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Shrinkage estimators are introduced for the scale parameter of the Rayleigh distribution by using two different shrinkage techniques. The mean squared error properties of the proposed estimator have been derived. The comparison of proposed classes of the estimators is made with the respective conventional unbiased estimators by means of mean squared error in the simulation study. Simulation results show that the proposed shrinkage estimators yield smaller mean squared error than the existence of unbiased estimators.

Keywords: Rayleigh distribution, scale parameter, shrinkage estimator, mean squared error

Introduction

Unbiased estimators are commonly used for making inferences about an unknown descriptive parameter(s) of a population. In cases, when unbiased estimators possess larger variances, biased estimators with smaller mean square error (MSE) are preferred. Searls (1964), Thompson (1968), Mehta and Srinivasan (1971), Das (1975), Srivastava et al. (1980), Rao and Singh (1982), Bhatnagar (1986), Singh and Katyar (1988), Rytgaard (1990), Jani (1991), Kourouklis (1994), Singh and Singh (1997), Singh and Shukla (2003), Singh and Saxena (2003), Prakash et al.

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(2006), Prakash (2009), Ebegil and Ozdemir (2016) and many others proposed numerous biased estimators as an alternative to the unbiased estimator.

Rayleigh distribution was introduced by Rayleigh (1880) after facing a problem in acoustics. It has a wide range of applications in almost every field of science where the phenomena of interest assume positive value. In this study, we deal with two different classes of shrinkage-type estimators for the scale parameter of the Rayleigh distribution. The proposed classes of the estimators will be compared with the usual unbiased estimators in terms of MSE by deriving optimal conditions. For this purpose, a simulation study will be conducted for numerical comparisons.

The Class of Shrinkage Estimators

Jani (1991) and Singh and Singh (1997) proposed two different shrinkage estimator classes for the scale parameter of the exponential and normal distributions. Shrinkage estimator class of Jani for the scale parameter of the exponential distribution is defined in Equation (1):

$$T_p = \theta_0 \left[1 + k \left(\frac{\theta_0}{\hat{\theta}} \right)^p \right], \tag{1}$$

where θ_0 is an a priori value of the parameter θ, k is a shrinkage factor that minimizes the MSE value, p is a nonzero real number, and $\hat{\theta}_0$ is the unbiased estimator of the parameter θ. Singh and Singh (1997) considered the estimation problem of population variance by adapting the same class defined in Equation (1) for a normal population. The class of estimators by Singh and Singh is given in Equation (2):

$$\hat{\sigma}^2_{(p)} = \sigma_0^2 \left[1 + w \left(\frac{s^2}{\sigma_0^2} \right)^p \right], \tag{2}$$

where σ_0^2 is an a priori value of the parameter σ^2, w is a shrinkage factor that minimizes the MSE value, p is a nonzero real number, and s^2 is an unbiased estimator of the parameter σ^2.

Similarly, Ebegil and Ozdemir (2016) proposed biased estimators for the shape parameter of the classical Pareto distribution using two different shrinkage techniques. One of the two classes proposed by Singh and Singh (1997) is given below:

$$\beta^*_{(p)} = \beta_0 + \left(\hat{\beta} - \beta_0\right) k_{(p)}, \tag{3}$$

where

$$k_{(p)} = \left(n-1\right)^p \frac{\left(n+p-1\right)!}{\left(n+2p-1\right)!} \tag{4}$$

and p is a nonzero real number.

Next, a shrinkage type of estimators will be developed having smaller MSEs as compared to the unbiased estimator for the scale parameter of Rayleigh distribution. Specifically, we develop the estimators in line to Equations (1)-(3).

