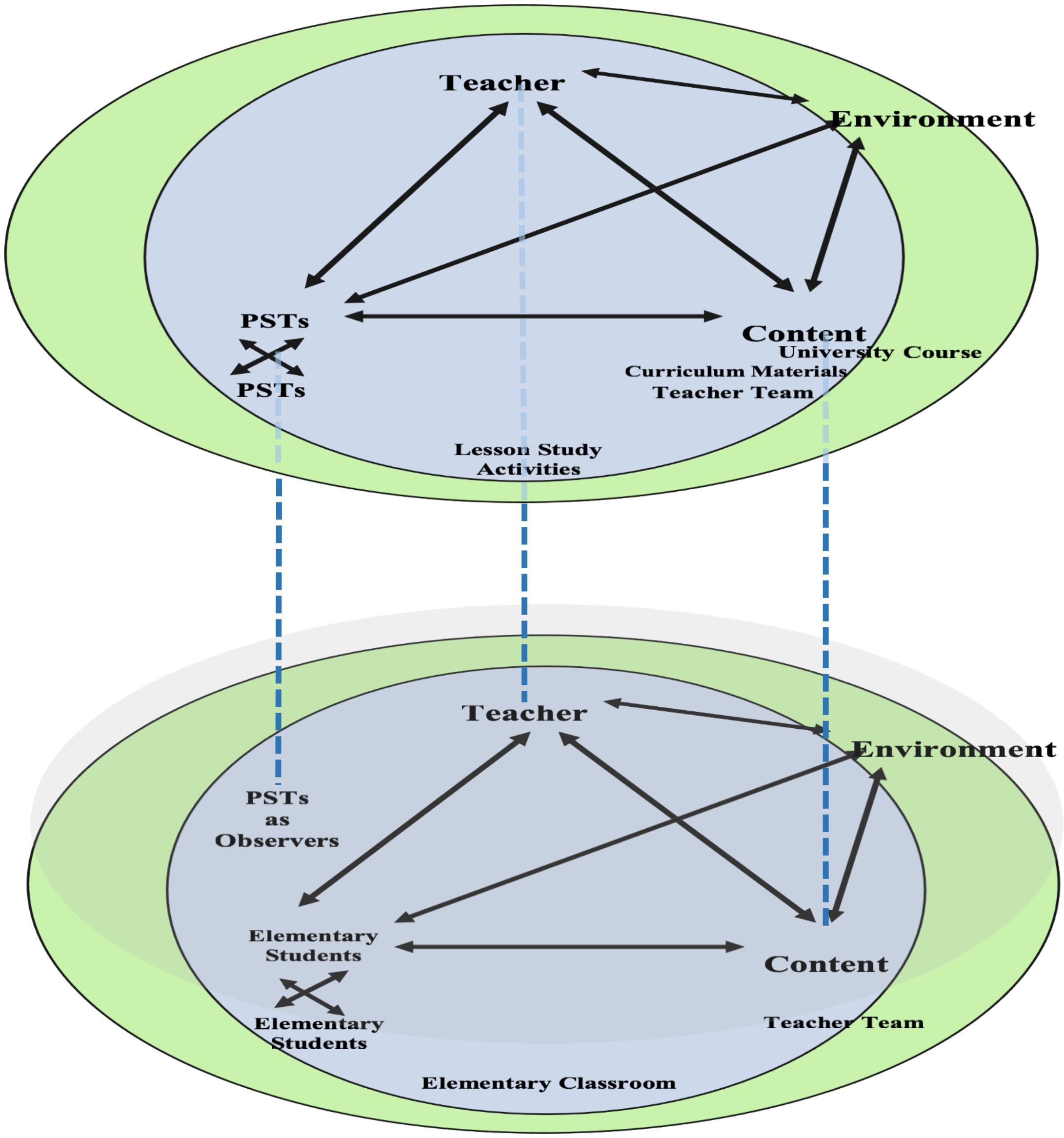


Figure 22. Alignment of Facets of Lesson Study Activities within the Model of Instruction as Interaction. Adapted from D. Cohen, S. W. Raudenbush, & D. L. Ball (2003). Resources, instruction and research. *Educational Evaluation and Policy Analysis*, 25(2), p. 124. Copyright 2003 by Sage Publishing.

University course. The primary mathematical activity of the university course was the APEX cycle (Ozgun-Koca et al. 2020) described in Chapter 4. This activity allowed students to explore mathematical ideas together, present their thinking to one another, and to explain (in written form) the understanding that they form during the process. The activity prompts emphasized flexible thinking, multiple solution paths, and visual representation. The cycle was repeated throughout the semester, so the PSTs came to view mathematical competence as the ability to express their own ideas, hear the ideas of others, and to build new understanding from these ideas. This builds mathematical *teaching* competence since, as Bass (2015) points out, "Teaching requires the skills to not only hear and validate and give space to…ideas, but also to help students reshape them in mathematically productive ways" (p. 636). As the course instructor, I made sure to encourage, support, and praise students who engaged in this work. I tried to, borrowing the term introduced by Abbey and embraced by her fellow PSTs (see page 100), "validate" their efforts in this challenging and unfamiliar mathematical activity.

In addition to the APEX cycle, where listening to the mathematical ideas of others was valued, the whole-class discussions of the lesson study activities were infused with the validation of ideas as well. Each discussion was structured using a protocol based on thinking routines developed and collected by Project Zero of the Harvard Graduate School of Education (2020). Each PST took a turn to share one idea that stood out to him or her (e.g. from the research article that the group studied). It could not be an idea that had been shared previously. Once each PST had a turn, each then had an opportunity to revoice something that was shared by another PST that resonated with him or her. For example, Abbey might say, "I liked what Jude said because…" Therefore, the PSTs were immersed in validating structures throughout the university course.

Curriculum materials. The *Eureka Math* (Great Minds, 2015) curriculum materials and the research article that the PSTs studied shared this emphasis on flexibility in understanding, multiple solution paths, mathematical representations, and classroom discourse. As reported in Chapter 4, the PSTs noticed and commented on this agreement as they studied the *Module 3 Overview* (Great Minds, 2015) and connected it to the recommendations of the research article. This was, of course, intentional. I selected the article because of its alignment in topic (multiplication facts) and its approach: relational versus instrumental understanding (Skemp, 2006) (e.g. knowing from memory versus memorization). This choice proved to be productive as judged by the PSTs' comments during the respective whole-class discussions. In particular, the discussions served as the opportunities for the PSTs to voice the dissonance between this new view of mathematics and their prior experiences.

Teacher team. The teacher team's decision to build the research lesson plan around a single application problem had consequences in terms of bringing the lesson plan into further alignment with the university course. By engineering more space in the plan to allow the elementary students to share their thinking and representations, the teacher team shined an even brighter light on validating instruction. Although the teacher team worked—for the most part—independently from the PSTs, their contribution helped create another layer of alignment. Again, this alignment was intentional. As a facilitator of the teacher team's lesson study cycle, I encouraged the teachers to look for ways to increase equity in their practice; and their adjustments to the lesson plan did just that.

Research lesson. As I became aware of the power of the alignment of these different components of the lesson study activities, I began to entertain an important modification to the normal lesson study structure. In a typical lesson study, a member of the teacher team that develops the lesson delivers the lesson to students; but I decided it would be more instructive for the PSTs if I taught the lesson. The decision created an element of discomfort for me, stemming from my dual role as teacher-educator and researcher. This is not uncommon.

Baumann and Duffy (2001) note that "…simultaneously engaging in teaching and research was a reinforcing, symbiotic phenomenon for some teacher researchers, others reported occasional tension between the roles of teacher and researcher" (p. 612). Lunenberg, Ponte, and Van de Ven (2007) use the term "practitioner research" to describe the type of work that I carried out with this study where "the researcher is highly involved in the object of the research" (p. 16). They specify that the method of study should be systematic; and I claim that this decision to diverge from the norms of lesson study practice does not violate this part of their definition. The decision for me to teach was made in order to increase the alignment of the lesson study with the experience of the PSTs in our university course. That is, I wanted the PSTs to see what they had been experiencing themselves as students in an elementary classroom. To ensure that they did, I taught the lesson. From the reflections and discussion following the lesson, it is clear to me that the PSTs noticed this alignment and it was important for their learning.

There are challenges to achieving the alignment described above. Some have to do with the logistical concerns of coordinating all of the moving parts involved the study, and I will discuss these shortly. At the moment, I will focus on one facet of the alignment that could be problematic for generalization and replication of this study: my role as the teacher in both the university course and the elementary research lesson, and my role as the facilitator of the teacher team's lesson study cycle. I offer a bit of background information to help clarify.

