January 2019

Numerical Approaches To A Thermoelastic Kirchhoff-Love Plate System

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NUMERICAL APPROACHES TO A THERMOELASTIC KIRCHHOFF-LOVE PLATE SYSTEM

by

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DISSERTATION

Submitted to the Graduate School,

of Wayne State University,

Detroit, Michigan

in partial fulfillment of the requirements

for the degree of

DOCTOR OF PHILOSOPHY

2019

MAJOR: MATHEMATICS

Approved By:

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Advisor Date

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DEDICATION

To My Parents and girlfriend

ACKNOWLEDGEMENTS

First, I would like give the most sincere thanks to my dissertation advisor Professor Zhimin Zhang. If not his colloquium in my college China University of Geosciences, I would never have the chance to enter the Ph.D program at Wayne State University. As to my advisor's huge efforts in mentoring and his infinite patience, those are the keys to the finish of this dissertation. Eternally grateful, I will always feel towards to your fully support and help.

Here I would like extend my sincere gratitude to Dr. Ming Sun, Pro. Hao Pan. They are visiting scholars from Shangdong University, and visiting periods 2016-2017, 2017-2018 respectively. When I lose all my confidence and all my guts to dive into studies, they tutored me, and guided me in a right path. Without their help, I might be dragged through past 6 years, and wasting those past 6 years. And special credits shall be given to Dr. Xiang Wan, he is a postdoc at out department. This dissertation is partially based on his research. If not his and Pro. Pan's proposal, perhaps I am still struggling in choosing the topic for dissertation.

To Department of Mathematics at Wayne State University, thank you for awarding me teaching assistantship in 2013, which enabled me come to the United States and totally changed my life. I wouldn't have gone this far without the department's financial support and generous assistance during my study. I appreciate all the classes, trainings and teaching experiences I received from the department. They are invaluable treasure and will affect me in the rest of my life. If not this experience, I would never find out how to de-puzzle the system, understanding how the world running.

Other than that, I wish thank Pro.Lin Mu, I jumped to weak galerkin method quite late, and asked her some stupid questions. She gave me all perfect and useful answers right away. Even once, she helped me locating a small yet unnoticeable bug in the code. I would thank Pro. Waixiang Cao, if not her questions and discussions, I could still be doubtful about my programming. Also I will put some credits on Pro. Waixiang Cao, if not her addressing my doubts, I probably still get stuck somewhere tricky.Thanks Pro. Hailong Guo, he is a lecturer at Melbourne University, and was my roommate, without his support mentally and inspiration, maybe I would have quit the Ph.D program.

Still I remembered those days in CSRC in summer 2016, without Dr. Yang Wang, Mr. Xiaobo Zhang and other pals, I would have endured a boring and burdensome summer days. There I had the chance to do the maintenance of their clusters and solved the routing problems.

Meanwhile, I would love to share my sincerest gratitude to Mrs. Yali Han, Mr. Jia Zhao, Mrs. Haiting Xia, Mrs. Jiawei Dong, they made me feel warm and hopeful. If not their accompany, the living should be much less colorful. Also, I would like to thank my beloved parents for their unconditional love, support, and encouragement for all the past years. I won't be this stronger without both of you as my inspiration. I love you guys all so much.

Finally, the essential thanks would be credited to my girlfriend, if not her support of 2018-2019, I could've literally endured a really harsh life in USA, and not even to mention finishing this dissertation.

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CHAPTER 1 INTRODUCTION

The kirchhoff-love plate is a mathematical model used to determine the stresses and deformations in thin plates subjected to forces and moments under two dimensional case. It is the extension of Euler-Bernoulli beam theory and was developed in 1888 by Love using assumptions proposed by Kirchhoff. The theory assumes that a mid-surface plane can be used to represent a three-dimensional form. There are three assumptions for Kirchhoff-Love plate theory:

- straight lines normal to the mid-surface remain straight after deformation.
- straight lines normal to the mid-surface remain normal to the mid-surface after deformation.
- the thickness of the plate does not change during a deformation.

For the thermoelastic Kirchhoff-love plate, additional assumptions are added[1]:

- strains can be linearly decomposed into elastic and thermal ones.

This thermoelastic Kirchhoff-love plate is a coupling system with parabolic-like properties.

Recently, numerous mathematical models has rised in engineering areas, such as continuum mechanics in system of equations with various physical quantities, that needs different numerical approximations compared to common ones. The finite element approximations of such problems with extra independent variables are called mixed finite element methods. This thesis first introduce the basic concepts from the theory of mixed finite element methods, and how to get variational form of our kirchhoff-love plate system. After that, it demonstrates how to solve the system with mixed finite element method and error estimates. For further theoretical details, the reader is referred to the monographs by Boffi, Brezzi and Fortin, Brenner and Scott, Ern and Guermond, Gatica and Girault and Raviart.

Since H^1 Galerkin method requires the C^1 continuity of the finite element, it has been attractive and widely used by researchers for parabolic and hyperbolic equations. Earlier mixed element finite element methods were studied in [2] [3] [4] for elliptic equations, [5] [6] for parabolic equations, and [7][8] for hyperbolic equations. However, mixed element method has the requirement of LBB consistency condition in general, and that limits the choice of finite element spaces. To overcome such difficulty, Pani proposed $H^1 - Galerkin$ method for parabolic problems in [9][10]. For hyperbolic problems, [11] reformulate the problem as a first-order system and propose least square approaches for solution and flux. As two different V_h and W_h are used, different polynomials orders could apply respectively. Besides, the main advantage over the standard mixed element method is that, it does not require LBB condition.

Discontinuous Galerkin method(DG) has been active for hyperbolic and nearly hyperbolic equations since Reed and Hill first introduced the DG in [12]. Since that time, DG method has also been applied to elliptic problems [13] and parabolic problems[6]. Bassi and Rebay[14], studied the variations and generalize this method, introduced the local discontinuous galerkin method(LDG). Meanwhile, interior penalty method for DG developed independently almost the same time in 1970's. The first DG method for acoustic wave equation with second formulation was proposed by Rivie$\grave{r}$ in [15] used a nonsymmetric interior penalty form. It needs extra stablization terms for optimal L^2 convergence rate. Symmetric Interior Discontinuous Galerkin(SIP-DG) was presented for the time-dependent wave equation in [16]. For SIP-DG, symmetrical discretization of the wave equation can guarantee the stiffness matrix is symmetric positive, semi-discrete formulation is energy-conserved for all time.

The rest of this dissertation is organized as follows:

Chapter 2 introduces the sobolev spaces, basic theorems and lemmas from inequalities and finite element spaces. Also reviews the Lagrange elements, $H(div)$ elements and Discontinuous Galerkin methods. In section 2.5, it introduces how to establish the thermoelastic Kirchhoff-Love plate system. introduces the thermoelastic equations and theoretical background of finite element method.

Chapter 3 reviews the general theory of mixed element methods and gives the equivalent variational form for the KL system in section 3.1 and 3.2 respectively. Then it demonstrates the proof of semi discrete and fully discrete error estimates and the existences and uniqueness of solutions under those two situations in Section 3.3 and 3.4 Then Chapter 3 shows the numerical experiments and conclusions. reviews the common mixed element method, and gives out the numerical approach to the thermoelastic kirchhoff-love plate system.

Chapter 4 mainly talks about the Interior Penalty- Discontinuous Galerkin method(IP-DG). First establish the corresponding variational form. Later semi and fully discrete analysis for IP-DG are analyzed.

Chapter 5 is about the main results of H^1 Galerkin method for the KL system. We establish the variational problem using H^1 Galerkin method, and shows the semi and fully discrete analysis outcomes. Then numerical experiments are conducted and numerical results are presented.

CHAPTER 2 PRELIMINARIES

2.1 Sobolev spaces

In this work, we assume $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary $\partial\Omega$. Let f denote the Lebesgue integrable function on the domain Ω.

For $1 \le p \le \infty$, let

$$\|f\|_{p,\Omega} = \left(\int_\Omega |f|^p d\mathbf{x} \right)^{1/p}$$

and if $p = \infty$,

$$\|f\|_{0,\infty,\Omega} = ess \sup_{\mathbf{x}\in\Omega} \{|f(\mathbf{x})|\}$$

Definition 2.1 ($W_p^k(\Omega)$ space and sobolev norms)**.**

$$W_p^k(\Omega) := \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega), |\alpha| \le k\}, \quad \text{for } 1 \le p \le \infty$$

With associated sobolev norm,

$$\|u\|_{W_p^k(\Omega)} := \left(\sum_{|\alpha|\le k} \|D^\alpha u\|_{L^p(\Omega)}^p \right)^{1/p} \quad \text{when } 1 \le p \le \infty$$

And

$$\|u\|_{W_\infty^k(\Omega)} := \left(\sum_{|\alpha|\le k} \|D^\alpha u\|_{L^\infty(\Omega)} \right) \quad \text{when } p = \infty$$

Also we have the semi norms

$$|u|_{W_p^k(\Omega)} := \left(\sum_{|\alpha|= k} \|D^\alpha u\|_{L^p(\Omega)}^p \right)^{1/p} \quad \text{when } 1 \le p \le \infty$$

And

$$|u|_{W^k_\infty(\Omega)} := \left(\sum_{|\alpha|=k} \|D^\alpha u\|_{L^\infty(\Omega)} \right) \quad \text{when } \ p = \infty$$

Definition 2.2 (Weak Derivative)**.** If $u(\mathbf{x}) \in L^2(\Omega)$ has a derivative of order α, α is a multi-index of non-negative integers and $|\alpha| = \sum_{i=1}^{n} \alpha_i$. Provided $v \in L^2(\Omega)$, and

$$\int_\Omega D^\alpha u(\mathbf{x}) \cdot v(\mathbf{x}) d\mathbf{x} = (-1)^{|\alpha|} \int_\Omega D^\alpha v(\mathbf{x}) \cdot u(\mathbf{x}) d\mathbf{x} \quad |\alpha| \leq k, \forall v(\mathbf{x}) \in C_0^\infty(\Omega)$$

If such v exists, then we define $D^\alpha u = v$.

When $p = 2$, $H^k(\Omega) = W^{k,2}(\Omega)$ called Hilbert space and the index p is omitted in their corresponding norms and seminorms. The corresponding inner product,

$$(u, v)_{W^k_2(\Omega)} := \sum_{|\alpha| \leq k} (D^\alpha u, D^\alpha v)$$

Remark. This dissertation will quite frequently refer to $H^1(\Omega)$ and $H^2(\Omega)$ spaces.

$$H^1(\Omega) = \{u \in L^2(\Omega) : \frac{\partial u}{\partial x_j} \in L^2(\Omega), j = 1, \cdots, n\}$$
$$H^2(\Omega) = \{u \in L^2(\Omega) : \frac{\partial u}{\partial x_j} \in L^2(\Omega), j = 1, \cdots, n; \frac{\partial^2 u}{\partial x_i \partial x_j} \in L^2(\Omega), i, j = 1, \cdots, n\}$$

Meanwhile, assuming $\partial\Omega$ is sufficiently smooth,

$$H^1_0(\Omega) = \{u \in H^1(\Omega), u = 0 \text{ on } \partial\Omega\}$$

Let $P_n(K)$ be the space of polynomials of degree less than or equal to n over K and $N(k)$ be the dimension of $P_k(K)$ with $N(k) = \frac{1}{2}(k+1)(k+2)$. We may denote $\|\cdot\|_{W^k_p(\Omega)}$ as

$\|\cdot\|_{k,p,\Omega}$, $|\cdot|_{W_p^k(\Omega)}$ as $|\cdot|_{k,p,\Omega}$, $\|\cdot\|_{H^k(\Omega)}$ as $\|\cdot\|_{k,\Omega}$ and $|\cdot|_{H^k(\Omega)}$ as $|\cdot|_{k,\Omega}$, the domain Ω can be abbreviated in the text.

2.2 Useful Inequalities

Throughout this article, the letter C or c, with or without subscript, denotes a generic constant which is independent of h and may not be the same at each occurence.

Lemma 2.1 (Green formula). *Let Ω be bounded with Lipschitz continuous boundary $\partial\Omega$, for any $u, v \in H^1(\Omega)$, then*

$$\int_\Omega u \cdot \partial_i v dx = -\int_\Omega \partial_i u \cdot v dx + \int_{\partial\Omega} uv \cdot \mathbf{n_i} ds, \quad i = 1, 2, \cdots, n$$

Also, replacing u by ∂u, we get,

$$\int_\Omega \nabla u \cdot \nabla v dx = -\int_\Omega \triangle u \cdot v dx + \int_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}} v ds, \quad u \in H^2(\Omega), \forall v \in H^1(\Omega)$$

Also,

$$\int_\Omega \nabla u \cdot \vec{v} dx = -\int_\Omega u \nabla \cdot \vec{v} dx + \int_{\partial\Omega} u \vec{v} \mathbf{n} ds, \quad u \in H^1(\Omega), \forall \vec{v} \in L^2(\Omega)$$

Lemma 2.2 (Holder inequality). *If $1 < p, q < \infty$, and $\frac{1}{p} + \frac{1}{q} = 1$, if $f \in L^p(\Omega), g \in L^q(\Omega)$, then*

$$|\int_\Omega f(x)g(x)dx| \leq (\int_\Omega |f(x)|^p dx)^{\frac{1}{p}} (\int_\Omega |g(x)|^q dx)^{\frac{1}{q}}$$

Lemma 2.3 (Young inequality). *For $\forall \epsilon > 0, a, b \in R$, then we have,*

$$ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon}$$

Lemma 2.4 (Gronwall inequality). *Let $u(t)$ be continuous on $[0,T]$, suppose that $u(t) \geq 0$ and $\Phi(t) \geq 0$, $u_0 \geq 0$ is a constant, if u satisfies the inequality:*

$$u(t) \leq u_0 + \int_0^t \Phi(\tau)u(\tau)d\tau, \quad \forall t \in [0,T]$$

then:

$$u(t) \leq u_0 \cdot exp(\int_0^t \Phi(\tau)d\tau), \quad \forall t \in [0,T] \tag{2.1}$$

Lemma 2.5 (Discreted Gronwall inequality). *let $\{u_n\}, \{\xi_n\}, \{\Phi_n\}$ be nonnegative series, if $u_0 \leq \xi_0$, $u_n \leq \xi_n + \sum_j \Phi_j u_j$, $n \geq 1$, then:*

$$u_n \leq \xi_n + \sum_{0<k\leq n} \xi_k \Phi_k exp(\sum_{k<j<n} \Phi_j), \quad n \geq 1 \tag{2.2}$$

among that, $\Phi_j \geq 0$, and Φ_n is nonnegative monotone nondecreasing.

2.3 Finite Element Spaces

Lemma 2.6 (Lax-Milgram theorem). *Let V be Hilbert space, $a(\cdot,\cdot) : V \times V \rightarrow$ is a bilinear form, and $L(\cdot)$ a linear form, and let those three conditions hold:*

1. *(Coercivity) $a(u,u) \geq \alpha \|u\|_V^2$, $\quad \forall u \in V$*
2. *(Continuity) $a(u,v) \leq C\|u\|_V \|v\|_V$, $\quad \forall u, v \in V$*
3. *$L(v) \leq D\|v\|_V$, $\quad \forall v \in V$*

Then the problem: Find $u \in V$, such that

$$a(u,v) = L(v), \quad \forall v \in V \tag{2.3}$$

is well-posed and there exists a unique solution if with following stability condition.

$$\|u\|_V \leq \frac{C}{\alpha}\|L\|_{V'}$$

where V' is the dual space of V, $\|\cdot\|_V$ means the norm defined on V.

Lemma 2.7 (Cea Lemma)**.** *If bilinear form $a(\cdot,\cdot)$ is continuous and coersive, let $L : V \to R$ be a bounded linear operator, V_h is finite dimensional subspace of V, consider the problem: Find $u_h \in V_h$ such that*

$$a(u_h, v_h) = L(v_h), \quad \forall v \in V_h \tag{2.4}$$

then there exists a constant C, such that

$$\|u - u_h\|_V \leq C \inf_{v_h \in V_h} \| u - v_h\|_V$$

where u and u_h are the solutions of Eq.(2.3) and Eq.(2.4) respectively.

Applying the Lax-Milgram lemma, we know the discrete problem Eq.(2.4) has a unique solution u_h.

Lemma 2.8 (First Strang Lemma)**.** *Let $V_h \subset V$, and let $a_h(\cdot,\cdot)$ is a continuous and coercive bilinear form in $V_h \times V_h$, there exists a constant α such that*

$$a_h(v_h, v_h) \geq \alpha\|v_h\|^2 \quad \forall v_h \in V_h$$

Then there exists a constant C such that

$$\|u-u_h\| \leq C\left(\inf_{v_h\in V_h}\| u-v_h\|_{V_h}+\sup_{w_h\in V_h}\frac{|a_h(u,w_h)-a(v_h,w_h)|}{\|w_h\|_{V_h}}\right)$$

If $a_h(u_h,v_h)=a(u_h,v_h)$, it is called conforming finite element method, otherwise nonconforming finite element method.

Definition 2.3 (Finite Element Method)**.** The finite element method, in its simplest form, constructs finite dimensional subspaces V_h, solve variational problems related to BVP, IVP. This V_h is called finite element space.

To construct V_h, a finite element triple $(\mathcal{T}_h,\Pi_h,\Sigma_h)$ is established. $\mathcal{T}_h$ is a triangulation established on $\bar{\Omega}$, the set $\bar{\Omega}$ is subdivided into a finite number of subsets K, called elements with following properties:

1. $\bar{\Omega}=\bigcup_K \in \mathcal{T}_h$.

2. For each $K\in\mathcal{T}_h$, the set K is closed and its interior $\mathring{K}$ is nonempty.

3. For each pair $K_1,K_2\in\mathcal{T}_h$, one has $\mathring{K}_1\bigcap\mathring{K}_2=\emptyset$.

4. For each $K\in\mathcal{T}_h$, the boundary ∂K is Lipschitz continuous.

Second, Π_h is a subspace of $C(K)$ with finite dimension n, can be chosen as $P_n(K)$ or $Q_n(K)$.

Third, we also assume, $\mathcal{T}_h$ satisfies following condition: For any $K\in\mathcal{T}_h$, let h_T denote the diameter of K and ρ_K denote the supremum of the diameter of the spheres inscribed in K. The mesh size of $\mathcal{T}_h$ is denoted by $h=\max\limits_{K\in\mathcal{T}_h} h_K$. We say $\mathcal{T}_h$ is regular if there exists a constant σ such that

$$\frac{h_K}{\rho_K} \leq \sigma, \quad \forall K \in \mathcal{T}_h$$

In the thesis, we always assume mesh $\mathcal{T}_h$ is regular.

A mesh $\mathcal{T}_h$ is called quasi-uniform mesh if there exists a constant $\mu \geq 0$ such that

$$\frac{h}{h_K} \leq \mu, \quad \forall K \in \mathcal{T}_h$$

2.3.1 Lagrange Elements

For any $K \in \mathcal{T}_h$, let a_j, be the vertices of K for $1 \leq j \leq 3$. For any $k > 0$, let

$$\Sigma_k(T) = \{x = \sum_{j=1}^{3} \lambda_j a_j; \sum_{j=1}^{3} \lambda_j = 1, \lambda_j \in \{0, \frac{1}{k}, \cdots, \frac{k-1}{k}, 1\}, 1 \leq j \leq 3\}$$

The C^0 finite element space of order k associated with mesh $\mathcal{T}_h$ is defined as

$$S^{h,k} = \{v \in C(\Omega) : v|_K \in P_k(K), \forall K \in \mathcal{T}_h\}$$

Then the C^0 Lagrange element of degree k is defined as $(K, P_k(K), \Sigma_k(K))$. Typical examples of C^0 Lagrange elements include linear element and quadratic element. For linear element, degree of freedom only contains vertices. For quadratic element, degrees of freedom include both vertices and edge centers. For cubic element, degrees of freedom include both vertices, element center and trisection points on each edge.

Remark. Note that the choice of $\Sigma_k(K)$ guarantees the continuity of v across the boundaries of elements in $\mathcal{T}_h$. Let N_h denote the set of all mesh nodes.

Corollary 2.8.1. *Let $I_h : C(\bar{\Omega}) \to S^{h,k}$ denotes the Lagrange interpolation operator, i.e. $I_h u = \Sigma u(\mathbf{x})\Phi(\mathbf{x}), \forall u \in C(\bar{\Omega})$, $\Phi(\mathbf{x})$ denotes the global basis function, $\mathbf{x}$ is the node point. Then $\forall v \in S^{h,k}$, v can be written as $v = \Sigma v(\mathbf{x})\Phi(\mathbf{x})$.*

Theorem 2.9. *Let $\mathcal{T}_h$ be a mesh of Ω. Let $k \geq 1$. Then a piecewise function $v \in C^\infty : \bar{\Omega} \to R$ over the mesh $\mathcal{T}_h$ belongs to $H^k(\Omega)$ if and only if $v \in C^{k-1}(\bar{\Omega})$.*

Remark.

$$\begin{aligned} C^0(\bar{\Omega}) &= \{v : v \text{ is a continuous function defined on } \bar{\Omega}\} \\ C^1(\bar{\Omega}) &= \{v : v \in C^0(\bar{\Omega}) : D^\alpha v \in C^0(\bar{\Omega})\} \end{aligned} \tag{2.5}$$

The C^0 Lagrange elements are often referred to as conforming elements.

2.3.2 Conforming $H(div)$ elements

Definition 2.4 ($H(div, \Omega)$)**.** If Ω is with Lipschitz boundary $\partial\Omega$

$$H(div, \Omega) := \{\vec{q} \in L^2(\Omega)^2 |\, div\, \vec{q} \in L^2(\Omega)\}$$

with respect to the inner product

$$(\vec{u}, \vec{q})_{div,\Omega} := (\vec{u}, \vec{q})_{0,\Omega} + (\nabla \cdot\, \vec{u}, \nabla \cdot\, \vec{q})_{0,\Omega}$$

The associated norm can be denoted as $\|\cdot\|_{div,\Omega}$.

Lemma 2.10. *Let T_h be a triangulation of Ω and let element $K \in \mathcal{T}_h$, $\vec{W} := \{\vec{q} = (q_1, q_2)' | q_i : K \to R, K \in \mathcal{T}_h\}$, $V_h := \{\vec{q_h} : \bar{\Omega} \to R \,|\, |\vec{q_h}|_K \in \vec{W}\}$ be given by previous definition respectively,* *Assume that*

$$\vec{W} \subset H(div, K), K \in \mathcal{T}_h$$

$[\mathbf{n} \cdot \vec{q}|_e] = 0$ *for all* $e = K_i \cap K_j, \vec{q} \in V_h$ *where* $[\mathbf{n} \cdot q|_e]$ *denotes the jump of* $\mathbf{n} \cdot q$ *across the boundary* e*. ie,*

$$[\mathbf{n} \cdot \vec{q}|_e] := \mathbf{n} \cdot \vec{q}|_{e \bigcap K_i} - \mathbf{n} \cdot \vec{q}|_{e \bigcap K_j}$$

Then $V_h \subset H(div, \Omega)$*.*

2.3.3 Raviart-Thomas elements

Let us first consider the case of simplicial triangulations $\mathcal{T}_h$ of Ω, for $K \subset \mathcal{T}_h$ and $k \in N_0$. we set

$$\Phi_k(\partial K) := \{\phi \in L^2(\partial K)| \ \ \phi|_e \in P_k(e), e \in \partial K\} \quad \text{when } d = 2$$

For $\forall \vec{q} \in RT_k(K)$, the degrees of freedom Σ_k are given by

$$\int_{\partial K} \vec{q} \cdot \vec{n}\, p_k ds, \quad p_k \in \Phi_k(\partial K)$$
$$\int_K \vec{q} \cdot \vec{p}_{k-1} d\mathbf{x}, \quad \vec{p}_{k-1} \in P_{k-1}(K)^d$$

We have:

$$dim\ \mathbf{RT}_k(K) = (k+1)(k+3), \quad d = 2$$

Lemma 2.11. *There exist a constant* $C > 0$*, independent of the mesh, such that*

$$\|u - u_h\|_{1,\Omega} \leq C \inf \|u - v\|_{1,\Omega}$$

Taking v as the Lagrange interpolation of u, we can get the following H^1 error estimate.

Remark. If the solution u of the equation is in the space $H^{r+1} \bigcap H_0^1(\Omega)$ and $u_h \in S_0^{h,k}$ is the solution, then

$$\|u - u_h\|_{1,\Omega} \leq Ch^k |u|_{k+1,\Omega}$$

Using the duality argument, we can prove the following L^2 error estimate:

$$\|u - u_h\|_{0,\Omega} \leq Ch^{k+1} |u|_{k+1,\Omega}$$

2.4 Mixed Element Method

Lemma 2.12 (LBB condition)**.** *If V and H are two Hilbert spaces, suppose that $a(\cdot,\cdot) : V \times V \to R$ and $b(\cdot,\cdot) : V \times H \to R$ are both continuous bilinear forms, and moreover that a is coercive on the kernel of b:*

$$a(v, v) \geq \alpha \|v\|_V^2$$

for $\forall v$ such that $b(v, q) = 0$ for all $q \in H$. If $b(\cdot,\cdot)$ satisfies the inf-sup or Ladyzhenskaya-Babuska-Brezzi condition

$$\sup_{v \in V, v \neq 0} \frac{b(v, q)}{\|v\|_V} \geq \beta \|q\|_H$$

Let $F \in V'$ and $G \in H'$, consider the variational problem to find $u \in V$ and $p \in H$ such that

$$\begin{aligned} a(u, v) + b(v, p) &= F(v) \quad \forall v \in V \\ b(u, q) &= G(q) \quad \forall q \in H \end{aligned} \tag{2.6}$$

Lemma 2.13. *If $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ satisfy:*

1. *$a(\cdot,\cdot)$ is continuous on $V \times V$ and coercive on M.*

2. *$b(\cdot,\cdot)$ is continuous on $V \times H$ and satisfy LBB condition.*

Then eqn(2.6) has a unique solution $(u,q) \in V \times H$, and there exists a constant C such that,

$$\|u\|_V + \|p\|_H \leq C(\|F\|_{V'} + \|G\|_{H'})$$

Here $M = \{v \in V|\, b(v,q) = 0, \forall q \in H\}$.

Consider the discrete problem of eqn(2.6), find $(u_h, p_h) \in V_h \times H_h$ such that

$$\begin{aligned} a_h(u_h, v_h) + b_h(v_h, p_h) = F(v_h) \quad \forall v_h \in V_h \\ b_h(u_h, q_h) = G(q_h) \quad \forall q_h \in H_h \end{aligned} \tag{2.7}$$

If V_h,H_h are conforming finite element spaces, then $a_h(u_h, v_h) = a(u_h, v_h)$ and $b_h(u_h, v_h) = b(u_h, v_h)$. Next, consider the discrete version of Lemma(2.13),

Lemma 2.14. *If $a_h(\cdot,\cdot)$ and $b_h(\cdot,\cdot)$ satisfy:*

1. *$a_h(\cdot,\cdot)$ is continuous on $V_h \times V_h$ and coercive on $M_h \times M_h$.*

2. *$b_h(\cdot,\cdot)$ is continuous on $V_h \times H_h$ and satisfy discrete LBB condition.*

Then Eq.(2.7) has a unique solution $(u_h, q_h) \in V_h \times H_h$, and there exists a constant β^* such that,

$$\sup_{v_h \in V_h} \frac{b_h(v_h, q_h)}{\|v_h\|_{V_h}} \geq \beta^* \|q_h\|_{H_h} \quad \forall q_h \in H_h$$

Here $M_h = \{v_h \in V_h | b_h(v_h, q_h) = 0, \forall q_h \in H_h\}$.

2.5 Discontinuous Galerkin Method

First consider the case of simplicial triangulations $\mathcal{T}_h$ of Ω, for $K \subset \mathcal{T}_h$ and $k \in N_0$. Define the DG space

$$\mathcal{S}_h = \{v \in L_2(\Omega) : v|_K \in P_k(K), \forall K \in \mathcal{T}_h\}$$

and $P_k(K)$ defines the space of polynomials degree not greater than k. Try to intrduce the Broken Sobolev space

$$H^s(\mathcal{T}_h) = \{v \in L_2(\Omega) : v|_K \in H^s(K), \forall K \in \mathcal{T}_h\}, s > \frac{1}{2}$$

Denote $\Gamma_h = \bigcup\{e \subset \partial K : K \in \mathcal{T}_h\}$, when $v \in H^s(\mathcal{T}_h)$, from trace theorem, $v \in L_2(\Gamma_h) \equiv \prod_{e \in \Gamma_h} L_2(e)$. Let $v \in H^s(\mathcal{T}_h), s > \frac{1}{2}$, K_1 and K_2 are two adjacent elements with intersection at $e = \partial K_1 \cap \partial K_2$. Use $v_i = v|_{\partial K_i}$ denotes the trace of function v restricted on edge e from element K_i, $\mathbf{n}_i = \mathbf{n}|_{\partial K_i}$ is outer normal vector. To deal with discontinuity across the interior edge $e \in \Gamma_h^0$, where $\Gamma_h^0 = \Gamma_h \backslash \partial\Omega$, $\Gamma_{h,D} = \Gamma_h^0 \bigcup \Gamma_D$, Γ_D is Dirichlet boundary. It is necessary to define jump $[v]$ and average $\{v\}$,

$$[v] = v_1\mathbf{n_1} + v_2\mathbf{n_2}, \quad \{v\} = \frac{1}{2}(v_1 + v_2), \quad e \in \Gamma_h^0$$

Consider the vector, $\vec{\tau} \in [H^s(T_h)^2]$, define:

$$[\tau] = \mathbf{v_1} \cdot \mathbf{n_1} + \mathbf{v_2} \cdot \mathbf{n_2}, \quad \{\tau\} = \frac{1}{2}(\tau_\mathbf{1} + \tau_\mathbf{2}), \quad e \in \Gamma_h^0$$

If $e \in \partial\Omega$, define:

$$[v] = v\mathbf{n}, \quad \{v\} = v; \quad [\vec{\tau}\,] = \tau \cdot \mathbf{n}, \quad \{\vec{\tau}\} = \vec{\tau}$$

Lemma 2.15. *Let* $(v, \vec{\tau}) \in H^s(\mathcal{T}_h) \times [H^s(\mathcal{T}_h)]^2, s > \frac{1}{2}$*, then there exist,*

$$\sum_{K\in\mathcal{T}_h} \int_{\partial K} v\vec{\tau} \cdot \mathbf{n} ds = \sum_{e\in\Gamma_h} \int_e \{\vec{\tau}\} \cdot [v] ds + \sum_{e\in\Gamma_h^0} \{v\} \cdot [\tau] ds \tag{2.8}$$

Proposition 1. For disconinuous galerkin approximation of elliptic problem, find $u_h \in S_h$, such that

$$a_h(u_h, v_h) = \langle f, v_h \rangle, \quad \forall v_h \in S_h$$

We may introduce the bilinear form

$$a_h(u_h, v_h) = (\nabla u, \nabla v)_w = \sum_{K\in\mathcal{T}_h} (\nabla u, \nabla v)_K - \sum_{e\in\Gamma_{h,D}} \int_e \langle \nabla u \rangle \cdot \mathbf{n}[v] dS + \epsilon \sum_{e\in\Gamma_{h,D}} \int_e \langle \nabla v \rangle \cdot \mathbf{n}[u] dS$$
$$+ \sum_{e\in\Gamma_{h,D}} \frac{\gamma}{h_e} \int_e [u][v] ds$$

where γ is the penalty parameter, $h_e = diam(e)$, ϵ can be $-1, 0, 1$.

Introduce the DG norm:

$$|||u|||^2 = \sum_{K\in\mathcal{T}_h} \|\nabla u\|_{0,K}^2 + \sum_{e\in\Gamma_{h,D}} h_e^{-1} \|[u]\|_{0,e}^2$$

Lemma 2.16. *There exists* $\beta > 0$*, independent of the mesh size, such that*

$$a_h(u_h, u_h) \geq \gamma |||u_h|||^2$$

Lemma 2.17. *For quasi-uniform meshes $\mathcal{T}_h$, there holds*

$$a_h(u_h, u_h) \geq C \max\{1, \beta\} |||u_h|||^2$$

with a stability constant $C > 0$ that is independent of the mesh size.

