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Well-Posedness And Symmetry Properties Of Free Boundary Problems For Some Non-Linear Degenerate Elliptic Second Order Partial Differential Equations

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WELL-POSEDNESS AND SYMMETRY PROPERTIES OF FREE BOUNDARY PROBLEMS FOR SOME NON-LINEAR DEGENERATE ELLIPTIC SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS

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DEDICATION

To my husband Ahmad and my three daughters Jana, Sara and Aya

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CHAPTER 1 INTRODUCTION

1.1 Phase Transition Free Boundary Problems

A free boundary problem is a problem for which a function u satisfying certain PDEs along with some boundary conditions on some different domains needs to be determined. These domains are a priori unknown; otherwise the problem would be a boundary value problem. Therefore, to solve a free boundary problem, one needs to find a solution u as well as to determine the a priori unknown boundaries of these domains. These boundaries are called free boundaries and they are subject to some prescribed conditions.

An example of free boundary problems is the phase transition type problem which models the heat-diffusion and the exchange of latent heat of phase transition. This problem goes back to around 1990 when Stephan introduced the Stephan Problem to model the solid-liquid transitions (see [St]). For this problem, in both of the solid and liquid regions, the temperature function u satifies the heat equations

$$\frac{\partial u}{\partial t} = \alpha_{\pm} \Delta u$$

.

Here t is the time, $\alpha_{\pm}$ is equal to α_{+} in the positive region and α_{-} in the negative region, the themal diffusivity constants in water and ice respectively.

Where we have assumed that there is no source or sink of heat.

If L is the latent heat of phase transition, $v(t)$ is the velocity at time t and ν the outer unit normal to the positive phase of the free boundary (i.e. Γ which is the interface between the water and the ice), then using the continuity of the temperature across the free boundary along with Fourier Law for heat flux, one can find that on the free boundary, u satisfies the condition:

$$\alpha_{+} u_{\nu}^{+} - \alpha_{-} u_{\nu}^{-} = L \ \ v \cdot \nu$$

The existence of a solution for this problem was proved by Lev Rubinstein in 1947 and since then this type of problem has been extensively studied.

1.2 The Problems we Study in Chapters 2 and 3

In this thesis we consider some perturbed free boundary problems associated with some non-linear degenerate non-proper elliptic second order partial differential operators $F(S, x, s)$. Here F is a real valued function having a domain in $\mathbb{S}_{n\times n} \times \mathbb{R}^n \times \mathbb{R}$, where $\mathbb{S}_{n\times n}$ is the set of $n \times n$ symmetric matrices with real entries and $\mathbb{R}^n$ is the n-dimensional real vector space. By degenerate elliptic we mean that the operator F is non-increasing in S and by non-proper we mean that we do not require F to be non-decreasing in s. Mainly, we study the well-posedeness and the bifurcation about the uniqueness of the problems in consideration as well as the stability of solutions. We also study the radial symmetry properties of a solution and the stability of its free boundary when a domain is slightly perturbed.

In chapter 2, we consider the boundary value problem:

$$\begin{cases} -\triangle_p u + Q(x)\beta_\varepsilon(u) = 0 & \text{in } \Omega \\ u(x) = \sigma(x) & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

Here $\triangle_p$ is the p-Laplace operator defined by:

$$\triangle_p u := \nabla \cdot (|\nabla u|^{p-2} \nabla u) \tag{1.2}$$

Ω is an open bounded domain in $\mathbb{R}^n$, $\sigma \in C(\partial\Omega)$ with $\min_{\partial\Omega} \sigma > \varepsilon > 0$, and

$\beta_\varepsilon(s) = \frac{1}{\varepsilon}\beta(\frac{s}{\varepsilon})$ where β is a smooth positive function supported on $(0, 1)$ and $\int_0^1 \beta(s)ds = 1$.

In section 2.4, we point out two solutions for the problem (1.1) and we study the existence of the third solution based on the size of the boundary data σ. Then, in sections 2.5 and 2.6, we prove a comparison principle for the corresponding evolution problem and we study the stability of the steady state solutions.

The boundary value problem (1.1) is a singular perturbation of the one phase transition free boundary problem:

$$\begin{cases} -\triangle_p u + Q(x)\chi_{\{u>0\}} = 0 & \text{in } \Omega \\ u(x) = \sigma(x) & \text{on } \partial\Omega \\ u_\nu = Q(x)^{\frac{1}{p}} & \text{on } \Gamma \end{cases} \tag{1.3}$$

Where Γ is the free boundary $\partial\{u>0\}$ and ν is the outer unit normal to the free boundary Γ.

The free boundary condition $u_\nu = Q(x)^{\frac{1}{p}}$ is verified in the weak sense:

$$\lim_{\epsilon\to 0}\int_{\partial\{u>\epsilon\}} (|\nabla u|^p - Q)\varphi \cdot \nu dH^{n-1} = 0$$

For every $\varphi \in W_0^{1,\infty}(\Omega, \mathbb{R}^n)$. Here ν is the normal to the smooth hyper-surface $\partial\{u>\epsilon\}$.

In chapter 3 we study the radial symmetry properties of a solution for the boundary value problem:

$$\begin{cases} -\Delta u + f(u) = 0 & \text{in } \Omega = B(z,R)\setminus \bar{B}_1 \\ u = 1 & \text{on } \partial B_1 \\ u = -1 & \text{on } \partial B(z,R) \end{cases} \tag{1.4}$$

Here, B_1 is the unit ball and $B(z,R)$ is a ball centered at z, of radius R and contains B_1 $f\colon [-1,\infty) \to \mathbb{R}$ is a C^1 function such that $f(s) \le 0$ and $f(s) = 0$ for $s \le 0$

While we do not require f to be monotone, we had to impose a lower bound condition on f' :

$$\min f' > \frac{-2(n+2)}{R^2}$$

.

Section 3.3 is devoted to the proof of the radial symmetry of a solution over a ring:

$$\begin{cases} -\Delta u + f(u) = 0 & \text{in } \Omega = B_R \setminus \bar{B}_1 \\ u = 1 & \text{on } \partial B_1 \\ u = -1 & \text{on } \partial B_R \end{cases} \tag{1.5}$$

To carry out the proof, we generalize the moving plane method argument to the case where the domain is a ring. We also provide a new proof for the radial symmetry without monotonicity.

Section 3.4 presents the approximate symmetry of a solution when the domain is a shifted ring (problem (1.4)), in which well-posedness of the parallel evolution, convergence of the evolution, and bounds by the evolutionary limit solutions are established.

Our study of the approximate radial symmetry of a solution of (1.4) was motivated by a real problem associated with a Radiographic Integrated Test Stand or RITS. In such a system, the following free boundary problem appears:

$$\begin{cases} \Delta u = f(u) & \text{in } \{u > 0\} \\ \Delta u = 0 & \text{in } \{u < 0\} \\ u_\nu^+ = u_\nu^- & \text{along } \mathcal{F} := \partial \{u > 0\} \\ u = 1 & \text{on } |x| = 1 \\ u = -1 & \text{on } |x| = R \end{cases} \tag{1.6}$$

where $\mathcal{F}$ is the *free boundary* and $f(s) < 0$ for $s > 0$.

In this problem, there is a need to guarantee that the free boundary does not touch the inner sphere in a RITS to prevent technical disaster and shutdown of the system. To do this, we attempt to show that the free boundary is approximately radially symmetric. More specifically we want to show that the free boundary is trapped between two balls away from the inner ball. The problem (1.4) can be seen as a perturbed version of this problem. In future work, we will extend our results to the singular version using approximation.

1.3 About the Singular Versions of our problems

In the paper [ACF], Alt, Cafarelli, and Friedman have studied a general quasi-linear one phase variational free boundary problem. They considered the problem of minimizing the functional:

$$J[u] := \int_\Omega \left(F(|\nabla u|^2) + \lambda^2 \chi_{\{u>0\}} \right) dx \qquad (\lambda > 0)$$

over the admissibke set $\mathcal{K} := \{v \in w^{1,2}(\Omega)$ with $v = u_0$ on $\partial\Omega$ in the trace sense $\}$

Here Ω is a Lipschitz domain in $\mathbb{R}^n$. and F is a convex increasing function defined on the positive real line with $F(0) = 0$.

The authors prove the existence of a minimizer which is also a weak solution in the positive domain and a subsolution is the whole domain Ω for the Euler-Lagrange equation

$$\nabla \cdot \big(f_p(\nabla u) \big) = 0 \tag{1.7}$$

where $f(p) := F(|p|^2)$ Then, with some extra conditions on f to make the opperator L uniformely elliptic, the authors prove the $C^{2+\alpha}$ regularity of a minimizer u in the positive domain. To do this, the authors first prove that $u \in C^{1+\alpha}$ by applying the Nash-de Giorgi estimate to an approximating sequence. Then the authors prove that for all $1 < i < n$, u_{x_i} solve some uniformely elliptic PDE with holder continuous leading coefficients.

To study the regularity of the minimizer across the free boundary,the authors first derive the free boundary condition:

$$\lim_{u>\epsilon} \left(\phi(|\nabla u|^2) - \lambda^2 \right) \eta \cdot \nu = 0 \tag{1.8}$$

for any $\eta \in W_0^{1,2}(\Omega, \mathbb{R}^n)$, where ν is the outward unit normal.

Then they prove holder continuity of the minimizer u across the free boundary using the Method of Morrey.

To prove the Lipschitz continuity of the minimizer, the authors derive a growth estimate away from the boundary:

$u(x_0) < Cd(x_0)$ where $d(x_0)$ is the distance of x_0 to the free boundary.

Then the authors prove the non-degeneray of a minimizer: For any small r, there exists a constant C such that

$$\frac{1}{r}\left(\fint_{B_r(x)} u^p\right)^{\frac{1}{p}} > \alpha\lambda$$

for all free boundary points x

Moreover, in the paper [DP], the authors Danielli and Petrosyan study the variational problem associated with the problem (1.3). In this problem the operator is degenerate (and not uniformely elliptic as in the case of ([ACF]). The authors establish, among other results, the uniform Lipschitz continuity of a minimizer. They run the proof by using and deriving some inequalities associated with the p-Laplace operator to obtaining the uniform local holder continuity as well as some growth condition and nondegeneray results away from the boundary.

CHAPTER 2 THE ONE-PHASE BIFURCATION FOR THE P-LAPLACIAN

2.1 Existence Arguments

There are various approaches in Mathematics to prove the existence of a weak solution for a given boundary value problem.

For example, for a linear and quasi-linear degenerate elliptic operators in the divergence form, one can use the Lax-Milgram theorem along with the Fredholm alternative to prove the existence and uniqueness of a weak solution. This approach uses Rietz Representation Theorem from functional analysis and some linear algebra results.

Moreover, when a comparison principle is available, a commonly used method to prove the existence of a weak solution for a Dirichlet problem is Perron's Method. This method proves that the function

$$u := \sup\{v;\ \text{for } v \text{ a continous subsolution of the given PDE in } \bar{\Omega}; v \leq g \text{ on } \partial\Omega\}$$

is indeed a solution.

Here g is the boundary data.

Another widely used approach to prove the existence of a solution is the variational approach. One can derive the corresponding functional of the PDE in consideration, and study the existence and uniqueness of a minimizer.

The last approach we list here is the "Mountain Pass Theorem" approach which can be used when we already have two solutions to prove the existence of a third one. This approach considers the corresponding functional for the PDE in consideration; but this time proves the existence of a saddle point function which ends up to be a solution for the PDE. The authors of the paper [CW] have followed this approach to prove the existence of a third solution (beside the trivial solution and the minimizer of the corresponding functional) for the singularely perturbed one-phase free boundary problem of phase transition for the Laplacian. In section (2.4) we prove the same result but this time for the p-Laplacian.

The idea behind this approach is, given a functional I on some normed vector space subject to some regularity conditions with $I[0] = 0$ and $I[v_2] < 0$, if we can show that I have some elevation in between (i.e. for $||v|| = r < ||v_2||$), then to travel along a pass from 0 to V_2, we must go up and down. This asserts the existence of a critical point for I. The theorem also proves that this critial point is a saddle point.

2.2 The p-Laplace Operator

Expanding the p-Laplace operator defined in (1.2) we obtain:

$$\triangle_p u = \sum_{i,j} \left(|\nabla u|^{p-2} + (p-2) u_{x_i} u_{x_j} \right) u_{x_i x_j}$$

We see that at the points x with $\nabla u(x) = 0$, the p-Laplace operator is singular when $1 < p < 2$, uniformely elliptic when $p = 2$, and degenerate when $p > 2$.

A function $u \in W^{1,p}(\Omega)$ is said to be a weak solution for $\triangle_p u = 0$ in a open bounded domain Ω if

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \eta dx = 0$$

for all $\eta \in W_0^{1,p}(\Omega)$.

The existence and uniqueness of a weak solution u for a Dirichlet problem associated with $\triangle_p u = 0$ with boundary condition $u - u_0 \in W_0^{1,p}(\Omega)$ can be proven by following a variational approach. (See for example Lindqvist notes in [L]).

There, it is shown that the only weak solution for $\triangle_p u = 0$ is the unique minimizer for the functional

$$I[v] := \int_\Omega |\nabla v|^p dx$$

over the admissible set

$$\mathcal{A} := \{u \in W^{1,p}(\Omega) \quad u - u_0 \in W_0^{1,p}(\Omega)\}$$

In the notes [L], Lindqvist also provides a proof for the Weak Comparison Principle for the p-Laplace operator:

Theorem 2.1. *If u and v are two continuous functions in $W^{1,p}(\Omega)$ such that $-\triangle_p u \leq 0$ (i.e. u is a subsolution), $-\triangle_p v \leq 0$ (i.e. v is a supersolution) and*

$$\textit{for every } \xi \in \partial\Omega\,,\ \limsup_{x\to\xi} u(x) \leq \liminf_{x\to\xi} v(x)$$

, then

$$u(x) \leq v(x) \textit{ for a.e. } x \in \Omega$$

2.2.1 The Optimal Interior Regularity of a p-Harmonic Function

Consider the linear elliptic operator:

$$Lv := \sum_{i,j} a^{ij}(x) u_{x_i x_j} + b(x) \cdot \nabla u = 0 \tag{2.1}$$

Where L is uniformely elliptic and the coefficients $a^{i,j}$ are uniformely bounded measurable functions of $x \in \Omega$.

DeGiorgi (1957), Nash (1958), and Moser (1961) have provided three different arguments to prove the local Holder continuity of a bounded weak solution $u \in W^{1,2}(\Omega)$ for $Lu = 0$ when the coefficients of L are simply Lebesgue Measurable.

Moreover, from the classical regularity theory (see chapter 6 Evan's book ([E]) for example), we know that when the leading coefficients of L are locally holder continuous, a bounded weak solution $u \in W^{1,2}(\Omega)$ for $Lu = 0$ is C^∞ in the open bounded domain in consideration.

This can be proven by deriving the $W^{2,2}(\Omega)$ estimate :

$$||u||_{W^{1,2}(\Omega)} \leq C||u||_{\mathcal{L}^2(\Omega)}$$

Then the linearity of the operator and the smoothness of the leading coefficients allow to reach the higher regularity result.

Now consider the quasi-linear elliptic operator in the divergence form:

$$Gu := \nabla \cdot \Big(A(|\nabla u|)\nabla u) \tag{2.2}$$

For a fixed $1 < p < \infty$, a function $u \in W^{1,p}(\Omega)$ with $A(|\nabla u|)$ is in $\mathcal{L}^q$ is called a weak solution for $Gu = 0$ if

$$\int_\Omega A(|\nabla u|)\nabla u \cdot \nabla \phi dx = 0$$

for all $\phi \in W_0^{1,p}(\Omega)$. Here q is such $\frac{1}{p} + \frac{1}{q} = 1$.

Similarely, when G is uniformely elliptic and the coefficients $A(|s|)$ is a smooth function of the vector s, then a solution for $Gu = 0$ is smooth.

This can be proved through linearization (See for example ([GT])). More specifically, if $u \in W^{1,p}(\Omega)$ a weak solution for $Gu = 0$, one can prove the following local $W^{2,p}(\Omega)$ estimate:

$$||u||_{w^{2,p}(K)} \leq C(K)||u||_{W^{1,p}(\Omega)}$$

for all $K \subset\subset \Omega$.

Then, expand $Gu = 0$ into the non-divergence form $\sum_{i,j} a^{i,j}(\nabla u)u_{x_i x_j}$.

If, for all fixed $1 < l < n$, we differentiate the equation $Gu = 0$ with respect to x_l, we can see that u_{x_l} satisfies the PDE

$$\sum_{i,j} a^{ij}(\nabla u)v_{x_i x_j} + \sum_k b^k v_l = 0 \tag{2.3}$$

where $b^k := \sum_{i,j} \left(\frac{\partial}{u_{x_i x_j}} a^{ij}_{y_k}(\nabla u) \right)$

Now to apply the above linear case result for $Lv = 0$ it suffices to prove the holder regularity of ∇u. Since this will immediately imply that the leading coefficients of 2.3 are holder continuous. But the holder continuity of ∇u follows from the Nash, Moser and Diorgi's Holder continuity result and the argument is complete

On the other hand, if G is degenerate, the optimal regularity of a solution for $Gu = 0$ is at most $C^{1,\alpha}$.

A prototype of such an operator is the p-Laplacian operator (1.2).

The arguments of Nash, Moser and Giorgi have been extended to prove the Holder continuity of a p-Harmonic function.

For example, in [L], Linqvist provided a proof for the local Holder continuity of a weak solution for $\triangle_p u = 0$ using, among other tools, Moser's iteration for $p > n$.

In particular, Lindqvist reaches Harnack's inequality which implies the validity of the Strong Maximum Principle.

Furthermore, many researchers followed various approaches to prove the optimal $C^{1,\alpha}$ regularity of a p-Harmonic function. (See for example [U], [E1], and [Le]).

2.3 About Our Result

Throughout this section we require the dimension n satisfy $n < \frac{p}{2-p}$ if $1 < p < 2$. Otherwise, n can be any positive integer.

In this chapter, we take on the task of establishing in the general case when $p \neq 2$ the results proved in [CW] for the Laplacian when $p = 2$. That is we study the existence of a Mountain pass solution based on the size of the boundary data, as well as the convergence of an evolution to stable solutions and the unstability of the Mountain Pass solution. The main difficulty in this generalization lies in the lack of sufficient regularity (at most $C^{1,\alpha}$) and the singular-degenerate nature of the p-Laplacian when $p \neq 2$. Thus we need to employ more techniques associated with the p-Laplacian, and in a case or two we have to make our conclusion slightly weaker. Nevertheless, we follow the overall scheme of approach used in [CW].

In section 2.4, we prove the bifurcation phenomenon through the Mountain Pass Theorem for the solutions of

$$\begin{cases} -\triangle_p u + Q(x)\beta_\varepsilon(u) = 0 & \text{in } \Omega \\ u(x) = \sigma(x) & \text{on } \partial\Omega \end{cases} \tag{2.4}$$

As we stated in the introduction, Ω is an open bounded domain in $\mathbb{R}^n$, $\sigma \in C(\partial\Omega)$ with $\min_{\partial\Omega} \sigma > \varepsilon > 0$, and $\beta_\varepsilon(s) = \frac{1}{\varepsilon}\beta(\frac{s}{\varepsilon})$ where β is a smooth positive function supported on $(0,1)$ and $\int_0^1 \beta(s)ds = 1$.

We denote by u_0 the solution of the Dirichlet problem:

$$\begin{cases} -\Delta_p u = 0 & \text{in } \Omega \\ u = \sigma & \text{on } \partial\Omega \end{cases} \tag{2.5}$$

Since $\min_{\partial\Omega} \sigma > \varepsilon$, then u_0 exists and its optimal regularity is $C^{1,\alpha}$ for some $\alpha \in (0,1)$, cf. [L]. We regard u_0 as a trivial solution of (2.4) in the sense $\beta_\varepsilon(u_0) \equiv 0$ due to $u_0 > \varepsilon$.

Moreover, we consider the corresponding variational problem of minimizing the functional:

$$J_{p,\epsilon}[u] = \int_\Omega \frac{1}{p}|\nabla u|^p + Q(x)\Gamma_\epsilon(u(x))dx \qquad (1 < p < \infty) \tag{2.6}$$

over the admissible set

$$\mathcal{A} = \{u \in W^{1,p}(\Omega), u - \sigma \in W_0^{1,p}(\Omega)\}$$

Where $\Gamma_\epsilon(s) = \Gamma(\frac{s}{\epsilon})$ for some Γ an anti-derivative of β such that $\Gamma(0) = 0$.

The functional $J_{p,\epsilon}[u]$ admit exactly one minimizer which turn out to be a solution for the problem (2.4). Denote this unique minimizer by u_2.

Since in the following we will fix the value of ε and will not use the notation J_p for a different purpose, we are going to abuse the notation by using J_p for the functional $J_{p,\varepsilon}$ from now on.

In section 2.4, we study the existence of a third solution for (2.4). We show that if the constant boundary data is sufficiently large, then u_0 is the only solution and therefore it coincides with the minimizer. However, when the boundary data is sufficiently small, the trivial solution u_0 and the minimizer u_2 are two different solutions. Moreover, we prove the existence of a third solution following a Mountain Pass Lemma approach.

In sections 2.5 and 2.6, we study the stability of the solutions for the problem (2.4). That is, we consider the parabolic version of our problem:

$$\begin{cases} w_t - \Delta_p w + Q(x)\beta_\epsilon(w) = 0 & \text{in } \Omega \times (0,\infty) \\ w(x,t) = \sigma(x) & \text{on } \partial\Omega \times (0,\infty) \\ w(x,0) = v_0(x) & \text{for } x \in \overline{\Omega} \end{cases} \tag{2.7}$$

In section 2.5 we prove the Parabolic Comparison Principle for the PDE of the problem (2.7). In section 2.6 we prove the convergence of the evolution (2.7) to stable solutions and we show that the Mountain Pass solution is unstable.