For nearly twenty-five years, in addition to my university teaching, I have taught within the WSU Math Corps (2020), a university-based outreach program that allows me to work with middle-school and high-school students from the surrounding community. This work has involved teaching high-school courses during the school year as well as directing an intense six-week summer camp for middle- and high-school students. In all of these programs, I have been surrounded by inspiring educators who are united in the belief that each child is a unique, and therefore precious, gift to be cherished. This has formed the basis of my own teaching philosophy, and every educational decision I make rests upon this belief. For example, treating students with care and expecting them to treat each other with care, listening to and valuing their mathematical ideas and expecting them to listen to and consider each other's thinking, are dispositions that I try to imbue my practice with. Over the course of the last seven years, I have facilitated multiple lesson study cycles for in-service primary, secondary, and university teachers. Through this work, I have observed and delivered many research lessons to elementary students and have found that the dispositions mentioned above resonate with these younger students equally well.

I present this background to explain that these experiences left me well-positioned to teach the research lesson in this third-grade classroom. As one of the key vertices in the model of instruction as interaction—in both the university and elementary classroom—the teachers are responsible for a great deal of the alignment of the pedagogical approaches as they facilitate the learning in their classrooms. I understand that other teacher educators might not feel comfortable or competent in taking on the role of teacher in both contexts. This is more likely the case for mathematics faculty who often teach the types of content courses for elementary school teachers and who do not typically have experience working with younger students. This should be considered, as substantially more coordination would be required between the different teachers if one wished to approach the same level of alignment that was possible in this study.

I must also note that my role of facilitator for the teacher team's lesson study cycle influenced my choice to teach the research lesson. As mentioned earlier, the teacher team was exploring and attempting to implement an ambitious, high-leverage practice: increasing equity through increasing student voice in classroom discussions. Facilitating a productive mathematical discussion is challenging. I believed that my teaching of the lesson that they had enhanced could serve as a model for the teacher team and give them all an opportunity to observe their students as they shared and discussed their mathematical insights.

In addition to the logistical challenges that my playing so many different roles within the lesson study activity posed, it also introduced relational and ethical complexities that needed to be acknowledged and navigated. Researching the multiple contexts in which I work, so-called "insider fieldwork" (Savin-Baden & Major, 2013), involved the potential for increased levels of bias or even selective inattention to anomalies worth investigating (Schön, 1983). My responsibilities to my university PSTs as their instructor, to the inservice teachers as the facilitator of their lesson study cycle, and to the elementary school students in whose classroom I delivered the research, all needed to be met while I sought to document, analyze, and theorize about what was happening (often simultaneously) across these contexts. I attempted to minimize bias and missed opportunities to think about practice by using what Schön (1983) termed "reflection-in-action" (p. 49). This involved, for example, reflecting on judgements about what I thought the PSTs *could or should be* learning through lesson study and what their comments and reflections showed that they *were* learning, and deciding whether I needed to address that difference. Issues of this type within a group could be compounded with issues between groups. For example, if the teacher team had decided to focus their lesson study on a topic outside the purview of the PSTs' university course or inappropriate for their level of experience, collaboration might be ruled out. Steering the teacher team towards a topic that would allow collaboration might compromise the team's autonomy. In the end, I was fortunate enough to not have to make these types of decisions, but these types of ethical concerns, inherent in insider fieldwork and when playing so many roles within a study, could be challenges to the alignment discussed above and to the study as a whole.

The Benefits and Challenges of the Research Lesson

The research lesson was the most important component of the lesson study activity in terms of creating a new vision of mathematics as well as what the PSTs of this study came to describe as "validating" instruction. As noted in the discussion of the broad themes presented in the previous chapter, the research lesson is the focal point of lesson study's lens. It is where the PSTs can see, with their own eyes, the new vision of mathematics and math class *with elementary students included*. The research lesson is the place where the much of the new knowledge that PSTs create during the cycle can be activated as they make sense of student thinking and evaluate the lesson against the specific learning goals. It is *the* complexity that the literature recommends PSTs engage in during their preparation programs. In short, this study shows that the research lesson is a crucial factor in PST learning.

This is not to say that the other components are not sites of significant learning; the considerable learning that occurred during the individual activities has been noted. This aligns with the findings of previous research involving modified lesson study cycles without a research lesson with elementary students. For example, "microteaching" activities, where PSTs deliver a collaboratively created lesson *to other PSTs*, have been shown to be effective learning activities (Fernandez, 2005). This modification is understandable based on the pre-service context, and many similar successful modifications of lesson study have been documented (Ponte, 2017). However, my study shows the importance of the research lesson observation to the considerable learning that was identified. I claim that Feiman-Nemser's (2001) recommendation for PSTs to have opportunities to analyze beliefs, increase subject-matter knowledge and knowledge of learners, build a beginning repertoire of teacher moves, and develop tools to study teaching are all addressed by the inclusion of the research lesson with elementary students. The research lesson serves as a capstone experience in terms of the mathematical knowledge for teaching and their fledgling knowledge of students gained through the *kyoziakenkyu.* It serves as an opportunity to observe preplanned and impromptu teacher responses to student thinking and allows the PSTs to study teaching as opposed to studying *about* teaching.

Thus, if the goals of a particular preparation program intersect with the goals foregrounded in this study, a lesson study activity including a research lesson observation with elementary school students seems to offer excellent opportunities to attain those goals. If that program seeks to instill or increase the belief in the effectiveness of a particular pedagogical approach, such as culturally responsive teaching (Hammond, 2015), teaching through problem solving (Takahashi, Lewis & Perry, 2013), or the "validating instruction" noted by the PSTs in this study, then the research lesson observation seems to be a crucial component in achieving this outcome. In his summary of research on beliefs, Philipp (2007) concluded:

> Mathematics educators generally agree on what beliefs are; we now face a greater challenge than defining beliefs: how to change teachers' beliefs. If beliefs are lenses through which we humans view the world, then the beliefs we hold filter what we see; yet what we see also affects our beliefs—creating a quandary. (p. 309)

He proposed that reflecting on their practice, especially the mathematical thinking of their students. This, Philipp (2007) claimed, is how beliefs would change. The conclusions of this study show that this change in beliefs regarding the efficacy of engaging, validating instruction is possible even at the earliest stages of pre-service preparation. It is important to note that PSTs at this early stage will require the intense study involved in the *kyozaikenkyu* stage of lesson study to notice and reflect on the student thinking that they observe during the research lesson. That is, this study suggests that the research lesson observation is necessary, but not sufficient, for the type of learning and the effects on beliefs reported here.

A research lesson brings together many things: content and pedagogical knowledge developed through the *kyozaikenkyu*, a lesson plan and the students for whom it was designed, teachers and students, etc. In the case of this modified lesson study activity, it also brought together for the first time (physically, at least) the teacher team and the PSTs.

Both of these groups met in the same classroom, on the same day, at the same time in order to observe the lesson they had both worked over the course of nearly three months. The logistics involved in coordinating the people, facilities, and timing of the research lesson were extremely difficult, but there were already pieces in place without which there could be no thought of a research lesson.

One important piece was my connection to the teaching team through a professional development program. I had worked with some of the members of the teaching team, including Mrs. Cuoco, the teacher whose classroom the research lesson occurred, for three years. More importantly, I was working with this team during the semester in which this study took place. Therefore, I had ready access to a classroom of elementary school students. My relationship with team members was built on mutual trust and respect built up over these years; thus, they accepted my request to integrate the PSTs into their lesson study practice. Likewise, based on my long-term relationship with the teachers at that particular school, their administration allowed the research lesson to take place, helped facilitate substitute coverage of the teacher team members' classes, and welcomed the PSTs. Even with these relationships in place, the research lesson was still difficult to realize.

These issues must be taken into by educators considering similar work. I find it important to acknowledge that I was privileged to have had access to an established team of elementary teachers engaged in lesson study, to their school, and to their students. The relationships that were responsible for this access took years to cultivate and their importance in making this work possible cannot be overstated.

The Learning Occluded by the Modified Lesson Study Activity

Although the learning that occurred within the lesson study activities was substantial, it must be noted that the activities did occlude other learning opportunities. It may be more precise to say that the lesson study activities *displaced* other learning. That is, the lesson study activities took a considerable amount of time away from the more typical university course activities. In total, I dedicated approximately eight class hours for the various in-class discussions; and this constituted roughly two weeks' worth of class time (or 13% of the class time). As mentioned earlier in the analysis of the LMT measures the mathematical knowledge for teaching, this displaced important topics from the syllabus, most notably the introductory study of rational numbers. Even though this important topic could be addressed in the second course of our university's two-course sequence, that meant that other topics from that second course, in turn, were displaced. This presents yet another difficult choice for teacher educators who must curate rich learning opportunities in what, in essence, is a zero-sum game. If something is included in a course or preparation program, something else must be excluded. Despite the learning detected via other methods, the lack of growth in the mathematical knowledge for teaching (Ball et al., 2008) as measured by the LMT-TKAS (Hill et al., 2007) is a concern. I hypothesized earlier that this lack of growth may have been due in part to the displacement of specific topics (e.g. fractions); but this does not explain the nearly complete absence of improvement in this crucial domain.

CHAPTER 6 LIMITATIONS, RECOMMENDATIONS FOR FURTHER RESEARCH, AND CONCLUSION

Limitations

This study was limited to a small number of participants over a short span of time. The participants were not randomly selected, but rather represented a nonprobability, purposive sample (Merriam, 1988). A randomized study that compared the learning of PSTs that participated in lesson study activities to those who did not, might provide a fuller picture of the effect of these activities. The use of thick description in the results was used to address this limitation, but, nonetheless, the small number of participants does present challenges to the generalizability of the results.

