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Two Sample Statistical Test for Location Parameters

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A class of distribution-free tests for the homogeneity of location parameters is proposed and compared with different competitors in terms of Pitman asymptotic relative efficiency. A numerical example is provided and a simulation study is made to check the performance of the tests.

Keywords: Distribution-free, U-statistics, asymptotic relative efficiency, simulation study

Introduction

Let $X_1, X_2, \dots, X_n$ and $Y_1, Y_2, \dots, Y_m$ be the two independent random samples from two populations with absolutely continuous cumulative distribution functions $F(x)$ and $F(x - \Delta)$, respectively. The parameter Δ is called the shift parameter. If $\Delta > 0$ then it means the Y's distribution is shifted towards right and Y's would tend to be larger than the X's. If $\Delta < 0$ then it means Y's distribution is shifted towards left and Y's would tend to be smaller than the X's. Now, we want to test the null hypothesis

$$H_0: \quad \Delta = 0$$

against the alternative hypothesis

$$H_1: \quad \Delta \neq 0$$

In the following sections, the problem is tested without making any parametric model assumption regarding the $F(.)$.

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Testing the homogeneity of location parameters plays an important role in real life situations, such as in the field of agriculture, business, drug-screening studies, engineering and in many other fields. Thus testing of equality of location parameters received considerable attention in the literature. Under the assumption of the Normality, the simplest parametric test that is t-test, which is applied to check the equality of location parameters. When the data is not Normal, the most familiar non-parametric tests are Wilcoxon-Mann-Whitney test (Wilcoxon (1945) and Mann and Whitney (1947)), which was further generalized by Kochar (1978), Deshpande and Kochar (1980, 1982) Stephenson and Ghosh (1985), Ahmad (1996), Kumar (1997, 2015), Xie and Priebe (2000), Öztürk (2001, 2002), Kumar et al. (2003), Kössler and Kumar (2008), Kössler (2010), Kumar and Chattopadhyay (2013), Kumar and Chawla (2016) and Kumar and Goyal (2018). For testing the equality of scale parameters one may refer to recent works of Shetty and Umarani (2010), Goyal and Kumar (2018, 2019) and references cited therein.

Below, a class of distribution-free tests based on sub sample extremes is introduced. The proposed class of tests is compared with different competitors in terms of Pitman asymptotic relative efficiency and is found that their performance is better than its competitor tests for short-tailed, symmetric and skewed distributions. A numerical example is provided to see the execution of the proposed tests. A simulation study is carried out to check the performance of proposed tests.

The Proposed Class of Tests

Let c be the fixed positive integer such that $c < \min(n,m)$ and define the following kernel:

$$\Phi(X_1, X_2, \ldots, X_c;\ Y_1, Y_2, \ldots, Y_c) = \begin{cases} 1 & \text{if } X_{1:c} < Y_{1:c} \text{ and } X_{c:c} < Y_{c:c} \\ 0 & \text{Otherwise} \end{cases}$$

where $X_{1:c}$ and $Y_{1:c}$ are the minimum values from the samples $X_1, X_2, \ldots, X_c$ and $Y_1, Y_2, \ldots, Y_c$ respectively and $X_{c:c}$ and $Y_{c:c}$ are the maximum values from the samples $X_1, X_2, \ldots, X_c$ and $Y_1, Y_2, \ldots, Y_c$, respectively.

The U-statistics associated with the kernel Φ(.) is defined as:

$$U_c = \left[\binom{n}{c}\binom{m}{c}\right]^{-1} \Sigma\Phi\left(X_{w_1}, X_{w_2}, \ldots, X_{w_c};\ Y_{z_1}, Y_{z_2}, \ldots, Y_{z_c}\right),$$

where the summation is extended over all possible combinations $(w_1, w_2, \dots, w_c)$ of c integers chosen from (1, …, n) and all possible combinations $(z_1, z_2, \dots, z_c)$ of c integers chosen from (1, …, m).

In particular, when $c = 1$ the test statistics U_c becomes the test statistics given by Wilcoxon Mann Whitney [19, 13].

Distribution of the Test Statistics

The expectation of U_c is:

$$E(U_c) = \left[\binom{n}{c}\binom{m}{c}\right]^{-1} \sum E\left[\Phi\left(X_{w_1}, X_{w_2}, \dots, X_{w_c};\ Y_{z_1}, Y_{z_2}, \dots, Y_{z_c}\right)\right]$$

$$= P[X_{1:c} < Y_{1:c} \ \text{ and } \ X_{c:c} < Y_{c:c}]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{v} (P[X_{1:c} \le t \ \text{ and } \ X_{c:c} \le v])dP[Y_{1:c} \le t \ \text{ and } \ Y_{c:c} \le v]$$

Under the null hypothesis H_0, the expectation of U_c, E(U_c), is:

$$\mathrm{E}(U_c) = \int_{-\infty}^{\infty}\int_{-\infty}^{v} \sum_{r=1}^{c} \frac{(c!)^2}{(r!)(c-r)!\,(c-2)!} \left(F(t)\right)^r \left(\left(F(v)\right) - \left(F(t)\right)\right)^{2c-r-2} dF(t)dF(v)$$

After some computations we get

$$E(U_c) = \begin{cases} \dfrac{1}{2}, & \text{if } c = 1 \\ \displaystyle\sum_{r=1}^{c} \sum_{m=0}^{2c-r-2} \frac{(c-1)(-1)^{2c-r+m-2}}{(4c-2m-2)} \binom{c}{r}\binom{2c-r-2}{m}, & \text{if } c \ge 2 \end{cases}$$

Using the results of Randles and Wolfe (1979), Chapter 3, p. 92, we have the following theorem, which provides the asymptotic distribution of the U_c.

