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The Wedge Family Of The Cohomology Of The C-Motivic Steenrod Algebra

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THE WEDGE FAMILY OF THE COHOMOLOGY OF THE $\mathbb{C}$-MOTIVIC STEENROD ALGEBRA

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DEDICATION

To my family and my teachers

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TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1 INTRODUCTION

1.1 The stable homotopy groups of sphere spectrum

The questions of determining the homotopy groups $[S^{n+k}, S^n]$ of spheres turns out to be one of the most important questions in algebraic topology. The groups $[S^{n+k}, S^n]$ are the groups of homotopy classes of continuous based maps $f : S^{n+k} \to S^n$ between two spheres which describe how spheres of different dimensions can wrap around each other.

The Freudenthal Suspension Theorem gives a fundamental relationship between the groups $[S^{n+k}, S^n]$ when k is fixed and n varies. To be precise, the suspension from S^n to S^{n+1} induces a group homomorphism

$$[S^{n+k}, S^n] \to [S^{n+k+1}, S^{n+1}].$$

This group homomorphism is isomorphic when $n > k + 1$. In other words, the groups $[S^{n+k}, S^n]$ depend only on k when $n > k + 1$.

Definition 1.1. When $n > k + 1$, the group $\pi_k = [S^{n+k}, S^n]$ is called the k-th stable homotopy group of spheres or k-stem.

Example 1.2. The group $\pi_0 = [S^n, S^n] \cong \mathbb{Z}$.

When $n \leq k + 1$ the group $[S^{n+k}, S^n]$ is called unstable. The stable homotopy groups have additional structure making them more amenable to compute than the unstable ones. In this thesis, we are mostly interested in the stable case. The unstable case is one of our long term projects which will be discussed in Chapter 6.

Example 1.3. When $k = 1$, the groups $[S^2, S^1] \cong 0$ and $[S^3, S^2] \cong \mathbb{Z}$ are unstable. The groups $[S^4, S^3] \cong [S^5, S^4] \cong [S^6, S^5] \cong \ldots \cong \mathbb{Z}/2$ are stable.

Theorem 1.4. *The group π_k is finite and abelian for all $k \geq 1$.*

Theorem 1.5. *[The Primary Decomposition Theorem] Let G be an abelian group of order $m > 1$ and let the unique factorization of m into distinct prime powers be*

$$m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$$

then

$$G \cong A_1 \times A_2 \times \cdots \times A_k$$

for $|A_i| = p_i^{\alpha_k}$.

Theorems 1.4 and 1.5 suggest to study π_k one prime at a time. In other words, we will compute the p-primary component of π_k for all primes p and then combine these components to obtain a uniquely determined finite abelian group which is defined in Theorem 1.5.

Remark 1.6. For abelian groups the Sylow p-groups are called the p-primary components. In this thesis we use the latter name.

In this thesis, we will focus on the 2-primary component $\hat{\pi_k}$ of π_k. For the study at odd primes, we suggest readers to see Ravenel's book [31].

The knowledge about the stable homotopy groups has important applications in the study of high-dimensional manifolds. One of the well known examples is the Kervaire invariant problem. Kervaire and Milnor reduced the classification of smooth structures to a computation of stable homotopy groups.

The Adams spectral sequence appears to be one of the most effective tools to compute the stable homotopy groups. The spectral sequence has been studied by J. F. Adams [1] [2], M. Mahowald [4] [22], M. Tangora [35], J. P. May [25] and others [6].

1.2 Motivation for this Thesis

In 1999, Morel and Voevodsky introduced motivic homotopy theory [27]. One of its consequences is the realization that almost any object studied in classical algebraic topology could be given a motivic analog. In particular, we can define the motivic Steenrod algebra $\mathbf{A}$ [40], the motivic stable homotopy groups of spheres [27] and the motivic Adams spectral sequence [12]. In the motivic perspective, there are many more non-zero classes in the motivic Adams spectral sequence, which allows the detection of otherwise elusive phenomena. Also, the additional motivic weight grading can eliminate possibilities which appear plausible in the classical perspective.

Let $\mathbb{M}_2$ denote the motivic cohomology of a point, which is isomorphic to $\mathbb{F}_2[\tau]$ where τ has bidegree $(0,1)$ [38]. The motivic Steenrod algebra $\mathbf{A}$ is the $\mathbb{M}_2$-algebra generated by elements Sq^{2k} and Sq^{2k-1} for all $k \geq 1$, of bidegrees $(2k,k)$ and $(2k-1,k-1)$ respectively, subject to Adem relations [40] [39]. Let $\mathrm{Ext}_{\mathbf{A}}(\mathbb{M}_2,\mathbb{M}_2)$ denote the cohomology of the motivic Steenrod algebra. To run the motivic Adams spectral sequence, one begins with $\mathrm{Ext}_{\mathbf{A}}(\mathbb{M}_2,\mathbb{M}_2)$. The cohomology $\mathrm{Ext}_{\mathbf{A}}(\mathbb{M}_2,\mathbb{M}_2)$ has an $\mathbb{M}_2$-algebra structure. Inverting τ in $\mathrm{Ext}_{\mathbf{A}}(\mathbb{M}_2,\mathbb{M}_2)$ gives the cohomology $\mathrm{Ext}_{\mathbf{A}_{cl}}(\mathbb{F}_2,\mathbb{F}_2)$ of the classical Steenrod algebra $\mathbf{A}_{cl}$ [18]. Given a classical element, there are many corresponding motivic elements. We typically want to find the corresponding element with the highest weight. For example, the classical element g corresponds to the motivic elements $\tau^k g$ for all $k \geq 1$. The element τg has weight 11, but there is no motivic element of weight 12 that corresponds to the classical element g.

The algebra $\mathrm{Ext}_{\mathbf{A}}(\mathbb{M}_2,\mathbb{M}_2)$ is infinitely generated and irregular. A natural approach is to look for systematic phenomena in $\mathrm{Ext}_{\mathbf{A}}(\mathbb{M}_2,\mathbb{M}_2)$. One potential candidate is the wedge family in $\mathrm{Ext}_{\mathbf{A}}(\mathbb{M}_2,\mathbb{M}_2)$.

The classical wedge family was studied by M. Mahowald and M. Tangora [21]. It is a subset of the cohomology $\mathrm{Ext}_{\mathbf{A}_{\mathrm{cl}}}(\mathbb{F}_2,\mathbb{F}_2)$ of the classical Steenrod algebra, consisting of non-zero elements $P^i g^j \lambda$ and $g^j t$ in which λ is in $\mathbf{\Lambda}$, t is in $\mathbf{T}$, $i \geq 0$ and $j \geq 0$. The sets $\mathbf{\Lambda}$ and $\mathbf{T}$ are specific subsets of $\mathrm{Ext}_{\mathbf{A}_{\mathrm{cl}}}(\mathbb{F}_2,\mathbb{F}_2)$. The wedge family gives an infinite wedge-shaped diagram inside the cohomology of the classical Steenrod algebra, which fills out an angle with vertex at g^2 in degree (40,8) (i.e. g^2 has stem 40 and Adams filtration 8), bounded above by the line $f = \frac{1}{2}s - 12$, parallel to the Adams edge [1], and bounded below by the line $s = 5f$, in which f is the Adams filtration and s is the stem. The wedge family is a large piece of $\mathrm{Ext}_{\mathbf{A}_{\mathrm{cl}}}(\mathbb{F}_2, \mathbb{F}_2)$ which is regular, of considerable size and easy to understand.

Using this idea we build the motivic version of the wedge. However, it appears to be more complicated than the classical one. The highest weights of the motivic wedge elements follow a somewhat irregular pattern. We will discuss this irregularity in more detail later.

Let $\mathbf{A}(2)$ denote the $\mathbb{M}_2$-subalgebra of $\mathbf{A}$ generated by $\mathrm{Sq}^1, \mathrm{Sq}^2$ and Sq^4. Let $\mathrm{Ext}_{\mathbf{A}(2)}(\mathbb{M}_2, \mathbb{M}_2)$ denote the cohomology of $\mathbf{A}(2)$. The finitely generated algebra $\mathrm{Ext}_{\mathbf{A}(2)}(\mathbb{M}_2, \mathbb{M}_2)$ is fully understood by [17]. We use a new technique of comparison to $\mathrm{Ext}_{\mathbf{A}(2)}(\mathbb{M}_2, \mathbb{M}_2)$ which makes the proof of the non-triviality of the wedge elements easy. We consider the ring homomorphism ϕ from $\mathrm{Ext}_{\mathbf{A}}(\mathbb{M}_2, \mathbb{M}_2)$ to $\mathrm{Ext}_{\mathbf{A}(2)}(\mathbb{M}_2, \mathbb{M}_2)$ induced by the inclusion from $\mathbf{A}(2)$ to $\mathbf{A}$. We use the map ϕ to detect structure in $\mathrm{Ext}_{\mathbf{A}}(\mathbb{M}_2, \mathbb{M}_2)$. Most of the elements studied in this article have non-zero images via ϕ [17]. Therefore, they are all non-trivial elements in $\mathrm{Ext}_{\mathbf{A}}(\mathbb{M}_2, \mathbb{M}_2)$.

We define set-valued operations $\mathbf{P}$ and $\mathbf{g}$ on $\mathrm{Ext}_{\mathbf{A}}(\mathbb{M}_2, \mathbb{M}_2)$. Classically, g is an element of the cohomology of the classical Steenrod algebra. However, this is not true motivically. Rather, τg is an element in $\mathrm{Ext}_{\mathbf{A}}(\mathbb{M}_2, \mathbb{M}_2)$, while g itself does not survive the motivic May spectral sequence. Consequently, multiplication by g does not make sense motivically. Also, P is not an element in $\mathrm{Ext}_{\mathbf{A}}(\mathbb{M}_2, \mathbb{M}_2)$ either. We instead consider the set-valued operations $\mathbf{P}$ and $\mathbf{g}$ whose actions can be seen as multiplications by P and g in $\mathrm{Ext}_{\mathbf{A}(2)}(\mathbb{M}_2, \mathbb{M}_2)$ respectively.

For any λ in $\mathrm{Ext}_{\mathbf{A}}(\mathbb{M}_2, \mathbb{M}_2)$, $i \geq 0$ and $j \geq 0$, let $\mathbf{P}^i\mathbf{g}^j\lambda$ be the set consisting of all elements x in $\mathrm{Ext}_{\mathbf{A}}(\mathbb{M}_2, \mathbb{M}_2)$ such that $\phi(x) = P^i g^j \phi(\lambda)$ in $\mathrm{Ext}_{\mathbf{A}(2)}(\mathbb{M}_2, \mathbb{M}_2)$.

We define the wedge family via the actions of $\mathbf{P}$ and $\mathbf{g}$. The wedge is the set consisting of all elements in $\mathbf{P}^i\mathbf{g}^j\lambda$ with $i \geq 0$ and $j \geq 0$, where λ is contained in a specific 16-element subset $\mathbf{\Lambda}$ of $\mathrm{Ext}_{\mathbf{A}}(\mathbb{M}_2, \mathbb{M}_2)$ to be defined in Table 5.

The motivic wedge family takes the same position and same shape as the classical one (Figure 1). However the vertex of the motivic wedge is at τg^2 in degree $(40, 8, 23)$ having weight 23. Note that g^2 in degree $(40, 8, 24)$ does not survive the motivic May spectral sequence [18]. Our main result, Theorem 5.8, states that the subsets $\mathbf{P}^i \mathbf{g}^j \lambda$ are non-empty and consist of non-zero elements for all λ in $\mathbf{\Lambda}$.

However, our main result is not optimal, in the sense that there exist elements of weight greater than the weight of elements in $\mathbf{P}^i \mathbf{g}^j \lambda$ for some values of i, j, and λ. Some such elements are listed in Table 6.

We can not even conjecture the optimal result in general. However, we know a bit more about elements in the set $e_0^t \mathbf{g}^k$ for $t \geq 0$ and $k \geq 0$, which are part of the wedge. We will show that $\tau e_0^t \mathbf{g}^k$ is non-empty for all $t \geq 0$ and $k \geq 0$. We do not know whether $e_0^t \mathbf{g}^k$ is non-empty in general, but we make the following conjecture.

Conjecture 1.7. The set $e_0^t \mathbf{g}^k$ is non-empty if and only if $k = (\sum_{i=1}^{t} 2^{n_i}) - t$ for some integers $n_i \geq 1$.

The conjecture is equivalent to the conjecture that $e_0 \mathbf{g}^k$ is non-empty if and only if $k+1$ is a power of 2, since

$$e_0^t \mathbf{g}^k \supseteq e_0 \mathbf{g}^{2^{n_1}-1} \cdots e_0 \mathbf{g}^{2^{n_t}-1}.$$

By explicit computations we know that $e_0, e_0\mathbf{g}$ and $e_0\mathbf{g}^3$ are non-empty and $e_0\mathbf{g}^2$ and $e_0\mathbf{g}^4$ are empty [18]. This means that the subsets $e_0\mathbf{g}^k$ are non-empty sometimes but empty other times. The analogous classical question is trivial, since $e_0^t g^k$ is a product of e_0^t and g^k.

1.3 Notation

We will use the following notation.

1. $\mathbb{M}_2 = \mathbb{F}_2[\tau]$ is the mod 2 motivic cohomology of a point, where τ has bidegree (0,1).

2. $\mathbf{A}$ is the mod 2 motivic Steenrod algebra over $\mathbb{C}$.

3. $\mathbf{A}(2)$ is the $\mathbb{M}_2$-subalgebra of $\mathbf{A}$ generated by $\mathrm{Sq}^1, \mathrm{Sq}^2$ and Sq^4.

4. $\mathbf{A}_{\mathrm{cl}}$ is the mod 2 classical Steenrod algebra.

5. Ext is the trigraded ring $\mathrm{Ext}_{\mathbf{A}}(\mathbb{M}_2, \mathbb{M}_2)$, the cohomology of the motivic Steenrod algebra.

6. $\mathrm{Ext}_{\mathbf{A}(2)}$ is the trigraded ring $\mathrm{Ext}_{\mathbf{A}(2)}(\mathbb{M}_2, \mathbb{M}_2)$, the cohomology of the $\mathbb{M}_2$-subalgebra of $\mathbf{A}$ generated by $\mathrm{Sq}^1, \mathrm{Sq}^2$ and Sq^4.

7. $\mathrm{Ext}_{\mathrm{cl}}$ is the bigraded ring $\mathrm{Ext}_{\mathbf{A}_{\mathrm{cl}}}(\mathbb{F}_2, \mathbb{F}_2)$, the cohomology of the classical Steenrod algebra.

8. We use the notation of [19] for elements in Ext.

9. We use the notation of [17] for elements in $\mathrm{Ext}_{\mathbf{A}(2)}$, except that we use a and n instead of α and ν respectively.

10. An element x in Ext has degree of the form (s, f, w) where:

    (a) f is the Adams filtration, i.e., the homological degree.

    (b) $s+f$ is the internal degree, i.e., corresponds to the first coordinate in the bidegrees of $\mathbf{A}$.

    (c) s is the stem, i.e., the internal degree minus the Adams filtration.

    (d) w is the motivic weight.

11. The Chow degree of an element of degree (s, f, w) is $s + f - 2w$.

12. The coweight of an element of degree (s, f, w) is $s - w$.

1.4 Organization

Here is a brief organization of this thesis.

Chapter 1. This chapter contains some background material to study the thesis. In particular, we introduce the history of the study of the homotopy groups of spheres and how it gives rise to the stable approach to study the homotopy groups of the sphere spectrum. We also give a brief introduction to the motivic perspective in studying homotopy groups which is the main purpose of this thesis.

Chapter 2. This chapter introduces the general approach to spectral sequences which is our main tool (Adams spectral sequence, May spectral sequence) in studying the homotopy groups of the sphere spectrum. We introduce how to construct a spectral sequence using exact couples or filtrations. We also discuss a little bit on recovering the desired objects from spectral sequences.

Chapter 3. In this chapter, we introduce the motivic computation program to study the homotopy groups of the motivic sphere spectrum. The motivic Adams spectral sequence, the motivic version of the classical Adams spectral sequence, is our main technique in this study.

Chapter 4. Chapter 4 studies the cohomology Ext of the motivic Steenrod algebra $\mathbf{A}$ serving as the E_2-term of the motivic Adams spectral sequence. The cohomology Ext is infinitely generated and irregular. We also discuss the cohomology of $\mathbf{A}(2)$, a subalgebra of $\mathbf{A}$ and the h_1-localization technique which are two of the most important tools in studying Ext.

Chapter 5. Chapter 5 is the main part of the thesis where we study the $\mathbb{C}$-motivic wedge family of the cohomology Ext of the motivic Steenrod algebra. The wedge is an infinite subset of Ext which is regular and easy enough to study. We also state two conjectures about two other families in Ext which rise naturally from the study of the wedge.

Chapter 6. Chapter 6 is a side project. We study the motivic Lambda algebra which is expected to be the motivic version of the classical Lambda algebra. Our purpose is to use the motivic Lambda algebra to study the unstable homotopy groups of the motivic sphere spectrum.

CHAPTER 2 SPECTRAL SEQUENCES

2.1 Definition

The well known definition of spectral sequences is as follows.