The study was completed within a single university semester and displaced other opportunities for learning. Lesson study is a learning structure that is based on intense, yet steady and unhurried, collaborative inquiry that results in gradual and incremental learning. The short time period of the study may have been inadequate to capture the breadth or the depth of the learning that was sparked by the lesson study activities.

Recommendations for Further Research

This study revealed much about the learning of PSTs within a modified lesson study activity, but it also generated new questions. I mention again that this study took place over an extremely short period time: one fifteen-week university semester. The first two recommendations for further research stem from this important fact.

Do Results Persist over Time?

First, it seems worth exploring if the learning and changes in beliefs that the analysis made visible persists and can be deepened or further solidified when PSTs engage in multiple cycles of lesson study throughout their preparation program—especially as they move into their field experiences. This is especially important with respect to the mathematical knowledge for teaching (Ball et al., 2008) which, as measured by the LMT-TKAS (Hill et al., 2007), was seemingly unaffected by the lesson study activities. This absence of growth on these important measures is worth exploring further. Chokshi and Fernadez (2004) warned:

> …lesson study can serve as the vehicle by which practioners can deepen their understanding of content…however, we would like to emphasize that learning content knowledge through lesson study is not an automatic process. (p. 521)

This study showed that their warning was warranted. As Ma (1999) pointed out, developing this content knowledge is crucial as "without solid support from subject matter knowledge, promising methods or new teaching conceptions cannot be successfully realized" (p. 33).

Thus, it would be helpful to track growth in the PSTs' mathematical knowledge for teaching (Ball et al., 2008) via the LMT-TKAS over a longer period of time as well as to identify opportunities within the cycle to deepen their understanding of content. The issue of alignment of the pedagogical approaches of individual preparation courses within a program could also be explored in this type of longitudinal study.

Secondly, further research is warranted to see if and how PSTs draw upon their experiences within lesson study once they are in classrooms of their own. This, of course, is the real question. Hiebert (2015) has shown that teacher preparation does matter in terms of how well novice teachers can explain the mathematics studied in their preparation programs. It is worth exploring if the positive affect towards mathematics, mathematical discussions, validating instruction, or the nascent research stance that was captured during this study travels with the PSTs into their future classrooms.

Does Increased Interaction with the Teacher Team Lead to Increased Learning?

The interaction of the PSTs and the teacher team members was limited in this study. It may be fruitful to explore the opportunities to learn, especially from the view of the community of practice framework, that come from engaging in more of the lesson study components together with the teacher team. What would more interaction between the groups afford and what might it occlude? A quote from Abbey following the research lesson shows that even casual interaction can be productive in terms of peripheral members feeling welcomed to the community of teaching:

> Abbey: [What struck me was] how thankful they were that we were there. Because, it's very discouraging when you tell people that you want to be a teacher. So, I think it was nice to be validated for something that you aspire to do. In a sense, people really appreciate something that you want to do rather than saying, "Oh, you're not going to make any money. Why do you want to do that?"
>
> Chris: And you felt that coming from the kids?
>
> Abbey: Um, more from the faculty.
>
> Chris: Oh, okay!
>
> Abbey: When you were still in the classroom and we were in [the post-lesson discussion room], she was like, "I'm so happy that you guys are here. I really can't thank you enough. You guys are going to be great teachers." Just *validation* (emphasis added). (Post-Lesson Discussion Transcript)

What are the Elements of Validating Instruction?

It is appropriate that Abbey's quote again brings up "validation." The notion of validating instruction (a term which she coined) was striking in terms of the universal embrace it received from the PSTs. I have pointed to some features of my own instruction, the instructional approach suggested by the *Eureka Math* curriculum materials (Great Minds, 2015), and the instructional decisions built into the lesson by the teacher team that the PSTs considered elements of "validating instruction," but it is clear from the response of the PSTs' and the elementary students alike that something important (and, sadly, atypical) was happening. This seems connected to Noddings' (2012) notion of care in education. Questions about this pedagogical approach, such as what, precisely, the elements of "validating instruction" that so captured the attention and imaginations of these students really are, are worthy of further study.

Conclusion

This study shows that pre-service teachers at the earliest stages of their preparation can learn a great deal through a modified lesson study activity. Through the lesson study activities designed for this study, novice members on the extreme periphery of the community of teaching practice were able to better engage with real challenges of teaching. These included anticipating and using the diverse thinking of students as resources for classroom discourse as well as negotiating the time constraints on planning and executing ambitious instruction. They were better positioned to deal with these challenges by creating new knowledge for action—in particular their knowledge of how students think about and discuss mathematics as a connected, sense-making activity. PSTs were able to negotiate their idea of mathematical competence and to imagine a math class that differed markedly from their own experience. This new vision of mathematics and math class, and the significant change in their beliefs in the efficacy of instruction—in particular engaging, validating instruction—was brought into sharper focus through the alignment of the various components and surrounding contexts of the lesson study. This study indicates that if the pedagogical approaches of these various components reinforce each other, they will help PSTs notice and reflect on what they learn and how they feel as students in their university courses and how that is mirrored in the experiences of the elementary students that they observe during the research lesson. The research lesson serves as an indispensable culminating experience of the activities where all of the new knowledge is called into action enabling the PSTs to learn in and from practice.

The lack of growth in Mathematical Knowledge for Teaching (Ball et al., 2008) over the course of the lesson study activities as measured by the LMT-TKAS (Hill et al., 2007) is concerning and deserves further study. In addition, the intense lesson study activity occupied a considerable amount of time in the university course's already brimming syllabus. The lesson study activity drew upon relationships and programs that took years to establish and nurture; and the logistics of bringing the learning and work products of both PSTs and an inservice teacher team together for a research lesson with elementary students were substantial. These elements should be considered before attempting to implement such an activity in a pre-service content course; but the depth and types of PST learning uncovered in this study make the proposition worth entertaining.

APPENDIX A

Table A1

Code Book

Name	Description	Files	References
Assessing Student Learning	Deals with formal assessments (e.g. quizzes and exams) as well as observational and inferential assessment of student learning.	5	20
Behind the Scenes	Aspects of teaching that are not visible to students. This includes lesson planning, content and curriculum material study, and thinking about teaching and learning in general—that will be deployed in a lesson.	2	27
Building Community Language	Includes developing language that is shared by the community (the community of PSTs, the broader community of teachers, etc.) especially mathematical terms.	7	27
Building Community Context	Captures elements that contextualize our work either in a particular classroom/school or classrooms/schools in general. Includes norms, materials, descriptions.	9	28
Changing View of Math Self	Items that reflect a change in how the PST views themselves as doers of mathematics and/or teachers of mathematics.	1	3
Classroom Discourse	Deals with the benefits of, facilitation of, and recommendation/guidance for classroom discussion	2	31
Collaboration	Deals with teachers working with other teachers (in general) and also our group collaborating with the Teacher Team. This latter sense is often one-sided as the PSTs received documents created by the team.	3	12
Comparing Contrasting with Own Experience	Applies to instances that compare or contrast an idea, feeling, practice, or teaching move with what they experienced as a student or expected as a PST.	7	50
Complexity of Live Instruction	Statements regarding the complex interactive nature of live instruction. Items deal with student actions/thinking and moves made by the instructor. Items regarding the preparation/anticipation of this complexity are also included.	2	37
Conjecturing about Student Thinking	This code captures both predictions and interpretations of what students will or did think	5	31

Name	Description	Files	References
	during a lesson. This code is also applied to broader conjectures about student thinking (e.g. appropriate age level for a particular topic).		
Connection to Our Course	Statements that reference (either explicitly or implicitly) content, practices, or discussions from our course.	8	31
Connections	PSTs commenting on or making connections between mathematical ideas and representations.	2	6
Content Knowledge	Items regarding mathematical content and justification are included.	6	53
Continual Learning	Applied to issues involving a long-term view of learning to teach. They can be explicit statements regarding learning over time or viewing their current class/activity as a first step in a continuing process.	4	19
Curriculum Applications	Applies to items regarding the application of knowledge explicitly suggested by the curricular materials.	3	3
Curriculum Guidance	PSTs mentions of recommendations (on teacher moves, instructional modes, explanation, scaffolding, etc.) included in the curriculum materials we examined.	7	52
Curriculum progression	Applied to instances involving the progression/sequencing of ideas within a lesson, across a unit, or over different grades. These ideas are motivated by the curriculum materials that the PSTs studied.	6	35
Depth of Understanding	Dealing with the depth of understanding that is expected of students or required by teachers with respect to content. In addition, displays of deepening understanding by the PSTs are grouped in this code.	6	42
Describing Student Actions	PSTs describing the actions of elementary students that they observe during a live or recorded lesson.	2	7
Early Stage of Preparation	Statements that explicitly or implicitly refer to the PSTs' being in the early stages, time-wise, of their preparation programs. This code is also applied when there is some reference to teaching being "down the road" from where they are now.	3	13
Enthusiasm	Expressions of excitement, interest or engagement by PSTs regarding teaching/learning.	4	27
Instructional Decisions	Deals with items where PSTs propose, reflect on, or question instructional decisions. These	2	32