Theorem 1. Let $N = n + m$, then the asymptotic distribution of $N^{\frac{1}{2}}[U_c - E(U_c)]$, as $N \to \infty$ in such a way that $\frac{n}{N} \to \lambda$ and $0 < \lambda < 1$ is Normal with mean zero and variance $\sigma^2(U_c)$, as

$$\sigma^2(U_c) = c^2\left(\frac{\xi_{10}}{\lambda} + \frac{\xi_{01}}{1-\lambda}\right).$$

where,

$$\xi_{10} = \mathrm{E}\left[\left(\Phi(x_0, X_2, \dots, X_c;\ Y_1, Y_2, \dots, Y_c)\right)^2\right] - (E(U_c))^2$$

and

$$\xi_{01} = \mathrm{E}\left[\left(\Phi(X_1, X_2, \dots, X_c;\ y_0, Y_2, \dots, Y_c)\right)^2\right] - (E(U_c))^2,$$

with,

$$\Phi(x_0, X_2, \dots, X_c;\ Y_1, Y_2, \dots, Y_c) = \mathrm{E}[\Phi(X_1, X_2, \dots, X_c;\ Y_1, Y_2, \dots, Y_c) | X_1 = x_0]$$

and

$$\Phi(X_1, X_2, \dots, X_c;\ y_0, Y_2, \dots, Y_c) = \mathrm{E}[\Phi(X_1, X_2, \dots, X_c;\ Y_1, Y_2, \dots, Y_c) | Y_1 = y_0].$$

Under H_0, after some computations, we find the asymptotic null variance of U_c as:

$$\sigma_0^2(U_c) = \frac{c^2(\Psi_c)}{\lambda(1-\lambda)},$$

where,

$$\Psi_c = \int_{-\infty}^{\infty} \mathrm{I}^2 \mathrm{dx} - \left(\mathrm{E}(U_c)\right)^2$$

with,

$$I = \int_{-\infty}^{\infty} \int_{-\infty}^{v} (P[\min(x, X_2, \dots, X_c) < t \quad \text{and} \quad \max(x, X_2, \dots, X_c) < v]) \times (dP[\min(X_1, X_2, \dots, X_c) \le t \quad \text{and} \quad \max(X_1, \dots, X_c) \le v])$$

$$= \int_{x}^{\infty} \int_{-\infty}^{x} \sum_{r=1}^{c-1} \frac{(c!)(c-1)!}{(r!)(c-1-r)!\,(c-2)!} \big(F(t)\big)^{r} \Big(\big(F(v)\big) - \big(F(t)\big)\Big)^{2c-r-3} dF(t)dF(v)$$
$$+ \int_{x}^{\infty} \int_{x}^{v} \sum_{r=0}^{c-1} \frac{(c!)(c-1)!}{(r!)(c-1-r)!\,(c-2)!} \big(F(t)\big)^{r} \Big(\big(F(v)\big) - \big(F(t)\big)\Big)^{2c-r-3} dF(t)dF(v)$$

After some computation we get,

$$\Psi_c = \begin{cases} \frac{1}{12}, & \text{if } c = 1 \\ A + B + 2D - \big(\mathrm{E}(U_c)\big)^2, & \text{if } c \geq 2 \end{cases}$$

where,

$$A = \sum_{r=1}^{c-1} \sum_{s=1}^{c-1} \left(\frac{c(c-1)}{(2c-1)}\right)^2 \left(\frac{(c-1)!}{(c-1-r)!\,r!}\right) \left(\frac{(c-1)!}{(c-1-s)!\,s!}\right) \left(1 + \frac{1}{4c-1} - \frac{1}{c}\right) \times \left(\frac{r!\,(2c-r-3)!}{(2c-2)!}\right) \left(\frac{s!\,(2c-s-3)!}{(2c-2)!}\right),$$

$$B = \sum_{l=0}^{2c-3} \sum_{m=0}^{2c-3} \frac{(-1)^{l+m}\big(c(c-1)\big)^2}{(l+1)(m+1)} \binom{2c-3}{l} \binom{2c-3}{m} \times \Bigg(\frac{1}{(2c-1)^2}\left(1 + \frac{1}{4c-1} - \frac{1}{c}\right) - \frac{1}{(2c-1)(2c-l-2)}\left(\frac{1}{l+2} - \frac{1}{2c+l+1} - \frac{1}{2c} + \frac{1}{4c-1}\right) - \frac{1}{(2c-1)(2c-m-2)}\left(\frac{1}{m+2} - \frac{1}{2c+m+1} - \frac{1}{2c} + \frac{1}{4c-1}\right) + \frac{1}{(2c-m-2)(2c-l-2)}\left(\frac{1}{l+m+3} - \frac{1}{2c+l+1} - \frac{1}{2c+m+1} + \frac{1}{4c-1}\right)\Bigg)$$