Lemma 2.18. *If $a_h(u, v)$ is bounded for norm $|||\cdot|||$ on $H^{1+s}(\mathcal{T}_h)$, then there exists a constant $C > 0$, such that*

$$|a_h(u, v)| \leq M |||u||| \cdot |||v|||, \quad \forall u, v \in H^{1+s}(\mathcal{T}_h), s \geq \frac{1}{2}$$

Definition 2.5. Let $u \in H^2(\Omega)$, the L^2 projection $P_h u \in S_h$ of u is defined by requiring that

$$a_h(P_h u, v) = a_h(u, v), \quad \forall v \in V_h$$

Lemma 2.19. *If additionally $u \in H^{k+1}(\Omega)$ for $k \geq 1$, then*

$$|||u - P_h u||| \leq C h^k \|u\|_{k+1}$$

$$\|u - P_h u\| \leq C h^{k+1} \|u\|_{k+1}$$

2.6 Thermoelastic Kirchhoff & Love Plate Model

Let $\Omega \in R^2$ be a bounded domain with a smooth boundary representing the midplane of a thermoelastic plate. u denotes the deflection and θ is the thermo moment based on plate thickness. If denoting the thickness of the plate as h, the complete domain of this plate is $\Omega \times (h/2, h/2)$ in R^3. Besides, assume the elasticity, thermal isotropy of this plate, and small heat flux. What is more, linearize this plate, and strains are composite of elastic and thermal parts.

If denoting the displacement vectors as $\mathbf{U} = (U_1, U_2, U_3)$ and $\mathbf{u} = (u_1, u_2, u_3)$. Then the stresses and strains represented as

$$\sigma = (\sigma_{ij}) \quad \text{and} \quad \epsilon = \frac{1}{2}(\nabla \mathbf{U} + (\nabla \mathbf{U})^T)$$

By the assumptions, the stresses and strains can be decomposed into elastic and thermal parts respectively, $\sigma = \sigma^{elastic} - \sigma^{thermal}$

$$\begin{cases} \sigma_{11} &= \dfrac{E}{1-\mu^2}(\epsilon_{11} + \mu\epsilon_{22}) - \dfrac{E}{1-\mu}\epsilon^T \\ \sigma_{22} &= \dfrac{E}{1-\mu^2}(\mu\epsilon_{11} + \epsilon_{22}) - \dfrac{E}{1-\mu}\epsilon^T \\ \sigma_{ij} &= \dfrac{E}{1+\mu}\epsilon_{ij} \quad (i \neq j) \\ \sigma_{33} &= 0 \end{cases} \tag{2.9}$$

Consider small displacement,

$$U_1 = u_1 - z\frac{\partial u_3}{\partial x}, \, U_2 = u_2 - z\frac{\partial u_3}{\partial y}, \, U_3 = u_3$$

Besides,

$$\begin{cases} \epsilon_{11} &= \dfrac{\partial u_1}{\partial x} - z\dfrac{\partial^2 u_3}{\partial x^2} \\ \epsilon_{22} &= \dfrac{\partial u_2}{\partial y} - z\dfrac{\partial^2 u_3}{\partial y^2} \\ \epsilon_{ij} &= \dfrac{1}{2}(\dfrac{\partial u_1}{\partial y} + \dfrac{\partial u_2}{\partial x} - 2z\dfrac{\partial^2 u_3}{\partial x \partial y}) \\ \epsilon_{13} &= \epsilon_{23} = \epsilon_{33} = 0 \end{cases} \tag{2.10}$$

For the bending component, u_3, replaced by w.

$$\rho h w_{tt} - \frac{\rho h^3}{12}\triangle w_{tt} + D(\triangle^2 w + \frac{1+\mu}{2}\triangle\theta) = f$$

Assume the change τ in the temperature is small compared to the reference temperature T_0 and $\epsilon^T = \alpha\tau$, now we have the thermal-strain-displacement relations,

$$\rho c\theta_t - \lambda_0\triangle\theta + \frac{12\lambda_0}{\rho c h^2}(\frac{h\lambda_1}{2} + 1)\theta + \frac{\alpha\gamma}{\rho c}\triangle w_t = 0$$

Here, for simplicity, we consider a simply supported plate held at the reference temperature at the boundary:

$$w = \triangle w = \theta = 0$$

The nonlinear Kirchhoff & Love thermoelastic plate system can be written as,

$$\begin{cases} u_{tt} - \triangle u_{tt} + a(-\triangle u)\triangle^2 u + \alpha \triangle \theta = f(-\triangle u) \\ \theta_t - \triangle \theta + \theta - \alpha \triangle u_t = 0 \\ u|_{\partial\Omega} = \triangle u|_{\partial\Omega} = \theta|_{\partial\Omega} = 0 \\ u(\mathbf{x}, 0) = u^0(\mathbf{x}), u_t(\mathbf{x}, 0) = u^1(\mathbf{x}), \theta(\mathbf{x}, 0) = \theta^0(\mathbf{x}) \end{cases}$$

Here we mainly consider:

$$\begin{cases} u_{tt} - \triangle u_{tt} + \triangle^2 u + \alpha \triangle \theta = f \\ \theta_t - \triangle \theta + \theta - \alpha \triangle u_t = g \\ u|_{\partial\Omega} = \triangle u|_{\partial\Omega} = \theta|_{\partial\Omega} = 0 \\ u(\mathbf{x}, 0) = u^0(\mathbf{x}), u_t(\mathbf{x}, 0) = u^1(\mathbf{x}), \theta(\mathbf{x}, 0) = \theta^0(\mathbf{x}) \end{cases}$$

CHAPTER 3 STANDARD MIXED ELEMENT METHOD

In this section, we introduce how to use mixed element method, with basic definitions and properties. The existence of a unique solution of semi-discrete mixed element method and error analysis are given in section 2. And the existence of a unique of its fully discrete method and error analysis are considered in section 3. Numerical results are presented in section 4.

3.1 Semi-discrete Mixed Element Formulation

Let $\Omega \subset R^2$, we consider the linearized thermoelastical kirchhoff-love plate equation system.

Problem(I): Find $(u,\theta) \in (W^{1,\infty}(0,T;L^2(\Omega)) \bigcap L^\infty(0,T;W^{2,4}(\Omega))) \times L^\infty(0,T;W^{2,4}(\Omega))$ such that for all $T>0$,

$$\begin{cases} u_{tt} - \triangle u_{tt} + \triangle^2 u + \alpha \triangle \theta = f, \quad (\mathbf{x},t) \in \ \Omega \times (0,T) \\ \theta_t - \triangle \theta + \theta - \alpha \triangle u_t = g, \quad (\mathbf{x},t) \in \ \Omega \times (0,T) \\ u = \triangle u = \theta = 0, \quad (\mathbf{x},t) \in \ \partial\Omega \times (0,T) \\ u(\mathbf{x},0) = u^0(\mathbf{x}), u_t(\mathbf{x},0) = u^1(\mathbf{x}), \theta(\mathbf{x},0) = \theta^0(\mathbf{x}), \quad \mathbf{x} \in \ \Omega \end{cases} \tag{3.1}$$

where u denotes the displacement of the plate, θ denotes the displacement caused by therm changes, g, u^0, u^1 and θ^0 are given functions. Let $q = u_t, \sigma = -\triangle u$, the original problem(I) would be transformed into:

Problem(I*): Find $(q,\sigma,u,\theta) \in L^\infty(0,T;H^1(\Omega)) \times L^\infty(0,T;H_0^1(\Omega)) \times L^\infty(0,T;H_0^1(\Omega)) \times L^\infty(0,T;H_0^1(\Omega))$ such that, for all $T>0$

$$\begin{cases} q_t - \triangle q_t - \triangle u + \triangle \theta = f, \quad (\mathbf{x}, t) \in \Omega \times [0, T] \\ \theta_t - \triangle \theta + \theta - \triangle q = g, \quad (\mathbf{x}, t) \in \Omega \times [0, T] \\ u = \sigma = \theta = 0, \quad (\mathbf{x}, t) \in \partial\Omega \times [0, T] \\ u(\mathbf{x}, 0) = u^0(\mathbf{x}), \sigma(\mathbf{x}, 0) = \triangle u^0(\mathbf{x}), q(\mathbf{x}, 0) = u^1(\mathbf{x}), \theta(\mathbf{x}, 0) = \theta^0(\mathbf{x}), \quad (\mathbf{x}, t) \in \Omega \end{cases} \tag{3.2}$$

3.1.1 Semi-discrete Mixed Element Formulation

To implement the mixed element method, we consider the following weak formulation: Problem(I**): Find $(q, \sigma, u, \theta) : [0, T] \to V \times V \times V \times V$ such that, $\forall T > 0$, $\forall p, l, w, r \in V$

$$\begin{cases} (u_t, p) - (q, p) = 0, \quad (\mathbf{x}, t) \in \Omega \times [0, T] \\ (\sigma, l) - (\nabla u, \nabla l) = 0, \quad (\mathbf{x}, t) \in \Omega \times [0, T] \\ (q_t, w) + (\nabla q_t, \nabla w) + (\nabla \sigma, \nabla w) - (\nabla \theta, \nabla w) = (f, w), \quad (\mathbf{x}, t) \in \Omega \times [0, T] \\ (\theta_t, r) + (\nabla \theta, \nabla r) + (\theta, r) + (\nabla q, \nabla r) = (g, r), \quad (\mathbf{x}, t) \in \Omega \times [0, T] \\ u = \sigma = \theta = 0, \quad (\mathbf{x}, t) \in \partial\Omega \times [0, T] \\ u(x, 0) = u^0(x), \sigma(x, 0) = \triangle u^0(x), q(x, 0) = u^1(x), \theta(x, 0) = \theta^0(\mathbf{x}), \quad (\mathbf{x}, t) \in \Omega \end{cases} \tag{3.3}$$

Theorem 3.1. *Problem I^{**} has a unique solution.*

Proof. The existence of the solution follows from the existence and regularity assumption. By defining $q = u_t, \sigma = -\triangle u$, it immediates comes up with a weak solution for I^{**}. To prove the uniqueness of the solution, we need to prove the stability first, let $q_0^i, \sigma_0^i, u_0^i, \theta_0^i \in$

H^2 be the initial data, and $q^i, \sigma^i, u^i, \theta^i$ be the corresponding weak solutions. Then

$$\|\sigma^1 - \sigma^2\|_{L^\infty(0,T;L^2(\Omega))} + \|u^1 - u^2\|_{L^\infty(0,T;L^2(\Omega))} \leq C(T)\|u_0^1 - u_0^2\| \tag{3.4}$$

where $C(T)$ is a positive constant. Next to prove the stability result.

Denote $\tilde{q} = q^1 - q^2$, $\tilde{\sigma} = \sigma^1 - \sigma^2$, $\tilde{u} = u^1 - u^2$, $\tilde{\theta} = \theta^1 - \theta^2$.

$$\begin{cases} (\triangle \tilde{u}_t, p) - (\triangle \tilde{q}, p) = 0 \\ (\tilde{\sigma}, l) - (\nabla \tilde{u}, \nabla l) = 0 \\ (\tilde{q}_t, w) + (\nabla \tilde{q}_t, \nabla w) + (\nabla \tilde{\sigma}, \nabla w) - (\nabla \tilde{\theta}, \nabla w) = 0 \\ (\tilde{\theta}_t, r) + (\nabla \tilde{\theta}, \nabla r) + (\tilde{\theta}, r) + (\nabla \tilde{q}, \nabla r) = 0 \end{cases} \tag{3.5}$$

Choose $p = \triangle \tilde{u}$, $l = \triangle \tilde{q}$, $w = \tilde{q}$ and $r = \tilde{\theta}$, and adding up all four equations,

$$\frac{1}{2}\frac{d}{dt}\|\tilde{\sigma}\|^2 + \frac{1}{2}\frac{d}{dt}\|\tilde{q}\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla \tilde{q}\|^2 + \frac{1}{2}\frac{d}{dt}\|\tilde{\theta}\|^2 + \|\nabla \tilde{\theta}\|^2 + \|\tilde{\theta}\|^2 = 0 \tag{3.6}$$

Taking the integral, thus we have the stability. This completes the proof of the theorem.

□

3.1.2 The Existence and Uniqueness Semi-discrete Mixed Element Formulation

In this section, we demonstrate on the existence and uniqueness of the solution of system. Now we consider the following semi-discrete form for problem(I^{**}).

Problem(I_h): Find $(q_h, \sigma_h, u_h, \theta_h) : [0, T] \to V_h \times V_h \times V_h \times V_h$ such that $\forall T > 0$, $\forall p_h, l_h, w_h, r_h \in$

V_h

$$\begin{cases} (a)\ (u_{ht}, p_h) - (q_h, p_h) = 0, (x,t) \in \Omega \times [0,T] \\ (b)\ (\sigma_h, l_h) - (\nabla u_h, \nabla l_h) = 0, (x,t) \in \Omega \times [0,T] \\ (c)\ (q_{ht}, w_h) + (\nabla q_{ht}, w_h) + (\nabla \sigma_h, \nabla w_h) - (\nabla \theta_h, \nabla w_h) = (f, w_h), (x,t) \in \Omega \times [0,T] \\ (d)\ (\theta_{ht}, r_h) + (\nabla \theta_h, \nabla r_h) + (\theta_h, r_h) + (\nabla q_h, \nabla r_h) = (g, r_h), (x,t) \in \Omega \times [0,T] \end{cases} \tag{3.7}$$

with given $q_h(0), \sigma_h(0), u_h(0), \theta_h(0)$ determined.

Theorem 3.2. *Problem I_h has a unique solution.*

Proof. Denote $\tilde{q}_h = q_h^1 - q_h^2$, $\tilde{\sigma}_h = \sigma_h^1 - \sigma_h^2$, $\tilde{u}_h = u_h^1 - u_h^2$, $\tilde{\theta}_h = \theta_h^1 - \theta_h^2$.

For problem (I_h), work on eqn (c) and (d), substitute by $\tilde{w}_h = -\triangle \tilde{q}_h$ and $\tilde{r}_h = -\triangle \tilde{\theta}_h$,

$$\begin{cases} (\triangle \tilde{u}_{ht}, p_h) - (\triangle \tilde{q}_h, p_h) = 0 \\ (\tilde{\sigma}_h, l) - (\nabla \tilde{u}_h, \nabla l_h) = 0 \\ (\tilde{q}_{ht}, w_h) + (\nabla \tilde{q}_{ht}, \nabla w_h) + (\nabla \tilde{\sigma}_h, \nabla w_h) - (\nabla \tilde{\theta}_h, \nabla w_h) = 0 \\ (\tilde{\theta}_{ht}, r_h) + (\nabla \tilde{\theta}_h, \nabla r_h) + (\tilde{\theta}_h, r_h) + (\nabla \tilde{q}_h, \nabla r_h) = 0 \end{cases} \tag{3.8}$$

Choose $p_h = \triangle \tilde{u}_h$, $l_h = \triangle \tilde{q}_h$, $w_h = \tilde{q}_h$ and $r_h = \tilde{\theta}_h$, and adding up all four equations,

$$\frac{1}{2}\frac{d}{dt}\|\tilde{\sigma}_h\|^2 + \frac{1}{2}\frac{d}{dt}\|\tilde{q}_h\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla \tilde{q}_h\|^2 + \frac{1}{2}\frac{d}{dt}\|\tilde{\theta}_h\|^2 + \|\nabla \tilde{\theta}_h\|^2 + \|\tilde{\theta}_h\|^2 = 0 \tag{3.9}$$

Taking the integral, thus we have the stability. This completes the proof of the theorem.

□

3.1.3 Semi Discrete Estimates

First we need to introduce the elliptic projection, find R_hq, $R_h\sigma$, R_hu, $R_h\theta \in V_h$ satisfying:

$$\begin{aligned} &(\nabla(q - R_hq), \nabla v) = 0, \quad \forall v_h \in V_h \qquad (\nabla(u - R_hu), \nabla v) = 0, \quad \forall v_h \in V_h \\ &(\nabla(\sigma - R_h\sigma), \nabla v) = 0, \quad \forall v_h \in V_h \qquad (\nabla(\theta - R_h\theta), \nabla v) = 0, \quad \forall v_h \in V_h \end{aligned} \tag{3.10}$$

The following estimates are well known from [4]: for $j = 0, 1$

$$\begin{aligned} \|u - R_hu\|_j &\le Ch^{k+1-j}\|u\|_{k+1} & \|q - R_hq\|_j &\le Ch^{k+1-j}\|q\|_{k+1} \\ \|(u - R_hu)_t\|_j &\le Ch^{k+1-j}\|u_t\|_{k+1} & \|(q - R_hq)_t\|_j &\le Ch^{k+1-j}\|q_t\|_{k+1} \\ \|\theta - R_h\theta\|_j &\le Ch^{k+1-j}\|\theta\|_{k+1} & \|\sigma - R_h\sigma\|_j &\le Ch^{k+1-j}\|\sigma\|_{k+1} \\ \|(\theta - R_h\theta)_t\|_j &\le Ch^{k+1-j}\|\theta_t\|_{k+1} & \|(\sigma - R_h\sigma)_t\|_j &\le Ch^{k+1-j}\|\sigma_t\|_{k+1} \end{aligned} \tag{3.11}$$

Subtracting (I^{**}) from I_h, we obtain:

$$\begin{cases} \text{(a) } (u_{ht} - u_t, p_h) - (q_h - q, p_h) = 0 \\ \text{(b) } (\sigma_h - \sigma, l_h) - (\nabla u_h - \nabla u, \nabla l_h) = 0 \\ \text{(c) } (q_{ht} - q_t, w_h) + (\nabla q_{ht} - \nabla q_t, w_h) + (\nabla \sigma_h - \nabla \sigma, \nabla w_h) - (\nabla \theta_h - \nabla \theta, \nabla w_h) = 0 \\ \text{(d) } (\theta_{ht} - \theta_t, r_h) + (\nabla \theta_h - \nabla \theta, \nabla r_h) + (\theta_h - \theta, r_h) + (\nabla q_h - \nabla q, \nabla r_h) = 0 \end{cases} \tag{3.12}$$

Denote:

$$\begin{aligned}(\partial_t(q_h - R_h q), q_h - R_h q) &= \frac{1}{2}\frac{d}{dt}\|q_h - R_h q\|^2 \\ (\partial_t \nabla(q_h - R_h q), \nabla(q_h - R_h q)) &= \frac{1}{2}\frac{d}{dt}\|\nabla(q_h - R_h q)\|^2 \\ (\partial_t(\sigma_h - R_h\sigma), \nabla\partial_t(u_h - R_h u)) &= (\sigma_h - R_h\sigma, -\partial_t \triangle_h (u_h - R_h u)) = \frac{1}{2}\frac{d}{dt}\|(\sigma_h - R_h\sigma)\|^2 \\ (\partial_t(\theta_h - R_h\theta), \theta_h - R_h\theta) &= \frac{1}{2}\frac{d}{dt}\|\theta_h - R_h\theta\|^2\end{aligned} \tag{3.13}$$

Take decomposition:

$$\begin{aligned} q_h - q &= q_h - R_h q + R_h q - q \qquad & u_h - u &= u_h - R_h u + R_h u - u \\ \sigma_h - \sigma &= \sigma_h - R_h\sigma + R_h\sigma - \sigma \qquad & \theta_h - \theta &= \theta_h - R_h\theta + R_h\theta - \theta \end{aligned} \tag{3.14}$$

Then the equation system can be written as:

(a) $(u_{ht} - R_h u_t, p_h) - (q_h - R_h q, p_h) = -(R_h u_t - u_t, p_h) + (R_h q - q, p_h)$

(b) $(\sigma_h - R_h\sigma, l_h) - (\nabla u_h - \nabla R_h u, \nabla l_h) = -(R_h\sigma - \sigma, l_h) + (\nabla R_h u - \nabla u, \nabla l_h)$

(c)
$$\begin{aligned}&(q_{ht} - R_h q_t, w_h) + (\nabla q_{ht} - \nabla R_h q_t, w_h) + (\nabla\sigma_h - \nabla R_h\sigma, \nabla w_h) - (\nabla R_h\theta_h - \nabla R_h\theta, \nabla w_h) \\ &= -(R_h q_{ht} - q_t, w_h) - (\nabla R_h q_{ht} - \nabla q_t, w_h) - (\nabla R_h\sigma_h - \nabla\sigma, \nabla w_h) \\ &+ (\nabla R_h\theta_h - \nabla\theta, \nabla w_h)\end{aligned}$$

(d)
$$\begin{aligned}&(\theta_{ht} - R_h\theta_t, r_h) + (\nabla\theta_h - \nabla R_h\theta, \nabla r_h) + (\theta_h - R_h\theta, r_h) + (\nabla q_h - \nabla R_h q, \nabla r_h) \\ &= -(\theta_{ht} - R_h\theta_t, r_h) - (\nabla\theta_h - \nabla R_h\theta, \nabla r_h) - (\theta_h - R_h\theta, r_h) - (\nabla q_h - \nabla R_h q, \nabla r_h)\end{aligned}$$

(3.15)

Theorem 3.3. *With* $u_h(0) = \tilde{u}_h(0)$*,* $\sigma_h(0) = \tilde{\sigma}_h(0)$*,* $\theta_h(0) = \tilde{\theta}_h(0)$*,* $q_h(0) = \tilde{q}_h(0)$*, the following estimate holds:*

$$\|q - q_h\| + \|u - u_h\| + \|\sigma - \sigma_h\| + \|\theta - \theta_h\| \leq Ch^k$$

C *depends on* $\|q_t\|_{L^\infty(H^{k+1})}$*,* $\|\theta_t\|_{L^\infty(H^{k+1})}$*,* $\|\theta\|_{L^\infty(H^{k+1})}$*,* $\|\sigma\|_{L^\infty(H^{k+1})}$*,* $|q\|_{L^\infty(H^{k+1})}$*,* $\|u_t\|_{L^\infty(H^{k+1})}$*.*

Proof. Take docomposition:

$$q_h - q = q_h - R_h q + R_h q - q = \xi_1 + \eta_1 \qquad u_h - u = u_h - R_h u + R_h u - u = \xi_3 + \eta_3$$

$$\sigma_h - \sigma = \sigma_h - R_h \sigma + R_h \sigma - \sigma = \xi_2 + \eta_2 \qquad \theta_h - \theta = \theta_h - R_h \theta + R_h \theta - \theta = \xi_4 + \eta_4$$

Choose $w_h = q_h - R_h q$ and $r_h = \theta_h - R_h \theta$, sum Eq.(c) and Eq.(d).

$$(\xi_{1,t}, \xi_1) + (\nabla\xi_{1,t}, \nabla\xi_1) + (\nabla\xi_2, \nabla\xi_1) + (\xi_{4,t}, \xi_4) + (\nabla\xi_{4,t}, \nabla\xi_4) + (\xi_4, \xi_4)$$
$$= -(\eta_{1,t}, \xi_1) - (\nabla\eta_{1,t}, \nabla\xi_1) - (\eta_{4,t}, \xi_4) - (\eta_4, \xi_4)$$

As to:

$$(\xi_{1,t}, \xi_1) = \frac{d}{dt}\|\xi_1\|^2$$
$$(\nabla\xi_{1,t}, \nabla\xi_1)_w = \frac{d}{dt}\|\nabla\xi_1\|^2$$
$$(\xi_{4,t}, \xi_4) = \frac{d}{dt}\|\xi_4\|^2$$
$$(\nabla\xi_4, \nabla\xi_4)_w = \|\nabla\xi_4^n\|^2, \quad (\xi_4, \xi_4) = \|\xi_4\|^2$$

Besides,

$$(\nabla\xi_2, \nabla\xi_1) = (\xi_{2,t}, \xi_2) + (\eta_{2,t}, \eta_2) + (\xi_{2,t}, \eta_2) + (\eta_{2,t}, \xi_2)$$
$$= (\xi_{2,t}, \xi_2) + (\eta_{2,t}, \eta_2) + \frac{d}{dt}(\xi_2, \eta_2)$$

Let's deal with the right hand side,

$$\|(\eta_{1,t}, \xi_1)\| \leq C\|\eta_{1,t}\|^2 + C\|\xi_1\|^2 \leq Ch^{2(k+1)}\|q_t\|_{k+1}^2 + C\|\xi_1\|^2$$

$$\|(\nabla\eta_{1,t}, \nabla\xi_1)\| \leq C\|\nabla\eta_{1,t}\|^2 + C\|\nabla\xi_1\|^2 \leq Ch^{2k}\|q_t\|_{k+1}^2 + C\|\xi_1\|^2$$

$$\|(\eta_{4,t}, \xi_4)\| \leq C\|\eta_{4,t}\|^2 + C\|\xi_4\|^2 \leq Ch^{2(k+1)}\|\theta_t\|_{k+1}^2 + C\|\xi_4\|^2$$

$$\|(\eta_4, \xi_4)\| \leq C\|\eta_4\|^2 + C\|\xi_4\|^2 \leq Ch^{2(k+1)}\|\theta\|_{k+1}^2 + C\|\xi_4\|^2$$

$$(\xi_{2,t}, \eta_2) = \frac{d}{dt}(\xi_2, \eta_2) - (\eta_{2,t}, \xi_2)$$

$$\|(\eta_{2,t}, \xi_2)\| \leq C\|\eta_{2,t}\|^2 + C\|\xi_2\|^2 \leq Ch^{2(k+1)}\|\sigma_t\|_{k+1}^2 + C\|\xi_2\|^2$$

When summing those right hand side, we will have:

$$\begin{aligned}
&\frac{d}{dt}\|\xi_1\|^2 + \frac{d}{dt}\|\nabla\xi_1\|^2 + \frac{d}{dt}\|\xi_2\|^2 + \frac{d}{dt}\|\xi_4\|^2 + \frac{d}{dt}\|\nabla\xi_4\|^2 + \|\xi_4\|^2 \\
&\quad = -(\eta_{1,t}, \xi_1) - (\nabla\eta_{1,t}, \nabla\xi_1) - (\eta_{4,t}, \xi_4) - (\eta_4, \xi_4) - \frac{d}{dt}\|\eta_2\|^2 - \frac{d}{dt}(\xi_2, \eta_2) \\
&\quad \leq Ch^{2k}(\|q_t\|_{k+1}^2 + \|\theta_t\|_{k+1}^2 + \|\theta\|_{k+1}^2 + \|\sigma_t\|_{k+1}^2) + C(\|\xi_1\|^2 + \|\xi_2\|^2 + \|\xi_4\|^2) + \frac{d}{dt}(\xi_2, \eta_2)
\end{aligned}$$

Using Cauchy-Schwatz inequality, and apply Gronwall inequality, integrate from 0 to t.

$$\begin{aligned}
&\|\xi_1\|^2 + \|\nabla\xi_1\|^2 + \|\xi_2\|^2 + \|\xi_4\|^2 + \|\nabla\xi_4\|^2 + \int_0^T \|\xi_4\|^2 ds \\
&\quad = \|\xi_1(0)\|^2 + \|\nabla\xi_1(0)\|^2 + \|\xi_2(0)\|^2 + \|\xi_4(0)\|^2 + \|\nabla\xi_4(0)\|^2 \\
&\quad + Ch^{2k}\int_0^T (\|q_t\|_{k+1}^2 + \|\theta_t\|_{k+1}^2 + \|\theta\|_{k+1}^2 + \|\sigma_t\|_{k+1}^2) ds \\
&\quad + C\epsilon\int_0^T (\|\xi_1\|^2 + \|\xi_2\|^2 + \|\xi_4\|^2) ds + C\epsilon\|\xi_2\|^2 + C\|\eta_2\|^2
\end{aligned}$$

Simplify further,

$$\begin{aligned}\|\xi_1\|^2 + \|\nabla\xi_1\|^2 + (1-C\epsilon)\|\xi_2\|^2 + \|\xi_4\|^2 + \|\nabla\xi_4\|^2 + \int_0^T \|\xi_4\|^2 ds \\ = \|\xi_1(0)\|^2 + \|\nabla\xi_1(0)\|^2 + \|\xi_2(0)\|^2 + \|\xi_4(0)\|^2 + \|\nabla\xi_4(0)\|^2 \\ + Ch^{2k}\int_0^T (\|q_t\|_{k+1}^2 + \|\theta_t\|_{k+1}^2 + \|\theta\|_{k+1}^2 + \|\sigma_t\|_{k+1}^2) ds \\ + C\epsilon\int_0^T (\|\xi_1\|^2 + \|\xi_2\|^2 + \|\xi_4\|^2) ds + Ch^4\|\sigma\|^2\end{aligned}$$

Since $\|\xi_1(0)\|, \|\xi_4(0)\|$ has the order of $O(h^{k+1})$ and $\|\nabla\xi_1(0)\|, \|\nabla\xi_4(0)\|$ has the order of $O(h^k)$. We will have the conclusion,

$$\|\xi_1\| + \|\xi_2\| + \|\xi_4\| \le Ch^k$$

Choose $p_h = \xi_3$, we can obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\xi_3\|^2 = (\xi_1, \xi_3) - (\eta_{3,t}, \xi_3) + (\eta_1, \xi_3) \\ \le C\|\xi_1\|^2 + C\|\xi_3\|^2 + C\|\eta_{3,t}\|^2 + C\|\eta_1\|^2\end{aligned} \tag{3.16}$$

Since

$$\begin{aligned}\|\eta_{3,t}\|^2 \le Ch^{2(k+1)}\|u_t\|_2^2 \\ \|\eta_1\|^2 \le Ch^{2(k+1)}\|q\|_2^2\end{aligned} \tag{3.17}$$

Using Cauchy-Schwatz inequality, and apply Gronwall inequality, integrate from 0 to t.

$$\frac{1}{2}\frac{d}{dt}\|\xi_3\|^2 \le \|\xi_3(0)\|^2 + Ch^{2k}\int_0^T (\|u_t\|_{k+1}^2 + \|q\|_{k+1}^2) ds + C\|\xi_3\|^2 \tag{3.18}$$

This completes the proof. □

3.2 Fully Discrete Mixed Element Formulation

3.2.1 The Existence and Uniqueness Full Discrete Mixed Element Formulation

For full discretization, we use the backward Euler method of first order accurate in time. For the backward Euler method, Let M be a positive integer, then $\Delta t = T/M$ be the step size of time, $t^i = i\Delta t, 0 \leq i \leq M$. Further, let $\psi^n = \psi(t^n)$ and $\partial_t \psi^n = (\psi^n - \psi^{n-1})/\Delta t$, for some continuous function $\psi \in C^0[0,T]$, and let $(Q^n, S^n, U^n, \Theta^n) \in V_h^n \times V_h^n \times V_h^n \times V_h^n$ be the mixed element approximation of $(q(t^n), \sigma(t^n), u(t^n), \theta(t^n))$. For each n, the different time interval is (t^n, t^{n+1}), the corresponding triangulation is $\mathcal{T}_h^n$, finite element space V_h^n. Then the fully discrete mixed finite element solution for problem (I^{**}) may be presented as follows:

Problem(I_h^n). Find $(Q^n, S^n, U^n, \Theta^n) \in V_h^n \times V_h^n \times V_h^n \times V_h^n$ such that, for $1 \leq n \leq M$, $\forall p_h, l_h, w_h, r_h \in V_h$

$$\begin{cases} (\overline{\partial} U^n, p_h) = (Q^n, p_h) \\ (S^n, l_h) - (\nabla U^n, \nabla l_h) = 0 \\ (\overline{\partial} Q^n, w_h) + (\nabla \overline{\partial} Q^n, w_h) + (\nabla S^n, \nabla w_h) - (\nabla \Theta^n, \nabla w_h) = (f^n, w_h) \\ (\overline{\partial} \Theta^n, r_h) + (\overline{\partial} \nabla \Theta^n, \nabla r_h) + (\Theta^n, r_h) + (\nabla Q^n, \nabla r_h) = (g^n, r_h) \end{cases}$$

Theorem 3.4. *Problem I_h^n has a unique solution $(Q^n, S^n, U^n, \Theta^n) \in V_h^n \times V_h^n \times V_h^n \times V_h^n$.*

Proof. Denote $\tilde{Q}^n = Q^{1,n} - Q^{2,n}$, $\tilde{S}^n = S^{1,n} - S^{2,n}$, $\tilde{U}^n = U^{1,n} - U^{2,n}$, $\tilde{\Theta}^n = \Theta^{1,n} - \Theta^{2,n}$.

