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On the Extension of Exponentiated Pareto Distribution

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In this study, an extended exponentiated Pareto distribution is proposed. Some statistical properties are derived. We consider maximum likelihood, least squares, weighted least squares and Bayesian estimators. A simulation study is implemented for investigating the accuracy of different estimators. An application of the proposed distribution to a real data is presented.

Keywords: Exponentiated Pareto distribution, maximum likelihood estimation, least squares method, Bayesian method

Introduction

The Pareto distribution can be used quite effectively in analyzing many lifetime data. A two-parameter distribution, called the exponentiated Pareto (EP) distribution has been proposed by Gupta et al. (1998) as a simple generalization or modification of the well-known standard Pareto distribution by introducing one more shape parameter. The EP distribution can be decreasing and upside-down bathtub shaped failure rates. The probability density function (pdf) of the EP distribution is given by

$$\mathrm{g}\left(x;\lambda,\theta\right)=\lambda\theta\left(1+x\right)^{-(\lambda+1)}\left[1-\left(1+x\right)^{-\lambda}\right]^{\theta-1};\quad x,\lambda,\theta>0, \tag{1}$$

where θ and λ are two shape parameters. The cumulative distribution function (cdf) of EP distribution is given by

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$$G(x;\lambda,\theta)=\left[1-(1+x)^{-\lambda}\right]^{\theta}. \tag{2}$$

Note that, if $\theta = 1$, the EP distribution reduces to the standard Pareto (SP) distribution of the second kind.

In recent times, several generated classes of distributions were introduced and studied, by several authors. Among them, Marshall-Olkin-G (Marshall & Olkin, 1997), beta-G (Eugene et al., 2002; Jones, 2004), gamma-G (Zografos & Balakrishanan, 2009), Kumaraswamy-G (Cordeiro & de Castro, 2011), generalized beta-G (Alexander et al., 2012), exponentiated generalized (Cordeiro et al., 2013), transformed-transformer (Alzaatreh et al., 2013), exponentiated *T-X* (Alzaghal et al., 2013), Weibull-G (Bourguignon et al., 2014), Type-1 half-logistic (Cordeiro et al., 2015), logistic-X (Tahir et al., 2016), Kumaraswamy Weibull-G (Hassan & Elgarhy, 2016b), exponentiated Weibull-G (Hassan & Elgarhy, 2016a), additive Weibull-G (Hassan & Hemeda, 2016), odd Lindley-G (Silva et al., 2017), Type II half logistic-G (Hassan, Elgarhy, & Shakil, 2017), generalized additive Weibull-G (Hassan, Hemeda et al., 2017), generalized odd log-logistic-G (Cordeiro et al., 2017), power Lindley-G (Hassan & Nassr, 2018b), inverse Weibull-G (Hassan & Nassr, 2018a), odd Lomax-G (Cordeiro et al., 2019), Type II generalized Topp-Leone-G (Hassan, Elgarhy, & Ahmad, 2019), and extended odd Weibull-G (Alizadeh et al., 2019).

For a baseline continuous cdf $G(x;\xi)$, Cordeiro et al. (2013) defined the exponentiated generalized (EG) class of distributions by

$$F(x;\alpha,\beta,\xi)=\left[1-\left(1-G(x;\xi)\right)^{\alpha}\right]^{\beta}, \tag{3}$$

where $\alpha > 0$ and $\beta > 0$ are two shape parameters whose role is to govern skewness and generate distributions with heavier lighter tails. The pdf corresponding to (3) is given by

$$f(x;\alpha,\beta,\xi)=\alpha\beta\, g(x;\xi)\left[1-G(x;\xi)\right]^{\alpha-1}\left[1-\left(1-G(x;\xi)\right)^{\alpha}\right]^{\beta-1}, \tag{4}$$

where, $g(x;\xi)$ is the baseline pdf. Setting $\alpha = 1$, the pdf (4) gives Lehmann Type I class. Also, for $\beta = 1$, the pdf (4) gives the Lehmann Type II class. So, the family (4) generalizes both Lehmann Types I and II classes. Furthermore, the baseline distribution $G(x;\xi)$ is a special case of (4) when $\alpha = \beta = 1$.

According to Cordeiro et al. (2013), the class of EG distributions shares an attractive physical interpretation whenever α and β are positive integers. Consider a device made of independent components in a parallel system with each component is made of independent subcomponents identically distributed according to G(x; ξ) in a series system. The device fails if all components fail and each component fails if any subcomponent fails. Let $X_{j1},\ldots, X_{j\alpha}$ denote the lifetimes of the subcomponents within the j^{th} component, $j = 1,\ldots, \beta$, with common cdf G(x; ξ). Let X_j denote the lifetime of the j^{th} component and let X denote the lifetime of the device. Therefore, the cdf of X is as follows:

$$\begin{aligned}
\mathrm{P}(X \le x) &= \mathrm{P}(X_1 \le x,\ldots, X_\beta \le x) = \mathrm{P}(X_1 \le x)^\beta = \left[1-\mathrm{P}(X_1 > x)\right]^\beta \\
&= \left[1-\mathrm{P}(X_{11} > x,\ldots, X_{1\alpha} > x)\right]^\beta = \left[1-\mathrm{P}(X_{11} > x)^\alpha\right]^\beta \\
&= \left[1-\left(1-\mathrm{P}(X_{11} < x)\right)^\alpha\right]^\beta
\end{aligned}$$

and the lifetime of the device obeys the EG family of distributions. So, the goal of the present work is introducing a new extension for EP distribution based on the EG family with more flexibility than the baseline (2). The extensions of classical distributions sometimes provide reasonable parametric fits to particular applications as in lifetimes and reliability studies. The new distribution has more sub-models when compared with baseline distribution and hence it allows us to study more comprehensive structural properties.

An Extended Exponentiated Pareto Distribution

We introduce an extended form for EP distribution, named as exponentiated generalized exponentiated Pareto (EGEP) distribution. The four-parameter EGEP distribution is particular case from EG class presented by Cordeiro et al. (2013). Some sub-models and some related distributions are presented. Further, the reliability, hazard rate, cumulative hazard rate and odds ratio functions are given. Asymptotic and possible shapes of the pdf, cdf, and the mode of EGEP are discussed.