Definition 2.1. A spectral sequence $E = \{E^n\}$ consists of a sequence of $\mathbb{Z}$-bigraded R modules $E^n = \{E^n_{p,q}\}$ together with differentials

$$d^n : E^n_{p,q} \to E^n_{p-n,q+n-1}$$

such that $E^{n+1} \cong H_*(E^n)$.

Let Z^1 be the kernel of d^1 and B^1 be the image of d^1. Then $E^2 = H_*(E^1) = Z^1/B^1$ and we can see d^2 as the map

$$d^2 : Z^1/B^1 \longrightarrow Z^1/B^1.$$

Repeating this process we can see E^r as Z^{r-1}/B^{r-1} and

$$d^r : Z^{r-1}/B^{r-1} \longrightarrow Z^{r-1}/B^{r-1}.$$

The map d^r has kernel Z^r/B^{r-1} and image B^r/B^{r-1}. Then we have the following sequence of submodules

$$0 = B^0 \subset B^1 \subset \ldots \subset Z^2 \subset Z^1 \subset Z^0 = E^1.$$

We denote $Z^\infty = \bigcap_{r=1}^\infty Z^r$, $B^\infty = \bigcup_{r=1}^\infty B^r$ and $E^\infty_{p,q} = Z^\infty_{p,q}/B^\infty_{p,q}$.

In practice, spectral sequences are sometimes trigraded depending on the objects we are dealing with. For example, the motivic May spectral sequence and the motivic Adams spectral sequence are trigraded but the classical ones are bigraded. Also, the grading of the differential d^n is not always as defined in Definition 2.5. Depending on the problems we are working with, we can choose a suitable grading for d^n. The grading of a spectral sequence is possibly different from that of another spectral sequence.

2.2 Constructing a spectral sequence

2.2.1 Exact couples

Nowadays, people usually construct a spectral sequence by using exact couples or filtrations.

Definition 2.2. Let D and E be modules over a ring R. An exact couple $\mathcal{C} = \langle D, E; i, j, k\rangle$ is a diagram

$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

in which i, j and k are module homomorphisms satisfying $Ker j = \operatorname{Im} i, \operatorname{Ker} k = \operatorname{Im} j$ and $\operatorname{Ker} i = \operatorname{Im} k$.

If $d = jk : E \to E$, then $d \circ d = (jk)(jk) = j(kj)k = 0$ since $kj = 0$. As a result, (E, d) has a chain complex structure and we can define the homology $H_*(E, d)$.

We construct $\mathcal{C}' = \langle D', E'; i', j', k'\rangle$ by letting

$$D' = i(D), \;\; E' = H_*(E, d),$$

$$i' = i|_{i(D)}, \quad j'(i(x)) = \overline{(j(x))} = j(x) + jk(E), \quad k'(\overline{y}) = k'(y + jk(E)) = k(y).$$

In other words, i' is a restriction of i and j' and k' are induced from j and k by passing to homology. It is easy to check that j' and k' are well-defined. We can also easily prove the following result.

Proposition 2.3. *$\mathcal{C}' = \langle D', E'; i', j', k'\rangle$ is an exact couple.*

By iterating the above construction we can define

$$\mathcal{C}^{(n)} = \langle D^n, E^n; i^{(n)}, j^{(n)}, k^{(n)}\rangle,$$

By denoting $d^n = j^{(n)}k^{(n)}$ we have the spectral sequence (E^n, d^n). We grade

$$|i| = (1, -1),\ |j| = (0, 0) \text{ and } |k| = (-1, 0).$$

As a result

$$|i^{(n)}| = (1, -1),\ \ |j^{(n)}| = (-n + 1, n - 1) \text{ and } \ |k^{(n)}| = (-1, 0).$$

Therefore $|d^n| = |j^{(n)}k^{(n)}| = (-n, n - 1)$, and we have

$$d^n : E^n_{p,q} \to E^n_{p-n,q+n-1}.$$

As we mentioned, this way of grading is not unique. Practitioners can choose another grading which is helpful to his/her computations.

2.2.2 Filtrations

In algebraic topology, it is usual that we want to compute the homology $H_*(A)$ of a chain complex A. However, this computation is not always easy to be carried out in high dimensions. An approach is that we first look for a "simpler" version E^0H_*A of $H_*(A)$ then recover the structure of $H_*(A)$ from the structure of $E^0H_*(A)$.

Definition 2.4. Let (A, d) be a $\mathbb{Z}$-graded complex of R-modules. An increasing filtration of A is a sequence of subcomplexes

$$\ldots \subset F_{p-1}A \subset F_pA \subset F_{p+1}A \subset \ldots$$

of A. The associated graded complex E^0A is the bigraded complex defined by

$$E^0_{p,q}A = (F_pA/F_{p-1}A)_{p+q}$$

with differential d^0 induced by d.

The inclusions $F_pA \hookrightarrow A$ induce the module homomorphisms $i_* : H_*(F_pA) \to H_*A$. The homology $H_*(A)$ is then filtered by

$$F_pH_*(A) = \operatorname{Im} i_*$$

and we can define E^0H_*A.

We observe that an element $\overline{x}$ in $E^0_{p,q}H_*A$ is the class of an element x at degree $p+q$ in A on the filtration F_pA with respect to the quotient by $F_{p-1}A$. In other words, $\overline{x} = x + F_{p-1}A$.

Denote $E^0_pA = F_pA/F_{p-1}A$. We have the following short exact sequence of chain complexes

$$0 \longrightarrow F_{p-1}A \xrightarrow{i} F_pA \xrightarrow{j} E^0_pA \longrightarrow 0$$

in which i is the inclusion and j is the surjection. This short exact sequence induces a long exact sequence

$$\ldots \longrightarrow H_n(F_{p-1}A) \xrightarrow{i_*} H_n(F_pA) \xrightarrow{j_*} H_n(E^0_pA) \xrightarrow{k_*} H_{n-1}(F_{p-1}A) \longrightarrow \ldots$$

Let $D^1_{p,q} = H_{p+q}(F_pA)$ and $E^1_{p,q} = H_{p+q}(E^0_pA)$. Then we have the following exact couple

$$\langle D^1, E^1; i_*, j_*, k_* \rangle.$$

As a result, we have the spectral sequence E^nA. The following theorem is a key result to the spectral sequence computations.

Theorem 2.5. *If $A = \bigcup_p F_pA$ and for each n there exists $s(n)$ such that $F_{s(n)}A_n = 0$, then we have the following isomorphism of modules*

$$E^\infty_{p,q}A \cong E^0_{p,q}H_*(A).$$

We say that the spectral sequence $\{E^n\}$ converges to H_*A and write

$$E^2_{p,q}A \Rightarrow H_{p+q}(A).$$

The key idea is that we will compute the module E^0H_*A via a spectral sequence and use it to recover H_*A. Unfortunately, there is no general way to recover H_*A from E^0H_*A. It depends on the chain complex A, the spectral sequence we are dealing with and the structure we want to recover. We will discuss more about it in the next Section.

The equivalence of two approaches. We have discussed two approaches to construct a spectral sequence. They are filtrations and exact couples. These two approaches give us the same spectral sequences.

Proposition 2.6. *[26, Proposition 2.11] For a filtered differential graded module, the spectral sequence associated to the filtration and the spectral sequence associated to the exact couple are the same.*

2.2.3 Recovering $H_*(A)$ from E^∞-page and the hidden extension problem

Roughly speaking, the E^∞-page is just a graded filtered version of $H_*(A)$. Unfortunately, these two objects are possibly very different from each other. The reason lies in the so-called hidden extension problem which is as follows.

Consider an R-module M and its submodule N. Then we have the following short exact sequence.

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0.$$

The module M plays as $H_*(A)$, the targeted object we want to compute. The filtration is $0 \subset N \subset M$ and then E^∞-page is $N/0 \oplus M/N$. The problem is that how to learn M from N and M/N. Unfortunately, there is no unique answer for M. Let us see the following example.

Example 2.7. Consider the following short exact sequence.

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow M \longrightarrow \mathbb{Z}/2 \longrightarrow 0.$$

The module M can be $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ or $\mathbb{Z}/4$.

2.2.4 Vector space structure

The following result plays a key result in our work with spectral sequences.

Definition 2.8. A module P is projective if for every surjective module homomorphism $f : A \to B$ and every module homomorphism $g : P \to B$, there exists a homomorphism $h : P \to A$ such that $f \circ h = g$.

$$\begin{tikzcd} & A \arrow[d, two heads, "f"] \\ P \arrow[ur, dashed, "\exists h"] \arrow[r, "g"] & B \end{tikzcd}$$

Example 2.9. Every vector space is a projective module.

Proposition 2.10. *The short exact sequence*

$$0 \longrightarrow N \xrightarrow{i} M \xrightarrow{j} P \longrightarrow 0$$

is split, i.e. $M \cong N \oplus P$*, if* P *is a projective module.*

Proof. Since P is a projective module and $j : M \to P$ is surjective, we have

$$\begin{tikzcd} & M \arrow[d, two heads, "j"] \\ P \arrow[ur, dashed, "\exists h"] \arrow[r, "i"] & P \end{tikzcd}$$

in which $i : P \to P$ is the identity map.

Since $j \circ h = i$, the map h is injective. Therefore, we can identify P with its image in M. Similarly, we identify N with its image in M. For y in $N \cap P$ since $P \cong M/N$, the element y has to be zero. Then $N \cap P = \{0\}$. For any x in M, since we identify P with $h(P)$ contained in M, the element $h(j(x))$ is in P. Consider the element $z = x - h(j(x))$ we have

$$j(z) = j(x) - j(h(j(x))) = j(x) - j(x) = 0,$$

in which the second identity is because $j \circ h = i$. As a result, z belongs to $ker(j) = im(i)$. Since we identify N with $i(n)$, we can see z as an element in N. Therefore, $x = z + h(j(x))$ which is a summation of an element in N and an element in P. We obtain the desired statement. $\square$

Proposition 2.11. *Suppose that V is a vector space and V has the filtration $V_0 \subset V_1 \subset \ldots \subset V_n = V$ in which V_i is a subspace of V for all $0 \leq i \leq n$. Then*

$$V = V_0 \oplus V_1/V_0 \oplus V_2/V_1 \oplus \ldots \oplus V_n/V_{n-1}.$$

Proof. Since any vector space is a projective module, by Proposition 2.12 we have

$$V_1 \cong V_0 \oplus V_1/V_0 \text{ and } V_2 \cong V_1 \oplus V_2/V_1.$$

It implies that

$$V_2 \cong V_0 \oplus V_1/V_0 \oplus V_2/V_1.$$

By repeating this argument for $V_3, V_4, \ldots$ we get

$$V = V_0 \oplus V_1/V_0 \oplus V_2/V_1 \oplus \ldots \oplus V_n/V_{n-1}$$

as desired. $\square$

Theorem 2.12. *If $H_*(A)$ is a vector space, then $H_*(A) \cong E^0 H_*(A)$ as vector spaces.*

Proof. It is straight forward from Proposition 2.11 and the definition of the associated graded algebra $E^0 H_*(A)$. $\square$

Ring/Product structure and Massey product. Unfortunately, we do not have a nice result like Theorem 2.12 to recover the product structure of $H_*(A)$ from the E^∞-page. Let us consider the following example.

Example 2.13. We filter the polynomial ring $A = \mathbb{R}[x]$ as follows.

$$\mathbb{R}[x] \supset \langle x \rangle \supset \langle x^2 \rangle \supset \langle 2x^2 \rangle \supset 0$$

in which $F_1 = \langle x \rangle$, $F_2 = \langle x^2 \rangle$ and $F_3 = \langle 2x^2 \rangle$.

We consider non-zero classes $\overline{x}$ and $\overline{2x}$ in $E_1^0 A = F_1/F_2$. The product $\overline{x} \cdot \overline{2x}$ is an element in $E_2^0 A = F_2/F_3$, because of degree reasons, then it is zero.

We say that the associated graded algebra $E^0R[x]$ with respect to the given filtration hides some of the product structure of $R[x]$. We do not have a general way to recover this product structure. In the May spectral sequence, Massey products are a very efficient tool which will be discussed in this thesis.

2.2.5 Summary

To compute a complicated homology H_*A which is hard to compute directly, we can filter A and then this filtration will give rise to a spectral sequence E^r having E^∞ isomorphic to the associated graded module E^0H_*A which is a filtered version of H_*A. If the E^1-page is infinitely generated, it is possible that we will never obtain the E^∞-page of spectral sequences (eg: May spectral sequence, Adams spectral sequence). However, we can restrict our computation to some finite range, then we will definitely obtain the E^∞-page up to the chosen range after finite steps.

In this section, we only introduce the spectral sequence with module structure. In practice, spectral sequences (eg: May spectral sequence, Adams spectral sequence) can enjoy multiplicative structure. In that case, the multiplication on E^∞ is induced from that on A. Unfortunately, spectral sequences usually do not preserve the multiplicative structure of A. Depending on the objects and the spectral sequences we are working on, we need suitable methods to recover the multiplicative and module structures of A. Moreover, there are higher structures in our study such as Massey products in the May spectral sequence and Toda brackets in the Adams spectral sequence. It is possible that we are not able to recover the targeted objects completely from spectral sequences. We just do it as much as possible.

Usually, we input some data to a spectral sequence to learn some output. However, there are cases that we use known outputs to study inputs via spectral sequences. This often happens to the Serre spectral sequence.

CHAPTER 3 THE MOTIVIC ADAMS SPECTRAL SEQUENCE

3.1 The motivic approach to homotopy theory

Model and Voevodsky study the motivic homotopy theory in [27]. The motivic homotopy theory is a homotopy theory for algebraic varieties. In other words, people use the techniques from homotopy theory to study algebraic varieties. The starting point is the category built out of smooth schemes over a field k. In this thesis, we are mostly interested in the complex base field $\mathbb{C}$. The case over the real base field $\mathbb{R}$ will be our future project. Unfortunately, this category is not good enough for homotopical purposes since it does not have homotopy colimits serving as gluing constructions.

There is a realization functor from the motivic homotopy theory over $\mathbb{C}$ to the ordinary homotopy theory. For any complex scheme X, there is an associated topological space $X_{\mathbb{C}}$ of $\mathbb{C}$-valued points. Almost any object studied in classical algebraic topology can be given a motivic analogue. In particular, we can define the motivic stable homotopy groups of spheres.

In [18] Isaksen shows that we have very good calculational control over this realization functor. The functor will help us not only deduce motivic facts from classical results but also deduce classical facts from motivic results which is the key idea of our research.

The cohomology of a point. To run the motivic Adams spectral sequence, we need to know the motivic cohomology $\mathbb{M}_2$ of a point.

Theorem 3.1 (Voevodsky). *The motivic cohomology* $\mathbb{M}_2$ *is the polynomial ring* $\mathbb{F}_2[\tau]$ *on one generator* τ *of bidegree* $(0, 1)$.

In a bidegree (p, q), we refer to p as the topological degree and q as the weight.

The polynomial ring $\mathbb{M}_2 = \mathbb{F}_2[\tau]$ is an $\mathbb{F}_2$-vector space. It is also an $\mathbf{A}$-module with trivial actions of Sq^i for all $i \geq 1$.

The motivic Adem relation. For the motivic Steenrod operations Sq^{2k} of bidegrees

$(2k,k)$ and Sq^{2k-1} of bidegrees $(2k-1,k-1)$ for $k \geq 1$ and $a < 2b$ the motivic Adem relations are as follows [40, §10].

$$\mathrm{Sq}^a\mathrm{Sq}^b = \sum_c \binom{b-1-c}{a-2c} \tau^? \mathrm{Sq}^{a+b-c}\mathrm{Sq}^c.$$

The symbol ? stands for either 0 or 1, depending on which value makes the formula balanced in weight. It is easy to prove that τ appears when a and b are even and c is odd. By removing τ from the motivic Adem relations, we obtain the classical Adem relation [28].

Example 3.2. We have $\mathrm{Sq}^2\mathrm{Sq}^2 = \tau\mathrm{Sq}^3\mathrm{Sq}^1$ motivically since the weight of $\mathrm{Sq}^3\mathrm{Sq}^1$ is 1 but the weight of $\mathrm{Sq}^2\mathrm{Sq}^2$ is 2. Classically, $\mathrm{Sq}^2\mathrm{Sq}^2 = \mathrm{Sq}^3\mathrm{Sq}^1$.

Remark 3.3. The right hand side of the Adem relation can contain a monomial term. For example, $\mathrm{Sq}^1\mathrm{Sq}^2 = \mathrm{Sq}^3$.

The motivic Steenrod algebra. The motivic Steenrod algebra $\mathbf{A}$ is the ring of stable cohomology operations on mod 2 motivic cohomology. In addition, this ring is generated by the Steenrod operations Sq^i over $\mathbb{M}_2$, subject to the motivic Adem relations. To be precise, we have the following result.