and

$$D = \sum_{r=1}^{c-1} \sum_{l=0}^{2c-3} \left(\frac{(-1)^l \big(c(c-1)\big)^2}{(l+1)} \right) \left(\frac{(c-1)!}{(c-1-r)!\, r!} \right) \times \binom{2c-3}{l} \left(\frac{r!\,(2c-r-3)!}{(2c-2)!} \right) \times \left(\frac{1}{(2c-1)^2} \left(1 + \frac{1}{4c-1} - \frac{1}{c} \right) - \frac{1}{(2c-1)(2c-l-2)} \left(\frac{1}{l+2} - \frac{1}{2c+l+1} - \frac{1}{2c} + \frac{1}{4c-1} \right) \right)$$

Pitman Asymptotic Relative Efficiency

In this Section, we calculated the Pitman efficacy of the U_c tests and compare it with several relative tests in the sense of Pitman asymptotic relative efficiency (ARE). The limiting efficacy of the test U_c under local alternatives $\xi = \frac{\vartheta}{\sqrt{\mathrm{N}}}$, is given as:

$$e^2(U_c) = \lim_{N \to \infty} \frac{\left[\frac{d}{d\xi} E(U_c)|_{\xi=0} \right]^2}{N {\sigma^2}_0(U_c)}.$$

with,

$$\frac{\mathrm{d}}{\mathrm{d}\xi} E(U_c)|_{\xi=0} = \begin{cases} \int_{-\infty}^{\infty} \big(f(x)\big)^2 dx, & c \geq 1 \\ \int_{-\infty}^{\infty} \int_{-\infty}^{y} \frac{\sqrt{N}}{c-1} \left(\frac{c!}{(c-2)!} \right)^2 \begin{pmatrix} -\big(F(y)-F(x)\big)^{c-1} \\ \times \big(f(y)-f(x)\big) \\ +F(y)^{c-1} f(y) \end{pmatrix} \times \big(F(y)-F(x)\big)^{c-2} \times dF(x)dF(x), & c \geq 2. \end{cases}$$

Now we shall compare the performance of the proposed class of tests, with respect to relative competitors of two sample location problem, in terms of the

Pitman ARE. The U_c tests are compared with Wilcoxon-Mann-Whitney (1945, 1947) test (WMW), Ahmad (1996) test ($A_{r,s}$), Kumar (1997) test (K_m), Öztürk (2001) test ($OZ_{r,s}$), Kumar (2015) test ($K_{c,d;i,j}$) and Kumar and Goyal (2018) test (KG_c). The values of Pitman asymptotic relative efficiencies (AREs) are presented in Tables 1-9.

If the values of asymptotic relative efficiency are greater than value one, this implies that the test U_c has larger efficiency w.r.t. the competing test. From the above tables of AREs, it can be concluded that:

(i) U_c test performs same or better than Wilcoxon-Mann-Whitney (1945, 1947) test (WMW) for U-quadratic, Uniform, Exponential, Gumbel, Beta (1, 2) and Beta (2, 2) distributions for all values of c. While in case of Normal distribution U_c tests performs same or better than Wilcoxon-Mann-Whitney (1945, 1947) test (WMW) for $c \leq 3$.

(ii) U_c test performs better than Ahmad (1996) test ($A_{r,s}$) for U-quadratic and Unifom distribution when $c > 2$, for Exponential distribution when $c > 1$, for Normal, Gumbel, Beta (1, 2) and Beta (2, 2) distributions for all values of c and in case of Cauchy distribution for $c = 1$.

(iii) U_c test performs better than Kumar (1997) test (K_m) for U-quadratic, Unifom, Exponential, Gumbel, Beta (1, 2) and Beta (2, 2) for all values of c. While in case of Normal distribution U_c tests performs better than Kumar (1997) test (K_m) for $c \leq 5$.

(iv) U_c test performs better than Kumar and Goyal (2018) test (KG_c) for U-quadratic and Unifom distribution for $c > 2$. For Exponential, Gumbel and Beta (1, 2) distributions for all values of c, U_c tests performs better than Kumar and Goyal (2018) test (KG_c). While in case of Cauchy distribution U_c tests performs better than Kumar and Goyal (2018) test (KG_c) for $c \leq 3$.

(v) U_c test performs same or better than Öztürk (2001) test ($OZ_{r,s}$) and Kumar (2015) test ($K_{c,d;i,j}$) for U-quadratic, Unifom, Exponential, Gumbel, Beta (1, 2) and Beta (2, 2) distributions for all values of $c \geq 3$. While in case of logistic distribution U_c tests performs same or better than Öztürk (2001) test ($OZ_{r,s}$) and Kumar (2015) test ($K_{c,d;i,j}$) for $c = 1$.

Thus, we conclude that the proposed tests U_c performs better in case of short-tailed, symmetric and skewed distributions.