The cdf of the EGEP distribution is obtained by taking G(x; ξ) to be cdf of EP distribution as follows:

$$\mathrm{F}(x;\phi)=\left\{1-\left[1-\left(1-(1+x)^{-\lambda}\right)^{\theta}\right]^{\alpha}\right\}^{\beta}; \quad x>0;\alpha,\beta,\lambda,\theta>0, \tag{5}$$

where $\phi = (\alpha, \beta, \lambda, \theta)$ is the set of parameters The pdf of EGEP distribution corresponding to (5) is written as follows:

$$\mathrm{f}(x;\phi)=\frac{\alpha\beta\lambda\theta}{(1+x)^{(\lambda+1)}}\left[1-(1+x)^{-\lambda}\right]^{\theta-1}\left[1-\left(1-(1+x)^{-\lambda}\right)^{\theta}\right]^{\alpha-1} \times\left\{1-\left[1-\left(1-(1+x)^{-\lambda}\right)^{\theta}\right]^{\alpha}\right\}^{\beta-1} \tag{6}$$

$X \sim$ EGEP($\alpha, \beta, \lambda, \theta$) denotes a random variable with the pdf (6). Figure 1 displays a variety of possible shapes of pdf of EGEP distribution for some selected values of parameters. Clearly, EGEP densities take various shapes such as unimodal, reversed J shaped, and right skewed.

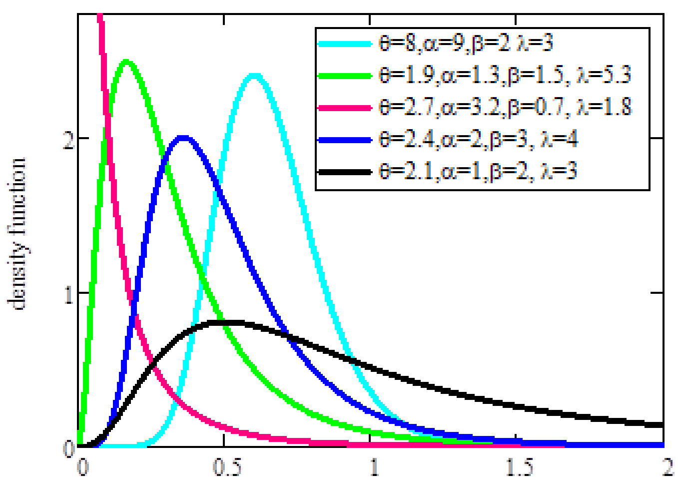

Figure 1. The pdf of EGEP distribution for different values of α, β, λ, and θ

Some Special Distributions

Here, we present special distributions to EGEP model as follows:

- For $\beta = 1$, the EGEP reduces to Generalized exponentiated Pareto (GEP) as a new model.
- For $\alpha = 1$, the EGEP reduces to the Exponentiated Pareto (EP) distribution with power parameter $\theta\beta$.
- For $\alpha = 1$ and $\beta = 1$, the EGEP reduces to EP distribution (see Gupta et al., 1998).
- For $\alpha = 1$, $\beta = 1$, and $\theta = 1$, the EGEP reduces to the Lomax model (see Lomax, 1954).
- For $\alpha = 1$, $\beta = 1$, $\lambda = 1$, and $\theta = 1$, the EGEP reduces to log-logistic model.

Some Related Distributions

Here, we provide some relations of the EGEP distribution to other distributions.

- For $Y = \ln(1 + X)$, $X \sim \text{EGEP}(\theta, \lambda, \alpha, \beta)$, the EGEP reduces to exponentiated Kumaraswamy exponential distribution (see Rodrigues & Silva, 2015).
- For $Y = \ln(1 + X)$, $X \sim \text{EGEP}(\theta, \lambda, 1, 1)$, the EGEP distribution reduces to exponentiated exponential distribution (Gupta & Kundu, 1999).
- For $Y = \ln(1 + X)$, $X \sim \text{EGEP}(\theta, \lambda, \alpha, 1)$, the EGEP distribution reduces to Kumaraswamy exponential distribution (Cordeiro et al., 2010).
- For $Y = (1 + X)^{-1}$, $X \sim \text{EGEP}(\theta, \lambda, \alpha, \beta)$, the EGEP distribution reduces to Kumaraswamy Kumaraswamy distribution (El-Sherpieny & Ahmed, 2014).
- For $Y = (1 + X)^{-1}$, $X \sim \text{EGEP}(\theta, \lambda, \alpha, 1)$, the EGEP distribution reduces to exponentiated Kumaraswamy distribution (Lemonte et al., 2013).
- For $Y = (1 + X)^{-1}$, $X \sim \text{EGEP}(\theta, \lambda, 1, 1)$, the EGEP distribution reduces to Kumaraswamy distribution.
- For $Y = X/\delta$, $X \sim \text{EGEP}(\theta, \lambda, 1, 1)$, the EGEP distribution reduces to exponentiated Lomax distribution (Abdul-Moniem & Abdel-Hameed, 2012).

Reliability Analysis

Expressions for the reliability function, hazard function, reversed hazard function, cumulative hazard rate and odds function are provided.

The reliability function, denoted by $\bar{\mathrm{F}}(x;\phi)$, is as follows:

$$\bar{\mathrm{F}}(x;\phi)=1-\left\{1-\left[1-\left(1-(1+x)^{-\lambda}\right)^{\theta}\right]^{\alpha}\right\}^{\beta}. \tag{7}$$

The hazard rate function (hrf), say $\mathrm{h}(x;\phi)$, is defined as follows:

$$\mathrm{h}(x;\phi)=\frac{\alpha\beta\lambda\theta(1+x)^{-(\lambda+1)}\left[1-(1+x)^{-\lambda}\right]^{\theta-1}\left[1-\left(1-(1+x)^{-\lambda}\right)^{\theta}\right]^{\alpha-1}}{1-\left\{1-\left[1-\left(1-(1+x)^{-\lambda}\right)^{\theta}\right]^{\alpha}\right\}^{\beta}} \times\left\{1-\left[1-\left(1-(1+x)^{-\lambda}\right)^{\theta}\right]^{\alpha}\right\}^{\beta-1} \tag{8}$$

Figure 2 displays a variety of possible shapes of hrf of EGEP distribution for some selected values of parameters. It can be deduced from Figure 2 that the shape of the hazard function of the EGEP distribution could be constant, decreasing, and upside-down (depending on the value of the parameters).