Theorem 3.4. *[40, §11] The motivic Steenrod algebra $\mathbf{A}$ is the $\mathbb{M}_2$-algebra generated by elements Sq^{2k} and Sq^{2k-1} for all $k \geq 1$, of bidegrees $(2k,k)$ and $(2k-1,k-1)$ respectively, and satisfy the motivic Adem relations.*

The dual of the motivic Steenrod algebra. We denote $\mathbf{A}_{*,*} = \mathrm{Hom}_{\mathbb{M}_2}(\mathbf{A}, \mathbb{M}_2)$ for the dual of the motivic Steenrod algebra. The algebra $\mathbf{A}_{*,*}$ has a very nice description which will be very helpful in studying the motivic May spectral sequence.

Theorem 3.5. *[40] The dual $\mathbf{A}_{*,*}$ of the motivic Steenrod algebra is equal to*

$$\frac{\mathbb{M}_2[\tau_0, \tau_1, \ldots, \xi_1, \xi_2, \ldots]}{\tau_i^2 = \tau\xi_{i+1}}$$

in which

1. *the generator* τ_i *has bidegree* $(2^{i+1}-1, 2^i-1)$,

2. *the generator* ξ_i *has bidegree* $(2^{i+1}-2, 2^i-1)$.

3.2 The motivic Adams spectral sequence

Let H be the mod 2 motivic Eilenberg-Mac Lane spectrum, i.e., the motivic spectrum that represents the mod 2 motivic cohomology. We construct an Adams resolution for the motivic sphere spectrum $S^{0,0}$ as follows.

$$\begin{tikzcd} & K_2 & K_1 & K_0 & \\ \cdots \arrow[r, "g_2"] & X_2 \arrow[u, "f_2"] \arrow[r, "g_1"] & X_1 \arrow[u, "f_1"] \arrow[r, "g_0"] & X_0 \arrow[u, "f_0"] \arrow[r, equal] & S^{0,0} \end{tikzcd}$$

where each K_i is a motivic finite type wedge of suspensions of H, $X_i \to K_i$ is surjective on mod 2 motivic cohomology, and X_{i+1} is the homotopy fiber of $X_i \to K_i$. Applying $\pi_* = \pi_{*,u}$, for each u, we have a long exact sequence

$$\pi_*(X_{s+1}) \xrightarrow{\pi_*(g_s)} \pi_*(X_s) \xrightarrow{\pi_*(f_s)} \pi_*(K_s) \xrightarrow{\delta_{s,*}} \pi_*(X_{s+1})$$

We denote $D_1 = \pi_*(X_s)$ and $E_1 = \pi_*(K_s)$. In particular, $D_1^{s,t} = \pi_{t-s}(X_s)$ and $E_1^{s,t} = \pi_{t-s}(K_s)$. We also denote,

$$i_1 = \pi_{t-s}(g_s) : D_1^{s+1,t+1} \longrightarrow D_1^{s,t},$$

$$j_1 = \pi_{t-s}(f_s) : D_1^{s,t} \longrightarrow E_1^{s,t},$$

and

$$k_1 = \delta_{s,t-s} : E_1^{s,t} \longrightarrow D_1^{s+1,t}.$$

We obtain the following exact couple which gives rise to the motivic Adams spectral sequence.

$$\begin{tikzcd}
D_1 \arrow[rr, "i_1"] & & D_1 \arrow[dl, "j_1"] \\
& E_1 \arrow[ul, "k_1"] &
\end{tikzcd}$$

Denote $d = j_1 k_1 : E_1 \longrightarrow E_1$. We have $E_2 = H(E_1, d)$, the homology of E_1 with respect to d. The E_2-term of the motivic Adams spectral sequence is isomorphic to $\mathrm{Ext}_{\mathbf{A}}(\mathbb{M}_2, \mathbb{M}_2)$.

Convergence. The motivic Adams spectral sequence converges to the homotopy groups of the H-nilpotent completion of the motivic sphere spectrum [12]. Classically, the E_∞-page $E_\infty^{s,t}$ of the classical Adams spectral sequence is the subquotient im $\pi_{t-s}(X_s)/\text{im } \pi_{t-s}(X_{s+1})$. After computing E_∞ we have to use other methods such as Toda brackets to solve the extension problems and recover the group $\pi_*(X)$ [18].

CHAPTER 4 THE COHOMOLOGY OF THE MOTIVIC STEENROD ALGEBRA

Adams suggests the study of the cohomology Ext_{cl} of the classical Steenrod algebra which appears to be the E_2-page of the classical Adams spectral sequence [2]. There are at least three different methods to compute Ext_{cl}. Adams computed Ext_{cl} as the homology of the cobar construction. The second method is using the May spectral sequence. The third one is computing the homology of the classical Lambda algebra. If we do it by hand, even with the minimal resolution of the Steenrod algebra, the cobar construction is very large and the computation is very slow and cumbersome. The same story happens to the computation of the homology of the classical Lambda algebra. However, nowadays with the use of modern computers, these computations become much easier.

In this thesis, we study the cohomology $\mathrm{Ext}_{\mathbf{A}}(\mathbb{M}_2, \mathbb{M}_2)$ of the motivic Steenrod algebra serving as the E_2-page of the motivic Adams spectral sequence which is mostly computed by the motivic May spectral sequence.

4.1 The algebra structure and vector space structure of $\mathbf{Ext}_{\mathbf{A}}(\mathbb{M}_2, \mathbb{M}_2)$

First of all, we recall that $\mathbb{M}_2$ has an $\mathbf{A}$-module structure with trivial actions of Sq^i for all $i \geq 1$. To construct $\mathrm{Ext}_{\mathbf{A}}(\mathbb{M}_2, \mathbb{M}_2)$ one begins with a projective resolution

$$\ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathbb{M}_2 \longrightarrow 0,$$

in which P_n is an $\mathbf{A}$-projective module for all $n \geq 0$. Then we remove $\mathbb{M}_2$ and form the cochain complex

$$0 \longrightarrow \mathrm{Hom}_{\mathbf{A}}(P_0, \mathbb{M}_2) \longrightarrow \mathrm{Hom}_{\mathbf{A}}(P_1, \mathbb{M}_2) \longrightarrow \ldots .$$

Finally, $\mathrm{Ext}_{\mathbf{A}}(\mathbb{M}_2, \mathbb{M}_2)$ is the cohomology of this chain complex.

Moreover, the Steenrod algebra $\mathbf{A}$ is an $\mathbb{F}_2$-vector space and we usually choose $P_n =$

$\mathbf{A} \otimes \mathbf{A} \otimes \cdots \otimes \mathbf{A}$ (n factors). In addition, $\mathbb{M}_2$ is an $\mathbb{F}_2$-vector space. It is easy to check that $\mathrm{Hom}_{\mathbf{A}}(P_n, \mathbb{M}_2)$ is an $\mathbb{F}_2$-vector space for all n. As a result, $\mathrm{Ext}_{\mathbf{A}}(\mathbb{M}_2, \mathbb{M}_2)$ is an $\mathbb{F}_2$-vector space. This vector space structure plays a key role in recovering the abelian group structure of $\mathrm{Ext}_{\mathbf{A}}(\mathbb{M}_2, \mathbb{M}_2)$ from the E_∞-page of the May spectral sequence.

4.2 The motivic May spectral sequence

We use the motivic May spectral sequence to compute the cohomology Ext of the $\mathbb{C}$-motivic Steenrod algebra. In addition, we will need some information from the cohomology of the classical Steenrod algebra which has been verified only by machine. We construct the motivic May spectral sequence by filtering the motivic Steenrod algebra. This filtration will induce a filtration on the dual of the Steenrod algebra and the cobar complex. The induced filtration will give rise to the motivic May spectral sequence.

To be more precise, we denote I to be the kernel of the augmentation map $\mathbf{A} \to \mathbb{M}_2$. Because of degree reasons, I is equal to the two-sided ideal of the motivic Steenrod algebra generated by Sq^i for all $i \geq 1$. Let $\mathrm{Gr}_I(A)$ denoted the associated graded algebra

$$\mathbf{A}/I \oplus I/I^2 \oplus I^2/I^3 \oplus \cdots .$$

The associated algebra $\mathrm{Gr}_I(A)$ is trigraded with two gradings from $\mathbf{A}$ and one from the I-adic valuation which is referred as the May filtration.

The motivic May spectral sequence has the form

$$E_2 = \mathrm{Ext}^{s,(a,b,c)}_{\mathrm{Gr}_I(\mathbf{A})}(\mathbb{M}_2, \mathbb{M}_2) \Rightarrow \mathrm{Ext}^{s,(b,c)}_{\mathbf{A}}(\mathbb{M}_2, \mathbb{M}_2).$$

Remark 4.1. The $\mathbf{A}$-module structure on $\mathbb{M}_2$ is trivial since every Sq^i acts by zero for all $i \geq 1$ because of degree reasons. Consequently, we can define $\mathrm{Ext}_{\mathrm{Gr}_I(\mathbf{A})}(\mathbb{M}_2, \mathbb{M}_2)$ and $\mathrm{Ext}_{\mathbf{A}}(\mathbb{M}_2, \mathbb{M}_2)$.

We can also obtain the motivic May spectral sequence by filtering the cobar complex in the first place.

Now we discuss how to study the E_2-term of the motivic May spectral sequence from its classical analog. Let I_{cl} be the ideal of the classical Steenrod algebra $\mathbf{A}_{cl}$ that is generated by Sq^i for all $i \geq 1$.

Proposition 4.2. *[12, Proposition 5.2]*

1. *The trigraded algebras* $\mathrm{Gr}_I(\mathbf{A})$ *and* $\mathrm{Gr}_{I_{cl}}(\mathbf{A}_{cl}) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\tau]$ *are isomorphic.*

2. *The quadruply-graded rings* $\mathrm{Ext}_{\mathrm{Gr}_{\mathrm{I}}(\mathbf{A})}(\mathbb{M}_2, \mathbb{M}_2)$ *and* $\mathrm{Ext}_{\mathrm{Gr}_{\mathrm{I_{cl}}}(\mathbf{A}_{\mathrm{cl}})}(\mathbb{F}_2, \mathbb{F}_2) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\tau]$ *are isomorphic.*

The classical E_2-term is studied in great detail by J.P. May [24]. It can be computed as the homology of the differential graded algebra

$$\mathbb{F}_2[h_{ij} | i > 0, j \geq 0],$$

with differential given by

$$d(h_{ij}) = \sum_{0<k<i} h_{kj} h_{i-k,k+j}.$$

By Proposition 4.2, the motivic E_2-term is the homology of the differential graded algebra

$$\mathbb{F}_2[\tau, h_{ij} | i > 0, j \geq 0]$$

where

1. τ has degree $(0, 0, 0, -1)$.

2. h_{i0} has degree $(i, 2^i - 2, 1, 2^{i-1} - 1)$.

3. h_{ij} has degree $(i, 2^j(2^i - 1) - 1, 1, 2^{j-1}(2^i - 1))$ if $j > 0$,

with differential given by

$$d(h_{ij}) = \sum_{0<k<i} h_{kj} h_{i-k,k+j}.$$

There is no τ in the above formula since the weights on the both sides are balanced. The motivic May differential is given by the same formula as the classical one together with $d(\tau) = 0$. However, the motivic and classical differentials are different in some cases. This happens because of the motivic weight. For example, classically we have $d_2(h_{20}^2) = h_{11}^3 + h_{10}^2 h_{12}$. Motivically, the element h_{11}^3 has weight 3 whereas h_{20}^2 and $h_{10}^2 h_{12}$ both have weight 2. As a result, $d_2(h_{20}^2) = \tau h_{11}^3 + h_{10}^2 h_{12}$ motivically. It leads to a very important consequence. Classically, $d_2(h_{11} h_{20}^2) = h_1^4$. Then h_1 is nilpotent in $\mathrm{Ext}_{cl}(\mathbb{F}_2, \mathbb{F}_2)$. However, motivically $d_2(h_{11} h_{20}^2) = \tau h_{11}^4$. Actually, h_1^n is not hit by any differentials in the motivic May spectral sequence for all n. In other words, it is not nilpotent in Ext.

Table 1: Some differences between the classical and motivic differentials.

Classical	Motivic
$d_2(b_{20}) = h_{11}^3 + h_{10}^2 h_{12}$	$d_2(b_{20}) = \tau h_{11}^3 + h_{10}^2 h_{12}$
$d_2(h_{11} b_{20}) = h_{11}^4$	$d_2(h_{11} b_{20}) = \tau h_{11}^4$

4.2.1 $\mathbb{F}_2$-vector space structure

By Theorem 2.11 the E_∞-page of the May spectral sequence preserves completely the $\mathbb{F}_2$-vector space structure of Ext. In other words, Ext$\cong E_\infty$ as $\mathbb{F}_2$-vector spaces.

4.2.2 Ring structure and $\mathbb{M}_2$-module

By ring structure we mean the product structure and by $\mathbb{M}_2$-module structure we mean the multiplication with τ. The May spectral sequence hides some of the product structure and $\mathbb{M}_2$-module structure in the sense that there exist elements x and y in Ext such that xy is non-zero in Ext but $\overline{xy} = 0$ in E_∞ or τx is non-zero in Ext but $\tau \overline{x}$ is zero in E_∞. In [18] Isaksen studied all possible hidden extensions by τ, h_0, h_1 and h_2. There are several tools but the four main are

1. Comparing with classical hidden extensions,

2. Shuffle relations with Massey products,

3. Steenrod operations,

4. The isomorphism at Chow degree zero.

We will discuss several examples using the above methods. In [7], Bruner uses a computer to compute the cohomology of the classical Steenrod algebra by constructing a minimal resolution. By looking at Bruner's computation, we can figure out a lot of classical hidden extensions which can be used to obtain motivic hidden extensions.

Example 4.3. Classically we have the hidden extension $h_0 \cdot e_0 g = h_0^4 x$ in Ext_{cl}. Motivically, we have $\tau^2 \cdot h_o e_0 g = h_0^4 x$ in Ext.

Example 4.4. We have $h_0 \cdot h_2^2 g = h_1^3 h_4 c_0$ in Ext.

Proof. We use the shuffle

$$h_1^3 h_4 \langle h_1, h_0, h_2^2 \rangle = \langle h_1^3 h_4, h_1, h_0 \rangle h_2^2$$

and the Massey products $\langle h_1, h_0, h_2^2 \rangle = c_0$ and $\langle h_1^3 h_4, h_1, h_0 \rangle = h_0 g$. □

4.3 The cohomology Ext

Relationship between motivic and classical calculations. The following theorem plays a key role in comparing the motivic and the classical computations, saying that they become the same after inverting τ.

Theorem 4.5. *[12, Proposition 3.5] There is an isomorphism of rings*

$$\mathrm{Ext} \otimes_{\mathbb{M}_2} \mathbb{M}_2[\tau^{-1}] \cong \mathrm{Ext}_{\mathrm{cl}} \otimes_{\mathbb{F}_2} \mathbb{F}_2[\tau, \tau^{-1}].$$

Roughly speaking, the elements x in Ext which are killed by some power of τ, i.e. $\tau^n x = 0$ for some n, will map to zero via the above isomorphism.

The motivic Adams line of slope $\frac{1}{2}$. Classically, Adams shows that Ext_{cl} vanishes above a certain line of slope $\frac{1}{2}$, called the Adams line, with the exception of the elements h_0^k in the 0-stem. Motivically, by inspection of the Adams chart [16] we see that Ext does not vanish above the same line of slope $\frac{1}{2}$. However, Ext vanishes above a line of slope 1. Further inspection shows that in a large range, all elements above the Adams line of slope $\frac{1}{2}$ are h_1-local, in the sense that they are h_1-divisible and support infinitely many multiplications by h_1.

Theorem 4.6. *[15, Theorem 1.1] Let $s > 0$, the map*

$$h_1 : \mathrm{Ext}_{\mathbf{A}}^{s,f,w}(\mathbb{M}_2, \mathbb{M}_2) \longrightarrow \mathrm{Ext}_{\mathbf{A}}^{s+1,f+1,w+1}(\mathbb{M}_2, \mathbb{M}_2)$$

is an isomorphism if $f \geq \frac{1}{2}s + 2$, and it is a surjection if $f \geq \frac{1}{2}s + \frac{1}{2}$.

4.4 The cohomology $\mathrm{Ext}_{\mathbf{A}(2)}(\mathbb{M}_2, \mathbb{M}_2)$

The cohomology $\mathrm{Ext}_{\mathbf{A}(2)}(\mathbf{M}_2, \mathbf{M}_2)$ is a finitely generated algebra which is understood fully by [17]. It plays as a “lighthouse" in our study of the cohomology Ext of the motivic Steenrod algebra. Specifically, we can study an element in Ext via its image in $\mathrm{Ext}_{\mathbf{A}(2)}(\mathbf{M}_2, \mathbf{M}_2)$.

The subalgebra $\mathbf{A}(2)$. We recall that the bigraded $\mathbb{M}_2$ is the polynimial ring $\mathbf{F}_2[\tau]$ on one generator τ having bidegree $(0, 1)$.

Definition 4.7. The algebra $\mathbf{A}(2)$ is the $\mathbb{M}_2$-subalgebra of the motivic Steenrod algebra $\mathbf{A}$ generated by $\mathrm{Sq}^1, \mathrm{Sq}^2$ and Sq^4.