2.4 Existence of a Third Solution for (2.4)

2.4.1 The Bifurcation Phenomenon

We first prove if the boundary data is small enough, then the minimizer is nontrivial. More precisely, set

$$\sigma_M = \max_{\partial\Omega} \sigma(x) \quad \text{and} \quad \sigma_m = \min_{\partial\Omega} \sigma(x).$$

and let u_0 and u_2 be as defined above in section(2.3).

We show here that σ_M is small enough, then $u_0 \neq u_2$.

To see this, pick $u \in W^{1,p}(\Omega)$ so that

$$\begin{cases} u = 0 & \text{in } \Omega_\delta \\ u = \sigma & \text{on } \partial\Omega, \quad \text{and} \\ -\triangle_p u = 0 & \text{in } \Omega\backslash\bar{\Omega}_\delta, \end{cases} \tag{2.8}$$

where $\Omega_\delta = \{x \in \Omega\colon dist(x, \partial\Omega) > \delta\}$ and $\delta > 0$ is a small constant independent of ε and σ so that $\int_{\Omega_\delta} Q(x)\,dx$ has a positive lower bound which is also independent of ε and σ. Using an approximating domain if necessary, we may assume Ω_δ possesses a smooth boundary. Clearly,

$$J_p(u_0) = \int_\Omega \frac{1}{p}|\nabla u_0|^p + Q(x)\,dx \geq \int_\Omega Q(x)\,dx.$$

It is well-known that

$$\int_{\Omega\backslash\Omega_\delta} |\nabla u|^p \leq C\sigma_M^p \delta^{1-p} \quad \text{for } C = C(n, p, \Omega),$$

so that

$$\begin{aligned} J_p(u) &\leq \int_{\Omega\backslash\Omega_\delta} \frac{1}{p}|\nabla u|^p + \int_{\Omega\backslash\Omega_\delta} Q(x)\,dx \\ &\leq C\sigma_M^p \delta^{1-p} + \int_{\Omega\backslash\Omega_\delta} Q(x)\,dx. \end{aligned}$$

So, for all small $\varepsilon > 0$,

$$J_p(u) - J_p(u_0) \le C\sigma_M^p \delta^{1-p} - \int_{\Omega_\delta} Q(x)\,dx < 0$$

if $\sigma_M \le \sigma_0$ for some small enough $\sigma_0 = \sigma_0(\delta, \Omega, Q)$.

On the other hand, if the boundary data σ is constant or $\sigma_M - \sigma_m << \sigma_m$, and if σ_m is sufficiently large, then we must have $u_2 = u_0$. In fact, if $u_2(x) \le \varepsilon$ somewhere, then somewhere in the sub-domain $\{u_2 > \varepsilon\}$ the gradient $|\nabla u_2|$ is in proportion to σ_m, and hence in a substantial subset of $\{u_2 > \varepsilon\}$ the gradient $|\nabla u_2|$ is sufficiently large due to the $C^{1,\alpha}$ regularity of u_2 in the set $\{u_2 > \varepsilon\}$. As a consequence,

$$J_p(u_2) \ge \int_{u_2 > \varepsilon} \frac{1}{p} |\nabla u_2|^p \,dx \ge J_p(u_0), \quad \text{if } \sigma_m \text{ is large enough,}$$

which is a contradiction.

Therefore this is a bifurcation phenomenon in the case of constant boundary data. If the boundary data is sufficiently large, there is only the trivial solution u_0, while if the boundary data decreases to a threshold, there are more than one solution, say u_0 and u_2. Moreover, we will prove there is a third solution u_1 in the latter case in the following.

2.4.2 Existence of a Third Solution

Let $\mathfrak{B} = W_0^{1,p}(\Omega)$. For every $v \in \mathfrak{B}$, we write $u = v + u_0$ and adopt the norm $\|v\|_{\mathfrak{B}} = \left(\int_\Omega |\nabla v|^p\right)^{\frac{1}{p}} = \left(\int_\Omega |\nabla u - \nabla u_0|^p\right)^{\frac{1}{p}}$. We define the functional

$$I[v] = J_p(u) - J_p(u_0) = \int_\Omega \frac{1}{p} |\nabla u|^p - \int_{\{u<\varepsilon\}} Q(x) \left(1 - \Gamma_\varepsilon(u)\right) - \int_\Omega \frac{1}{p} |\nabla u_0|^p \tag{2.9}$$

Set $v_2 = u_2 - u_0$. Clearly, $I[0] = 0$ and $I[v_2] \le 0$ on account of the definition of u_2 as a minimizer of J_p. If $I[v_2] < 0$ which is the case if σ_M is small, we will apply the Mountain Pass Lemma to prove there exists a critical point of the functional I which is a weak solution of the problem (2.4).

The Fréchet derivative of I at $v \in \mathfrak{B}$ is given by

$$I'[v]\varphi = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi + Q(x)\beta_\varepsilon(u)\varphi \quad \varphi \in \mathfrak{B} \tag{2.10}$$

which is obviously in the dual space $\mathfrak{B}^*$ of $\mathfrak{B}$ in light of the Hölder's inequality. Equivalently

$$I'[v] = -\triangle_p (v + u_0) + Q(x)\beta_\varepsilon(v + u_0) \in \mathfrak{B}^*. \tag{2.11}$$

We see that I' is Lipschitz continuous on any bounded subset of $\mathfrak{B}$ with Lipschitz constant depending on ε, p, and $\sup Q$. In fact, for any v, w, and $\varphi \in \mathfrak{B}$,

$$\begin{aligned}
&|I'[v]\varphi - I'[w]\varphi| = |\int_\Omega |\nabla v + \nabla u_0|^{p-2}(\nabla v + \nabla u_0)\cdot\nabla\varphi + Q(x)\beta_\varepsilon(v+u_0)\varphi(x) \\
&- |\nabla w + \nabla u_0|^{p-2}(\nabla w + \nabla u_0)\cdot\nabla\varphi - Q(x)\beta_\varepsilon(w+u_0)\varphi(x)| \\
&\le \left|\int_\Omega |\nabla v + \nabla u_0|^{p-2}(\nabla v + \nabla u_0)\cdot\nabla\varphi - |\nabla w + \nabla u_0|^{p-2}(\nabla w + \nabla u_0)\cdot\nabla\varphi\right| \\
&+ \left|\int_\Omega Q(x)\beta_\varepsilon(v+u_0) - Q(x)\beta_\varepsilon(w+u_0)\varphi(x)\right|
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\left|\int_\Omega Q(x)\beta_\varepsilon(v+u_0) - Q(x)\beta_\varepsilon(w+u_0)\right| \\
&= \left|\int_\Omega Q(x)\int_0^1 \beta_\varepsilon'((1-t)w + tv + u_0)\,dt\,(v(x) - w(x))\,dx\right| \\
&\le \sup|\beta_\varepsilon'| \int_\Omega |Q(x)\,(v(x) - w(x))|\;dx \\
&\le \frac{C}{\varepsilon^2}\left(\int_\Omega Q^{p'}(x)\right)^{\frac{1}{p'}}\left(\int_\Omega |v(x) - w(x)|^p\,dx\right)^{\frac{1}{p}}
\end{aligned}$$

and

$$\begin{aligned}
&\left|\int_\Omega |\nabla v+\nabla u_0|^{p-2}(\nabla v+\nabla u_0)\cdot\nabla\varphi-|\nabla w+\nabla u_0|^{p-2}(\nabla w+\nabla u_0)\cdot\nabla\varphi\right|\\
&\le\left|\int_\Omega |\nabla v+\nabla u_0|^{p-2}(\nabla v-\nabla w)\cdot\nabla\varphi\right|\\
&\quad+\left|\int_\Omega \left(|\nabla v+\nabla u_0|^{p-2}-|\nabla w+\nabla u_0|^{p-2}\right)(\nabla w+\nabla u_0)\cdot\nabla\varphi\right|.
\end{aligned}$$

In addition,

$$\begin{aligned}
&\left|\int_\Omega |\nabla v+\nabla u_0|^{p-2}(\nabla v-\nabla w)\cdot\nabla\varphi\right|\\
&\le\left(\int_\Omega |\nabla v+\nabla u_0|^p\right)^{\frac{p-2}{p}}\left(\int_\Omega|\nabla\varphi|^p\right)^{\frac{1}{p}}\left(\int_\Omega|\nabla v-\nabla w|^p\right)^{\frac{1}{p}},
\end{aligned}$$

and

$$\begin{aligned}
&\left|\int_\Omega \left(|\nabla v+\nabla u_0|^{p-2}-|\nabla w+\nabla u_0|^{p-2}\right)(\nabla w+\nabla u_0)\cdot\nabla\varphi\right|\\
&\le C(p)\int_\Omega\left(|\nabla v+\nabla u_0|^{p-3}+|\nabla w+\nabla u_0|^{p-3}\right)|\nabla v-\nabla w||\nabla w+\nabla u_0||\nabla\varphi|\\
&\le C(p)\left(\|\nabla v\|_{L^p}+\|\nabla w\|_{L^p}+\|\nabla u_0\|_{L^p}\right)^{p-2}\|\nabla v-\nabla w\|_{L^p(\Omega)}\|\nabla\varphi\|_{L^p(\Omega)}.
\end{aligned}$$

Therefore I' is Lipschitz continuous on bounded subsets of $\mathfrak{B}$.

We note that $f\in\mathfrak{B}^*$ if there exist f^0, f^1, f^2, ..., $f^n\in L^{p'}(\Omega)$, where $\frac{1}{p}+\frac{1}{p'}=1$, such that

$$<f,u>=\int_\Omega f^0u+\sum_{i=1}^n f^iu_{x_i}\quad\text{holds for all } u\in\mathfrak{B}\text{; and}\tag{2.12}$$

$$\|f\|_{\mathfrak{B}^*}=\inf\left\{\left(\int_\Omega\sum_{i=0}^n|f^i|^{p'}\,dx\right)^{\frac{1}{p'}}:\text{ (2.12) holds.}\right\}\tag{2.13}$$

Particularly, $L^{p'}(\Omega)\subset\mathfrak{B}^*$.

Next we justify the Palais-Smale condition on the functional I. Suppose $\{v_k\} \subset \mathfrak{B}$ is a Palais-Smale sequence in the sense that

$$|I[v_k]| \leq M \quad \text{and} \quad I'[v_k] \to 0 \quad \text{in } \mathfrak{B}^*$$

for some $M > 0$. Let $u_k = v_k + u_0 \in W^{1,p}(\Omega)$, $k = 1, 2, 3, \ldots$.

We observe that $Q(x)\beta_\varepsilon(v + u_0) \in W_0^{1,p}(\Omega)$, since $Q \in W^{2,p}(\Omega)$ is continuous and bounded, β_ε and β'_ε are smooth and supported in $[0, \varepsilon]$, and $v + u_0 > \varepsilon$ near $\partial\Omega$. If $p \geq 2$ and $p \neq n$, the mapping $v \mapsto Q(x)\beta_\varepsilon(v + u_0)$ is a compact map from $W_0^{1,p}(\Omega)$ to $\mathfrak{B}^*$ due to the fact $W_0^{1,p}(\Omega) \subset\subset L^p(\Omega) \subseteq L^{p'}(\Omega) \subset \mathfrak{B}^*$ ($p' = \frac{p}{p-1} \leq p$) following from the Rellich-Kondrachov Compactness Theorem when $p < n$ and from the Morrey's inequality and Arzela-Ascoli Theorem when $n < p \leq \infty$. When $p \geq 2$ and $p = n$, we may take a $\tilde{p} < p$ such that $\tilde{p}^* := \frac{n\tilde{p}}{n-\tilde{p}} > p'$. Then the mapping $v \mapsto Q(x)\beta_\varepsilon(v + u_0)$ is again a compact map from $W_0^{1,p}(\Omega)$ to $\mathfrak{B}^*$, since $W_0^{1,p}(\Omega) \subset W_0^{1,\tilde{p}}(\Omega) \subset\subset L^{p'}(\Omega) \subset \mathfrak{B}^*$. When $1 < p < 2$, the condition $n < \frac{p}{2-p}$ on the dimension n implies the Hölder conjugate p' is less than the Sobolev conjugate $p* = \frac{np}{n-p}$ when $n \geq 2$. The mapping $v \mapsto Q(x)\beta_\varepsilon(v + u_0)$ is again a compact map from $W_0^{1,p}(\Omega)$ to $\mathfrak{B}^*$ as a result of the Rellich-Kondrachov Compactness Theorem $W_0^{1,p}(\Omega) \subset\subset L^{p'}(\Omega) \subset \mathfrak{B}^*$. If $n = 1$ and $1 < p < 2$, the compactness of the mapping follows from the Morrey's inequality and Arzela-Ascoli Theorem. In all the allowed cases, the mapping $v \mapsto Q(x)\beta_\varepsilon(v + u_0)$ is a compact map from $W_0^{1,p}(\Omega)$ to $L^{p'}(\Omega) \subset \mathfrak{B}^*$. Then there exists $f \in L^{p'}(\Omega) \subset \mathfrak{B}^*$ such that for a subsequence, still denoted by $\{v_k\}$, of $\{v_k\}$, it holds that

$$Q(x)\beta_\varepsilon(u_k) \to -f \quad \text{in } L^{p'}(\Omega).$$

We recall that

$$|I'[v_k]\varphi| = \sup_{\|\varphi\|_{\mathfrak{B}} \leq 1} \left| \int_\Omega |\nabla u_k|^{p-2} \nabla u_k \cdot \nabla\varphi + Q(x)\beta_\varepsilon(u_k)\varphi \right| \to 0.$$

As a consequence,

$$\sup_{\|\varphi\|_{\mathfrak{B}}\le M}\left|\int_\Omega |\nabla u_k|^{p-2}\nabla u_k\cdot\nabla\varphi - f\varphi\right| \to 0 \quad \text{for any } M\ge 0. \tag{2.14}$$

Obviously, that $\{I[v_k]\}$ is bounded implies that a subsequence of $\{v_k\}$, still denoted by $\{v_k\}$ by abusing the notation without confusion, converges weakly in $\mathfrak{B} = W_0^{1,p}(\Omega)$. In particular,

$$\int_\Omega fv_k - fv_m \to 0 \quad \text{as } k,\, m\to\infty.$$

Then by setting $\varphi = v_k - v_m = u_k - u_m$ in (2.14), we get

$$\left|\int_\Omega \left(|\nabla u_k|^{p-2}\nabla u_k - |\nabla u_m|^{p-2}\nabla u_m\right)\cdot\nabla(u_k-u_m)\right| \to 0 \quad \text{as } k,\, m\to\infty, \tag{2.15}$$

since

$$\|u_k-u_m\|_{\mathfrak{B}}^p = \|v_k - v_m\|_{\mathfrak{B}}^p \le 2pM + 2J_p[u_0].$$

In particular, if $p=2$, $\{v_k\}$ is a Cauchy sequence in $W_0^{1,2}(\Omega)$ and hence converges. We will apply the following elementary inequalities, with $a=\nabla u_m$ and $b=\nabla u_k$, associated with the p-Laplacian, [L], to the general case $p\neq 2$:

$$< |b|^{p-2}b - |a|^{p-2}a,\, b-a >\ge (p-1)|b-a|^2(1+|a|^2+|b|^2)^{\frac{p-2}{2}},\ \ 1\le p\le 2; \tag{2.16}$$

$$\text{and} \quad < |b|^{p-2}b - |a|^{p-2}a,\, b-a >\ge 2^{2-p}|b-a|^p,\ \ p\ge 2. \tag{2.17}$$

We assume first $1<p<2$. Let $K = 2pM + 2J_p[u_0]$. Then the first elementary inequality (2.16) implies

$$\begin{aligned}&(p-1)\int_\Omega |\nabla u_k - \nabla u_m|^2\left(1+|\nabla u_k|^2+|\nabla u_m|^2\right)^{\frac{p-2}{2}}\\ \le &\int_\Omega \left(|\nabla u_k|^{p-2}\nabla u_k - |\nabla u_m|^{p-2}\nabla u_m\right)\cdot\nabla(u_k-u_m)\to 0\end{aligned}$$

Meanwhile the Hölder's inequality implies

$$\begin{aligned}
&\int_\Omega |\nabla v_k - \nabla v_m|^p = \int_\Omega |\nabla u_k - \nabla u_m|^p \\
&\le \left(\int_\Omega |\nabla u_k - \nabla u_m|^2 \left(1 + |\nabla u_k|^2 + |\nabla u_m|^2\right)^{\frac{p-2}{2}}\right)^{\frac{p}{2}} \left(\int_\Omega \left(1 + |\nabla u_k|^2 + |\nabla u_m|^2\right)^{\frac{p}{2}}\right)^{\frac{2-p}{2}} \\
&\le C(p)\left(|\Omega| + K\right)^{\frac{2-p}{2}} \left(\int_\Omega |\nabla u_k - \nabla u_m|^2 \left(1 + |\nabla u_k|^2 + |\nabla u_m|^2\right)^{\frac{p-2}{2}}\right)^{\frac{p}{2}}
\end{aligned}$$

Therefore, $\{v_k\}$ is a Cauchy sequence in $\mathfrak{B}$ and hence converges.

Suppose $p > 2$. The second elementary inequality (2.17) implies

$$\begin{aligned}
&\int_\Omega |\nabla v_k - \nabla v_m|^p = \int_\Omega |\nabla u_k - \nabla u_m|^p \\
&\le 2^{p-2} \int_\Omega \left(|\nabla u_k|^{p-2}\nabla u_k - |\nabla u_m|^{p-2}\nabla u_m\right) \cdot \left(\nabla u_k - \nabla u_m\right),
\end{aligned}$$

which in turn implies $\{v_k\}$ is a Cauchy sequence in $\mathfrak{B}$ and hence converges, on account of (2.15). The Palais-Smale condition is verified for $1 < p < \infty$ for the functional I on the Banach space $W_0^{1,p}(\Omega)$.

Before we continue the main proof, let us state an elementary result closely related to the p-Laplacian, which follows readily from the Fundamental Theorem of Calculus.

Lemma 2.2. *For any a and $b \in \mathbb{R}^n$ $(n \ge 1)$, it holds*

$$|b|^p \ge |a|^p + p < |a|^{p-2}a, b - a > + C(p)|b - a|^p \quad (p \ge 2) \tag{2.18}$$

where $C(p) > 0$.

If $1 < p < 2$, then

$$|b|^p \ge |a|^p + p < |a|^{p-2}a, b - a > + C(p)|b - a|^2 \int_0^1 \int_0^t |(1-s)a + sb|^{p-2}\, dsdt, \tag{2.19}$$

where $C(p) = p(p-1)$.

We are now in a position to show there is a closed mountain ridge around the origin of the Banach space $\mathfrak{B}$ that separates v_2 from the origin with the energy I as the elevation function, which is the content of the following lemma.

Lemma 2.3. *For all small $\varepsilon > 0$ such that $C\varepsilon \leq \frac{1}{2}\sigma_m$ for a large universal constant C, there exist positive constants δ and a independent of ε, such that, for every v in $\mathfrak{B}$ with $\|v\|_{\mathfrak{B}} = \delta$, the inequality $I[v] \geq a$ holds.*

Proof. It suffices to prove $I[v] \geq a > 0$ for every $v \in C_0^\infty(\Omega)$ with $\|v\|_{\mathfrak{B}} = \delta$ for δ small enough, as I is continuous on $\mathfrak{B}$, and $C_0^\infty(\Omega)$ is dense in $\mathfrak{B}$.

Let $\Lambda = \{x \in \Omega\colon u(x) \leq \varepsilon\}$, where $u = v + u_0$. We claim that $\Lambda = \emptyset$ if δ is small enough. If not, we may pick $z \in \Lambda$. Let $\mathcal{AC}([a,b], S)$ be the set of absolutely continuous functions $\gamma\colon [a,b] \to S$, where $S \subseteq \mathbb{R}^n$. For each $\gamma \in \mathcal{AC}([a,b], S)$, we define its length to be $L(\gamma) = \int_a^b |\gamma'(t)|\, dt$. For $x_0 \in \partial\Omega$, we define the distance from x_0 to z to be

$$d(x_0, z) = \inf\{L(\gamma) : \gamma \in \mathcal{AC}([0,1], \bar{\Omega}),\ \text{s.t. } \gamma(0) = x_0,\ \text{and } \gamma(1) = z\}$$

As shown in [CW], there is a minimizing path γ_{x_0} for the distance $d(x_0, z)$.