Name	Description	Files	References
	decisions might involve particular sources (e.g. curriculum materials, planning discussions, observations) or reflect their own opinions.		
Lesson-Teaching Effect on Students	Involves the connection between what is taught (or how it is taught/presented) and the students.	2	43
Live Lesson	Any items that refer to the live research lesson.	1	32
Math Phobia Animosity	Items that involve fear, anxiety, or anger related to mathematics. These can be personal or general statements.	1	6
Memorization	Captures statements regarding the memorization of mathematical procedures or facts (although they are often in contrast to a conceptual understanding, they need not be framed negatively).	5	15
Multiple Solution Paths	PST comments on the different solution paths, representations, and tools that are available or used.	7	46
My Future Classroom	What PSTs plan to do when they begin teaching (i.e. in their own classrooms). These statements are necessarily personal (e.g. "my classroom", "I will…") as opposed to general (e.g. "Teachers should…").	1	15
New Knowledge for Action	Statements that explicitly connect new knowledge or awarenesses (developed in class, through curriculum study, or planning discussions) with some aspect of teaching.	2	25
New View of Mathematics	Applied to statements by PSTs that reflect a new way of viewing mathematics. For example, viewing mathematics as connected and understandable versus a collection of isolated facts is one such shift. The new view can also be related to how mathematics can be presented to and experienced by students.	1	20
Opening up Classrooms	Statements that deal with making practice visible to others. In addition, reflections and hypotheses about teaching that are possible because a classroom has been opened up to the PSTs are also grouped with this code.	3	17
Past Teachers	Statements that recall past teachers' actions or characteristics. This also captures statements that refer to "how I was taught" rather than specific teachers.	4	6
Presenting Content	This code is applied to statements about how mathematics was or can/should be presented.	2	28
Proposing Teacher	Statements that suggest teaching moves in	4	16

Name	Description	Files	References
Move	response to students' thinking/moves either from a recorded/live lessons or anticipated student thinking/moves (while planning).		
Representations	Statements about mathematical representations or manipulatives.	6	27
Research Stance	These statements reflect a willingness to experiment with teaching—to try something and observe how it plays out in a lesson.	2	10
Resources	Statements that involve or refer to resources (such as lesson plans, research, etc.) available to the PSTs and in-service teachers.	4	22
Student Engagement	Statements regarding observations of ,and planning activities to increase, student engagement.	1	29
Teacher Challenge (Different Curricula)	This code is applied to statements regarding the challenge of juggling competing visions or approaches that may be found in different curricula.	1	2
Teacher Challenges (Autonomy)	This code captures instances where PSTs comment on teacher autonomy. This includes statements regarding teacher-initiated decisions/activities and their ability to implement them.	6	10
Teacher Challenges (Competency)	Statements regarding the competency of teachers (including perceptions of people outside the profession).	3	3
Teacher Challenges (Diversity of Learners)	Deals with statements regarding the diversity of student thinking, preparation, and needs and how these affect planning and enacting instruction.	6	24
Teacher Challenges (Funding)	This code is applied to statements that involve school funding and how it affects teaching/learning.	1	4
Teacher Challenges (Management)	Issues of management and building a productive classroom culture. These can be broad or specific to a particular lesson.	2	9
Teacher Challenges (MKT)	Statements regarding the demands on teachers' mathematical knowledge for teaching. The cases can deal with teachers in general or instances where PSTs need to apply their own MKT	4	13
Teacher Challenges (Pay)	Statements regarding teacher pay (specifically the effects of lower salaries for American teachers).	1	2
Teacher Challenges	Issues related to the (real or perceived) low	2	3

Name	Description	Files	References
(Status)	status of teachers and the effects on motivation, dedication, and initiative.		
Teacher Challenges (Structural)	Issues involving structural challenges in mathematics education (e.g. teacher isolation)	2	4
Teacher Challenges (Time)	Issues relating to the time pressures experienced by teachers and their effects on, for example, planning and enacting instruction or improving one's practice.	6	17
Teacher learning and student performance	Instances where PSTs connect what they've learned to being able to help students better understand or better engage with mathematics.	7	29
Teacher Student Relationships	Issues of teacher-student relationships—build ing them, maintaining them, etc. In addition, this code is applied to statements involving the affordances of solid relationships.	4	18
Teaching and Learning US Culture	Cultural issues of schooling in the United States. This includes perceptions of teachers, of students, and of mathematics.	7	24
Understanding Student Thinking	Statements regarding student thinking and the teacher's understanding of that that thinking. E.g. how to make student thinking more visible through instruction and assessment, anticipating student thinking.	8	34
Validating Students	Statements regarding the validation of students and their thinking.	1	7

APPENDIX B

***Eureka Math Grade 3 Module 3 Teacher Guide* Materials**

Grade 3 • Module 3

Multiplication and Division with Units of 0, 1, 6–9, and Multiples of 10

OVERVIEW

This 25-day module builds directly on students' work with multiplication and division in Module 1. At this point, Module 1 instruction coupled with fluency practice in Module 2 has students well on their way to meeting the Grade 3 fluency expectation for multiplying and dividing within 100 (**3.OA.7**). Module 3 extends the study of factors from 2, 3, 4, 5, and 10 to include all units from 0 to 10, as well as multiples of 10 within 100. Similar to the organization of Module 1, the introduction of new factors in Module 3 spreads across topics. This allows students to build fluency with facts involving a particular unit before moving on. The factors are sequenced to facilitate systematic instruction with increasingly sophisticated strategies and patterns.

Topic A begins by revisiting the commutative property. Students study familiar facts from Module 1 to identify known facts using units of 6, 7, 8, and 9 (**3.OA.5**, **3.OA.7**). They realize that they already know more than half of their facts by recognizing, for example, that if they know 2 × 8, they also know 8 × 2 through commutativity. This begins a study of arithmetic patterns that becomes an increasingly prominent theme in the module (**3.OA.9**). The subsequent lesson carries this study a step further; students apply the commutative property to relate 5 × 8 and 8 × 5 and then add one more group of 8 to solve 6 × 8 and, by extension, 8 × 6. The final lesson in this topic builds fluency with familiar multiplication and division facts, preparing students for the work ahead by introducing the use of a letter to represent the unknown in various positions (**3.OA.3**, **3.OA.4**).

Topic B introduces units of 6 and 7, factors that are well suited to Level 2 skip-counting strategies and to the Level 3 distributive property strategy, already familiar from Module 1. Students learn to compose up to and then over the next ten. For example, to solve a fact using units of 7, they might count 7, 14, and then mentally add 14 + 6 + 1 to make 21. This strategy previews the associative property using addition and illuminates patterns as students apply count-bys to solve problems. In the next lesson, students apply the distributive property (familiar from Module 1) as a strategy to multiply and divide. They decompose larger unknown facts into smaller known facts to solve. For example, 48 ÷ 6 becomes (30 ÷ 6) + (18 ÷ 6), or 5 + 3 (**3.OA.5**, **3.OA.7**). Topic B's final lesson emphasizes word problems, providing opportunities to analyze and model. Students apply the skill of using a letter to represent the unknown in various positions within multiplication and division problems (**3.OA.3**, **3.OA.4**, **3.OA.7**).

Topic C anticipates the formal introduction of the associative property with a lesson focused on making use of structure to problem solve. Students learn the conventional order for performing operations when parentheses are and are not present in an equation (**3.OA.8**). With this student knowledge in place, the associative property emerges in the next lessons as a strategy to multiply using units up to 8 (**3.OA.5**). Units of 6 and 8 are particularly useful for presenting this Level 3 strategy. Rewriting 6 as 2 × 3 or 8 as 2 × 4 makes shifts in grouping readily apparent (see example on next page) and also utilizes the familiar factors 2, 3, and 4 as students learn the new material. The following strategy may be used to solve a problem like 8 × 5:

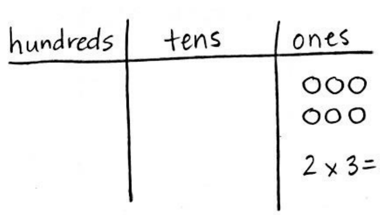
$$8 \times 5 = (4 \times 2) \times 5$$
$$8 \times 5 = 4 \times (2 \times 5)$$
$$8 \times 5 = 4 \times 10$$

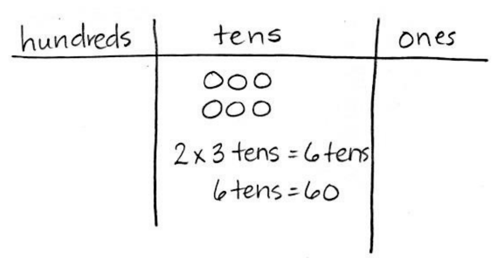
In the final lesson of Topic C, students relate division to multiplication using units up to 8. They understand division as both a quantity divided into equal groups and an unknown factor problem for which—given the large size of units—skip-counting to solve can be more efficient than dividing (**3.OA.3**, **3.OA.4**, **3.OA.7**).