For problem (I_h^n), work on eqn (c) and (d), substitute by $\tilde{w}_h = -\triangle \tilde{Q}^n$ and $\tilde{r}_h = -\triangle \tilde{\Theta}^n$,

$$\begin{cases} (\frac{\triangle\tilde{U}^n - \triangle\tilde{U}^{n-1}}{\tau}, p_h) - (\triangle\tilde{Q}^n, p_h) = 0 \\ (\tilde{S}^n, l_h) - (\nabla\tilde{U}^n, \nabla l_h) = 0 \\ (\frac{\tilde{Q}^n - \tilde{Q}^{n-1}}{\tau}, w_h) + (\frac{\nabla\tilde{Q}^n - \nabla\tilde{Q}^{n-1}}{\tau}, \nabla w_h) + (\nabla\tilde{S}^n, \nabla w_h) - (\nabla\tilde{\Theta}^n, \nabla w_h) = 0 \\ (\frac{\tilde{\Theta}^n - \tilde{\Theta}^{n-1}}{\tau}, r_h) + (\nabla\tilde{\Theta}^n, \nabla r_h) + (\tilde{\Theta}^n, r_h) + (\nabla\tilde{Q}^n, \nabla r_h) = 0 \end{cases}$$

Choose $p_h = \triangle\tilde{U}^n$, $l_h = \triangle\tilde{Q}^n$, $w_h = \tilde{Q}^n$ and $r_h = \tilde{\Theta}^n$, and adding up all four equations,

$$\begin{aligned} (\frac{\triangle\tilde{U}^n - \triangle\tilde{U}^{n-1}}{\tau}, \triangle\tilde{U}^n) - (\triangle\tilde{Q}^n, \triangle\tilde{U}^n) + (\tilde{S}^n, \triangle\tilde{Q}^n) - (\nabla\tilde{U}^n, \nabla\triangle\tilde{Q}^n) \\ + (\frac{\tilde{Q}^n - \tilde{Q}^{n-1}}{\tau}, \tilde{Q}^n) + (\frac{\nabla\tilde{Q}^n - \nabla\tilde{Q}^{n-1}}{\tau}, \nabla\tilde{Q}^n) + (\nabla\tilde{S}^n, \nabla\tilde{Q}^n) - (\nabla\tilde{\Theta}^n, \nabla\tilde{Q}^n) \\ + (\frac{\tilde{\Theta}^n - \tilde{\Theta}^{n-1}}{\tau}, \tilde{\Theta}^n) + (\nabla\tilde{\Theta}^n, \nabla\tilde{\Theta}^n) + (\tilde{\Theta}^n, \tilde{\Theta}^n) + (\nabla\tilde{Q}^n, \nabla\tilde{\Theta}^n) = 0 \end{aligned} \tag{3.19}$$

As the fact,

$$\begin{aligned} (\frac{\triangle\tilde{U}^n - \triangle\tilde{U}^{n-1}}{\tau}, \triangle\tilde{U}^n) &\geq \frac{\|\triangle\tilde{U}^n\|^2 - \|\triangle\tilde{U}^{n-1}\|^2}{2\tau} \\ (\frac{\tilde{Q}^n - \tilde{Q}^{n-1}}{\tau}, \tilde{Q}^n) &\geq \frac{\|\tilde{Q}^n\|^2 - \|\tilde{Q}^{n-1}\|^2}{2\tau} \\ (\frac{\nabla\tilde{Q}^n - \nabla\tilde{Q}^{n-1}}{\tau}, \nabla\tilde{Q}^n) &\geq \frac{\|\nabla\tilde{Q}^n\|^2 - \|\nabla\tilde{Q}^{n-1}\|^2}{2\tau} \\ (\frac{\tilde{\Theta}^n - \tilde{\Theta}^{n-1}}{\tau}, \tilde{\Theta}^n) &\geq \frac{\|\tilde{\Theta}^n\|^2 - \|\tilde{\Theta}^{n-1}\|^2}{2\tau} \end{aligned}$$

Simplify eqn(4.10):

$$\begin{aligned} \frac{\|\triangle\tilde{U}^n\|^2 - \|\triangle\tilde{U}^{n-1}\|^2}{2\tau} + \frac{\|\tilde{Q}^n\|^2 - \|\tilde{Q}^{n-1}\|^2}{2\tau} + \frac{\|\nabla\tilde{Q}^n\|^2 - \|\nabla\tilde{Q}^{n-1}\|^2}{2\tau} + \frac{\|\tilde{\Theta}^n\|^2 - \|\tilde{\Theta}^{n-1}\|^2}{2\tau} \\ + \|\nabla\tilde{\Theta}^n\|^2 + \|\tilde{\Theta}^n\|^2 \leq 0 \end{aligned} \tag{3.20}$$

Sum 4.11 for $n = 1 \cdots M$, then

$$\begin{aligned} &\frac{\|\triangle\tilde{U}^M\|^2 - \|\triangle\tilde{U}^0\|^2}{2\tau} + \frac{\|\tilde{Q}^M\|^2 - \|\tilde{Q}^0\|^2}{2\tau} + \frac{\|\nabla\tilde{Q}^M\|^2 - \|\nabla\tilde{Q}^0\|^2}{2\tau} + \frac{\|\tilde{\Theta}^M\|^2 - \|\tilde{\Theta}^0\|^2}{2\tau} \\ &\quad + \sum_{i=1}^{M}(\|\nabla\tilde{\Theta}^i\|^2 + \|\tilde{\Theta}^i\|^2) \leq 0 \end{aligned} \tag{3.21}$$

By discrete Gronwall inequality, thus we have the stability for fully discrete form. This completes the proof of the theorem. □

3.2.2 Fully Discrete Error Estimates

For the error estimate, we need to introduce the decomposition.

$$\begin{aligned} q(t^n) - Q^n &= q(t^n) - R_h q(t^n) + R_h q(t^n) - Q^n = \eta_1 + \xi_1 \\ \sigma(t^n) - S^n &= \sigma(t^n) - R_h \sigma(t^n) + R_h \sigma(t^n) - S^n = \eta_2 + \xi_2 \\ u(t^n) - U^n &= u(t^n) - R_h u(t^n) + R_h u(t^n) - U^n = \eta_3 + \xi_3 \\ \theta(t^n) - \Theta^n &= \theta(t^n) - R_h u(t^n) + R_h u(t^n) - \Theta^n = \eta_4 + \xi_4 \end{aligned} \tag{3.22}$$

We may denote $\phi(t^n)$ as ϕ^n, here ϕ can be q, σ, u, θ. R_h is elliptical projection defined previously.

Theorem 3.5. *Let* $(Q^n, S^n, U^n, \Theta^n)$ *and* (q, σ, u, θ) *are the solutions of* (I_h^n) *and* (I^{**})*. If* $q_t \in L^\infty(H^{k+1})$, $\theta_t \in L^\infty(H^{k+1})$, $u_t \in L^\infty(H^{k+1})$, $u_{tt} \in L^2(H^{k+1})$, $q_{tt} \in L^2(H^{k+1})$, $\sigma_{tt} \in L^2(H^{k+1})$, $\theta_{tt} \in L^2(H^{k+1})$*. Then for* $\forall n \geq 0$*, there exist:*

$$\|Q^n - q(t^n)\| + \|\Theta^n - \theta(t^n)\| + \|S^n - \sigma(t^n)\| \leq C_1(h^k + \tau)$$

$$\|U^n - u(t^n)\| \leq C_2(h^{k+1} + \tau)$$

Proof. Using the system of equation at $t = t^n$, we obtain

$$\begin{cases} (\overline{\partial} U^n, p_h^n) = (Q^n, p_h^n) \\ (S^n, l_h^n) - (\nabla U^n, \nabla l_h^n) = 0 \\ (\overline{\partial} Q^n, w_h^n) + (\nabla \overline{\partial} Q^n, w_h^n) + (\nabla S^n, \nabla w_h^n) - (\nabla \Theta^n, \nabla w_h^n) = (F, w_h^n) \\ (\overline{\partial} \Theta^n, r_h^n) + (\overline{\partial} \nabla \Theta^n, \nabla r_h^n) + (\Theta^n, r_h^n) + (\nabla Q^n, \nabla r_h^n) = (G, r_h^n) \end{cases} \tag{3.23}$$

Easily we will have:

$$\begin{cases} (\overline{\partial} \xi_3^n, p_h^n) - (\xi_1^n, p_h^n) = -(\overline{\partial} \eta_3^n, p_h^n) + (\eta_1^n, p_h^n) \\ (\xi_2^n, l_h^n) - (\nabla \xi_3^n, \nabla l_h^n) = -(\eta_2^n, l_h^n) + (\nabla \eta_3^n, \nabla l_h^n) \\ (\overline{\partial} \xi_1^n, w_h^n) + (\nabla \overline{\partial} \xi_1^n, \nabla w_h^n) + (\nabla \xi_2^n, \nabla w_h^n) - (\nabla \xi_4^n, \nabla w_h^n) = -(\overline{\partial} \eta_1^n + \pi_1^n, w_h^n) - (\nabla \overline{\partial} \eta_1^n + \pi_2^n, \nabla w_h^n) \\ (\overline{\partial} \xi_4^n, r_h^n) + (\overline{\partial} \nabla \xi_4^n, \nabla r_h^n) + (\xi_4^n, r_h^n) + (\nabla \xi_1^n, \nabla r_h^n) = -(\overline{\partial} \eta_4^n + \pi_3^n, r_h^n) - (\eta_4^n, r_h^n) \end{cases} \tag{3.24}$$

Take $w_h = \xi_1$ and $r_h = \xi_4$ respectively:

$$\begin{cases} (\overline{\partial} \xi_1^n, \xi_1^n) + (\nabla \overline{\partial} \xi_1^n, \xi_1^n) + (\nabla \xi_2^n, \nabla \xi_1^n) - (\nabla \xi_4^n, \nabla \xi_1^n) = -(\overline{\partial} \eta_1^n + \pi_1^n, \xi_1^n) - (\nabla \overline{\partial} \eta_1^n + \pi_2^n, \nabla \xi_1^n) \\ (\overline{\partial} \xi_4^n, \xi_4^n) + (\overline{\partial} \nabla \xi_4^n, \nabla \xi_4^n) + (\xi_4^n, \xi_4^n) + (\nabla \xi_1^n, \nabla \xi_4^n) = -(\overline{\partial} \eta_4^n + \pi_3^n, \xi_4^n) - (\eta_4^n, \xi_4^n) \end{cases} \tag{3.25}$$

Sum those two equations:

$$\begin{aligned} &(\overline{\partial} \xi_1^n, \xi_1^n) + (\nabla \overline{\partial} \xi_1^n, \nabla \xi_1^n) + (\nabla \xi_2^n, \nabla \xi_1^n) + (\overline{\partial} \xi_4^n, \xi_4^n) + (\overline{\partial} \nabla \xi_4^n, \nabla \xi_4^n) + (\xi_4^n, \xi_4^n) \\ &\quad = -(\overline{\partial} \eta_1^n + \pi_1^n, \xi_1^n) - (\nabla \overline{\partial} \eta_1^n + \pi_2^n, \nabla \xi_1^n) - (\overline{\partial} \eta_4^n + \pi_3^n, \xi_4^n) - (\eta_4^n, \xi_4^n) \end{aligned} \tag{3.26}$$

Remark. The following inequalities and identities will be useful.

$$
\begin{aligned}
&(\overline{\partial}\xi_1^n, \xi_1^n) \geq \frac{\|\xi_1^n\|^2 - \|\xi_1^{n-1}\|^2}{2\tau}, \quad (\nabla\overline{\partial}\xi_1^n, \nabla\xi_1^n) \geq \frac{\|\nabla\xi_1^n\|^2 - \|\nabla\xi_1^{n-1}\|^2}{2\tau} \\
&(\overline{\partial}\xi_4^n, \xi_4^n) \geq \frac{\|\xi_4^n\|^2 - \|\xi_4^{n-1}\|^2}{2\tau}, \quad (\nabla\xi_4^n, \nabla\xi_4^n) = \|\nabla\xi_4^n\|^2, \quad (\xi_4^n, \xi_4^n) = \|\xi_4^n\|^2
\end{aligned} \tag{3.27}
$$

Let's deal with the right hand side, If applying Cauchy Schwartz inequality,

$$
\begin{aligned}
\|(\overline{\partial}\eta_1^n, \xi_1^n)\| &\leq C\|\overline{\partial}\eta_1^n\|^2 + C\|\xi_1^2\|^2 \leq \frac{Ch^{2(k+1)}}{\tau}\int_{t_{n-1}}^{t_n} \|q_t^n\|_{k+1}^2 + C\|\xi_1^2\|^2 \\
\|(\nabla\overline{\partial}\eta_1^n, \nabla\xi_1^n)\| &\leq C\|\nabla\overline{\partial}\eta_1^n\|^2 + C\|\nabla\xi_1^n\|^2 \leq \frac{Ch^{2k}}{\tau}\int_{t_{n-1}}^{t_n} \|q_t^n\|_{k+1}^2 + C\|\nabla\xi_1^n\|^2 \\
\|(\overline{\partial}\eta_4^n, \xi_4^n)\| &\leq C\|\overline{\partial}\eta_4^n\|^2 + C\|\xi_4^2\|^2 \leq \frac{Ch^{2(k+1)}}{\tau}\int_{t_{n-1}}^{t_n} \|\theta_t^n\|_{k+1}^2 + C\|\xi_4^n\|^2 \\
\|(\eta_4^n, \xi_4^n)\| &\leq C\|\eta_4^n\|^2 + C\|\xi_4^n\|^2 \leq Ch^{2(k+1)}\|\theta^n\|_{k+1}^2 + C\|\xi_4^n\|^2 \\
(\overline{\partial}\xi_2^n, \eta_2^n) &= \overline{\partial}(\xi_2^n, \eta_2^n) - (\overline{\partial}\eta_2^n, \xi_2^n) \\
\|(\overline{\partial}\eta_2^n, \xi_2^n)\| &\leq C\|\overline{\partial}\eta_2^n\|^2 + C\|\xi_2^2\|^2 \leq \frac{Ch^{2(k+1)}}{\tau}\int_{t_{n-1}}^{t_n} \|\sigma_t^n\|_{k+1}^2 + C\|\xi_2^n\|^2 \\
\|(\pi_1^n, \xi_1^n)\| &\leq C\|\pi_1^n\|^2 + C\|\xi_1^2\|^2 \leq C\tau\int_{t_{n-1}}^{t_n} \|q_{tt}\|^2 + C\|\xi_1^2\|^2 \\
\|(\pi_2^n, \xi_1^n)\| &\leq C\|\pi_2^n\|^2 + C\|\xi_1^2\|^2 \leq C\tau\int_{t_{n-1}}^{t_n} \|\sigma_{tt}\|^2 + C\|\xi_1^2\|^2 \\
\|(\pi_3^n, \xi_4^n)\| &\leq C\|\pi_3^n\|^2 + C\|\xi_1^2\|^2 \leq C\tau\int_{t_{n-1}}^{t_n} \|\theta_{tt}\|^2 + C\|\xi_1^2\|^2
\end{aligned} \tag{3.28}
$$

Also, we can transform the term,

$$
\begin{aligned}
(\nabla\xi_2^n, \nabla\xi_1^n) &= (\overline{\partial}\xi_2^n, \xi_2^n) + (\overline{\partial}\eta_2^n, \eta_2^n) + (\overline{\partial}\xi_2^n, \eta_2^n) + (\overline{\partial}\eta_2^n, \xi_2^n) \\
&= (\overline{\partial}\xi_2^n, \xi_2^n) + (\overline{\partial}\eta_2^n, \eta_2^n) + \overline{\partial}(\xi_2^n, \eta_2^n)
\end{aligned} \tag{3.29}
$$

When summing those right hand side, we will have:

$$\|\xi_1^n\|^2 + \|\nabla\xi_1^n\|^2 + \|\nabla\xi_2\|^2 + \|\xi_4^n\|^2 + \tau\sum_0^n(\|\xi_4^n\|^2 + \tau\|\nabla\xi_4^n\|^2)$$
$$\le \|\xi_1^0\|^2 + \|\nabla\xi_1^0\|^2 + \|\xi_4^0\|^2 + Ch^{2k}(\int_0^T \|q_t\|_{k+1}^2 + \|\theta_t\|_{k+1}^2)$$
$$+ C\tau^2\int_0^T(\|q_{tt}\|^2 + \|\sigma_{tt}\|^2 + \|\theta_{tt}\|^2) + C\tau\sum_0^n(\|\xi_1^n\|^2 + \|\xi_4^n\|^2) \tag{3.30}$$

Besides,

$$\|R_hq^0 - q^0\| + h^k\|R_hq^0 - q^0\|_1 \le Ch^{k+1}\|q\|_{k+1}$$
$$\|R_h\sigma^0 - \sigma^0\| + h^k\|R_h\sigma^0 - \sigma^0\|_1 \le Ch^{k+1}\|\sigma\|_{k+1}$$
$$\|R_hu^0 - u^0\| + h^k\|R_hu^0 - u^0\|_1 \le Ch^{k+1}\|u\|_{k+1}$$
$$\|R_h\theta^0 - \theta^0\| + h^k\|R_h\theta^0 - \theta^0\|_1 \le Ch^{k+1}\|\theta\|_{k+1} \tag{3.31}$$

Take $p_h = \xi_3$, we can obtain

$$\|\xi_3^n\|^2 \le \|\xi_3^0\|^2 + Ch^{2(k+1)}\int_0^T \|u_t\|_{k+1}^2 ds + C\tau^2\int_0^T \|u_{tt}\|^2 ds + C\tau\sum_0^n(\|\xi_1^n\|^2 + \|\xi_3^n\|^2) \tag{3.32}$$

According to the discrete Gronwall inequality, that proofs the theorem. □

3.3 Numerical Examples

Consider

$$\begin{cases} u_{tt} - \triangle u_{tt} + \triangle^2 u + \alpha\triangle\theta = f \\ \theta_t - \triangle\theta + \theta - \alpha\triangle u_t = g \\ u|_{\partial\Omega} = \triangle u|_{\partial\Omega} = \theta|_{\partial\Omega} = 0 \\ u(\mathbf{x}, 0) = u^0(\mathbf{x}), u_t(\mathbf{x}, 0) = u^1(\mathbf{x}), \theta(\mathbf{x}, 0) = \theta^0(\mathbf{x}) \end{cases}$$

And $\alpha = 1$, $\Omega = [0,1] \times [0,1]$, $T = 1$, with exact solution

$$\begin{cases} u = sin(\pi x)sin(\pi y)e^{-t} \\ \theta = sin(2\pi x)sin(2\pi y)e^{-t} \end{cases}$$

$\mathcal{T}_h$ is regular pattern triangular mesh, and mixed element space is $(P_1(K)\times P_1(K)\times P_1(K)\times P_1(K))$ used to solve the problem. The convergence curves of L^2 error of the solution are depicted in the figures. From the plots, we can clearly observe the L^2 convergence rate for those four variables (q, σ, u, θ) are $O(h^2), O(h^2), O(h^2), O(h^2)$, which has a higher convergence rate than theoretical ones.

L^2-Convergence rate of at time $t = 0.2$.

N	$\|q^n - Q_h^n\|_0$	Order	$\|\sigma^n - \Sigma_h^n\|_0$	Order	$\|u^n - U_h^n\|_0$	Order	$\|\theta^n - \Theta_h^n\|_0$	Order
2	2.065535		4.269080		0.463385		3.378281	
4	1.119665	0.88	3.313670	0.37	0.210791	1.14	1.493514	1.18
8	0.341691	1.71	1.464374	1.18	0.044783	2.23	0.473359	1.66
16	0.091543	1.90	0.515730	1.51	0.011396	1.97	0.12637	1.91
32	0.023339	1.97	0.178029	1.53	0.002801	2.02	0.032157	1.97
64	0.005887	1.99	0.061717	1.53	0.000701	2.00	0.008090	1.99

Table 1: The numerical test for MFEM convergence rates $t = 0.2$

L^2-Convergence rate of at time $t = 0.4$.

N	$\|q^n - Q_h^n\|_0$	Order	$\|\sigma^n - \Sigma_h^n\|_0$	Order	$\|u^n - U_h^n\|_0$	Order	$\|\theta^n - \Theta_h^n\|_0$	Order
2	1.094914		7.393774		0.695839		2.551924	
4	0.777217	0.49	4.137935	0.84	0.367084	0.92	1.149737	1.15
8	0.311103	1.32	1.492817	1.47	0.103911	1.82	0.383152	1.59
16	0.085048	1.87	0.521902	1.52	0.026937	1.95	0.104057	1.88
32	0.022087	1.95	0.157846	1.73	0.006779	1.99	0.026956	1.95
64	0.005585	1.98	0.052072	1.60	0.001698	2.00	0.006807	1.99

Table 2: The numerical test for MFEM convergence rates $t = 0.4$

L^2-Convergence rate of at time $t = 0.8$.

N	$\|q^n - Q_h^n\|_0$	Order	$\|\sigma^n - \Sigma_h^n\|_0$	Order	$\|u^n - U_h^n\|_0$	Order	$\|\theta^n - \Theta_h^n\|_0$	Order
2	0.628412		5.309609		0.447831		0.330048	
4	0.638675	-0.02	4.010646	0.40	0.307582	0.54	0.482643	-0.55
8	0.240155	1.41	1.430877	1.49	0.105983	1.54	0.209806	1.20
16	0.06458	1.89	0.444102	1.69	0.029738	1.83	0.059779	1.81
32	0.016836	1.94	0.132804	1.74	0.007612	1.97	0.015558	1.94
64	0.004265	1.98	0.040809	1.71	0.001917	1.99	0.003944	1.98

Table 3: The numerical test for MFEM convergence rates $t = 0.8$

L^2-Convergence rate of at time $t = 1.0$.

N	$\|q^n - Q_h^n\|_0$	Order	$\|\sigma^n - \Sigma_h^n\|_0$	Order	$\|u^n - U_h^n\|_0$	Order	$\|\theta^n - \Theta_h^n\|_0$	Order
4	0.749048		1.271249		0.162699		0.617435	
8	0.306067	1.29	0.789878	0.69	0.052576	1.63	0.258817	1.25
16	0.086848	1.82	0.306119	1.37	0.014797	1.83	0.076604	1.76
32	0.022782	1.93	0.082206	1.90	0.00381	1.96	0.020485	1.90
64	0.005779	1.98	0.21296e-1	1.95	0.000959	1.99	0.005227	1.97

Table 4: The numerical test for MFEM convergence rates $t = 1.0$

We can observe that the L^2 error of $(q_h, \sigma_h, u_h, \theta_h)$ converges at the rate of 2 that is more than expected. The following four figures, at different time, are using $log - log$ plot, then the slope is equivalent to the convergence rate, in absolute meaning.

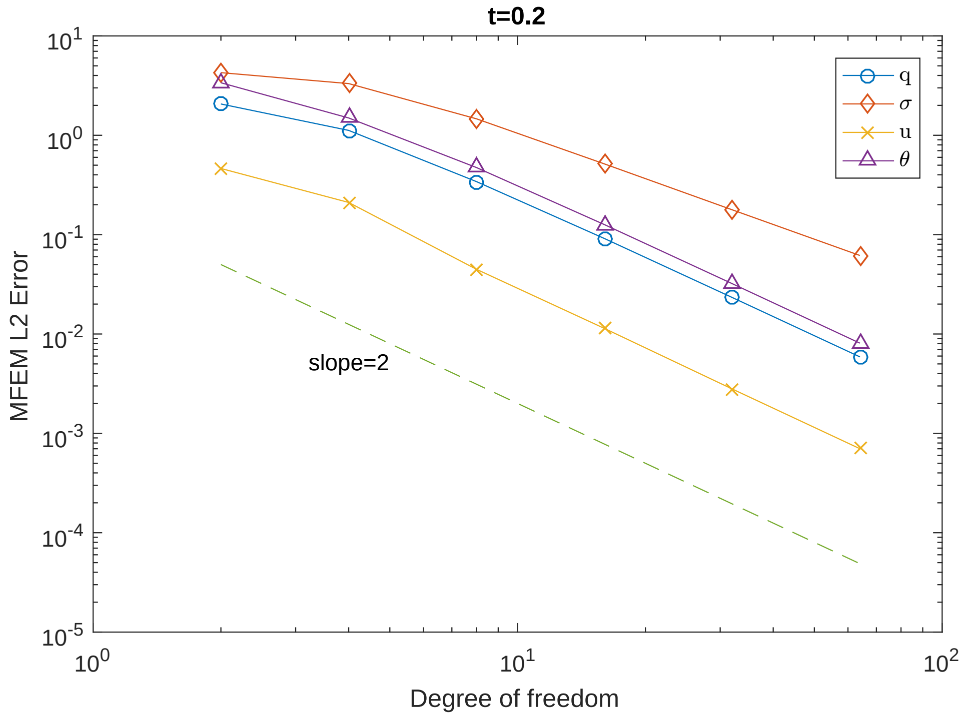

Figure 1: Mixed element method L^2 convergence rate, time $t = 0.2$

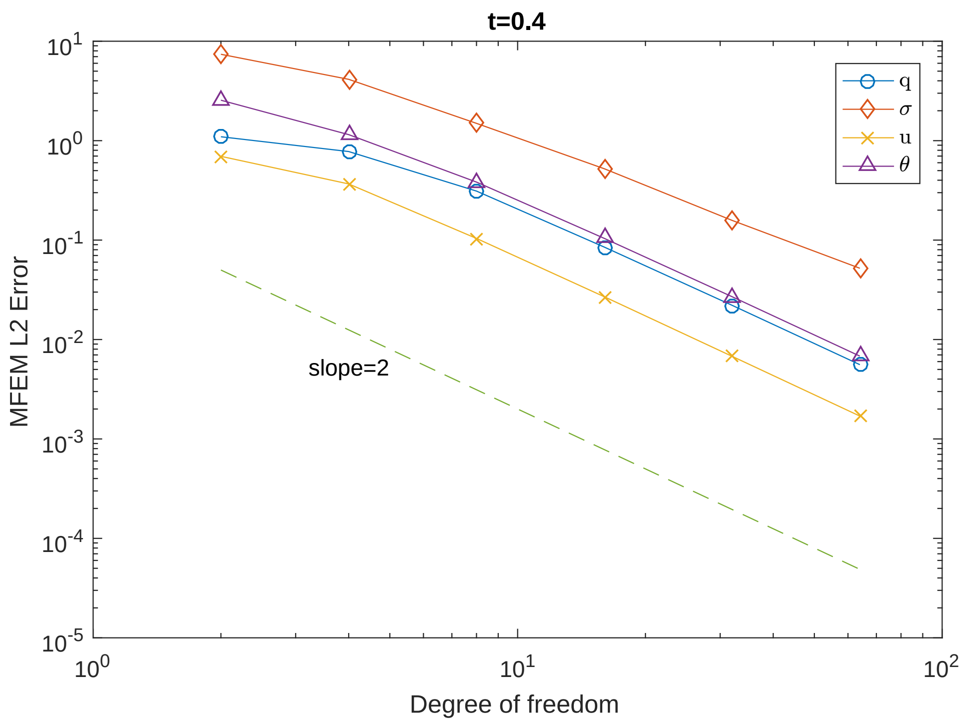

Figure 2: Mixed element method L^2 convergence rate, time $t = 0.4$

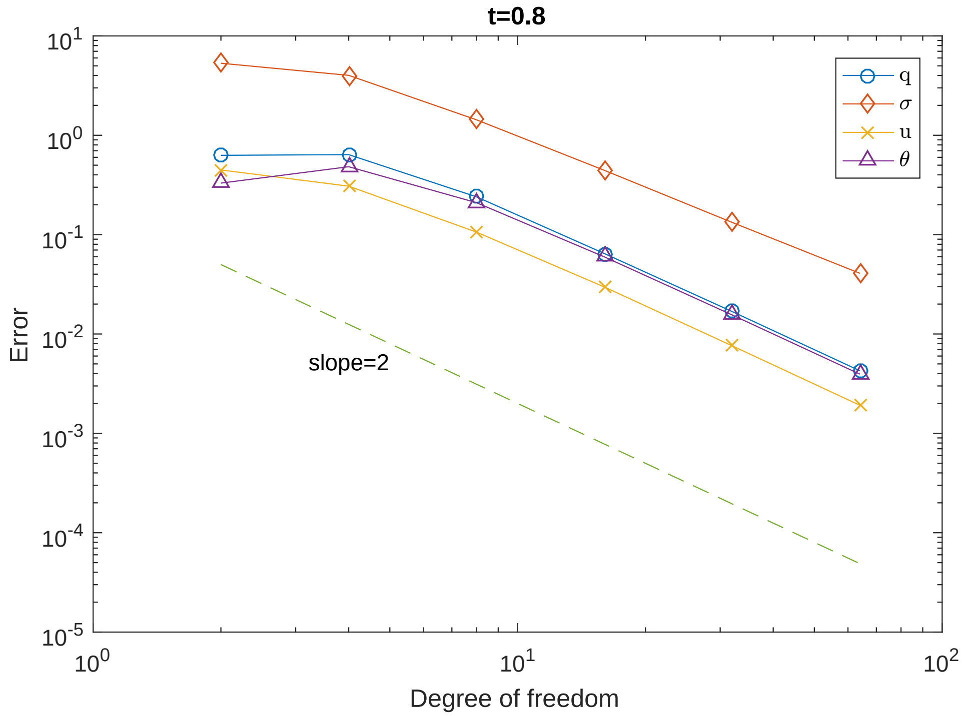

Figure 3: Mixed element method L^2 convergence rate, time $t = 0.8$

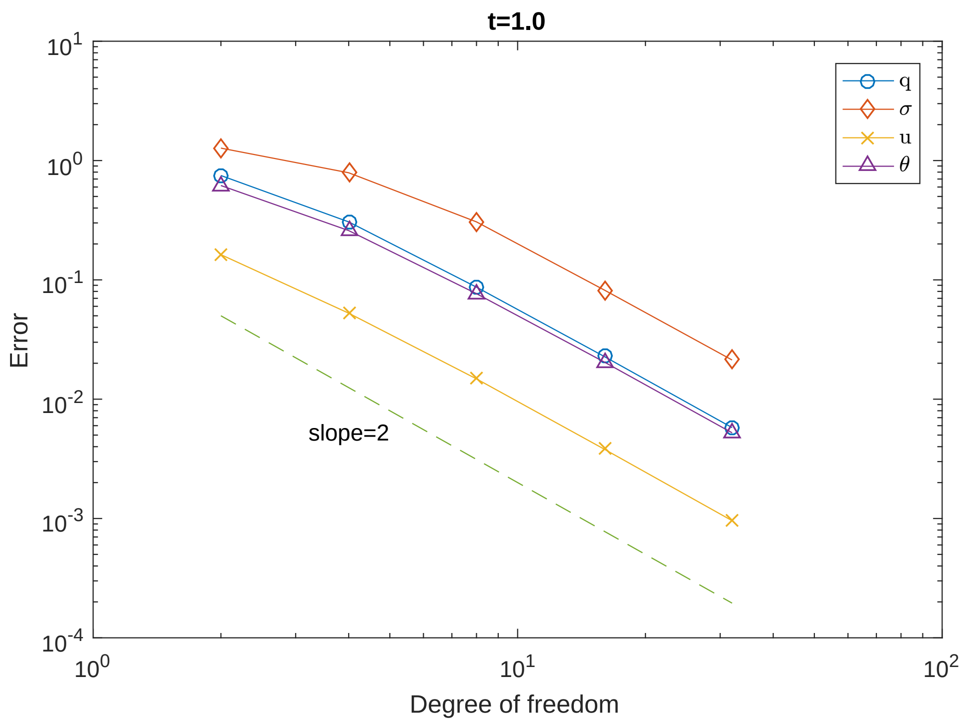

Figure 4: Mixed element method L^2 convergence rate, time $t = 1.0$

CHAPTER 4 IP-DG METHOD

In this section, we introduce how to use IP-DG mixed element method, with basic definitions and properties. The existence of a unique solution of semi-discrete mixed element method and error analysis are given in section 2. And the existence of a unique of its fully discrete method and error analysis are considered in section 3. Numerical results are presented in section 4.

Let $\Omega \subset R^2$, we consider the linearized thermoelastical Kirchhoff-Love plate equation system.

Problem(I): Find $(u,\theta) \in (W^{1,\infty}(0,T;L^2(\Omega)) \bigcap L^\infty(0,T;W^{2,4}(\Omega))) \times L^\infty(0,T;W^{2,4}(\Omega))$ such that for all $T>0$,

$$\begin{cases} u_{tt} - \triangle u_{tt} + \triangle^2 u + \alpha\triangle\theta = f, \quad (\mathbf{x},t) \in \Omega\times(0,T) \\ \theta_t - \triangle\theta + \theta - \alpha\triangle u_t = g, \quad (\mathbf{x},t) \in \Omega\times(0,T) \\ u = \triangle u = \theta = 0, \quad (\mathbf{x},t) \in \partial\Omega\times(0,T) \\ u(\mathbf{x},0) = u^0(\mathbf{x}), u_t(\mathbf{x},0) = u^1(\mathbf{x}), \theta(\mathbf{x},0) = \theta^0(\mathbf{x}), \quad \mathbf{x} \in \Omega \end{cases} \tag{4.1}$$

where u denotes the displacement of the plate, θ denotes the displacement caused by thermol changes, g, u^0, u^1 and θ^0 are given functions.