An Illustrative Example

To see the execution of the proposed test, we study the data given by Russell, et al. (1973). In this experiment, initial hemodynamics was studied over 50 patients with acute myocardial infraction classified according to anterior or inferior location. Here we are interested to see that there is significant difference for stroke index (which is measured in milliliters per beat per square meter of body surface area) for two locations that are anterior and inferior. The value of the test statistics and p-value are given in Table 10.

Table 1. AREs of U_c w.r.t. different tests for U- quadratic distribution.

Tests	c					
	1	2	3	4	5	6
WMW	1.00000	1.46150	2.90684	4.63505	6.44362	8.27489
$A_{1,2}$	1.06666	1.55893	3.10063	4.94406	6.87319	8.82655
$A_{1,3}$	0.81026	1.18421	2.35532	3.75564	5.22106	6.70488
$A_{2,3}$	0.61096	0.89293	1.77598	2.83186	3.93683	5.05568
$A_{3,3}$	0.47970	0.70109	1.39442	2.22345	3.09103	3.96950
K_1	3.26434	4.77085	9.48893	15.13043	21.03421	27.01213
K_2	5.22772	7.64034	15.19616	24.23080	33.68548	43.25888
K_3	6.99542	10.22384	20.33458	32.42418	45.07586	57.88640
KG_2	2.47510	3.61738	7.19474	11.47227	15.94866	20.48127
KG_3	1.04601	1.52874	3.04058	4.84831	6.74009	8.65562
KG_4	0.53755	0.78563	1.56258	2.49159	3.46380	4.44821
$OZ_{1,2}$	1.00000	1.46150	2.90684	4.63505	6.44362	8.27489
$OZ_{1,3}$	0.68422	1.00000	1.98893	3.17142	4.40889	5.66190
$OZ_{2,3}$	0.47049	0.68763	1.36766	2.18078	3.03171	3.89332
$OZ_{3,3}$	0.34401	0.50278	1.00000	1.59453	2.21670	2.84669
$K_{1,3:1,2}$	2.17672	3.18129	6.32739	10.08920	14.02599	18.01217
$K_{1,3:1,3}$	0.68422	1.00000	1.98893	3.17142	4.40889	5.66190
$K_{1,4;1,4}$	0.47049	0.68763	1.36766	2.18078	3.03171	3.89332

Table 2. AREs of U_c w.r.t. different tests for Uniform distribution.

Tests	c					
	1	2	3	4	5	6
WMW	1.00000	1.22807	1.84063	2.50072	3.16673	3.83333
$A_{1,2}$	1.06666	1.30994	1.96334	2.66744	3.37784	4.08889
$A_{1,3}$	0.96428	1.18421	1.77490	2.41142	3.05364	3.69644
$A_{2,3}$	0.85333	1.04795	1.57067	2.13395	2.70227	3.27111
$A_{3,3}$	0.75757	0.93035	1.39442	1.89449	2.39903	2.90404
K_1	1.70129	2.08931	3.13147	4.25448	5.38755	6.52165
K_2	1.96884	2.41788	3.62393	4.92354	6.23480	7.54725
K_3	2.11917	2.60249	3.90062	5.29947	6.71084	8.12350
KG_2	1.08580	1.33344	1.99857	2.71530	3.43845	4.16226
KG_3	0.82143	1.00877	1.51196	2.05418	2.60126	3.14883
KG_4	0.61935	0.76061	1.14001	1.54884	1.96134	2.37420
$OZ_{1,2}$	1.00000	1.22807	1.84063	2.50072	3.16673	3.83333
$OZ_{1,3}$	0.81428	1.00000	1.49880	2.03630	2.57862	3.12143
$OZ_{2,3}$	0.65714	0.80701	1.20956	1.64333	2.08099	2.51905
$OZ_{3,3}$	0.54329	0.66719	1.00000	1.35862	1.72045	2.08261
$K_{1,3:1,2}$	1.45712	1.78945	2.68204	3.64387	4.61432	5.58565
$K_{1,3:1,3}$	0.81429	1.00000	1.49880	2.03631	2.57863	3.12144
$K_{1,4;1,4}$	0.65714	0.80701	1.20956	1.64333	2.08099	2.51904

Table 3. AREs of U_c w.r.t. different tests for Exponential distribution.

Tests	c					
	1	2	3	4	5	6
WMW	1.00000	1.22807	1.84063	2.50072	3.16673	3.83333
$A_{1,2}$	0.80000	1.60000	2.40000	3.20000	4.00000	4.80000
$A_{1,3}$	1.28571	2.57142	3.85714	5.14285	6.42857	7.71428
$A_{2,3}$	1.77777	3.55555	5.33333	7.11111	8.88888	10.66667
$A_{3,3}$	2.27272	4.54545	6.81818	9.09090	11.36360	13.63636
K_1	1.70129	2.08931	3.13147	4.25448	5.38755	6.52165
K_2	1.96881	2.41784	3.62387	4.92347	6.23470	1.96881
K_3	2.11917	2.60249	3.90062	5.29947	6.71084	8.12350
KG_2	1.93031	2.37056	3.55301	4.82719	6.11279	7.39955
KG_3	2.51563	3.08938	4.63037	6.29093	7.96635	9.64329
KG_4	2.99171	3.67403	5.50665	7.48146	9.47395	11.46825
$OZ_{1,2}$	1.00000	1.22807	1.84063	2.50072	3.16673	3.83333
$OZ_{1,3}$	0.81428	1.00000	1.49880	2.03630	2.57862	3.12143
$OZ_{2,3}$	0.65714	0.80701	1.20956	1.64333	2.08099	2.51905
$OZ_{3,3}$	0.54329	0.66719	1.00000	1.35862	1.72045	2.08261
$K_{1,3:1,2}$	1.45714	1.78947	2.68207	3.64392	4.61438	5.58573
$K_{1,3:1,3}$	0.81428	1.00000	1.49880	2.03630	2.57862	3.12143
$K_{1,4;1,4}$	0.65714	0.80701	1.20956	1.64333	2.08099	2.51905