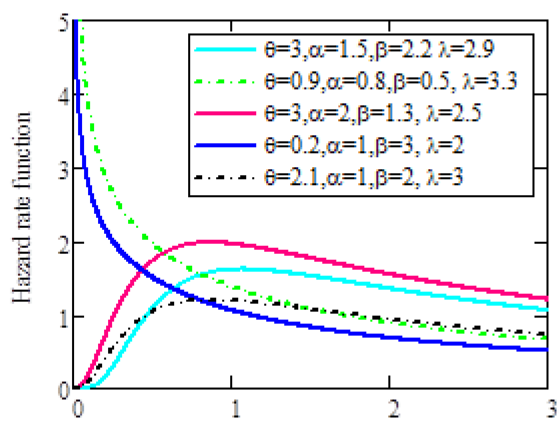

Figure 2. Hazard rate function of EGEP distribution for different values of α, β, λ, and θ

Furthermore, the reversed hazard rate function, say r(x; ϕ); the cumulative hazard rate, say H(x; ϕ); and the odds function, say O(x; ϕ); are given, respectively, as follows:

$$\mathrm{r}\left(x;\phi\right)=\frac{\alpha\beta\lambda\theta\left(1+x\right)^{-(\lambda+1)}\left[1-\left(1+x\right)^{-\lambda}\right]^{\theta-1}\left[1-\left(1-\left(1+x\right)^{-\lambda}\right)^{\theta}\right]^{\alpha-1}}{1-\left(1-\left(1-\left(1+x\right)^{-\lambda}\right)^{\theta}\right)^{\alpha}},$$

$$\mathrm{H}\left(x;\phi\right)=-\ln\left[\overline{\mathrm{F}}\left(x;\phi\right)\right]=-\ln\left[1-\left\{1-\left[1-\left(1-\left(1+x\right)^{-\lambda}\right)^{\theta}\right]^{\alpha}\right\}^{\beta}\right],$$

and

$$\mathrm{O}\left(x;\phi\right)=\left(\left\{1-\left[1-\left(1-\left(1+x\right)^{-\lambda}\right)^{\theta}\right]^{\alpha}\right\}^{-\beta}-1\right)^{-1}.$$

Asymptotic & Shapes

We discuss the asymptotic and possible shapes of the cdf (5), pdf (6), reliability function (7), and hazard rate function (8) as follows:

i) When $x \to 0$,

$$\lim_{x\to 0}\mathrm{f}\left(x;\phi\right)=\lim_{x\to 0}\mathrm{F}\left(x;\phi\right)=\lim_{x\to 0}\mathrm{h}\left(x;\phi\right)=0,\ \text{and}\ \lim_{x\to 0}\overline{\mathrm{F}}\left(x;\phi\right)=1.$$

ii) When $x \to \infty$,

$$\lim_{x\to\infty}\mathrm{f}\left(x;\phi\right)=\lim_{x\to\infty}\mathrm{F}\left(x;\phi\right)=\lim_{x\to\infty}\mathrm{h}\left(x;\phi\right)=0,\ \text{and}\ \lim_{x\to\infty}\overline{\mathrm{F}}\left(x;\phi\right)=1.$$

Mode of EGEP Distribution

The mode x_0 of the EGEP distribution is obtained by solving the equation f′(x) = 0, where

$$\begin{aligned} f'(x) = \alpha\beta\lambda\theta\big[&A'(x)B(x)C(x)D(x) + A(x)B'(x)C(x)D(x) \\ &+ A(x)B(x)C'(x)D(x) + A(x)B(x)C(x)D'(x)\big] = 0 \end{aligned} \tag{9}$$

where

$$A(x) = (1+x)^{-(\lambda+1)},\ B(x) = \left[1-(1+x)^{-\lambda}\right]^{\theta-1},\ C(x) = \left[1-\left(1-(1+x)^{-\lambda}\right)^{\theta}\right]^{\alpha-1},$$

$$D(x) = \left[1-\left(1-\left(1-(1+x)^{-\lambda}\right)^{\theta}\right)^{\alpha}\right]^{\beta-1},\ A'(x) = -(\lambda+1)(1+x)^{-(\lambda+2)},$$

$$B'(x) = \lambda(\theta-1)(1+x)^{-(\lambda+1)}\left[1-(1+x)^{-\lambda}\right]^{\theta-2},$$

$$C'(x) = -\theta\lambda(\alpha-1)(1+x)^{-(\lambda+1)}\left[1-(1+x)^{-\lambda}\right]^{\theta-1}\left[1-\left(1-(1+x)^{-\lambda}\right)^{\theta}\right]^{\alpha-2},$$

$$\begin{aligned} D'(x) = \frac{\alpha\lambda\theta(\beta-1)}{(1+x)^{(\lambda+1)}}&\left[1-(1+x)^{-\lambda}\right]^{\theta-1}\left[1-\left(1-(1+x)^{-\lambda}\right)^{\theta}\right]^{\alpha-1} \\ &\times\left[1-\left(1-\left(1-(1+x)^{-\lambda}\right)^{\theta}\right)^{\alpha}\right]^{\beta-2} \end{aligned}$$

Clearly, the root of the equation (9) is not available in closed form. So, numerical calculation is adopted to solve this equation for some values of parameters.

Expansion for Density Function

We consider the power series expansion, for any real non-integer z:

$$(1-z)^{\beta-1} = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(\beta)}{\Gamma(\beta-k)k!} z^k\ , \tag{10}$$

which is valid for $|z| < 1$. Then, by applying (10) in (6), the pdf of EGEP distribution, where β is a real non-integer, becomes

$$\mathrm{f}\left(x;\phi\right)=\sum_{k=0}^{\infty}\frac{\left(-1\right)^{k}\alpha\lambda\theta\Gamma\left(\beta+1\right)}{\Gamma\left(\beta-k\right)k!\left(1+x\right)^{\left(\lambda+1\right)}}\left[1-\left(1+x\right)^{-\lambda}\right]^{\theta-1}\times\left[1-\left(1-\left(1+x\right)^{-\lambda}\right)^{\theta}\right]^{\alpha+\alpha k-1} \tag{11}$$

Again, using the binomial expansion (10) in (11),

$$\mathrm{f}\left(x;\phi\right)=\sum_{k,j=0}^{\infty}w_{k,j}\lambda\theta\left(1+x\right)^{-\left(\lambda+1\right)}\left[1-\left(1+x\right)^{-\lambda}\right]^{\theta+\theta j-1}, \tag{12}$$

where

$$w_{k,j}=\frac{\left(-1\right)^{k+j}\alpha\beta\Gamma\left(\beta\right)\Gamma\left(\alpha+\alpha k\right)}{j!\Gamma\left(\beta-k\right)\Gamma\left(\alpha+\alpha k-j\right)k!}.$$

Further, (12) can be rewritten as follows:

$$\mathrm{f}\left(x;\phi\right)=\sum_{k,j=0}^{\infty}w_{k,j}^{*}\,\mathrm{g}\left(x;\lambda,\theta\left(j+1\right)\right), \tag{13}$$

where

$$w_{k,j}^{*}=\frac{w_{k,j}}{j+1}$$

and $\mathrm{g}(x;\lambda,\theta(j+1))$ is the pdf of EP distribution with parameters λ and $\theta(j+1)$. Further, the cdf corresponding to (13) takes the following form:

$$\mathrm{F}\left(x;\phi\right)=\sum_{k,j=0}^{\infty}w_{k,j}^{*}\,\mathrm{G}\left(x;\lambda,\theta\left(j+1\right)\right), \tag{14}$$

where $\mathrm{G}(x;\lambda,\theta(j+1))$ is the cdf of EP distribution with parameters λ and $\theta(j+1)$.