Suppose the domain Ω is convex or star-like about z. For any $x_0 \in \partial\Omega$, let $\gamma = \gamma_{x_0}$ be a minimizing path of $d(x_0, z)$. Then it is clear that γ is a straight line segment and $\gamma(t) \neq z$ for $t \in [0, 1)$. Furthermore, for any two distinct points x_1 and $x_2 \in \partial\Omega$, the corresponding minimizing paths do not intersect in $\Omega\backslash\{z\}$. For this reason, we can carry out the following computation. Clearly $v(x_0) = 0$ and $v(\gamma(1)) = \varepsilon - u_0(\gamma(1)) \leq \varepsilon - \sigma_m < 0$. So the Fundamental Theorem of Calculus

$$v(\gamma(1)) - v(\gamma(0)) = \int_0^1 \nabla v(\gamma(t)) \cdot \gamma'(t) dt$$

implies

$$\sigma_m - \varepsilon \leq \int_0^1 |\nabla v(\gamma(t))||\gamma'(t)| dt. \tag{2.20}$$

For each $x_0 \in \partial\Omega$, let $e(x_0)$ be the unit vector in the direction of $x_0 - z$ and $\nu(x_0)$ the outer normal to $\partial\Omega$ at x_0. Then $\nu(x_0) \cdot e(x_0) > 0$ everywhere on $\partial\Omega$. Hence the above inequality (2.20) implies

$$
\begin{aligned}
&(\sigma_m - \varepsilon) \int_{\partial\Omega} \nu(x_0) \cdot e(x_0)\, dH^{n-1}(x_0) \\
&\leq \int_{\partial\Omega} \int_0^1 |\nabla v(\gamma(t))||\gamma'(t)|\, dt\, \nu(x_0) \cdot e(x_0)\, dH^{n-1}(x_0) \\
&\leq \int_{\partial\Omega} \left(\int_0^1 |\gamma'(t)|\, dt \right)^{\frac{1}{p'}} \left(\int_0^1 |\nabla v(\gamma(t))|^p |\gamma'(t)|\, dt \right)^{\frac{1}{p}} \nu(x_0) \cdot e(x_0)\, dH^{n-1}(x_0), \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{where } \frac{1}{p} + \frac{1}{p'} = 1, \\
&= \int_{\partial\Omega} L(\gamma_{x_0})^{\frac{1}{p'}} \left(\int_0^1 |\nabla v(\gamma(t))|^p |\gamma'(t)|\, dt \right)^{\frac{1}{p}} \nu(x_0) \cdot e(x_0)\, dH^{n-1}(x_0) \\
&\leq \left(\int_{\partial\Omega} L(\gamma_{x_0}) \nu(x_0) \cdot e(x_0)\, dH^{n-1} \right)^{\frac{1}{p'}} \left(\int_{\partial\Omega} \int_0^1 |\nabla v(\gamma(t))|^p |\gamma'(t)| \nu \cdot e\, dt\, dH^{n-1} \right)^{\frac{1}{p}} \\
&= C|\Omega|^{\frac{1}{p'}} \left(\int_\Omega |\nabla v|^p\, dx \right)^{\frac{1}{p}} \\
&\leq C|\{u > \varepsilon\}|^{\frac{1}{p'}} \delta \leq C|\{u > 0\}|^{\frac{1}{p'}} \delta,
\end{aligned}
$$

where the second and third inequalities are due to the application of the Hölder's inequality, and the constant C depends on n and p. The second equality follows from the two representation formulas

$$|\Omega| = C(n) \int_{\partial\Omega} L(\gamma_{x_0}) \nu(x_0) \cdot e(x_0)\, dH^{n-1}(x_0)$$

and

$$\int_\Omega |\nabla v(x)|^p\, dx = C(n) \int_{\partial\Omega} \int_0^1 |\nabla v(\gamma_{x_0}(t))|^p \left|\gamma_{x_0}'(t)\right| \nu(x_0) \cdot e(x_0)\, dt\, dH^{n-1}(x_0).$$

If we take δ sufficiently small and independent of ε in the preceding inequality

$$(\sigma_m - \varepsilon) \int_{\partial\Omega} \nu(x_0) \cdot e(x_0) \, dH^{n-1}(x_0) \le C|\{u > 0\}|^{\frac{1}{p'}} \delta,$$

the measure $|\{u > 0\}|$ of the positive domain would be greater than that of Ω, which is impossible, provided that

$$\int_{\partial\Omega} \nu(x_0) \cdot e(x_0) \, dH^{n-1}(x_0) \ge C, \tag{2.21}$$

for a constant C which depends on n, p and $|\Omega|$, but not on z or v. Hence Λ must be empty. So we need to justify the inequality (2.21). To fulfil that condition, for $e = e(x_0)$, we set $l(e, z) = l(e) = L(\gamma_{x_0})$. Then

$$\int_{\partial\Omega} \nu(x_0) \cdot e(x_0) \, dH^{n-1}(x_0) = \int_{e \in \partial B} (l(e))^{n-1} \, d\sigma(e),$$

where B is the unit ball about z and $d\sigma(e)$ is the surface area element on the unit sphere ∂B which is invariant under rotation and reflection. Clearly,

$$\left(\int_{\partial B} (l(e))^{n-1} \, d\sigma(e) \right)^{\frac{2}{n-1}} \ge C(n) \int_{\partial B} l^2(e) \, d\sigma(e)$$

Consequently, in order to prove (2.21), we need only to prove

$$\int_{\partial B} l^2(e) \, d\sigma(e) \ge C(n, p, |\Omega|). \tag{2.22}$$

Next, we show the integral on the left-hand-side of (2.22) is minimal if Ω is a ball while its measure is kept unchanged. In fact, this is almost obvious if one notices the following fact. Let π be any hyperplane passing through z, and x_1 and x_2 be the points on $\partial\Omega$ which lie on a line perpendicular to π. Let x_1^* and x_2^* be the points on the boundary $\partial\Omega_\pi$, where Ω_π is the symmetrized image of Ω about the hyperplane π, which lie on the line $\overline{x_1 x_2}$. Let

$2a = |\overline{x_1x_2}| = |\overline{x_1^*x_2^*}|$ and d be the distance from z to the line $\overline{x_1x_2}$. Then for some t in $-a \le t \le a$, it holds that

$$L^2(\gamma_{x_1}) + L^2(\gamma_{x_2}) = \left(d^2 + (a-t)^2\right) + \left(d^2 + (a+t)^2\right) \ge 2(d^2 + a^2) = 2\left(L^*(\gamma_{x_1^*})\right)^2.$$

As a consequence, if Ω^* is the symmetrized ball with measure equal to that of Ω, then

$$\int_{\partial B} l^2(e)\, d\sigma(e) \ge \int_{\partial B} (l^*(e))^2\, d\sigma(e) = C(n, |\Omega|),$$

where l^* is the length from z to a point on the boundary $\partial\Omega^*$ which is constant. This finishes the proof of the fact that $\Lambda = \emptyset$.

In case the domain Ω is not convex, the minimizing paths of $d(x_1, z)$ and $d(x_2, z)$ for distinct x_1, $x_2 \in \partial\Omega$ may partially coincide. We form the set $\mathcal{DA}(\partial\Omega)$ of the points x_0 on $\partial\Omega$ so that a minimizing path γ of $d(x_0, z)$ satisfies $\gamma(t) \in \Omega\backslash\{z\}$ for $t \in (0, 1)$. We call a point in $\mathcal{DA}(\partial\Omega)$ a **directly accessible** boundary point. Let Ω_1 be the union of these minimizing paths for the directly accessible boundary points. It is not difficult to see that $|\Omega_1| > 0$ and hence $H^{n-1}(\mathcal{DA}(\partial\Omega)) > 0$. Then we may apply the above computation to the star-like domain Ω_1 with minimal modification. We have

$$(\sigma_m - C\varepsilon) \int_{\partial\Omega} \nu(x_0) \cdot e(x_0)\, dH^{n-1}(x_0) \le C|\Omega_1|^{\frac{1}{p'}}\delta \le C|\Omega|^{\frac{1}{p'}}\delta. \tag{2.23}$$

For small enough δ, this raises a contradiction $|\Omega| > |\Omega|$. So $\Lambda = \emptyset$.

Finally we prove that $\|v\|_{\mathfrak{B}} = \delta$ implies

$$I[v] = \int_\Omega \frac{1}{p}|\nabla v + \nabla u_0|^p - \frac{1}{p}|\nabla u_0|^p \ge a \text{ for a certain } a > 0. \tag{2.24}$$

If $p \geq 2$, then the elementary inequality (2.18) implies that

$$
\begin{aligned}
I[v] &= \int_\Omega \frac{1}{p} |\nabla v + \nabla u_0|^p - \frac{1}{p} |\nabla u_0|^p \\
&\geq \int_\Omega < |\nabla u_0|^{p-2} \nabla u_0, \nabla v > + C(p) |\nabla v|^p \\
&= C(p) \int_\Omega |\nabla v|^p = C(p)\delta^p > 0,
\end{aligned}
$$

while if $1 < p < 2$, then the elementary inequality (2.19) implies

$$
\begin{aligned}
I[v] &\geq p(p-1) \int_\Omega |\nabla v|^2 \int_0^1 \int_0^t \frac{1}{|\nabla u_0 + s\nabla v|^{2-p}} \, dsdtdx \\
&\geq p(p-1) \int_\Omega |\nabla v|^2 \int_0^1 \int_0^t \frac{1}{(|\nabla u_0| + s |\nabla v|)^{2-p}} \, dsdtdx.
\end{aligned}
$$

If $\int_\Omega |\nabla u_0|^p = 0$, then $I[v] = \frac{1}{p}\delta^p > 0$. So in the following, we assume $\int_\Omega |\nabla u_0|^p > 0$.

Let $S = S_\lambda = \{x \in \Omega \colon |\nabla v| > \lambda\delta\}$, where the constant $\lambda = \lambda(p, |\Omega|)$ is to be taken. Then

$$
\begin{aligned}
\delta^p &= \int_\Omega |\nabla v|^p = \int_{\{|\nabla v| \leq \lambda\delta\}} |\nabla v|^p + \int_S |\nabla v|^p \\
&\leq (\lambda\delta)^p |\Omega| + \int_S |\nabla v|^p
\end{aligned}
$$

and hence

$$
\int_S |\nabla v|^p \geq \delta^p \left(1 - \lambda^p |\Omega|\right) \geq \frac{1}{2}\delta^p, \quad \text{if } \lambda \text{ satisfies } \frac{1}{4} < \lambda^p |\Omega| \leq \frac{1}{2}.
$$

Meanwhile, for $1 < p < 2$, it holds that

$$
\begin{aligned}
I[v] &\geq C(p) \int_S |\nabla v|^2 \int_0^1 \int_0^t \frac{1}{(|\nabla u_0| + s |\nabla v|)^{2-p}} \, dsdtdx \\
&= C(p) \left(\int_{S \cap \{|\nabla u_0| \leq |\nabla v|\}} |\nabla v|^2 \int_0^1 \int_0^t \frac{1}{(|\nabla u_0| + s|\nabla v|)^{2-p}} \, dsdtdx \right. \\
&\quad \left. + \int_{S \cap \{|\nabla u_0| > |\nabla v|\}} |\nabla v|^2 \int_0^1 \int_0^t \frac{1}{(|\nabla u_0| + s |\nabla v|)^{2-p}} \, dsdtdx \right).
\end{aligned}
$$

The first integral on the right satisfies

$$\begin{aligned}
&\int_{S\cap\{|\nabla u_0|\leq|\nabla v|\}} |\nabla v|^2 \int_0^1\int_0^t \frac{1}{(|\nabla u_0|+s|\nabla v|)^{2-p}}\,dsdtdx\\
&\geq \int_{S\cap\{|\nabla u_0|\leq|\nabla v|\}} |\nabla v|^p \int_0^1\int_0^t \frac{1}{(1+s)^{2-p}}\,dsdtdx\\
&= C(p)\int_{S\cap\{|\nabla u_0|\leq|\nabla v|\}} |\nabla v|^p\ dx,
\end{aligned}$$

while the second integral on the right satisfies

$$\begin{aligned}
&\int_{S\cap\{|\nabla u_0|>|\nabla v|\}} |\nabla v|^2 \int_0^1\int_0^t \frac{1}{(|\nabla u_0|+s\,|\nabla v|)^{2-p}}\,dsdtdx\\
&\geq \int_{S\cap\{|\nabla u_0|>|\nabla v|\}} \frac{|\nabla v|^2}{|\nabla u_0|^{2-p}} \int_0^1\int_0^t \frac{ds\,dt}{(1+s)^{2-p}}\,dx\\
&= C(p)\int_{S\cap\{|\nabla u_0|>|\nabla v|\}} \frac{|\nabla v|^2}{|\nabla u_0|^{2-p}}\,dx.
\end{aligned}$$

The Hölder's inequality applied with exponents $\frac{2}{p}$ and $\frac{2}{2-p}$ implies that

$$\int_{S\cap\{|\nabla u_0|>|\nabla v|\}} |\nabla v|^p \leq \left(\int_{S\cap\{|\nabla u_0|>|\nabla v|\}} \frac{|\nabla v|^2}{|\nabla u_0|^{2-p}}\right)^{\frac{p}{2}} \left(\int_{S\cap\{|\nabla u_0|>|\nabla v|\}} |\nabla u_0|^p\right)^{\frac{2-p}{2}},$$

or equivalently

$$\begin{aligned}
\int_{S\cap\{|\nabla u_0|>|\nabla v|\}} \frac{|\nabla v|^2}{|\nabla u_0|^{2-p}} &\geq \frac{\left(\int_{S\cap\{|\nabla u_0|>|\nabla v|\}} |\nabla v|^p\right)^{\frac{2}{p}}}{\left(\int_{S\cap\{|\nabla u_0|>|\nabla v|\}} |\nabla u_0|^p\right)^{\frac{2-p}{p}}}\\
&\geq \frac{\left(\int_{S\cap\{|\nabla u_0|>|\nabla v|\}} |\nabla v|^p\right)^{\frac{2}{p}}}{\left(\int_\Omega |\nabla u_0|^p\right)^{\frac{2-p}{p}}}.
\end{aligned}$$

Consequently,

$$
\begin{aligned}
I[v] &\geq C(p)\int_{S\cap\{|\nabla u_0|\leq|\nabla v|\}} |\nabla v|^p + C(p)\frac{\left(\int_{S\cap\{|\nabla u_0|>|\nabla v|\}}|\nabla v|^p\right)^{\frac{2}{p}}}{\left(\int_\Omega |\nabla u_0|^p\right)^{\frac{2-p}{p}}}\\
&\geq C(p)\left(\int_{S\cap\{|\nabla u_0|\leq|\nabla v|\}} |\nabla v|^p\right)^{\frac{2}{p}} + C(p)\frac{\left(\int_{S\cap\{|\nabla u_0|>|\nabla v|\}}|\nabla v|^p\right)^{\frac{2}{p}}}{\left(\int_\Omega |\nabla u_0|^p\right)^{\frac{2-p}{p}}}, \quad \text{as } \delta \text{ is small}\\
&\geq C(p)A(u_0)\left(\left(\int_{S\cap\{|\nabla u_0|\leq|\nabla v|\}} |\nabla v|^p\right)^{\frac{2}{p}} + \left(\int_{S\cap\{|\nabla u_0|>|\nabla v|\}} |\nabla v|^p\right)^{\frac{2}{p}}\right)\\
&\geq C(p)A(u_0)\left(\int_S |\nabla v|^p\right)^{\frac{2}{p}} = C(p)A(u_0)\delta^2,
\end{aligned}
$$

where the last inequality is a consequence of the elementary inequality

$$
a^{\frac{2}{p}} + b^{\frac{2}{p}} \geq C(p)\,(a+b)^{\frac{2}{p}} \quad \text{for} \;\; a, b \geq 0,
$$

and the constant

$$
A(u_0) = \min\left\{1, \frac{1}{\left(\int_\Omega |\nabla u_0|^p\right)^{\frac{2-p}{p}}}\right\}.
$$

So we have proved $I[v] \geq a > 0$ for some $a > 0$ whenever $v \in C_0^\infty(\Omega)$ satisfies $\|v\|_{\mathfrak{B}} = \delta$, for any $p \in (1, \infty)$. □

Let

$$
\mathcal{G} = \{\gamma \in C([0,1], H) : \gamma(0) = 0 \text{ and } \gamma(1) = v_2\}
$$

and

$$
c = \inf_{\gamma\in\mathcal{G}} \max_{0\leq t\leq 1} I[\gamma(t)].
$$

The verified Palais-Smale condition and the preceding lemma allow us to apply the Mountain Pass Theorem as stated, for example, in [J] to conclude that there is a $v_1 \in \mathfrak{B}$ such that $I[v_1] = c$, and $I'[v_1] = 0$ in $\mathfrak{B}^*$. That is

$$\int_\Omega |\nabla u_1|^{p-2} \nabla u_1 \cdot \nabla\varphi + Q(x)\beta_\varepsilon(u_1)\varphi dx = 0$$

for any $\varphi \in \mathfrak{B} = W_0^{1,p}(\Omega)$, where $u_1 = v_1 + u_0$. So u_1 is a weak solution of the problem (**??**) and (**??**). In essence, the Mountain Pass Theorem is a way to produce a saddle point solution. Therefore, in general, u_1 tends to be an unstable solution in contrast to the stable solutions u_0 and u_2.

In this subsection, we have proved the following theorem.

Theorem 2.4. *If $\varepsilon << \sigma_m$ and $J_p(u_2) < J_p(u_0)$, then there exists a third weak solution u_1 of the problem (2.4). Moreover, $J_p(u_1) \geq J_p(u_0) + a$, where a is independent of ε.*

2.5 A Comparison Principle for the Corresponding Evolution Problem

In this section, we prove a comparison theorem for the following evolution problem.

$$\begin{cases} w_t - \triangle_p w + \alpha(x,w) = 0 & \text{in } \Omega \times (0,T) \\ w(x,t) = \sigma(x) & \text{on } \partial\Omega \times (0,T) \\ w(x,0) = v_0(x) & \text{for } x \in \bar{\Omega}, \end{cases} \tag{2.25}$$

where $T > 0$ may be finite or infinite, and α is a continuous function satisfying $0 \leq \alpha(x,w) \leq Kw$ and

$$|\alpha(x,r_2) - \alpha(x,r_1)| \leq K\,|r_2 - r_1|$$

for all $x \in \Omega$, r_1 and $r_2 \in \mathbb{R}$, and some $K \geq 0$. Let us introduce the notation $H_p w = w_t - \triangle_p w + \alpha(x,w)$. We recall a weak sub-solution $w \in L^2(0,T;W^{1,p}(\Omega))$ satisfies

$$\int_V w\varphi \,\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_V -w\varphi_t + |\nabla w|^{p-2}\nabla w \cdot \nabla\varphi + \alpha(x,w)\varphi \leq 0$$

for any region $V \subset\subset \Omega$ and any test function $\varphi \in L^2_0(0,T;W^{1,p}(\Omega))$ such that $\varphi_t \in L^2(\Omega \times \mathbb{R}_T)$ and $\varphi \geq 0$ in $\Omega \times \mathbb{R}_T$, where $L^2_0(0,T;W^{1,p}(\Omega))$ is the subset of $L^2(0,T;W^{1,p}(\Omega))$ that contains functions which is equal zero on the boundary of $\Omega \times \mathbb{R}_T$, where $\mathbb{R}_T = [0,T]$. For convenience, we let $\mathfrak{T}_+$ denote this set of test functions in the following.

In particular, it holds that

$$\int_0^T \int_\Omega -w\varphi_t + < |\nabla w|^{p-2}\nabla w, \nabla\varphi > + \alpha(x,w)\varphi \leq 0$$

for any test function $\varphi \in L^2_0(0,T;W^{1,p}(\Omega))$ such that $\varphi_t \in L^2(\Omega \times \mathbb{R}_T)$ and $\varphi \geq 0$ in $\Omega \times \mathbb{R}_T$.

The comparison principle for weak sub- and super-solutions is stated as follows.

Theorem 2.5. *Suppose w_1 and w_2 are weak sub- and super-solutions of the evolutionary problem (2.25) respectively with $w_1 \leq w_2$ on the parabolic boundary $(\bar{\Omega} \times \{0\}) \cup (\partial\Omega \times (0,+\infty))$. Then $w_1 \leq w_2$ in $\mathcal{D} := \Omega \times \mathbb{R}_T$.*

Uniqueness of a weak solution of (2.25) follows from the comparison principle, Theorem 2.5, immediately.