Topic D introduces units of 9 over three days, with students exploring a variety of arithmetic patterns that become engaging strategies for quickly learning facts with automaticity (**3.OA.3**, **3.OA.7**, **3.OA.9**). Nines are placed late in the module so that students have enough experience with multiplication and division to recognize, analyze, and apply the rich patterns found in the manipulation of units of 9. As with other topics, the sequence ends with interpreting the unknown factor to solve multiplication and division problems (**3.OA.3**, **3.OA.4**, **3.OA.5**, **3.OA.7**).

In Topic E, students begin by working with facts using units of 0 and 1. From a procedural standpoint, these are simple facts that require little time for students to master; however, understanding the concept of nothing (zero) is more complex, particularly as it relates to division. This unique combination of simple and complex explains the late introduction of 0 and 1 in the sequence of factors. Students study the results of multiplying and dividing with units of 0 and 1 to identify relationships and patterns (**3.OA.7**, **3.OA.9**).
The topic closes with a lesson devoted to two-step problems involving all four operations (**3.OA.8**). In this lesson, students work with equations involving unknown quantities and apply the rounding skills learned in Module 2 to make estimations that help them assess the reasonableness of their solutions (**3.OA.8**).

In Topic F, students multiply by multiples of 10 (**3.NBT.3**). To solve a fact like 2 × 30, they first model the basic fact 2 × 3 on the place value chart. Place value understanding helps them to notice that the product shifts one place value to the left when multiplied by 10: 2 × 3 tens can be found by simply locating the same basic fact in the tens column.





In the subsequent lesson, place value understanding becomes more abstract as students model place value strategies using the associative property (**3.NBT.3**, **3.OA.5**). 2 × 30 = 2 × (3 × 10) = (2 × 3) × 10. The final lesson focuses on solving two-step word problems involving multiples of 10 and equations with unknown quantities (**3.OA.8**). As in the final lesson of Topic E, students estimate to assess the reasonableness of their solutions (**3.OA.8**).

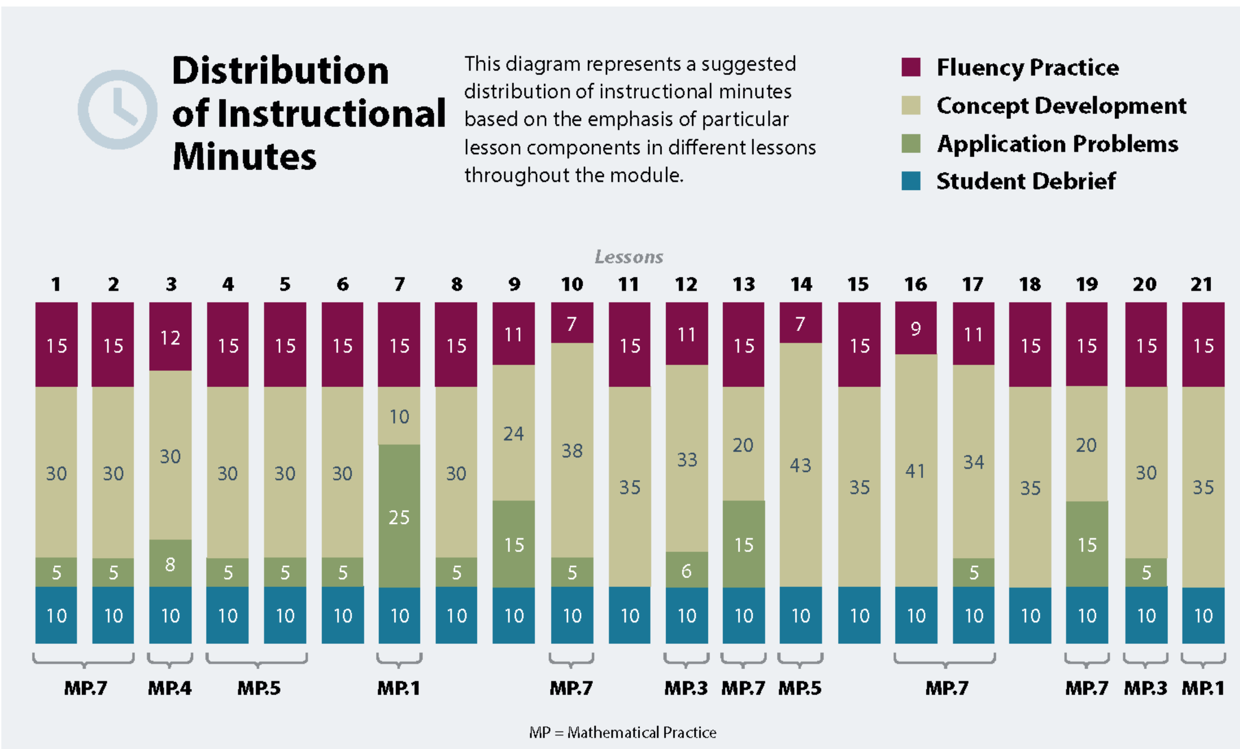
Notes on Pacing for Differentiation

If pacing is a challenge, consider the following modifications and omissions.

Omit Lessons 6 and 10. Both lessons involve using the distributive property with multiplication and division, a recurring objective in Module 3. Within later distributive property lessons, incorporate units of 6 and 7.

Omit Lesson 11, a problem solving lesson involving multiplication and division. Lesson 11 shares an objective with Lesson 15 and is also similar to Lesson 7.

Omit Lesson 13. Study its essential understandings, and embed them into the delivery of Lesson 14's Concept Development. Modify Lesson 14 by omitting Part 1 of the Concept Development, a part which relies on the foundation of Lesson 13.



Focus Grade Level Standards

Represent and solve problems involving multiplication and division.[1]

3.OA.3 Use multiplication and division within 100 to solve word problems in situations involving equal groups, arrays, and measurement quantities, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem. (See Glossary, Table 2.)

3.OA.4 Determine the unknown whole number in a multiplication or division equation relating three whole numbers. *For example, determine the unknown number that makes the equation true in each of the equations 8 × ? = 48, 5 = _ ÷ 3, 6 × 6 = ?.*

Understand properties of multiplication and the relationship between multiplication and division.[2]

3.OA.5 Apply properties of operations as strategies to multiply and divide. (Students need not use formal terms for these properties.) *Examples: If 6 × 4 = 24 is known, then 4 × 6 = 24 is also known. (Commutative property of multiplication.) 3 × 5 × 2 can be found by 3 × 5 = 15, then 15 × 2 = 30, or by 5 × 2 = 10, then 3 × 10 = 30. (Associative property of multiplication.) Knowing that 8 × 5 = 40 and 8 × 2 = 16, one can find 8 × 7 as 8 × (5 + 2) = (8 × 5) + (8 × 2) = 40 + 16 = 56. (Distributive property.)*

Multiply and divide within 100.[3]

3.OA.7 Fluently multiply and divide within 100, using strategies such as the relationship between multiplication and division (e.g., knowing that 8 × 5 = 40, one knows 40 ÷ 5 = 8) or properties of operations. By the end of Grade 3, know from memory all products of two one-digit numbers.

Solve problems involving the four operations, and identify and explain patterns in arithmetic.[4]

3.OA.8 Solve two-step word problems using the four operations. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding.
(This standard is limited to problems posed with whole numbers and having whole-number answers; students should know how to perform operations in the conventional order when there are no parentheses to specify a particular order, i.e., Order of Operations.)

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[1]The balance of this cluster is addressed in Module 1.

[2]The balance of this cluster is addressed in Module 1.

[3]From this point forward, fluency practice with multiplication and division facts is part of the students' on-going experience.

[4]After being fully taught in Module 3, this standard (as well as 3.OA.3) continues to be practiced throughout the remainder of the school year.

3.OA.9 Identify arithmetic patterns (including patterns in the addition table or multiplication table), and explain them using properties of operations. *For example, observe that 4 times a number is always even, and explain why 4 times a number can be decomposed into two equal addends.*

Use place value understanding and properties of operations to perform multi-digit arithmetic. (A range of algorithms may be used.)[5]

3.NBT.3 Multiply one-digit whole numbers by multiples of 10 in the range 10–90 (e.g., 9 × 80, 5 × 60) using strategies based on place value and properties of operations.

Foundational Standards

2.OA.3 Determine whether a group of objects (up to 20) has an odd or even number of members, e.g., by pairing objects or counting them by 2s; write an equation to express an even number as a sum of two equal addends.

2.OA.4 Use addition to find the total number of objects arranged in rectangular arrays with up to 5 rows and up to 5 columns; write an equation to express the total as a sum of equal addends.

2.NBT.2 Count within 1000; skip-count by 5s, 10s, and 100s.