Moments and Moments of the Residual Life

Here, we derive the expression for ordinary and moments of residual life of EGEP distribution.

The r^{th} moment for the EGEP is derived as follows:

$$\mu_r' = \int_0^\infty x^r \sum_{k,j=0}^{\infty} w_{k,j}^* \, \mathrm{g}\big(x;\lambda,\theta(j+1)\big)dx .$$

Hence, after simplification, the r^{th} moment for EGPF takes the form

$$\mu_r' = \sum_{k,j=0}^{\infty} \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} w_{k,j}^* \theta(j+1) \mathrm{B}\left(1-\frac{i}{\lambda}, \theta(j+1)\right). \tag{15}$$

Some important measures, such as mean, variance, cumulants, skewness, and kurtosis, can be derived from (15). Setting $r = 1$ and $r = 2$ in (15), the mean and variance of EGEP are obtained as

$$\mathrm{E}(X) = \sum_{k,j=0}^{\infty} w_{k,j}^* \left[\theta(j+1)\mathrm{B}\left(1-\frac{1}{\lambda}, \theta(j+1)\right) - 1\right]$$

and

$$\mathrm{Var}(X) = \sum_{k,j=0}^{\infty} w_{k,j}^* \left[1 - 2\theta(j+1)\mathrm{B}\left(1-\frac{1}{\lambda}, \theta(j+1)\right) + \theta(j+1)\mathrm{B}\left(1-\frac{2}{\lambda}, \theta(j+1)\right)\right] - \left\{\sum_{k,j=0}^{\infty} w_{k,j}^* \left[\theta(j+1)\mathrm{B}\left(1-\frac{1}{\lambda}, \theta(j+1)\right) - 1\right]\right\}^2$$

The residual life plays an important role in life testing situations and reliability theory. The n^{th} moment of the residual life, $\mathrm{m}_n(t) = \mathrm{E}[(X - t)^n \mid X > t]$, $n = 1, 2, \ldots$ uniquely determines $\mathrm{F}(x)$ (see Navarro et al., 1998). The n^{th} moment of the residual life is defined by

$$\mathrm{m}_n(t) = \frac{1}{\overline{\mathrm{F}}(t)} \int_t^\infty (x-t)^n \, \mathrm{f}(x) dx .$$

Using pdf (13), the n^{th} moment of the residual life of EGEP distribution is as follows:

$$\mathrm{m}_n(t)=\frac{1}{\bar{\bar{\mathrm{F}}}(t;\phi)}\sum_{k,j=0}^{\infty}\sum_{r=0}^{n}(-1)^{n-r}\binom{n}{r}t^{n-r}w_{k,j}^{*}\int_{t}^{\infty}x^{r}\,\mathrm{g}\big(x;\lambda,\theta(j+1)\big)dx\,.$$

By using the binomial expansion and the upper incomplete gamma function, we obtain

$$\mathrm{m}_n(t)=\frac{1}{\bar{\bar{\mathrm{F}}}(t;\phi)}\sum_{k,j=0}^{\infty}\sum_{r=0}^{n}\sum_{h=0}^{r}\Bigg[(-1)^{n-r}\binom{n}{r}\binom{r}{h}t^{n-r}w_{k,j}^{*}\lambda\theta(j+1)$$
$$\times\mathrm{B}\left((1+t)^{-\lambda},1-\frac{r}{\lambda},\theta+\theta j\right)\Bigg]$$

where

$$\mathrm{B}\left((1+t)^{-\lambda},1-\frac{r}{\lambda},\theta+\theta j\right)$$

is the incomplete beta function.

Rényi Entropy

For a certain random phenomenon under study, it is important to quantify the uncertainty associated with the random variable of interest. The Rényi entropy is one of the most popular measures used to quantify the variability of random variable X. The Rényi entropy of X with density (6), say $\mathrm{I_R}(\rho)$, for $\rho > 0$ and $\rho \neq 1$ is given by

$$\mathrm{I_R}(\rho)=\frac{1}{1-\rho}\log\int_{-\infty}^{\infty}\mathrm{f}(x)^{\rho}\,dx\,.$$

Let

$$\mathrm{f}\left(x;\phi\right)^{\rho}=\left(\alpha\beta\lambda\theta\right)^{\rho}\left(1+x\right)^{-\rho(\lambda+1)}\left[1-\left(1+x\right)^{-\lambda}\right]^{\rho(\theta-1)}$$
$$\times\left[1-\left(1-\left(1+x\right)^{-\lambda}\right)^{\theta}\right]^{\rho(\alpha-1)}\left\{1-\left[1-\left(1-\left(1+x\right)^{-\lambda}\right)^{\theta}\right]^{\alpha}\right\}^{\rho(\beta-1)} \quad (16)$$

Using the binomial expansion (10) twice in (16), we obtain

$$\mathrm{f}\left(x;\phi\right)^{\rho}=\left(\alpha\beta\lambda\theta\right)^{\rho}\sum_{k,j=0}^{\infty}\xi_{k,j}\left(1+x\right)^{-\rho(\lambda+1)}\left[1-\left(1+x\right)^{-\lambda}\right]^{\rho(\theta-1)+\theta j},$$

where

$$\xi_{k,j}=\frac{(-1)^{k+j}\left(\rho\left(\beta-1\right)\right)!\left(\rho\left(\alpha-1\right)+\alpha k\right)!}{\left(\rho\left(\beta-1\right)-k\right)!\left(\rho\left(\alpha-1\right)+\alpha k-j\right)!k!\,j!}.$$

Hence, after some simplification, we have Rényi entropy as follows:

$$\mathrm{I}_{\mathrm{R}}\left(\rho\right)=\frac{1}{1-\rho}\log\left[\sum_{k,j=0}^{\infty}\xi_{k,j}\int_{0}^{\infty}\left(\alpha\beta\lambda\theta\right)^{\rho}\left(1+x\right)^{-\rho(\lambda+1)}\left[1-\left(1+x\right)^{-\lambda}\right]^{\rho(\theta-1)+\theta j}dx\right].$$

Hence, the Rényi entropy of X can be expressed as follows:

$$\mathrm{I}_{\mathrm{R}}\left(\rho\right)=\frac{1}{1-\rho}\log\left[\sum_{k,j=0}^{\infty}\xi_{k,j}\left(\alpha\beta\theta\right)^{\rho}\lambda^{\rho-1}\,\mathrm{B}\left(\frac{\rho\left(\lambda+1\right)-1}{\lambda},\rho\left(\theta-1\right)+\theta j+1\right)\right].$$

Parameter Estimation

The estimators of the EGEP model parameters are obtained using maximum likelihood (ML), least squares (LS), weighted least squares (WLS) and Bayesian methods.