Lemma 2.6. *For $T > 0$ small enough, if $H_p w_1 \le 0 \le H_p w_2$ in the weak sense in $\Omega \times \mathbb{R}_T$ and $w_1 < w_2$ on $\partial_p(\Omega \times \mathbb{R}_T)$, then $w_1 \le w_2$ in $\Omega \times \mathbb{R}_T$.*

Proof. For any given small number $\delta > 0$, we define a new function $\tilde{w}_1$ by

$$\tilde{w}_1(x,t) = w_1(x,t) - \frac{\delta}{T-t},$$

where $x \in \bar{\Omega}$ and $0 \le t < T$. In order to prove $w_1 \le w_2$ in $\Omega \times \mathbb{R}_T$, it suffices to prove $\tilde{w}_1 \le w_2$ in $\Omega \times \mathbb{R}_T$ for all small $\delta > 0$. Clearly, $\tilde{w}_1 < w_2$ on $\partial_p(\Omega \times \mathbb{R}_T)$, and $\lim_{t \to T} \tilde{w}_1(x,t) = -\infty$ uniformly on Ω. Moreover, the following holds for any $\varphi \in \mathfrak{T}_+$:

$$\begin{aligned}
&\int_0^T \int_\Omega -\tilde{w}_1 \varphi_t + < |\nabla \tilde{w}_1|^{p-2} \nabla \tilde{w}_1, \nabla \varphi > + \alpha(x, \tilde{w}_1)\varphi \\
&= \int_0^T \int_\Omega -w_1 \varphi_t + < |\nabla w_1|^{p-2} \nabla w_1, \nabla \varphi > + \frac{\delta}{T-t}\varphi_t + \left(\alpha(x,\tilde{w}_1) - \alpha(x, w_1)\right)\varphi \\
&\le \int_0^T \int_\Omega \frac{\delta}{T-t}\varphi_t + K\frac{\delta}{T-t}\varphi, \quad \text{as } w_1 \text{ is a weak sub-solution} \\
&= \int_0^T \int_\Omega \left(-\frac{\delta}{(T-t)^2} + K\frac{\delta}{T-t}\right)\varphi \\
&\le \int_0^T \int_\Omega -\frac{\delta}{2(T-t)^2}\varphi, \quad \text{for } T \le \frac{1}{2K} \text{ so that } 2K \le \frac{1}{T-t} \\
&< 0,
\end{aligned}$$

i. e.

$$H_p \tilde{w}_1 \le -\frac{\delta}{2(T-t)^2} \le -\frac{\delta}{2T^2} < 0 \ \text{ in the weak sense.}$$

That is, if we abuse the notation a little by denoting $\tilde{w}_1$ by w_1 in the following for convenience, it holds for any $\varphi \in \mathfrak{T}_+$,

$$\int_0^T \int_\Omega -w_1 \varphi_t + < |\nabla w_1|^{p-2} \nabla w_1, \nabla \varphi > + \alpha(x, w_1)\varphi \le \int_0^T \int_\Omega -\frac{\delta}{2T^2}\varphi < 0.$$

Meanwhile, for any $\varphi \in \mathfrak{T}_+$, w_2 satisfies

$$\int_0^T \int_\Omega -w_2\varphi_t + < |\nabla w_2|^{p-2}\nabla w_2, \nabla\varphi > +\alpha(x, w_2)\varphi \geq 0.$$

Define, for $j = 1, 2$, $v_j(x,t) = e^{-\lambda t} w_j(x,t)$, where the constant $\lambda > 2K$. Then $w_j(x,t) = e^{\lambda t} v_j(x,t)$, and it is clear that $w_1 \leq w_2$ in $\Omega \times \mathbb{R}_T$ is equivalent to $v_1 \leq v_2$ in $\Omega \times \mathbb{R}_T$. In addition, for any $\varphi \in \mathfrak{T}_+$, the following inequalities hold:

$$\int_0^T \int_\Omega -e^{\lambda t} v_1\varphi_t + e^{\lambda(p-1)t} < |\nabla v_1|^{p-2}\nabla v_1, \nabla\varphi > +\alpha(x, e^{\lambda t} v_1)\varphi \leq -\int_0^T \int_\Omega \frac{\delta}{2T^2}\varphi$$

$$\text{and} \quad \int_0^T \int_\Omega -e^{\lambda t} v_2\varphi_t + e^{\lambda(p-1)t} < |\nabla v_2|^{p-2}\nabla v_2, \nabla\varphi > +\alpha(x, e^{\lambda t} v_2)\varphi \geq 0.$$

Consequently, it holds for any $\varphi \in \mathfrak{T}_+$

$$\int_0^T \int_\Omega -e^{\lambda t}(v_1 - v_2)\varphi_t + e^{\lambda(p-1)t} < |\nabla v_1|^{p-2}\nabla v_1 - |\nabla v_2|^{p-2}\nabla v_2, \nabla\varphi >$$
$$+ \left(\alpha(x, e^{\lambda t} v_1) - \alpha(x, e^{\lambda t} v_2)\right)\varphi \leq -\int_0^T \int_\Omega \frac{\delta}{2T^2}\varphi.$$

We take $\varphi = (v_1 - v_2)^+ = \max\{v_1 - v_2, 0\}$ as the test function, since it vanishes on the boundary of $\Omega \times \mathbb{R}_T$. Then

$$\int_0^T \int_{\{v_1 > v_2\}} -e^{\lambda t}(v_1 - v_2)(v_1 - v_2)_t + e^{\lambda(p-1)t} < |\nabla v_1|^{p-2}\nabla v_1 - |\nabla v_2|^{p-2}\nabla v_2, \nabla v_1 - \nabla v_2 >$$
$$+ \left(\alpha(x, e^{\lambda t} v_1) - \alpha(x, e^{\lambda t} v_2)\right)(v_1 - v_2) \leq -\frac{\delta}{2T^2}\int_0^T \int_{\{v_1 > v_2\}} (v_1 - v_2).$$

Since

$\{v_1 > v_2\} \subset \Omega \times (0,T)$ due to the facts $v_1 \leq v_2$ on $\partial_p(\Omega \times \mathbb{R}_T)$ and $v_1 \to -\infty$ as $t \uparrow T$, the divergence theorem implies

$$\int_0^T \int_{\{v_1>v_2\}} -e^{\lambda t}(v_1-v_2)(v_1-v_2)_t = \int_0^T \int_{\{v_1>v_2\}} \lambda e^{\lambda t}\frac{1}{2}(v_1-v_2)^2.$$

On the other hand,

$$\left(\alpha(x,e^{\lambda t}v_1)-\alpha(x,e^{\lambda t}v_2)\right)(v_1-v_2) \geq -Ke^{\lambda t}(v_1-v_2)^2 \text{ on } \{v_1>v_2\}.$$

As a consequence, it holds that

$$\int_0^T \int_{\{v_1>v_2\}} \left(\frac{\lambda}{2}-K\right)e^{\lambda t}(v_1-v_2)^2 + e^{\lambda(p-1)t} < |\nabla v_1|^{p-2}\nabla v_1 - |\nabla v_2|^{p-2}\nabla v_2, \nabla v_1-\nabla v_2 >$$
$$\leq -\frac{\delta}{2T^2}\int_0^T \int_{\{v_1>v_2\}} (v_1-v_2).$$

We call into play two elementary inequalities ([L]) associated with the p-Laplacian:

$$< |b|^{p-2}b-|a|^{p-2}a, b-a > \geq (p-1)|b-a|^2\left(1+|b|^2+|a|^2\right)^{\frac{p-2}{2}} \quad (1\leq p\leq 2),$$

and

$$< |b|^{p-2}b-|a|^{p-2}a, b-a > \geq 2^{2-p}|b-a|^p \ \ (p\geq 2) \text{ for any } a,\, b\in\mathbb{R}^n.$$

By applying them with $b=\nabla v_1$ and $a=\nabla v_2$ in the preceding inequalities, we obtain

$$\int_0^T \int_{\{v_1>v_2\}} \left(\frac{\lambda}{2}-K\right)e^{\lambda t}(v_1-v_2)^2 + (p-1)e^{\lambda(p-1)t}\left|\nabla v_1-\nabla v_2\right|^2\left(1+|\nabla v_1|^2+|\nabla v_2|^2\right)^{\frac{p-2}{2}}$$
$$\leq -\frac{\delta}{2T^2}\int_0^T \int_{\{v_1>v_2\}} (v_1-v_2) \quad \text{for } 1<p<2$$

and

$$\int_0^T \int_{\{v_1 > v_2\}} \left(\frac{\lambda}{2} - K\right) e^{\lambda t}(v_1 - v_2)^2 + 2^{2-p} e^{\lambda(p-1)t} |\nabla v_1 - \nabla v_2|^p$$
$$\leq -\frac{\delta}{2T^2} \int_0^T \int_{\{v_1 > v_2\}} (v_1 - v_2) \quad \text{for } p \geq 2.$$

One can easily see in either case the respective inequality is true only if the measure of the set $\{v_1 > v_2\}$ is zero. The proof is complete. □

In the next lemma, we show the strict inequality on the boundary data can be relaxed to a non-strict one.

Lemma 2.7. *For $T > 0$ sufficiently small, if $H_p w_1 \leq 0 \leq H_p w_2$ in the weak sense in $\Omega \times \mathbb{R}_T$ and $w_1 \leq w_2$ on $\partial_p(\Omega \times \mathbb{R}_T)$, then $w_1 \leq w_2$ on $\overline{\Omega \times \mathbb{R}_T}$.*

Proof. For any $\delta > 0$, take $\tilde{\delta} > 0$ such that $\tilde{\delta} \leq \frac{\delta}{4K}$ and define

$$\tilde{w}_1(x,t) = w_1(x,t) - \delta t - \tilde{\delta} \quad (x,t) \in \bar{\Omega} \times \mathbb{R}^n.$$

Then $\tilde{w}_1 < w_1 \leq w_2$ on $\partial_p(\Omega \times \mathbb{R}^n)$, and for any $\varphi \in \mathfrak{T}_+$, the following holds:

$$\int_0^T \int_\Omega -\tilde{w}_1 \varphi_t + < |\nabla \tilde{w}_1|^{p-2} \nabla \tilde{w}_1, \nabla \varphi > + \alpha(x, \tilde{w})\varphi$$
$$= \int_0^T \int_\Omega -w_1 \varphi_t + < |\nabla w_1|^{p-2} \nabla w_1, \nabla \varphi > + \alpha(x, w_1)\varphi$$
$$- \delta\varphi + \Big(\alpha(x, w_1 - \delta t - \tilde{\delta}) - \alpha(x, w_1)\Big)\varphi$$
$$\leq \int_0^T \int_\Omega -\delta\varphi + K\left(\delta t + \tilde{\delta}\right)\varphi \leq \int_0^T \int_\Omega -\delta\varphi + K\left(\delta T + \tilde{\delta}\right)\varphi$$
$$\leq \int_0^T \int_\Omega \left(-\delta + \frac{\delta}{2} + \frac{\delta}{4}\right)\varphi \quad \text{for } T \text{ small}$$
$$= -\frac{\delta}{4} \int_0^T \int_\Omega \varphi.$$

The preceding lemma implies $\tilde{w}_1 \leq w_2$ in $\overline{\Omega \times \mathbb{R}_T}$ for small T and for any small $\delta > 0$, and whence the conclusion of this lemma. □

Now the parabolic comparison principle, Theorem 2.5, follows from the preceding lemma quite easily as shown by the following argument: Let $T_0 > 0$ be any small value of T in the preceding lemma so that the conclusion of the preceding lemma holds. Then $w_1 \leq w_2$ on $\overline{\Omega \times (0, T_0)}$. In particular, $w_1 \leq w_2$ on $\partial_p(\Omega \times (T_0, 2T_0))$. The preceding lemma may be applied again to conclude that $w_1 \leq w_2$ on $\overline{\Omega \times (T_0, 2T_0)}$. And so on. This recursion allows us to conclude that $w_1 \leq w_2$ on $\overline{\Omega \times \mathbb{R}_T}$.

2.6 Convergence of the Evolution

Define $\mathfrak{S}$ to be the set of weak solutions of the stationary problem (2.4). The p-harmonic function u_0 is the maximum element in $\mathfrak{S}$, while u_2 denotes the least solution which may be constructed as the infimum of super-solutions. We also use the term *non-minimal solution* with the same definition in [CW]. That is, u is a non-minimal solution of the problem (2.4) if it is a viscosity solution but not a local minimizer in the sense that for any $\delta > 0$, there exists v in the admissible set of the functional J_p with $v = \sigma$ on $\partial\Omega$ such that $\|v - u\|_{L^\infty} < \delta$, and $J_p(v) < J_p(u)$.

In this section, we consider the evolutionary problem defined in (2.7) and will apply the parabolic comparison principle, Theorem 2.5, proved in Section 2.5 to prove the following convergence of evolution theorem. The reader may just note that the parabolic problem (2.25) includes the above problem (2.7) as a special case so that the comparison principle (2.5) applies in this case.

Theorem 2.8. *If the initial data v_0 falls into any of the categories specified below, the corresponding conclusion of convergence holds.*

1. *If $v_0 \leq u_2$ on $\bar{\Omega}$, then $\lim_{t\to+\infty} w(x,t) = u_2(x)$ locally uniformly for $x \in \bar{\Omega}$;*

2. *Define*

   $$\bar{u}_2(x) = \inf_{u\in\mathfrak{S}, u\geq u_2, u\neq u_2} u(x),\ x \in \bar{\Omega}.$$

   If $\bar{u}_2 > u_2$, then for v_0 such that $u_2 < v_0 < \bar{u}_2$, $\lim_{t\to+\infty} w(x,t) = u_2(x)$ locally uniformly for $x \in \bar{\Omega}$;

3. *Define $\bar{u}_0(x) = \sup_{u\in\mathfrak{S}, u\leq u_0, u\neq u_0} u(x)$, $x \in \bar{\Omega}$. If $\bar{u}_0 < u_0$, then for v_0 such that $\bar{u}_0 < v_0 < u_0$, $\lim_{t\to+\infty} w(x,t) = u_0(x)$ locally uniformly for $x \in \bar{\Omega}$;*

4. *If $v_0 \geq u_0$ in $\bar{\Omega}$, then $\lim_{t\to+\infty} w(x,t) = u_0(x)$ locally uniformly for $x \in \bar{\Omega}$;*

5. *Suppose u_1 is a non-minimal solution of (2.4). For any small $\delta > 0$, there exists v_0 such*

that $\|v_0 - u_1\|_{L^\infty(\Omega)} < \delta$ *and the solution* w *of the problem (2.7) does not satisfy*

$$\lim_{t\to\infty} w(x,t) = u_1(x) \ \textit{ in } \Omega.$$

Proof. We first take care of case 4. We may take new initial data a smooth function $\tilde{v}_0$ so that $D^2\tilde{v}_0 < -KI$ and $|\nabla \tilde{v}_0| \geq \delta > 0$ on $\bar{\Omega}$. According to the parabolic comparison principle (2.5), it suffices to prove the solution $\tilde{w}$ generated by the initial data $\tilde{v}_0$ converges locally uniformly to u_0 if we also take $\tilde{v}_0$ large than v_0, which can easily be done. So we use v_0 and w for the new functions $\tilde{v}_0$ and $\tilde{w}$ without any confusion.

For any $V \subset\subset \Omega$ and any nonnegative function φ which is independent of the time variable t and supported in V, it holds that

$$\begin{aligned}\int_V |\nabla v_0|^{p-2}\nabla v_0 \cdot \nabla\varphi &= \int_V -div\left(|\nabla v_0|^{p-2}\nabla v_0\right)\varphi \\ &\geq \int_V M\varphi \quad \text{for some } M = M(n,p,K,\delta) > 0.\end{aligned}$$

The Hölder continuity of ∇w up to $t = 0$ as stated in [DiB], then implies

$$\int_V |\nabla w|^{p-2}\nabla w \cdot \nabla\varphi \geq \frac{M}{2}\int_V \varphi$$

for any small t in $(0, t_0)$, and any nonnegative function φ which is independent of t, supported in V and subject to the condition

$$\frac{\int_V |\nabla\varphi|}{\int_V \varphi} \leq A \tag{2.26}$$

for a fixed constant $A > 0$ and some $t_0 > 0$ dependent on A. Then the sub-solution condition on w

$$\int_V w\varphi\Big|_{t=t_2} - \int_V w\varphi\Big|_{t=t_1} + \int_{t_1}^{t_2}\int_V |\nabla w|^{p-2}\nabla w \cdot \nabla\varphi \leq 0$$

implies that

$$\int_V w\varphi\Big|_{t=t_2} - \int_V w\varphi\Big|_{t=t_1} \leq -\frac{M}{2}(t_2 - t_1)\int_V \varphi$$

for any small $t_2 > t_1$ in $(0, t_0)$, and any nonnegative function φ which is independent of t, supported in V and subject to (2.26). In particular, $\int_V w\varphi|_{t_1}^{t_2} \leq 0$ for any nonnegative function φ independent of t, supported in V and subject to (2.26). So

$$w(x, t_2) \leq w(x, t_1)$$

for any $x \in \Omega$ and $0 \leq t_1 \leq t_2$. Then the parabolic comparison principle readily implies w is decreasing in t for t in $[0, \infty)$. Therefore $w(x, t) \to u^\infty(x)$ locally uniformly as $t \to \infty$ and hence u^∞ is a solution of (2.4). Furthermore, the parabolic comparison principle also implies $w(x, t) \geq u_0(x)$ at any time $t > 0$. Consequently, $u^\infty = u_0$ as u_0 is the greatest solution of (2.4).

Next, we briefly explain the proof for case 1. We may take a new smooth initial data $\tilde{v}_0$ such that $\tilde{v}_0$ is very large negative, $D^2\tilde{v}_0 \geq KI$ and $|\nabla\tilde{v}_0| \geq \delta$ on $\bar{\Omega}$ for large constant $K > 0$ and constant $\delta > 0$. It suffices to prove the solution $\tilde{w}$ generated by the initial data $\tilde{v}_0$ converges to u_2 locally uniformly on $\bar{\Omega}$ as $t \to \infty$. Following a computation exactly parallel to that in case 4, we can prove w is increasing in t in $[0, \infty)$. So w converges locally uniformly to a solution u^∞ of (2.4). As $u^\infty \leq u_2$ and u_2 is the least solution of (2.4), we conclude $u^\infty = u_2$.

In case 2, we may replace v_0 by a strict super-solution of $\triangle_p v - Q\beta_\varepsilon(v) = 0$ in $\bar{\Omega}$ between u_2 and $\bar{u}_2$, by employing the fact that u_2 is the infimum of super-solutions of (2.4). Using v_0 as the initial data, we obtain a solution $w(x, t)$ of (2.7). Then one argues as in case 4 that for any $V \subset\subset \Omega$, there exist constants $A > 0$ and $t_0 > 0$ such that for $t_1 < t_2$ with t_1, $t_2 \in [0, t_0)$, $\int_V w\varphi\,|_{t_1}^{t_2} \leq 0$ for any nonnegative function φ independent of t, supported in V and subject to the condition $\frac{\int_V |\nabla\varphi|}{\int_V \varphi} \leq A$. As a consequence, $w(x, t_1) \geq w(x, t_2)$ $(x \in \Omega)$. Then the parabolic comparison principle implies w is decreasing in t over $[0, +\infty)$.

Therefore $w(x,t)$ converges locally uniformly to some function u^∞ as $t \to \infty$ which solves (2.4). Clearly $u_2(x) \le w(x,t) \le \bar{u}_2(x)$ from which $u_2(x) \le u^\infty(x) \le \bar{u}_2(x)$ follows. As w is decreasing in t and $v_0 \neq \bar{u}_2$, $u^\infty \neq \bar{u}_2$. Hence $u^\infty = u_2$.

The proof of case 3 is parallel to that of case 2 with the switch of sub- and super-solutions. Hence we skip it.

In case 5, we pick v_0 with $\|v_0 - u_1\|_{L^\infty} < \delta$ and $J_p(v_0) < J_p(u_1)$. Let w be the solution of (2.7) with v_0 as the initial data. Clearly, we may change the value of v_0 slightly if necessary so that it is not a solution of the equation

$$-\nabla \cdot \left(\left(\varepsilon + |\nabla u|^2\right)^{p/2-1} \nabla u \right) + Q(x)\beta(u) = 0$$

for any small $\varepsilon > 0$.

Let w^ε be the smooth solution of the uniformly parabolic boundary-value problem

$$\begin{cases} w_t - \nabla \cdot \left(\left(\varepsilon + |\nabla w|^2\right)^{p/2-1} \nabla w \right) + Q\beta(w) = 0 & \text{in } \Omega \times (0, +\infty) \\ w(x,t) = \sigma(x) & \text{on } \partial\Omega \times (0, +\infty) \\ w(x,0) = v_0(x) & \text{on } \bar{\Omega}. \end{cases}$$

w^ε converges to w in $W^{1,p}(\Omega)$ for every $t \in [0, \infty)$ as $\varepsilon \to 0$.

We define the functional

$$J_{\varepsilon,p}(u) = \frac{1}{p} \int_\Omega \left(\varepsilon + |\nabla u|^2\right)^{p/2} + Q(x)\Gamma(u)\, dx.$$

It is easy to see that

$$\int_0^t \int_\Omega (w_t^\varepsilon)^2 - \nabla \cdot \left(\left(\varepsilon + |\nabla w^\varepsilon|^2\right)^{p/2-1} \nabla w^\varepsilon \right) w_t^\varepsilon + Q\beta(w^\varepsilon) w_t^\varepsilon = 0.$$

As $w_t^\varepsilon = 0$ on $\partial\Omega \times (0, \infty)$, we get

$$\int_0^t \int_\Omega (w_t^\varepsilon)^2 + \left(\varepsilon + |\nabla w^\varepsilon|^2\right)^{p/2-1} \nabla w^\varepsilon \cdot \nabla w_t^\varepsilon + Q(x)\Gamma(w^\varepsilon)_t = 0,$$

which implies

$$\int_0^t \int_\Omega (w_t^\varepsilon)^2 + \frac{1}{p}\left(\left(\varepsilon + |\nabla w^\varepsilon|^2\right)^{p/2}\right)_t + Q(x)\Gamma(w^\varepsilon)_t = 0.$$

Consequently, it holds

$$\begin{aligned} &\int_0^t \int_\Omega (w_t^\varepsilon)^2 + \frac{1}{p}\int_\Omega \left(\varepsilon + |\nabla w^\varepsilon(x,t)|^2\right)^{p/2} + Q\Gamma(w^\varepsilon(x,t)) \\ &= \frac{1}{p}\int_\Omega \left(\varepsilon + |\nabla w^\varepsilon(x,0)|^2\right)^{p/2} + Q\Gamma(w^\varepsilon(x,0)) \end{aligned}$$

i. e.

$$\int_0^t \int_\Omega (w_t^\varepsilon)^2 + J_{\varepsilon,p}(w^\varepsilon(\cdot,t)) = J_{\varepsilon,p}(w^\varepsilon(\cdot,0)).$$

Therefore

$$J_{\varepsilon,p}(w^\varepsilon(\cdot,t) \leq J_{\varepsilon,p}(v_0),$$

which in turn implies

$$J_p(w(\cdot,t) \leq J_p(v_0) < J_p(u_1).$$

In conclusion, w does not converge to u_1 as $t \to \infty$. $\square$

CHAPTER 3 SYMMETRY OF A NONLINEAR ELLIPTIC PROBLEM OVER A RING

3.1 the Moving Plane Method

A widely used tool in proving radial symmetry is the Moving Plane Method first found by Alexandroff and then used by Serrin in [S] and Gidas, Ni, and Nirenberg in [GNN].

in [S], Serrin considers the overdetermined problem:

$$\begin{cases} \Delta u = -1 & \text{in } \Omega \\ u = 0 \,,\ \frac{\partial u}{\partial n} = C & \text{on } \partial\Omega \end{cases} \tag{3.1}$$

Where Ω is a bounded open connected domain with smooth boundary and C is a constant.