Table 4. AREs of U_c w.r.t. different tests for Normal distribution.

Tests	c					
	1	2	3	4	5	6
WMW	1.00000	1.02202	1.03173	0.99511	0.94223	0.88794
$A_{1,2}$	1.06666	1.09015	1.10052	1.06145	1.00504	0.94714
$A_{1,3}$	1.15869	1.18421	1.19546	1.15303	1.09175	1.02886
$A_{2,3}$	1.25520	1.28285	1.29504	1.24907	1.18269	1.11456
$A_{3,3}$	1.35152	1.38129	1.39442	1.34492	1.27345	1.20008
K_1	1.09420	1.11830	1.12893	1.08885	1.03099	0.97159
K_2	1.14780	1.17307	1.18423	1.14219	1.08149	1.01918
K_3	1.18328	1.20934	1.22084	1.17750	1.11493	1.05069
KG_2	0.96516	0.98641	0.99579	0.96044	0.90940	0.85701
KG_3	0.96348	0.98470	0.99406	0.95877	0.90782	0.85552
KG_4	0.96257	0.98377	0.99313	0.95787	0.90697	0.85472
$OZ_{1,2}$	1.00000	1.02202	1.03173	0.99511	0.94223	0.88794
$OZ_{1,3}$	0.97845	1.00000	1.00950	0.97367	0.92193	0.86881
$OZ_{2,3}$	0.96662	0.98791	0.99730	0.96189	0.91078	0.85831
$OZ_{3,3}$	0.96923	0.99058	1.00000	0.96450	0.91324	0.86063
$K_{1,3:1,2}$	1.05456	1.07778	1.08803	1.04940	0.99364	0.93639
$K_{1,3:1,3}$	0.97845	1.00000	1.00950	0.97367	0.92193	0.86881
$K_{1,4;1,4}$	0.96662	0.98791	0.99730	0.96180	0.91078	0.85831

Table 5. AREs of U_c w.r.t. different tests for Logistic distribution.

Tests	c					
	1	2	3	4	5	6
WMW	1.00000	0.94793	0.88765	0.80516	0.72452	0.65386
$A_{1,2}$	1.06666	1.01112	0.94683	0.85883	0.77282	0.69745
$A_{1,3}$	1.19047	1.12849	1.05673	0.95852	0.86252	0.77840
$A_{2,3}$	1.33333	1.26391	1.18353	1.07354	0.96602	0.87181
$A_{3,3}$	1.48484	1.40753	1.31802	1.19554	1.07580	0.97088
K_1	1.02918	0.97559	0.91355	0.82865	0.74566	0.67294
K_2	1.05880	1.00367	0.93985	0.85250	0.76712	0.69230
K_3	1.08120	1.02491	0.95973	0.87054	0.78335	0.70695
KG_2	1.03130	0.97761	0.91544	0.83036	0.74720	0.67433
KG_3	1.11806	1.05984	0.99245	0.90021	0.81006	0.73105
KG_4	1.19865	1.13624	1.06399	0.96511	0.86845	0.78375
$OZ_{1,2}$	1.00000	0.94793	0.88765	0.80516	0.72452	0.65386
$OZ_{1,3}$	1.00528	0.95294	0.89235	0.80942	0.72835	0.65732
$OZ_{2,3}$	1.02678	0.97332	0.91142	0.82672	0.74392	0.67137
$OZ_{3,3}$	1.06484	1.00940	0.94521	0.85737	0.77150	0.69626
$K_{1,3:1,2}$	1.01190	0.95921	0.89822	0.81474	0.73314	0.66164
$K_{1,3:1,3}$	1.00528	0.95294	0.89235	0.80942	0.72835	0.65732
$K_{1,4;1,4}$	1.02678	0.97332	0.91142	0.82672	0.74392	0.67137

Table 6. AREs of U_c w.r.t. different tests for Cauchy distribution.