4.1 Semi-discrete IP-DG Mixed Element Formulation

4.1.1 Semi-discrete IP-DG Mixed Element Formulation

Let $q = u_t, \sigma = -\triangle u$, the original problem(I) would be transformed into:

Problem(I*): Find $(q,\sigma,u,\theta) \in L^\infty(0,T;H_0^1(\Omega)) \times L^\infty(0,T;H_0^1(\Omega)) \times L^\infty(0,T;H_0^1(\Omega)) \times$

$L^\infty(0,T;H_0^1(\Omega))$ such that, for all $T>0$

$$\begin{cases} q_t - \triangle q_t - \triangle u + \triangle\theta = f, \quad (\mathbf{x},t)\in\ \Omega\times(0,T) \\ \theta_t - \triangle\theta + \theta - \triangle q = g, \quad (\mathbf{x},t)\in\ \Omega\times(0,T) \\ u=\sigma=\theta=0, \quad (\mathbf{x},t)\in\ \partial\Omega\times(0,T) \\ u(\mathbf{x},0)=u^0(\mathbf{x}), \sigma(\mathbf{x},0)=\triangle u^0(\mathbf{x}), q(\mathbf{x},0)=u^1(\mathbf{x}), \theta(\mathbf{x},0)=\theta^0(\mathbf{x}), \quad \mathbf{x}\in\ \Omega \end{cases}$$

Definition 4.1. The bilinear form $a_h(u,v)$ contains the parameter ϵ defined as follows taking the value $-1,0,1$, and is symmetric when $\epsilon=-1$ and it is nonsymmetric otherwise.

$$a_\epsilon(u,v)=(\nabla u,\nabla v)_w=\sum_{K\in\mathcal{T}_h}(\nabla u,\nabla v)_K-\sum_{e\in\Gamma_{h,D}}\int_e\langle\nabla u\rangle\cdot\mathbf{n}[v]dS+\epsilon\sum_{e\in\Gamma_{h,D}}\int_e\langle\nabla v\rangle\cdot\mathbf{n}[u]dS$$
$$+\sum_{e\in\Gamma_{h,D}}\frac{\gamma}{h_e}\int_e[u][v]ds \quad \forall u,v\in V \tag{4.2}$$

Remark. Notation:

Average,

$$\langle v\rangle=\frac{1}{2}(v^+ + v^-)$$

Jump,

$$[v]=(v^+ - v^-)$$

Lemma 4.1. *For DG bilinear form, $|||\cdot|||_V$ is coervive if there is a positive constant κ such that:*

$$\forall v\in V, \kappa|||v|||_V^2\le a_\epsilon(v,v)$$

If $\epsilon=1$, then obviously $\kappa=1$, or $\kappa=\frac{1}{2}$ if choosing γ large enough for $\epsilon=0$ or 1.

To implement the mixed element method, we consider the following weak formulation:

Problem(I**): Find $(q, \sigma, u, \theta) : [0, T] \to V \times V \times V \times V$ such that, $\forall T > 0$, $\forall p, l, w, r \in V$

$$\begin{cases} (u_t, p) - (q, p) = 0, (x,t) \in \Omega \times [0,T] \\ (\sigma, l) - (\nabla u, \nabla l)_w = 0, (x,t) \in \Omega \times [0,T] \\ (q_t, w) + (\nabla q_t, \nabla w)_w + (\nabla \sigma, \nabla w)_w - (\nabla \theta, \nabla w)_w = l_1(w), (x,t) \in \Omega \times [0,T] \\ (\theta_t, r) + (\nabla \theta, \nabla r)_w + (\theta, r) + (\nabla q, \nabla r)_w = l_2(r), (x,t) \in \Omega \times [0,T] \\ u = \sigma = \theta = 0, (x,t) \in \partial\Omega \times [0,T] \\ u(x,0) = u^0(x), \sigma(x,0) = \triangle u^0(x), q(x,0) = u^1(x), \theta(x,0) = \theta^0, (x,t) \in \Omega \times [0,T] \end{cases}$$

where $V = L^\infty(0, T; H_0^1(\Omega))$. Now we consider the following semi-discrete form for problem(I^{**}).

Theorem 4.2. *Problem I^{**} has a unique solution.*

Proof. To prove this theorem, just need to prove the bilinear form is coercive and continuous.

The existence of the solution follows from the existence and regularity assumption. By defining $q = u_t, \sigma = -\triangle u$, it immediates comes up with a weak solution for I^{**}. To prove the uniqueness of the solution, we need to prove the stability first, let $q_0^i, \sigma_0^i, u_0^i, \theta_0^i \in H^3$ be the initial data, and $q^i, \sigma^i, u^i, \theta^i$ be the corresponding weak solutions. Then

$$\|\sigma^1 - \sigma^2\|_{L^\infty(0,T;L^2(\Omega))} + \|u^1 - u^2\|_{L^\infty(0,T;L^2(\Omega))} \le C(T)\|u_0^1 - u_0^2\| \tag{4.3}$$

where $C(T)$ is a positive constant. Next to prove the stability result.

Denote $\tilde{q} = q^1 - q^2$, $\tilde{\sigma} = \sigma^1 - \sigma^2$, $\tilde{u} = u^1 - u^2$, $\tilde{\theta} = \theta^1 - \theta^2$.

$$\begin{cases} (\triangle \tilde{u}_t, p) - (\triangle \tilde{q}, p) = 0 \\ (\tilde{\sigma}, l) - (\nabla \tilde{u}, \nabla l)_w = 0 \\ (\tilde{q}_t, w) + (\nabla \tilde{q}_t, \nabla w)_w + (\nabla \tilde{\sigma}, \nabla w)_w - (\nabla \tilde{\theta}, \nabla w)_w = 0 \\ (\tilde{\theta}_t, r) + (\nabla \tilde{\theta}, \nabla r)_w + (\tilde{\theta}, r) + (\nabla \tilde{q}, \nabla r)_w = 0 \end{cases}$$

Choose $p = \triangle \tilde{u}$, $l = \triangle \tilde{q}$, $w = \tilde{q}$ and $r = \tilde{\theta}$, and adding up all four equations,

$$\frac{1}{2}\frac{d}{dt}\|\tilde{\sigma}\|^2 + \frac{1}{2}\frac{d}{dt}\|\tilde{q}\|^2 + \frac{1}{2}\frac{d}{dt}|||\tilde{q}|||^2 + \frac{1}{2}\frac{d}{dt}\|\tilde{\theta}\|^2 + \left|\left|\left|\tilde{\theta}\right|\right|\right|^2 + \|\tilde{\theta}\|^2 = 0$$

Taking the integral, thus we have the stability. This completes the proof of the theorem.

□

4.1.2 The Existence and Uniqueness Semi-discrete Mixed Element Formulation

In this section, we demonstrate on the existence and uniqueness of the solution of system. Now we consider the following semi-discrete form for problem(I^{**}). Problem(I_h): Find $(q_h, \sigma_h, u_h, \theta_h) : [0, T] \to S_h \times S_h \times S_h \times S_h$ such that, $\forall T > 0$, $\forall p_h, l_h, w_h, r_h \in S_h$

$$\begin{cases} (a)\ (u_{h,t}, p_h) - (q_h, p_h) = 0 \\ (b)\ (\sigma_h, l_h) - (\nabla u_h, \nabla l_h)_w = 0 \\ (c)\ (q_{h,t}, w_h) + (\nabla q_{h,t}, \nabla w_h)_w + (\nabla \sigma_h, \nabla w_h)_w - (\nabla \theta_h, \nabla w_h)_w = l_1(w_h) \\ (d)\ (\theta_{h,t}, r_h) + (\nabla \theta_h, \nabla r_h)_w + (\theta_h, r_h) + (\nabla q_h, \nabla r_h)_w = l_2(r_h) \end{cases} \tag{4.4}$$

with given $q_h(0), \sigma_h(0), u_h(0), \theta_h(0)$ determined, $l_1(w_h)$ and $l_2(r_h)$ are two linear forms can be written as $l_1(w_h) = \langle f_h, w_h \rangle$ and $l_2(r_h) = \langle g_h, r_h \rangle$ respectively, S_h is the broken space defined in Chapter 2, and $S_h \subset V$.

Theorem 4.3. *Problem I_h has a unique solution.*

Proof. Denote $\tilde{q}_h = q_h^1 - q_h^2$, $\tilde{\sigma}_h = \sigma_h^1 - \sigma_h^2$, $\tilde{u}_h = u_h^1 - u_h^2$, $\tilde{\theta}_h = \theta_h^1 - \theta_h^2$.

For problem (I_h), work on Eq.(4.5c) and Eq.(4.5d), substitute by $\tilde{w}_h = -\triangle \tilde{q}_h$ and $\tilde{r}_h = -\triangle \tilde{\theta}_h$,

$$\begin{cases} (\triangle \tilde{u}_{ht}, p_h) - (\triangle \tilde{q}_h, p_h) = 0 \\ (\tilde{\sigma}_h, l) - (\nabla \tilde{u}_h, \nabla l_h)_w = 0 \\ (\tilde{q}_{ht}, w_h) + (\nabla \tilde{q}_{ht}, \nabla w_h)_w + (\nabla \tilde{\sigma}_h, \nabla w_h)_w - (\nabla \tilde{\theta}_h, \nabla w_h)_w = 0 \\ (\tilde{\theta}_{ht}, r_h) + (\nabla \tilde{\theta}_h, \nabla r_h)_w + (\tilde{\theta}_h, r_h) + (\nabla \tilde{q}_h, \nabla r_h)_w = 0 \end{cases} \tag{4.5}$$

Choose $p_h = \triangle \tilde{u}_h$, $l_h = \triangle \tilde{q}_h$, $w_h = \tilde{q}_h$ and $r_h = \tilde{\theta}_h$, and adding up all four equations,

$$\frac{1}{2}\frac{d}{dt}\|\sigma_h\|^2 + \frac{1}{2}\frac{d}{dt}\|\tilde{q}_h\|^2 + \frac{1}{2}\frac{d}{dt}|||\tilde{q}_h|||^2 + \frac{1}{2}\frac{d}{dt}\|\tilde{\theta}_h\|^2 + \left|\left|\left|\tilde{\theta}_h\right|\right|\right|^2 + \|\tilde{\theta}_h\|^2 = 0$$

Taking the integral, thus we have the stability. This completes the proof of the theorem.

□

4.1.3 Semi Discrete Estimates

First we need to introduce the L^2 projection: find $P_h q$, $P_h \sigma$, $P_h u$, $P_h \theta \in \mathcal{S}_h$ satisfying:

$$(\nabla(q - P_h q), \nabla v) = 0, \quad \forall v_h \in V_h \qquad (\nabla(u - P_h u), \nabla v) = 0, \quad \forall v_h \in V_h$$

$$(\nabla(\sigma - P_h \sigma), \nabla v) = 0, \quad \forall v_h \in V_h \qquad (\nabla(\theta - P_h \theta), \nabla v) = 0, \quad \forall v_h \in V_h$$

The following estimates are well known from [15]: for $j = 0, 1$

$$\|u - P_h u\|_j \leq C h^{k+1-j} \|u\|_{k+1} \qquad \|q - P_h q\|_j \leq C h^{k+1-j} \|q\|_{k+1}$$

$$\|(u - P_h u)_t\|_j \leq C h^{k+1-j} \|u_t\|_{k+1} \qquad \|(q - P_h q)_t\|_j \leq C h^{k+1-j} \|q_t\|_{k+1}$$

$$\|\theta - P_h \theta\|_j \leq C h^{k+1-j} \|\theta\|_{k+1} \qquad \|\sigma - P_h \sigma\|_j \leq C h^{k+1-j} \|\sigma\|_{k+1}$$

$$\|(\theta - P_h \theta)_t\|_j \leq C h^{k+1-j} \|\theta_t\|_{k+1} \qquad \|(\sigma - P_h \sigma)_t\|_j \leq C h^{k+1-j} \|\sigma_t\|_{k+1}$$

Subtracting (I^{**}) from I_h, we obtain:

$$\begin{cases} \text{(a) } (u_{ht} - u_t, p_h) - (q_h - q, p_h) = 0 \\ \text{(b) } (\sigma_h - \sigma, l_h) - (\nabla u_h - \nabla u, \nabla l_h)_w = 0 \\ \text{(c) } (q_{ht} - q_t, w_h) + (\nabla q_{ht} - \nabla q_t, w_h)_w + (\nabla \sigma_h - \nabla \sigma, \nabla w_h)_w - (\nabla \theta_h - \nabla \theta, \nabla w_h)_w = 0 \\ \text{(d) } (\theta_{ht} - \theta_t, r_h) + (\nabla \theta_h - \nabla \theta, \nabla r_h)_w + (\theta_h - \theta, r_h) + (\nabla q_h - \nabla q, \nabla r_h)_w = 0 \end{cases}$$

Denote:

$$
\begin{aligned}
(\partial_t(q_h - P_h q), q_h - P_h q) &= \frac{1}{2}\frac{d}{dt}\|q_h - P_h q\|^2 \\
(\partial_t \nabla(q_h - P_h q), \nabla(q_h - P_h q))_w &= \frac{1}{2}\frac{d}{dt}|||(q_h - P_h q)|||^2 \\
(\partial_t(\sigma_h - P_h\sigma), \nabla\partial_t(u_h - P_h u)) &= (\sigma_h - P_h\sigma, -\partial_t\triangle_h(u_h - P_h u)) = \frac{1}{2}\frac{d}{dt}\|(\sigma_h - P_h\sigma)\|^2 \\
(\partial_t(\theta_h - P_h\theta), \theta_h - P_h\theta) &= \frac{1}{2}\frac{d}{dt}\|\theta_h - P_h\theta\|^2
\end{aligned}
$$

Theorem 4.4. *With $u_h(0) = \tilde{u}_h(0)$, $\sigma_h(0) = \tilde{\sigma}_h(0)$, $\theta_h(0) = \tilde{\theta}_h(0)$, $q_h(0) = \tilde{q}_h(0)$, the following estimate holds:*

$$
\|q - q_h\| + \|u - u_h\| + \|\sigma - \sigma_h\| + \|\theta - \theta_h\| \leq Ch^k
$$

C depends on $\|q_t\|_{L^\infty(H^{k+1})}$, $\|\theta_t\|_{L^\infty(H^{k+1})}$, $\|\theta\|_{L^\infty(H^{k+1})}$, $\|\sigma\|_{L^\infty(H^{k+1})}$, $\|q\|_{L^\infty(H^{k+1})}$, $\|u_t\|_{L^\infty(H^{k+1})}$.

Proof. Take docomposition:

$$
q_h - q = q_h - P_h q + P_h q - q = \xi_1 + \eta_1 \qquad u_h - u = u_h - P_h u + P_h u - u = \xi_3 + \eta_3
$$

$$
\sigma_h - \sigma = \sigma_h - P_h\sigma + P_h\sigma - \sigma = \xi_2 + \eta_2 \qquad \theta_h - \theta = \theta_h - P_h\theta + P_h\theta - \theta = \xi_4 + \eta_4
$$

Then the equation system can be written as:

(a) $(u_{ht} - P_h u_t, p_h) - (q_h - P_h q, p_h) = -(P_h u_t - u_t, p_h) + (P_h q - q, p_h)$

(b) $(\sigma_h - P_h \sigma, l_h) - (\nabla u_h - \nabla p_h u, \nabla l_h)_w = -(P_h \sigma - \sigma, l_h)$

(c)
$$(q_{ht} - P_h q_t, w_h) + (\nabla q_{ht} - \nabla P_h q_t, w_h)_w + (\nabla \sigma_h - \nabla p_h \sigma, \nabla w_h)_w - (\nabla P_h \theta_h - \nabla P_h \theta, \nabla w_h)_w$$
$$= -(P_h q_{ht} - q_t, w_h) - (\nabla P_h q_{ht} - \nabla q_t, w_h)_w$$

(d)
$$(\theta_{ht} - P_h \theta_t, r_h) + (\nabla \theta_h - \nabla P_h \theta, \nabla r_h)_w + (\theta_h - P_h \theta, r_h) + (\nabla q_h - \nabla P_h q, \nabla r_h)_w$$
$$= -(\theta_{ht} - P_h \theta_t, r_h) - (\theta_h - P_h \theta, r_h)$$

Take $w_h = \xi_1$ and $r_h = \xi_4$ respectively:

$$\begin{cases} (\xi_{1,t}, \xi_1) + (\nabla \xi_{1,t}, \nabla \xi_1)_w + (\nabla \xi_2, \nabla \xi_1)_w - (\nabla \xi_4, \nabla \xi_1)_w = -(\eta_{1,t}, \xi_1) - (\nabla \eta_{1,t}, \nabla \xi_1) \\ (\xi_{4,t}, \xi_4) + (\nabla \xi_{4,t}, \nabla \xi_4)_w + (\xi_4, \xi_4) + (\nabla \xi_1, \nabla \xi_4)_w = -(\eta_{4,t}, \xi_4) - (\eta_4, \xi_4) \end{cases}$$

Sum those two equations:

$$(\xi_{1,t}, \xi_1) + (\nabla \xi_{1,t}, \nabla \xi_1)_w + (\nabla \xi_2, \nabla \xi_1)_w + (\xi_{4,t}, \xi_4) + (\nabla \xi_{4,t}, \nabla \xi_4)_w + (\xi_4, \xi_4)$$
$$= -(\eta_{1,t}, \xi_1) - (\nabla \eta_{1,t}, \nabla \xi_1)_w - (\eta_{4,t}, \xi_4) - (\eta_4, \xi_4)$$

As to:

$$(\xi_{1,t}, \xi_1) = \frac{d}{dt}\|\xi_1\|^2$$
$$(\nabla \xi_{1,t}, \nabla \xi_1)_w = \frac{d}{dt}|||\xi_1|||^2$$
$$(\xi_{4,t}, \xi_4) = \frac{d}{dt}\|\xi_4\|^2$$
$$(\nabla \xi_4, \nabla \xi_4)_w = |||\xi_4^n|||^2, \quad (\xi_4, \xi_4) = \|\xi_4\|^2$$

Besides,

$$(\nabla\xi_2, \nabla\xi_1)_w = (\xi_{2,t}, \xi_2) + (\eta_{2,t}, \eta_2) + (\xi_{2,t}, \eta_2) + (\eta_{2,t}, \xi_2)$$
$$= (\xi_{2,t}, \xi_2) + (\eta_{2,t}, \eta_2) + \frac{d}{dt}(\xi_2, \eta_2)$$

Let's deal with the right hand side,

$$\|(\eta_{1,t}, \xi_1)\| \le C\|\eta_{1,t}\|_0^2 + C\|\xi_1\|_0^2 \le Ch^{2(k+1)}\|q_t\|_{k+1}^2 + C\|\xi_1\|_0^2$$

$$\|(\nabla\eta_{1,t}, \nabla\xi_1)\| \le C\left|\left|\left|\eta_{1,t}\right|\right|\right|^2 + C|||\xi_1|||^2 \le Ch^{2k}\|q_t\|_{k+1}^2 + C\|\xi_1\|^2$$

$$\|(\eta_{4,t}, \xi_4)\| \le C\|\eta_{4,t}\|_0^2 + C\|\xi_4\|_0^2 \le Ch^{2(k+1)}\|\theta_t\|_{k+1}^2 + C\|\xi_4\|_0^2$$

$$\|(\eta_4, \xi_4)\| \le C\|\eta_4\|_0^2 + C\|\xi_4\|_0^2 \le Ch^{2(k+1)}\|\theta\|_{k+1}^2 + C\|\xi_4\|_0^2$$

$$(\xi_{2,t}, \eta_2) = \frac{d}{dt}(\xi_2, \eta_2) - (\eta_{2,t}, \xi_2)$$

$$\|(\eta_{2,t}, \xi_2)\| \le C\|\eta_{2,t}\|_0^2 + C\|\xi_2\|_0^2 \le Ch^{2(k+1)}\|\sigma_t\|_{k+1}^2 + C\|\xi_2\|_0^2$$

When summing those right hand side, we will have:

$$\frac{d}{dt}\|\xi_1\|^2 + \frac{d}{dt}|||\xi_1|||^2 + \frac{d}{dt}\|\xi_2\|^2 + \frac{d}{dt}\|\xi_4\|^2 + \frac{d}{dt}|||\xi_4|||^2 + \|\xi_4\|^2$$
$$= -(\eta_{1,t}, \xi_1) - (\nabla\eta_{1,t}, \nabla\xi_1)_w - (\eta_{4,t}, \xi_4) - (\eta_4, \xi_4) - \frac{d}{dt}\|\eta_2\|^2 - \frac{d}{dt}(\xi_2, \eta_2)$$
$$= Ch^{2k}(\|q_t\|_{k+1}^2 + \|\theta_t\|_{k+1}^2 + \|\theta\|_{k+1}^2 + \|\sigma_t\|_{k+1}^2) + C(\|\xi_1\|^2 + \|\xi_2\|^2 + \|\xi_4\|^2) + \frac{d}{dt}(\xi_2, \eta_2)$$

Taking integral of both sides,

$$
\begin{aligned}
&\|\xi_1\|^2 + |||\xi_1|||^2 + \|\xi_2\|^2 + \|\xi_4\|^2 + |||\xi_4|||^2 + \int_0^T \|\xi_4\|^2 ds \\
&\quad = \|\xi_1(0)\|^2 + |||\xi_1(0)|||^2 + \|\xi_2(0)\|^2 + \|\xi_4(0)\|^2 + |||\xi_4(0)|||^2 \\
&\quad + Ch^{2k} \int_0^T (\|q_t\|_{k+1}^2 + \|\theta_t\|_{k+1}^2 + \|\theta\|_{k+1}^2 + \|\sigma_t\|_{k+1}^2) ds \\
&\quad + C\epsilon \int_0^T (\|\xi_1\|^2 + \|\xi_2\|^2 + \|\xi_4\|^2) ds + C\epsilon \|\xi_2\|^2 + C\|\eta_2\|^2
\end{aligned}
$$

Simplify further,

$$
\begin{aligned}
&\|\xi_1\|^2 + |||\xi_1|||^2 + (1 - C\epsilon)\|\xi_2\|^2 + \|\xi_4\|^2 + |||\xi_4|||^2 + \int_0^T \|\xi_4\|^2 ds \\
&\quad = \|\xi_1(0)\|^2 + |||\xi_1(0)|||^2 + \|\xi_2(0)\|^2 + \|\xi_4(0)\|^2 + |||\xi_4(0)|||^2 \\
&\quad + Ch^{2k} \int_0^T (\|q_t\|_{k+1}^2 + \|\theta_t\|_{k+1}^2 + \|\theta\|_{k+1}^2 + \|\sigma_t\|_{k+1}^2) ds \\
&\quad + C\epsilon \int_0^T (\|\xi_1\|^2 + \|\xi_2\|^2 + \|\xi_4\|^2) ds + Ch^{2(k+1)} \|\sigma\|_{k+1}^2
\end{aligned}
$$

Since $\|\xi_1(0)\|, |||\xi_1(0)|||, \|\xi_2(0)\|, \|\xi_4\|, |||\xi_4(0)|||$ has the order of $O(h^2)$. We will have the conclusion,

$$
\|\xi_1\| + \|\xi_2\| + \|\xi_4\| \leq Ch^k
$$

Choose $p_h = \xi_3$, we can obtain

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\xi_3\|^2 &= (\xi_1, \xi_3) - (\eta_{3,t}, \xi_3) + (\eta_1, \xi_3) \\
&\leq C\|\xi_1\|^2 + C\|\xi_3\|^2 + C\|\eta_{3,t}\|^2 + C\|\eta_1\|^2
\end{aligned}
\tag{4.6}
$$

Since

$$\|\eta_{3,t}\|^2 \le Ch^{2(k+1)}\|u_t\|_2^2$$
$$\|\eta_1\|^2 \le Ch^{2(k+1)}\|q\|_2^2 \tag{4.7}$$

Using Cauchy-Schwatz inequality, and apply Gronwall inequality, integrate from 0 to T.

$$\|\xi_3\|^2 \le \|\xi_3(0)\|^2 + Ch^{2(k+1)}\int_0^T(\|u_t\|_{k+1}^2 + \|q\|_{k+1}^2)ds + C\int_0^T \|\xi_3\|^2 ds \tag{4.8}$$

This completes the proof. □

4.2 Fully Discrete Mixed Element Formulation

4.2.1 The Existence and Uniqueness Fully Discrete Mixed Element Formulation

For full discretization, we use the backward Euler method of first order accurate in time. For the backward Euler method, Let M be a positive integer, then $\Delta t = T/M$ be the step size of time, $t^i = i\Delta t, 0 \le i \le M$. Further, let $\psi^n = \psi(t^n)$ and $\partial_t\psi^n = (\psi^n - \psi^{n-1})/\Delta t$, for some continuous function $\psi \in C^0[0,T]$, and let $(Q^n, S^n, U^n, \Theta^n) \in \mathcal{S}_h^n \times \mathcal{S}_h^n \times \mathcal{S}_h^n \times \mathcal{S}_h^n$ be the mixed element approximation of $(q(t^n), \sigma(t^n), u(t^n), \theta(t^n))$. For each n, the different time interval is (t^n, t^{n+1}), the corresponding triangulation is $\mathcal{T}_h^n$, broken space $\mathcal{S}_h^n$. Then the fully discrete mixed finite element solution for problem (I^{**}) may be presented as follows: Problem(I_h^n). Find $(Q^n, S^n, U^n, \Theta^n) \in \mathcal{S}_h^n \times \mathcal{S}_h^n \times \mathcal{S}_h^n \times \mathcal{S}_h^n$ such that, for $1 \le n \le M$, $\forall p_h, l_h, w_h, r_h \in \mathcal{S}_h^n$

$$\begin{cases} \text{(a) } (\overline{\partial}U^n, p_h^n) = (Q^n, p_h^n) \\ \text{(b) } (S^n, l_h^n) - (\nabla U^n, \nabla l_h^n)_w = 0 \\ \text{(c) } (\overline{\partial}Q^n, w_h^n) + (\nabla\overline{\partial}Q^n, w_h^n)_w + (\nabla S^n, \nabla w_h^n)_w - (\nabla\Theta^n, \nabla w_h^n)_w = (f^n, w_h^n) \\ \text{(d) } (\overline{\partial}\Theta^n, r_h^n) + (\overline{\partial}\nabla\Theta^n, \nabla r_h^n)_w + (\Theta^n, r_h^n) + (\nabla Q^n, \nabla r_h^n)_w = (g^n, r_h^n) \end{cases} \tag{4.9}$$

Theorem 4.5. *Problem I_h^n has a unique solution $(Q^n, S^n, U^n, \Theta^n) \in \mathcal{S}_h^n \times \mathcal{S}_h^n \times \mathcal{S}_h^n \times \mathcal{S}_h^n$.*

Proof. Denote $\tilde{Q}^n = Q^{1,n} - Q^{2,n}$, $\tilde{S}^n = S^{1,n} - S^{2,n}$, $\tilde{U}^n = U^{1,n} - U^{2,n}$, $\tilde{\Theta}^n = \Theta^{1,n} - \Theta^{2,n}$.

For problem (I_h^n), work on Eq.(4.9c) and Eq.(4.9c), substitute by $\tilde{w}_h = -\triangle\tilde{Q}^n$ and $\tilde{r}_h = -\triangle\tilde{\Theta}^n$,

$$\begin{cases} (\dfrac{\triangle\tilde{U}^n - \triangle\tilde{U}^{n-1}}{\tau}, p_h) - (\triangle\tilde{Q}^n, p_h) = 0 \\ (\tilde{S}^n, l_h) - (\nabla\tilde{U}^n, \nabla l_h)_w = 0 \\ (\dfrac{\tilde{Q}^n - \tilde{Q}^{n-1}}{\tau}, w_h) + (\dfrac{\nabla\tilde{Q}^n - \nabla\tilde{Q}^{n-1}}{\tau}, \nabla w_h)_w + (\nabla\tilde{S}^n, \nabla w_h)_w - (\nabla\tilde{\Theta}^n, \nabla w_h)_w = 0 \\ (\dfrac{\tilde{\Theta}^n - \tilde{\Theta}^{n-1}}{\tau}, r_h) + (\nabla\tilde{\Theta}^n, \nabla r_h)_w + (\tilde{\Theta}^n, r_h) + (\nabla\tilde{Q}^n, \nabla r_h)_w = 0 \end{cases}$$

Choose $p_h = \triangle\tilde{U}^n$, $l_h = \triangle\tilde{Q}^n$, $w_h = \tilde{Q}^n$ and $r_h = \tilde{\Theta}^n$, and adding up all four equations,

$$\begin{aligned} (\frac{\triangle\tilde{U}^n - \triangle\tilde{U}^{n-1}}{\tau}, \triangle\tilde{U}^n) - (\triangle\tilde{Q}^n, \triangle\tilde{U}^n) + (\tilde{S}^n, \triangle\tilde{Q}^n) - (\nabla\tilde{U}^n, \nabla\triangle\tilde{Q}^n)_w \\ + (\frac{\tilde{Q}^n - \tilde{Q}^{n-1}}{\tau}, \tilde{Q}^n) + (\frac{\nabla\tilde{Q}^n - \nabla\tilde{Q}^{n-1}}{\tau}, \nabla\tilde{Q}^n) \\ + (\nabla\tilde{S}^n, \nabla\tilde{Q}^n)_w - (\nabla\tilde{\Theta}^n, \nabla\tilde{Q}^n)_w + (\frac{\tilde{\Theta}^n - \tilde{\Theta}^{n-1}}{\tau}, \tilde{\Theta}^n) \\ + (\nabla\tilde{\Theta}^n, \nabla\tilde{\Theta}^n) + (\tilde{\Theta}^n, \tilde{\Theta}^n) + (\nabla\tilde{Q}^n, \nabla\tilde{\Theta}^n)_w = 0 \end{aligned} \tag{4.10}$$

As the fact,

$$
\begin{aligned}
(\frac{\triangle\tilde{U}^n-\triangle\tilde{U}^{n-1}}{\tau},\triangle\tilde{U}^n) &\geq \frac{\|\triangle\tilde{U}^n\|^2-\|\triangle\tilde{U}^{n-1}\|^2}{2\tau}\\
(\frac{\tilde{Q}^n-\tilde{Q}^{n-1}}{\tau},\tilde{Q}^n) &\geq \frac{\|\tilde{Q}^n\|^2-\|\tilde{Q}^{n-1}\|^2}{2\tau}\\
(\frac{\nabla\tilde{Q}^n-\nabla\tilde{Q}^{n-1}}{\tau},\nabla\tilde{Q}^n)_w &\geq \frac{\left|\left|\left|\tilde{Q}^n\right|\right|\right|^2-\left|\left|\left|\tilde{Q}^{n-1}\right|\right|\right|^2}{2\tau}\\
(\frac{\tilde{\Theta}^n-\tilde{\Theta}^{n-1}}{\tau},\tilde{\Theta}^n) &\geq \frac{\|\tilde{\Theta}^n\|^2-\|\tilde{\Theta}^{n-1}\|^2}{2\tau}
\end{aligned}
$$

Simplify eqn(4.10):

$$
\begin{aligned}
&\frac{\|\triangle\tilde{U}^n\|^2-\|\triangle\tilde{U}^{n-1}\|^2}{2\tau}+\frac{\|\tilde{Q}^n\|^2-\|\tilde{Q}^{n-1}\|^2}{2\tau}+\frac{\left|\left|\left|\tilde{Q}^n\right|\right|\right|^2-\left|\left|\left|\tilde{Q}^{n-1}\right|\right|\right|^2}{2\tau}+\frac{\|\tilde{\Theta}^n\|^2-\|\tilde{\Theta}^{n-1}\|^2}{2\tau}\\
&\quad+\left|\left|\left|\tilde{\Theta}^n\right|\right|\right|^2+\|\tilde{\Theta}^n\|^2\leq 0
\end{aligned}
\tag{4.11}
$$

Sum Eq.(4.11) for $n=1\cdots M$, then

$$
\begin{aligned}
&\frac{\|\triangle\tilde{U}^M\|^2-\|\triangle\tilde{U}^0\|^2}{2\tau}+\frac{\|\tilde{Q}^M\|^2-\|\tilde{Q}^0\|^2}{2\tau}+\frac{\left|\left|\left|\tilde{Q}^M\right|\right|\right|^2-\left|\left|\left|\tilde{Q}^0\right|\right|\right|^2}{2\tau}+\frac{\|\tilde{\Theta}^M\|^2-\|\tilde{\Theta}^0\|^2}{2\tau}\\
&\quad+\sum_{i=1}^{M}(\left|\left|\left|\tilde{\Theta}^i\right|\right|\right|^2+\|\tilde{\Theta}^i\|^2)\leq 0
\end{aligned}
\tag{4.12}
$$

By discrete Gronwall inequality, thus we have the stability for fully discrete form. This completes the proof of the theorem. □

4.2.2 Fully Discrete Estimates

For the error estimate, we need to introduce the decomposition.

$$
\begin{aligned}
q^n - Q^n &= q^n - P_h q^n + P_h q^n - Q^n = \eta^n + \xi^n \\
\sigma^n - S^n &= \sigma^n - P_h \sigma^n + P_h \sigma^n - S^n = \tau^n + \psi^n \\
u^n - U^n &= u^n - P_h u^n + P_h u^n - U^n = \delta^n + \gamma^n \\
\theta^n - \Theta^n &= \theta^n - P_h \theta^n + P_h \theta^n - \Theta^n = \rho^n + \zeta^n
\end{aligned}
\tag{4.13}
$$

We may denote $\phi(t^n)$ as ϕ^n, here ϕ can be q, σ, u, θ. P_h is L^2 projection defined previously.