Following a Moving Plane Method argument, Serrin proved that if there exists a function $u \in C^2(\bar{\Omega})$ a solution for (3.1) then Ω must be a ball, say $\Omega = B(x_0, R)$ and u has the specific form:

$$u(x) = \frac{R^2 - |x - x_0|^2}{2n}$$

for all x in the domain. Serrin also extend his result to show that if we replace the linear PDE $\Delta u = -1$ by the nonlinear PDE

$$a(u, |\nabla u|)\Delta u + h(u, |\nabla u|)u_{x_i}u_{x_j}u_{x_ix_j} = f(u, |\nabla u|)$$

then the same result still holds.

in [GNN], the authors use the same techniques to study the radial symmetry of positive solutions u of elliptic equations in an open bounded domain Ω satisfying $u \equiv 0$ on $\partial\Omega$. The method in fact studies the monotonicity in every fixed direction and relies heavily on some versions of Hopf's Lemma and the Strong Maximum Principle. Formally, the radial symmetry can be reached by proving, for every fixed direction γ, the monotonicty of a solution in both of the γ and $-\gamma$ directions starting from the boundary of the domain up to the middle of the domain.

The procedure starts with fixing a unit vector direction γ. Then Letting T_λ be the hyperplane $\gamma \cdot x = \lambda$ and λ_0 be the largest value of λ such that T_λ has a non-empty intersection with $\bar{\Omega}$.

For simplicity and WLOG, assume that $\gamma = e_1$. The method moves the hyperplane T_λ in the $-e_1$ direction creating a subset $\Sigma(\lambda) := \bar{\Omega} \cap \{x_1 > \lambda\}$. Denote by $\Sigma^{'}(\lambda)$ the reflection of $\Sigma(\lambda)$ with respect to the hyperplane T_λ and by $x^\lambda \in \Sigma^{'}(\lambda)$ the reflection of $x \in \Sigma(\lambda)$. The hyperplane moves while at the same time proving that

$$u_{x_1} < 0 \text{ and } u(x) < u(x^\lambda) \text{ in } \Sigma(\lambda) \tag{3.2}$$

The hyperplane stops moving at a value μ where we cannot reflect in the domain Ω anymore establishing the inequlaity (3.2) for $\lambda = \mu$.

In addition, the method also implies that if $u_{x_1} = 0$ at some point on $\Omega \cap T_\mu$ then necessarely u is symmetric in the plane T_μ and the domain Ω is symmetric in T_μ.

This process relies heavily on two main tools:

First of all, to enable the start of the movement of the hyperplane T_λ from its original position T_{λ_0}, the authors prove Hopf's lemma for the PDE $\Delta u = f(u)$.

Second, to keep moving the hyperplane all the way up to the maximum reflection, the authors make use of the Hopf's lemma and the Strong Maximum Principle for the Linearized PDE: $-\Delta v(x) + c(x)v(x) = 0$.

3.1.1 Monotonicity on Some Special Domains

Monotonicity over a Ball Clearly, The result above implies in particular that if $u > 0$ is $C^2(\bar{\Omega})$ solution of the boundary value problem:

$$\begin{cases} \Delta u = f(u) & \text{in } B(0,R) \\ u = 0 & \text{on } \partial B(0,R) \end{cases} \tag{3.3}$$

then, u must be monotonically radially symmetric.

Here $B(0,R)$ is the ball centered at 0 and of radius R.

Symmetry over a Ring Now lets consider a similar problem but this time over a ring:

$$\begin{cases} \Delta u = f(u) & \text{in } B(0,R) \\ u = 0 & \text{on } \partial B(0,R) \\ u = 1 & \text{on } \partial B(0,R) \end{cases} \tag{3.4}$$

Given a positive function u in $C^2(\bar{\Omega})$ solution for (3.3), for a fixed direction x_1, the Moving Plane Method stop in the middle of the ring achieving monotonicity only in the first half of the ring.

Therefore, the standard Moving Plane Method argument does not apply in the ring case and a new version of it is needed.

Moreover, one does not expect a positive solution for (3.3) to be monotone. This is because there is no uniqueness of a solution for (3.3), which can easily be seen. For example, suppose λ is an eigenvalue of $(-\Delta)$ with an eigenfunction w on the region $\Omega = B_R \backslash \bar{B}_r$. That is

$$\begin{cases} \Delta w &= -\lambda w & \text{in } \Omega \\ w &= 0 & \text{on } \partial B_r \cup \partial B_R \end{cases}$$

If u is a solution of the Dirichlet problem

$$\begin{cases} \Delta u &= -\lambda u & \text{in } \Omega \\ u &= 1 & \text{on } \partial B_r \\ u &= -1 & \text{on } \partial B_R, \end{cases}$$

so is $u + w$. This does not happen for the primary eigenvalue but occurs for other eigenvalues according to the classical Courant's nodal set theorem.

The non-uniqness implies that the Comparison Principle does not hold in general. This opens the possibility for a solution u to attain its maximum and/or minimum inside the ring which implies non-monotonicity. When applies, the standard moving plane method proves that this situation cannot happen and we indeed have radial monotonicity. However, as we mentioned earlier, in ring like domain, the standard moving plane method does not work and non-monotonicity is possible in this case.

Existing results of symmetry or asymptotic symmetry of a solution over a ring-like domain depends on the assumption that the right-hand-side f is non-decreasing. The reader may refer to [HPP], [HP] and the references therein. Since we do not assume the monotonicity of f, our method as well as results are new in the study of radial symmetry of a solution and may be applied in a broader scope in studying symmetry problems.

3.2 Symmetry and Approximate Symmetry Without Monotonicity

Let Ω be the domain between two concentric spheres $|x| = 1$ and $|x| = R$ for some large radius R. Assume $u \in C^2(\bar{\Omega})$ is a solution of the boundary value problem

$$\begin{cases} \Delta u = f(u) & \text{in } \Omega = B_R \backslash \bar{B}_1, \\ u = 1 & \text{on } |x| = 1, \\ u = -1 & \text{on } |x| = R. \end{cases} \tag{3.5}$$

The function $f\colon \mathbb{R}^+ \to \mathbb{R}$ is a C^1 function satisfying $f(s) \leq 0$. We study the radial symmetry of a solution of this boundary value problem.

In section 3.3, we prove the radial symmetry of a solution of the boundary value problem (3.5) under a not-too-negative condition on f'.

As we mentioned in section (3.1.1), the non-uniqueness, and therefore the lack of a Comparison Principle opens the possibility for a solution of (3.5) to be non-monotone.

For this reason, to prove the radial symmetry of a solution for (3.5) we play the trick of adding to u a dominating radially symmetric function ϕ such that the resulting sum function $\tilde{u} := u + \phi$ is expected to attain both of its maximum and minimum on the boundaries of our ring domain. To prove that $\tilde{u}$ is monotonically radially symmetric we had to write a new version of the moving plane method argument. This is because the standard version stops in the middle of the ring and hence does not reach the expected radial symmetry of $\tilde{u}$. Finally, this result immediately implies that u is is also radially symmetric.

Moreover, in section 3.4, we consider the problem (3.5) when the bigger sphere shifts its center a little from the origin

$$\begin{cases} \Delta u = f(u) & \text{in } \Omega = B_R(Z) \backslash \bar{B}_1, \\ u = 1 & \text{on } |x| = 1, \\ u = -1 & \text{on } |x - Z| = R, \end{cases} \tag{3.6}$$

where $|Z| = \delta$ is small. The boundary of the positive set $\mathcal{F} := \partial \{u > 0\}$ in each problem is the *free boundary* of a solution u.

We prove the approximate radial symmetry of the free boundary of a solution of problem (3.6).

In order to prove the approximate symmetry of a solution when the domain is shifted from a ring, we are, in a sense, forced to employ a technique of using evolutionary limits to bound the solution. The reason is the lack of an elliptic comparison principle and the uniqueness of a solution as stated above, and meanwhile we come to realize the validity of a parabolic comparison principle. We have not seen such an approach in the literature except the joint work [CW] of one of the authors with Luis A. Caffarelli, in which the authors use a similar evolutionary view to examine the stability of a solution of an elliptic free boundary problem. Construction of the evolutionary limits depends on an existence theorem of a solution for the corresponding parabolic initial-boundary-value problem and locally uniform convergence of the evolution. In proving the existence theorem for an evolution, we are helped with an iteration rather than the widely used Perron's method, since the solution produced from that method may not be regular enough. This evolutionary approach to a problem in a steady state seems promising to us in application in the study of other PDE or free boundary problems.

The main results of this chapter are the following two theorems regarding to problems

(3.5) and (3.6).

Theorem 3.1. *Let* $R > 1$ *and* $\Omega = B_R \backslash \bar{B}_1$ *be the domain of a ring or shell. Suppose* $f\colon \mathbb{R}_+ \to \mathbb{R}$ *is a* C^1 *function such that* $f(s) \leq 0$ *and* $\inf_{\mathbb{R}_+} f'(s) > -\frac{4(n+2)}{R^2}$.

Then a solution $u \in C^2(\overline{B_R \backslash B_1})$ *of (3.5) is radially symmetric in the sense* $u(x) = u(y)$ *if* $x, y \in \Omega$ *with* $|x| = |y|$.

The definition of a *stable solution* in the statement of the second theorem is given in Definition 3.9.

Theorem 3.2. *Suppose* $R > 1$, *and* $f\colon \mathbb{R}_+ \to \mathbb{R}$ *is a* C^1 *function such that* $f(s) \leq 0$ *and* $\inf_{\mathbb{R}_+} f'(s) > -\frac{2(n+2)}{R^2}$. *Let* $u \in C^2(\overline{B_R(Z) \backslash B_1})$ *be a stable solutions of (3.6) with free boundary* $\mathcal{F}$, *where* $|Z| = \delta$.

Then there exists a constant $\delta_0 > 0$ *such that for every constant* δ *in* $0 < \delta \leq \delta_0$, *there is a solution* $u_0 \in C^2(\overline{B_R \backslash \bar{B}_1})$ *of (3.5) with free boundary* $\mathcal{F}_0$ *so that*

$$|u(x) - u_0(x)| \leq C\delta \text{ in } (B_R(Z) \cap B_R) \backslash B_1, \text{ and}$$

$$dist(\mathcal{F}, \mathcal{F}_0) < C|Z| = C\delta$$

for a constant $C = C(n, R, \inf f')$ *which is independent of* δ. *The latter estimate, in other words, states that the free boundary* $\mathcal{F}$ *is in the shell between two concentric spheres of thickness* $2C\delta$, *as Theorem 3.1 implies* $\mathcal{F}_0$ *is a sphere. In particular, the free boundary* $\mathcal{F}$ *keeps a positive distance from the boundary of the domain* $\partial\Omega$.

3.3 Symmetry over a Ring for Problem (3.5)

In this section, one considers the following boundary value problem.

$$\begin{cases} \Delta u = f(u) & \text{in } 1 \leq |x| \leq R \\ u = 1 & \text{on } |x| = 1 \\ u = 0 & \text{on } |x| = R \end{cases} \tag{3.7}$$

One assumes R is large, $u \in C^2(\overline{B_R \backslash B_1})$, and $f\colon \mathbb{R}_+ \to \mathbb{R}$ is a C^1 function such that $f(s) \leq 0$ and $\inf_{\mathbb{R}_+} f'(s) > -\frac{2(n+2)}{R^2}$. Let $\Omega = B_R \backslash \bar{B}_1$ be the domain of a ring or shell. We note the non-essential difference in the boundary value of a solution between the problems 3.5 and 3.7.

The goal of this section is to prove Theorem 3.1 which is equivalent to the following theorem.

Theorem 3.3. *Let $R > 1$ and $\Omega = B_R \backslash \bar{B}_1$ be the domain of a ring or shell. Suppose $f\colon \mathbb{R}_+ \to \mathbb{R}$ is a C^1 function such that $f(s) \leq 0$ and $\inf_{\mathbb{R}_+} f'(s) > -\frac{2(n+2)}{R^2}$.*

Then a solution $u \in C^2(\overline{B_R \backslash B_1})$ of (3.7) is radially symmetric in the sense $u(x) = u(y)$ if x, $y \in \Omega$ with $|x| = |y|$.

Firstly, one constructs an auxiliary dominating radially symmetric function. For any number $A > 0$, an alternating sequence $\{a_k\}_{k=0}^{\infty}$ is defined recursively by

$$a_0 > 0, \quad a_{k+1} = -\frac{A a_k}{2(n+2k)(k+1)}.$$

One defines an analytic function ϕ on $\mathbb{R}$ by a power series

$$\phi(s) = \sum_{k=0}^{\infty} a_k s^{2k},$$

which is obviously uniformly convergent on any bounded subset of $\mathbb{R}$. A direct computation shows that

$$\phi''(s) + \frac{n-1}{s}\phi'(s) = -A\phi(s) \ \ (s \in \mathbb{R}),$$

which implies

$$\Delta\phi(|x|) = -A\phi(|x|) \ \ (x \in \mathbb{R}^n \setminus \{0\}).$$

In addition,

$$\begin{aligned} \phi'(s) &= \sum_{j=1}^{\infty} 2(2j-1)a_{2j-1}\left(1 - \frac{As^2}{2(n+4j-2)(2j-1)}\right)s^{4j-3} \\ &< 0 \quad \text{if} \ \ s < \sqrt{\frac{2(n+2)}{A}} \end{aligned}$$

Moreover, if one requires

$$-\inf_{\mathbb{R}_+} f'(s) < A < \frac{2(n+2)}{R^2},$$

then for $s \leq R$ it holds

$$\begin{aligned} \phi'(s) &\leq 2a_1\left(1 - \frac{As^2}{2(n+2)}\right) \\ &\leq -\frac{Aa_0}{n}\left(1 - \frac{AR^2}{2(n+2)}\right) \end{aligned}$$

We will apply the well-known moving plane method which plays the key role in [S] and [GNN] to the function

$$\tilde{u}(x) = u(x) + C\phi(|x|) \tag{3.8}$$

in Ω for positive constants A and C. We pick the value of C so that

$$C \geq \frac{n}{Aa_0\left(1 - \frac{AR^2}{2(n+2)}\right)} \sup_{\Omega} |\nabla u(x)| \,.$$

Then $\tilde{u}_r(x) \leq 0$ for all $x \in \Omega$, i. e. $\tilde{u}$ is radially decreasing.

For any domain $\mathcal{D}$ in consideration, $\nu(x_0)$ denotes the outer unit normal to $\partial\mathcal{D}$ at a point $x_0 \in \partial\mathcal{D}$.

In order to prove u is radially symmetric in Ω, it suffices to prove $\tilde{u}$ is radially symmetric in the ring Ω, which is equivalent to that $\tilde{u}$ is symmetric in every hyperplane through the origin. Without loss of generality, one takes the direction $\nu = e_1$ and starts to prove $\tilde{u}$ is symmetric in the hyperplane $x_1 = 0$.

For the sake of completeness of this work, we include here the version of Hopf's lemma and Strong Maximum Principle that we will use in the proof.

Theorem 3.4. ***Hopf's Lemma***

Suppose $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ is a solution of the differential inequality

$$\Delta u(x) + c(x)u(x) \geq 0$$

in Ω, where $c \in C(\Omega)$. Assume further $u(x) < 0$ in Ω, $x_0 \in \partial\Omega$ such that $u(x_0) = 0$, and there is a ball $B \subset \Omega$ that touches $\partial\Omega$ at x_0.

Then

$$u_\nu(x_0) > 0$$

for the unit outer normal ν at x_0 to $\partial\Omega$.

For a proof of the Hopf's lemma, the reader may refer to [E] for the case $c(x) \leq 0$ and [GNN] for the case $c(x) > 0$.

Theorem 3.5. ***Strong Maximum Principle***

Suppose Ω is connected and $u \in C^2(\Omega)$ is a solution of the differential inequality

$$\Delta u(x) + c(x)u(x) \geq 0$$

in Ω, where $c \in C(\Omega)$, and $u(x) \leq 0$ in Ω.

If $u(x_0) = 0$ at a point x_0 in Ω, then $u(x) \equiv 0$ in Ω.

For any $\lambda \geq 0$, let T_λ be the hyperplane $x_1 = \lambda$, $x^\lambda = (2\lambda - x_1, x_2, \ldots, x_n)$ be the mirror image of $x = (x_1, x_2, \ldots, x_n)$ in T_λ, $\Sigma(\lambda) = \Omega \cap \{x \colon x_1 > \lambda\}$, $\Pi(\lambda) = \big\{x \in \Sigma(\lambda) \colon x^\lambda \in \Omega\big\}$,

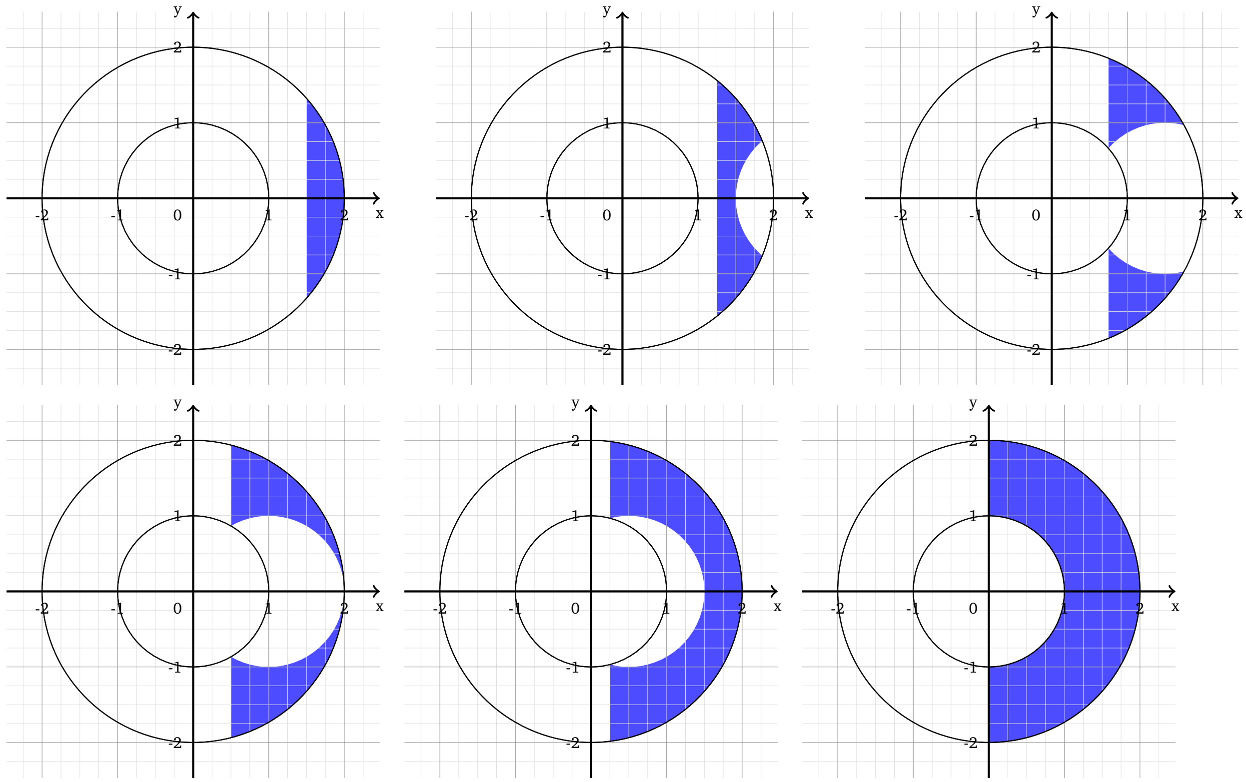

Figure 1: $\Pi(\lambda)$ for $R = 2$, $\lambda = 1.5, 1.25, 1, 0.75, 0.5, 0.25, 0$, respectively

$\Sigma'(\lambda)$ the reflection of $\Sigma(\lambda)$ in T_λ, and $\Pi'(\lambda) = \Sigma'(\lambda) \cap \Omega$ the reflection of $\Pi(\lambda)$ in T_λ. Figure 1 provides some snapshots of the domain $\Pi(\lambda)$, shaded in blue, during the motion of the hyperplane at different values of λ when the outer radius of the ring $R = 2$.

If one notices that u is super-harmonic in Ω and attains its minimum on the sphere $|x| = R$, it is obvious the following lemma is true.

Lemma 3.6. *Suppose* $x_0 \in \partial B_R$ *with* $\nu_1(x_0) > 0$.

Then there exists $\delta > 0$ *such that*

$$u_{x_1} < 0 \text{ and hence } \tilde{u}_{x_1} < 0$$

in $\Omega \cap \{x\colon |x - x_0| < \delta\}$.

The next lemma allows one to move the hyperplane T_λ for $\lambda > 0$ in the negative x_1-axis direction.

Lemma 3.7. *Fix some* λ *in* $0 \le \lambda < R$. *Assume*

$$\tilde{u}_{x_1}(x) \le 0 \quad \textit{in } \Sigma(\lambda) \quad \textit{and } \tilde{u}(x) \le \tilde{u}(x^\lambda) \;\; \textit{in } \Pi(\lambda),$$

but $\tilde{u}(x) \not\equiv \tilde{u}(x^\lambda)$ *in* $\Pi(\lambda)$.

Then $\tilde{u}(x) < \tilde{u}(x^\lambda)$ *in* $\Pi(\lambda)$ *and* $\tilde{u}_{x_1}(x) < 0$ *on* $\Omega \cap T_\lambda$.