Tests	c					
	1	2	3	4	5	6
WMW	1.00000	0.89112	0.60960	0.39099	0.25375	0.17051
$A_{1,2}$	1.06666	0.95053	0.65025	0.41706	0.27067	0.18187
$A_{1,3}$	1.34089	1.19490	0.81742	0.52428	0.34026	0.22863
$A_{2,3}$	1.76138	1.56961	1.07375	0.68869	0.44696	0.30033
$A_{3,3}$	2.32854	2.07502	1.41950	0.91044	0.59088	0.39704
K_1	0.79598	0.70932	0.48523	0.31122	0.20198	0.13572
K_2	0.74300	0.66211	0.45294	0.29051	0.18854	0.12669
K_3	0.72062	0.64216	0.43929	0.28175	0.18286	0.12287
KG_2	1.93032	1.72015	1.17674	0.75474	0.48983	0.32914
KG_3	3.67621	3.27596	2.24105	1.43737	0.93286	0.62683
KG_4	6.27830	5.59475	3.82730	2.45477	1.59316	1.07051
$OZ_{1,2}$	1.00000	0.89112	0.60960	0.39099	0.25375	0.17051
$OZ_{1,3}$	1.13231	1.00903	0.69026	0.44272	0.28733	0.19307
$OZ_{2,3}$	1.35642	1.20874	0.82688	0.53035	0.34420	0.23128
$OZ_{3,3}$	1.66990	1.48809	1.01798	0.65292	0.42375	0.28473
$K_{1,3:1,2}$	0.85698	0.76367	0.52242	0.33507	0.21746	0.14612
$K_{1,3:1,3}$	1.35642	1.20874	0.82688	0.53035	0.34420	0.23128
$K_{1,4;1,4}$	1.35642	1.20874	0.82688	0.53035	0.34420	0.23128

Table 7. AREs of U_c w.r.t. different tests for Gumbel distribution.

Tests	c					
	1	2	3	4	5	6
WMW	1.00000	1.03192	1.06626	1.05076	1.01374	0.97107
$A_{1,2}$	1.34999	1.39309	1.43945	1.41852	1.36855	1.31095
$A_{1,3}$	1.71428	1.76900	1.82788	1.80130	1.73784	1.66470
$A_{2,3}$	2.08333	2.14983	2.22138	2.18908	2.11196	2.02307
$A_{3,3}$	2.45454	2.53289	2.61718	2.57913	2.48827	2.38354
K_1	1.11845	1.15415	1.19257	1.17523	1.13382	1.08610
K_2	1.18080	1.21849	1.25904	1.24074	1.19703	1.14665
K_3	1.22102	1.25999	1.30193	1.28300	1.23780	1.18570
KG_2	1.08580	1.12046	1.15775	1.14092	1.10072	1.05439
KG_3	1.18358	1.22136	1.26200	1.24366	1.19984	1.14934
KG_4	1.26895	1.30946	1.35304	1.33336	1.28639	1.23225
$OZ_{1,2}$	1.00000	1.03192	1.06626	1.05076	1.01374	0.97107
$OZ_{1,3}$	0.96906	1.00000	1.03328	1.01825	0.98238	0.94103
$OZ_{2,3}$	0.94628	0.97649	1.00899	0.99432	0.95929	0.91891
$OZ_{3,3}$	0.93785	0.96779	1.00000	0.98546	0.95074	0.91072
$K_{1,3:1,2}$	1.07055	1.10472	1.14149	1.12489	1.08526	1.03950
$K_{1,3:1,3}$	0.96906	1.00000	1.03328	1.01825	0.98238	0.94108
$K_{1,4;1,4}$	0.94628	0.97649	1.00899	0.99432	0.95929	0.91891

Table 8. AREs of U_c w.r.t. different tests for Beta (1, 2) distribution of first kind.

Tests	c					
	1	2	3	4	5	6
WMW	1.00000	1.19323	1.69235	2.20351	2.69991	3.18328
$A_{1,2}$	1.66666	1.98872	2.82059	3.67251	4.49985	5.30546
$A_{1,3}$	2.05078	2.44706	3.47065	4.51892	5.53693	6.52821
$A_{2,3}$	2.29687	2.74070	3.88713	5.06119	6.20136	7.31159
$A_{3,3}$	2.46734	2.94411	4.17562	5.43682	6.66161	7.85425
K_1	1.57609	1.88064	2.66731	3.47294	4.25531	5.01714
K_2	1.79425	2.14096	3.03652	3.95367	4.84434	5.71163
K_3	1.91759	2.28813	3.24525	4.22544	5.17734	6.10424
KG_2	1.22140	1.45742	2.06705	2.69138	3.29769	3.88807
KG_3	1.12073	1.33729	1.89667	2.46954	3.02587	3.56759
KG_4	1.00712	1.20172	1.70440	2.21920	2.71913	3.20594
$OZ_{1,2}$	1.00000	1.19323	1.69235	2.20351	2.69991	3.18328
$OZ_{1,3}$	0.83805	1.00000	1.41829	1.84667	2.26268	2.66777
$OZ_{2,3}$	0.69636	0.83092	1.17850	1.53445	1.88013	2.21673
$OZ_{3,3}$	0.59089	0.70507	1.00000	1.30203	1.59535	1.88097
$K_{1,3:1,2}$	1.37731	1.64345	2.33090	3.03492	3.71862	4.38437
$K_{1,3:1,3}$	0.83806	1.00000	1.41829	1.84667	2.26269	2.66778
$K_{1,4;1,4}$	0.69636	0.83092	1.17850	1.53445	1.88013	2.21673

Table 9. AREs of U_c w.r.t. different tests for Beta (2, 2) distribution of first kind.