Comparison to $\mathrm{Ext}_{\mathbf{A}(2)}$. The inclusion $\mathbf{A}(2) \hookrightarrow \mathbf{A}$ induces a homomorphism ϕ : Ext $\to \mathrm{Ext}_{\mathbf{A}(2)}$ which allows us to detect some structure in Ext via $\mathrm{Ext}_{\mathbf{A}(2)}$. We emphasize that

$\mathrm{Ext}_{\mathbf{A}(2)}$ is described completely in [17, Theorem 4.13]. Table 2 gives some values of ϕ that we will need.

Table 2: Some values of the map $\phi : \mathrm{Ext} \longrightarrow \mathrm{Ext}_{\mathbf{A}(2)}$.

Ext	$\mathrm{Ext}_{\mathbf{A}(2)}$	(s, f, w)
i	Pn	$(23, 7, 12)$
k	dn	$(29, 7, 16)$
r	n^2	$(30, 6, 16)$
m	ng	$(35, 7, 20)$
$\Delta h_1 d_0$	$\Delta h_1 \cdot d_0$	$(39, 9, 21)$
τg^2	$\tau \cdot g^2$	$(40, 8, 23)$
$\tau \Delta h_1 g$	$\tau \cdot \Delta h_1 \cdot g$	$(45, 9, 24)$
$h_2 g^j$	$h_2 \cdot g^j$	$(20j + 3, 4j + 1, 12j + 2)$
$P^i d_0$	$P^i \cdot d_0$	$(8i + 14, 4i + 4, 4i + 8)$
$P^i e_0$	$P^i \cdot e_0$	$(8i + 17, 4i + 4, 4i + 10)$

Remark 4.8. In some cases, $\phi(x)$ is decomposable in $\mathrm{Ext}_{\mathbf{A}(2)}$ when x is indecomposable in Ext. For example, the element $\Delta h_1 d_0$ in Ext is indecomposable but $\phi(\Delta h_1 d_0) = \Delta h_1 \cdot d_0$ is the product of Δh_1 and d_0 in $\mathrm{Ext}_{\mathbf{A}(2)}$.

Remark 4.9. We know values of ϕ by comparing the May spectral sequence computing Ext and the May spectral sequence computing $\mathrm{Ext}_{\mathbf{A}(2)}$.

4.5 Comparison to the h_1-localization of Ext

If we invert h_1 on Ext, then $\mathrm{Ext}[h_1^{-1}]$ becomes simpler. We can use $\mathrm{Ext}[h_1^{-1}]$ to detect some structure in Ext. The following theorems describe $\mathrm{Ext}[h_1^{-1}]$ and $\mathrm{Ext}_{\mathbf{A}(2)}[h_1^{-1}]$.

Definition 4.10 (Direct Limit)**.** Let $\langle I, \leq \rangle$ be a directed set. Let $\{A_i : i \in I\}$ be a family of modules indexed by I and $f_{i,j} : A_i \to A_j$ be a homomorphism for all $i \leq j$ with the following properties:

1. $f_{i,i}$ is the identity of A_i, and

2. $f_{i,k} = f_{j,k} \circ f_{i,j}$ for all $i \leq j \leq k$.

The pair $\langle A_i, f_{i,j} \rangle$ is called a direct system over I.

The direct limit $\varinjlim A_i$ of the direct system $\langle A_i, f_{i,j} \rangle$ is the unique up to isomorphism module L satisfying the following universal mapping property: there are maps $f_i : A_i \to L$ such that $f_i = f_j \circ f_{i,j}$ for every pair $i \leq j$, and if there is a module C together with maps $\tau_i : A_i \to C$ such that $\tau_i = \tau_j \circ f_{i,j}$ for every pair $i \leq j$, then there is a unique module homomorphism $\tau : L \to C$ with $\tau_i = \tau \circ f_i$.

$$\begin{tikzcd} A_i \arrow[r, "\tau_i"] \arrow[d, "f_i"] & C \\ L \arrow[ur, "\tau"] & \end{tikzcd}$$

The construction of the direct limit is as follows. Let M be the direct sum of the A_i, and let N be the submodule of M generated by all elements of the form $a - f_{ij}(a)$ for all $i \leq j$ and all $a \in A_i$. Then M/N, together with f_i the compositions of the natural maps $A_i \to M \to M/N$, satisfy the mapping property for the direct limit.

Lemma 4.11. *Let $\varinjlim A_i$ be the direct limit of a directed system of modules. Then for any element x of $\varinjlim A_i$, there exists an element a in some A_i such that $x = f_i(a)$.*

Proof. It is straight forward from the construction of the direct limit. $\square$

Definition 4.12. The h_1-localization $\mathrm{Ext}[h_1^{-1}]$ of Ext is the direct limit of the sequence

$$\mathrm{Ext} \xrightarrow{h_1} \mathrm{Ext} \xrightarrow{h_1} \mathrm{Ext} \xrightarrow{h_1} \dots$$

in which the map h_1 is multiplication with h_1.

Roughly speaking, if two elements x and y in Ext satisfy $h_1^n x = h_1^m y$ for some n and m then x and y become identical in $\mathrm{Ext}[h_1^{-1}]$. The most important observation is that all elements x which are killed by some power of h_1 in Ext, i.e. $h_1^n x = 0$, become zero in $\mathrm{Ext}[h_1^{-1}]$.

Theorem 4.13. *[14, Theorem 1.1] The h_1-localization* $\mathrm{Ext}[h_1^{-1}]$ *is a polynomial algebra over* $\mathbb{F}_2[h_1^{\pm 1}]$ *on generators* v_1^4 *and* v_n *for* $n \geq 2$, *where:*

- *the element* v_1^4 *has degree* $(8, 4, 4)$.
- *the element* v_n *has degree* $(2^{n+1} - 2, 1, 2^n - 1)$.

Theorem 4.14. *[14, Proposition 3.7] The h_1-localization* $\mathrm{Ext}_{\mathbf{A}(2)}[h_1^{-1}]$ *is a polynomial algebra*

$$\mathrm{Ext}_{\mathbf{A}(2)}[h_1^{-1}] \cong \mathbb{F}_2[h_1^{\pm 1}, a_1, v_1^4, v_2]$$

in which a_1 *has degree* $(11, 3, 7)$; v_1^4 *has degree* $(8, 4, 4)$; *and* v_2 *has degree* $(6, 1, 3)$.

We can use h_1-localization to prove the non-existence of certain elements x in Ext. Guillou and Isaksen [14, §5] used the May spectral sequence analysis of $\mathrm{Ext}[h_1^{-1}]$ to determine the localization map

$$L : \mathrm{Ext} \longrightarrow \mathrm{Ext}[h_1^{-1}]$$

in a range. Some values of L are given in Table 3 [14, Table 13].

Table 3: Some values of the localization map $L : \mathrm{Ext} \to \mathrm{Ext}[h_1^{-1}]$

x	$L(x)$
$P^k h_1$	$h_1 v_1^{4k}$
$P^k d_0$	$h_1^2 v_1^{4k} v_2^2$
$P^k e_0$	$h_1^3 v_1^{4k} v_3$
$e_0 g$	$h_1^7 v_4$

There is also a localization map $L : \mathrm{Ext}_{\mathbf{A}(2)} \longrightarrow \mathrm{Ext}_{\mathbf{A}(2)}[h_1^{-1}]$ [14, §5.1]. The following diagram is commutative [14].

$$\begin{tikzcd}
\mathrm{Ext} \arrow[r, "\phi"] \arrow[d, "L"] & \mathrm{Ext}_{\mathbf{A}(2)} \arrow[d, "L"] \\
\mathrm{Ext}[h_1^{-1}] \arrow[r, "\phi"] & \mathrm{Ext}_{\mathbf{A}(2)}[h_1^{-1}]
\end{tikzcd}$$

Definition 4.15. Let t be a non-negative integer. We define $\alpha(t)$ to be the number of 1's in the binary expansion of t.

Lemma 4.16. *Let t, k and s be non-negative integers, $s \geq 1$. We have*

- $\alpha(t) \leq t$.
- $\alpha(t+k) \leq \alpha(t) + \alpha(k)$.
- $\alpha(2^s t) = \alpha(t)$.

Proof. Suppose that $t = \sum_{i=1}^{n} 2^{m_i}$ in which $m_i \geq 0$ and $m_i \neq m_j$ if $i \neq j$. Consequently, $\alpha(t) = n$. Since $t = \sum_{i=1}^{n} 2^{m_i} \geq 1 \cdot n = n$, we obtain the first inequality.

With the above t we suppose further that $k = \sum_{i=1}^{p} 2^{q_i}$ in which $q_i \geq 0$ and $q_i \neq q_j$ if $i \neq j$. Consequently, $\alpha(k) = p$. We have

$$t + k = \sum_{i=1}^{n} 2^{m_i} + \sum_{j=1}^{p} 2^{q_j}$$

where the right hand side has $n+p$ powers of 2. If there is no pair (m_i, q_j) such that $m_i = q_j$, then $\alpha(t+k) = n + p = \alpha(t) + \alpha(k)$. If there exists at least one pair (m_i, q_j) such that $m_i = q_j = c$, since $2^{m_i} + 2^{q_j} = 2^{c+1}$ we have $\alpha(t+k) < n + p = \alpha(t) + \alpha(k)$. Therefore, $\alpha(t+k) \leq \alpha(t) + \alpha(k)$.

The last identity can be proven by the observation that if $t = \sum_{i=1}^{n} 2^{m_i}$, then $2^s t = \sum_{i=1}^{n} 2^{m_i+s}$. $\square$

Lemma 4.17. *The map $\phi : \mathrm{Ext}[h_1^{-1}] \longrightarrow \mathrm{Ext}_{\mathbf{A}(2)}[h_1^{-1}]$ takes v_1^4 to v_1^4 and for all $n \geq 2$, ϕ maps v_n to $h_1^{-3(2^{n-2}-1)} a_1^{2^{n-2}-1} v_2$.*

Proof. This statement is stated as Conjecture 5.5 in [14] by Guillou and Isaksen. They also prove that if a "$\mathbb{C}$-motivic modular forms" spectrum exists, then the conjecture holds [14, Proposition 6.4]. This spectrum has recently been constructed by Gheorghe, Isaksen, Krause and Ricka [13, §5], so we obtain the desired statement. □

Lemma 4.18. *The image of*

$$\phi : \mathrm{Ext}[h_1^{-1}] \longrightarrow \mathrm{Ext}_{\mathbf{A}(2)}[h_1^{-1}]$$

is spanned by the monomials $h_1^d v_1^{4a} v_2^b a_1^c$ where a, b and c are non-negative integers for which $\alpha(b+c) \leq b$ and d is an integer.

Proof. Denote by $\mathcal{G}$ the $\mathbb{M}_2$-submodule of $\mathrm{Ext}_{\mathbf{A}(2)}[h_1^{-1}]$ spanned by the monomials $h_1^d v_1^{4a} v_2^b a_1^c$ where a, b and c are non-negative integers for which $\alpha(b+c) \leq b$ and d is an integer.

Using Lemma 4.17 to get

$$\phi : v_1^{4a} \prod_{j \in J} v_j \longmapsto h_1^{-3\sum_{j \in J}(2^{j-2}-1)} v_1^{4a} v_2^m a_1^{\sum_{j \in J}(2^{j-2}-1)}$$

in which J is a sequence $(j_1, \ldots, j_m)$ of length m such that $j_k \geq 2$ (repeats are allowed).

Consequently, the image of ϕ equals the $\mathbb{M}_2$-submodule $\mathcal{H}$ of $\mathrm{Ext}_{\mathbf{A}(2)}[h_1^{-1}]$ spanned by the monomials of the form $h_1^d v_1^{4a} v_2^m a_1^{\sum_{j \in J}(2^{j-2}-1)}$ in which J is a sequence $(j_1, \ldots, j_m)$ of length m such that $j_k \geq 2$ (repeats are allowed).

Since J has length m,

$$\alpha(m + \sum_{j \in J}(2^{j-2} - 1)) = \alpha(\sum_{j \in J} 2^{j-2}) \leq m.$$

As a result, $\mathcal{H}$ is contained in $\mathcal{G}$.

Conversely, for any monomial $h_1^d v_1^{4a} v_2^b a_1^c$ for which $\alpha(b+c) \leq b$, we can suppose that $b + c = \sum_{j \in J} 2^j$ where J is a sequence $(j_1, \ldots, j_r)$ of length $r \leq b$ such that $j_k \geq 0$ for k in $\{1, \ldots, r\}$. By replacing 2^j by $2^{j-1} + 2^{j-1}$ as necessary, we can rewrite $b + c$ as

$$b + c = \sum_{i \in I} 2^i$$

where I is a sequence $(i_1, \ldots, i_b)$ of length b such that $i_k \geq 0$ for k in $\{1, \ldots, b\}$. Then

$$c = \sum_{i \in I} 2^i - b = \sum_{i \in I} (2^i - 1).$$

This shows that $\mathcal{G}$ is contained in $\mathcal{H}$. $\square$

4.6 Comparison via Chow degree zero isomorphism

We remind that the Chow degree of an element of degree (s, f, w) is $s + f - 2w$. The part of Ext at Chow degree 0 is isomorphic to $\mathrm{Ext}_{\mathrm{cl}}$.

Theorem 4.19. *[18, Theorem 2.1.12] There is an isomorphism from* $\mathrm{Ext}_{\mathrm{cl}}$ *to the subalgebra of* Ext *consisting of elements in degrees* (s, f, w) *such that* $s + f - 2w = 0$*. This isomorphism takes classical elements of degrees* (s, f) *to motivic elements of degrees* $(2s+f, f, s+f)$*, and it preserves all higher structure including products, squaring operations, and Massey products.*

In other words,

$$\mathrm{Ext}|_{s+f-2w=0} \cong \mathrm{Ext}_{\mathrm{cl}}.$$

Some elements of Ext at Chow degree zero and their corresponding elements in $\mathrm{Ext}_{\mathrm{cl}}$ via the Chow degree zero isomorphism are shown in the following table.

Table 4: Some elements of Ext and their corresponding elements via the Chow degree zero isomorphism.

Ext	$\mathrm{Ext}_{\mathrm{cl}}$
h_1	h_0
h_2	h_1
h_n	h_{n-1}
$h_2 g^j$	$P^j h_1$

We will show in the next chapter that the element $h_2 g^j$ in Ext corresponds to the element $h_1 P^j$ in Ext_{cl} via the Chow isomorphism.

CHAPTER 5 THE MOTIVIC WEDGE FAMILY

The classical wedge family was studied by M. Mahowald and M. Tangora [21]. It is a subset of the cohomology $\mathrm{Ext}_{\mathbf{A}_{\mathrm{cl}}}(\mathbb{F}_2, \mathbb{F}_2)$ of the classical Steenrod algebra, consisting of non-zero elements $P^i g^j \lambda$ and $g^j t$ in which λ is in $\mathbf{\Lambda}$, t is in $\mathbf{T}$, $i \geq 0$ and $j \geq 0$. The sets $\mathbf{\Lambda}$ and $\mathbf{T}$ are specific subsets of $\mathrm{Ext}_{\mathbf{A}_{\mathrm{cl}}}(\mathbb{F}_2, \mathbb{F}_2)$. The wedge family gives an infinite wedge-shaped diagram inside the cohomology of the classical Steenrod algebra, which fills out an angle with vertex at g^2 in degree (40,8) (i.e. g^2 has stem 40 and Adams filtration 8), bounded above by the line $f = \frac{1}{2}s - 12$, parallel to the Adams edge [1], and bounded below by the line $s = 5f$, in which f is the Adams filtration and s is the stem. The wedge family is a large piece of $\mathrm{Ext}_{\mathbf{A}_{\mathrm{cl}}}(\mathbb{F}_2, \mathbb{F}_2)$ which is regular, of considerable size and easy to understand. Using this idea we build the motivic version of the wedge.

Recall that the inclusion $\mathbf{A}(2) \hookrightarrow \mathbf{A}$ induces a homomorphism of algebras $\phi : \mathrm{Ext} \to \mathrm{Ext}_{\mathbf{A}(2)}$.

Definition 5.1. For any λ in Ext, $i \geq 0$ and $j \geq 0$, $\mathbf{P}^i \mathbf{g}^j \lambda$ is the set which consists of all elements x in $\mathrm{Ext}^{s,f,w}$ such that $\phi(x) = P^i g^j \phi(\lambda)$ having degree (s, f, w) in $\mathrm{Ext}_{\mathbf{A}(2)}$.

Example 5.2. The set $\mathbf{g}^2 r$ contains m^2 because $\phi(m^2) = g^2 n^2 = g^2 \phi(r)$.

Remark 5.3. We differentiate $\mathbf{P}$ and $\mathbf{g}$ with P and g. By the bold $\mathbf{P}$ and $\mathbf{g}$ we mean set-valued operations from Ext to Ext. Remember that $\mathbf{P}$ and $\mathbf{g}$ do not exist in Ext as elements. By P and g, we mean elements in $\mathrm{Ext}_{\mathbf{A}(2)}$.

Remark 5.4. We sometimes write the symbols $\mathbf{P}$ and $\mathbf{g}$ in a different order for consistency with standard notation. For example:

- By $e_0 \mathbf{g}^2$ we mean $\mathbf{g}^2(e_0)$. The set $e_0 \mathbf{g}^2$ is empty (See Corollary 5.30).
- By $\tau \Delta h_1 \mathbf{g}^{j+1}$ we mean the set $\mathbf{g}^j(\tau \Delta h_1 g)$.
- By $\tau \mathbf{P}^i \mathbf{g}^{j+1}$ we mean $\mathbf{P}^i \mathbf{g}^{j+1}(\tau)$.