3.OA.1 Interpret products of whole numbers, e.g., interpret 5 × 7 as the total number of objects in 5 groups of 7 objects each. *For example, describe a context in which a total number of objects can be expressed as 5 × 7.*

3.OA.2 Interpret whole-number quotients of whole numbers, e.g., interpret 56 ÷ 8 as the number of objects in each share when 56 objects are partitioned equally into 8 shares, or as a number of shares when 56 objects are partitioned into equal shares of 8 objects each. *For example, describe a context in which a number of shares or a number of groups can be expressed as 56 ÷ 8.*

3.OA.6 Understand division as an unknown-factor problem. *For example, find 32 ÷ 8 by finding the number that makes 32 when multiplied by 8.*

Focus Standards for Mathematical Practice

MP.1 **Make sense of problems and persevere in solving them.** Students engage in exploratory lessons to discover and interpret patterns, and they apply their observations to solving multi-step word problems involving all four operations.

MP.3 **Construct viable arguments and critique the reasoning of others.** As students compare solution strategies, they construct arguments and critique the reasoning of their peers. This practice is particularly exemplified in daily Application Problems and problem-solving specific lessons in which students share and explain their work with one another.

MP.4 **Model with mathematics**. Students use arrays, tape diagrams, and equations to represent word problem situations.

[5] The balance of this cluster is addressed in Module 2.

MP.5 **Use appropriate tools strategically**. Students analyze problems and select the appropriate tools and pathways to solutions. This is particularly evident as students select problem-solving strategies and use arithmetic properties as simplifying strategies when appropriate.

MP.7 **Look for and make use of structure**. In this module, patterns emerge as tools for problem solving. For example, students make use of structure as they utilize the distributive property to establish the 9 = 10 – 1 pattern, or when they check the solution to a fact using units of 9 by making sure the sum of the digits in the product adds up to 9. They make use of the relationship between multiplication and division as they determine unknown factors and interpret their meanings.

Overview of Module Topics and Lesson Objectives

Standards		Topics and Objectives	Days
3.OA.4 **3.OA.5** **3.OA.7** **3.OA.9** 3.OA.1 3.OA.2 3.OA.3 3.OA.6	A	**The Properties of Multiplication and Division** Lesson 1: Study commutativity to find known facts of 6, 7, 8, and 9. Lesson 2: Apply the distributive and commutative properties to relate multiplication facts $5 \times n + n$ to $6 \times n$ and $n \times 6$ where n is the size of the unit. Lesson 3: Multiply and divide with familiar facts using a letter to represent the unknown.	3
3.OA.3 **3.OA.4** **3.OA.5** **3.OA.7** 3.OA.1 3.OA.2 3.OA.6	B	**Multiplication and Division Using Units of 6 and 7** Lesson 4: Count by units of 6 to multiply and divide using number bonds to decompose. Lesson 5: Count by units of 7 to multiply and divide using number bonds to decompose. Lesson 6: Use the distributive property as a strategy to multiply and divide using units of 6 and 7. Lesson 7: Interpret the unknown in multiplication and division to model and solve problems using units of 6 and 7.	4
3.OA.3 **3.OA.4** **3.OA.5** **3.OA.7** 3.OA.1 3.OA.2 3.OA.6 3.OA.8	C	**Multiplication and Division Using Units up to 8** Lesson 8: Understand the function of parentheses and apply to solving problems. Lesson 9: Model the associative property as a strategy to multiply. Lesson 10: Use the distributive property as a strategy to multiply and divide. Lesson 11: Interpret the unknown in multiplication and division to model and solve problems.	4

Standards		Topics and Objectives	Days
		Mid-Module Assessment: Topics A–C (assessment ½ day, return ½ day, remediation or further applications 1 day)	2
3.OA.3 **3.OA.4** **3.OA.5** **3.OA.7** **3.OA.9** 3.OA.1 3.OA.2 3.OA.6	D	**Multiplication and Division Using Units of 9** Lesson 12: Apply the distributive property and the fact 9 = 10 – 1 as a strategy to multiply. Lessons 13–14: Identify and use arithmetic patterns to multiply. Lesson 15: Interpret the unknown in multiplication and division to model and solve problems.	4
3.OA.3 **3.OA.7** **3.OA.8** **3.OA.9** 3.OA.1 3.OA.2 3.OA.4 3.OA.6	E	**Analysis of Patterns and Problem Solving Including Units of 0 and 1** Lesson 16: Reason about and explain arithmetic patterns using units of 0 and 1 as they relate to multiplication and division. Lesson 17: Identify patterns in multiplication and division facts using the multiplication table. Lesson 18: Solve two-step word problems involving all four operations and assess the reasonableness of solutions.	3
3.OA.5 **3.OA.8** **3.OA.9** **3.NBT.3** 3.OA.1	F	**Multiplication of Single-Digit Factors and Multiples of 10** Lesson 19: Multiply by multiples of 10 using the place value chart. Lesson 20: Use place value strategies and the associative property $n \times (m \times 10) = (n \times m) \times 10$ (where n and m are less than 10) to multiply by multiples of 10. Lesson 21: Solve two-step word problems involving multiplying single-digit factors and multiples of 10.	3
		End-of-Module Assessment: Topics A–F (assessment ½ day, return ½ day, remediation or further application 1 day)	2
Total Number of Instructional Days			**25**

Terminology

New or Recently Introduced Terms

- Multiple (specifically with reference to naming multiples of 9 and 10, e.g., 20, 30, 40, etc.)
- Product (the quantity resulting from multiplying two or more numbers together)

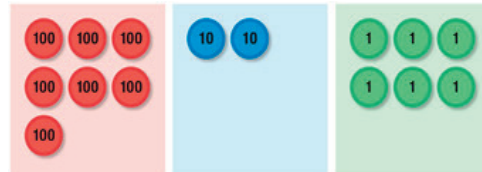
Familiar Terms and Symbols[6]

- Array (a set of numbers or objects that follow a specific pattern)
- Commutative property (e.g., 2 × 3 = 3 × 2)
- Distribute (with reference to the distributive property; e.g., in 12 × 3 = (10 × 3) + (2 × 3), the 3 is the multiplier for each part of the decomposition)
- Divide, division (partitioning a total into equal groups to show how many equal groups add up to a specific number, e.g., 15 ÷ 5 = 3)
- Equal groups (with reference to multiplication and division; one factor is the number of objects in a group, and the other is a multiplier that indicates the number of groups)
- Equation (a statement that two expressions are equal, e.g., 3 × 4 = 12)
- Even number (a whole number whose last digit is 0, 2, 4, 6, or 8)
- Expression (a number, or any combination of sums, differences, products, or divisions of numbers that evaluates to a number, e.g., 8 × 3, 15 ÷ 3)
- Factors (numbers that are multiplied to obtain a product)
- Multiply, multiplication (an operation showing how many times a number is added to itself, e.g., 5 × 3 = 15)
- Number bond (model used to show part–part–whole relationships)
- Number sentence (an equation or inequality for which both expressions are numerical and can be evaluated to a single number, e.g., 21 > 7 × 2, 5 ÷ 5 = 1)
- Odd number (a number that is not even)
- Ones, twos, threes, etc. (units of one, two, or three)
- Parentheses (the symbols ( ) used around a fact or numbers within an equation, expression, or number sentence)
- Quotient (the answer when one number is divided by another)
- Row, column (in reference to rectangular arrays)
- Tape diagram (a method for modeling problems)
- Unit (one segment of a partitioned tape diagram)
- Unknown (the *missing* factor or quantity in multiplication or division)
- Value (how much)

[6]These are terms and symbols students have used or seen previously.

Suggested Tools and Representations

- Array
- Number bond (model used to show part–part–whole relationships)
- Place value disks (pictured at right)
- Tape diagram (a method for modeling problems)



Scaffolds[7]

The scaffolds integrated into *A Story of Units* give alternatives for how students access information as well as express and demonstrate their learning. Strategically placed margin notes are provided within each lesson elaborating on the use of specific scaffolds at applicable times. They address many needs presented by English language learners, students with disabilities, students performing above grade level, and students performing below grade level. Many of the suggestions are organized by Universal Design for Learning (UDL) principles and are applicable to more than one population. To read more about the approach to differentiated instruction in *A Story of Units,* please refer to "How to Implement *A Story of Units*."

Assessment Summary

Type	Administered	Format	Standards Addressed
Mid-Module Assessment Task	After Topic C	Constructed response with rubric	3.OA.3 3.OA.4 3.OA.5 3.OA.7 3.OA.9
End-of-Module Assessment Task	After Topic F	Constructed response and timed fluency with rubric	3.OA.3 3.OA.4 3.OA.5 3.OA.7 3.OA.8 3.OA.9 3.NBT.3

[7]Students with disabilities may require Braille, large print, audio, or special digital files. Please visit the website www.p12.nysed.gov/specialed/aim for specific information on how to obtain student materials that satisfy the National Instructional Materials Accessibility Standard (NIMAS) format.