Proof. On $\overline{\Pi'(\lambda)}$, one defines the functions

$$v(x) = u(x^\lambda), \quad \tilde{v}(x) = \tilde{u}(x^\lambda) = u(x^\lambda) + C\phi(|x^\lambda|),$$
$$\text{and} \;\; h(x) = C\phi(|x^\lambda|) - C\phi(|x|) \le 0.$$

Define $w(x) = \tilde{v}(x) - \tilde{u}(x)$ on $\overline{\Pi'(\lambda)}$. Then $w(x) \le 0$ in $\Pi'(\lambda)$ and w satisfies

$$\Delta w + c(x)w = -\int_0^1 f'((1-t)u + tv)\, dt\, h + \Delta h$$

for

$$c(x) = -\int_0^1 f'((1-t)u + tv)\, dt$$

which is a continuous function on Ω, due to the equality

$$\begin{aligned}\Delta(v - u + h) &= f(v) - f(u) + \Delta h \\ &= \int_0^1 f'((1-t)u + tv)\, dt\, (v - u) + \Delta h.\end{aligned}$$

As a consequence,

$$\begin{aligned}\Delta w + c(x)w &\ge -\inf_{\mathbb{R}} f'(s)\, h + \Delta h \\ &\ge Ah + \Delta h \\ &= 0\end{aligned}$$

as

$$\Delta h(x) = \Delta\left(C\phi(|x^\lambda|)\right) - \Delta\left(C\phi(|x|)\right) = -AC\phi(|x^\lambda|) + AC\phi(|x|)$$
$$= -Ah(x).$$

Notice that $w(x) = 0$ on $T_\lambda \cap \bar{\Omega}$ and $w(x) \leq 0$ elsewhere on $\partial\Pi'(\lambda)$. Then the Strong Maximum Principle implies $w < 0$ in $\Pi'(\lambda)$, and the Hopf's Lemma implies $w_{x_1}(x) > 0$ on $T_\lambda \cap \Omega$. These mean

$$\tilde{v}(x) < \tilde{u}(x) \text{ in } \Pi'(\lambda), \text{ or equivalently } \tilde{u}(x) < \tilde{u}(x^\lambda) \text{ in } \Pi(\lambda)$$

and $\tilde{u}_{x_1}(x) < 0$ on $\Omega \cap T_\lambda$, since $w_{x_1}(x) = -\tilde{u}_{x_1}(x^\lambda) - \tilde{u}_{x_1}(x) = -2\tilde{u}_{x_1}(x)$ on $T_\lambda \cap \Omega$. □

The main Theorem (3.3) follows from the following theorem by considering all possible directions along which a hyperplane is moved.

Theorem 3.8. *For any λ in $0 < \lambda < R$, it holds that*

$$\tilde{u}_{x_1}(x) < 0 \text{ in } \Sigma(\lambda) \text{ and } \tilde{u}(x) < \tilde{u}(x^\lambda) \text{ in } \Pi(\lambda). \tag{3.9}$$

In particular, $\tilde{u}_{x_1}(x) < 0$ in $\Omega \cap \{x_1 > 0\}$.

Consequently, $\tilde{u}(x)$ is symmetric with respect to the hyperplane $x_1 = 0$.

Proof. We define the set $\mathcal{A}$ as

$$\mathcal{A} = \left\{\lambda \in (0, R)\colon \tilde{u}_{x_1}(x) < 0 \text{ in } \Sigma(\lambda) \text{ and } \tilde{u}(x) < \tilde{u}(x^\lambda) \text{ in } \Pi(\lambda)\right\}.$$

Firstly, one notices that Lemma 3.6 implies there exists some λ close to R in $0 < \lambda < R$ which is in $\mathcal{A}$.

Let $\mu = \inf \mathcal{A}$. Since (3.9) holds for all $\lambda > \mu$, we have by continuity that

$$\tilde{u}_{x_1}(x) < 0 \text{ in } \Sigma(\mu) \text{ and } \tilde{u}(x) \leq \tilde{u}(x^\lambda) \text{ in } \Pi(\mu).$$

We claim that $\mu = 0$.

Suppose $\mu > 0$. For any $x_0 \in (\partial B_R \cap \{x_1 > \mu\})$ such that $x_0^\mu \in \Omega$, it holds that $-1 + \phi(R) = \min_{\bar{\Omega}} \tilde{u} = \tilde{u}(x_0) < \tilde{u}(x_0^\mu)$. So $\tilde{u}(x) \not\equiv \tilde{u}(x^\lambda)$ in $\Pi(\mu)$. Lemma 3.7 then implies

$$\tilde{u}(x) < \tilde{u}(x^\mu) \text{ in } \Pi(\mu) \text{ and } \tilde{u}_{x_1}(x) < 0 \text{ on } \Omega \cap T_\mu.$$

That is, (3.9) holds for $\lambda = \mu$.

At every point $x_0 \in \partial\Omega \cap T_\mu$, Lemma 3.6 states there is a $\varepsilon > 0$ such that

$$\tilde{u}_{x_1} < 0 \text{ in } \Omega \cap \{|x - x_0| < \varepsilon\},$$

as T_μ is not perpendicular to $\partial\Omega$. Here one notices that the situation when $|x_0| = 1$ is parallel to that in Lemma 3.6 and a similar conclusion holds. Since $\partial\Omega \cap T_\mu$ is compact, there is an $\varepsilon > 0$ such that

$$\tilde{u}_{x_1} < 0 \text{ in } \Omega \cap \{x_1 > \mu - \varepsilon\} \cap N_\varepsilon(\partial\Omega \cap T_\mu),$$

where $N_\varepsilon(S)$ denotes the ε-neighborhood of a set $S \in \mathbb{R}^n$. On the other hand, since $\tilde{u}_{x_1} < 0$ on $\Omega \cap T_\mu$, one gets by continuity of $\tilde{u}_{x_1}$ that

$$\tilde{u}_{x_1} < 0 \text{ in } \Omega \cap \{x_1 > \mu - \varepsilon\} \setminus N_\varepsilon(\partial\Omega \cap T_\mu)$$

so long as the value of ε is taken smaller if necessary. In all, for this $\varepsilon > 0$,

$$\tilde{u}_{x_1} < 0 \text{ in } \Omega \cap \{x_1 > \mu - \varepsilon\}. \tag{3.10}$$

As $\mu = \inf \mathcal{A}$, $\exists \{\lambda^j\}$ such that $0 < \lambda^j < \mu$ and

$$\exists x_j \in \Pi(\lambda^j) \text{ such that } \tilde{u}(x_j) \geq \tilde{u}(x_j^{\lambda^j}) \text{ for every } j.$$

Without loss of generality, we assume $x_j \to \tilde{x}$ for some $\tilde{x} \in \overline{\Pi(\mu)}$. Clearly $x_j^{\lambda^j} \to \tilde{x}^\mu$ and hence $\tilde{u}(\tilde{x}) \geq \tilde{u}(\tilde{x}^\mu)$. Since (3.9) holds for $\lambda = \mu$, we must have $\tilde{x} \in \partial\Pi(\mu)$. There are four possibilities, $|\tilde{x}| = 1$, $|\tilde{x}| = R$, $\tilde{x} \in T_\mu \cap \Omega$, and $x \in (\partial\Pi(\mu) \backslash T_\mu) \cap \Omega$. One first notes that it is impossible that $|\tilde{x}| = 1$ but $\tilde{x} \notin T_\mu$, since otherwise $\left|(x^j)^\mu\right| < 1$ holds for sufficiently large j due to $\mu > 0$. If $|\tilde{x}| = R$, then $\tilde{x}^\mu \in \Omega$ or $|\tilde{x}^\mu| = 1$, and since $\tilde{u}$ is radially decreasing,

$$\tilde{u}(\tilde{x}) = \min_{\bar{\Omega}} \tilde{u} < \tilde{u}(\tilde{x}^\mu), \quad \text{which is a contradiction.}$$

Similarly, we get a contradiction when $\tilde{x} \in (\partial\Pi(\mu) \backslash T_\mu) \cap \Omega$, since, in this case, $|\tilde{x}^\mu| = 1$ and the fact $\tilde{u}$ is radially decreasing imply

$$\tilde{u}(\tilde{x}) < \max_{\bar{\Omega}} \tilde{u} = \tilde{u}(\tilde{x}^\mu).$$

Therefore $\tilde{x} \in T_\mu \cap \bar{\Omega}$ and $\tilde{x}^\mu = \tilde{x}$. On the other hand, for large j, the segment $[x_j, x_j^{\lambda^j}] \subset \Omega$ and therefore $\exists y_j \in [x_j, x_j^{\lambda^j}]$ such that $u_{x_1}(y_j) \geq 0$ according to the Mean Value Theorem. Since $y_j \to \tilde{x}$, we get $u_{x_1}(\tilde{x}) \geq 0$ which is in contradiction to (3.10).

Thus $\mu = 0$ and (3.9) holds for all λ in $0 < \lambda < R$. By continuity, it holds that $\tilde{u}_{x_1}(x) \leq 0$ and $\tilde{u}(x) \leq \tilde{u}(x^0)$ in $\Sigma(0)$, where x^0 is the reflection of x in the hyperplane $x_1 = 0$.

If one moves the hyperplane along the positive x_1-axis direction from the other side of the ring Ω, the above argument shows that $\tilde{u}(x) \geq \tilde{u}(x^0)$ and hence $\tilde{u}$ and therefore u are symmetric about the hyperplane $x_1 = 0$. $\square$

The main theorem 3.3 of this section follows readily from the preceding theorem.

3.4 Stability of the Free Boundary for Problem (3.6)

This section is devoted to the proof of Theorem 3.2. Let $\Omega = B_R(Z)\backslash\bar{B}_1$ be a slight deformation of the ring $B_R\backslash\bar{B}_1$ with $|Z| = \delta > 0$ being sufficiently small. Now one considers the following boundary value problem.

$$\begin{cases} \Delta u = f(u) & \text{in } \Omega \\ u = 1 & \text{on } |x| = 1 \\ u = -1 & \text{on } |x - Z| = R \end{cases} \tag{3.11}$$

One assumes $R > 1$, $u \in C^2(\overline{\Omega})$, and $f : \mathbb{R} \to \mathbb{R}$ is a C^3 function such that $f(s) \le 0$, $f(s) = 0$ if $s \le 0$, and $\inf_{\mathbb{R}_+} f'(s) > -\frac{2(n+2)}{R^2}$. We consider only the stability of the free boundaries of what we call *stable solutions* in a strong sense defined below.

Definition 3.9. *A solution u of (3.11) is **stable** if for any $\varepsilon > 0$, there exist functions v_1 and v_2 in $C^2(\bar{\Omega})$ that satisfy*

$$u - \varepsilon \le v_1 \le u \le v_2 \le u + \varepsilon \ \ \textit{on } \bar{\Omega}, \tag{3.12}$$

$$-\Delta v_1 + f(v_1) < -\varepsilon \ \ \textit{and} \ \ -\Delta v_2 + f(v_2) > \varepsilon \ \ \textit{in } \Omega\textit{, simultaneously.} \tag{3.13}$$

Remark 3.10. *When the domain is a ring and $f(s) \equiv 0$, it is easy to construct the sub- and super-solutions v_1 and v_2. One may readily perturb the domain to a ring-like one such as Ω and construct corresponding sub- and super-solutions over Ω that satisfy the requirements in the above definition. The reader is referred to the following proof for detailed computation.*

In other words, a stable solution u is a uniform supremum of strict subsolutions and a uniform infimum of strict supersolutions. Compared to the concentric case when $Z = 0$, the center of the exterior sphere drifts away from the origin a bit. Our goal in this section is to prove in this situation the free boundary of u drifts away from its original position also by a bit. In mathematical terms, we are to prove the stability of the free boundary. We will also give an estimate of the drift of the free boundary. However, for this seemingly clear fact, we need to prove it through a delicate evolution with quite a few technicalities.

The reason we go through this quite troublesome process lies in the observation there is no comparison principle and hence no uniqueness for the elliptic problem when the nonlinear term $f(u)$ is negative. Nevertheless, there is a comparison principle for the corresponding evolution. Meanwhile, the reader may have realized that the practical reason why we study this problem on approximate radial symmetry has already been mentioned in the introduction.

We first state the parabolic comparison principle which is needed in the coming proof. Consider the initial-boundary value problem

$$\begin{cases} Hw := w_t - \Delta w + \alpha(x,w) = 0 & \text{in } \Omega \times (0,\infty) \\ w(x,t) = \sigma(x,t) \ \text{ on } \partial\Omega \times (0,\infty), & w(x,0) = v_0(x) \ \text{ for } x \in \bar{\Omega} \end{cases} \tag{3.14}$$

where α is a C^1 function that satisfies the condition $0 \le \alpha(x,w) \le Cw$, and Ω is a bounded domain with smooth boundary. This problem includes two important cases that we will apply the comparison principle to, the case when $\alpha = f(w)$ and the other when $\alpha = f'(w)z$ where z is one of the first order derivatives of w.

Theorem 3.11. *Suppose two functions w_1 and w_2 satisfy $Hw_1 \le 0 \le Hw_2$ in the viscosity sense as continuous functions or in the weak sense as H^1-functions in $\Omega \times \mathbb{R}^+$ and $w_1 \le w_2$ on the parabolic boundary $\partial_p(\Omega \times \mathbb{R}^+)$. Then $w_1 \le w_2$ in $\Omega \times \mathbb{R}^+$. Here $\mathbb{R}^+ = (0,\infty)$.*

Proof. The proof is done with the introduction of the new functions

$$\tilde{w}_j(x,t) = e^{-\lambda t}\left(w_j(x,t) - \frac{\delta}{T-t}\right), \quad j = 1,2,$$

for any fixed small $T > 0$ and some large constant λ, cf. Theorem 3.1 [CW] and Lemma 6.3 [LW]. □

Now let B_{R_1} be the largest ball inscribed in $B_R(Z)$ with the origin as its center and B_{R_2} be the smallest ball circumscribing $B_R(Z)$ with the origin as its center. Also, let $\mathcal{R} = B_R \backslash \bar{B}_1$

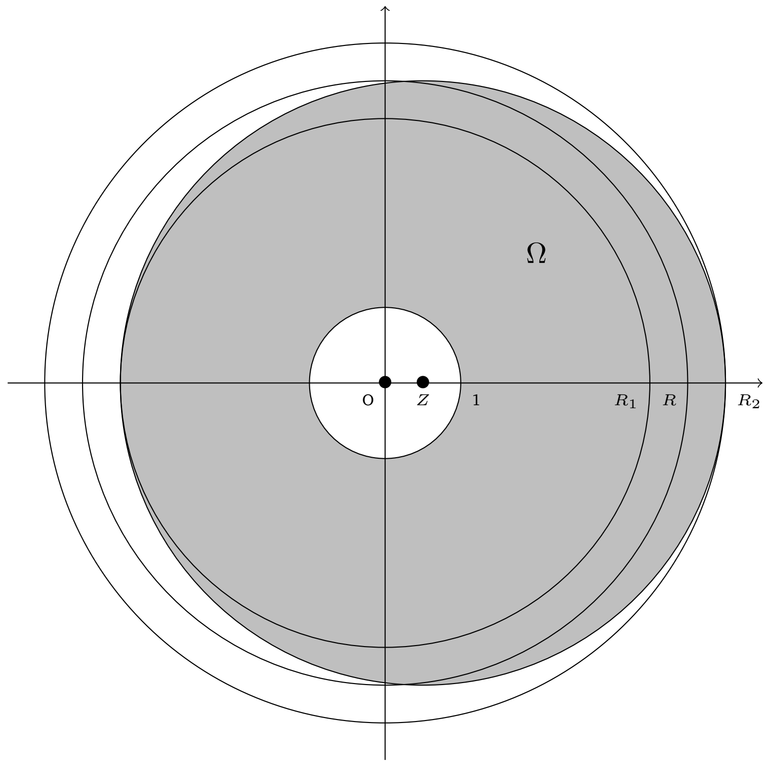

Figure 2: The spheres B_1, B_{R_1}, $B_R(Z)$, B_R, and B_{R_2} for $\delta = 0.5$ and $R = 4$

be a concentric ring, $\Omega_1 = B_{R_1} \setminus \bar{B}_1$ and $\Omega_2 = B_{R_2} \setminus \bar{B}_1$. Figure 2 illustrates the two-dimensional sections of these spheres and the domain Ω as shaded in gray.

Let u be a stable solution of the free boundary problem (3.11). Fix a small number $\varepsilon = K\delta$ for a relative large universal constant $K > 0$ in (3.12) and (3.13). Let v_1 and v_2 be as in the definition of the stable solution u in Ω. It is not difficult to see that, in accordance with the definition of v_1 and v_2, $v_1 < u < v_2$ on $\partial\Omega$.

In the following, we will construct a function v_{01} (resp. v_{00} and v_{02}) a strict subsolution (resp. strict supersolutions) of our problem on the perfect ring Ω_1 (resp.$\mathcal{R}$ and Ω_2) such that

$$\begin{aligned} &u - C\delta \leq v_{01} \leq u \text{ in } \Omega_1, \text{ and, } u \leq v_{02} \leq u + C\delta \text{ in } \Omega \\ &v_{01} \leq v_{00} \text{ in } \Omega_1 \text{ and, } v_{00} \leq v_{02} \text{ in } \mathcal{R} \end{aligned} \tag{3.15}$$

for a constant C.

Then we will use v_{01} (resp. v_{00} and v_{02}) as initial data of the parabolic version of our problem on $\Omega_1 \times (0, \infty)$ (resp. $\mathcal{R} \times (0, \infty)$ and $\Omega_2 \times (0, \infty)$) to construct solutions of the respective evolution.

Finally, we prove convergence of the evolution with each initial data to a steady state which gives desired solutions u_1, u_0, and u_2 of the elliptic problems on Ω_1, $\mathcal{R}$, and Ω_2. The solutions u_1 and u_2 will give the lower and upper bounds for the solution u of (3.6), while u_0 will be a radially symmetric approximation of u. In particular, the free boundary of u_0 is an approximation of that of u.

3.4.1 Construction of a solution of our problem on the perfect ring Ω_1

Construction of a strict subharmonic function in Ω satisfying the boundary conditions associated with our problem One takes $\phi_0 \colon \mathcal{R} \to \mathbb{R}$ defined by

$$\phi_0(x) = Ae^{\lambda|x|} + B \ \ (1 \leq |x| \leq R)$$

where the constants $\lambda < 0$, $A > 0$ and B satisfy the conditions

$$\begin{cases} Ae^{\lambda} + B = 1 \\ Ae^{\lambda R} + B = -1 \end{cases}$$

Then for a suitable value of $\lambda < 0$, it holds that

$$\begin{aligned} -\Delta\phi_0 + f(\phi_0) \leq -\Delta\phi_0 = -A\left(\lambda^2 + \lambda\frac{n-1}{|x|}\right)e^{\lambda|x|} = -\frac{2e^{\lambda|x|}}{e^{\lambda} - e^{\lambda R}}\left(\lambda^2 + \lambda\frac{n-1}{|x|}\right) \\ \leq -\frac{2e^{\lambda R}}{e^{\lambda} - e^{\lambda R}}\left(\lambda^2 + \lambda\frac{n-1}{|x|}\right) = -\mu < -2\varepsilon \end{aligned}$$

in $\mathcal{R}$ for a constant $\mu > 0$, $\phi_0 = -1$ on ∂B_R, and $\phi_0 = 1$ on ∂B_1, if we take δ_0 such that $0 < \delta_0 = K^{-1}\varepsilon < \frac{1}{2K}\mu$.

Let $\tilde{\phi}$ denote the translation of ϕ_0 to the ring $B_R(Z)\backslash\bar{B}_1(Z)$. That is $\tilde{\phi}$ satisfies

$$-\Delta\tilde{\phi} + f(\tilde{\phi}) < -\mu$$

in $B_R(Z)\backslash\bar{B}_1(Z)$ for the constant $\mu > 2\varepsilon$, $\tilde{\phi} = -1$ on $\partial B_R(Z)$, and $\tilde{\phi} = 1$ on $\partial B_1(Z)$.