- The same convention is applied for $\tau\mathbf{g}^k$, $\tau e_0^t\mathbf{g}^k$ and many others.

Remark 5.5. From Definition 5.1 we have $\mathbf{P}^i\mathbf{g}^j x \cdot \mathbf{P}^a\mathbf{g}^b y \subseteq \mathbf{P}^{i+a}\mathbf{g}^{j+b}xy$. However, the inverse inclusion is not correct generally. For example, by low dimension calculation [18] we have

$$e_0 \cdot \tau^2\mathbf{g} = \{e_0\}\{\tau^2 g\} \subsetneq \tau^2 e_0\mathbf{g} = \{\tau^2 e_0 g, \tau^2 e_0 g + h_0^3 x\}.$$

Definition 5.6. We define $\mathbf{\Lambda}$ to be the following sixteen elements of Ext.

Table 5: Sixteen elements of the set $\mathbf{\Lambda}$

element	(s,f,w)	element	(s,f,w)
τg^2	(40,8,23)	$d_0 r$	(44,10,24)
$\tau\Delta h_1 g$	(45,9,24)	$d_0 m$	(49,11,28)
gr	(50,10,28)	$\tau e_0^2 g$	(54,12,31)
gm	(55,11,32)	$\tau\Delta h_1 e_0^2$	(59,13,32)
$\tau\Delta h_1 e_0$	(42,9,22)	$d_0 l$	(46,11,26)
$e_0 r$	(47,10,26)	τe_0^3	(51,12,29)
$e_0 m$	(52,11,30)	$\tau\Delta h_1 d_0 e_0$	(56,13,30)
$\tau e_0 g^2$	(57,12,33)	$d_0 e_0 r$	(61,14,34)

Remark 5.7. The elements in the set $\mathbf{\Lambda}$ are not optimal in the sense that there may exist elements of weight greater than the weight of elements in $\mathbf{\Lambda}$ (see Table 6). For example, the element τe_0^3 in $\mathbf{\Lambda}$ has weight 29 but the element e_0^3 in Ext has weight 30. The reason for this choice is that our proof for Theorem 5.8 works for τe_0^3 but does not work for e_0^3. The same story happens to $\tau\Delta h_1 e_0$, $\tau e_0^2 g$, $\tau\Delta h_1 e_0$ and $\tau\Delta h_1 d_0 e_0$.

The following theorem is our main result.

Theorem 5.8. *For any λ in $\mathbf{\Lambda}$, $i \geq 0$, $j \geq 0$ and $k \geq 0$, the set $\tau^k\mathbf{P}^i\mathbf{g}^j\lambda$ is non-empty and consists of non-zero elements.*

Combining all elements of $\tau^k\mathbf{P}^i\mathbf{g}^j\lambda$ with $i \geq 0$, $j \geq 0$, $k \geq 0$ and λ in $\mathbf{\Lambda}$, we obtain an infinite wedge-shaped diagram, filling out the angle with vertex at τg^2 in degree $(40,8,23)$, bounded above by the line $f = \frac{1}{2}s - 12$ parallel to the Adams edge [15], and bounded below by the line $s = 5f$ in Ext (Figure 1). We call this set the wedge. To be precise, we have the following definition.

Definition 5.9. For $i \geq 0, j \geq 0, k \geq 0$, and $\lambda \in \mathbf{\Lambda}$ the set

$$\{x \in \text{Ext} : x \in \tau^k \mathbf{P}^i \mathbf{g}^j \lambda\}$$

is called the wedge family of the cohomology of the motivic Steenrod algebra.

We need a couple of preliminary results before proving Theorem 5.8.

Lemma 5.10. *The sets $\mathbf{P}^i d_0$, $\mathbf{P}^i e_0$ and $\mathbf{P}^i \Delta h_1 e_0$ are non-empty for $i \geq 0$.*

Proof. Since d_0, e_0 and $\Delta h_1 e_0$ are generators in Ext, the statement is trivial when $i = 0$.

We now consider the case $i > 0$. The Adams periodicity operator P^i is an isomorphism on Ext in specified ranges [20, Theorem 1.4]. Since the element d_0 lies in these ranges, then $\mathbf{P}^i d_0$ contains the element $P^i d_0$. Therefore, the set $\mathbf{P}^i d_0$ is non-empty. The same argument is applied for $\mathbf{P}^i e_0$ and $\mathbf{P}^i \Delta h_1 e_0$. $\square$

Lemma 5.11. *Let x be an element in* Ext *such that $h_1^3 \phi(x) = 0$. Then $\mathbf{P}^{i+1}\mathbf{g}x$ contains the non-empty set $\mathbf{P}^i d_0^2 x$ for all $i \geq 0$. As a result, $\mathbf{P}^{i+1}\mathbf{g}x$ is non-empty.*

Proof. Since $\mathbf{P}^i d_0$ is non-empty by Lemma 5.10, the set $\mathbf{P}^i d_0^2 x$ is non-empty. Consider an element β in $\mathbf{P}^i d_0^2 x$. Since $\phi(d_0) = d$ and $d^2 = Pg + h_1^3 \cdot \Delta h_1$ in $\text{Ext}_{\mathbf{A}(2)}$ [17, Table 8], we have

$$\begin{aligned} \phi(\beta) = P^i d^2 \phi(x) &= P^i(Pg + h_1^3 \cdot \Delta h_1)\phi(x) \\ &= P^{i+1} g\phi(x) + P^i \Delta h_1 \cdot h_1^3 \phi(x) = P^{i+1} g \phi(x) \end{aligned}$$

Consequently, $\mathbf{P}^{i+1}\mathbf{g}x$ contains the element β of $\mathbf{P}^i d_0^2 x$. $\square$

Lemma 5.12. *Consider $j \geq 2$ and suppose that $j = 2^r(2k+1)$ for $r \geq 0$ and $k \geq 0$. We have the differential: $d_{2^{r+2}}(P^j) = h_0^5 x_j$ for some x_j in the classical May spectral sequence.*

Proof. Since $j \geq 2$, we do not consider $r = 0$ and $k = 0$. When $r = 0$ and $k \geq 1$ we have

$$\begin{aligned} d_4(P^{2k+1}) &= d_4(P) \cdot P^{2k} + d_4(P^{2k}) \cdot P \\ &= d_4(P) \cdot P^{2k} \\ &= h_0^4 h_3 \cdot P^{2k}. \end{aligned}$$

In the E_4 page of the classical May spectral sequence we have

$$P^2 \cdot h_3 = h_0^2 i + \tau P h_1 d_0.$$

Then

$$\begin{aligned} h_0^4 h_3 \cdot P^{2k} &= h_0^4 \cdot P^{2k-2} \cdot P^2 h_3 \\ &= h_0^4 \cdot P^{2k-2} \cdot (h_0^2 i + \tau P h_1 d_0) \\ &= h_0^5 \cdot h_0 P^{2k-2} i. \end{aligned}$$

When $r \geq 1$ we have

$$\begin{aligned} d_{2^{r+2}}(P^{2^r(2k+1)}) &= d_{2^{r+2}}(P^{2^r(2k)} \cdot P^{2^r}) \\ &= d_{2^{r+2}}(P^{2^r}) \cdot P^{2^r(2k)} + d_{2^{r+2}}(P^{2^r(2k)}) \cdot P^{2^r} \\ &= d_{2^{r+2}}(P^{2^r}) \cdot P^{2^{r+1}k}. \end{aligned}$$

Using Nakamura's formula [29, §4 page 14] we get

$$\begin{aligned} d_{2^{r+2}}(P^{2^r}) \cdot P^{2^{r+1}k} &= h_0^{2^{r+2}} h_{r+3} \cdot P^{2^{r+1}k} \\ &= h_0^5 \cdot h_0^{2^{r+2}-5} h_{r+3} P^{2^{r+1}k}. \end{aligned}$$

□

We denote by $\tilde{x}_j$ the motivic element of Chow degree zero corresponding to x_j (defined in Lemma 5.12) via the Chow degree zero isomorphism in Theorem 4.19.

Remark 5.13. In the proof of Lemma 5.12, we actually show that $d_{2^{r+2}}(P^j) = h_0^6 y_j$ for some y_j in the classical May spectral sequence. However, in order to prepare for Lemma 5.14, we prefer the statement stated in Lemma 5.12.

Lemma 5.14. *In* Ext*, for* $j \geq 2$*, the Massey product* $\langle h_2, h_1, h_1^4 \tilde{x}_j \rangle$ *equals* $h_2 g^j$*.*

Proof. The motivic elements $h_2 g^j, h_2, h_1$ and $h_1^4 \tilde{x}_j$ all have Chow degree zero. They correspond to classical elements $P^j h_1$, h_1, h_0 and $h_0^4 x_j$ via the Chow degree zero isomorphism in Theorem 4.19.

Classically we have $P^j h_1 = \langle h_1, h_0, h_0^4 x_j \rangle$. We obtain the desired identity by the Chow degree zero isomorphism. □

Remark 5.15. The g in $h_2 g^j$ in the above argument is not the operator $\mathbf{g}$. We write $h_2 g^j$ for the element of Ext which corresponds to the classical element $P^j h_1$ via the Chow degree zero isomorphism in Theorem 4.19.

Lemma 5.16. *The sets* $\mathbf{g}^j m$*,* $\mathbf{g}^j l$ *and* $\mathbf{g}^j r$ *are non-empty for all* $j \geq 0$*.*

Proof. For $j = 0$ the set $\mathbf{g}^j m$ contains m, $\mathbf{g}^j l$ contains l and $\mathbf{g}^j r$ contains r. For $j = 1$ the set $\mathbf{g}^j m$ contains gm, $\mathbf{g}^j l$ contains $e_0 m$ and $\mathbf{g}^j r$ contains gr. For $j \geq 2$ we have

$$h_2 \langle h_1, h_1^4 \tilde{x}_j, m \rangle = \langle h_2, h_1, h_1^4 \tilde{x}_j \rangle \cdot m = h_2 g^j \cdot m$$

in which the last identity is by Lemma 5.14. Consider an element β in $\langle h_1, h_1^4 \tilde{x}_j, m \rangle$. We apply ϕ to get

$$h_2 \phi(\beta) = h_2 g^j \phi(m) = h_2 n g^{j+1}.$$

By inspection of $\mathrm{Ext}_{\mathbf{A(2)}}$, we have

$$\phi(\beta) = n g^{j+1}.$$

Therefore $\mathbf{g}^j m$ contains β, and is non-empty.

The same argument is applied to $\mathbf{g}^j l$ and $\mathbf{g}^j r$. □

Lemma 5.17. *The set* $\tau \Delta h_1 \mathbf{g}^{j+1}$ *is non-empty for all* $j \geq 0$*.*

Proof. The set $\tau\Delta h_1\mathbf{g}$ is non-empty since it contains $\tau\Delta h_1 g$.

When $j \geq 1$, the set $\tau\Delta h_1\mathbf{g}^{j+1} \supseteq r \cdot \mathbf{g}^{j-1}m$ is non-empty since $\mathbf{g}^j m$ is non-empty by Lemma 5.16. Here we are using the identity $\tau\Delta h_1 g \cdot g = \phi(r)\cdot\phi(m)$ in $\mathrm{Ext}_{\mathbf{A}(2)}$. □

Lemma 5.18. *The set $\tau\mathbf{g}^j$ is non-empty for any $j \geq 0$.*

Proof. The claim for $j = 0$ and $j = 1$ is proven by explicit low dimension calculation [18] [12]. By Lemma 5.14

$$\langle \tau, h_1^4\tilde{x}_j, h_1\rangle h_2 = \tau\langle h_1^4\tilde{x}_j, h_1, h_2\rangle = \tau h_2 g^j.$$

Consider an element γ in $\langle \tau, h_1^4\tilde{x}_j, h_1\rangle$. We apply ϕ to get

$$\phi(\gamma)h_2 = \tau h_2 g^j.$$

By inspection of $\mathrm{Ext}_{\mathbf{A}(2)}$ [17, Theorem 4.13],

$$\phi(\gamma) = \tau g^j.$$

Therefore $\tau\mathbf{g}^j$ contains γ, and is non-empty. □

Lemma 5.19. *The set $\tau\mathbf{P}^i\mathbf{g}^{j+1}$ is non-empty for $i \geq 0$ and $j \geq 0$.*

Proof. The case $i = 0$ is established in Lemma 5.18. Now we assume $i > 0$. When $j = 0$, by Lemma 5.10 the set $\mathbf{P}^{i-1}d_0$ is non-empty. Consequently, $\tau\mathbf{P}^{i-1}d_0^2$ is non-empty. We consider an element x in $\tau\mathbf{P}^{i-1}d_0^2$. The set $\mathbf{P}^i(\tau g)$ contains x because

$$\phi(x) = \tau P^{i-1}d_0^2 = \tau P^{i-1}(Pg + h_1^3\Delta h_1) = \tau P^i g.$$

When $j \geq 1$, consider x in $\tau\mathbf{g}^j$. Since $h_1^3\phi(x) = h_1^3\cdot\tau g^j = 0$ in $\mathrm{Ext}_{\mathbf{A}(2)}$, $\mathbf{P}^i\mathbf{g}x$ is non-empty by Lemma 5.11. Therefore $\tau\mathbf{P}^i\mathbf{g}^{j+1} = \mathbf{P}^i\mathbf{g}x$ is non-empty. □

Example 5.20. The set $\tau\mathbf{P}\mathbf{g}$ contains τd_0^2.

Now we can prove Theorem 5.8.

Proof. We will prove that $\mathbf{P}^i\mathbf{g}^j\lambda$ is non-empty. For all $i \geq 0$ and $j \geq 0$ we have the following inclusions of sets.

$$\mathbf{P}^i\mathbf{g}^j\tau\Delta h_1 e_0 \supseteq \tau\mathbf{g}^j \cdot \mathbf{P}^i\Delta h_1 e_0,$$

$$\mathbf{P}^i\mathbf{g}^j e_0 r \supseteq \mathbf{P}^i e_0 \cdot \mathbf{g}^j r,$$

$$\mathbf{P}^i\mathbf{g}^j e_0 m \supseteq \mathbf{P}^i e_0 \cdot \mathbf{g}^j m,$$

$$\mathbf{P}^i\mathbf{g}^j\tau e_0 g^2 \supseteq \mathbf{P}^i e_0 \cdot \tau\mathbf{g}^{j+2},$$

$$\mathbf{P}^i\mathbf{g}^j d_0 r \supseteq \mathbf{P}^i d_0 \cdot \mathbf{g}^j r,$$

$$\mathbf{P}^i\mathbf{g}^j d_0 m \supseteq \mathbf{P}^i d_0 \cdot \mathbf{g}^j m,$$

$$\mathbf{P}^i\mathbf{g}^j\tau e_0^2 g \supseteq \mathbf{P}^i e_0^2 \cdot \tau\mathbf{g}^{j+1},$$

$$\mathbf{P}^i\mathbf{g}^j\tau\Delta h_1 e_0^2 \supseteq \tau\mathbf{g}^j \cdot \mathbf{P}^i\Delta h_1 e_0 \cdot e_0,$$

$$\mathbf{P}^i\mathbf{g}^j d_0 l \supseteq \mathbf{P}^i d_0 \cdot \mathbf{g}^j l,$$

$$\mathbf{P}^i\mathbf{g}^j\tau e_0^3 \supseteq \tau\mathbf{g}^j \cdot \mathbf{P}^i e_0^3,$$

$$\mathbf{P}^i\mathbf{g}^j\tau\Delta h_1 d_0 e_0 \supseteq \tau\mathbf{g}^j \cdot \mathbf{P}^i\Delta h_1 e_0 \cdot d_0,$$

$$\mathbf{P}^i\mathbf{g}^j d_0 e r \supseteq \mathbf{P}^i d_0 \cdot \mathbf{g}^j r \cdot e_0.$$

The set $\tau\mathbf{g}^j \cdot \mathbf{P}^i\Delta h_1 e_0$ consists of all products $x \cdot y$ in which x is an element of $\tau\mathbf{g}^j$ and y is an element of $\mathbf{P}^i\Delta h_1 e_0$. The same interpretation is applied for other sets on the right hand side.

The sets on the right hand side are all non-empty because of Lemmas 5.10, 5.16 and 5.18.

For example, since $\tau\mathbf{g}^j$ and $\mathbf{P}^i\Delta h_1 e_0$ are non-empty,

$$\mathbf{P}^i\mathbf{g}^j\tau\Delta h_1 e_0 \supseteq \tau\mathbf{g}^j \cdot \mathbf{P}^i\Delta h_1 e_0$$

is non-empty. Therefore, the sets on the left are all non-empty.

Several values of λ remain.

Now consider $\lambda = \tau g^2$. The set $\mathbf{P}^i\mathbf{g}^j(\tau g^2)$, or $\tau\mathbf{P}^i\mathbf{g}^{j+2}$, is non-empty by Lemma 5.19.