Maximum Likelihood Estimators

Let $X_1, X_2, \ldots, X_n$ be a simple random sample from the EGEP distribution with set of parameters $\phi = (\alpha, \lambda, \theta, \beta)$. The likelihood function based on the observed random sample of size n from density (6) is given by

$$\mathrm{L}\left(\phi \mid x\right) = \left(\alpha\beta\lambda\theta\right)^n \prod_{i=1}^{n} S_i^{-(\lambda+1)} Q_i^{\theta-1} \left[1 - Q_i^{\theta}\right]^{\alpha-1} \left\{1 - \left[1 - Q_i^{\theta}\right]^{\alpha}\right\}^{\beta-1}, \qquad (17)$$

where $S_i = (1 + x_i)$ and $Q_i = \left(1 - S_i^{-\lambda}\right)$. The natural logarithm likelihood function, denoted by $\ln l$, is obtained as follows:

$$\ln l = n\ln\alpha + n\ln\beta + n\ln\lambda + n\ln\theta - \left(\lambda+1\right)\sum_{i=1}^{n}\ln S_i + \left(\theta-1\right)\sum_{i=1}^{n}\ln\left(Q_i\right)$$
$$+\left(\alpha-1\right)\sum_{i=1}^{n}\ln\left[1-Q_i^{\theta}\right] + \left(\beta-1\right)\sum_{i=1}^{n}\ln\left\{1-\left[1-Q_i^{\theta}\right]^{\alpha}\right\}$$

The partial derivatives of the log-likelihood function with respect to the unknown parameters are given by

$$\frac{\partial \ln l}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n}\ln S_i + \left(\theta-1\right)\sum_{i=1}^{n} Q_i^{-1} S_i^{-\lambda}\ln S_i - \theta\left(\alpha-1\right)\sum_{i=1}^{n}\left[1-Q_i^{\theta}\right]^{-1} Q_i^{\theta-1} S_i^{-\lambda}\ln S_i$$
$$+\alpha\theta\left(\beta-1\right)\sum_{i=1}^{n}\left\{1-\left[1-Q_i^{\theta}\right]^{\alpha}\right\}^{-1}\left[1-Q_i^{\theta}\right]^{\alpha-1} Q_i^{\theta-1} S_i^{-\lambda}\ln S_i,$$

$$\frac{\partial \ln l}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\ln\left[1-Q_i^{\theta}\right] - \left(\beta-1\right)\sum_{i=1}^{n}\left\{1-\left[1-Q_i^{\theta}\right]^{\alpha}\right\}^{-1}\left[1-Q_i^{\theta}\right]^{\alpha}\ln\left[1-Q_i^{\theta}\right],$$

$$\frac{\partial \ln l}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^{n}\ln\left(Q_i\right) - \left(\alpha-1\right)\sum_{i=1}^{n}\left[1-Q_i^{\theta}\right]^{-1} Q_i^{\theta}\ln\left(Q_i\right)$$
$$+\alpha\left(\beta-1\right)\sum_{i=1}^{n}\ln\left\{1-\left[1-Q_i^{\theta}\right]^{\alpha}\right\}^{-1}\left[1-Q_i^{\theta}\right]^{\alpha-1} Q_i^{\theta}\ln\left(Q_i\right),$$

and

$$\frac{\partial \ln l}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\ln\left\{1-\left[1-Q_i^{\theta}\right]^{\alpha}\right\}$$

The ML estimators of the model parameters are determined by solving numerically the non-linear equations $\partial \ln l / \partial \alpha = 0, \partial \ln l / \partial \lambda = 0, \partial \ln l / \partial \theta = 0$, and $\partial \ln l / \partial \beta = 0$ simultaneously.

Least Squares and Weighted Least Squares Estimators

Suppose $X_1, X_2, \ldots, X_n$ is a random sample of size n from EGEP distribution and suppose $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ denotes the corresponding ordered sample. According to Johnson et al. (1995), the expectation and the variance of distribution are independent of the unknown parameter and are given by

$$\mathrm{E}\left[\mathrm{F}\left(X_{i:n}\right)\right] = \frac{i}{n+1} \quad \text{and} \quad \mathrm{Var}\left[\mathrm{F}\left(X_{i:n}\right)\right] = \frac{i\left(n-i+1\right)}{\left(n+1\right)^2\left(n+2\right)},$$

where $[\mathrm{F}(X_{i:n})]$ is the cdf for any distribution and $X_{i:n}$ is the i^{th} order statistic. Then LS estimators can be obtained by minimizing the sum of squares errors,

$$\sum_{i=1}^{n}\left[\mathrm{F}\left(X_{i:n}\right) - \frac{i}{n+1}\right]^2,$$

with respect to the unknown parameters. So, the LS estimators of the unknown parameters α, λ, θ, and β of the EGEP model can be obtained by minimizing the following quantity:

$$\sum_{i=1}^{n}\left[\left\{1-\left[1-\left(1-\left(1+x_{i:n}\right)^{-\lambda}\right)^{\theta}\right]^{\alpha}\right\}^{\beta} - \frac{i}{n+1}\right]^2$$

with respect to α, λ, θ, and β. WLS estimators can be obtained by minimizing the sum of squares errors

$$\sum_{i=1}^{n}\frac{1}{\mathrm{Var}\left(\mathrm{F}\left(X_{i:n}\right)\right)}\left[\mathrm{F}\left(X_{i:n}\right) - \mathrm{E}\left(\mathrm{F}\left(X_{i:n}\right)\right)\right]^2$$

with respect to the unknown parameters α, λ, θ, and β. Therefore, the WLS estimators can be obtained by minimizing the following quantity

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[\left\{1-\left[1-\left(1-(1+x_{i:n})^{-\lambda}\right)^{\theta}\right]^{\alpha}\right\}^{\beta}-\frac{i}{n+1}\right]^2$$

with respect to the parameters α, λ, θ, and β.