Tests	c					
	1	2	3	4	5	6
WMW	1.00000	1.08872	1.27356	1.40489	1.49534	1.56123
$A_{1,2}$	1.06666	1.16130	1.35847	1.49855	1.59503	1.66531
$A_{1,3}$	1.08905	1.18567	1.38698	1.53000	1.62850	1.70026
$A_{2,3}$	1.09416	1.19124	1.39349	1.53718	1.63614	1.70824
$A_{3,3}$	1.09489	1.19204	1.39442	1.53821	1.63724	1.70939
K_1	1.26494	1.37717	1.61096	1.77711	1.89153	1.97487
K_2	1.37826	1.50055	1.75531	1.93631	2.06098	2.15179
K_3	1.44579	1.57406	1.84131	2.03118	2.16195	2.25721
KG_2	0.92333	1.00526	1.17593	1.29719	1.38070	1.44154
KG_3	0.80237	0.87355	1.02187	1.12724	1.19981	1.25268
KG_4	0.70325	0.76564	0.89564	0.98799	1.05160	1.09794
$OZ_{1,2}$	1.00000	1.08872	1.27356	1.40489	1.49534	1.56123
$OZ_{1,3}$	0.91850	1.00000	1.16978	1.29040	1.37348	1.43400
$OZ_{2,3}$	0.84260	0.91736	1.07311	1.18376	1.25998	1.31549
$OZ_{3,3}$	0.78519	0.85486	1.00000	1.10311	1.17414	1.22587
$K_{1,3:1,2}$	1.16812	1.27176	1.48768	1.64108	1.74674	1.82370
$K_{1,3:1,3}$	0.95306	1.03762	1.21379	1.33895	1.42515	1.48795
$K_{1,4;1,4}$	0.84260	0.91736	1.07311	1.18376	1.25998	1.31549

Table 10. Computed values of the test U_c and the corresponding p-values.

c	U_c	p-value
1	0.7089840	0.0096420
2	0.6127184	0.0043070
3	0.6110307	0.0040010
4	0.6177080	0.0046880
5	0.6266630	0.0057360

Simulation study

In this Section, Monte Carlo simulation technique is used to check the performance of the proposed test. We have calculated the estimated power and level of significance of the proposed test by using the Monte Carlo simulation technique for sample sizes n and m and with n, m = 10, 20, 30 and 40. The random samples are generated of sizes n and m from Normal distribution. The location parameter considered are $\Delta = 0.3$, 0.6 and 0.9 and calculation is based over the 10,000 repetitions. The power is estimated at fixed level of significance at 5% and 1% and is given in Table 11 and Table 12 respectively. The estimated level of significance for 5% and 1% are given in Table 13 and 14 respectively.

From the Tables 11-14, we can make the following observations:

(i) From Table 11, we observe that the location change of order 0.9 is detected at random sample of size $(n, m) = (30, 40)$ and $c = 1$ with approximately 95% power of the proposed test.

(ii) From Table 12, we found that location change of order 0.9 is detected at random sample of size $(n, m) = (30, 40)$ and $c = 4$ with approximately 95% power of the proposed test.

(iii) From Tables 13 and 14, we observe that as sample size increases then the size of the test is achieved for both 5% and 1% level of significance.

Table 11. Estimated power of the proposed test for Normal distribution for 5% level of significance.

		c				
n,m	**Δ**	1	2	3	4	5
10,10	0.3	0.1007	0.1428	0.1598	0.1805	0.2086
	0.6	0.2406	0.3315	0.3594	0.3711	0.3959
	0.9	0.4665	0.5767	0.3594	0.5907	0.6142
10,20	0.3	0.1112	0.1487	0.1735	0.1868	0.2114
	0.6	0.3058	0.3950	0.4159	0.4231	0.4221
	0.9	0.5792	0.6739	0.6919	0.6929	0.6950
10,30	0.3	0.1222	0.1589	0.1840	0.1980	0.1999
	0.6	0.3488	0.4239	0.4472	0.4485	0.4470
	0.9	0.6490	0.7167	0.7312	0.7350	0.7490
10,40	0.3	0.1262	0.1729	0.1872	0.1975	0.2183
	0.6	0.3616	0.4386	0.4661	0.4745	0.4823
	0.9	0.6793	0.7369	0.7599	0.7665	0.7733
20,20	0.3	0.1481	0.1903	0.2068	0.2310	0.2460
	0.6	0.4299	0.5121	0.5324	0.5374	0.5400
	0.9	0.7766	0.8326	0.8369	0.8394	0.8401
20,30	0.3	0.1662	0.2223	0.2221	0.2300	0.2401
	0.6	0.5208	0.5734	0.6353	0.5900	0.6139
	0.9	0.8421	0.8826	0.9233	0.8950	0.8971
20,40	0.3	0.1815	0.2229	0.2320	0.2412	0.2472
	0.6	0.5560	0.6158	0.6353	0.6672	0.6690
	0.9	0.8845	0.9188	0.9233	0.9320	0.9381
30,30	0.3	0.1954	0.2414	0.2520	0.2732	0.2815
	0.6	0.6067	0.6694	0.6950	0.7051	0.7063
	0.9	0.9169	0.9380	0.9470	0.9491	0.9381
30,40	0.3	0.2199	0.2648	0.2804	0.2930	0.2815
	0.6	0.6530	0.7191	0.7340	0.7591	0.7163
	0.9	0.9508	0.9647	0.9665	0.9671	0.9510
40,40	0.3	0.2471	0.2830	0.3040	0.3190	0.3211
	0.6	0.7420	0.7789	0.7940	0.8140	0.8146
	0.9	0.9740	0.9808	0.9840	0.9856	0.9866