Besides,

$$
\begin{aligned}
\pi^n &= q^n - \bar{\partial} q^n & \phi^n &= u^n - \bar{\partial} u^n \\
\nu^n &= \theta^n - \bar{\partial} \theta^n & \kappa^n &= \sigma^n - \bar{\partial} \sigma^n
\end{aligned}
$$

Theorem 4.6. *Let $(Q^n, S^n, U^n, \Theta^n)$ and (q, σ, u, θ) are the solutions of (I_h^n) and (I^{**}). If $q_t \in L^\infty(H^{k+1})$, $\theta_t \in L^\infty(H^{k+1})$, $u_t \in L^\infty(H^{k+1})$, $u_{tt} \in L^2(H^{k+1})$, $q_{tt} \in L^2(H^{k+1})$, $\sigma_{tt} \in L^2(H^{k+1})$, $\theta_{tt} \in L^2(H^{k+1})$. Besides,*

$$
\begin{aligned}
\|P_h q^0 - q^0\| + h^k \|P_h q^0 - q^0\|_1 &\leq C h^{k+1} \|q\|_{k+1} \\
\|P_h \sigma^0 - \sigma^0\| + h^k \|P_h \sigma^0 - \sigma^0\|_1 &\leq C h^{k+1} \|\sigma\|_{k+1} \\
\|P_h u^0 - u^0\| + h^k \|P_h u^0 - u^0\|_1 &\leq C h^{k+1} \|u\|_{k+1} \\
\|P_h \theta^0 - \theta^0\| + h^k \|P_h \theta^0 - \theta^0\|_1 &\leq C h^{k+1} \|\theta\|_{k+1}
\end{aligned}
\tag{4.14}
$$

Then for $\forall n \geq 0$*, there exist:*

$$\|Q^n - q(t^n)\| + \|\Theta^n - \theta(t^n)\| + \|S^n - \sigma(t^n)\| \leq C_1(h^k + \tau)$$

$$\|U^n - u(t^n)\| \leq C_2(h^{k+1} + \tau)$$

Proof. Subtracting (I_h^n) from (I^{**}),

$$\begin{cases} (a)\ (\bar{\partial}\gamma^n, p_h) - (\xi^n, p_h) = -(\bar{\partial}\delta^n + \pi^n, p_h) + (\rho^n, p_h) \\ (b)\ (\psi^n, l_h) - (\nabla\gamma^n, \nabla l_h)_w = -(\tau^n, l_h) - (\nabla\delta^n, \nabla l_h)_w \\ (c)\ (\bar{\partial}\xi^n, w_h) + (\bar{\partial}\nabla\xi^n, \nabla w_h)_w + (\nabla\psi^n, \nabla w_h) - (\nabla\zeta^n, \nabla w_h)_w = -(\bar{\partial}\eta^n + \pi^n, w_h) - (\bar{\partial}\nabla\eta^n + \pi^n, \nabla w_h)_w \\ (d)\ (\bar{\partial}\zeta^n, r_h) + (\nabla\zeta^n, \nabla r_h)_w + (\zeta^n, r_h) + (\nabla\psi^n, \nabla r_h)_w = -(\bar{\partial}\rho^n + \nu^n, r_h) - (\rho^n, r_h) \end{cases} \tag{4.15}$$

As we have the fact,

$$-(\psi^n, \triangle\xi^n) = (\bar{\partial}\psi^n + \bar{\partial}\tau^n + \kappa^n + \triangle\eta^n, \psi^n)$$

Choose $w_h = \xi^n$ and $r_h = \zeta^n$, and substitute into Eq.(4.15c) and Eq.(4.15d). Thus,

$$\begin{aligned} &(\bar{\partial}\xi^n, \xi^n) + (\bar{\partial}\nabla\xi^n, \nabla\xi^n)_w + (\bar{\partial}\psi^n, \psi^n) - (\nabla\zeta^n, \nabla\xi^n)_w \\ &\quad = -(\bar{\partial}\eta^n + \pi^n, \xi^n) - (\bar{\partial}\nabla\eta^n + \nabla\pi^n, \nabla\xi^n)_w - (\bar{\partial}\tau^n + \kappa^n + \triangle\eta^n, \psi^n) \\ &\quad = -(\bar{\partial}\eta^n + \pi^n, \xi^n) - (\bar{\partial}\nabla\eta^n + \nabla\pi^n, \nabla\xi^n)_w - (\bar{\partial}\tau^n + \kappa^n, \psi^n) \end{aligned} \tag{4.16}$$

$$(\bar{\partial}\zeta^n, \zeta^n) + (\nabla\zeta^n, \nabla\zeta^n)_w + (\zeta^n, \zeta^n) + (\nabla\psi^n, \nabla\zeta^n)_w = -(\bar{\partial}\rho^n + \nu^n, \zeta^n) - (\rho^n, \zeta^n) \tag{4.17}$$

Take the sum of Eq.(4.16) and Eq.(4.17),

$$(\bar{\partial}\xi^n, \xi^n) + (\bar{\partial}\nabla\xi^n, \nabla\xi^n)_w + (\bar{\partial}\psi^n, \psi^n) + (\bar{\partial}\zeta^n, \zeta^n) + (\nabla\zeta^n, \nabla\zeta^n)_w + (\zeta^n, \zeta^n)$$
$$= -(\bar{\partial}\eta^n + \pi^n, \xi^n) - (\bar{\partial}\nabla\eta^n + \pi^n, \nabla\xi^n)_w - (\bar{\partial}\tau^n + \kappa^n, \psi^n) - (\bar{\partial}\rho^n + \nu^n, \zeta^n) - (\rho^n, \zeta^n) \tag{4.18}$$

To simplify Eq.(4.18):

$$\frac{1}{2}\bar{\partial}\|\xi^n\|^2 + \frac{1}{2}\bar{\partial}\|\!|\xi^n|\!\|^2 + \frac{1}{2}\bar{\partial}\|\psi^n\|^2 + \frac{1}{2}\bar{\partial}\|\zeta^n\|^2 + \|\!|\zeta^n|\!\|^2 + \|\zeta^n\|^2$$
$$\le C(\|\bar{\partial}\eta^n\|^2 + \|\pi^n\|^2 + \left|\!\left|\!\left|\bar{\partial}\eta^n\right|\!\right|\!\right|^2 + \|\!|\pi^n|\!\|^2 + \|\bar{\partial}\tau^n\|^2 + \|\kappa^n\|^2 + \|\bar{\partial}\rho^n\|^2 \tag{4.19}$$
$$+ \|\nu^n\|^2 + \|\rho^n\|^2 + \|\xi^n\|^2 + \|\!|\xi^n|\!\|^2 + \|\psi^n\|^2 + \|\zeta^n\|^2)$$

Consider the RHS.

$$\|\bar{\partial}\eta^n\|^2 \le \frac{Ch^{2(k+1)}}{\tau}\int_{t_{n-1}}^{t_n}\|q_t\|_{k+1}^2 ds \tag{4.20a}$$

$$\|\pi^n\|^2 \le c\tau\int_{t_{n-1}}^{t_n}\|q_{tt}\|^2 ds \tag{4.20b}$$

$$|\!|\!|\bar{\partial}\eta^n|\!|\!|^2 \le C\|\bar{\partial}\nabla\eta^n\|^2 \le \frac{Ch^{2k}}{\tau}\int_{t_{n-1}}^{t_n}\|q_t\|_{k+1}^2 ds \tag{4.20c}$$

$$|\!|\!|\pi^n|\!|\!|^2 \le C\|\nabla\pi^n\|^2 \le C\tau\int_{t_{n-1}}^{t_n}\|q_{tt}\|_1^2 ds \tag{4.20d}$$

$$\|\bar{\partial}\tau^n\|^2 \le \frac{Ch^{2(k+1)}}{\tau}\int_{t_{n-1}}^{t_n}\|\sigma_t\|_{k+1}^2 ds \tag{4.20e}$$

$$\|\kappa^n\|^2 \le c\tau\int_{t_{n-1}}^{t_n}\|\sigma_{tt}\|^2 ds \tag{4.20f}$$

$$\|\bar{\partial}\rho^n\|^2 \le \frac{Ch^{2(k+1)}}{\tau}\int_{t_{n-1}}^{t_n}\|\theta_t\|_{k+1}^2 ds \tag{4.20g}$$

$$\|\nu^n\|^2 \le c\tau\int_{t_{n-1}}^{t_n}\|\theta_{tt}\|^2 ds \tag{4.20h}$$

$$\|\rho^n\|^2 \le Ch^{2(k+1)}\|\theta^n\|_{k+1} \tag{4.20i}$$

Substitute (4.20) into (4.19).

$$\begin{aligned}
&\frac{1}{2}\bar{\partial}\|\xi^n\|^2 + \frac{1}{2}\bar{\partial}|\!|\!|\xi^n|\!|\!|^2 + \frac{1}{2}\bar{\partial}\|\psi^n\|^2 + \frac{1}{2}\bar{\partial}\|\zeta^n\|^2 + |\!|\!|\zeta^n|\!|\!|^2 + \|\zeta^n\|^2 \\
&\le \frac{Ch^{2k}}{\tau}\int_{t_{n-1}}^{t_n}(\|q_t\|_{k+1}^2 + \|\sigma_t\|_{k+1}^2 + \|\theta_t\|_{k+1}^2)ds + C\tau\int_{t_{n-1}}^{t_n}(\|q_{tt}\|^2 + \|q_{tt}\|_1^2 + \|\sigma_{tt}\|^2 + \|\theta_{tt}\|^2)ds \\
&+ C\|\rho^n\| + C(\|\xi^n\|^2 + |\!|\!|\xi^n|\!|\!|^2 + \|\psi^n\|^2 + \|\zeta^n\|^2)
\end{aligned} \tag{4.21}$$

Sum (4.21) for $n = 1, \cdots, M$.

$$\|\xi^M\|^2 + \left|\!\left|\!\left|\xi^M\right|\!\right|\!\right|^2 + \|\psi^M\|^2 + \|\zeta^M\|^2 + \sum_{n=1}^{M}(|\!|\!|\zeta^n|\!|\!|^2 + \|\zeta^n\|^2)$$
$$\leq \|\xi^0\|^2 + \left|\!\left|\!\left|\xi^0\right|\!\right|\!\right|^2 + \|\psi^0\|^2 + \|\zeta^0\|^2 + Ch^{2k}\int_0^T (\|q_t\|_{k+1}^2 + \|\sigma_t\|_{k+1}^2 + \|\theta_t\|_{k+1}^2)ds$$
$$+ C\tau^2 \int_0^T (\|q_{tt}\|^2 + \|q_{tt}\|_1^2 + \|\sigma_{tt}\|^2 + \|\theta_{tt}\|^2)ds$$
$$+ C\sum_{n=0}^{M}\|\rho^n\|^2 + C\tau\sum_{n=0}^{M}(\|\xi^n\|^2 + |\!|\!|\xi^n|\!|\!|^2 + \|\psi^n\|^2 + \|\zeta^n\|^2) \tag{4.22}$$

At least, we need to guarantee $1 - C\tau > 0$ to apply discrete Gronwall inequality.

$$(1 - C\tau)\|\xi^M\|^2 + (1 - C\tau)\left|\!\left|\!\left|\xi^M\right|\!\right|\!\right|^2 + (1 - C\tau)\|\psi^M\|^2 + (1 - C\tau)\|\zeta^M\|^2 + \sum_{n=1}^{M}(|\!|\!|\zeta^n|\!|\!|^2 + \|\zeta^n\|^2)$$
$$\leq \|\xi^0\|^2 + \left|\!\left|\!\left|\xi^0\right|\!\right|\!\right|^2 + \|\psi^0\|^2 + \|\zeta^0\|^2 + Ch^{2k}\int_0^T (\|q_t\|_{k+1}^2 + \|\sigma_t\|_{k+1}^2 + \|\theta_t\|_{k+1}^2)ds$$
$$+ C\tau^2 \int_0^T (\|q_{tt}\|^2 + \|q_{tt}\|_1^2 + \|\sigma_{tt}\|^2 + \|\theta_{tt}\|^2)ds$$
$$+ CMh^{2(k+1)}\|\theta\|_{k+1}^2 + C\tau\sum_{n=0}^{M-1}(\|\xi^n\|^2 + |\!|\!|\xi^n|\!|\!|^2 + \|\psi^n\|^2 + \|\zeta^n\|^2) \tag{4.23}$$

In the numerical experiment, we use $\tau = h^2$, then $Mh^{2(k+1)} = h^{2k}$. Applying the discrete Gronwall inequality, we have the estimate,

$$\|\xi^M\|^2 + \|\psi^M\|^2 + \|\zeta^M\|^2 \leq C(h^{2k} + \tau^2)$$

Take $p_h = \gamma$, we can obtain

$$\|\gamma^n\|^2 \leq \|\gamma^0\|^2 + Ch^{2(k+1)}\int_0^T \|u_t\|_{k+1}^2 ds + C\tau^2\int_0^T \|u_{tt}\|^2 ds + C\tau\sum_{n=0}^{M}(\|\xi^n\|^2 + \|\gamma^n\|^2) \tag{4.24}$$

According to the discrete Gronwall inequality, that proves the theorem. □

4.3 Numerical Examples

Consider Consider

$$\begin{cases} u_{tt} - \triangle u_{tt} + \triangle^2 u + \alpha \triangle \theta = f \\ \theta_t - \triangle \theta + \theta - \alpha \triangle u_t = g \\ u|_{\partial\Omega} = \triangle u|_{\partial\Omega} = \theta|_{\partial\Omega} = 0 \\ u(\mathbf{x}, 0) = u^0(\mathbf{x}), u_t(\mathbf{x}, 0) = u^1(\mathbf{x}), \theta(\mathbf{x}, 0) = \theta^0(\mathbf{x}) \end{cases}$$

And $\alpha = 1$, $\Omega = [0, 1] \times [0, 1]$, $T = 1$, with exact solution

$$\begin{cases} u = sin(\pi x) sin(\pi y) e^{-t} \\ \theta = sin(2\pi x) sin(2\pi y) e^{-t} \end{cases}$$

the linearlized Kirchhoff-Love plate system with zero boundary condition on a unit square $\Omega = [0, 1] \times [0, 1]$ regular pattern triangular mesh will be utilized, and $(DG_1(K) \times DG_1(K) \times DG_1(K) \times DG_1(K))$ is used to solve the problem. The convergence curves of L^2 error of the solution are depicted in the figures. From the plots, we can clearly observe the convergence rate for those variables are $O(h^{1.5}), O(h^{0.5}), O(h^2), O(h^{1.5})$, which has a higher convergence rate than theory.

L^2-Convergence rate of at time $t = 0.4$.

N	$\|q^n - Q_h^n\|_0$	Order	$\|\sigma^n - \Sigma_h^n\|_0$	Order	$\|u^n - U_h^n\|_0$	Order	$\|\theta^n - \Theta_h^n\|_0$	Order
2	0.155971		4.653860		0.128919		0.331690	
4	0.078748	0.99	1.147516	2.02	0.021667	2.57	0.126256	1.39
8	0.024106	1.71	0.794004	0.53	0.005217	2.05	0.030676	2.04
16	0.010124	1.25	0.501864	0.66	0.001250	2.06	0.011094	1.47
32	0.003556	1.51	0.355609	0.50	0.000315	1.99	0.003732	1.57

Table 5: The numerical test for SIP-DG convergence rates $t = 0.4$

L^2-Convergence rate of at time $t = 0.8$.

N	$\|q^n - Q_h^n\|_0$	Order	$\|\sigma^n - \Sigma_h^n\|_0$	Order	$\|u^n - U_h^n\|_0$	Order	$\|\theta^n - \Theta_h^n\|_0$	Order
2	0.113938		1.901486		0.077915		0.128894	
4	0.022832	2.32	0.390783	2.28	0.014178	2.46	0.080740	0.67
8	0.008757	1.38	0.087337	2.16	0.003472	2.03	0.022539	1.84
16	0.003640	1.27	0.072964	0.26	0.001070	1.70	0.005677	1.99
32	0.001390	1.39	0.052947	0.46	0.000291	1.88	0.001713	1.73

Table 6: The numerical test for SIP-DG convergence rates $t = 0.8$

L^2-Convergence rate of at time $t = 1.0$.

N	$\|q^n - Q_h^n\|_0$	Order	$\|\sigma^n - \Sigma_h^n\|_0$	Order	$\|u^n - U_h^n\|_0$	Order	$\|\theta^n - \Theta_h^n\|_0$	Order
2	0.098843		1.396027		0.059813		0.119414	
4	0.018618	2.41	0.289390	2.27	0.011609	2.37	0.066128	0.85
8	0.004650	2.00	0.084935	1.77	0.003007	1.95	0.016115	2.04
16	0.003176	0.55	0.026624	1.67	0.000793	1.92	0.004653	1.79
32	0.001266	1.33	0.025795	0.05	0.000202	1.98	0.001488	1.65

Table 7: The numerical test for SIP-DG convergence rates $t = 1.0$

The following three figures are for SIP-DG, at different time $t = 0.4, 0.8, 1.0$, are using $log - log$ plot, then the slope is equivalent to the convergence rate, in absolute meaning.

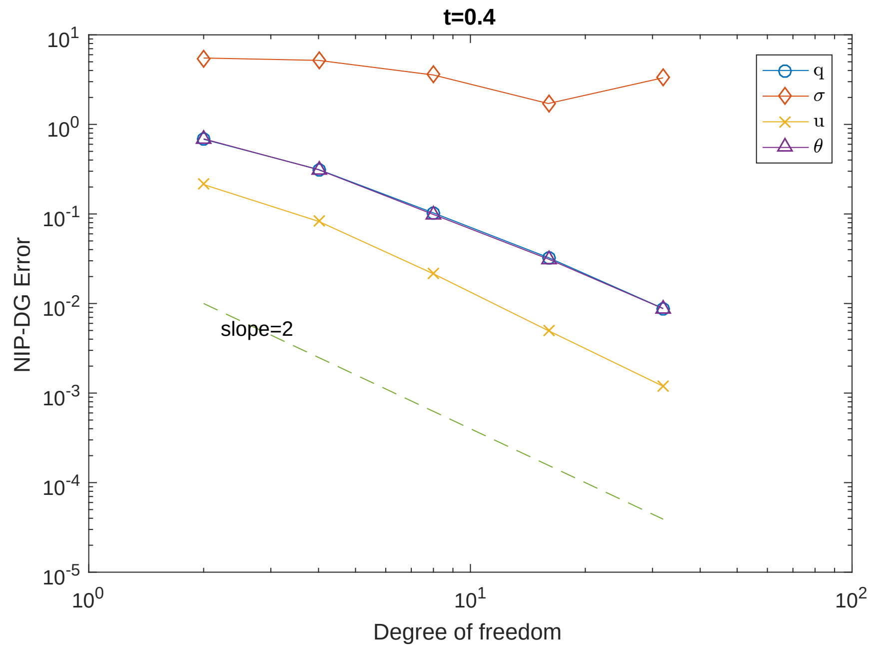

Figure 5: SIP-DG L^2 convergence rate, time $t = 0.4$

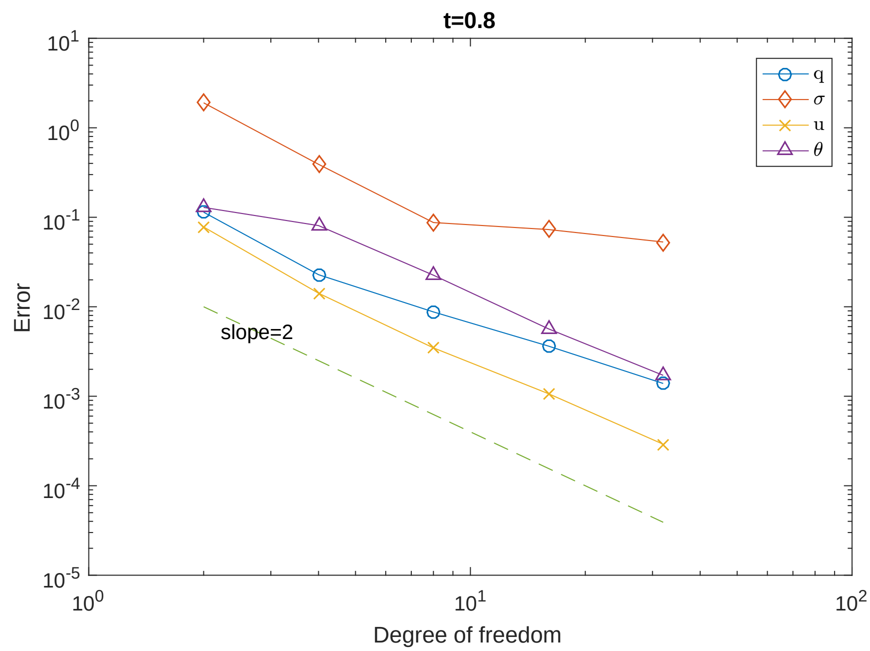

Figure 6: SIP-DG L^2 convergence rate, time $t = 0.8$

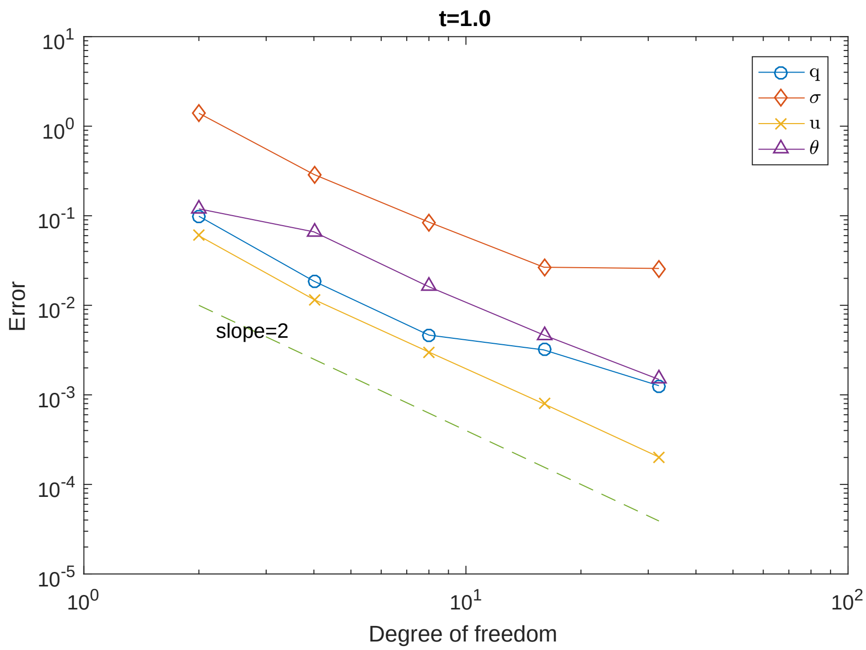

Figure 7: SIP-DG L^2 convergence rate, time $t = 1.0$

NSIP-DG L^2-Convergence rate of at time $t = 0.4$.

N	$\|q^n - Q_h^n\|_0$	Order	$\|\sigma^n - \Sigma_h^n\|_0$	Order	$\|u^n - U_h^n\|_0$	Order	$\|\theta^n - \Theta_h^n\|_0$	Order
4	0.682449		5.511690		0.212949		0.689707	
8	0.313659	1.12	5.187170	0.09	0.082745	1.36	0.311912	1.14
16	0.102740	1.61	3.562770	0.54	0.021483	1.95	0.098739	1.66
32	0.032585	1.66	1.710790	1.06	0.004980	2.11	0.031392	1.65
64	0.008806	1.89	3.316620	-0.96	0.001191	2.06	0.008768	1.84

Table 8: The numerical test for NSIP-DG convergence rates $t = 0.4$

NSIP-DG L^2-Convergence rate of at time $t = 0.8$.

N	$\|q^n - Q_h^n\|_0$	Order	$\|\sigma^n - \Sigma_h^n\|_0$	Order	$\|u^n - U_h^n\|_0$	Order	$\|\theta^n - \Theta_h^n\|_0$	Order
4	0.257494		5.471160		0.241653		0.128610	
8	0.182394	0.50	2.189670	1.32	0.080026	1.59	0.138881	-0.11
16	0.070341	1.37	0.835353	1.39	0.018095	2.14	0.064290	1.11
32	0.020312	1.79	0.471876	0.82	0.004216	2.10	0.019019	1.76
64	0.005468	1.89	0.286252	0.72	0.001070	1.98	0.005209	1.87

Table 9: The numerical test for NSIP-DG convergence rates $t = 0.8$

NSIP-DG L^2-Convergence rate of at time $t = 1.0$.

N	$\|q^n - Q_h^n\|_0$	Order	$\|\sigma^n - \Sigma_h^n\|_0$	Order	$\|u^n - U_h^n\|_0$	Order	$\|\theta^n - \Theta_h^n\|_0$	Order
4	0.448340		4.009150		0.161125		0.369467	
8	0.219525	1.03	1.701410	1.24	0.047769	1.75	0.201786	0.87
16	0.059488	1.88	0.773496	1.14	0.012396	1.95	0.057879	1.80
32	0.016482	1.85	0.341843	1.18	0.003011	2.04	0.016152	1.84
64	0.004679	1.82	0.185809	0.88	0.000706	2.09	0.004563	1.82

Table 10: The numerical test for NSIP-DG convergence rates $t = 1.0$

The following three figures are for NSIP-DG, at different time $t = 0.4, 0.8, 1.0$, are using $log - log$ plot, then the slope is equivalent to the convergence rate, in absolute meaning.

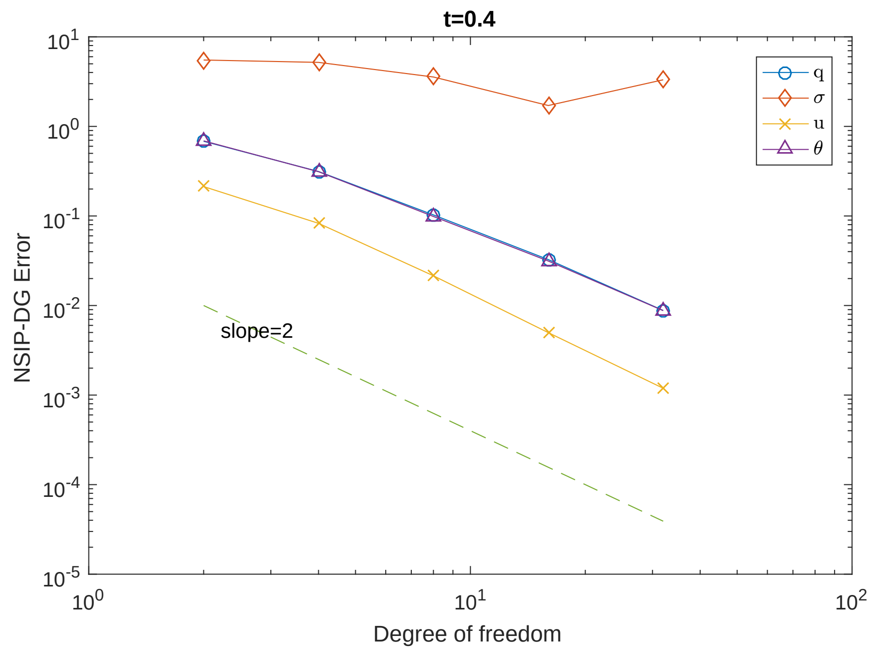

Figure 8: NSIP-DG L^2 convergence rate, time $t = 0.4$

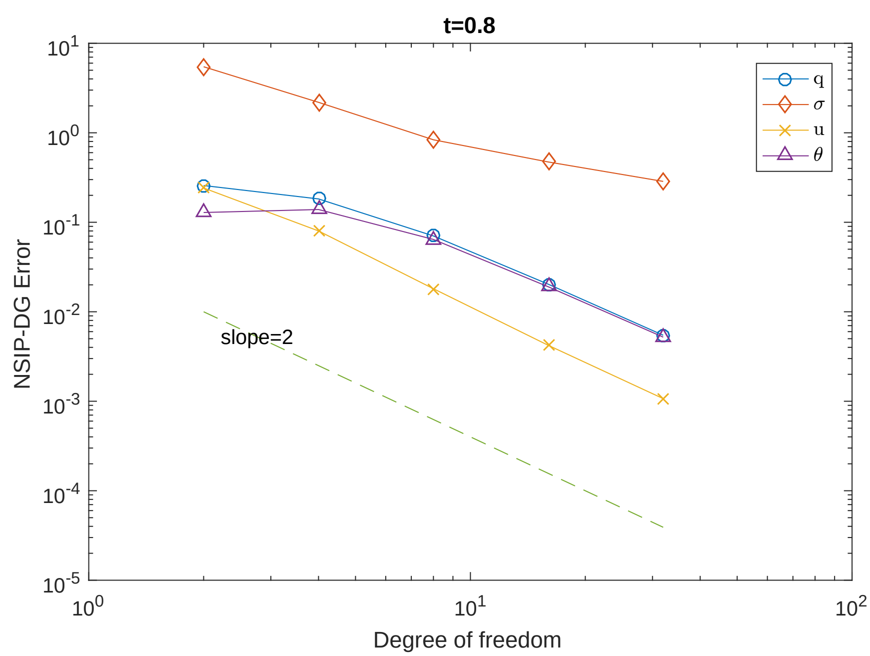

Figure 9: NSIP-DG L^2 convergence rate, time $t = 0.8$

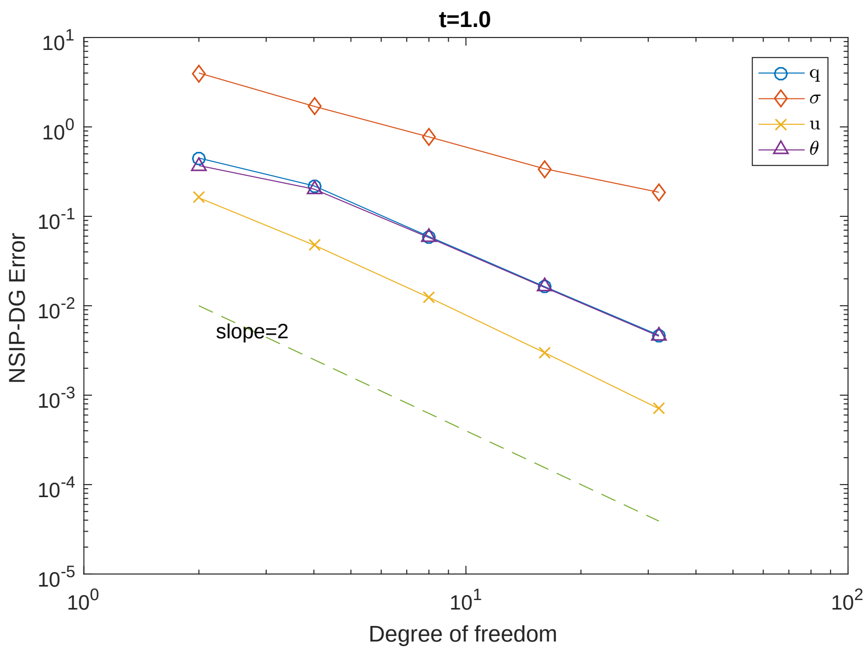

Figure 10: NSIP-DG L^2 convergence rate, time $t = 1.0$

4.4 Conclusions

In this chapter, we used both symmetric and nonsymmetric interior penalty discontinuous galerkin methods to approach the thermoelastic Kirchhoff-Love plate system. First prove the existence and uniqueness of the problem for both NSIP-DG and SIP-DG, the only difference is the coercive coefficient. If applying sufficiently smooth solutions, for SIP-DG, the penalty needs to be large enough to guarantee stability, and the a-priori L^2 error for $q_h, \sigma_h, u_h, \theta_h$ are at the rate of $O(h^k + \tau), O(h^{k-1} + \tau), O(h^{k+1} + \tau), O(h^{k+1} + \tau)$, for for NSIP-DG, the penalty needs to be set nonnegative, thus set 4, and the a-priori L^2 error for $q_h, \sigma_h, u_h, \theta_h$ are at the rate of $O(h^{k+1} + \tau), O(h^k + \tau), O(h^{k+1} + \tau), O(h^{k+1} + \tau)$, therefore showing a greater convergence performance than SIP-DG. Here h is the mesh size, and τ is the time step, k denotes the polynomial degree.

Since the SIP-DG bilinear form guarantees symmetric, continuous, coercive and adjoint consistent properties, that leads to the stability of the numerical scheme. And this method conserves energy for stability property all time, that transits to the convergence proof. however the main issue is the penalty needs to be large enough. NSIP-DG has a less requirement on penalty term.

Remark. The convergence results holds for fully DG method where the underlying bilinear form is symmetric, continuous, coercive and adjoint consistent.