Bayesian Estimation and MCMC Approach

The Bayesian estimator using squared error loss (SEL) function under the assumption of uniform priors of the population parameters for EGEP distribution is obtained. We consider the Bayesian estimation (BE) under the assumption that the random variables α, λ, θ, and β independently distributed with non-informative type of priors

$$\mathrm{g}(\alpha)\propto\alpha^{-1},\mathrm{g}(\lambda)\propto\lambda^{-1},\mathrm{g}(\beta)\propto\beta^{-1},\text{ and }\mathrm{g}(\theta)\propto\theta^{-1}.$$

Hence, the joint prior pdf of the unknown parameters can be expressed by

$$\pi^*(\alpha,\lambda,\beta,\theta)\propto(\alpha\lambda\beta\theta)^{-1}. \tag{18}$$

Combining (17) and (18) to obtain the posterior density of $\phi = (\alpha, \lambda, \theta, \beta)$ given the data as follows:

$$\pi(\phi\,|\,x)\propto(\alpha\beta\lambda\theta)^{n-1}\prod_{i=1}^{n}S_i^{-(\lambda+1)}\left(1-S_i^{-\lambda}\right)^{\theta-1}\left[1-\left(1-S_i^{-\lambda}\right)^{\theta}\right]^{\alpha-1}\times\left\{1-\left[1-\left(1-S_i^{-\lambda}\right)^{\theta}\right]^{\alpha}\right\}^{\beta-1} \tag{19}$$

Therefore, the Bayesian estimator of parameters, say $\phi = (\alpha, \lambda, \beta, \theta)$ under SEL function; denoted by $\tilde{u}_{(SEL)}(\phi)$ can be calculated through the following equations as follows:

$$\tilde{u}_{(\mathrm{SEL})}(\phi)=\mathrm{E}(\phi\,|\,\underline{x})=\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\phi\,\mathrm{L}(\phi\,|\,x)\,\pi(\phi\,|\,x)\,d\alpha\, d\lambda\, d\theta\, d\beta\,. \tag{20}$$

Generally, the ratio of four integrals given by equations (19) and (20) cannot be obtained in a closed form. In this case, we use the MCMC technique to generate samples from the posterior distributions and then compute the Bayesian estimators of the individual parameters. The conditional posterior densities of α, λ, θ, and β are as follows:

$$\pi_1(\alpha \mid x) \propto \alpha^{n-2} \prod_{i=1}^{n} \left[1-\left(1-S_i^{-\lambda}\right)^{\theta}\right]^{\alpha-1} \left\{1-\left[1-\left(1-S_i^{-\lambda}\right)^{\theta}\right]^{\alpha}\right\}^{\beta-1}$$

$$\pi_2(\lambda \mid x) \propto \lambda^{n-2} \prod_{i=1}^{n} S_i^{-(\lambda+1)} \left(1-S_i^{-\lambda}\right)^{\theta-1} \left[1-\left(1-S_i^{-\lambda}\right)^{\theta}\right]^{\alpha-1} \left\{1-\left[1-\left(1-S_i^{-\lambda}\right)^{\theta}\right]^{\alpha}\right\}^{\beta-1}$$

$$\pi_3(\theta \mid x) \propto \theta^{n-2} \prod_{i=1}^{n} \left(1-S_i^{-\lambda}\right)^{\theta-1} \left[1-\left(1-S_i^{-\lambda}\right)^{\theta}\right]^{\alpha-1} \left\{1-\left[1-\left(1-S_i^{-\lambda}\right)^{\theta}\right]^{\alpha}\right\}^{\beta-1}$$

and

$$\pi_4(\beta \mid x) \propto \beta^{n-2} \prod_{i=1}^{n} \left\{1-\left[1-\left(1-S_i^{-\lambda}\right)^{\theta}\right]^{\alpha}\right\}^{\beta-1}$$

The Bayesian estimators are computed using the idea of Markov Chain Monte Carlo (MCMC) method based on Gibbs sampling. Therefore, to generate from this distribution, we use the Metropolis-Hastings method. To run the Gibbs sampler algorithm, we start with the ML estimates. We then draw samples from various full conditionals, in run, using the most recent values of all other conditioning variables unless some systematic pattern of convergence is achieved.

Gibbs Sampling Algorithm

The algorithm of Gibbs sampling is described as follows:

Step 1: Start with ($\alpha^{(0)} = \alpha$, $\lambda^{(0)} = \lambda$, $\theta^{(0)} = \theta$, $\beta^{(0)} = \beta$) and set $I = 1$.
Step 2: Generate α^I from $\pi_1(\alpha \mid x)$.
Step 3: Generate λ^I from $\pi_2(\lambda \mid x)$.
Step 4: Generate θ^I from $\pi_3(\theta \mid x)$.
Step 5: Generate β^I from $\pi_4(\beta \mid x)$.
Step 6: Compute α^I, λ^I, θ^I, and β^I.
Step 7: Set $I = I + 1$.
Step 8: Repeat steps 2-6 N times.

Step 9: We obtain the Bayes MCMC point estimate of $\phi_q \equiv (\phi_1 = \alpha, \phi_2 = \lambda, \phi_3 = \theta, \phi_4 = \beta)$, $q = 1, 2, 3,$ and 4 as

$$\mathrm{E}\left(\phi_q \mid \text{data}\right) \propto \frac{1}{N-M} \sum_{i=M+1}^{N} \phi_q^{(i)} ,$$

where M is the burn-in period (that is, a number of iterations before the stationary distribution is achieved) and the posterior variance of ϕ becomes

$$\mathrm{Var}\left(\phi_q \mid \text{data}\right) \propto \frac{1}{N-M} \sum_{i=M+1}^{N} \left[\phi_q^{(i)} - \hat{\mathrm{E}}\left(\phi_q \mid \text{data}\right)\right]^2 .$$

Simulation Study

The derived expressions for the estimators are too complicated to study analytically. Consequently, a numerical study is performed to compare the different estimators. The performances of the different estimates are compared in terms of their mean square error (MSE) and standard error (SE). The numerical procedure is described below:

- 10000 random samples of sizes 10, 20, 30, and 50 are generated from the EGEP distribution.
- Four sets of parameter values are selected as:
  Case I $\equiv (\alpha = 0.4, \lambda = 0.7, \theta = 0.25, \beta = 1.2)$,
  Case II $\equiv (\alpha = 0.5, \lambda = 0.7, \theta = 0.25, \beta = 1.5)$,
  Case III $\equiv (\alpha = 1, \lambda = 2.6, \theta = 2, \beta = 0.75)$, and
  Case IV $\equiv (\alpha = 0.4, \lambda = 0.5, \theta = 0.7, \beta = 0.75)$,
- The ML, LS and WLS estimates of the unknown parameters are obtained.
- MSEs and SEs of different estimates of unknown parameters are computed.