Table 12. Estimated power of the proposed test for Normal distribution for 1% level of significance.

		c				
n,m	**Δ**	1	2	3	4	5
10,10	0.3	0.0181	0.0570	0.0760	0.0919	0.0923
	0.6	0.0672	0.1635	0.2030	0.2253	0.2274
	0.9	0.1805	0.3617	0.3981	0.4181	0.4480
10,20	0.3	0.0278	0.0581	0.0763	0.0923	0.0966
	0.6	0.1157	0.2104	0.2426	0.2711	0.2883
	0.9	0.3040	0.4745	0.5076	0.5374	0.5520
10,30	0.3	0.0328	0.0678	0.0847	0.0953	0.0970
	0.6	0.1373	0.2344	0.2600	0.2784	0.2950
	0.9	0.3598	0.5192	0.5515	0.5553	0.5680
10,40	0.3	0.0345	0.0679	0.0832	0.0966	0.0970
	0.6	0.1486	0.2464	0.2820	0.2966	0.3197
	0.9	0.3942	0.5496	0.5804	0.5990	0.6299
20,20	0.3	0.0444	0.0812	0.0900	0.0980	0.1179
	0.6	0.1987	0.3083	0.3374	0.3393	0.4412
	0.9	0.5168	0.6585	0.6814	0.7844	0.7988
20,30	0.3	0.0564	0.0894	0.0904	0.1100	0.1179
	0.6	0.2558	0.3681	0.3951	0.4251	0.4412
	0.9	0.6292	0.7524	0.7612	0.7813	0.8001
20,40	0.3	0.0577	0.0983	0.0985	0.1190	0.1271
	0.6	0.2891	0.4089	0.4160	0.4414	0.4731
	0.9	0.6900	0.7990	0.8115	0.8305	0.8449
30,30	0.3	0.0613	0.1096	0.1220	0.1357	0.1496
	0.6	0.3463	0.4711	0.4960	0.5147	0.5316
	0.9	0.7629	0.8450	0.8750	0.8951	0.9013
30,40	0.3	0.0779	0.1223	0.1420	0.1432	0.1509
	0.6	0.4026	0.5210	0.5530	0.5713	0.5799
	0.9	0.8320	0.8990	0.9050	0.9111	0.9271
40,40	0.3	0.0942	0.1410	0.1469	0.1501	0.1599
	0.6	0.4815	0.5815	0.5980	0.6110	0.6310
	0.9	0.8915	0.9366	0.9460	0.9561	0.9711

Table 13. Estimated size of the test at 5% level of significance.

	c				
n,m	1	2	3	4	5
10,10	0.0515	0.0491	0.0539	0.0590	0.0620
10,20	0.0495	0.0509	0.0493	0.0557	0.0590
10,30	0.0502	0.0500	0.0491	0.0539	0.0571
10,40	0.0505	0.0470	0.0490	0.0530	0.0558
20,20	0.0503	0.0486	0.0494	0.0529	0.0547
20,30	0.0493	0.0503	0.0500	0.0513	0.0530
20,40	0.0502	0.0498	0.0510	0.0509	0.0520
30,30	0.0495	0.0500	0.0510	0.0511	0.0519
30,40	0.0496	0.0513	0.0520	0.0507	0.0512
40,40	0.0499	0.0501	0.0509	0.0503	0.0505

Table 14. Estimated size of the test at 1% level of significance.

	c				
n,m	1	2	3	4	5
10,10	0.0071	0.0130	0.0194	0.0285	0.0396
10,20	0.0084	0.1250	0.0183	0.0229	0.0270
10,30	0.0081	0.0114	0.0148	0.0156	0.0217
10,40	0.0083	0.0115	0.0146	0.0151	0.0190
20,20	0.0087	0.0111	0.0149	0.0149	0.0171
20,30	0.0091	0.0090	0.0135	0.0140	0.0163
20,40	0.0095	0.0113	0.0120	0.0131	0.0161
30,30	0.0101	0.0092	0.0121	0.0121	0.0151
30,40	0.0095	0.0091	0.0139	0.0120	0.0130
40,40	0.0106	0.0101	0.0119	0.0110	0.0120

Conclusion

The proposed class of tests performs same or better than its competitor tests for short tailed, symmetric and skewed distributions using the criteria of Pitman asymptotic relative efficiency. By simulation study, we observe that the location change of order 0.9 is detected at random samples of size $(n, m) = (30, 40)$ and $c = 1$ with approximately 95% power. Moreover, with the increase in sample observations, size of the test is achieved for both 5% and 1% levels of significance.

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