Next consider $\lambda = gr$. The case $i = 0$ is established in Lemma 5.16. We consider $i > 0$. Since $\mathbf{g}^j r$ is non-empty by Lemma 5.16, we consider any element x in $\mathbf{g}^j r$. Since $h_1^3\phi(x) = h_1^3 n^2 \cdot g^j = 0$ [17, Theorem 4.13], then $\mathbf{P}^i\mathbf{g}^j\lambda = \mathbf{P}^i\mathbf{g}x$ is non-empty by Lemma 5.11. The same argument is applied for $\lambda = gm$.

Finally, consider $\lambda = \tau\Delta h_1 g$. The case $i = 0$ is established in Lemma 5.17. We consider $i > 0$. When $j > 0$, since $\tau\Delta h_1\mathbf{g}^j$ is non-empty by Lemma 5.17, we consider any element x in $\tau\Delta h_1\mathbf{g}^j$. Since $h_1^3\phi(x) = h_1^3\tau\Delta h_1 \cdot g^j = 0$ [17, Theorem 4.13], then $\mathbf{P}^i\mathbf{g}^j\lambda = \mathbf{P}^i\mathbf{g}x$ is non-empty by Lemma 5.11. When $j = 0$, since

$$\begin{aligned}\phi(\tau P^{i-1}d_0 \cdot d_0\Delta h_1) &= \tau P^{i-1}d_0^2\Delta h_1 \\ &= \tau P^{i-1}(Pg + h_1^3\Delta h_1)\Delta h_1 \\ &= P^i\tau\Delta h_1 g,\end{aligned}$$

$\mathbf{P}^i\tau\Delta h_1 g$ contains $\tau P^{i-1}d_0 \cdot \Delta h_1 d_0$, so it is non-empty. Therefore, the set $\tau^k\mathbf{P}^i\mathbf{g}^j\lambda$ is non-empty. The non-triviality of elements in $\tau^k\mathbf{P}^i\mathbf{g}^j\lambda$ is obtained by comparison to $\mathrm{Ext}_{\mathbf{A}(2)}$. $\square$

The multiplicative structure of the wedge family. We are mostly interested in the $\mathbb{M}_2$-module structure of the wedge. However, it also has multiplicative structure. To be precise, the wedge is closed under multiplication.

Proposition 5.21. *For $i \geq 0, j \geq 0, k \geq 0$ and $\lambda \in \mathbf{\Lambda}$, the set*

$$\{x \in \mathrm{Ext} : x \in \tau^k\mathbf{P}^i\mathbf{g}^j\lambda\}$$

is closed under multiplication.

Proof. It is sufficient to prove that for $i \geq 0, j \geq 0, k \geq 0$ and $\lambda \in \mathbf{\Lambda}$ the subset

$$\{\tau^k P^i g^j \phi(\lambda)\}$$

of $\mathrm{Ext}_{\mathbf{A}(2)}$ is closed under multiplication. This is done by using [17, Theorem 4.13]. □

5.1 The $e_0^t \mathbf{g}^k$ family

The wedge family is not optimal in the sense that there exist elements of weight greater than the weight of elements in $\mathbf{\Lambda}$. For example, the wedge element $\tau e_0^2 g$ in $\mathbf{\Lambda}$ being of weight 31 is not optimal because the element $e_0^2 g$ in Ext is of weight 32. Table 6 lists all such elements in $\mathbf{\Lambda}$. The analysis of elements in $\mathbf{\Lambda}$ leads us to the study of two families $e_0^t \mathbf{g}^k$ and $\Delta h_1 e_0^t \mathbf{g}^k$. Unfortunately, these two families have complicated behavior. We state two conjectures on them. If these two conjectures are correct, then we obtain two other systematic phenomena in Ext.

Table 6:

element of the wedge	weight	element of higher weight	weight
$\tau \Delta h_1 e_0$	22	$\Delta h_1 e_0$	23
τe_0^3	29	e_0^3	30
$\tau \Delta h_1 d_0 e_0$	30	$\Delta h_1 d_0 \cdot e_0$	31
$\tau e_0^2 g$	31	$e_0 \cdot e_0 g$	32
$\tau \Delta h_1 e_0^2$	32	$\Delta h_1 e_0 \cdot e_0$	33

Remark 5.22. By $\mathbf{g}^j$ we mean $\mathbf{g}^j(1)$ which is understood in the sense of Definition 5.1.

Lemma 5.23. *The set $\mathbf{g}^j$ is empty for all $j \geq 0$.*

Proof. We prove the statement via contradiction. Suppose that $\mathbf{g}^j$ is non-empty. Consider any element x in $\mathbf{g}^j$. Since x maps to the non-zero element g^j in $\mathrm{Ext}_{\mathbf{A}(2)}$, x is non-zero.

Furthermore, because x has Chow degree zero, x corresponds to a classical element at degree $(8j, 4j)$ in $\mathrm{Ext}_{\mathrm{cl}}$ via the Chow degree zero isomorphism. However, $\mathrm{Ext}_{\mathrm{cl}}$ is zero in degrees $(8j, 4j)$ for all non-negative integers j [1]. Therefore, x does not exist. □

Lemma 5.24. *The set $d_0\mathbf{g}$ contains e_0^2.*

Proof. We have $\phi(e_0^2) = e_0^2 = gd$. The last identity is because $e_0^2 = gd$ in $\mathrm{Ext}_{\mathbf{A}(2)}$. □

We study the behavior of the sets $\mathbf{P}^i\mathbf{g}^j\tau e_0^3$, $\mathbf{P}^i\mathbf{g}^j\tau\Delta h_1 e_0$, $\mathbf{P}^i\mathbf{g}^j\tau\Delta h_1 d_0 e_0$, $\mathbf{P}^i\mathbf{g}^j\tau e_0^2 g$ and $\mathbf{P}^i\mathbf{g}^j\tau\Delta h_1 e_0^2$ for $i \geq 0$ and $j \geq 0$.

Theorem 5.25. *The sets $\mathbf{P}^i\mathbf{g}^j\tau e_0^3$, $\mathbf{P}^i\mathbf{g}^j\tau\Delta h_1 e_0$, $\mathbf{P}^i\mathbf{g}^j\tau\Delta h_1 d_0 e_0$, $\mathbf{P}^i\mathbf{g}^j\tau e_0^2 g$ and $\mathbf{P}^i\mathbf{g}^j\tau\Delta h_1 e_0^2$ contain an element divisible by τ if*

- *$i \geq 0$ and $j = 0$, or*
- *$i \geq j \geq 1$, or*
- *$1 \leq i < j \leq 3i$.*

Proof. Consider $\mathbf{P}^i\mathbf{g}^j\tau e_0^3$. When $i = 0$ and $j = 0$, the element τe_0^3 is divisible by τ. When $i \geq 1$ and $j = 0$, the set $\tau\mathbf{P}^i e_0^3$ contains the element $\tau \cdot P^i e_0^3$ which is divisible by τ. Apply Example 5.20 and Lemma 5.24 to get:

- When $i \geq j \geq 1$, the set $\tau\mathbf{P}^i\mathbf{g}^j e_0^3$ contains the element $\tau \cdot d_0^{2j} \cdot P^{i-j} e_0^3$ which is divisible by τ.
- When $1 \leq i < j \leq 3i$, the set $\tau\mathbf{P}^i\mathbf{g}^j e_0^3$ contains the element $\tau \cdot d_0^{3i-j} \cdot e_0^{2(j-i)+3}$ which is divisible by τ.

The same argument can be used for $\mathbf{P}^i\mathbf{g}^j\tau\Delta h_1 e_0$, $\mathbf{P}^i\mathbf{g}^j\tau\Delta h_1 d_0 e_0$, $\mathbf{P}^i\mathbf{g}^j\tau e_0^2 g$ and $\mathbf{P}^i\mathbf{g}^j\tau\Delta h_1 e_0^2$. □

There are unknown cases from Theorem 5.25. Consider $\mathbf{P}^i\mathbf{g}^j\tau e_0^3$. When $i=0$ and $j \geq 1$, the set $\tau e_0^3\mathbf{g}^j$ is not known fully. When $3i < j$, the set $\mathbf{P}^i\mathbf{g}^j\tau e_0^3$ contains the set $\tau e_0^{4i+3}\mathbf{g}^{j-3i}$ which is not known fully. The same story happens to $\mathbf{P}^i\mathbf{g}^j\tau\Delta h_1 e_0$, $\mathbf{P}^i\mathbf{g}^j\tau\Delta h_1 d_0 e_0$, $\mathbf{P}^i\mathbf{g}^j\tau e_0^2 g$ and $\mathbf{P}^i\mathbf{g}^j\tau\Delta h_1 e_0^2$ which leads us to an observation that there are two sets $\Delta h_1 e_0^t\mathbf{g}^k$ and $e_0^t\mathbf{g}^k$ for $k \geq 0$ and $t \geq 1$ which are not known fully.

Remark 5.26. By "not known fully" we mean that we do not know if the sets $\Delta h_1 e_0^t\mathbf{g}^k$ and $e_0^t\mathbf{g}^k$ are non-empty in general. The low dimension calculations show that they are empty with some values of t and k and non-empty with some other values.

Remark 5.27. Since Δh_1 is not an element of Ext, $\Delta h_1\mathbf{g}^k$ is not defined. Therefore, we do not consider the set $\Delta h_1 e_0^t\mathbf{g}^k$ when $t=0$. We do not consider the set $e^t\mathbf{g}^k$ when $t=0$ either because the set $\mathbf{g}^k$ is known to be empty by Lemma 5.23.

Lemma 5.28. *If $e_0^t\mathbf{g}^k$ is non-empty, then $e_0^t\mathbf{g}^k$ consists of elements which are non-zero in the h_1-localization* $\mathrm{Ext}[h_1^{-1}]$.

Proof. For any element x in $e_0^t\mathbf{g}^k$ and any non-negative integer n, we have $\phi(h_1^n x) = h_1^n e_0^t g^k$ which is non-zero in $\mathrm{Ext}_{\mathbf{A}(2)}$ [17, Theorem 4.13]. Consequently, $h_1^n x$ is non-zero in Ext. In other words, x is non-zero in the h_1-localization $\mathrm{Ext}_{\mathbf{A}}[h_1^{-1}]$. □

Proposition 5.29. *Let t and k be non-negative integers. If $\alpha(t+k) > t$, then $e_0^t\mathbf{g}^k$ is empty.*

Proof. (Via contradiction) Suppose that $e_0^t\mathbf{g}^k$ is non-empty. As a result, its elements survive the h_1-localization by Lemma 5.28. Note that elements of $e_0^t\mathbf{g}^k$ have Chow degree t and coweight $(7t+8k)$. By Theorem 4.13, after considering Chow degrees, any element of $e_0^t\mathbf{g}^k$ maps to a summation of monomials of the form

$$v_1^{4n}v_2^m \prod_{i=1}^{t-4n-m} v_{m_i}$$

in $\mathrm{Ext}_{\mathbf{A}}[h_1^{-1}]$ for some n, m and $m_i \geq 3$. By comparing coweights, we have

$$7t + 8k = 4n + 3m + \sum_{i=1}^{t-4n-m} (2^{m_i} - 1).$$

Then

$$8t + 8k = 8n + 4m + \sum_{i=1}^{t-4n-m} 2^{m_i}.$$

Since $m_i \geq 3$, m has to be even, i.e., $m = 2m'$ for some non-negative integer m'. We obtain

$$t + k = n + m' + \sum_{i=1}^{t-4n-m} 2^{m_i - 3}.$$

By Lemma 4.16,

$$\alpha(t+k) \leq \alpha(n) + \alpha(m') + t - 4n - m = t + (\alpha(n) - 4n) + (\alpha(m') - 2m') \leq t.$$

□

Corollary 5.30. *If $e_0\mathbf{g}^k$ is non-empty, then $k = 2^n - 1$ for some non-negative integer n.*

Proof. Since $e_0\mathbf{g}^k$ is non-empty, $\alpha(1+k) \leq 1$ by Proposition 5.29. Then $1 + k = 2^n$ for some non-negative integer n. □

We state the following conjecture.

Conjecture 5.31. The set $e_0\mathbf{g}^k$ is non-empty if and only if $k = 2^n - 1$ for some non-negative integer n.

We mention some evidence supporting the conjecture. The elements $e_0 g$ and $e_0 g^3$ survive in Ext (by explicit computations). Also, the conjecture fits nicely with the properties of the h_1-localization of Ext [14].

Theorem 5.32. *Suppose that $e_0\mathbf{g}^{2^n-1}$ is non-empty for every non-negative integer n. Then $e_0^t\mathbf{g}^k$ is non-empty if and only if $k = (\sum_{i=1}^t 2^{n_i}) - t$ for some non-negative integers n_i.*

Proof. If $e_0^t\mathbf{g}^k$ is non-empty, then by Proposition 5.29 we have $\alpha(k+t) \leq t$. As a result, $k + t = \sum_{i=1}^t 2^{n_i}$ for some non-negative integers n_i. In other words,

$$k = (\sum_{i=1}^{t} 2^{n_i}) - t.$$

Conversely, if $k = (\sum_{i=1}^{t} 2^{n_i}) - t$, since $e_0\mathbf{g}^{2^{n_i}-1}$ is non-empty for all n_i then

$$e_0^t\mathbf{g}^k \supseteq e_0\mathbf{g}^{2^{n_1}-1} \cdots e_0\mathbf{g}^{2^{n_t}-1}$$

is non-empty. $\square$

Remark 5.33. The condition $k = (\sum_{i=1}^{t} 2^{n_i}) - t$ is equivalent to $\alpha(k+t) \leq t$. In practice, we use the latter condition rather than the former one.

5.2 The $\Delta h_1 e_0^t \mathbf{g}^k$ family

Proposition 5.34. *If $\alpha(1+k+t) > t$ for $t \geq 1$ and $k \geq 0$, then the set $\Delta h_1 e_0^t \mathbf{g}^k$ is empty.*

Proof. (Via contradiction) Recall the following commutative diagram [14]

$$\begin{tikzcd}
\mathrm{Ext} \arrow[r, "\phi"] \arrow[d, "L"] & \mathrm{Ext}_{\mathbf{A}(2)} \arrow[d, "L"] \\
\mathrm{Ext}[h_1^{-1}] \arrow[r, "\phi"] & \mathrm{Ext}_{\mathbf{A}(2)}[h_1^{-1}]
\end{tikzcd}$$

Suppose that $\Delta h_1 e_0^t \mathbf{g}^k$ is non-empty. Then it contains an element x. The element x maps to the element $\Delta h_1 e_0^t g^k$ in $\mathrm{Ext}_{\mathbf{A}(2)}$, surviving h_1-localization. The element $\Delta h_1 e_0^t g^k$ maps to

$$h_1^{-2k-5} v_1^4 a_1^{2+2k+t} v_2^t + h_1^{-2k+1} v_2^{4+t} a_1^{2k+t}$$

in $\mathrm{Ext}_{\mathbf{A}(2)}[h_1^{-1}]$ via L.

Since $\alpha(1+k+t) > t$, the term $h_1^{-2k-5} v_1^4 a_1^{2+2k+t} v_2^t$ is not in the image of $\phi : \mathrm{Ext}[h_1^{-1}] \longrightarrow \mathrm{Ext}_{\mathbf{A}(2)}[h_1^{-1}]$ by Lemma 4.18. $\square$

Lemma 5.35. *For any integer $k \geq 0$, there is no element x in* Ext *such that $\phi(x) = \Delta h_1 g^k$ in $\mathrm{Ext}_{\mathbf{A}(2)}$.*

Proof. We apply the same argument as in Proposition 5.34. $\square$

By Proposition 5.34, a necessary condition for the set $\Delta h_1 e_0 \mathbf{g}^j$ to be non-empty is $\alpha(2+j) \le 1$, or $j = 2^n - 2$ for some non-negative integer n. Unfortunately, we do not know if it is sufficient. We state the following conjecture.

Conjecture 5.36. The set $\Delta h_1 e_0 \mathbf{g}^j$ is non-empty if and only if $j = 2^n - 2$ for some non-negative integer n.

Theorem 5.37. *Suppose that* $e_0 \mathbf{g}^{2^n-1}$ *and* $\Delta h_1 e_0 \mathbf{g}^{2^n-2}$ *are non-empty for every non-negative integer* n*. Then* $\Delta h_1 e_0^t \mathbf{g}^k$ *is non-empty if and only if* $k = (\sum_{i=1}^t 2^{n_i}) - t - 1$ *for some non-negative integers* n_i*.*

Proof. If $\Delta h_1 e_0^t \mathbf{g}^k$ is non-empty, then $\alpha(1+k+t) \le t$ or $k = (\sum_{i=1}^t 2^{n_i}) - t - 1$ for some non-negative integers n_i.

Conversely, if $k = (\sum_{i=1}^t 2^{n_i}) - t - 1$ for some non-negative integers n_i, then $\Delta h_1 e_0^t \mathbf{g}^k$ contains the set

$$\Delta h_1 e_0 \mathbf{g}^{2^{n_{i_1}}-2} e_0 \mathbf{g}^{2^{n_{i_2}}-1} \cdot \ldots e_0 \mathbf{g}^{2^{n_{i_t}}-1}$$

which is non-empty. $\square$

5.3 The wedge at filtrations $f = 4k$ and $f = 4k+1$ for $k \ge 2$

At filtrations $f = 4k+2$ and $f = 4k+3$ for $k \ge 2$, the wedge is optimal in the sense that all elements are of the greatest weight. At filtrations $f = 4k$ and $f = 4k+1$ for $k \ge 2$, the wedge is not optimal in the sense that there exist elements in Ext which are of weight greater than the weight of the wedge elements.