Now, for each x in $\Omega = B_R(Z)\backslash\bar{B}_1$, define $\tilde{x} = \tau(x)$ in $B_R(Z)\backslash\bar{B}_1(Z)$ in the following way. Write $e = \frac{x}{|x|}$. If

$$x = (1-\lambda)e + \lambda q \ \ (0 \leq \lambda \leq 1)$$

where q is the point of intersection of the ray from the origin in the direction of e with the sphere $\partial B_R(Z)$, then

$$\tilde{x} = \tau(x) = (1-\lambda)(Z+e) + \lambda p = Z + (1-\lambda)e + \lambda Re,$$

where p is the point of intersection of the ray from the point Z in the direction of e with the sphere $\partial B_R(Z)$. Clearly, the mapping $x \mapsto \tilde{x}$ is a one-to-one function from Ω onto $B_R(Z)\backslash\bar{B}_1(Z)$. Suppose $q = te$. Then from $|q - Z| = R$ one can get

$$t = \sigma(x) := \sqrt{\delta^2\mu^2 + (R^2 - \delta^2)} - \delta\mu,$$

where $\mu = e \cdot e_1 = x_1/|x|$, and consequently

$$\lambda = \frac{|x| - 1}{t - 1}.$$

Hence

$$\tilde{x} = \tau(x) = -\delta e_1 + \left(\frac{t - |x|}{t-1} + \frac{|x|-1}{t-1}R\right)e.$$

Finally we define the function $\phi\colon \Omega \to \mathbb{R}$ by

$$\phi(x) = \tilde{\phi}(\tilde{x}).$$

We claim that ϕ satisfies the conditions

$$-\Delta\phi + f(\phi) < -\varepsilon$$

in Ω, $\phi = -1$ on $\partial B_R(Z)$, and $\phi = 1$ on ∂B_1. In fact, the boundary conditions are obvious. As for the differential inequality, one first writes $\tau = (\tau^1, \tau^2, \ldots, \tau^n)$. Then

$$\phi_{x_i} = \tilde{\phi}_{\tilde{x}_k} \tau^k_{x_i}$$

and

$$\phi_{x_i x_j} = \tilde{\phi}_{\tilde{x}_k \tilde{x}_l} \tau^k_{x_i} \tau^l_{x_j} + \tilde{\phi}_{\tilde{x}_k} \tau^k_{x_i x_j}.$$

Here and in the following the summation convention is adopted. Consequently

$$\phi_{x_i x_i} = \tilde{\phi}_{\tilde{x}_k \tilde{x}_l} \tau^k_{x_i} \tau^l_{x_i} + \tilde{\phi}_{\tilde{x}_k} \tau^k_{x_i x_i}$$

and hence

$$-\Delta\phi = - < D^2\tilde{\phi}\tau_{x_i}, \tau_{x_i} > -\tilde{\phi}_{\tilde{x}_k} \Delta\tau^k.$$

Decompose τ as

$$\tau(x) = x + \psi(x), \quad \text{where} \quad \psi(x) = \tau(x) - x.$$

Then

$$\psi(x) = \tilde{x} - x = -\delta e_1 + \frac{|x| - 1}{t - 1}\,(R - t)\, e.$$

For any fixed $x \in \Omega$, it is clear that

$$R - t = \sigma(0) - \sigma(\delta) = -\sigma'(\zeta)\delta$$

for some $\zeta \in (0, \delta)$, and hence

$$|R - t| \leq 2\delta$$

as

$$|\sigma'(\zeta)| = \left| \frac{\delta\mu^2 - \delta}{\sqrt{\delta^2\mu^2 + (R^2 - \delta^2)}} - \mu \right| \leq 2$$

for sufficiently small δ. Moreover, one readily gets

$$\mu_{x_i} = \frac{\delta_{1i}}{|x|} - \frac{x^1 x^i}{|x|^3}$$

and

$$t_{x_i} = \sigma_{x_i} = \left(\frac{\delta\mu}{\sqrt{\delta^2\mu^2 + (R^2 - \delta^2)}} - 1 \right) \delta\mu_{x_i},$$

from which one also gets

$$\mu_{x_i x_i} = -2\delta_{1i} \frac{x^i}{|x|^3} - \frac{x^1}{|x|^3} + 3\frac{x^1 (x^i)^2}{|x|^5}$$

and

$$t_{x_i x_i} = \frac{R^2 - \delta^2}{\left(\delta^2\mu^2 + (R^2 - \delta^2)\right)^2} \delta^2 \mu_{x_i}^2 + \left(\frac{\delta\mu}{\sqrt{\delta^2\mu^2 + (R^2 - \delta^2)}} - 1 \right) \delta\mu_{x_i x_i}.$$

Clearly,

$$|\mu_{x_i}| \leq \frac{C}{|x|} \leq C \ \text{ in } \ \Omega,$$

and hence

$$|t_{x_i}| \leq C\delta \ \text{ in } \ \Omega.$$

Now

$$\psi_{x_i} = \beta_{x_i} (R - t)\, e - \beta t_{x_i} e + \beta (R - t)\, e_{x_i}, \tag{3.16}$$

where $\beta = (|x| - 1) / (t - 1)$. Evidently $\beta \in [0, 1]$ is bounded, and

$$|\beta_{x_i}| = \left| \frac{\frac{x_i}{|x|}(t-1) - (|x|-1)t_{x_i}}{(t-1)^2} \right| \leq \frac{1}{t-1} + \frac{|x|-1}{(t-1)^2} C\delta \leq C$$

in Ω. In addition, that

$$e_{x_i} = \frac{1}{|x|}e_i - \frac{x_i}{|x|^2}e$$

implies $|e_{x_i}| \leq C$ in $|x| \geq 1$. Then one deduces from (3.16) that

$$|\psi_{x_i}| \leq C\delta \text{ in } \Omega.$$

Next, one readily gets

$$\psi_{x_ix_i} = \beta_{x_ix_i}(R-t)\,e - \beta t_{x_ix_i}e + \beta(R-t)\,e_{x_ix_i} - 2\left(\beta_{x_i}t_{x_i}e + \beta t_{x_i}e_{x_i} - \beta_{x_i}(R-t)\,e_{x_i}\right) \quad (3.17)$$

It is clear from the formula of $\mu_{x_ix_i}$ that it is bounded on Ω, which helps to imply from the formula of $t_{x_ix_i}$ that $|t_{x_ix_i}| \leq C\delta$ on Ω. Meanwhile, one may compute the formula of $\beta_{x_ix_i}$:

$$\beta_{x_ix_i} = \left(\frac{1}{|x|} - \frac{x_i^2}{|x|^3}\right)\frac{1}{t-1} - \frac{x_i}{|x|}\frac{t_{x_i}}{(t-1)^2} - \frac{x_i}{|x|}\frac{1}{(t-1)^2} + 2\frac{(|x|-1)}{(t-1)^3}t_{x_i}^2 - \frac{|x|-1}{(t-1)^2}t_{x_ix_i}.$$

This formula shows that $|\beta_{x_ix_i}| \leq C$ in Ω on account of the estimates on t_{x_i} and $t_{x_ix_i}$. Similarly, one gets the formula of $e_{x_ix_i}$

$$e_{x_ix_i} = -\frac{2x_i}{|x|^3}e_i - \frac{1}{|x|^2}e + 2\frac{x_i^2}{|x|^4}e$$

and deduce from which that $|e_{x_ix_i}| \leq C$ in Ω. Then the formula (3.17) readily implies $|\psi_{x_ix_i}| \leq C\delta$ on account of the estimates on $R-t$, β, e, β_{x_i}, t_{x_i}, e_{x_i}, $\beta_{x_ix_i}$, $t_{x_ix_i}$ and $e_{x_ix_i}$, which in turn implies $\left|\Delta\psi^k\right| \leq C\delta$ for each $k = 1, \ldots, n$. Computation based on the definition of $\tilde{\phi}$ that

$$\tilde{\phi}(x) = Ae^{\lambda|x+\delta e_1|} + B$$

and the formulas that determine the values of A, B and λ helps one to conclude that $\tilde{\phi}_{x_k}$ and $\tilde{\phi}_{x_kx_l}$ are bounded on $\overline{B_R(Z)}\backslash B_1(Z)$. Combining all the preceding estimates, one concludes that

$$
\begin{aligned}
-\Delta\phi + f(\phi) &= -\Delta\tilde{\phi} + f(\tilde{\phi}) - 2\sum_i < D^2\tilde{\phi}e_i, \psi_{x_i} > - \sum_i < D^2\tilde{\phi}\psi_{x_i}, \psi_{x_i} > -\tilde{\phi}_{x_k}\Delta\psi^k \\
&< -\mu - 2\sum_i < D^2\tilde{\phi}e_i, \psi_{x_i} > - \sum_i < D^2\tilde{\phi}\psi_{x_i}, \psi_{x_i} > -\tilde{\phi}_{x_k}\Delta\psi^k \\
&< -\mu + C\delta \\
&< -\frac{1}{2}\mu, \text{ if we take } K > 2C \\
&< -\varepsilon
\end{aligned}
$$

for all $\delta \leq \delta_0$. So the claim is proved.

Construction of a strict subsolution of $\Delta u = f(u)$ on Ω satisfying the boundary conditions associated with our problem and the condition $u - \varepsilon \leq v_1 \leq u$ on Ω First replace the subsolution v_1 by $\omega_1 := v_1 - C_1\delta$, where $C_1 > \frac{4R}{R-\delta_0}\sup|\nabla u|$. The new function ω_1 satisfies the following conditions:

$$
\begin{cases}
u - (\varepsilon + C_1\delta) \leq \omega_1 \leq u - C_1\delta \text{ in } \Omega \\
-\Delta\omega_1 + f(\omega_1) < -(\varepsilon - C_0C_1\delta) < 0 \text{ in } \Omega \\
\omega_1 < -1 \text{ on } \partial B_R(Z), \text{ and } \omega_1 < 1 \text{ on } \partial B_1
\end{cases}
$$

for $C_0 = -\inf_{\mathbb{R}} f'(s) > 0$, if K is sufficiently large.

If one checks carefully our proof in the preceding subsection, it is proved that $-\Delta\tilde{\phi} < -\mu$ and $-\Delta\phi < -\varepsilon$. Then on $\partial\Omega$, $u = \phi$, and in Ω

$$\Delta(u - \phi) = f(u) - \Delta\phi \leq f(u) - \varepsilon \leq 0.$$

Then the Minimum Principle for super-harmonic functions implies that $u \geq \phi$ on $\bar{\Omega}$.

We are in a position to replace the sub-solution ω_1 by $\tilde{v}_1 := \max\{\omega_1, \phi\}$ which is also a sub-solution of the problem. Moreover, $\tilde{v}_1$ takes constant values on the exterior and interior spheres respectively. Without any possible confusion, we simply write v_1 for $\tilde{v}_1$

in the following. Since v_1 differs from ϕ on a precompact set, we may mollify it near the boundary of the set. The mollified function v_1 verifies $v_1 \in C^2(\bar{\Omega})$,

$$\begin{cases} u - (\varepsilon + 2C_1\delta) \leq v_1 \leq u - \frac{C_1}{2}\delta \ \text{ in } \Omega \\ -\Delta v_1 + f(v_1) < -(\varepsilon - 2C_0C_1\delta) < 0 \ \text{ in } \Omega \\ v_1 = -1 \ \text{ on } \partial B_R(Z), \ \text{ and } \ v_1 = 1 \ \text{ on } \partial B_1 \end{cases}$$

provided K is sufficiently large.

Construction of a function v_{01} strict subsolution of $\Delta u = f(u)$ on Ω_1 satisfying the boundary conditions associated with our problem and the condition $u - \varepsilon \leq v_{01} \leq u$ on Ω_1 We are ready to define a function $v_0 := v_{01} \in C^2(\overline{\Omega}_1)$ that satisfies

$$\begin{cases} u - C\delta < v_0 \leq u \quad \text{in } \Omega_1 \\ -\Delta v_0 + f(v_0) < 0 \ \text{ in } \Omega_1 \\ v_0 = -1 \ \text{ on } \partial B_{R_1} \text{ and } \ v_0 = 1 \ \text{ on } \ \partial B_1 \end{cases} \tag{3.18}$$

as the initial data for the evolution based on the strict sub-solutionv_1, where we use and will use in the following v_0 for v_{01} to avoid the use of disturbing double subscripts.

For $x \in \overline{\Omega}_1$, if one can write it as

$$x = (1 - \lambda)e + \lambda q,$$

where $e = x/|x|$ and $q = R_1 e$ is the point of intersection of the ray from the origin in the direction of e with the sphere ∂B_{R_1}, then one defines

$$x^* = (1 - \lambda)e + \lambda p,$$

where p is the point of intersection of the ray from the origin in the direction of e with the sphere $\partial B_R(Z)$. Clearly, the mapping $x \mapsto x^*$ is a bijection from $\overline{B_{R_1}} \backslash B_1$ onto $\overline{B_R(Z)} \backslash B_1$.

Write $p = te$ for $t > 0$. The condition $|p + \delta e_1| = R$ implies that

$$t = \sigma(x) := \sqrt{\delta^2\mu^2 + (R^2 - \delta^2)} - \delta\mu.$$

Also, we know $\lambda = \frac{|x|-1}{R_1-1}$. So

$$x^* = \varphi(x) := \left(\frac{R_1 - |x|}{R_1 - 1} + \frac{|x| - 1}{R_1 - 1} t \right) e.$$

Set in Ω_1

$$\psi(x) = \varphi(x) - x = x^* - x = \frac{|x| - 1}{R_1 - 1} (t - R_1) e.$$

We introduce the notation

$$\beta(x) = \frac{|x| - 1}{R_1 - 1}.$$

Then

$$\psi(x) = \beta(x) (\sigma(x) - R_1) e.$$

Now one can define

$$v_0(x) = v_1(x^*) \quad (x \in \overline{\Omega}_1)$$

we claim that v_0 satisfies the conditions (3.18).

The regularity and boundary conditions are evident.

To see that $u - C\delta < v_0 \le u$ in Ω_1, we write

$$v_0(x) - u(x) = v_1(x^*) - u(x) = \big(v_1(x^*) - u(x^*)\big) - \big(u(x) - u(x^*)\big),$$

which implies

$$v_0(x) - u(x) \le -\frac{C_1}{2}\delta + \sup |\nabla u||x - x^*| \le -\frac{C_1}{2}\delta + \frac{2\delta R}{R_1} \sup |\nabla u| < 0 \tag{3.19}$$

and

$$v_0(x) - u(x) \geq -\varepsilon - 4C_1\delta = -\left(K - 4C_1\right)\delta. \tag{3.20}$$

Here we note that the global gradient estimate of u implies $\sup|\nabla u|$ is controlled by n, R, and f.

Finally, we verify the differential inequality.

Obviously β and e are bounded. The term

$$\begin{aligned}
&\sigma(x) - R_1 \\
&= \sqrt{\delta^2\mu^2 + (R^2 - \delta^2)} - \delta\mu - R_1 \\
&= \sqrt{\delta^2\mu^2 + (R^2 - \delta^2)} - \delta\mu - (R - \delta) \\
&=: \tau(\delta)
\end{aligned}$$

for any fixed $x \in \Omega_1$. As $\tau(0) = 0$ and

$$|\tau'(\delta)| = \left|\frac{\mu^2\delta - 2\delta}{\sqrt{\delta^2\mu^2 + (R^2 - \delta^2)}} - \mu + 1\right| \leq C,$$

one concludes

$$|\sigma - R| \leq C\delta.$$

One easily gets

$$\psi_{x_i} = \beta_{x_i}\left(\sigma - R_1\right)e + \beta\sigma_{x_i}e + \beta\left(\sigma - R_1\right)e_{x_i}$$

and

$$\psi_{x_ix_i} = \beta_{x_ix_i}\left(\sigma - R_1\right)e + \beta\sigma_{x_ix_i}e + \beta\left(\sigma - R_1\right)e_{x_ix_i} + 2\left(\beta_{x_i}\sigma_{x_i}e + \beta_{x_i}\left(\sigma - R_1\right)e_{x_i} + \beta\sigma_{x_i}e_{x_i}\right). \tag{3.21}$$

Set $\mu(x) = e \cdot e_1 = \frac{x^1}{|x|}$. Then

$$\mu_{x_i} = \frac{\delta_{1i}}{|x|} - \frac{x^1x^i}{|x|^3}$$

and

$$\sigma_{x_i} = \frac{\delta^2 \mu \mu_{x_i}}{\sqrt{\delta^2\mu^2 + (R^2-\delta^2)}} - \delta\mu_{x_i} = \left(\frac{\delta\mu}{\sqrt{\delta^2\mu^2 + (R^2-\delta^2)}} - 1 \right) \delta\mu_{x_i}.$$

Also

$$\beta_{x_i} = -\frac{1}{R_1 - 1}\frac{x^i}{|x|}$$
$$\text{and } \; e_{x_i} = \frac{1}{|x|}e^i - \frac{x^i}{|x|^2}e.$$

As $|\mu_{x_i}| \le C$ in Ω_1, it holds $|\sigma_{x_i}| \le C\delta$ in Ω_1. Also one observes $|\beta_{x_i}| \le C$ and $|e_{x_i}| \le C$ in Ω_1. Consequently, it holds

$$|\psi_{x_i}(x)| \le C\delta \quad (x \in \Omega_1).$$

Further computation shows that

$$\beta_{x_i x_i} = -\frac{1}{(R_1-1)|x|} + \frac{x_i^2}{(R_1-1)|x|^3}$$

and

$$e_{x_i x_i} = -\frac{2x^i}{|x|^3}e_i - \frac{1}{|x|^3}x + \frac{3x_i^2}{|x|^5}x,$$

which imply that

$$|\beta_{x_i x_i}|, \quad |e_{x_i x_i}| \le C$$

in Ω_1. By computing

$$\mu_{x_i x_i} = -2\frac{\delta_{1i}x^i}{|x|^3} - \frac{x^1}{|x|^3} + 3\frac{x^1(x^i)^2}{|x|^5},$$

and

$$\sigma_{x_i x_i} = \left(\frac{\delta\mu}{\sqrt{\delta^2\mu^2 + (R^2-\delta^2)}} - 1 \right) \delta\mu_{x_i x_i} + \frac{R^2-\delta^2}{\left(\sqrt{\delta^2\mu^2 + (R^2-\delta^2)}\right)^3}\delta^2\mu_{x_i}^2,$$

one concludes $|\mu_{x_i x_i}| \le C$ in Ω_1 and hence

$$|\sigma_{x_i x_i}(x)| \le C\delta \quad (x \in \Omega_1).$$

The above estimates and the formula (3.21) of $\psi_{x_i x_i}$ imply that

$$|\psi_{x_i x_i}(x)| \leq C\delta \quad (x \in \Omega_1)$$

Since

$$v_{0,x_i} = v_{1,x_k^*}\varphi^k_{x_i}$$

and

$$v_{0,x_i x_i} = \sum_{k,l} v_{1,x_k^* x_l^*}\varphi^k_{x_i}\varphi^l_{x_i} + \sum_k v_{1,x_k^*}\varphi^k_{x_i x_i},$$

one gets

$$-\Delta v_0 = - < D^2 v_1 \varphi_{x_i}, \varphi_{x_i} > - \sum_k v_{1,x_k^*}\Delta\varphi^k.$$

As $\varphi_{x_i} = e_i + \psi_{x_i}$, one further gets from the above formula

$$-\Delta v_0 = -\Delta v_1 - 2 < D^2 v_1 e_i, \psi_{x_i} > - < D^2 v_1 \psi_{x_i}, \psi_{x_i} > - \sum_k v_{1,x_k^*}\Delta\psi^k.$$

So

$$\begin{aligned} -\Delta v_0 + f(v_0) &= -\Delta v_1 + f(v_1) - 2 < D^2 v_1 e_i, \psi_{x_i} > - < D^2 v_1 \psi_{x_i}, \psi_{x_i} > - \sum_k v_{1,x_k^*}\Delta\psi^k \\ &< -(\varepsilon - 2C_0C_1\delta) + C\delta + C\delta \\ &< -C\delta, \quad \text{for a new constant } C \text{ if } K \text{ is sufficiently large.} \\ &< 0 \end{aligned} \tag{3.22}$$

for all $\delta \leq \delta_0$, on account of the estimates on e_i, ψ_{x_i} and $\psi_{x_i x_i}$.

The inequalities in (3.19), (3.20) and (3.22) yield to the desired result (3.18).

Construction of $w_1(x,t)$ a solution of the parabolic version of our problem on $\Omega_1 \times (0,\infty)$

Using v_0 as the initial data, we are going to solve the following initial-boundary-value problem

$$\begin{cases} w_t - \Delta w + f(w) = 0 \text{ in } \Omega_1 \times (0, \infty) \\ w(x,t) = -1 \text{ on } \partial B_{R_1} \times (0,\infty), \quad w(x,t) = 1 \text{ on } \partial B_1 \times (0,\infty) \\ w(x,0) = v_0(x) \text{ for } x \in \overline{\Omega_1} \end{cases} \tag{3.23}$$

For convenience, one sets $\mathcal{D}_1 := \Omega_1 \times (0,\infty)$ and let $\partial_p \mathcal{D}_1$ be its parabolic boundary.

Lemma 3.12. *There is a solution w_1 of the evolution (3.23).*

Proof. We prove an existence theorem for the following initial-boundary-value problem rewritten from (3.23).

$$\begin{cases} w_t - \Delta w + f(w) = 0 & \text{in } \mathcal{D}_1 \\ w(x,t) = v_0(x) & \text{on } \partial_p \mathcal{D}_1, \end{cases} \tag{3.24}$$

where $v_0 \in C(\partial_p \mathcal{D}_1)$ is described as before. As f is not proper in the sense it is not a nondecreasing function, one may introduce a function $v(x,t) = e^{-\lambda t} w(x,t)$ in $\mathcal{D}_1$ for a large constant $\lambda >> \frac{2(n+2)}{R^2}$. The function w is a solution of (3.24) if and only if the new function v is a solution of the initial-boundary-value problem

$$\begin{cases} v_t - \Delta v + g(t,v) = 0 & \text{in } \mathcal{D}_1 \\ v(x,t) = -e^{-\lambda t} \text{ on } \partial B_{R_1} \times (0,\infty), \quad v(x,t) = e^{-\lambda t} \text{ on } \partial B_1 \times (0,\infty) & \\ v(x,0) = v_0(x) & \text{on } \bar{\Omega}_1, \end{cases}$$

where $g(t,v) = \lambda v + e^{-\lambda t} f(e^{\lambda t} v)$ is a C^3 function that is proper, namely g is increasing in v. In addition, $g(t,0) = 0$ for any t. For simplicity of notation, one may set $\sigma(t)$ be the lateral boundary data of v. Writing w for v in the above problem, we are to prove the existence of a solution of the initial-boundary-value problem

$$\begin{cases} w_t - \Delta w + g(t,w) = 0 & \text{in } \mathcal{D}_1 \\ w(x,t) = \sigma(t) & \text{on } (\partial B_{R_1} \cup \partial B_1) \times (0,\infty) \\ w(x,0) = v_0(x) & \text{on } \bar{\Omega}_1, \end{cases} \tag{3.25}$$

The solution of this problem should be well-known. However, as we have not found a proof of the exact problem in the literature, we outline a proof for the reader's convenience. Our proof is different from the usual Perron's method used to attack the existence problem for an elliptic or parabolic equation. Rather, we employed an iterative process to finish the game.