3 GRADE

Mathematics Curriculum

Topic A

The Properties of Multiplication and Division

3.OA.4, 3.OA.5, 3.OA.7, 3.OA.9, 3.OA.1, 3.OA.2, 3.OA.3, 3.OA.6

Focus Standards:	3.OA.4	Determine the unknown whole number in a multiplication or division equation relating three whole numbers. *For example, determine the unknown number that makes the equation true in each of the equations 8 × ? = 48, 5 = _ ÷ 3, 6 × 6 = ?*
	3.OA.5	Apply properties of operations as strategies to multiply and divide. (Students need not use formal terms for these properties.) *Examples: If 6 × 4 = 24 is known, then 4 × 6 = 24 is also known. (Commutative property of multiplication.) 3 × 5 × 2 can be found by 3 × 5 = 15, then 15 × 2 = 30, or by 5 × 2 = 10, then 3 × 10 = 30. (Associative property of multiplication.) Knowing that 8 × 5 = 40 and 8 × 2 = 16, one can find 8 × 7 as 8 × (5 + 2) = (8 × 5) + (8 × 2) = 40 + 16 = 56. (Distributive property.)*
	3.OA.7	Fluently multiply and divide within 100, using strategies such as the relationship between multiplication and division (e.g., knowing that 8 × 5 = 40, one knows 40 ÷ 5 = 8) or properties of operations. By the end of Grade 3, know from memory all products of two one-digit numbers.
	3.OA.9	Identify arithmetic patterns (including patterns in the addition table or multiplication table), and explain them using properties of operations. *For example, observe that 4 times a number is always even, and explain why 4 times a number can be decomposed into two equal addends*.
Instructional Days:	3	
Coherence -Links from:	G2–M6	Foundations of Multiplication and Division
	G3–M1	Properties of Multiplication and Division and Solving Problems with Units of 2–5 and 10
-Links to:	G3–M4	Multiplication and Area
	G4–M3	Multi-Digit Multiplication and Division

In Lesson 1, students study the commutativity of familiar Module 1 facts that use units of 2, 3, 4, 5, and 10 to discover facts that they already know using units of 6, 7, 8, and 9. For example, students recognize that if they know 3 × 6 = 18, then they also know 6 × 3 = 18. They write out familiar facts and those known through commutativity, organizing them in rows and columns to form the beginning of a table through which arithmetic patterns become visible. Students finish this lesson encouraged about the work to come after seeing that they already know more than half of their facts.

In Lesson 2, students apply commutativity in conjunction with the $n + 1$ strategy to solve unknown facts. For example, students relate 5 × 8 and 8 × 5 and then add one more group of 8 to solve 6 × 8 and, by extension, 8 × 6. Adding one more group to a known fact in order to find an unknown fact continues to bridge student understanding in Module 1 and Module 3 as students are reminded of their prior work with the distributive property.

Lesson 3 introduces using a letter to represent the unknown in various positions within multiplication and division problems. In Module 1, students represented the unknown on tape diagrams, and occasionally in equations, using a question mark. This lesson uses familiar facts to introduce the new abstraction of using a letter as a placeholder.

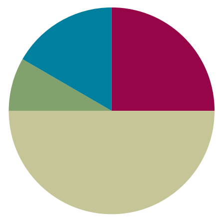
A Teaching Sequence Toward Mastery of The Properties of Multiplication and Division	
Objective 1:	**Study commutativity to find known facts of 6, 7, 8, and 9. (Lesson 1)**
Objective 2:	**Apply the distributive and commutative properties to relate multiplication facts $5 \times n + n$ to $6 \times n$ and $n \times 6$ where n is the size of the unit. (Lesson 2)**
Objective 3:	**Multiply and divide with familiar facts using a letter to represent the unknown. (Lesson 3)**

Lesson 1

Objective: Study commutativity to find known facts of 6, 7, 8, and 9.

Suggested Lesson Structure

Fluency Practice	(15 minutes)
Application Problem	(5 minutes)
Concept Development	(30 minutes)
Student Debrief	(10 minutes)
Total Time	**(60 minutes)**



Fluency Practice (15 minutes)

- Sprint: Mixed Multiplication **3.OA.7** (9 minutes)
- Group Counting **3.OA.1** (3 minutes)
- Commutative Property of Multiplication **3.OA.5** (3 minutes)

Sprint: Mixed Multiplication (9 minutes)

Materials: (S) Mixed Multiplication Sprint

Note: This Sprint reviews familiar multiplication facts from Module 1 and prepares students for today's lesson on using commutativity with known facts to find unknown facts.

Group Counting (3 minutes)

Note: Group counting reviews interpreting multiplication as repeated addition. Counting by sixes, sevens, eights, and nines in this activity anticipates multiplication using those units later in the module.

Direct students to count forward and backward, occasionally changing the direction of the count:

- Sixes to 60
- Sevens to 70
- Eights to 80
- Nines to 90

NOTES ON MULTIPLE MEANS OF ENGAGEMENT:

Group Counting in Module 3 no longer explicitly includes twos, threes, fours, and fives. However, consider including those units if the class has not yet mastered those facts.

Whisper/talking, hum/talking, or think/talking by threes and fours can also work as a scaffold to build fluency with sixes and eights.

Commutative Property of Multiplication (3 minutes)

Materials: (S) Personal white board

Note: This activity reviews the commutative property from Module 1 and anticipates its use in today's lesson.

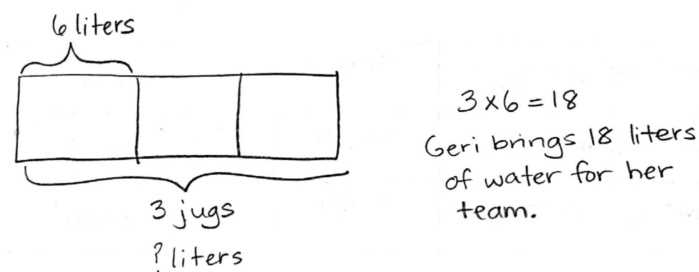
T: (Project array with 3 groups of 2 circles.) Write two multiplication sentences and two division sentences for this array.

S: (Write 3 × 2 = 6, 2 × 3 = 6, 6 ÷ 2 = 3, and 6 ÷ 3 = 2.)

Continue with the following suggested sequence: 2 groups of 9, 3 groups of 7, and 5 groups of 8.

Application Problem (5 minutes)

Geri brings 3 water jugs to her soccer game to share with teammates. Each jug contains 6 liters of water. How many liters of water does Geri bring?



NOTES ON MULTIPLE MEANS OF ENGAGEMENT:

Extend for students working above grade level with a related word problem with larger factors.

For example, "Kelly drinks 3 liters of water each day. How many liters of water does she drink in a week?"

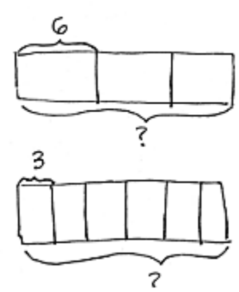
Note: This problem reviews multiplication using units of three. It leads into the discussion of commutativity in the Concept Development.

Concept Development (30 minutes)

Materials: (S) Personal white board, Problem Set

Part 1: Explore commutativity as it relates to multiplication.

Draw or project the tape diagrams shown to the right.



T: Talk to your partner. Which tape diagram represents the Application Problem? How do you know? (Allow time for discussion.)

T: Draw both tape diagrams on your personal white board. Write a multiplication sentence for each. (Allow time for students to work and finish.)

T: How are the multiplication sentences related?

S: They use the same numbers. → Both have a product of 18. → They use the same factors but in a different order. The product is the same.

MP.7

T: This is an example of the commutative property that we studied in Module 1. What does this property tell us about the product and its factors?

S: Even if the order of the factors changes, the product stays the same!

T: Earlier in the year, we learned our threes, including 3 × 6. If we know 3 × 6, what other fact do we know?

S: 6 × 3.

T: What is the product of both 3 × 6 and 6 × 3?

S: 18.

T: To show that 3 × 6 and 6 × 3 equal the same amount, we can write 3 × 6 = 6 × 3. (Model.)

NOTES ON MULTIPLE MEANS OF ENGAGEMENT:

Review the commutative property by exploring arrays—concrete or pictorial. Review 3 twos is 2 threes, for example, by 6 students standing in 2 rows of 3, and then 3 rows of 2.

When drawing the array, use color to differentiate 6 threes from 3 sixes.

T: Using commutativity as a strategy, we know many more facts than just the ones we've practiced!

Continue with the following suggested sequence:

- 2 × 7 = 7 × 2
- 5 eights = 8 fives
- 4 nines = 9 fours

Part 2: Use the multiplication chart to find known facts through commutativity.

T: Problem 1(a) on your Problem Set shows a multiplication chart. The shaded numbers along the left column and the top are factors. The numbers inside the chart are products. Each un-shaded box represents the product of one multiplication fact. Find the total number of facts on your multiplication chart. (Allow time for students to count.) How many facts are on the chart?

S: 100 facts.

T: Let's use the chart to locate the product of 3 and 6. Put your finger on the row labeled 3, and slide it to the right until it's also in the column labeled 6. The number in the square where the row and column meet is the product, which has been done for you. Using the chart, what is the product of 3 and 6?

S: 18.

T: Let's now locate the product of 6 and 3. Find the square where the row for 6 and the column for 3 meet. Use commutativity to write the product of 6 and 3 in that square on your chart.

S: (Write 18.)