CHAPTER 5 H1-GALERKIN MIXED ELEMENT METHOD

In this section, we introduce how to use H1-Galerkin mixed element method, with basic definitions and properties. The existence of a unique solution of semi-discrete mixed element method and error analysis are given in section 2. And the existence of a unique of its fully discrete method and error analysis are considered in section 3. Numerical results are presented in section 4.

5.1 Semi-discrete Mixed Element Formulation

5.1.1 Semi-discrete Mixed Element Formulation

Let $\Omega \subset R^2$, we consider the linearized thermoelastical kirchhoff-love plate equation system.

Problem(I): Find $(u,\theta) \in (W^{1,\infty}(0,T;L^2(\Omega)) \bigcap L^\infty(0,T;W^{2,4}(\Omega))) \times L^\infty(0,T;W^{2,4}(\Omega))$ such that for all $T > 0$,

$$\begin{cases} u_{tt} - \triangle u_{tt} + \triangle^2 u + \alpha \triangle \theta = f, \quad (\mathbf{x},t) \in \Omega \times (0,T) \\ \theta_t - \triangle \theta + \theta - \alpha \triangle u_t = g, \quad (\mathbf{x},t) \in \Omega \times (0,T) \\ u = \triangle u = \theta = 0, \quad (\mathbf{x},t) \in \partial\Omega \times (0,T) \\ u(\mathbf{x},0) = u^0(\mathbf{x}), u_t(\mathbf{x},0) = u^1(\mathbf{x}), \theta(\mathbf{x},0) = \theta^0(\mathbf{x}), \quad \mathbf{x} \in \Omega \end{cases} \tag{5.1}$$

where u denotes the displacement of the plate, θ denotes the displacement caused by therm changes, g, u^0, u^1 and θ^0 are given functions. Introduce the intermediate variables, the original problem would be transformed into:

Problem(I*): Find $(q,p,\sigma,\vec{r},\vec{s},\vec{w},u,\theta) \in L^\infty(0,T;H_0^1(\Omega)) \times L^\infty(0,T;H_0^1(\Omega)) \times L^\infty(0,T;H_0^1(\Omega)) \times$

$L^\infty(0,T;H(div,\Omega))\times L^\infty(0,T;H(div,\Omega))\times L^\infty(0,T;H(div,\Omega))\times L^\infty(0,T;H_0^1(\Omega))\times L^\infty(0,T;H_0^1(\Omega))$

such that, for all $T>0$,

$$\begin{cases} (a) \quad q = u_t, \quad (\mathbf{x},t)\in\ \Omega\times(0,T) \\ (b) \quad p = \sigma_t, \quad (\mathbf{x},t)\in\ \Omega\times(0,T) \\ (c) \quad \nabla u = \vec{r}, \quad (\mathbf{x},t)\in\ \Omega\times(0,T) \\ (d) \quad \nabla\sigma = \vec{s}, \quad (\mathbf{x},t)\in\ \Omega\times(0,T) \\ (e) \quad \sigma + \nabla\cdot\vec{r} = 0, \quad (\mathbf{x},t)\in\ \Omega\times(0,T) \\ (f) \quad \nabla\theta = \vec{w}, \quad (\mathbf{x},t)\in\ \Omega\times(0,T) \\ (g) \quad q_t + p_t - \nabla\cdot\vec{s} + \nabla\cdot\vec{w} = f, \quad (\mathbf{x},t)\in\ \Omega\times(0,T) \\ (h) \quad \theta_t - \nabla\cdot\vec{w} + \theta + p = g, \quad (\mathbf{x},t)\in\ \Omega\times(0,T) \end{cases} \tag{5.2}$$

To implement the mixed element method, we consider the following weak formulation:

Problem(I**): Find $(q,p,\sigma,\vec{r},\vec{s},\vec{w},u,\theta)\in L^\infty(0,T;H_0^1(\Omega))\times L^\infty(0,T;H_0^1(\Omega))\times L^\infty(0,T;H_0^1(\Omega))\times$

$L^\infty(0,T;H(div,\Omega))\times L^\infty(0,T;H(div,\Omega))\times L^\infty(0,T;H(div,\Omega))\times L^\infty(0,T;H_0^1(\Omega))\times L^\infty(0,T;H_0^1(\Omega))$

such that, for all $T>0$

$$
\begin{cases}
(a) \quad (q, \chi) = (u_t, \chi) \\
(b) \quad (p, \chi) = (\sigma_t, \chi) \\
(c) \quad (\nabla u, \nabla \chi) = (\vec{r}, \nabla \chi) \\
(d) \quad (\nabla \sigma, \nabla \chi) = (\vec{s}, \nabla \chi) \\
(e) \quad (\sigma, \nabla \cdot \vec{\chi}) + (\nabla \cdot \vec{r}, \nabla \cdot \vec{\chi}) = 0 \\
(f) \quad (\nabla \theta, \nabla \chi) = (\vec{w}, \nabla \chi) \\
(g) \quad (q_t, \nabla \cdot \vec{\chi}) + (p_t, \nabla \cdot \vec{\chi}) - (\nabla \cdot \vec{s}, \nabla \cdot \vec{\chi}) + (\nabla \cdot \vec{w}, \nabla \cdot \vec{\chi}) = (f, \nabla \cdot \vec{\chi}) \\
(h) \quad (\theta_t, \nabla \cdot \vec{\chi}) - (\nabla \cdot \vec{w}, \nabla \cdot \vec{\chi}) + (\theta, \nabla \cdot \vec{\chi}) + (p, \nabla \cdot \vec{\chi}) = (g, \nabla \cdot \vec{\chi})
\end{cases}
\tag{5.3}
$$

However we need to change the variance form. As:

$$
\begin{aligned}
(p_t, \nabla \cdot \vec{\chi}) &= -(\nabla p_t, \vec{\chi}) = -(\nabla \sigma_{tt}, \vec{\chi}) = -(\vec{s}_{tt}, \vec{\chi}) \\
(q_t, \nabla \cdot \vec{\chi}) &= -(\nabla q_t, \vec{\chi}) = -(\nabla u_{tt}, \vec{\chi}) = -(\vec{r}_{tt}, \vec{\chi}) \\
(\theta, \nabla \cdot \vec{\chi}) &= -(\nabla \theta, \vec{\chi}) = -(\vec{w}, \vec{\chi}) \\
(\theta_t, \nabla \cdot \vec{\chi}) &= -(\nabla \theta_t, \vec{\chi}) = -(\vec{w}_t, \vec{\chi})
\end{aligned}
\tag{5.4}
$$

Then the last two equations of (I^{**})can be transformed into:

$$
\begin{cases}
(\vec{r}_{tt}, \vec{\chi}) + (\vec{s}_{tt}, \vec{\chi}) + (\nabla \cdot \vec{s}, \nabla \cdot \vec{\chi}) - (\nabla \cdot \vec{w}, \nabla \cdot \vec{\chi}) = -\langle f, \nabla \cdot \vec{\chi} \rangle \\
(\vec{w}_t, \vec{\chi}) + (\nabla \cdot \vec{w}, \nabla \cdot \vec{\chi}) + (\vec{w}, \vec{\chi}) + (\vec{s}_t, \vec{\chi}) = -\langle g, \nabla \cdot \vec{\chi} \rangle
\end{cases}
\tag{5.5}
$$

Theorem 5.1. *Problem I^{**} is equivalent to Problem I^{*}*

Proof. $(\Longleftarrow)$. Suppose $(q, p, \sigma, \vec{r}, \vec{s}, \vec{w}, u, \theta) \in V \times V \times V \times \mathbf{W} \times \mathbf{W} \times \mathbf{W} \times V \times V$ is the solution of problem (I^*), it is obviously a solution of (I^{**}).

$(\Longrightarrow)$. From (5.3) Eq.(5.3a) and Eq.(5.3b), easy to know $q = u_t$ and $p = \sigma_t$.

Since $\vec{r}, \vec{s}, \vec{w}$ can be rewritten into,

$$\begin{aligned} \vec{r} &= \nabla \xi_1 + \vec{\phi}_1 \\ \vec{s} &= \nabla \xi_2 + \vec{\phi}_2 \\ \vec{w} &= \nabla \xi_3 + \vec{\phi}_3 \end{aligned} \tag{5.6}$$

where $\xi_1, \xi_2, \xi_3 \in L^\infty(0, T; H^2(\Omega) \cap H_0^1(\Omega))$, $\vec{\phi}_1, \vec{\phi}_2, \vec{\phi}_3 \in L^\infty(0, T; H(div, \Omega))$, and $\nabla \cdot \vec{\phi}_i = 0 \quad (i = 1, 2, 3)$., substitute into eqn$(c)$, eqn$(d)$ and eqn(f), apply Green formula,

$$\begin{aligned} (\nabla \xi_1, \nabla \chi) &= (\nabla u, \nabla \chi) \\ (\nabla \xi_2, \nabla \chi) &= (\nabla \sigma, \nabla \chi) \\ (\nabla \xi_3, \nabla \chi) &= (\nabla \theta, \nabla \chi) \end{aligned} \tag{5.7}$$

Then,

$$\begin{aligned} \vec{r} &= \nabla u + \vec{\phi}_1 \\ \vec{s} &= \nabla \sigma + \vec{\phi}_2 \\ \vec{w} &= \nabla \theta + \vec{\phi}_3 \end{aligned} \tag{5.8}$$

Substitute into the last two equations:

$$-(u_{tt}, \nabla \cdot \vec{\chi}) + (\vec{\phi}_{1,tt}, \vec{\chi}) - (\sigma_{tt}, \nabla \cdot \vec{\chi}) + (\vec{\phi}_{2,tt}, \vec{\chi}) + (\nabla \cdot \vec{s} - \nabla \cdot \vec{w}, \nabla \cdot \vec{\chi}) = -\langle f, \nabla \cdot \vec{\chi} \rangle$$

$$-(\theta_t, \nabla \cdot \vec{\chi}) + (\vec{\phi_{3,t}}, \vec{\chi}) + (\nabla \cdot \vec{w}, \nabla \cdot \vec{\chi}) - (\theta, \nabla \cdot \vec{\chi}) + (\vec{\phi}_3, \vec{\chi}) - (\sigma_t, \nabla \cdot \vec{\chi}) + (\vec{\phi}_{2,t}, \vec{\chi})$$

$$= -\langle g, \nabla \cdot \vec{\chi} \rangle \tag{5.9}$$

Choose $\vec{\chi} = \phi_{1,t} + \phi_{2,t}$ in the first equation.

$$\frac{d}{dt}(\phi_{1,t} + \phi_{2,t}, \phi_{1,t} + \phi_{2,t}) = (\phi_{1,tt} + \phi_{2,tt}, \phi_{1,t} + \phi_{2,t}) = 0$$

That leads to

$$(\phi_{1,t} + \phi_{2,t}, \phi_{1,t} + \phi_{2,t}) = (\phi_{1,t}(0) + \phi_{2,t}(0), \phi_{1,t}(0) + \phi_{2,t}(0))$$

Choose the initial value,

$$\vec{r}_t(0) = \nabla u_t, \quad \vec{s}_t(0) = \nabla \sigma_t$$

Then,

$$\vec{\phi}_{1,t}(0) = 0, \quad \vec{\phi}_{2,t}(0) = 0$$

And we can get, $\phi_{1,t} + \phi_{2,t} = 0$. Multiplying both sides by $\phi_1 + \phi_2$, integral over time $[0, t]$,

$$(\phi_1 + \phi_2, \phi_1 + \phi_2) = (\phi_1(0) + \phi_2(0), \phi_1(0) + \phi_2(0)) \tag{5.10}$$

Choose the initial value,

$$\vec{r}(0) = \nabla u, \quad \vec{s}(0) = \nabla \sigma$$

Thus,

$$\phi_1(0) = 0, \quad \phi_2(0) = 0$$

Then, from eqn(5.10),

$$\phi_1 + \phi_2 = 0$$

$$\begin{aligned} &-(u_{tt}, \nabla \cdot \vec{\chi}) - (\sigma_{tt}, \nabla \cdot \vec{\chi}) + (\nabla \cdot \vec{s} - \nabla \cdot \vec{w}, \nabla \cdot \vec{\chi}) = -\langle f, \nabla \cdot \vec{\chi} \rangle \\ &-(\theta_t, \nabla \cdot \vec{\chi}) + (\nabla \cdot \vec{w}, \nabla \cdot \vec{\chi}) - (\theta, \nabla \cdot \vec{\chi}) - (p, \nabla \cdot \vec{\chi}) = -\langle g, \nabla \cdot \vec{\chi} \rangle \end{aligned} \tag{5.11}$$

For any $p_t + q_t - f \in L^\infty(0, T; L^2(\Omega))$, $\exists F \in L^\infty(0, T; H(div, \Omega))$ and for any $\theta_t + \theta + p - g \in L^\infty(0, T; L^2(\Omega))$, $\exists G \in L^\infty(0, T; H(div, \Omega))$ such that,

$$\begin{aligned} &\nabla \cdot F = p_t + q_t - f \in L^2(\Omega) \\ &\nabla \cdot G = \theta_t + \theta + p - g \end{aligned} \tag{5.12}$$

Substitute into the equation,

$$\begin{aligned} (\nabla \cdot F, \nabla \cdot \vec{\chi}) - (\nabla \cdot \vec{s} - \nabla \cdot \vec{w}, \nabla \cdot \vec{\chi}) = 0 \\ (\nabla \cdot G, \nabla \cdot \vec{\chi}) - (\nabla \cdot \vec{w}, \nabla \cdot \vec{\chi}) = 0 \end{aligned} \tag{5.13}$$

As $\vec{\chi}$ can be chosen from $H(div)$, such that

$$\begin{aligned} \nabla \cdot F - (\nabla \cdot \vec{s} - \nabla \cdot \vec{w}) = 0 \\ \nabla \cdot G - (\nabla \cdot \vec{w}) = 0 \end{aligned} \tag{5.14}$$

Then we have following conclusion

$$
\begin{aligned}
q_t + p_t - \nabla \cdot \vec{s} + \nabla \cdot \vec{w} = f \\
\theta_t - \nabla \cdot \vec{w} + \theta + p = g
\end{aligned}
\tag{5.15}
$$

That proves ($\Longrightarrow$). □

Theorem 5.2. *Problem I^{**} has a unique solution.*

Proof. The existence of the solution follows from the existence and regularity assumption. By defining $q = u_t, p = \sigma_t, \vec{r} = \nabla u, \vec{s} = \nabla \sigma, \vec{w} = \nabla \theta$, it immediates comes up with a weak solution for I^{**}.

To prove the uniqueness of the solution, we need to prove the stability first.

let $(p_0^i, q_0^i, \sigma_0^i, \vec{r}_0^i, \vec{r}_0^i, \vec{w}_0^i, u_0^i, \theta_0^i)$ be the initial data, and $(p^i, q^i, \sigma^i, \vec{r}^i, \vec{r}^i, \vec{w}^i, u^i, \theta^i)$ be the corresponding weak solutions.

Denote $\tilde{q} = q^1 - q^2$, $\tilde{p} = p^1 - p^2$, $\tilde{\sigma} = \sigma^1 - \sigma^2$, $\tilde{u} = u^1 - u^2$, $\tilde{\theta} = \theta^1 - \theta^2$, $\tilde{\vec{r}} = \vec{r}^1 - \vec{r}^2$, $\tilde{\vec{s}} = \vec{s}^1 - \vec{s}^2$, $\tilde{\vec{w}} = \vec{w}^1 - \vec{w}^2$.

$$\begin{cases} (a)\ (\tilde{q}, \chi) = (\tilde{u}_t, \chi) \\ (b)\ (\tilde{p}, \chi) = (\tilde{\sigma}_t, \chi) \\ (c)\ (\nabla \tilde{u}, \vec{\chi}) = (\vec{\tilde{r}}, \vec{\chi}) \\ (d)\ (\nabla \tilde{\sigma}, \vec{\chi}) = (\vec{\tilde{s}}, \vec{\chi}) \\ (e)\ (\tilde{\sigma}, \chi) + (\nabla \cdot \vec{\tilde{r}}, \chi) = 0 \\ (f)\ (\nabla \tilde{\theta}, \vec{\chi}) = (\vec{\tilde{w}}, \vec{\chi}) \\ (g)\ (\vec{\tilde{r}}_{tt}, \vec{\chi}) + (\vec{\tilde{s}}_{tt}, \vec{\chi}) + (\nabla \cdot \vec{\tilde{s}}, \nabla \cdot \vec{\chi}) - (\nabla \cdot \vec{\tilde{w}}, \nabla \cdot \vec{\chi}) = 0 \\ (h)\ (\vec{\tilde{w}}_t, \vec{\chi}) + (\nabla \cdot \vec{\tilde{w}}, \nabla \cdot \vec{\chi}) + (\vec{\tilde{w}}, \vec{\chi}) + (\vec{\tilde{s}}_t, \vec{\chi}) = 0 \end{cases} \tag{5.16}$$

Choose last two test functions, $\vec{\chi} = \vec{\tilde{r}}_t$ and $\vec{\chi} = \vec{\tilde{w}}$, and adding up those two equations,

$$\frac{1}{2}\frac{d}{dt}\|\tilde{\sigma}_t\|^2 + \frac{1}{2}\frac{d}{dt}\|\vec{\tilde{s}}_t\|^2 + \|\nabla \cdot \vec{\tilde{s}}\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla \vec{\tilde{w}}\|^2 + \|\nabla \cdot \vec{\tilde{w}}\|^2 + \|\nabla \vec{\tilde{w}}\|^2 = 0$$

Taking the integral, thus we have the stability. This completes the proof of the theorem.

□

Now we consider the following semi-discrete form for problem(I^{**}).

Problem(I_h): Find $(q_h, p_h, \sigma_h, \vec{r}_h, \vec{s}_h, \vec{w}_h, u_h, \theta_h) : [0, T] \to V_h \times V_h \times V_h \times \vec{W}_h \times \vec{W}_h \times \vec{W}_h \times V_h \times V_h$

$$\begin{cases} (a)\ (q_h, \chi) = (u_{h,t}, \chi) \\ (b)\ (p_h, \chi) = (\sigma_{h,t}, \chi) \\ (c)\ (\nabla u_h, \vec{\chi}_h) = (\vec{r}_h, \vec{\chi}_h) \\ (d)\ (\nabla \sigma_h, \vec{\chi}_h) = (\vec{s}_h, \vec{\chi}_h) \\ (e)\ (\sigma_h, \chi_h) + (\nabla \cdot \vec{r}_h, \chi_h) = 0 \\ (f)\ (\nabla \theta_h, \overrightarrow{\chi_h}) = (\vec{w}_h, \overrightarrow{\chi_h}) \\ (g)\ (q_{h,t}, \nabla \cdot \vec{\chi}_h) + (p_{h,t}, \nabla \cdot \vec{\chi}_h) - (\nabla \cdot \vec{s}_h, \nabla \cdot \vec{\chi}_h) + (\nabla \cdot \vec{w}_h, \nabla \cdot \vec{\chi}_h) = (f_h, \nabla \cdot \vec{\chi}_h) \\ (h)\ (\theta_{h,t}, \nabla \cdot \vec{\chi}_h) - (\nabla \cdot \vec{w}_h, \nabla \cdot \vec{\chi}_h) + (\theta_h, \nabla \cdot \vec{\chi}_h) + (p_h, \nabla \cdot \vec{\chi}_h) = (g_h, \nabla \cdot \vec{\chi}_h) \end{cases} \tag{5.17}$$

with given $\sigma_h(0), u_h(0), \theta_h(0)$ determined.

Corespondingly, Eq.(5.17g) and Eq.(5.17h) can be transformed into,

$$\begin{cases} (\vec{r}_{h,tt}, \vec{\chi}_h) + (\vec{s}_{h,tt}, \vec{\chi}_h) + (\nabla \cdot \vec{s}_h, \nabla \cdot \vec{\chi}_h) - (\nabla \cdot \vec{w}_h, \nabla \cdot \vec{\chi}_h) = -\langle f_h, \nabla \cdot \vec{\chi}_h \rangle \\ (\vec{w}_{h,t}, \vec{\chi}_h) + (\nabla \cdot \vec{w}_h, \nabla \cdot \vec{\chi}_h) + (\vec{w}_h, \vec{\chi}_h) + (\nabla \cdot \vec{r}_{h,t}, \nabla \cdot \vec{\chi}_h) = -\langle g_h, \nabla \cdot \vec{\chi}_h \rangle \end{cases} \tag{5.18}$$

5.1.2 The Existence and Uniqueness Semi-discrete Mixed Element Formulation

In this section, we demonstrate on the existence and uniqueness of the solution of system.

Theorem 5.3. *Problem I_h has a unique solution.*

Proof. The existence of the solution follows from the existence and regularity assumption.

By defining $q_h = u_{ht}, p_h = \sigma_{ht}, \vec{r}_h = \nabla u_h, \vec{s}_h = \nabla \sigma_h, \vec{w}_h = \nabla \theta_h$, it immediates comes up with a weak solution for I_h.

To prove the uniqueness of the solution, we need to prove the stability first.

let $(p^i_{h0}, q^i_{h0}, \sigma^i_{h0}, \vec{r}^i_{h0}, \vec{r}^i_{h0}, \vec{w}^i_{h0}, u^i_{h0}, \theta^i_{h0})$ be the initial data, and $(p^i_h, q^i_h, \sigma^i_h, \vec{r}^i_h, \vec{r}^i_h, \vec{w}^i_h, u^i_h, \theta^i_h)$ be the corresponding weak solutions.

Denote $\tilde{q}_h = q^1_h - q^2_h$, $\tilde{p}_h = p^1_h - p^2_h$, $\tilde{\sigma}_h = \sigma^1_h - \sigma^2_h$, $\tilde{u}_h = u^1_h - u^2_h$, $\tilde{\theta}_h = \theta^1_h - \theta^2_h$, $\tilde{\vec{r}}_h = \vec{r}^1_h - \vec{r}^2_h$, $\tilde{\vec{s}}_h = \vec{s}^1_h - \vec{s}^2_h$, $\tilde{\vec{w}}_h = \vec{w}^1_h - \vec{w}^2_h$.

$$\begin{cases} (a)\ (\tilde{q}_h, \chi_h) = (\tilde{u}_{ht}, \chi) \\ (b)\ (\tilde{p}_h, \chi_h) = (\tilde{\sigma}_{ht}, \chi_h) \\ (c)\ (\nabla \tilde{u}_h, \vec{\chi}_h) = (\tilde{\vec{r}}_h, \vec{\chi}_h) \\ (d)\ (\nabla \tilde{\sigma}_h, \vec{\chi}_h) = (\tilde{\vec{s}}_h, \vec{\chi}_h) \\ (e)\ (\tilde{\sigma}_h, \chi_h) + (\nabla \cdot \tilde{\vec{r}}_h, \chi_h) = 0 \\ (f)\ (\nabla \tilde{\theta}_h, \vec{\chi}_h) = (\tilde{\vec{w}}_h, \vec{\chi}_h) \\ (g)\ (\tilde{\vec{r}}_{htt}, \vec{\chi}_h) + (\tilde{\vec{s}}_{htt}, \vec{\chi}_h) + (\nabla \cdot \tilde{\vec{s}}_h, \nabla \cdot \vec{\chi}_h) - (\nabla \cdot \tilde{\vec{w}}_h, \nabla \cdot \vec{\chi}_h) = 0 \\ (h)\ (\tilde{\vec{w}}_{ht}, \vec{\chi}_h) + (\nabla \cdot \ \tilde{\vec{w}}_h, \nabla \cdot \vec{\chi}_h) + (\tilde{\vec{w}}_h, \vec{\chi}_h) + (\tilde{\vec{s}}_{ht}, \vec{\chi}_h) = 0 \end{cases} \tag{5.19}$$

Choose last two test functions, $\vec{\chi} = \tilde{\vec{r}}_t$ and $\vec{\chi} = \tilde{\vec{w}}$, and adding up tbose two equations,

$$\frac{1}{2}\frac{d}{dt}\|\tilde{\sigma}_{ht}\|^2 + \frac{1}{2}\frac{d}{dt}\|\tilde{\vec{s}}_{ht}\|^2 + \|\nabla \cdot \tilde{\vec{s}}_h\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla \tilde{\vec{w}}_h\|^2 + \|\nabla \cdot \tilde{\vec{w}}\|^2 + \|\nabla \tilde{\vec{w}}_h\|^2 = 0$$

Taking the integral, thus we have the stability. This completes the proof of the theorem.

□

5.1.3 Semi Discrete Estimates

First we need to introduce the elliptic projection and Raviart-Thomas Projection: find R_hq, R_hp, $R_h\sigma$, $R_h\theta$, $R_hu \in V_h$ and $\Pi_h\vec{r}$, $\Pi_h\vec{s}$, $\Pi_h\vec{w} \in \vec{W}_h$ satisfying:

$$
\begin{aligned}
&(\nabla(q - R_hq), \nabla v_h) = 0, \quad \forall v_h \in V_h && (\nabla(u - R_hu), \nabla v_h) = 0, \quad \forall v_h \in V_h \\
&(\nabla(\sigma - R_h\sigma), \nabla v_h) = 0, \quad \forall v_h \in V_h && (\nabla(\theta - R_h\theta), \nabla v_h) = 0, \quad \forall v_h \in V_h \\
&(\nabla(p - R_hp), \nabla v_h) = 0, \quad \forall v_h \in V_h && (\nabla \cdot (\vec{s} - \Pi_h\vec{s}), \nabla \cdot v_h) = 0, \quad \forall v_h \in \vec{W}_h \\
&(\nabla \cdot (\vec{r} - \Pi_h\vec{r}), \nabla \cdot v_h) = 0, \quad \forall v_h \in \vec{W}_h && (\nabla \cdot (\vec{w} - \Pi_h\vec{w}), \nabla \cdot v_h) = 0, \quad \forall v_h \in \vec{W}_h
\end{aligned}
\tag{5.20}
$$

Subtracting (I^{**}) from I_h, we obtain:

$$
\begin{cases}
(u_{h,t} - u_t, \chi_h) - (q_h - q, \chi_h) = 0 \\
(\sigma_{h,t} - \sigma_t, \chi_h) - (p_h - p, \chi_h) = 0 \\
(\nabla(u_h - u), \vec{\chi_h}) - (\vec{r}_h - \vec{r}, \vec{\chi_h}) = 0 \\
(\nabla(\sigma_h - \sigma), \vec{\chi_h}) = (\vec{s}_h - \vec{s}, \vec{\chi_h}) \\
(\sigma_h - \sigma, \chi_h) + (\nabla \cdot (\vec{r}_h - \vec{r}), \chi_h) = 0 \\
(\nabla(\theta_h - \theta), \overrightarrow{\chi_h}) - (\vec{w_h} - \vec{w}, \overrightarrow{\chi_h}) = 0 \\
(\vec{r}_{h,tt} - \vec{r}_{tt}, \vec{\chi}_h) + (\vec{s}_{h,t} - \vec{s}_t, \vec{\chi}_h) + (\nabla \cdot (\vec{s}_h - \vec{s}), \nabla \cdot \vec{\chi}_h) - (\nabla \cdot (\vec{w}_h - \vec{w}), \nabla \cdot \vec{\chi}_h) = 0 \\
(\vec{w}_{h,t} - \vec{w}, \vec{\chi}_h) + (\nabla \cdot (\vec{w}_h - \vec{w}), \nabla \cdot \vec{\chi}_h) + (\vec{w}_h - \vec{w}, \vec{\chi}_h) + (\vec{s}_{h,t} - \vec{s}_t, \vec{\chi}_h) = 0
\end{cases}
\tag{5.21}
$$

Assume:

$$q - q_h = (u - R_h u) + (R_h u - u_h) = \xi_1 + \eta_1 \tag{5.22}$$

$$p - p_h = (p - R_h p) + (R_h p - p_h) = \xi_2 + \eta_2 \tag{5.23}$$

$$\sigma - \sigma_h = (\sigma - R_h \sigma) + (R_h \sigma - \sigma_h) = \xi_3 + \eta_3 \tag{5.24}$$

$$\vec{r} - \vec{r}_h = (\vec{r} - \Pi_h \vec{r}) + (\Pi_h \vec{r} - \vec{r}_h) = \vec{\xi}_4 + \vec{\eta}_4 \tag{5.25}$$

$$\vec{s} - \vec{s}_h = (\vec{s} - \Pi_h \vec{s}) + (\Pi_h \vec{s} - \vec{s}_h) = \vec{\xi}_5 + \vec{\eta}_5 \tag{5.26}$$

$$\vec{w} - \vec{w}_h = (\vec{w} - \Pi_h \vec{w}) + (\Pi_h \vec{w} - \vec{w}_h) = \vec{\xi}_6 + \vec{\eta}_6 \tag{5.27}$$

$$u - u_h = (u - R_h u) + (R_h u - u_h) = \xi_7 + \eta_7 \tag{5.28}$$

$$\theta - \theta_h = (\theta - R_h \theta) + (R_h \theta - \theta_h) = \xi_8 + \eta_8 \tag{5.29}$$

The following estimates are well known[9] for Raviart-Thomas projection operator Π_h : $\mathbf{W} \to \vec{W}_h$, if $\forall \vec{w}_h \in \vec{W}_h$, $\mathbf{q} \in \mathbf{W}$

$$\begin{aligned} (\nabla \cdot (\mathbf{q} - \Pi_h \mathbf{q}), \nabla \cdot \vec{w}_h) &= 0 \\ \|\mathbf{q} - \Pi_h \mathbf{q}\| &\leq C h^{m+1} \|\mathbf{q}\|_{m+1,\Omega} \\ \|\nabla \cdot (\mathbf{q} - \Pi_h \mathbf{q})\| &\leq C h^{m} \|\mathbf{q}\|_{m+1} \end{aligned} \tag{5.30}$$

And elliptical projection operator $R_h : V \to V_h$, for $\forall v_h \in V_h$, $p \in V$

$$\begin{aligned} (\nabla (p - R_h p), \nabla v_h) &= 0 \\ \|p - R_h p\| + h \|\nabla p - \nabla R_h p\| &\leq C h^{k+1} \|p\|_{k+1} \end{aligned} \tag{5.31}$$

Remark. V_h and $\vec{W}_h$ are the subspaces of V and $\mathbf{W}$, use V_h as P_k space, and $\vec{W}_h$ as RT_m

space. R_h is the finite element interpolation operator on V_h and Π_h is the interpolation operator on $\vec{W}_h$.

Then the equation system can be written as:

$$
\begin{aligned}
&(a)\ (\xi_{7,t},\chi_h)-(\xi_1,\chi_h)=-(\eta_{7,t},\chi_h)+(\eta_1,\chi_h)\\
&(b)\ (\xi_{3,t},\chi_h)-(\xi_2,\chi_h)=-(\eta_{3,t},\chi_h)+(\eta_2,\chi_h)\\
&(c)\ (\nabla\xi_7,\vec{\chi}_h)-(\vec{\xi}_4,\vec{\chi}_h)=-(\nabla\eta_7,\vec{\chi}_h)+(\vec{\eta}_4,\vec{\chi}_h)\\
&(d)\ (\nabla\xi_3,\vec{\chi}_h)-(\vec{\xi}_5,\vec{\chi}_h)=-(\nabla\eta_3,\vec{\chi}_h)+(\vec{\eta}_5,\vec{\chi}_h)\\
&(e)\ (\xi_3,\chi_h)+(\nabla\cdot\vec{\xi}_4,\chi_h)=-(\eta_3,\chi_h)+(\nabla\cdot\vec{\eta}_4,\chi_h)\\
&(f)\ (\nabla\xi_8,\overrightarrow{\chi_h})-(\vec{\xi}_6,\overrightarrow{\chi_h})=-(\nabla\eta_8,\overrightarrow{\chi_h})+(\vec{\eta}_6,\overrightarrow{\chi_h})\\
&(g)\ (\vec{\xi}_{4,tt},\vec{\chi}_h)+(\vec{\xi}_{5,tt},\vec{\chi}_h)+(\nabla\cdot\vec{\xi}_5,\nabla\cdot\vec{\chi}_h)-(\nabla\cdot\vec{\xi}_6,\nabla\cdot\vec{\chi}_h)=-(\vec{\eta}_{4,tt},\vec{\chi}_h)-(\vec{\eta}_{5,tt},\vec{\chi}_h)\\
&\qquad-(\nabla\cdot\vec{\eta}_5,\nabla\cdot\vec{\chi}_h)+(\nabla\cdot\vec{\eta}_6,\nabla\cdot\vec{\chi}_h)\\
&(h)\ (\vec{\xi}_{6,t},\vec{\chi}_h)+(\nabla\cdot\vec{\xi}_6,\nabla\cdot\vec{\chi}_h)+(\vec{\xi}_6,\vec{\chi}_h)+(\nabla\cdot\vec{\xi}_{4,t},\vec{\chi}_h)=-(\vec{\eta}_{6,t},\vec{\chi}_h)-(\nabla\cdot\vec{\eta}_6,\nabla\cdot\vec{\chi}_h)\\
&\qquad-(\vec{\eta}_6,\vec{\chi}_h)-(\nabla\cdot\vec{\eta}_{4,t},\nabla\cdot\vec{\chi}_h)
\end{aligned}
\tag{5.32}
$$

Theorem 5.4. *Given* $q_h(0)=q(0)$, $p_h(0)=p(0)$, $\vec{r}_h(0)=\vec{r}(0)$, $\vec{s}_h(0)=\vec{s}(0)$, $\vec{w}_h(0)=\vec{w}(0)$, $u_h(0)=u(0)$, $\sigma_h(0)=\sigma(0)$, $\theta_h(0)=\theta(0)$, *the following estimate holds:*

$$\|\vec{r}-\vec{r}_h\|+\|\vec{w}-\vec{w}_h\|+\|\sigma-\sigma_h\|+\|\vec{s}-\vec{s}_h\|\le Ch^{\min(m,k)}$$

$$\|p-p_h\|+\|q-q_h\|+\|u-u_h\|+\|\theta-\theta_h\|\le Ch^{\min(m+1,k+1)}$$

where C *depends on* $\|\vec{r}_{tt}\|_{L^\infty(H^{m+1})}$, $\|\vec{w}_t\|_{L^\infty(H^{m+1})}$, $\|\vec{w}\|_{L^\infty(H^{m+1})}$, $\|\vec{r}_t\|_{L^\infty(H^{m+1})}$, $\|\vec{s}_{tt}\|_{L^\infty(H^{m+1})}$,

$\|\vec{w}_0\|_{L^\infty(H^{m+1})}$, $\|\vec{r}_0\|_{L^\infty(H^{m+1})}$, $\|\sigma_0\|_{L^\infty(H^{k+1})}$, $\|\sigma_t\|_{L^\infty(H^{k+1})}$, $\|\vec{r}\|_{L^\infty(H^{m+1})}$, $\|u\|_{L^\infty(H^{k+1})}$.