One first picks a function $w^0 \in C^2(\bar{\mathcal{D}}_1)$ and proceeds to solve the initial-boundary-value problem

$$\begin{cases} w^1_t - \Delta w^1 + g(t, w^0) = 0 & \text{in } \mathcal{D}_1 \\ w^1(x,t) = \sigma(t) & \text{on } (\partial\Omega_1) \times (0, \infty) \\ w^1(x,0) = v_0(x) & \text{on } \bar{\Omega}_1, \end{cases} \tag{3.26}$$

for the unknown function w^1. This problem can be solved first on the cylinder $\mathcal{D}_{2T} := \Omega_1 \times (0, 2T]$ for a small T:

$$\begin{cases} w^1_t - \Delta w^1 + g(t, w^0) = 0 & \text{in } \mathcal{D}_{2T} \\ w^1(x,t) = \sigma(t) & \text{on } (\partial\Omega_1) \times (0, 2T] \\ w^1(x,0) = v_0(x) & \text{on } \bar{\Omega}_1, \end{cases}$$

One then proceeds solving the problem on the cylinder $\Omega_1 \times [T, 3T]$ with the proper initial-boundary data. The parabolic comparison principle then implies the solutions obtained on the cylinders $\mathcal{D}_{2T}$ and $\Omega_1 \times (T, 3T]$ coincide on the overlapping part of the two cylinders. And one moves on to the cylinders $\Omega_1 \times (2T, 4T]$, $\Omega \times (3T, 5T]$, etc. In the end, one finds a unique solution $w \in C^2(\mathcal{D}_1)$ of (3.26) which is C^2 up to the vertical boundary. In order to show w is C^1 down to the bottom $\Omega_1 \times \{t = 0\}$, one just differentiates the equation with respect to t to find that $v := w_t$ verifies the conditions

$$\begin{cases} v_t - \Delta v + g_t(t, w^0) + g_w(t, w^0) w^0_t = 0 & \text{in } \mathcal{D}_1 \\ v(x,t) = \sigma'(t) & \text{on } \partial\Omega_1 \times (0, \infty) \\ v(x,0) = \Delta v_0(x) - g(0, w^0(x,0)) & \text{on } \bar{\Omega}_1, \end{cases}$$

from which the classical regularity theory of linear equations shows v is continuous down to the bottom. Next, employing the same scheme, one may proceed to solve for each $k = 1, 2, ...$ the initial-boundary-value problem

$$\begin{cases} w_t^{k+1} - \Delta w^{k+1} + g(t, w^k) = 0 & \text{in } \mathcal{D}_1 \\ w^{k+1}(x,t) = \sigma(t) & \text{on } (\partial\Omega_1) \times (0, \infty) \\ w^{k+1}(x,0) = v_0(x) & \text{on } \bar{\Omega}_1. \end{cases}$$

The functions w^k are C^2 up to the lateral sides, and w_t is continuous down to the bottom.

Let $v^k = w^{k+1} - w^k$. Then v^k solves the initial-boundary-value problem

$$\begin{cases} v_t^k - \Delta v^k + g(t, w^k) - g(t, w^{k-1}) = 0 & \text{in } \mathcal{D}_1 \\ v^k = 0 & \text{on } \partial_p \mathcal{D}_1, \end{cases}$$

or equivalently,

$$\begin{cases} v_t^k - \Delta v^k + \tilde{g}(t,x) v^{k-1} = 0 & \text{in } \mathcal{D}_1 \\ v^k = 0 & \text{on } \partial_p \mathcal{D}_1, \end{cases} \tag{3.27}$$

where $\tilde{g}(t,x) = \int_0^1 g_w(t, (1-\mu)w^{k-1} + \mu w^k)\, d\mu$.

From here, one easily gets

$$\int_{\Omega_1} \frac{1}{2} \left(v^k(x,T)\right)^2 + \int_0^T \int_{\Omega_1} \left|\nabla v^k\right|^2 = -\int_0^T \int_{\Omega_1} \tilde{g}(t,x) v^k v^{k-1}, \tag{3.28}$$

which implies

$$\frac{1}{2} \int_{\Omega_1} \left(v^k(x,T)\right)^2 dx \leq \left(\int_0^T \int_{\Omega_1} \frac{1}{2} \left(v^k\right)^2 dx\, dt \int_0^T \int_{\Omega_1} 2\tilde{g}^2 \left(v^{k-1}\right)^2 dx\, dt \right)^{1/2}$$

The latter inequality leads to the estimates

$$\int_0^T \int_{\Omega_1} \frac{1}{2} \left(v^k\right)^2 \leq CT^2 \int_0^T \int_{\Omega_1} \frac{1}{2} \left(v^{k-1}\right)^2,$$

and hence

$$\int_0^T \int_{\Omega_1} \frac{1}{2} \left(v^k\right)^2 \le \lambda \int_0^T \int_{\Omega_1} \frac{1}{2} \left(v^{k-1}\right)^2$$

for some $\lambda \in [0, 1)$ if T is small enough. The inequality (3.28) also gives

$$\begin{aligned}\int_0^T \int_{\Omega_1} \left|\nabla v^k\right|^2 &\le - \int_0^T \int_{\Omega_1} \tilde{g}(t,x) v^k v^{k-1} \\ &\le \left(\int_0^T \int_{\Omega_1} \frac{1}{2} \left(v^k\right)^2 \int_0^T \int_{\Omega_1} \tilde{g}^2 \left(v^{k-1}\right)^2 \right)^{1/2} \\ &\le \lambda \int_0^T \int_{\Omega_1} \frac{1}{2} \left(v^{k-1}\right)^2,\end{aligned}$$

if one takes the value of T smaller and a new value of $\lambda \in [0, 1)$ if necessary. So $\left\{w^k\right\}$ is a Cauchy sequence with respect to the norm

$$\|w^k\|_2 = \left(\int_0^T \int_{\Omega_1} \left(w^k\right)^2 + \left|\nabla w^k\right|^2 \right)^{1/2}.$$

The equation (3.27) then implies the boundedness of w_t in the operator norm $\|w_t\|$. As a consequence, a subsequence of $\left\{w^k\right\}$, which we will also denote by $\left\{w^k\right\}$, converges to a certain w^∞ in the norm $\|\cdot\|_2$, and the time derivatives $\left\{w_t^k\right\}$ converges weakly to w_t^∞. Hence w^∞ is a weak solution of (3.25) on $\bar{\Omega}_1 \times [0, T]$. Repeating this process on the time intervals $[\frac{T}{2}, \frac{3T}{2}]$, $[T, 2T]$, $[\frac{3T}{2}, \frac{5T}{2}]$, ..., and employing the parabolic comparison principle, one can find a solution of (3.25) in $\mathcal{D}_1$. The classical regularity theory then implies $w^\infty \in C^2(\mathcal{D}_1) \cap C(\bar{\mathcal{D}}_1)$ ([LSU], [CLW], etc). In fact, w^∞ is C^2 up to the vertical lateral boundary. Moreover, as we did before, one can see $v := w_t^\infty$ solves the linear problem

$$\begin{cases} v_t - \Delta v + g_t(t, w^\infty) + g_w(t, w^\infty) v = 0 & \text{in } \mathcal{D}_1 \\ v(x,t) = \sigma'(t) & \text{on } \partial\Omega_1 \times (0, \infty) \\ v(x, 0) = \Delta v_0(x) - g(0, w^\infty(x, 0)) & \text{on } \bar{\Omega}_1, \end{cases}$$

Then $w_t^\infty = v$ is continuous down to the bottom. We set $w_1 = e^{\lambda t} w^\infty$, and this is the solution we started to obtain. The proof is complete. □

Convergence of the evolution to a steady state We prove the convergence of the evolution (3.23) to a steady state.

Lemma 3.13.

$$\lim_{t\to\infty} w_1(x,t) = u_1(x)$$

locally uniformly on $\bar{\Omega}_1$ *for some function* u_1*. As a consequence,* u_1 *solves the boundary value problem*

$$\begin{cases} \Delta u = f(u) & \textit{in } 1 \le |x| \le R_1 \\ u = 1 & \textit{on } |x| = 1 \\ u = -1 & \textit{on } |x| = R_1 \end{cases}$$

and satisfies

$$u(x) - C\delta \le u_1(x) \le u(x) \;\; \textit{in } \Omega_1.$$

Proof. Set $z(x,t) = w_{1,t}(x,t)$ on $\bar{\mathcal{D}}_1$. Then z solves the linear initial-boundary-value problem

$$\begin{cases} z_t - \Delta z + f'(w_1)z = 0 & \text{in } \mathcal{D}_1 \\ z(x,t) = 0 & \text{on } \partial\Omega_1 \times (0,\infty) \\ z(x,0) = \Delta v_0(x) - f(v_0) & \text{on } \bar{\Omega}_1, \end{cases}$$

Notice that $z \ge 0$ on $\partial_p \mathcal{D}_1$. As $v(x,t) \equiv 0$ is a sub-solution of the above problem with zero initial-boundary data, the parabolic comparison principle implies $z \ge 0$ on $\bar{\mathcal{D}}_1$. Since u is a solution of the evolutionary equation

$$u_t - \Delta u + f(u) = 0$$

in $\mathcal{D}_1$ and $u \ge w_1$ on $\partial_p \mathcal{D}_1$, we conclude $w_1(x,t) \le u(x)$ for all $x \in \bar{\Omega}_1$ and $t \ge 0$. Therefore

$$\lim_{t\to+\infty} w_1(x,t) = u_1(x) \le u(x)$$

monotonically for some function u_1 on $\bar{\Omega}_1$. According to either Theorem 3 in [C1] or Theorem 1 in [C2], it holds that

$$\|\nabla w_1\|_{L^\infty(\Omega' \times (0,\infty))} \leq C\left(\|v_0\|_{L^\infty(\bar{\Omega})}, \Omega'\right).$$

for any subdomain $\Omega' \subset\subset \Omega_1$. Therefore $w_1(x,t)$ converges to u_1 as $t \to +\infty$ locally uniformly on $\bar{\Omega}_1$. The proof is complete, if one further notices the boundary value of $w_1(x,t)$ is independent of t, and the monotonicity of w_1 in t along with the fact the initial data v_0 satisfies the inequality

$$u(x) - C\delta \leq v_0(x) \leq u(x)$$

in Ω_1. □

Lemma 3.14. $u_1 \in C^2(\bar{\Omega}_1)$.

Proof. In the preceding proof, we pointed out that $w_1 \in C^2(\bar{\Omega}_1 \times (0,\infty))$. As a consequence, u_1 is Lipschitz continuous up to the boundary $\partial\Omega_1$. The classical theory of the Possion's equation (e. g. [GT]) implies u_1 is C^2 up to the boundary. □

3.4.2 Construction of a solution on the perfect rings $\mathcal{R}$ and Ω_2 respectively

Following the same steps we can construct u_0 and u_2 solutions of our problem on $\mathcal{R}$ and Ω_2 respectively. we outline the construction of the initial data v_{00} and v_{02}.

1. Construct a strict superharmonic function in Ω satisfying the boundary conditions associated with our problem

2. Construct a strict supersolution v_2 of $\Delta u = f(u)$ on Ω satisfying the boundary conditions associated with our problem and the condition $v_2 - C\delta \leq u \leq v_2$ on Ω

3. Construct a strict supersolution v_{02} of $\Delta u = f(u)$ on Ω_2 satisfying the boundary conditions associated with our problem and the condition $v_{02} - C\delta \leq u \leq v_{02}$ on Ω_2

4. Extend v_{02} to $\mathcal{R}$ such that $v_{02} \equiv -1$ on $\mathcal{R} \setminus \Omega_1$. The construction of v_{00} is similar.

The remaining of the argument concerning the existence of a solution of the parabolic problems and the convergence of the evolution, as well as the proof of the above steps, are similar to the ones in the previous subsection. For this reason, we omit the details and just state the results in the following lemmas to avoid making this paper unnecessarily long.

Lemma 3.15. *Let $\mathcal{D}_2 = \Omega_2 \times (0, \infty)$. There exists a solution $w_2 \in C^2(\bar{\Omega}_2 \times (0, \infty)) \cap C(\bar{\Omega}_2 \times [0, \infty))$ of the initial-boundary-value problem*

$$\begin{cases} w_t - \Delta w + f(w) = 0 & \text{in } \mathcal{D}_2 \\ w(x,t) = v_{02}(x) & \text{on } \partial_p \mathcal{D}_2, \end{cases} \tag{3.29}$$

where $v_{0,2} \in C^2(\bar{\Omega}_2)$ satisfies

$$\begin{cases} u \leq v_{02} \leq u + C\delta \ \text{ in } \Omega_2 \\ -\Delta v_{02} + f(v_{02}) > \varepsilon > 0 \ \text{ in } \Omega_2 \\ v_{0,2} = -1 \ \text{ on } \partial B_{R_2} \text{ and } \ v_{02} = 1 \ \text{ on } \ \partial B_1 \end{cases} \tag{3.30}$$

Lemma 3.16.

$$\lim_{t \to \infty} w_2(x,t) = u_2(x)$$

locally uniformly and monotonically on $\bar{\Omega}_1$. As a consequence, u_2 solves the boundary value problem

$$\begin{cases} \Delta u = f(u) & \text{in } 1 \leq |x| \leq R_2 \\ u = 1 & \text{on } |x| = 1 \\ u = -1 & \text{on } |x| = R_2 \end{cases}$$

and satisfies

$$u(x) \leq u_2(x) \leq u(x) + C\delta \ \text{ in } \Omega.$$

Lemma 3.17. $u_2 \in C^2(\bar{\Omega}_2)$.

Similarly we have:

Lemma 3.18. *Let* $\mathcal{D} = \mathcal{R} \times (0, \infty)$. *There exists a solution* $w \in C^2(\bar{\mathcal{R}} \times (0, \infty)) \cap C(\bar{\mathcal{R}} \times [0, \infty))$ *of the initial-boundary-value problem*

$$\begin{cases} w_t - \Delta w + f(w) = 0 & \text{in } \mathcal{D} \\ w(x,t) = v_{00}(x) & \text{on } \partial_p \mathcal{D}, \end{cases}$$

where $v_{00} \in C^2(\bar{\Omega}_2)$ *satisfies*

$$\begin{cases} v_{01} \leq v_{00} \leq v_{02} \ \text{ in } \mathcal{R} \\ -\Delta v_{00} + f(v_{00}) > \varepsilon > 0 \ \text{ in } \mathcal{R} \\ v_{00} = -1 \ \text{ on } \partial\mathcal{R} \text{ and } \ v_{00} = 1 \ \text{ on } \ \partial B_1 \end{cases}$$

Lemma 3.19.

$$\lim_{t \to \infty} w(x,t) = u_0(x)$$

locally uniformly and monotonically on $\bar{\Omega}_1$. *As a consequence,* u_0 *solves the boundary value problem*

$$\begin{cases} \Delta u = f(u) & \text{in } 1 \leq |x| \leq R \\ u = 1 & \text{on } |x| = 1 \\ u = -1 & \text{on } |x| = R \end{cases}$$

and satisfies

$$u_1(x) \leq u_0(x) \ \text{ in } \Omega_1, \text{ and } \ u_0(x) \leq u_2(x) \ \text{ in } \mathcal{R}.$$

Lemma 3.20. $u_0 \in C^2(\bar{\mathcal{R}})$.

Applying the result of radial symmetry in the preceding section, we conclude that

Theorem 3.21. *The solutions* u_i, $i = 0, 1, 2$, *are radially symmetric functions on* $\mathcal{R}$ *and* Ω_i, $i = 1, 2$, *respectively. In particular, the free boundaries,* $\mathcal{F}_i = \partial\{u_i > 0\}$, $i = 0, 1, 2$, *are spheres with the center at the origin.*

3.4.3 Comparison and Stability

The following lemma states the non-degeneracy of u_2 in the positive domain.

Lemma 3.22. *Let $d(x)$ be the distance from x to $\mathcal{F}_2$. Then*

$$u_2(x) \geq Cd(x) \quad \textit{in} \ \{u_2 > 0\}\,.$$

Proof. One notices that u_2 is super-harmonic in the positive domain $\{u_2 > 0\}$ and the fact $\mathcal{F}_2$ is a sphere with the origin as its center. Recalling the boundary estimates for a non-negative harmonic function (e. g. [**?**], Lemma 6 and proof), one gets the estimate for u_2 in the positive domain by comparing u_2 to the harmonic function in $\{u_2 > 0\}$ with the same boundary data as u_2. □

It is a simple fact that even if two functions are uniformly very close to each other, their boundaries of zero sets, i. e. the "*free boundaries*", in general may be far away from each other. Nevertheless, the non-degeneracy of u_2 just established helps us to prove in our problem the following lemma that states the free boundary $\mathcal{F}_1$ is indeed close to the other free boundary $\mathcal{F}_2$.

Lemma 3.23.

$$dist(\mathcal{F}_1, \mathcal{F}_2) := \sup_{x \in \mathcal{F}_1} dist(x, \mathcal{F}_2) \leq C\delta.$$

Proof. It is known from the previous results, Lemmas 3.13 and 3.16, that $u_1 \leq u_2 \leq u_1 + C\delta$ on $B_{R_1} \backslash \bar{B}_1$. The non-degeneracy of u_2 proved in the preceding lemma implies that

$$u_2(x) \geq Cd(x)$$

holds on $\mathcal{F}_1$.

On $\mathcal{F}_1$,

$$u_1(x) + C\delta = C\delta \geq u_2(x) \geq Cd(x),$$

which implies $d(x) \leq C\delta$ for a new constant $C > 0$. That is

$$dist(\mathcal{F}_1, \mathcal{F}_2) \leq C\delta$$

□

We summarize the part of results of the Lemmas 3.13, 3.19 and 3.16 on the order of the solutions u_1, u_0, u and u_2 on respective domains in the following theorem.

Theorem 3.24. *Let u_i, $i = 0, 1, 2$, be as constructed in Lemmas 3.19, 3.13 and 3.16.*

Then $u_1 \leq u$ in Ω_1, $u \leq u_2$ in Ω, $u_1 \leq u_0$ in Ω_1, and $u_0 \leq u_2$ in $\mathcal{R}$.

In particular, we have

$$|u(x) - u_0(x)| < C\delta \ \ (x \in \Omega \cap \mathcal{R})$$

and the inclusion of the positive sets as stated below.

$$\{u_1 > 0\} \subseteq \{u > 0\} \subseteq \{u_2 > 0\} \ \ \textit{and}$$

$$\{u_1 > 0\} \subseteq \{u_0 > 0\} \subseteq \{u_2 > 0\}\,.$$

Proof. The first conclusion is evident from the lemmas mentioned. We need only to point out that

$$|u(x) - u_0(x)| < C\delta \ \ (x \in \Omega \cap \mathcal{R})$$

follows from the estimates in the Lemmas 3.13 and 3.16 and the first conclusion of this theorem. The inclusion of the sets is clear from the first conclusion. □

And in the end by applying Lemma 3.23, we have the desired approximate radial symmetry of u.

Theorem 3.25. *Let u be as in Theorem 3.2, u_i $(i = 1, 2)$ be as Lemmas 3.13 and 3.16, and $\mathcal{F}$, $\mathcal{F}_i$ $(i = 1, 2)$ be their respective free boundaries.*

Then

$$dist(\mathcal{F}, \mathcal{F}_0) \leq dist(\mathcal{F}_1, \mathcal{F}_2) < C|Z| = C\delta.$$

Proof. This theorem follows immediately from the inclusion of sets in the preceding theorem and Lemma 3.23. □

The proof of Theorem 3.2 is complete.

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ABSTRACT

WELL-POSEDNESS AND SYMMETRY PROPERTIES OF FREE BOUNDARY PROBLEMS FOR SOME NON-LINEAR DEGENERATE ELLIPTIC SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS

by

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Advisor: Dr. Pei-Yong Wang

Major: Mathematics

Degree: Doctor of Philosophy

In the first part of this thesis, a bifurcation about the uniqueness of a solution of a singularly perturbed free boundary problem of phase transition associated with the p-Laplacian, subject to given boundary condition is proved in the first chapter. We show this phenomenon by proving the existence of a third solution through the Mountain Pass Lemma when the boundary data decreases below a threshold. In the second chapter and third chapter, we prove the convergence of an evolution to stable solutions, and show the Mountain Pass solution is unstable in this sense.

In the second part of this thesis, we study a singularly perturbed free boundary problem arising from a real problem associated with a Radiographic Integrated Test Stand and concerning a solution of the equation $\Delta u = f(u)$ in a domain Ω subject to constant boundary data, where the function f in general is not monotone. In chapter 4, we let the domain Ω be a perfect ring and we incorporate a new idea of radial correction into the classical moving plane method to prove the radial symmetry of a solution. In chapter 5, we let the domain Ω be slightly shifted from a ring and we establish the stability of the solution by showing the approximate radial symmetry of the free boundary and the solution. For this purpose, we complete the proof via an evolutionary point of view, as an elliptic comparison principle is false, nevertheless a parabolic one holds.

AUTOBIOGRAPHICAL STATEMENT

Alaa Haj Ali has completed her first two years of her Bachelor's of Science degree in Mathematics at the Lebanese University in Beirut, Lebanon before she was transferred to the University of Michigan-Dearborn to earn her Bachelor's degree in Mathematics with a minor in Computer Science in August 2012. After this, she went on to pursue a phD in Mathematics at Wayne State University under the supervision of professor Peiyong Wang. Alaa's research interests include nonlinear partial differential equation and free boundary problems. Among other awards, she received the Thomas C. Rumble Fellowship during her last academic year 2018-2019. She will graduate in May 2019.