T: We can use commutativity to solve many new facts and fill in the products on the chart. On the chart, write the products for all the facts that we've already studied. Then, fill in those you can solve using commutativity. (Allow time for students to work.)

T: Shade in the facts you completed. (Allow time for students to work.) How many are left to learn?

S: 16.

T: Look carefully at those 16 facts. Are there any that you will be able to solve using the commutative property once you know one?

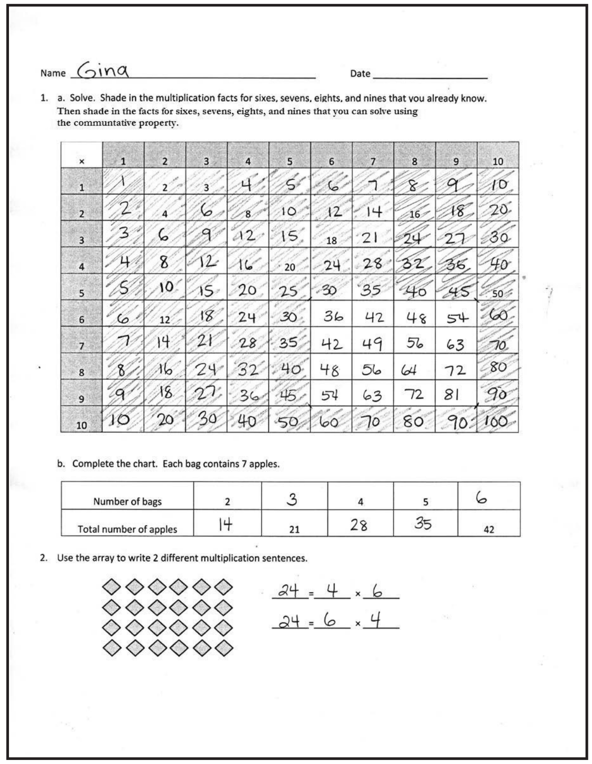
S: Yes! There are 12 facts that we can use the commutative property to solve. That means we only need to know 6 of them.

T: Really, there are only 10 new facts to learn before you know all the facts up to 10 × 10.

Problem Set (10 minutes)

Students should do their personal best to complete the Problem Set within the allotted 10 minutes. Some problems do not specify a method for solving. This is an intentional reduction of scaffolding that invokes MP.5, Use Appropriate Tools Strategically. Students should solve these problems using the RDW approach used for Application Problems.

For some classes, it may be appropriate to modify the assignment by specifying which problems students should work on first. With this option, let the purposeful sequencing of the Problem Set guide selections so that problems continue to be scaffolded. Balance word problems with other problem types to ensure a range of practice. Consider assigning incomplete problems for homework or at another time during the day.



Name Gina Date ____________

1. a. Solve. Shade in the multiplication facts for sixes, sevens, eights, and nines that you already know. Then shade in the facts for sixes, sevens, eights, and nines that you can solve using the communtative property.

×	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

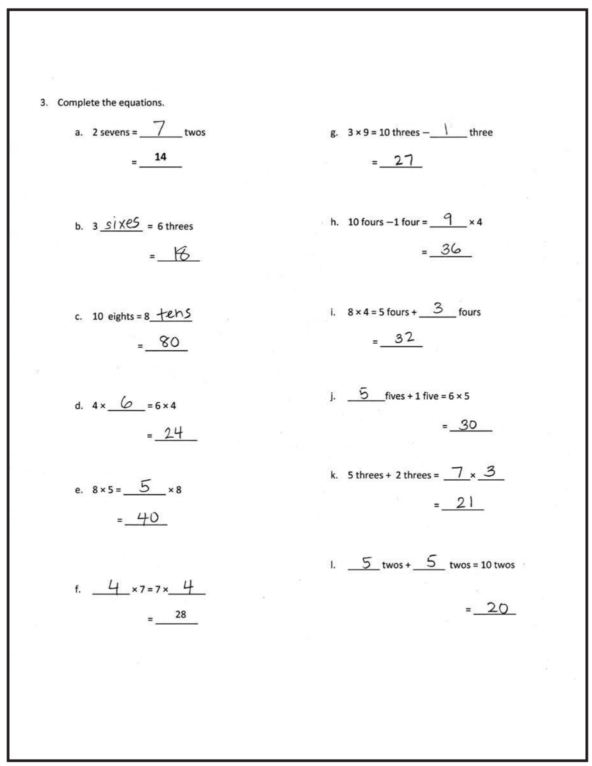
b. Complete the chart. Each bag contains 7 apples.

Number of bags	2	3	4	5	6
Total number of apples	14	21	28	35	42

2. Use the array to write 2 different multiplication sentences.

24 = 4 × 6

24 = 6 × 4



3. Complete the equations.

a. 2 sevens = 7 twos = 14

b. 3 sixes = 6 threes = 18

c. 10 eights = 8 tens = 80

d. 4 × 6 = 6 × 4 = 24

e. 8 × 5 = 5 × 8 = 40

f. 4 × 7 = 7 × 4 = 28

g. 3 × 9 = 10 threes – 1 three = 27

h. 10 fours – 1 four = 9 × 4 = 36

i. 8 × 4 = 5 fours + 3 fours = 32

j. 5 fives + 1 five = 6 × 5 = 30

k. 5 threes + 2 threes = 7 × 3 = 21

l. 5 twos + 5 twos = 10 twos = 20

Student Debrief (10 minutes)

Lesson Objective: Study commutativity to find known facts of 6, 7, 8, and 9.

The Student Debrief is intended to invite reflection and active processing of the total lesson experience.

Invite students to review their solutions for the Problem Set. They should check work by comparing answers with a partner before going over answers as a class. Look for misconceptions or misunderstandings that can be addressed in the Debrief. Guide students in a conversation to debrief the Problem Set and process the lesson.

Any combination of the questions below may be used to lead the discussion.

- How did commutativity help you solve more facts than you thought you knew in Problem 1(a)?
- Invite students to share their processes for finding the multiplication facts for the array in Problem 2.

- In Problems 3(a), 3(b), and 3(c), what do you notice about the words and numbers on each side of the equal sign? How are they related?
- How did you know to subtract 1 three in Problem 3(g)? What would that problem look like rewritten as an equation?

Exit Ticket (3 minutes)

After the Student Debrief, instruct students to complete the Exit Ticket. A review of their work will help with assessing students' understanding of the concepts that were presented in today's lesson and planning more effectively for future lessons. The questions may be read aloud to the students.

NOTES ON MULTIPLE MEANS OF ACTION AND EXPRESSION:

English language learners and others benefit from reviewing *commutative property* and *commutativity* during the Debrief. Allow students to explain the property to a partner in their first language, and/or record the term with an example in a personal math dictionary.

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ABSTRACT

UNDERSTANDING PRE-SERVICE ELEMENTARY MATHEMATICS TEACHER LEARNING IN AN EARLY LESSON STUDY EXPERIENCE

by

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As the most important in-school factor in student learning, elementary school teachers must be able to offer all students the quality mathematical learning experiences that they deserve; and the opportunities to learn that pre-service teachers (PSTs) encounter during their preparation programs impact their ability to do so. Content courses are crucial components of the mathematical education of elementary teachers and can be sites for the early development of Mathematical Knowledge for Teaching (MKT) (Ball, Thames, & Phelps, 2008). These courses, often taken early in their preparation programs, can provide PSTs opportunities to create a new vision of mathematics, build new knowledge for action, engage in the complexity of instruction, and prepare to learn in and from their own practice.

In this study, PSTs participated in a modified form of lesson study in a mathematics content course. Lesson study has become an important lever for improving teaching and learning. More common in inservice contexts, teacher educators have begun to use lesson study in pre-service preparation programs—typically in methods courses. This study sought to add to the knowledge base by examining the learning afforded and occluded by a modified lesson study activity at an early stage of a university preparation program.

This mixed-methods interpretive case study (*n*=11) analyzed the learning that occurred within the modified lesson study activity. While PSTs' scores on the Learning Mathematics for Teaching—Teacher Knowledge Assessment System (Hill, Sleep, Lewis, & Ball, 2007) showed little gain, significant changes in PSTs' beliefs regarding teaching efficacy were observed on the Mathematics Teaching Efficacy Belief Instrument (Enochs, Smith, & Huinker, 2000). Additionally, discussion transcripts and written reflections of PSTs indicated that the lesson study cycle acted as a powerful focusing lens that allowed PSTs to see their previous and current experiences with mathematics instruction juxtaposed with that of elementary students during the research lesson. This lens also allowed them to focus and activate their new MKT (Ball et al., 2008) as they prepared for and observed student thinking made visible through the more equitable and validating instruction that they designed in collaboration with an established lesson study teacher team.

AUTOBIOGRAPHICAL STATEMENT

Christopher Nazelli is a Senior Lecturer in the Department of Mathematics at Wayne State University and the Director of Curriculum and Instruction for Wayne State University's Center for Excellence and Equity in Mathematics. Mr. Nazelli serves within the WSU Math Corps, a nationally recognized program that brings over four hundred middle- and high-school students to Wayne State's campus each summer.

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