Table 1. MSEs, SEs, and MC errors of EGEP estimates for Case I and Case II

			Case I				Case II			
n	**Method**	**Properties**	***α*=0.4**	***λ*=0.7**	***θ*=0.25**	***β*=1.2**	***α*=0.5**	***λ*=0.7**	***θ*=0.25**	***β*=1.5**
10	ML	MSE	1.697	1.149	0.630	0.608	0.763	1.419	1.016	1.811
		SE	0.121	0.101	0.076	0.075	0.083	0.108	0.095	0.130
	LS	MSE	0.116	0.492	0.075	0.230	0.136	0.624	0.134	0.481
		SE	0.034	0.064	0.024	0.042	0.037	0.071	0.033	0.063
	WLS	MSE	0.122	0.556	0.076	0.206	0.135	0.741	0.156	0.495
		SE	0.035	0.068	0.025	0.040	0.037	0.076	0.036	0.065
	BE	MSE	0.344	0.594	0.535	0.333	0.202	0.555	0.480	0.794
		SE	0.005	0.049	9.300*	0.021	0.003	0.047	8.900*	0.061
		MC Error	1.134*	0.015	2.803*	6.411*	0.888*	0.015	2.735*	0.019
20	ML	MSE	0.702	0.704	0.078	0.188	0.429	0.792	0.274	0.903
		SE	0.039	0.040	0.013	0.022	0.032	0.041	0.024	0.048
	LS	MSE	0.059	0.327	0.045	0.141	0.062	0.449	0.081	0.347
		SE	0.012	0.026	9.044*	0.013	0.120	0.029	0.013	0.024
	WLS	MSE	0.080	0.361	0.060	0.146	0.061	0.529	0.085	0.358
		SE	0.014	0.027	0.011	0.014	0.012	0.032	0.013	0.026
	BE	MSE	0.189	0.148	0.351	0.074	0.087	0.195	0.312	0.467
		SE	0.005	0.026	9.600*	0.034	1.500*	0.044	0.011	0.013
		MC Error	2.248*	0.012	4.241*	0.015	0.584*	0.020	4.752*	5.490*
30	ML	MSE	0.440	0.542	0.045	0.128	0.202	0.502	0.142	0.468
		SE	0.021	0.023	6.217*	0.011	0.015	0.022	0.011	0.022
	LS	MSE	0.038	0.263	0.038	0.133	0.040	0.365	0.065	0.306
		SE	6.266*	0.015	5.420*	8.263*	6.162*	0.017	7.244*	0.014
	WLS	MSE	0.037	0.275	0.048	0.136	0.046	0.382	0.071	0.313
		SE	6.249*	0.016	6.161*	8.230*	6.820*	0.018	7.625*	0.014
	BE	MSE	0.159	0.020	0.296	0.045	0.062	7.870*	0.226	0.240
		SE	0.004	0.008	0.006	0.014	0.500*	7.100*	5.200*	9.200*
		MC Error	0.165*	4.424*	3.447*	7.548*	0.182*	3.843*	2.711*	4.907*
50	ML	MSE	0.218	0.335	0.037	0.097	0.149	0.307	0.079	0.354
		SE	9.118*	0.011	3.151*	4.681*	7.665*	0.010	4.680*	0.010
	LS	MSE	0.019	0.172	0.036	0.122	0.031	0.230	0.051	0.279
		SE	2.584*	7.505*	3.062*	3.906*	3.067*	8.328*	3.631*	6.461*
	WLS	MSE	0.021	0.173	0.045	0.136	0.031	0.226	0.052	0.268
		SE	2.718*	7.498*	3.338*	4.073*	3.164*	8.274*	3.712*	6.812*
	BE	MSE	0.022	6.570*	0.119	3.730*	0.370*	0.260*	0.083	0.033
		SE	0.300*	0.009	0.004	6.500*	0.200*	2.400*	2.800*	1.400*
		MC Error	0.163*	6.632*	2.937*	4.585*	0.080*	1.638*	1.898*	9.899*

Note: * Indicates that the value is multiplied by 10^{-3}

Table 2. MSEs, SEs, and MC errors of EGEP estimates for Case III and Case IV

			Case III				Case IV			
n	**Method**	**Properties**	***α*=1**	***λ*=2.6**	***θ*=2**	***β*=0.75**	***α*=0.4**	***λ*=0.9**	***θ*=0.7**	***β*=0.75**
10	ML	MSE	0.164	1.142	2.615	2.673	1.259	1.066	0.542	1.176
		SE	0.040	0.087	0.158	0.145	0.106	0.095	0.073	0.086
	LS	MSE	0.145	0.928	2.020	1.278	0.041	0.673	0.132	0.692
		SE	0.033	0.074	0.139	0.108	0.018	0.067	0.029	0.077
	WLS	MSE	0.145	0.905	2.232	1.269	0.060	0.854	0.136	0.746
		SE	0.033	0.071	0.148	0.106	0.023	0.076	0.029	0.080
	BE	MSE	0.612	1.051	0.686	0.979	0.341	0.452	0.381	0.325
		SE	5.850*	0.206	0.231	0.027	8.900*	0.055	0.018	0.018
		MC Error	1.329*	8.162*	0.065	0.073	2.661*	0.017	5.438*	5.363*
20	ML	MSE	0.060	0.521	1.480	0.840	0.955	0.735	0.195	0.591
		SE	0.012	0.030	0.060	0.041	0.047	0.040	0.022	0.031
	LS	MSE	0.094	0.537	1.081	0.604	0.023	0.494	0.090	0.250
		SE	0.011	0.022	0.052	0.037	6.063*	0.026	9.708*	0.021
	WLS	MSE	0.085	0.525	1.113	0.668	0.046	0.705	0.079	0.319
		SE	0.011	0.022	0.053	0.039	0.010	0.034	9.380*	0.024
	BE	MSE	0.579	0.329	0.142	0.923	0.268	0.257	0.281	0.237
		SE	0.016	0.117	0.076	0.034	0.010	0.043	0.021	0.011
		MC Error	6.806*	0.015	0.052	0.034	4.546*	0.019	9.202*	4.719*
30	ML	MSE	0.041	0.342	1.290	0.590	0.310	0.497	0.123	0.293
		SE	6.466*	0.017	0.038	0.023	0.018	0.022	0.011	0.014
	LS	MSE	0.083	0.476	0.788	0.427	0.021	0.454	0.090	0.199
		SE	5.914*	0.013	0.029	0.021	3.654*	0.016	5.835*	0.012
	WLS	MSE	0.069	0.445	0.632	0.789	0.039	0.694	0.076	0.208
		SE	5.270*	0.012	0.027	0.028	5.895*	0.021	5.669*	9.454*
	BE	MSE	0.039	0.473	0.087	0.793	0.135	0.034	0.086	0.037
		SE	1.297*	0.045	0.031	0.019	0.800*	0.021	0.011	7.800*
		MC Error	0.373*	0.024	0.017	0.011	0.368*	0.012	6.042*	4.252*
50	ML	MSE	0.041	0.298	1.305	0.522	0.223	0.347	0.067	0.180
		SE	3.698*	9.216*	0.023	0.013	9.185*	0.011	4.786*	6.670*
	LS	MSE	0.069	0.442	0.623	0.575	0.023	0.395	0.077	0.157
		SE	2.801*	5.658*	0.015	0.015	2.471*	9.006*	3.113*	5.953*
	WLS	MSE	0.054	0.381	0.432	0.298	0.023	0.576	0.065	0.196
		SE	2.504*	5.256*	0.014	0.010	2.541*	0.012	3.197*	7.415*
	BE	MSE	0.037	0.431	0.049	0.784	9.860*	5.660*	0.014	2.890*
		SE	3.499*	0.090	0.030	9.603*	0.200*	9.400*	6.200*	7.300*
		MC Error	2.228*	0.064	0.021	6.768*	0.896*	6.633*	4.341*	5.192*