Proof. Take Eq.(5.32g) and Eq.(5.32h) test function as $\vec{\xi}_{4,t}$ and $\vec{\xi}_6$

$$\frac{1}{2}\frac{d}{dt}\|\vec{\xi}_{4,t}\|^2 + \frac{1}{2}\frac{d}{dt}\|\vec{\xi}_6\|^2 + \|\nabla\cdot\vec{\xi}_6\|^2 + \|\vec{\eta}_6\|^2 + (\vec{\xi}_{5,tt}, \vec{\xi}_{4,t}) + (\nabla\cdot\vec{\xi}_5, \nabla\cdot\vec{\xi}_{4,t})$$
$$= -(\vec{\eta}_{4,tt}, \vec{\xi}_{4,t}) - (\vec{\eta}_{6,t}, \vec{\xi}_6) - (\nabla\cdot\vec{\eta}_6, \nabla\cdot\vec{\xi}_6) - (\vec{\eta}_6, \vec{\xi}_6) - (\nabla\cdot\vec{\eta}_{4,t}, \nabla\cdot\vec{\xi}_6) - (\vec{\eta}_{5,tt}, \vec{\xi}_{4,t}) - (\nabla\cdot\vec{\eta}_5, \nabla\cdot\vec{\xi}_{4,t}) \tag{5.33}$$

We have the identity

$$\vec{s} = \nabla\sigma = \nabla(-\nabla\cdot)\vec{r} = -\triangle\vec{r}$$

That means:

$$(\vec{\xi}_{5,tt} + \vec{\eta}_{5,tt}, \vec{\xi}_{4,t}) = (\triangle\vec{\xi}_{4,tt}, \xi_{4,t}) = \frac{1}{2}\frac{d}{dt}\|\nabla\vec{\xi}_{4,t}\|^2 \tag{5.34}$$

$$(\nabla\cdot(\vec{\xi}_5 + \vec{\eta}_5), \nabla\cdot(\vec{\xi}_{4,t} + \vec{\eta}_{4,t})) = (\triangle(\xi_3 + \eta_3), -(\xi_3 + \eta_3)_t) = (\nabla\xi_3, \nabla\xi_{3,t}) = \frac{1}{2}\frac{d}{dt}\|\vec{\xi}_3\|^2 \tag{5.35}$$

Then we need to estimate:

$$\frac{1}{2}\frac{d}{dt}\|\vec{\xi}_{4,t}\|^2 + \frac{1}{2}\frac{d}{dt}\|\vec{\xi}_6\|^2 + \|\nabla\cdot\vec{\xi}_6\|^2 + \|\vec{\xi}_6\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla\vec{\xi}_{4,t}\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla\vec{\xi}_3\|^2$$
$$= -(\vec{\eta}_{4,tt}, \vec{\xi}_{4,t}) - (\vec{\eta}_{6,t}, \vec{\xi}_6) - (\nabla\cdot\vec{\eta}_6, \nabla\cdot\vec{\xi}_6) - (\vec{\eta}_6, \vec{\xi}_6) - (\nabla\cdot\vec{\eta}_{4,t}, \nabla\cdot\vec{\xi}_6) + (\vec{\eta}_{5,tt}, \vec{\xi}_{4,t})$$
$$+ (\nabla\cdot\vec{\xi}_5, \nabla\cdot\vec{\eta}_{4,t}) \tag{5.36}$$

If taking integral from 0 to t:

$$\|\vec{\xi}_{4,t}\|^2 + \|\vec{\xi}_6\|^2 + \int_0^t \|\nabla\cdot\vec{\xi}_6\|^2 ds + \int_0^t \|\vec{\xi}_6\|^2 ds + \|\nabla\vec{\xi}_{4,t}\|^2 + \|\nabla\vec{\xi}_3\|^2$$
$$= \int_0^t (-(\vec{\eta}_{4,tt},\vec{\xi}_{4,t}) - (\vec{\eta}_{6,t},\vec{\xi}_6) - (\nabla\cdot\vec{\eta}_6,\nabla\cdot\vec{\xi}_6) - (\vec{\eta}_6,\vec{\xi}_6) - (\nabla\cdot\vec{\eta}_{4,t},\nabla\cdot\vec{\xi}_6) + (\vec{\eta}_{5,tt},\vec{\xi}_{4,t})$$
$$+ (\nabla\cdot\vec{\xi}_5,\nabla\cdot\vec{\eta}_{4,t}))ds + \|\vec{\xi}_{4,t}(0)\|^2 + \|\vec{\xi}_6(0)\|^2 + \|\nabla\vec{\xi}_{4,t}(0)\|^2 + \|\nabla\vec{\xi}_3(0)\|^2$$
$$\le I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + \|\vec{\xi}_{4,t}(0)\|^2 + \|\vec{\xi}_6(0)\|^2 + \|\nabla\vec{\xi}_{4,t}(0)\|^2 + \|\nabla\vec{\xi}_3(0)\|^2 \tag{5.37}$$

Working on the RHS:

$$\begin{aligned}
I_1 &\le C\int_0^t \|\vec{\eta}_{4,tt}\|^2 ds + \epsilon\int_0^t \|\vec{\xi}_{4,t}\|^2 ds \\
I_2 &\le C\int_0^t \|\vec{\eta}_{6,t}\|^2 ds + \epsilon\int_0^t \|\vec{\xi}_6\|^2 ds \\
I_3 &\le C\int_0^t \|\nabla\cdot\vec{\eta}_6\|^2 ds + \epsilon\int_0^t \|\nabla\cdot\vec{\xi}_6\|^2 ds \\
I_4 &\le C\int_0^t \|\vec{\eta}_6\|^2 ds + \epsilon\int_0^t \|\vec{\xi}_6\|^2 ds \\
I_5 &\le C\int_0^t \|\nabla\cdot\vec{\eta}_{4,t}\|^2 ds + \epsilon\int_0^t \|\nabla\cdot\vec{\xi}_6\|^2 ds \\
I_6 &\le C\int_0^t \|\vec{\eta}_{5,tt}\|^2 ds + \epsilon\int_0^t \|\vec{\xi}_{4,t}\|^2 ds \\
I_7 &\le C\int_0^t \|\vec{\eta}_{4,t}\|^2 ds + \epsilon\int_0^t \|\vec{\xi}_5\|^2 ds
\end{aligned} \tag{5.38}$$

Since for the initial values

$$\|R_h\sigma_0 - \sigma_0\| + h^k\|R_h\sigma_0 - \sigma_0\|_1 \le Ch^{k+1}\|\sigma_0\|_{k+1}$$
$$\|\Pi_h\vec{r}_0 - \vec{r}_0\| + h^r\|\Pi_h\vec{r}_0 - \vec{r}_0\|_{H(div)} \le Ch^{m+1}\|\vec{r}_0\|_{m+1}$$
$$\|\Pi_h\vec{w}_0 - \vec{w}_0\| + h^r\|\Pi_h\vec{w}_0 - \vec{w}_0\|_{H(div)} \le Ch^{m+1}\|\vec{w}_0\|_{m+1}$$

As:

$$\|\vec{\xi}_{4,t}\|^2 + \|\vec{\xi}_6\|^2 + \int_0^t \|\nabla\cdot\vec{\xi}_6\|^2 ds + \int_0^t \|\vec{\xi}_6\|^2 ds + \|\nabla\vec{\xi}_{4,t}\|^2 + \|\nabla\vec{\xi}_3\|^2$$
$$= \int_0^t (-(\vec{\eta}_{4,tt}, \vec{\xi}_{4,t}) - (\vec{\eta}_{6,t}, \vec{\xi}_6) - (\nabla\cdot\vec{\eta}_6, \nabla\cdot\vec{\xi}_6) - (\vec{\eta}_6, \vec{\xi}_6) - (\nabla\cdot\vec{\eta}_{4,t}, \nabla\cdot\vec{\xi}_6) + (\vec{\eta}_{5,tt}, \vec{\xi}_{4,t})$$
$$+ (\nabla\cdot\vec{\xi}_5, \nabla\cdot\vec{\eta}_{4,t}))ds + \|\vec{\xi}_{4,t}(0)\|^2 + \|\vec{\xi}_6(0)\|^2 + \|\nabla\vec{\xi}_{4,t}(0)\|^2 + \|\nabla\vec{\xi}_3(0)\|^2$$
$$\le C\int_0^t \|\vec{\eta}_{4,tt}\|^2 ds + \epsilon\int_0^t \|\vec{\xi}_{4,t}\|^2 ds + C\int_0^t \|\vec{\eta}_{6,t}\|^2 ds + \epsilon\int_0^t \|\vec{\xi}_6\|^2 ds + C\int_0^t \|\nabla\cdot\vec{\eta}_6\|^2 ds$$
$$+ \epsilon\int_0^t \|\nabla\cdot\vec{\xi}_6\|^2 ds + C\int_0^t \|\vec{\eta}_6\|^2 ds + \epsilon\int_0^t \|\vec{\xi}_6\|^2 ds + C\int_0^t \|\nabla\cdot\vec{\eta}_{4,t}\|^2 ds + \epsilon\int_0^t \|\nabla\cdot\vec{\xi}_6\|^2 ds$$
$$+ C\int_0^t \|\vec{\eta}_{5,tt}\|^2 ds + \epsilon\int_0^t \|\vec{\xi}_{4,t}\|^2 ds + C\int_0^t \|\vec{\eta}_{4,t}\|^2 ds + \epsilon\int_0^t \|\vec{\xi}_5\|^2 ds$$
$$+ \|\vec{\xi}_{4,t}(0)\|^2 + \|\vec{\xi}_6(0)\|^2 + \|\nabla\vec{\xi}_{4,t}(0)\|^2 + \|\nabla\vec{\xi}_3(0)\|^2 \tag{5.39}$$

In the RHS, integral including ξ_i can be controlled by the LHS. Then,

$$\|\vec{\xi}_{4,t}\|^2 + \|\vec{\xi}_6\|^2 + \int_0^t \|\nabla\cdot\vec{\xi}_6\|^2 ds + \int_0^t \|\vec{\xi}_6\|^2 ds + \|\nabla\vec{\xi}_{4,t}\|^2 + \|\nabla\vec{\xi}_3\|^2$$
$$\le C\int_0^t \|\vec{\eta}_{4,tt}\|^2 ds + C\int_0^t \|\vec{\eta}_{6,t}\|^2 ds + C\int_0^t \|\nabla\cdot\vec{\eta}_6\|^2 ds$$
$$+ C\int_0^t \|\vec{\eta}_6\|^2 ds + C\int_0^t \|\nabla\cdot\vec{\eta}_{4,t}\|^2 ds + C\int_0^t \|\vec{\eta}_{5,tt}\|^2 ds \tag{5.40}$$
$$+ C\int_0^t \|\vec{\eta}_{4,t}\|^2 ds + \|\vec{\xi}_{4,t}(0)\|^2 + \|\vec{\xi}_6(0)\|^2 + \|\nabla\vec{\xi}_{4,t}(0)\|^2 + \|\nabla\xi_3(0)\|^2$$

Try to estimate terms with η_i,

$$\begin{aligned}
\int_0^t \|\vec{\eta}_{4,tt}\|^2 ds &\le Ch^{2(m+1)} \int_0^t \|\vec{r}_{tt}\|^2_{m+1} ds \\
\int_0^t \|\vec{\eta}_{6,t}\|^2 ds &\le Ch^{2(m+1)} \int_0^t \|\vec{w}_t\|^2_{m+1} \\
\int_0^t \|\nabla \cdot \vec{\eta}_6\|^2 ds &\le Ch^{2m} \int_0^t \|\vec{w}_t\|^2_{m+1} \\
\int_0^t \|\vec{\eta}_6\|^2 ds &\le Ch^{2(m+1)} \int_0^t \|\vec{w}\|^2_{m+1} \\
\int_0^t \|\vec{\eta}_{4,t}\|^2 ds &\le Ch^{2(m+1)} \int_0^t \|\vec{r}_t\|^2_{m+1} \\
\int_0^t \|\vec{\eta}_{5,tt}\|^2 ds &\le Ch^{2(m+1)} \int_0^t \|\vec{s}_{tt}\|^2_{m+1}
\end{aligned} \tag{5.41}$$

Substitute those inequalities,

$$\begin{aligned}
\|\vec{\xi}_{4,t}\|^2 + \|\vec{\xi}_6\|^2 + \|\nabla\vec{\xi}_{4,t}\|^2 + \|\nabla\vec{\xi}_3\|^2 \le Ch^{\min 2(m,k)} \int_0^t (&\|\vec{r}_{tt}\|^2_{m+1} + \|\vec{w}_t\|^2_{m+1} + \|\vec{w}\|^2_{m+1} \\
&+ \|\vec{r}_t\|^2_{m+1} + \|\vec{s}_{tt}\|^2_{m+1} + \|\vec{w}_0\|^2_{m+1} + \|\vec{r}_0\|^2_{m+1} + \|\sigma_0\|^2_{k+1}) ds
\end{aligned} \tag{5.42}$$

From Eq.(5.32b), choose $\chi_h = \xi_2$

$$\begin{aligned}
\|\xi_2\|^2 &= (\eta_{3,t}, \xi_2) + (\xi_{3,t}, \xi_2) - (\eta_2, \xi_2) \\
&\le C\|\eta_{3,t}\|^2 + C\|\xi_{3,t}\|^2 + \epsilon\|\xi_2\|^2 + C\|\eta_2\|^2
\end{aligned} \tag{5.43}$$

That leads to

$$\begin{aligned}
(1-\epsilon)\|\xi_2\|^2 &\le C\|\eta_{3,t}\|^2 + C\|\xi_{3,t}\|^2 + C\|\eta_2\|^2 \\
&\le Ch^{2(k+1)}\|\sigma_t\|^2_{k+1} + Ch^{2\min(m+1,k+1)} + Ch^{2(k+1)}\|p\|^2 \\
&\le Ch^{2\min(m+1,k+1)}
\end{aligned} \tag{5.44}$$

From Eq.(5.32c), choose $\chi_h = \xi_7$

$$\begin{aligned} \|\nabla \xi_7\|^2 &= (\vec{\xi_4} + \vec{\eta_4}, \xi_7) \\ &\leq C\|\vec{\xi_4}\|^2 + C\|\vec{\eta_4}\|^2 + \epsilon\|\xi_7\|^2 \end{aligned} \tag{5.45}$$

That leads to

$$\begin{aligned} (1-\epsilon)\|\nabla \xi_7\|^2 &\leq C\|\vec{\xi_4}\|^2 + C\|\vec{\eta_4}\|^2 \\ &\leq Ch^{2\min(m,k)} + Ch^{2(m+1)}\|\vec{r}\|_{m+1} \\ &\leq Ch^{2\min(m,k)} \end{aligned} \tag{5.46}$$

Similarily for Eq.(5.32f), choose $\chi_h = \xi_8$

$$\begin{aligned} (1-\epsilon)\|\nabla \xi_8\|^2 &\leq C\|\vec{\xi_6}\|^2 + C\|\vec{\eta_6}\|^2 \\ &\leq Ch^{2\min(m,k)} + Ch^{2(m+1)}\|\vec{w}\|_{m+1} \\ &\leq Ch^{2\min(m,k)} \end{aligned} \tag{5.47}$$

In Eq.(5.32a), choose $\chi_h = \xi_1$

$$\begin{aligned} (1-\epsilon)\|\xi_1\|^2 &\leq C\|\vec{\xi}_{7,t}\|^2 + C\|\vec{\eta}_{7,t}\|^2 \\ &\leq Ch^{2\min(m+1,k+1)} \end{aligned} \tag{5.48}$$

In Eq.(5.32d), choose $\chi_h = \nabla \xi_5$

$$(1-\epsilon)\|\vec{\xi_5}\|^2 \leq Ch^{2\min(m,k)} \tag{5.49}$$

which proves the theorem. □

5.2 Fully discrete error

5.2.1 The Existence and Uniqueness Fully Discrete Mixed Element Formulation

For fully discretization, we use the backward Euler method of first order accurate in time. For the backward Euler method, Let M be a positive integer, then $\Delta t = T/M$ be the step size of time, $t^i = i\Delta t, 0 \leq i \leq M$. Further, let $\psi^n = \psi(t^n)$ and $\partial_t \psi^n = (\psi^n - \psi^{n-1})/\Delta t$, for some continuous function $\psi \in C^0[0,T]$. Find Find $(Q^n, P^n, \Sigma^n, \mathbf{\Upsilon}^n, \mathbf{S}^n, \mathbf{\Lambda}^n, U^n, \Theta^n) \in V_h^n \times V_h^n \times V_h^n \times \mathbf{W}_h^n \times \mathbf{W}_h^n \times \mathbf{W}_h^n \times V_h^n \times V_h^n$ be the mixed element approximation of $(q(t^n), p(t^n), \sigma(t^n), \vec{r}(t^n), \vec{r}(s^n), \vec{w}(t^n), u(t^n), \theta(t^n))$. For each n, the different time interval is (t^n, t^{n+1}), the corresponding triangulation is $\mathcal{T}_h^n$, finite element spaces are V_h^n and $\mathbf{W}_h^n$. Then the fully discrete mixed finite element solution for problem (I^{**}) may be presented as follows:

Problem(I_h^n). Find Find $(Q^n, P^n, \Sigma^n, \mathbf{\Upsilon}^n, \mathbf{S}^n, \mathbf{\Lambda}^n, U^n, \Theta^n)$ such that, for $1 \leq n \leq M$, If denoting:

$$\overline{\partial} v^n = (v^{n+1} - v^n)/\tau$$

$$\begin{cases} (\overline{\partial} U^n, \chi_h^n) = (Q_h^n, \chi_h^n) \\ (P_h^n, \chi_h^n) = (\overline{\partial}\Sigma_h^n, \chi_h^n) \\ (\nabla U_h^n, \vec{\chi}_h) = (\boldsymbol{\Upsilon}_h^n, \vec{\chi}_h^n) \\ (\nabla \Sigma_h^n, \vec{\chi}_h^n) = (\mathbf{S}_h^n, \vec{\chi}_h^n) \\ (\Sigma_h^n, \chi_h) + (\nabla \cdot \boldsymbol{\Upsilon}_h^n, \vec{\chi}_h^n) = 0 \\ (\nabla \Theta_h, \vec{\chi}_h^n) = (\boldsymbol{\Lambda}_h^n, \vec{\chi}_h^n) \\ (\overline{\partial} Q_h^n, \nabla \cdot \vec{\chi}_h^n) + (\overline{\partial} P_h^n, \nabla \cdot \vec{\chi}_h^n) - (\nabla \cdot \mathbf{S}_h^n, \nabla \cdot \vec{\chi}_h) + (\nabla \cdot \boldsymbol{\Lambda}_h^n, \nabla \cdot \vec{\chi}_h^n) = F \\ (\overline{\partial}\Theta_h^n, \nabla \cdot \vec{\chi}_h^n) - (\nabla \cdot \boldsymbol{\Lambda}_h^n, \nabla \cdot \vec{\chi}_h^n) + (\Theta_h^n, \nabla \cdot \vec{\chi}_h^n) + (P_h, \nabla \cdot \vec{\chi}_h^n) = G \end{cases} \tag{5.50}$$

Theorem 5.5. *Problem I_h^n has a unique solution* $(Q_h^n, P_h^n, \Sigma_h^n, \boldsymbol{\Upsilon}_h^n, \mathbf{S}_h^n, \boldsymbol{\Lambda}_h^n, U_h^n, \Theta_h^n) \in V_h^n \times V_h^n \times V_h^n \times \mathbf{W}_h^n \times \mathbf{W}_h^n \times \mathbf{W}_h^n \times V_h^n \times V_h^n$.

Proof. The existence of the solution follows from the existence and regularity assumption. By defining $q_h = u_{ht}, p_h = \sigma_{ht}, \vec{r}_h = \nabla u_h, \vec{s}_h = \nabla \sigma_h, \vec{w}_h = \nabla \theta_h$, it immediates comes up with a weak solution for I_h.

To prove the uniqueness of the solution, we need to prove the stability first.

let $(p_{h0}^i, q_{h0}^i, \sigma_{h0}^i, \vec{r}_{h0}^i, \vec{s}_{h0}^i, \vec{w}_{h0}^i, u_{h0}^i, \theta_{h0}^i)$ be the initial data, and $(p_h^i, q_h^i, \sigma_h^i, \vec{r}_h^i, \vec{r}_h^i, \vec{w}_h^i, u_h^i, \theta_h^i)$ be the corresponding weak solutions.

Denote $\tilde{Q}^n = Q^{1,n} - Q^{2,n}$, $\tilde{P}^n = P^{1,n} - P^{2,n}$, $\tilde{\Sigma}_h = \Sigma^{1,n} - \Sigma^{2,n}$, $\tilde{U}_h = U^{1,n} - U^{2,n}$, $\tilde{\Theta}_h = \Theta^{1,n} - \Theta^{2,n}$, $\tilde{\boldsymbol{\Upsilon}}^n = \boldsymbol{\Upsilon}^{1,n} - \boldsymbol{\Upsilon}^{2,n}$, $\tilde{\mathbf{S}}^n = \mathbf{S}^{1,n} - \mathbf{S}^{2,n}$, $\tilde{\boldsymbol{\Lambda}}^n = \boldsymbol{\Lambda}^{1,n} - \boldsymbol{\Lambda}^{2,n}$.

$$\begin{cases}(\tilde{Q}^n,\chi_h)=(\dfrac{\tilde{U}^n-\tilde{U}^{n-1}}{\tau},\chi)\\(\tilde{P}^n,\chi_h)=(\dfrac{\tilde{\Sigma}^n-\tilde{\Sigma}^{n-1}}{\tau},\chi_h)\\(\nabla\tilde{U}^n,\vec{\chi}_h)=(\tilde{\mathbf{\Upsilon}}^n,\vec{\chi}_h)\\(\nabla\tilde{\Sigma}^n,\vec{\chi}_h)=(\tilde{\mathbf{S}}^n,\vec{\chi}_h)\\(\tilde{\Sigma}^n,\chi_h)+(\nabla\cdot\tilde{\mathbf{\Upsilon}}^n,\chi_h)=0\\(\nabla\tilde{\Theta}^n,\vec{\chi}_h)=(\tilde{\mathbf{\Lambda}}^n,\vec{\chi}_h)\\(\dfrac{\tilde{\mathbf{\Upsilon}}^{n+1}-2\tilde{\mathbf{\Upsilon}}^n+\tilde{\mathbf{\Upsilon}}^{n-1}}{\tau^2},\vec{\chi}_h)+(\dfrac{\tilde{\mathbf{S}}^{n+1}-2\tilde{\mathbf{S}}^n+\tilde{\mathbf{S}}^{n-1}}{\tau^2},\vec{\chi}_h)+(\nabla\cdot\tilde{\mathbf{S}}_h,\nabla\cdot\vec{\chi}_h)-(\nabla\cdot\tilde{\mathbf{\Lambda}}_h,\nabla\cdot\vec{\chi}_h)=0\\(\dfrac{\tilde{\mathbf{\Lambda}}^n-\tilde{\mathbf{\Lambda}}^{n-1}}{\tau},\vec{\chi}_h)+(\nabla\cdot\tilde{\mathbf{\Lambda}}^n,\nabla\cdot\vec{\chi}_h)+(\tilde{\mathbf{\Lambda}}^n,\vec{\chi}_h)+(\dfrac{\tilde{\mathbf{S}}^n-\tilde{\mathbf{S}}^{n-1}}{\tau},\vec{\chi}_h)=0\end{cases}\tag{5.51}$$

Choose last two test functions, $\vec{\chi}=\dfrac{\tilde{\mathbf{\Upsilon}}^{n+1}-\tilde{\mathbf{\Upsilon}}^n}{\tau}$ and $\vec{\chi}=\tilde{\mathbf{\Lambda}}^n$, and adding up tbose two equations,

$$\frac{\|\tilde{\Sigma}^{n+1}\|^2-\|\tilde{\Sigma}^n\|^2-\|\tilde{\Sigma}^1\|^2+\|\tilde{\Sigma}^0\|^2}{\tau^2}+\frac{\|\tilde{\mathbf{S}}^{n+1}\|^2-\|\tilde{\mathbf{S}}^n\|^2-\|\tilde{\mathbf{S}}^1\|^2+\|\tilde{\mathbf{S}}^0\|^2}{\tau^2}+\frac{\|\tilde{\mathbf{\Lambda}}^n\|^2-\|\tilde{\mathbf{\Lambda}}^0\|^2}{\tau}$$
$$+\sum_{i=0}^{M}(\|\nabla\cdot\tilde{\mathbf{S}}^i\|^2+\|\nabla\cdot\tilde{\mathbf{\Lambda}}^i\|^2+\|\nabla\tilde{\mathbf{S}}^i\|^2)=0\tag{5.52}$$

Using discrete Gronwall inequality, thus we have the stability. This completes the proof of the theorem. □

5.2.2 Fully Discrete Error Estimates

For the error estimate, we write,

$$
\begin{aligned}
q^n - Q^n &= (q^n - R_h q^n) + (R_h q^n - Q^n) = \eta_1 + \xi_1 \\
p^n - p^n &= (p^n - R_h p^n) + (R_h p^n - P^n) = \eta_2 + \xi_2 \\
\sigma^n - \Sigma^n &= (\sigma - R_h \sigma^n) + (R_h \sigma^n - \Sigma^n) = \eta_3 + \xi_3 \\
\vec{r}^n - \mathbf{\Upsilon}^n &= (\vec{r}^n - \Pi_h \vec{r}^n) + (\Pi_h \vec{r}^n - \mathbf{\Upsilon}^n) = \vec{\eta}_4 + \vec{\xi}_4 \\
\vec{s}^n - \mathbf{S}^n &= (\vec{s}^n - \Pi_h \vec{s}^n) + (\Pi_h \vec{s}^n - \mathbf{S}^n) = \vec{\eta}_5 + \vec{\xi}_5 \\
\vec{w}^n - \mathbf{\Lambda}^n &= (\vec{w}^n - \Pi_h \vec{w}^n) + (\Pi_h \vec{w}^n - \mathbf{\Lambda}^n) = \vec{\eta}_6 + \vec{\xi}_6 \\
u^n - U^n &= (u^n - R_h u^n) + (R_h u^n - U^n) = \eta_7 + \xi_7 \\
\theta^n - \Theta^n &= (\theta^n - R_h \theta^n) + (R_h \theta^n - \Theta^n) = \eta_8 + \xi_8
\end{aligned}
\tag{5.53}
$$

Theorem 5.6. *If* $(Q_h^0, P_h^0, \Sigma_h^0, \mathbf{\Upsilon}_h^0, \mathbf{S}_h^0, \mathbf{\Lambda}_h^0, U_h^0, \Theta_h^0)$ *determined, there exist,*

$$
\|\vec{r}^j - \mathbf{\Upsilon}^j\| + \|\vec{s}^j - \mathbf{S}^j\| + \|\vec{w}^j - \mathbf{\Lambda}^j\| + \|\sigma^j - \Sigma^j\| \le C(h^{\min(m,k)} + \tau)
$$
$$
\|q^j - Q^j\| + \|p^j - P^j\| + \|u^j - U^j\| + \|\theta^j - \Theta^j\| \le C(h^{\min(r+1,k+1)} + \tau)
$$

where C *depends on* $\|\vec{r}_{tt}\|_{L^\infty(H^{m+1})}$, $\|\vec{w}_t\|_{L^\infty(H^{m+1})}$, $\|\vec{w}\|_{L^\infty(H^{m+1})}$, $\|\vec{r}_t\|_{L^\infty(H^{m+1})}$, $\|\vec{s}_{tt}\|_{L^\infty(H^{m+1})}$, $\|\vec{w}_0\|_{L^\infty(H^{m+1})}$, $\|\vec{r}_0\|_{L^\infty(H^{m+1})}$, $\|\sigma_0\|_{L^\infty(H^{k+1})}$, $\|\sigma_t\|_{L^\infty(H^{k+1})}$, $\|\vec{r}\|_{L^\infty(H^{m+1})}$, $\|u\|_{L^\infty(H^{k+1})}$.