Theorem 5.38. *Suppose that* $e_0 \mathbf{g}^{2^n-1}$ *is non-empty for every non-negative integer* n*. Then at filtration* $f = 4k$ *for* $k \ge 2$ *the set* $\tau e_0^s \mathbf{g}^{k-s}$ *contains an element divisible by* τ *if* $s \ge \alpha(k)$ *and does not contain any element divisible by* τ *if* $s < \alpha(k)$*.*

Proof. If $s \geq \alpha(k)$, by Theorem 5.32 and Remark 5.33 the set $e_0^s\mathbf{g}^{k-s}$ contains an element x. Then $\tau e_0^s\mathbf{g}^{k-s}$ contains the element $\tau \cdot x$ divisible by τ.

If $s < \alpha(k)$, we suppose that $\tau e_0^s\mathbf{g}^{k-s}$ contains an element $\tau \cdot y$ divisible by τ. The element $\tau \cdot y$ maps to $\tau e_0^s g^{k-s}$ in $\mathrm{Ext}_{\mathbf{A}(2)}$. Then y maps to $e_0^s g^{k-s}$ in $\mathrm{Ext}_{\mathbf{A}(2)}$. In other words, y is an element of the set $e_0^s\mathbf{g}^{k-s}$. However, since $s < \alpha(k)$, the set $e_0^s\mathbf{g}^{k-s}$ is empty by Proposition 5.29. □

Theorem 5.39. *Suppose that $e_0\mathbf{g}^{2^n-1}$ and $\Delta h_1 e_0\mathbf{g}^{2^n-2}$ are non-empty for every non-negative integer n. Then at filtration $f = 4k+1$ for $k \geq 2$ the set $\tau\Delta h_1 e_0^s\mathbf{g}^{k-s-1}$ contains an element divisible by τ if $s \geq \alpha(k)$ and does not contain any element divisible by τ if $s < \alpha(k)$.*

Proof. If $s \geq \alpha(k)$, then by Theorem 5.37 the set $\Delta h_1 e_0^s\mathbf{g}^{k-s-1}$ contains an element x. Then $\tau\Delta h_1 e_0^s\mathbf{g}^{k-s-1}$ contains the element $\tau \cdot x$ divisible by τ.

If $s < \alpha(k)$, we suppose that $\tau\Delta h_1 e_0^s\mathbf{g}^{k-s-1}$ contains an element $\tau \cdot y$ divisible by τ. The element $\tau \cdot y$ maps to $\tau\Delta h_1 e_0^s g^{k-s-1}$ in $\mathrm{Ext}_{\mathbf{A}(2)}$. Then the element y maps to $\Delta h_1 e_0^s g^{k-s-1}$ in $\mathrm{Ext}_{\mathbf{A}(2)}$. In other words, y is an element of the set $\Delta h_1 e_0^s\mathbf{g}^{k-s-1}$. However, since $s < \alpha(k)$, the set $\Delta h_1 e_0^s\mathbf{g}^{k-s-1}$ is empty by Proposition 5.34. □

Corollary 5.40. *At filtration $f = 4k+1$ and $f = 4k$ for $k \geq 2$ and $s < \alpha(k)$, all elements in the sets $\tau e_0^s\mathbf{g}^{k-s}$ and $\tau\Delta h_1 e_0^s\mathbf{g}^{k-s-1}$ are optimal.*

5.4 The wedge chart

This chart shows the wedge from its vertex to stem 70.

- Solid dots and open circles indicate copies of $\mathbb{M}_2$.
- Solid dots indicate elements which behave irregularly, as in Propositions 5.29 and 5.34.
- Open circles indicate elements which behave regularly.

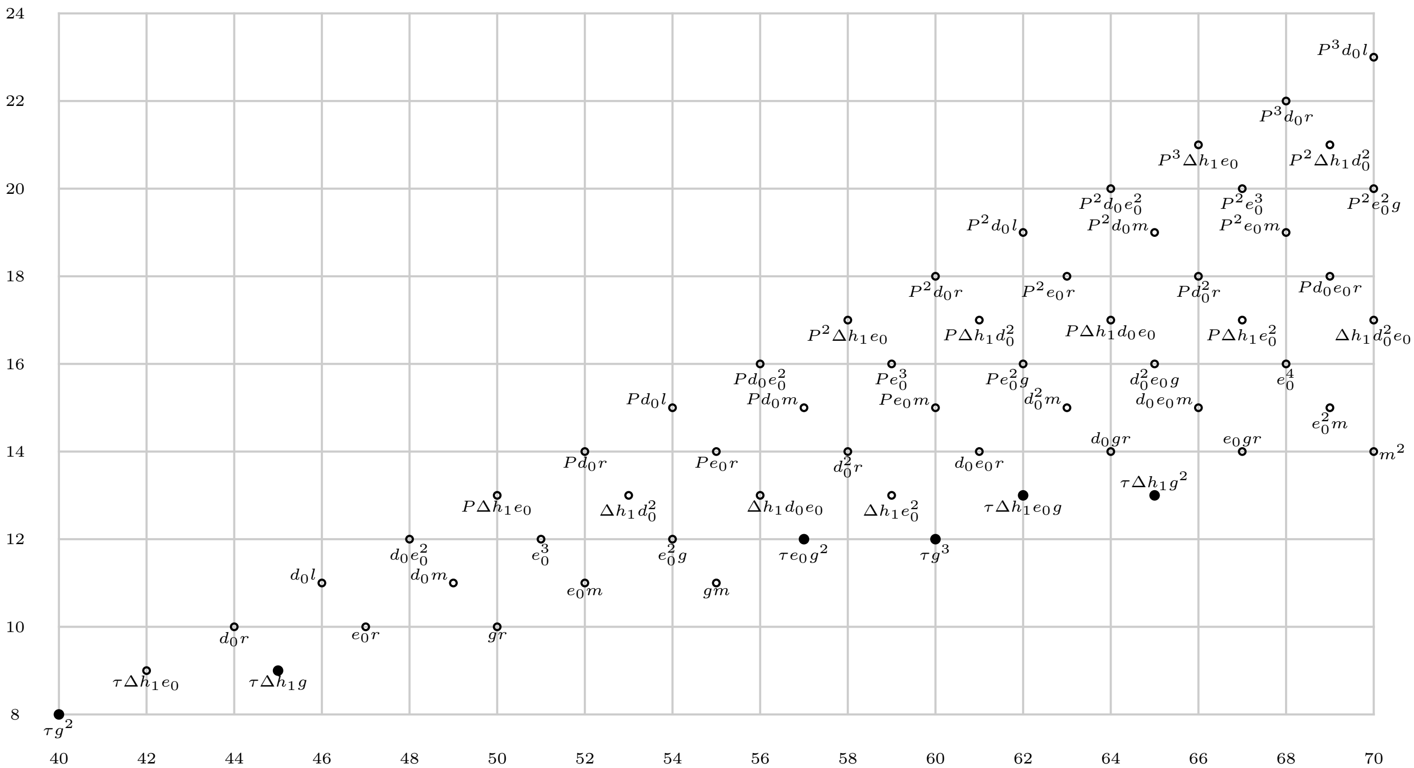

Figure 1: The $\mathbb{C}$-motivic wedge through the 70-stem

CHAPTER 6 THE $\mathbb{C}$-MOTIVIC UNSTABLE ADAMS SPECTRAL SEQUENCE PROGRAM

6.1 The classical unstable Adams spectral sequence

We recall that the suspension from S^n to S^{n+1} induces a group homomorphism

$$[S^{n+k}, S^n] \to [S^{n+k+1}, S^{n+1}].$$

This group homomorphism is not isomorphic when $n \leq k+1$ in general. In that case, the group $\pi_{n+k}(S^n) = [S^{n+k}, S^n]$ is called the unstable homotopy groups of spheres. Unfortunately, computing the unstable homotopy groups is much harder than that for the stable ones. One of the reasons is that the unstable homotopy groups do not have many structures allowing computations.

Much of the knowledge about the unstable homotopy groups comes from homology groups via the Hurewicz theorem, the Serre spectral sequence, the Hopf fibrations and the EHP spectral sequence.

We discuss how to use the Hopf fibration

$$S^1 \hookrightarrow S^3 \longrightarrow S^2$$

to compute the group $\pi_3(S^2)$. The above Hopf fibration gives rise to the following long exact sequences of homotopy groups

$$\begin{tikzcd} \ldots \pi_3(S^1) \arrow[r] & \pi_3(S^3) \arrow[r] & \pi_3(S^2) \arrow[dll, "\delta"] \\ \pi_2(S^1) \arrow[r] & \pi_2(S^3) \arrow[r] & \ldots \end{tikzcd}$$

Since $\pi_3(S^1) \cong 0$, $\pi_2(S^1) \cong 0$, $\pi_2(S^3) \cong 0$ and $\pi_3(S^3) \cong \mathbb{Z}$ we have the following short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \pi_3(S^2) \longrightarrow 0 \longrightarrow 0.$$

Then $\pi_3(S^2)/\mathbb{Z} \cong 0$ or $\pi_3(S^2) \cong \mathbb{Z}$.

Remark 6.1. We have

- $\pi_m(S^1) \cong 0$ for all $m > 1$.

- $\pi_m(S^n) \cong 0$ for all $m < n$.

- $\pi_n(S^n) \cong \mathbb{Z}$ for all $n \geq 1$.

Unfortunately, there are only four Hopf fibrations of spheres. Also, the use of the Serre spectral sequence or the EHP spectral sequence to compute the homotopy groups of spheres is complicated and not able to carry to high dimensions.

The success of the classical Adams spectral sequence for stable homotopy groups suggests to look for an unstable version for the Adams spectral sequence.

6.1.1 The classical unstable Adams spectral sequence

We denote $\pi_*(X) = \oplus_{k\geq 0}\pi_k(X)$.

The each space X being simply connected, Curtis [9] defines a spectral sequence which converges to $\pi_*(X)$. Then Rector [32] defines a mod-p version of this spectral sequence converging to the p-primary component of the group $\pi_*(S^n)$ and it appears to be a good candidate for an Unstable Adams spectral sequence.

6.1.2 The classical Lambda algebra

The classical Lambda algebra is defined by Bousfield et al. However, in this thesis we prefer using John Wang's description for the Lambda algebra.

Definition 6.2. The classical (mod-2) Lambda algebra $\mathbf{\Lambda}$ is an associative bigraded $\mathbb{F}_2$-algebra with generators $\lambda_n \in \mathbf{\Lambda}^{1,n+1}$ $(n \geq 0)$ and relations

$$\lambda_i\lambda_{2i+1+n} = \sum_{j\geq 0} \binom{n-j-1}{j} \lambda_{i+n-j}\lambda_{2i+1+j}$$

for $i, n \geq 0$ with differential

$$d(\lambda_n) = \sum_{j\geq 1} \binom{n-j}{j} \lambda_{n-j}\lambda_{j-1}.$$

Remark 6.3. The Lambda algebra is not commutative.

We can define the mod-p Lambda algebra. However, as mentioned before, in this thesis we always choose $p = 2$.

Remark 6.4. For the purpose of studying the homotopy groups of spheres, we would like to use the bigrading $\mathbf{\Lambda}^{n,1}$ instead of $\mathbf{\Lambda}^{1,n+1}$ as defined in Definition 6.2.

The Lambda algebra has chain complex structure. Its homology is isomorphic to the E_2-page of the classical Adams spectral sequence for the sphere spectrum or the cohomology Ext_{cl} of the classical Steenrod algebra.

Theorem 6.5 (Bousfield et al). *The homology of $\mathbf{\Lambda}$ with respect to the differential d is isomorphic to the cohomology of the classical Steenrod algebra,*

$$H(\mathbf{\Lambda}) = \mathrm{Ext}_{cl}.$$

The defining relations of the Lambda algebra suggest the following definition.

Definition 6.6. A monomial $\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_s}$ in $\mathbf{\Lambda}$ is admissible if $2i_r \geq i_{r+1}$ for $1 \leq r < s$. $\mathbf{\Lambda}(n)$ is the subcomplex of $\mathbf{\Lambda}$ spanned by the admissible monomials with $i_1 < n$.

Theorem 6.7 (Rector). *The homology of $\mathbf{\Lambda}(n)$ is isomorphic to the E_2-page of the (classical) unstable Adams spectral sequence converging to $\pi_*(S^n)$,*

$$H(\mathbf{\Lambda}(n)) \cong E_2(S^n).$$

6.1.3 Computing the homology of the Lambda algebra

Now we discuss how to compute $H(\mathbf{\Lambda})$ and $H(\mathbf{\Lambda}(n))$. Unfortunately, because of the non-commutativity of $\mathbf{\Lambda}$, the computation by hand is very ineffective at high dimensions or with large values of n. The key to this computation is to choose the right basis. We will discuss Curtis's idea to set aside the vast majority of monomials which are irrelevant.

One of the advantages of the Lambda algebra over the May spectral sequence in computing the E_2 page of the Adams spectral sequence is that all the product structure of the E_2 shows up. (We discussed in the chapter about spectral sequences how spectral sequences lose product structure.)

6.2 Looking for a $\mathbb{C}$-motivic Lambda algebra

Our main purpose is to construct a $\mathbb{C}$-motivic version of the unstable Adams spectral sequence converging to the homotopy groups $\pi_*(S^{p,q})$ of the motivic sphere $S^{p,q}$. However, we still do not know how to do it. One easier problem, suggested by Theorem 6.7, is to define a $\mathbb{C}$-motivic version $\mathbf{\Lambda}^{\mathbb{C}}$ of the classical Lambda algebra. We hope that the homology of $\mathbf{\Lambda}^{\mathbb{C}}$ gives information about the E_2-page of the desired $\mathbb{C}$-motivic unstable Adams spectral sequence. The author wants to thank Eva Belmont for her useful suggestion to this problem.

The desired motivic Lambda algebra $\mathbf{\Lambda}^{\mathbb{C}}$ should have its homology isomorphic to the cohomology Ext of the motivic Steenrod algebra. We emphasize that we already know Ext up to at least the 70th stem [18]. The main problem of this section is as follows.

Problem 6.8. Define a $\mathbb{C}$-motivic version $\mathbf{\Lambda}^{\mathbb{C}}$ of the classical Lambda algebra such that

$$H(\mathbf{\Lambda}^{\mathbb{C}}) \cong \mathrm{Ext}.$$

Priddy suggests another approach to construct the classical Lambda algebra. Unfortunately, it does not work in the $\mathbb{C}$-motivc context where we work on the $\mathbb{M}_2$-algebra structure rather than the $\mathbb{F}_2$-algebra structure as in the classical case. However, this approach suggests a very interesting algebra which is expected to be the desired motivic Lambda algebra. With an abuse of names, we will call it the motivic Lambda algebra $\mathbf{\Lambda}^{\mathbb{C}}$.

We recall that $\mathbb{M}_2$ is the polynomial ring $\mathbb{F}_2[\tau]$ with one generator τ having bidegree

$(0,1)$.

Definition 6.9. The motivic (mod-2) Lambda algebra $\mathbf{\Lambda}^{\mathbb{C}}$ is a trigraded $\mathbb{M}_2$-algebra with generators $\lambda_n \in \mathbf{\Lambda}^{n,1,\lfloor\frac{n+1}{2}\rfloor}$ $(n \geq 0)$ and the relations

$$\lambda_i \lambda_{2i+1+n} = \sum_{j\geq 0} \binom{n-j-1}{j} \tau^{?} \lambda_{i+n-j} \lambda_{2i+1+j}$$

for $i, n \geq 0$ with the differential $d(\lambda_0) = 0$ and

$$d(\lambda_n) = \sum_{j\geq 1} \binom{n-j}{j} \lambda_{n-j} \lambda_{j-1}.$$

Remark 6.10. Using Priddy's method, we assign λ_n to Sq^{n+1} having motivic weight $\lfloor\frac{n+1}{2}\rfloor$. We refer to the third degree of $\mathbf{\Lambda}^{\mathbb{C}}$ as its weight.

Remark 6.11. If $n < k$, then $\binom{n}{k} = 0$.

The motivic Lambda algebra is very similar to the classical Lambda algebra. There are two differences. First of all, the motivic Lambda algebra is an $\mathbb{M}_2$-algebra. Secondly, it is trigraded. As a result, τ appears in the relations. The symbol ? stands for either 0 or 1, depending on which value makes the formula balanced in weight. The element τ does not appear in the formula for the differential because the weights on both sides are already balanced.

Unfortunately, we still do not know if the homology of $\mathbf{\Lambda}^{\mathbb{C}}$ is isomorphic to Ext. We have the following conjecture.