Note: * Indicates that the value is multiplied by 10^{-3}

The following conclusions can be detected about the performance of different estimates:

- In all cases, the MSEs of Bayesian estimates are better than MSEs of ML, LS, and WLS estimates (see Tables 1 and 2).
- For all methods of estimations, it is clear that MSEs, SEs, and MC error decrease as sample size increases (see Tables 1 and 2).
- The MSEs and SEs of LS estimates, for all parameters values, are the smallest among the other estimates in almost all of the cases (see for example Table 1).
- For fixed value of $\lambda = 0.7$, $\theta = 0.25$ and as the value of shape parameters α, β increases, the MSEs and SEs for estimates λ, θ are increasing based on ML, LS and WLS methods but decreasing based on BE. Also, the MSEs and SEs for estimates α, are decreasing based on ML and Bayesian methods but increasing based on LS and WLS methods. In addition, the MSEs and SEs for estimates are increasing based on ML, LS, and WLS methods while decreasing based on BE (see Table 1).
- When the values of α, θ, and λ increase and the shape parameter β decrease, the MSEs and SEs for estimates of the unknown parameters based on ML, LS, WLS, and BE methods are increasing in approximately most of situations (see Tables 1 and 2).
- For the Case III of parameters ($\alpha = 1$, $\lambda = 2.6$, $\theta = 2$, $\beta = 0.75$) the Bayesian estimates have good statistical properties than the other cases of parameters (see Tables 1 and 2).

Application

We provide an application to real data set to illustrate the importance and flexibility of the EGEP distribution. The real data represent the survival times, in weeks, of 33 patients suffering from acute Myelogeneous Leukaemia. The data were analyzed by Feigl and Zelen (1965). The data are: 65, 156, 100, 134, 16, 108, 121, 4, 39, 143, 56, 26, 22, 1, 1, 5, 65, 56, 65, 17, 7, 16, 22, 3, 4, 2, 3, 8, 4, 3, 30, 4, 43.

We compare the fit of EGEP distribution to sub-models; EP and SP distributions; with competitive model, namely, additive Weibull (AW) (Almalki & Yuan, 2013).

In order to compare the models, we consider some goodness-of-fit measures including: $-2L$ where L is the maximized log-likelihood, Akaike information criterion (AIC), Bayesian information criterion (BIC), and the corrected Akaike information criterion (CAIC). Generally, the smaller values of these statistics, the better fit to real data. Additionally, the histogram plots and the estimated pdf of the models for the data set are shown. Furthermore, the plots of empirical cdf and estimated cdf of models are displayed.

Table 3. ML estimates, –2*L*, AIC, BIC, and CAIC for acute Myelogeneous Leukaemia data

Model	ML estimates	–2*L*	AIC	BIC	CAIC
EGEP	$\alpha = 1.000$	303.234	311.234	317.097	319.420
	$\beta = 4.205$				
	$\theta = 1.000$				
	$\lambda = 0.710$				
AW	$a = 2.003$	305.071	316.425	321.203	326.112
	$b = 5.642$				
	$c = 3.351$				
	$d = 1.830$				
EP	$\theta = 5.271$	323.696	325.696	327.162	327.763
	$\lambda = 6.032$				
SP	$\lambda = 6.032$	394.686	398.686	401.617	402.789

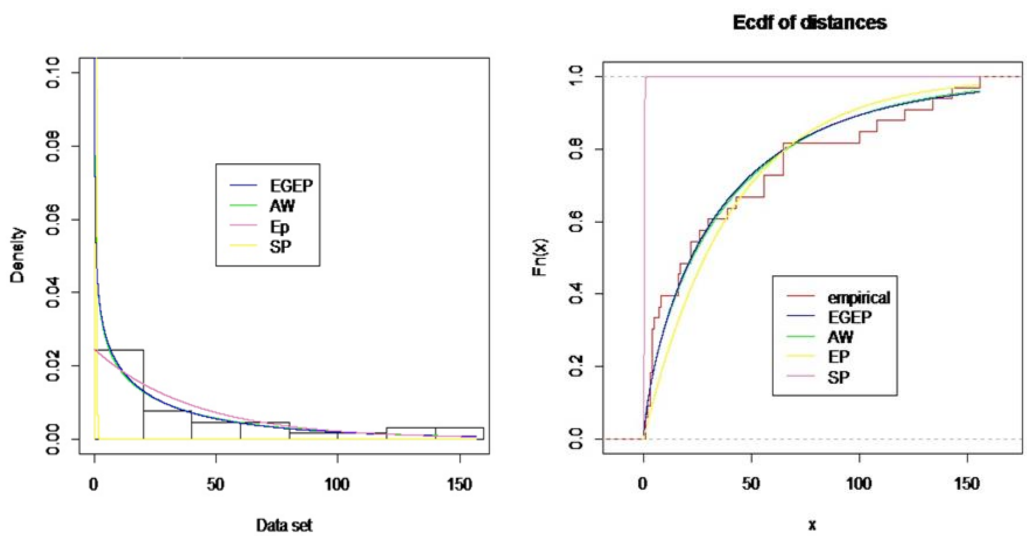

Figure 3. Estimated pdf and cdf of the models for the real data set

In general, the model with minimum values for these statistics could be chosen as the best model to fit the data. Table 3 lists the ML estimates of the model parameters and the values of statistics; –2*L*, AIC, BIC, and CAIC for the fitted models to the real data set. It is noted from Table 3 that the EGEP distribution gives the lowest values for the –2*L*, AIC, BIC, and CAIC statistics among all fitted models. Thus, the EGEP distribution could be chosen as the best models.

The results show that the EGEP distribution provides a significantly better fit than the other three models. Plots of the estimated pdf and cdf of the EGEP, AW, EP, and SP models fitted to this data set are given in Figure 3. The figures indicate that the EGEP distribution is superior to the other distributions in terms of model fitting.

Concluding Remarks

We introduce and study a new lifetime distribution, the so-called the exponentiated generalized exponentiated Pareto distribution as a generalization of the Pareto and the exponentiated Pareto distributions. A detailed study on the statistical properties of the new distribution is presented. The ordinary moments, Rényi entropy and moments of residual are derived. Estimation of parameters is approached by maximum likelihood, least squares, weighted least squares and Bayesian methods. A simulation study is implemented for investigating the accuracy of different estimates for different sample sizes. An application to real data shows the superiority of the proposed model than some other models.

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