Proof.

$$(a)\ (\overline{\partial}\xi_7^n,\chi_h)-(\xi_1^n,\chi_h)=-(\overline{\partial}\eta_7^n,\chi_h)+(\eta_1^n,\chi_h)$$

$$(b)\ (\overline{\partial}\xi_3^n,\chi_h)-(\xi_2^n,\chi_h)=-\overline{\partial}(\eta_3^n,\chi_h)+(\eta_2^n,\chi_h)$$

$$(c)\ (\nabla\xi_7^n,\vec{\chi}_h)-(\vec{\xi}_4^n,\vec{\chi}_h)=-(\nabla\eta_7^n,\vec{\chi}_h)+(\vec{\eta}_4^n,\vec{\chi}_h)$$

$$(d)\ (\nabla\xi_3^n,\vec{\chi}_h)-(\vec{\xi}_5^n,\vec{\chi}_h)=-(\nabla\eta_3^n,\vec{\chi}_h)+(\vec{\eta}_5^n,\vec{\chi}_h)$$

$$(e)\ (\xi_3^n,\chi_h)+(\nabla\cdot\vec{\xi}_4^n,\chi_h)=-(\eta_3^n,\chi_h)+(\nabla\cdot\vec{\eta}_4^n,\chi_h)$$

$$(f)\ (\nabla\xi_8^n,\vec{\chi}_h)-(\vec{\xi}_6^n,\vec{\chi}_h)=-(\nabla\eta_8^n,\vec{\chi}_h)+(\vec{\eta}_6,\vec{\chi}_h)$$

$$\begin{aligned}(g)\ &(\vec{\xi}_{4,tt}^n,\vec{\chi}_h)+(\vec{\xi}_{5,tt}^n,\vec{\chi}_h)+(\nabla\cdot\vec{\xi}_5^n,\nabla\cdot\vec{\chi}_h)-(\nabla\cdot\vec{\xi}_6^n,\nabla\cdot\vec{\chi}_h)=-(\vec{\eta}_{4,tt}^n,\vec{\chi}_h)-(\vec{\eta}_{5,tt}^n,\vec{\chi}_h)\\&-(\nabla\cdot\vec{\eta}_5^n,\nabla\cdot\vec{\chi}_h)+(\nabla\cdot\vec{\eta}_6^n,\nabla\cdot\vec{\chi}_h)\\(h)\ &(\vec{\xi}_{6,t}^n,\vec{\chi}_h)+(\nabla\cdot\vec{\xi}_6^n,\nabla\cdot\vec{\chi}_h)+(\vec{\xi}_6^n,\vec{\chi}_h)+(\nabla\cdot\vec{\xi}_{4,t}^n,\vec{\chi}_h)=-(\vec{\eta}_{6,t}^n,\vec{\chi}_h)-(\nabla\cdot\vec{\eta}_6^n,\nabla\cdot\vec{\chi}_h)\\&-(\vec{\eta}_6^n,\vec{\chi}_h)-(\nabla\cdot\vec{\eta}_{4,t}^n,\nabla\cdot\vec{\chi}_h)\end{aligned}\tag{5.54}$$

working on the last two equations:

$$\begin{aligned}&(\overline{\partial}^2\vec{\xi}_4^n,\overline{\partial}\vec{\xi}_4)+(\overline{\partial}^2\vec{\xi}_5^n,\overline{\partial}^2\vec{\xi}_4^n)+(\nabla\cdot\vec{\xi}_5^n,\nabla\cdot\overline{\partial}^2\vec{\xi}_4^n)+(\overline{\partial}\vec{\xi}_6^n,\overline{\partial}\vec{\xi}_6^n)+(\nabla\cdot\vec{\xi}_6^n,\nabla\cdot\vec{\xi}_6^n)+(\vec{\xi}_6^n,\vec{\xi}_6^n)\\&=-(\overline{\partial}^2\vec{\eta}_4^n,\overline{\partial}\vec{\xi}_4^n)-(\overline{\partial}^2\vec{\eta}_5^n,\overline{\partial}^2\vec{\xi}_4^n)-(\nabla\cdot\vec{\eta}_5^n,\nabla\cdot\overline{\partial}^2\vec{\xi}_4^n)-(\overline{\partial}\vec{\eta}_6^n,\overline{\partial}\vec{\xi}_6^n)-(\nabla\cdot\vec{\eta}_6^n,\nabla\cdot\vec{\xi}_6^n)\\&-(\vec{\eta}_6^n,\vec{\xi}_6^n)\end{aligned}\tag{5.55}$$

As $\mathbf{S}=\nabla\sigma=\nabla(-\nabla\cdot)\vec{r}=-\triangle\vec{r}$ and $\nabla\cdot\mathbf{S}=\triangle\sigma$ Thus,

$$(\overline{\partial}^2\vec{\xi}_5^n+\overline{\partial}^2\vec{\eta}_5^n,\overline{\partial}\vec{\xi}_4^n)=-(\triangle(\overline{\partial}^2\vec{\xi}_4^n+\overline{\partial}^2\vec{\eta}_4^n),\overline{\partial}\vec{\xi}_4^n)$$

$$(\nabla\cdot(\vec{\xi}_5^n+\vec{\eta}_5^n),\overline{\partial}\vec{\xi}_4^n)=(\triangle(\xi_3^n+\eta_3^n),\overline{\partial}\vec{\xi}_4^n)$$

Then we have:

$$(\overline{\partial}^2\vec{\xi}^n_4,\overline{\partial}\vec{\xi}^n_4)+(\nabla\overline{\partial}^2\vec{\xi}^n_4,\nabla\overline{\partial}\vec{\xi}^n_4)+(\nabla\xi^n_3,\nabla\cdot\overline{\partial}\vec{\xi}^n_4)+(\overline{\partial}\vec{\xi}^n_6,\overline{\partial}\vec{\xi}^n_6)+(\nabla\cdot\vec{\xi}^n_6,\nabla\cdot\vec{\xi}^n_6)+(\vec{\xi}^n_6,\vec{\xi}^n_6)$$
$$=-(\overline{\partial}^2\vec{\eta}^n_4,\overline{\partial}\vec{\xi}^n_4)-(\nabla\overline{\partial}^2\vec{\eta}^n_4,\nabla\overline{\partial}\vec{\xi}^n_4)-(\overline{\partial}\vec{\eta}^n_6,\overline{\partial}\vec{\xi}^n_6)-(\nabla\cdot\vec{\eta}^n_6,\nabla\cdot\vec{\xi}^n_6)-(\vec{\eta}^n_6,\vec{\xi}^n_6)$$

Besides,

$$\begin{aligned}(R_hq^n_t-Q^n_t)&=(R_hq^n_t-\overline{\partial}q^n)+(\overline{\partial}q^n-Q^n_t)\\(R_h\sigma^n_t-\Sigma^n_t)&=(R_h\sigma^n_t-\overline{\partial}\sigma^n)+(\overline{\partial}\sigma^n-\Sigma^n_t)\\(R_hp^n_t-P^n_t)&=(R_hp^n_t-\overline{\partial}p^n)+(\overline{\partial}p^n-P^n_t)\\(R_h\theta^n_t-\Theta^n_t)&=(R_h\theta^n_t-\overline{\partial}\theta^n)+(\overline{\partial}\theta^n-\Theta^n_t)\end{aligned}\tag{5.56}$$

Noted:

$$(\overline{\partial}^2\vec{\xi}^n_4,\overline{\partial}\vec{\xi}_4)^n\geq\frac{1}{2}\overline{\partial}\|\overline{\partial}\vec{\xi}^n_4\|^2,(\nabla\overline{\partial}^2\vec{\xi}^n_4,\nabla\overline{\partial}\vec{\xi}^n_4)\geq\frac{1}{2}\overline{\partial}\|\nabla\overline{\partial}\vec{\xi}^n_4\|^2$$

Then it comes to,

$$\begin{aligned}&\frac{1}{2}\overline{\partial}\|\overline{\partial}\vec{\xi}^n_4\|^2+\frac{1}{2}\overline{\partial}\|\nabla\overline{\partial}\vec{\xi}^n_4\|^2+\|\overline{\partial}\vec{\xi}^n_6\|^2+\|\nabla\cdot\vec{\xi}^n_6\|^2+\|\vec{\xi}^n_6\|^2\\&\leq\|(\overline{\partial}^2\vec{\eta}^n_4,\overline{\partial}\vec{\xi}^n_4)\|+\|(\nabla\overline{\partial}^2\vec{\eta}^n_4,\nabla\overline{\partial}\vec{\xi}^n_4)\|+\|(\overline{\partial}\vec{\eta}^n_6,\overline{\partial}\vec{\xi}^n_6)\|+\|(\nabla\cdot\vec{\eta}^n_6,\nabla\cdot\vec{\xi}^n_6)\|+\|(\vec{\eta}^n_6,\vec{\xi}^n_6)\|\\&+\|(\nabla\xi^n_3,\nabla\cdot\overline{\partial}\vec{\xi}_4\|\\&=I_1+I_2+I_3+I_4+I_5+I_6\end{aligned}\tag{5.57}$$

Consider the RHS.

$$
\begin{aligned}
I_1 &\leq C\|\overline{\partial}^2\vec{\eta}_4^n\|^2 + C\|\overline{\partial}\vec{\xi}_4^n\|^2 \\
I_2 &\leq C\|\nabla\overline{\partial}^2\vec{\eta}_4^n\|^2 + C\|\nabla\overline{\partial}\vec{\xi}_4^n\|^2 \\
I_3 &\leq C\|\overline{\partial}\vec{\eta}_6^n\|^2 + C\|\overline{\partial}\vec{\xi}_6^n\|^2 \\
I_4 &\leq C\|\nabla\cdot\vec{\eta}_6^n\|^2 + C\|\nabla\cdot\vec{\xi}_6^n\|^2 \\
I_5 &\leq C\|\vec{\eta}_6^n\|^2 + C\|\vec{\xi}_6^n\|^2 \\
I_6 &\leq C\|\nabla\xi_3^n\|^2 + C\|\nabla\cdot\overline{\partial}\vec{\xi}_4^n\|^2
\end{aligned}
\tag{5.58}
$$

And,

$$
\|\overline{\partial}\vec{\eta}_6^n\|^2 \leq \frac{Ch^{2(m+1)}}{\tau}\int_{t_{n-1}}^{t_n}\|\vec{w}_t\|_{m+1}^2 ds, \quad \|\vec{\eta}_6\|^2 \leq Ch^{2(m+1)}\|\vec{w}\|_{m+1}^2 ds
$$

$$
\|\nabla\overline{\partial}^2\vec{\eta}_4^n\|^2 \leq \frac{Ch^{2m}}{\tau}\int_{t_{n-1}}^{t_n}\|\nabla\vec{w}_{tt}\|_{m+1}^2 ds + C\tau\int_{t_{n-1}}^{t_n}\|\nabla\vec{w}_{tt}\|^2 ds
$$

$$
\|\overline{\partial}^2\vec{\eta}_4^n\|^2 \leq \frac{Ch^{2(m+1)}}{\tau}\int_{t_{n-1}}^{t_n}\|\vec{r}_{tt}\|_{m+1}^2 ds + C\tau\int_{t_{n-1}}^{t_n}\|\nabla\vec{r}_{tt}\|^2 ds
$$

The estimate inequality becomes,

$$
\begin{aligned}
&\|\overline{\partial}\vec{\xi}_4^n\|^2 + \|\nabla\overline{\partial}\vec{\xi}_4^n\|^2 + \tau\sum_{j=0}^{n}(\|\overline{\partial}\vec{\xi}_6^j\|^2 + \|\nabla\cdot\vec{\xi}_6^j\|^2 + \|\vec{\xi}_6^j\|^2) \\
&\quad \leq Ch^{2(m+1)}\int_0^T\|\vec{r}_{tt}\|_{m+1}^2 ds + C\tau^2\int_0^T\|\nabla\vec{r}_{tt}\|^2 ds + Ch^{2m}\int_0^T\|\nabla\vec{w}_{tt}\|_{r+1}^2 ds \\
&\quad + C\tau^2\int_0^T\|\nabla\vec{w}_{tt}\|^2 ds + Ch^{2(m+1)}\int_0^T\|\vec{w}_t\|_{m+1}^2 ds + \tau\sum_{j=0}^{n}(\|\nabla\cdot\vec{\eta}_6^j\|^2 + \|\vec{\eta}_6^j\|^2) \\
&\quad + \sum_{j=0}^{n}(C\|\nabla\xi_3^j\|^2 + C\|\nabla\cdot\overline{\partial}\vec{\xi}_4^j\|^2)
\end{aligned}
$$

From Eq.(5.54b), choose $\chi_h = \xi_2^n$

$$\begin{aligned}\|\xi_2^n\|^2 &= (\overline{\partial}\eta_3^n, \xi_2^n) + (\overline{\partial}\sigma^n - R_h\sigma_t^n, \xi_2^n) + (\overline{\partial}\xi_3^n, \xi_2^n) - (\eta_2^n, \xi_2^n) \\ &\leq C\|\overline{\partial}\eta_3^n\|^2 + C\|\overline{\partial}\xi_3^n\|^2 + \epsilon\|\xi_2^n\|^2 + C\|\eta_2^n\|^2 + \|\overline{\partial}\sigma^n - R_h\sigma_t^n\|^2\end{aligned} \tag{5.59}$$

That leads to

$$\begin{aligned}(1-\epsilon)\|\xi_2^n\|^2 + \sum_{j=0}^{n-1}\|\xi_2^j\|^2 &\leq C\sum_{j=0}^{n-1}(\|\overline{\partial}\eta_3^n\|^2 + C\|\overline{\partial}\xi_3^n\|^2 + C\|\eta_2^n\|^2 + \|\overline{\partial}\sigma^n - R_h\sigma_t^n\|^2) \\ &\leq Ch^{2(k+1)}\|\sigma_t\|_{k+1}^2 + Ch^{2\min(m+1,k+1)} + Ch^{2(k+1)}\|p\|_{k+1}^2 + C\tau^2\|\sigma_{tt}\|^2 \\ &\leq Ch^{2\min(m+1,k+1)} + C\tau^2\end{aligned} \tag{5.60}$$

From Eq.(5.54c), choose $\chi_h = \xi_7^n$

$$\begin{aligned}\|\nabla\xi_7^n\|^2 &= (\vec{\xi}_4^n + \vec{\eta}_4^n, \nabla\xi_7^n) \\ &\leq C\|\vec{\xi}_4^n\|^2 + C\|\vec{\eta}_4^n\|^2 + \epsilon\|\nabla\xi_7^n\|^2\end{aligned} \tag{5.61}$$

That leads to

$$\begin{aligned}(1-\epsilon)\|\nabla\xi_7^n\|^2 + \sum_{j=0}^{n-1}\|\nabla\xi_7^j\|^2 &\leq C\sum_{j=0}^{n}(\|\vec{\xi}_4^j\|^2 + C\|\vec{\eta}_4^j\|^2) \\ &\leq Ch^{2\min(m,k)} + Ch^{2(m+1)}\|\vec{r}\|_{m+1} + C\tau^2 \\ &\leq Ch^{2\min(m,k)} + C\tau^2\end{aligned} \tag{5.62}$$

Similarily for Eq.(5.54f), choose $\chi_h = \xi_8^n$

$$
\begin{aligned}
(1-\epsilon)\|\nabla \xi_8^n\|^2 + \sum_{j=0}^{n-1} \|\nabla \xi_8^j\|^2 &\le C \sum_{j=0}^{n-1} (\|\vec{\xi}_6^j\|^2 + C\|\vec{\eta}_6^j\|^2) \\
&\le Ch^{2\min(m,k)} + Ch^{2(m+1)}\|\vec{w}\|_{m+1} + C\tau^2 \\
&\le Ch^{2\min(m,k)} + C\tau^2
\end{aligned}
\tag{5.63}
$$

In Eq.(5.54a), choose $\chi_h = \xi_1^n$

$$
\begin{aligned}
(1-\epsilon)\|\xi_1^n\|^2 + \sum_{j=0}^{n-1} \|\xi_1^j\|^2 \le C \sum_{j=0}^{n-1} (\|\overline{\partial}\vec{\xi}_7^j\|^2 + C\|\overline{\partial}\vec{\eta}_7^j\|^2 + C\|\overline{\partial}u^j - R_h u_t^j\|^2) \\
\le Ch^{2\min(m+1,k+1)} + C\tau^2\|u_{tt}\|^2
\end{aligned}
\tag{5.64}
$$

In Eq.(5.54d), choose $\chi_h = \nabla \xi_5^n$

$$
(1-\epsilon)\|\xi_5^n\|^2 + \sum_{j=0}^{n-1} \|\xi_5^j\|^2 \le Ch^{2\min(m,k)} + C\tau^2 \tag{5.65}
$$

That proves the theorem. $\square$

5.3 Numerical Examples

Consider

$$
\begin{cases}
u_{tt} - \triangle u_{tt} + \triangle^2 u + \alpha \triangle \theta = f \\
\theta_t - \triangle \theta + \theta - \alpha \triangle u_t = g \\
u|_{\partial\Omega} = \triangle u|_{\partial\Omega} = \theta|_{\partial\Omega} = 0 \\
u(\mathbf{x},0) = u^0(\mathbf{x}), u_t(\mathbf{x},0) = u^1(\mathbf{x}), \theta(\mathbf{x},0) = \theta^0(\mathbf{x})
\end{cases}
$$

And $\alpha = 1$, $\Omega = [0,1] \times [0,1]$, $T = 1$, with exact solution

$$\begin{cases} u = sin(\pi x)sin(\pi y)e^{-t} \\ \theta = sin(2\pi x)sin(2\pi y)e^{-t} \end{cases}$$

$\mathcal{T}_h$ is a regular pattern triangular mesh, and $P_1(K)\times P_1(K)\times P_1(K)\times RT_0(K)\times RT_0(K)\times RT_0(K)\times P_1(K)\times P_1(K)\times P_1(K)$ is used to solve the problem. The convergence curves of L^2 error of the solution are depicted in the Fig. From the plots, we can clearly observe the convergence rate for those variables are $O(h^2)$ for $\|\sigma^n - \Sigma_h^n\|$, $\|\vec{r}^n - \Upsilon_h^n\|$, $\|\vec{s}^n - \vec{S}_h^n\|$, $\|\vec{w}^n - \vec{\Lambda}_h^n\|$ which has a higher convergence rate than predicted, and $\|q^n - Q_h^n\|$, $\|p^n - P_h^n\|$, $\|u^n - U_h^n\|$, $\|\theta^n - \Theta_h^n\|$ has the optimal convergence rates.

L^2-Convergence rate of at time $t = 0.2$.

N	$\|q^n - Q_h^n\|_0$	Order	$\|p^n - P_h^n\|_0$	Order	$\|\sigma^n - \Sigma_h^n\|_0$	Order	$\|\vec{r}^n - \Upsilon_h^n\|_0$	Order
2	3.442934		5.086444		0.782822		0.130887	
4	0.826153	2.06	1.790665	1.51	0.226591	1.79	0.031477	2.06
8	0.207379	1.99	0.484065	1.89	0.060803	1.90	0.007829	2.01
16	0.051023	2.02	0.123298	1.97	0.015008	2.02	0.001874	2.06
32	0.012759	2.00	0.031109	1.99	0.003768	1.99	0.000457	2.04
64	0.003186	2.00	0.007827	1.99	0.000941	2.00	0.000117	1.97

Table 11: The numerical test for H^1-Galerkin convergence rates $t = 0.2$

L^2-Convergence rate of at time $t = 0.2$.

N	$\|\vec{s}^n - \vec{S}_h^n\|_0$	Order	$\|\vec{w}^n - \vec{\Lambda}_h^n\|_0$	Order	$\|u^n - U_h^n\|_0$	Order	$\|\theta^n - \Theta_h^n\|_0$	Order
2	2.087481		3.764852		0.147388		1.08	
4	0.789527	1.40	0.831732	2.18	0.031073	2.25	0.252065	2.10
8	0.231109	1.77	0.21314	1.96	0.007949	1.97	0.065222	1.95
16	0.054428	2.09	0.053476	1.99	0.001864	2.09	0.016438	1.99
32	0.013275	2.04	0.013898	1.94	0.000478	1.96	0.004443	1.89
64	0.003605	1.88	0.003349	2.05	0.000116	2.04	0.0010306491	2.11

Table 12: The numerical test for H^1-Galerkin convergence rates $t = 0.2$

L^2-Convergence rate of at time $t = 0.4$.

N	$\|q^n - Q_h^n\|_0$	Order	$\|p^n - P_h^n\|_0$	Order	$\|\sigma^n - \Sigma_h^n\|_0$	Order	$\|\vec{r}^n - \Upsilon_h^n\|_0$	Order
2	5.945092		3.186198		1.44		0.28	
4	1.456603	2.03	1.287566	1.31	0.529754	1.45	0.096680	1.55
8	0.367174	1.91	0.36380	1.82	0.145848	1.86	0.026637	1.86
16	0.09052	2.02	0.097493	1.90	0.036675	1.99	0.006664	2.00
32	0.02264	2.00	0.024712	1.98	0.009222	1.99	0.001676	1.99
64	0.005656	2.00	0.006221	1.99	0.002306	2.00	0.000419	2.00

Table 13: The numerical test for H^1-Galerkin convergence rates $t = 0.4$

L^2-Convergence rate of at time $t = 0.4$.

N	$\|\vec{s}^n - \vec{S}_h^n\|_0$	Order	$\|\vec{w}^n - \vec{\Lambda}_h^n\|_0$	Order	$\|u^n - U_h^n\|_0$	Order	$\|\theta^n - \Theta_h^n\|_0$	Order
2	2.205385		3.233270		1.215843		1.011703	
4	1.233742	0.84	0.742617	2.12	0.252495	2.27	0.241503	2.07
8	0.368538	1.74	0.192296	1.95	0.062246	2.02	0.062355	1.95
16	0.093481	1.98	0.049189	1.97	0.014797	2.07	0.015853	1.98
32	0.023775	1.98	0.012332	2.00	0.003696	2.00	0.003974	2.00
64	0.005983	1.99	0.003087	2.00	0.000921	2.00	0.000994	2.00

Table 14: The numerical test for H^1-Galerkin convergence rates $t = 0.4$

L^2-Convergence rate of at time $t = 0.8$.

N	$\|q^n - Q_h^n\|_0$	Order	$\|p^n - P_h^n\|_0$	Order	$\|\sigma^n - \Sigma_h^n\|_0$	Order	$\|\vec{r}^n - \Upsilon_h^n\|_0$	Order
2	9.561965		0.864762		1.178455		0.241432	
4	2.435440	1.97	0.972961	-0.17	0.440979	1.42	0.088987	1.44
8	0.618396	1.98	0.358710	1.44	0.124427	1.83	0.025732	1.79
16	0.153153	2.01	0.097117	1.89	0.033662	1.89	0.007001	1.88
32	0.038326	2.00	0.025057	1.95	0.008505	1.98	0.001771	1.98
64	0.009576	2.00	0.006314	1.99	0.002137	1.99	0.000445	1.99

Table 15: The numerical test for H^1-Galerkin convergence rates $t = 0.8$

L^2-Convergence rate of at time $t = 0.8$.

N	$\|\vec{s}^n - \vec{S}_h^n\|_0$	Order	$\|\vec{w}^n - \vec{\Lambda}_h^n\|_0$	Order	$\|u^n - U_h^n\|_0$	Order	$\|\theta^n - \Theta_h^n\|_0$	Order
2	2.541767		1.632467		4.749412		0.849611	
4	1.142430	1.15	0.199947	3.03	1.041882	2.19	0.191329	2.15
8	0.349771	1.71	0.046496	2.10	0.258533	2.01	0.047360	2.01
16	0.098373	1.83	0.012801	1.86	0.062097	2.06	0.011921	1.99
32	0.025149	1.97	0.003242	1.98	0.015516	2.00	0.002981	2.00
64	0.006352	1.99	0.000816	1.99	0.003869	2.00	0.000745	2.00

Table 16: The numerical test for H^1-Galerkin convergence rates $t = 0.8$

L^2-Convergence rate of at time $t = 1.0$.

N	$\|q^n - Q_h^n\|_0$	Order	$\|p^n - P_h^n\|_0$	Order	$\|\sigma^n - \Sigma_h^n\|_0$	Order	$\|\vec{r}^n - \Upsilon_h^n\|_0$	Order
2	10.841409		1.246099		0.874368		0.177767	
4	2.812890	1.95	1.012360	0.30	0.230242	1.93	0.044937	1.98
8	0.709857	1.99	0.371736	1.45	0.057012	2.01	0.011341	1.99
16	0.177888	2.00	0.103674	1.84	0.014707	1.95	0.002925	1.96
32	0.044499	2.00	0.026688	1.96	0.003725	1.98	0.000740	1.98
64	0.01141	1.96	0.006509	2.04	0.000980	1.93	0.000189	1.96

Table 17: The numerical test for H^1-Galerkin convergence rates $t = 1.0$

L^2-Convergence rate of at time $t = 1.0$.

N	$\|\vec{s}^n - \vec{S}_h^n\|_0$	Order	$\|\vec{w}^n - \vec{\Lambda}_h^n\|_0$	Order	$\|u^n - U_h^n\|_0$	Order	$\|\theta^n - \Theta_h^n\|_0$	Order
2	1.689953		1.213424		6.985828		0.845894	
4	0.421024	2.01	0.096748	3.65	1.570450	2.15	0.197625	2.10
8	0.168368	1.32	0.031449	1.62	0.380347	2.05	0.049362	2.00
16	0.053842	1.64	0.009655	1.70	0.094300	2.01	0.012381	2.00
32	0.014687	1.87	0.002551	1.92	0.023525	2.00	0.003099	2.00
64	0.00367175	2.00	0.000607	2.07	0.006358	1.89	0.000794	1.96

Table 18: The numerical test for H^1-Galerkin convergence rates $t = 1.0$

The following four figures, at different time, are using $log - log$ plot, then the slope is equivalent to the convergence rate, in absolute meaning.

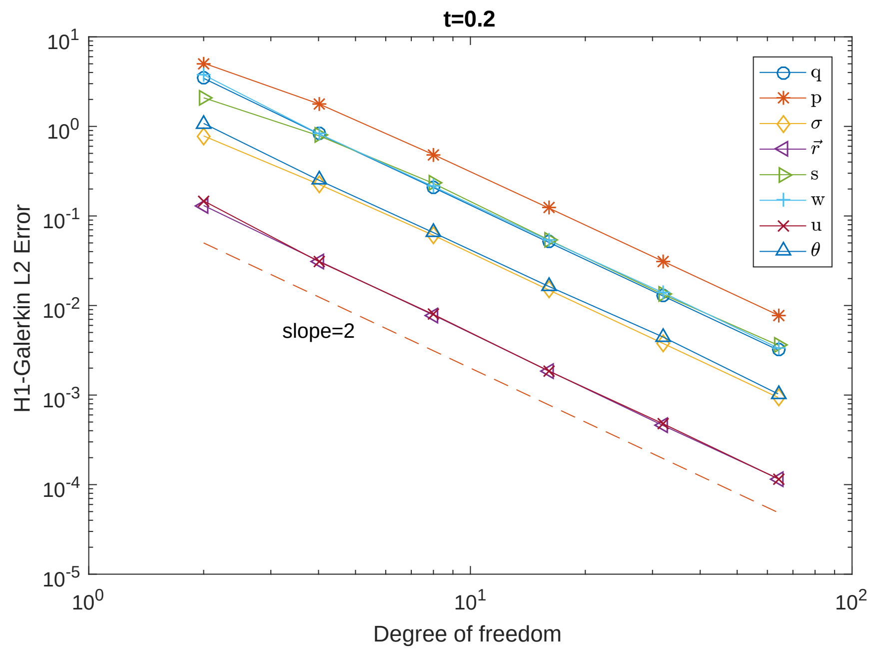

Figure 11: H1 Galerkin method L^2 convergence rate, time $t = 0.2$

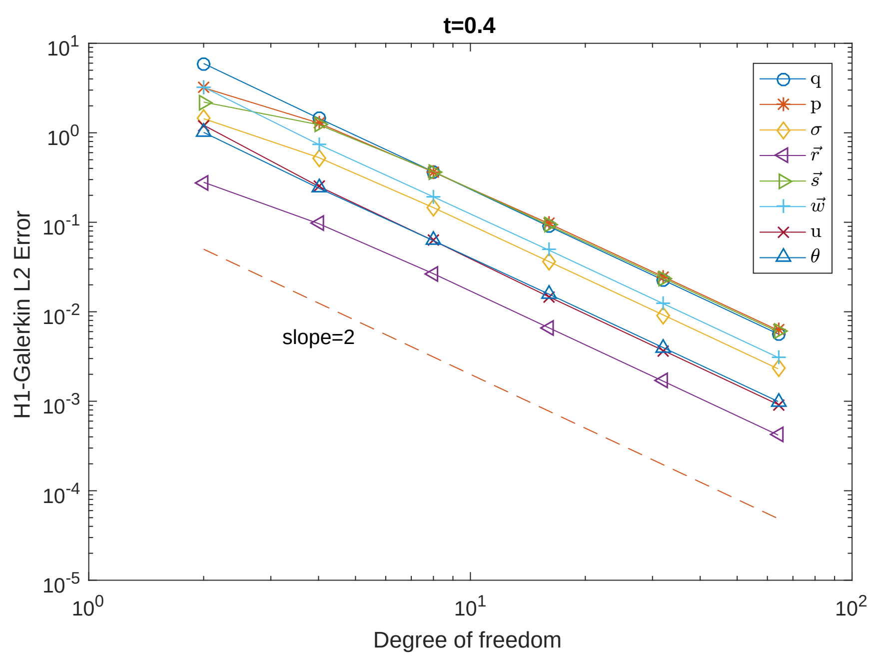

Figure 12: H1 Galerkin method L^2 convergence rate, time $t = 0.4$

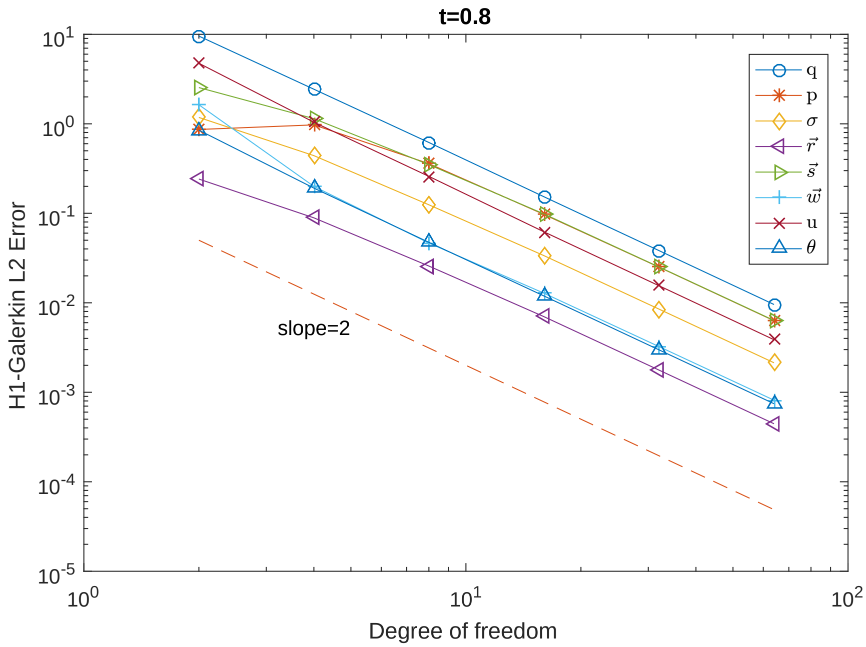

Figure 13: H1 Galerkin method L^2 convergence rate, time $t = 0.8$

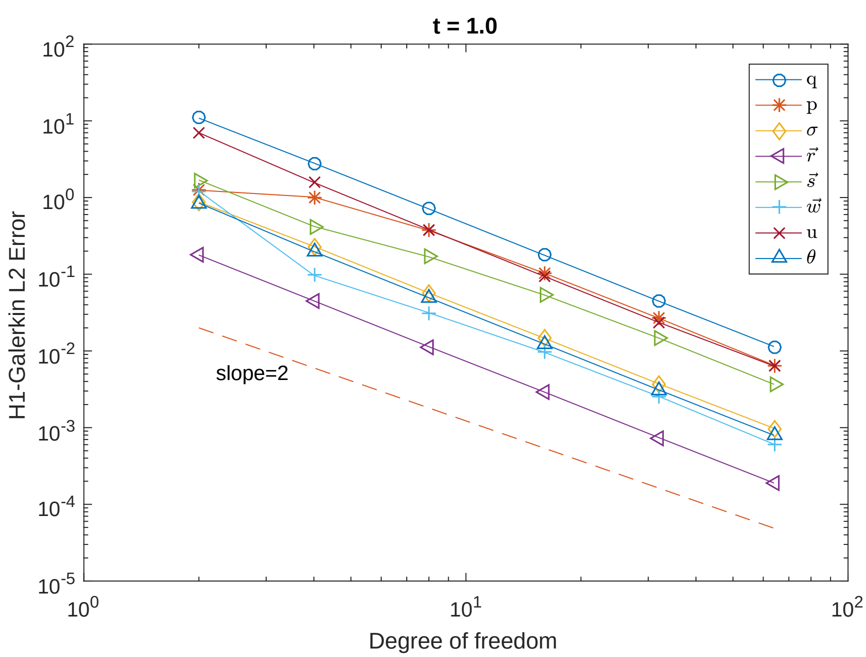

Figure 14: H1 Galerkin method L^2 convergence rate, time $t = 1.0$

CHAPTER 6 CONCLUSION

This thesis introduces a Thermoelastic Kirchhoff-Love plate system. Traditional mixed element method, $H^1 - Galerkin$ method and IP-DG method have been applied to this system of equations. And above three discrete schemes are all based on mixed forms with extra variables. The first challenge is how to assign the extra variables with boundary values and initial values.

IP-DG method is applied, SIP-DG and NSIP-DG schemes are both implemented, they differ in the penalty parameters. SIP-DG has an advantage over the other DG method, that is the underlying bilinear form guarantees symmetric, continuous, coercive and adjoint consistent properties. However requires a larger penalty parameter than NSIP-DG. From numerical experiment, we can find out NSIP-DG can achieve a better convergence performance.

In the mixed element method, how to prove the LBB condition is also problem. To address this issue, $H^1 - Galerkin$ method comes to stage. However when analyzing the semi discrete and fully discrete error estimates, $H^1 - Galerkin$ method is much more complicated than the traditional mixed element method. And $H^1 - Galerkin$ method shows a better performance than the other two.

Those three methods are observed higher accuracy than the theoretical ones. That is due to time dependent laplacian terms. However, the system implemented those methods is still linear. It is a future work to solve the KL system with nonlinear terms.

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ABSTRACT

Numerical Approaches to A Thermoelastic Kirchhoff-Love Plate System

by

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May 2019

Advisor: Dr. Zhimin Zhang

Major: Mathematics

Degree: Doctor of Philosophy

In this work, theory background of the sobolev spaces and finite element spaces are reviewed first. Then the details of how the thermoelastic Kirchhoff-Love(KL) plates numerically established are presented. Later we approaches to the thermoelastic KL system numerically with mixed element method, H^1-Galerkin method and interior penalty discontinuous galerkin method(IP-DG).

What is more, the IP-DG also applied to solve this KL system numerically. The well-posedness, existence, uniqueness and convergence properties are theoretical analyzed. The gain of the convergence rate is also $O(h^k)$, that is 1 less than the observed convergence rate.

When discussing the H^1-Galerkin method, the main advantages over traditional mixed element method, is LBB condition naturally inherent. It is proved that the existence and uniqueness of solutions for such discrete scheme. Furthermore, the semi discrete and fully discrete error estimates details are proposed to show the theoretical convergence rate is $O(h^k)$, which is also 1 lesser the observed convergence rate $O(h^{k+1})$. And optimal convergence rate $O(h^{k+1})$ can be obtained only for some variables.

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