Conjecture 6.12. The homology of the motivic Lambda algebra is isomorphic to the cohomology of the motivic Steenrod algebra,

$$H(\mathbf{\Lambda}^{\mathbb{C}}) \cong \mathrm{Ext}.$$

We use the shorthand a for λ_a and $a\ b$ for $\lambda_a \lambda_b$. In the classical case we have $1\ 0\ 2\ 1 = 1\ 1\ 1\ 1$ but motivically we have $1\ 0\ 2\ 1 = \tau(1\ 1\ 1\ 1)$ because of weight reasons. It leads to a very important differential.

Proposition 6.13. *We have*

$$d(2\ 2\ 1) = \tau(1\ 1\ 1\ 1).$$

Proof. It is straight forward from the formula for the differential and the relation $1\ 0\ 2\ 1 = \tau(1\ 1\ 1\ 1)$. $\square$

If Conjecture 6.12 is correct, then the homology class $\overline{\lambda_1}$, a generator, in $H(\mathbf{\Lambda}^{\mathbb{C}})$ maps to the element h_1 in Ext because of degree reasons. The above differential implies that the class $\tau(\overline{\lambda_1})^4 = 0$ in $H(\mathbf{\Lambda}^{\mathbb{C}})$. As a result, $\tau h_1^4 = 0$ in Ext which fits perfectly our knowledge about Ext. Unfortunately, this is the only motivic fact we know on $H(\mathbf{\Lambda}^{\mathbb{C}})$ which fits our knowledge about Ext.

Lemma 6.14. *For any integer x, we denote $p(x)$ to be the exponent of 2 in the prime factorization of x. For any integers $n \geq 1$ and $k > 0$ such that $2^n > k$ we have*

$$p(2^n - k) = p(k).$$

Proof. We suppose that $k = 2^m y$ for some integers $m \geq 0$ and y odd. Then $2^n - k = 2^m(2^{n-m} - y)$. Since $2^n > k$ then $n > m$. It implies that $(2^{n-m} - y)$ is odd. Consequently, $p(2^n - k) = p(k) = m$. $\square$

Lemma 6.15. *For a positive integer n we suppose that 2^s is the highest power of 2 which is not greater than n, i.e. $2^s \leq n \leq 2^{s+1} - 1$. Then*

1. $p(n!) = \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{2^2} \rfloor + \dots \lfloor \frac{n}{2^s} \rfloor$.

2. *For $x = (n+1)(n+2)\dots(2n)$ we have*

$$p(x) \geq \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{2^2} \rfloor + \dots \lfloor \frac{n}{2^s} \rfloor + 1.$$

Proof. The first statement is straight forward from the observation that from 1 to n there are exactly $\lfloor \frac{n}{2^k} \rfloor$ numbers which are divisible by 2^k.

To prove the second statement we first observe that in n consecutive positive integers there are at least $\lfloor \frac{n}{2^k} \rfloor$ numbers which are divisible by 2^k for $k \leq s$. Consequently, we have

$$p(x) \geq \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{2^2} \rfloor + \dots \lfloor \frac{n}{2^s} \rfloor.$$

Since $2^s \leq n \leq 2^{s+1} - 1$ then $n+1 \leq 2^{s+1} \leq 2n$. The power 2^{n+1} contributes one more factor 2 to x. It then implies the desired inequality. $\square$

Proposition 6.16. *For all $n \geq 0$ we have*

$$d(\lambda_{2^n-1}) = 0.$$

Proof. Since $d(\lambda_0) = 0$ we only have to prove the statement for $n \geq 1$. By the differential formula we have

$$d(\lambda_{2^n-1}) = \sum_{j\geq 1} \binom{2^n - 1 - j}{j} \lambda_{2^n-1-j}\lambda_{j-1}.$$

We only have to consider non-zero coefficients $\binom{2^n-1-j}{j}$ on the right hand side. We will show that $\binom{2^n-1-j}{j}$ is an even integer for all n. As a result, all coefficients on the right hand side of the above formula are zero. We have

$$\binom{2^n - 1 - j}{j} = \frac{(2^n - 2j)(2^n - 2j + 1)\cdots(2^n - j - 1)}{(j)!}.$$

By Lemma 6.14 we have $p(2^n - 2j) = p(2j), \dots, p(2^n - j - 1) = p(j + 1)$. Then

$$p[(2^n - 2j)(2^n - 2j + 1)\cdots(2^n - j - 1)] = p[(2j)(2j + 2)\dots(j + 1)].$$

By Lemma 6.15 we have $p[(2j)(2j + 2)\dots(j + 1)] \geq p(j!) + 1$. Then the number $\binom{2^n-1-j}{j}$ is even. $\square$

Proposition 6.16 gives a class of generators of $H(\mathbf{\Lambda}^{\mathbb{C}})$ containing only one element λ_i for some i. By degree reasons, if Conjecture 6.12 is correct, then the generators λ_{2^n-1} of $H(\mathbf{\Lambda}^{\mathbb{C}})$ correspond to the generators h_n of Ext. The proof of Proposition 6.16 also works for classical case.

Definition 6.17. A monomial $\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_s}$ in $\mathbf{\Lambda}$ is admissible if $2i_r \geq i_{r+1}$ for $1 \leq r < s$. $\mathbf{\Lambda}^{\mathbb{C}}(n)$ is the subcomplex of $\mathbf{\Lambda}^{\mathbb{C}}$ spanned by the admissible monomials with $i_1 < n$.

If $\mathbf{\Lambda}^{\mathbb{C}}$ satisfies Problem 6.8, then we can compute the homology of $\mathbf{\Lambda}^{\mathbb{C}}(n)$. We expect that this homology is isomorphic to the E_2-page of a motivic unstable Adams spectral sequence which we are looking for.

Problem 6.18. Compute $H(\mathbf{\Lambda}^{\mathbb{C}}(n))$.

What can we learn from $H(\mathbf{\Lambda}^{\mathbb{C}}(n))$? Classically, the subcomplex $\mathbf{\Lambda}(n)$ is the input to compute $\pi_*(S^n)$. However, the motivic spheres $S^{p,q}$ are bigraded. As a result, in order to compute $\pi_* S^{p,q}$ we need a bigraded input. There is a possibility that $H(\mathbf{\Lambda}^{\mathbb{C}}(n))$ may give information about $\oplus_q \pi_* S^{p,q}$ but it is still an unknown problem.

REFERENCES

[1] J.F. Adams, *A periodicity theorem in homological algebra*, Proc. Cambridge Philos. Soc. **62** (1966), 365-377.

[2] J.F. Adams, *On the structure and applications of the Steenrod algebra*, Comment. Math. Helv. **32** (1958), 180-214.

[3] D. Barnes, D. Poduska and P. Shick, *Unstable Adams spectral sequence charts*, Adams Memorial Symposium on Algebraic Topology, 2 (Manchester, 1990), 87-118, London Math. Soc. Lecture Note Ser., **176**, Cambridge Univ. Press, Cambridge, 1992.

[4] M. G. Barratt, M. Mahowald and M. C. Tangora, *Some differentials in the Adams spectral sequence*, II, Topology 9 (1970), 309-316.

[5] A. Bousfield, E. Curtis, D. Kan, D. Quillen, D. Rector and J. Schlesinger, *The mod-p lower central series and the Adam spectral sequence*, Topology **5** (1966), 331-342.

[6] Robert Bruner, *A new differential in the Adams spectral sequence*, Topology **23** (1984), no. 3, 271-276.

[7] Robert Bruner, *The cohomology of the mod 2 Steenrod algebra: A computer calculation*, Wayne State University Research Report **37** (1997).

[8] E. Curtis, P. Goerss, M. Mahowald and R. Milgram, *Calculations of Unstable Adams E_2 Terms of Spheres*, Algebraic topology (Seattle, Wash., 1985), 208-266, Lecture Notes in Math., 1286, Springer, Berlin, 1987.

[9] Edward Curtis, *Some relations between homotopy and homology*, Ann. of Math. (2) **82** (1965), 386-413.

[10] D. Dummit and R. Foote, *Abstract Algebra*, John Wiley & Sons, Inc., Hoboken, NJ, 2004.

[11] D. Dugger and D. Isaksen, *Motivic cell structures*, Algebr. Geom. **5** (2005), 615-652.

[12] D. Dugger and D. Isaksen, *The motivic Adams spectral sequence*, Geom. Topol. **14** (2010), no. 2, 967-1014.

[13] B. Gheorghe, D. Isaksen, A. Krause and N. Ricka, $\mathbb{C}$-*motivic modular forms*, preprint (2018) arXiv:1810.11050.

[14] B. J. Guillou and D. C. Isaksen, *The η-local motivic sphere*, Jounal of Pure and Applied Algebra **219** (2015) 4728-4756.

[15] B. J. Guillou and D. C. Isaksen, *The motivic Adams vanishing line of slope $\frac{1}{2}$*, New York J. Math. **21** (2015), 533-545.

[16] Daniel Isaksen, *Classical and motivic Adams charts*, s.wayne.edu/isaksen/adams-charts.

[17] Daniel Isaksen, *The cohomology of motivic* $\mathbf{A}$*(2)*, Homology Homotopy Appl. **11** (2009), no.2, 251-274.

[18] Daniel Isaksen, *Stable Stems*, Mem. Amer. Math. Soc., **262** (2019), no. 1269.

[19] D. Isaksen, G. Wang, Z. Xu, *More stable stems*, preprint (2020), arXiv:2001.04511.

[20] Ang Li, *The v_1-periodic region of the* $\mathbb{C}$-*motivic* Ext, preprint (2019), arXiv:1912.03111.

[21] M. Mahowald and M. C. Tangora, *An infinite subalgebra of* $\mathrm{Ext}_A(Z_2, Z_2)$, Trans. Amer. Math. Soc. **132** (1968) 263-274.

[22] M. Mahowald, and M. C. Tangora, *Some differentials in the Adams spectral sequence*, Topology **6** 1967, 349-369.

[23] H. Margolis, S. Priddy and M. Tangora, *Another systematic phenomenon in the cohomology of the Steenrod algebra*, Topology **10** (1971) 43-46.

[24] J. P. May, *The cohomology of restricted Lie algebras and of Hopf algebras*, Bull. Amer. Math. Soc. **71** (1965) 372-377.

[25] J. P. May, *The cohomology of restricted Lie algebras and of Hopf algebras; application to the Steenrod algebra*, Thesis (Ph.D.)-Princeton University. 1964.

[26] John McCleary, *A user's guide to spectral sequences*, Cambridge Studies in Advanced Mathematics, 58. Cambridge University Press, Cambridge, 2001.

[27] F. Morel and V. Voevodsky, $\mathbf{A}^1$*-homotopy theory of schemes*, Inst. Hautes Etudes Sci. Publ. Math. No. **90** (1999), 45-143 (2001).

[28] R. Mosher and M. Tangora, *Cohomology Operations and Applications in Homotopy Theory*, Harper & Row, Publishers, New York-London 1968.

[29] O. Nakamura, *On the squaring operations in the May spectral sequence*, Mem. Fac. Sci. Kyushu Univ. Ser. A **26** (1972), no. 2, 293-308.

[30] S. Priddy, *Koszul Resolutions*, Trans. Amer. Math. Soc. **152** (1970), 39-60.

[31] Douglas Ravenel, *Complex Cobordism and Stable Homotopy Groups of Spheres*, Pure and Applied Mathematics, **121**. Academic Press, Inc., Orlando, FL, 1986.

[32] David Rector, *An Unstable Adams spectral sequence*, Topology **5** (1966), 343-346.

[33] Martin Tangora, *Computing with the lambda algebra*, AstÃĺrisque No. **192** (1990), 6, 79-89.

[34] Martin Tangora, *Generating Curtis tables*, Algebraic topology (Proc. Conf., Univ. British Columbia, Vancouver, B.C., 1977), pp. 243-253, Lecture Notes in Math., **673**, Springer, Berlin, 1978.

[35] Martin Tangora, *On the cohomology of the Steenrod algebra*, Thesis (Ph.D.)-Northwestern University. 1966.

[36] Martin Tangora, *Some remarks on the lambda algebras*, Geometric applications of homotopy theory (Proc. Conf., Evanston, Ill., 1977), II, pp. 476-487, Lecture Notes in Math., **658**, Springer, Berlin, 1978.

[37] Hieu Thai, *The* $\mathbb{C}$*-motivic wedge family of the cohomology of the motivic Steenrod algebra*, Homology Homotopy Appl. (accepted).

[38] Vladimir Voevodsky, *Motivic cohomology with $\boldsymbol{Z}/2$-coefficients,* Publ. Math. Inst. Hautes ÂťEtudes Sci. **98** (2003) 59-104.

[39] Vladimir Voevodsky, *Motivic Eilenberg-Maclane spaces,* Publ. Math. Inst. Hautes Etudes Sci. **112** (2010) 1-99.

[40] Vladimir Voevodsky, *Reduced power operations in motivic cohomology*, Publ. Math. Inst. Hautes Etudes Sci. **98** (2003) 1-57.

[41] John Wang, *On the cohomology of the mod-2 Steenrod algebra and the non-existence of elements of Hopf invariant one*, Illinois J. Math. **11** (1967) 480-490.

[42] Charles Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, **38**. Cambridge University Press, Cambridge, 1994.

[43] A. Zachariou, *A Subalgebra of* $\mathrm{Ext}_A^{**}(\mathbb{Z}_2, \mathbb{Z}_2)$, Bull. Amer. Math. Soc. **73** (1967), 647-648.

ABSTRACT

THE WEDGE FAMILY OF THE COHOMOLOGY OF THE $\mathbb{C}$-MOTIVIC STEENROD ALGEBRA

by

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Computing the stable homotopy groups of the sphere spectrum is one of the most important problems of stable homotopy theory. Focusing on the 2-complete stable homotopy groups instead of the integral homotopy groups, the Adams spectral sequence appears to be one of the most effective tools to compute the homotopy groups. The spectral sequence has been studied by J. F. Adams, M. Mahowald, M. Tangora, J. P. May and others.

In 1999, Morel and Voevodsky introduced motivic homotopy theory. One of its consequences is the realization that almost any object studied in classical algebraic topology could be given a motivic analog. In particular, we can define the motivic Steenrod algebra $\mathbf{A}$, the motivic stable homotopy groups of spheres [27] and the motivic Adams spectral sequence. In the motivic perspective, there are many more non-zero classes in the motivic Adams spectral sequence, which allows the detection of otherwise elusive phenomena. Also, the additional motivic weight grading can eliminate possibilities which appear plausible in the classical perspective.

To run the motivic Adams spectral sequence, one begins with $\mathrm{Ext}_{\mathbf{A}}(\mathbb{M}_2, \mathbb{M}_2)$. The algebra $\mathrm{Ext}_{\mathbf{A}}(\mathbb{M}_2, \mathbb{M}_2)$ is infinitely generated and irregular. A natural approach is to look for systematic phenomena in $\mathrm{Ext}_{\mathbf{A}}(\mathbb{M}_2, \mathbb{M}_2)$. One potential candidate is the wedge family in $\mathrm{Ext}_{\mathbf{A}}(\mathbb{M}_2, \mathbb{M}_2)$.

The classical wedge family was studied by M. Mahowald and M. Tangora [21]. It is a subset of the cohomology $\mathrm{Ext}_{\mathbf{A}_{\mathrm{cl}}}(\mathbb{F}_2, \mathbb{F}_2)$ of the classical Steenrod algebra, consisting of non-zero elements $P^i g^j \lambda$ and $g^j t$ in which λ is in $\mathbf{\Lambda}$, t is in $\mathbf{T}$, $i \geq 0$ and $j \geq 0$. The sets $\mathbf{\Lambda}$ and

$\mathbf{T}$ are specific subsets of $\mathrm{Ext}_{\mathbf{A}_{\mathrm{cl}}}(\mathbb{F}_2, \mathbb{F}_2)$. The wedge family gives an infinite wedge-shaped diagram inside the cohomology of the classical Steenrod algebra, which fills out an angle with vertex at g^2 in degree (40,8) (i.e. g^2 has stem 40 and Adams filtration 8), bounded above by the line $f = \frac{1}{2}s - 12$, parallel to the Adams edge [1], and bounded below by the line $s = 5f$, in which f is the Adams filtration and s is the stem. The wedge family is a large piece of $\mathrm{Ext}_{\mathbf{A}_{\mathrm{cl}}}(\mathbb{F}_2, \mathbb{F}_2)$ which is regular, of considerable size and easy to understand.

Using this idea we build the motivic version of the wedge. However, it appears to be more complicated than the classical one. The motivic wedge family takes the same position and same shape as the classical one. However the vertex of the motivic wedge is at τg^2 in degree $(40, 8, 23)$ having weight 23. Note that g^2 in degree $(40, 8, 24)$ does not survive the motivic May spectral sequence [18]. Our main result, Theorem 5.8, states that the subsets $\tau^k \mathbf{P}^i \mathbf{g}^j \lambda$ are non-empty and consist of non-zero elements for all λ in $\mathbf{\Lambda}$.

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Publications and Preprints

1. Hieu Thai, *The $\mathbb{C}$-motivic wedge family of the cohomology of the motivic Steenrod algebra*, Homology Homotopy Appl. (2020, accepted).
2. (with C. T. Le) *JensenâĂŹs functional equation on the symmetric group S_n*, Aequationes Math. **82** (2011), no. 3, 269-276.