Let X be a random variable following the Rayleigh distribution. The probability density function (pdf) of X is given by

$$\mathrm{f}\left(x\right) = \frac{x}{\theta} e^{\frac{-x^2}{2\theta}}; x \geq 0, \theta > 0, \tag{5}$$

where, θ is the scale parameter of the Rayleigh distribution. The maximum likelihood estimator (MLE) for the scale parameter θ of the Rayleigh distribution is

$$\hat{\theta}_{\mathrm{MLE}} = \frac{\sum_{i=1}^{n} X_i^2}{2n} = \frac{Y}{2n}.$$

It is evident that Y has G(n, 2θ) and the pdf of Y is given by

$$\mathrm{f}\left(y\right) = \frac{1}{\Gamma\left(n\right)\left(2\theta\right)^n} y^{n-1} e^{\frac{-y}{2\theta}}.$$

The estimator $\hat{\theta}_{\mathrm{MLE}}$ is unbiased having variance

$$\mathrm{Var}\left(\hat{\theta}_{\mathrm{MLE}}\right) = \frac{\theta^2}{n}.$$

The Proposed Shrinkage-Type Estimators

Two different classes of shrinkage-type estimators for the scale parameter of the Rayleigh distribution are derived and discussed below.

Theorem 1. Following Singh and Singh (1997), the class of shrinkage estimators for the scale parameter of Rayleigh distribution is given by

$$\theta_p^* = \theta_0 + k_{(p)}\left(\hat{\theta}_{\mathrm{MLE}} - \theta_0\right). \tag{6}$$

The MSE and bias of θ_p^* are defined in Equation (7) and Equation (8), respectively:

$$\mathrm{MSE}\left(\theta_p^*\right) = \theta^2\left[\frac{k_{(p)}^2}{n} + \left(k_{(p)} - 1\right)^2\left(1-\lambda^{-1}\right)^2\right], \tag{7}$$

$$\mathrm{Bias}\left(\theta_p^*\right) = \left(1 - k_{(p)}\right)\left(\theta_0 - \theta\right), \tag{8}$$

where

$$k_{(p)} = \frac{(n+p-1)!\,n^p}{(n+2p-1)!} \quad \text{and} \quad \lambda = \frac{\theta}{\theta_0}.$$

Proof. From Equation (2), we have

$$\theta_p^* = \theta_0\left[1 + k\left(\frac{\left(\hat{\theta}_{\mathrm{MLE}}\right)^p}{\theta_0^p}\right)\right]. \tag{9}$$

Following equations will be used to derive the MSE of the class of estimator in Equation (9).

$$\mathrm{E}\left[\left(\hat{\theta}_{\mathrm{MLE}}\right)^{jp}\right] = K_{1(jp)}\theta^{jp}, \left(j = 1, 2\right) \tag{10}$$

and

$$K_{1(jp)} = \frac{(n + jp - 1)!}{n^{jp}(n-1)!}. \tag{11}$$

The MSE of the θ_p^* estimator is defined as

$$\mathrm{MSE}\left(\theta_p^*\right) = \mathrm{E}\left[\theta_p^* - \theta\right]^2 \tag{12}$$

Using Equations (9)-(11) in Equation (12), after necessary calculations, we get

$$\mathrm{MSE}\left(\theta_p^*\right) = \theta^2\left[\left(1-\lambda^{-1}\right)^2 + k^2\lambda^{2p-2}K_{1(2p)} + 2kK_{1(p)}\lambda^{p-2}\left(1-\lambda\right)\right]. \tag{13}$$

To obtain optimum value of k, Equation (13) is differentiated with respect to k and set equal to zero. After simplification, we get

$$k = \frac{K_{(1p)}}{K_{(2p)}}\lambda^{-p}\left(\lambda - 1\right), \tag{14}$$

which minimizes the MSE of Equation (13). After substituting the required values in Equation (14), k is defined as

$$k = k(p)\left(\frac{\theta}{\theta_0}\right)^p\left(\frac{\theta}{\theta_0} - 1\right). \tag{15}$$

The shrinking parameter k is obtained as a function of the parameter θ. In practice, it is impossible to obtain a true value of the parameter θ. Therefore, the unknown parameter of Equation (15) is replaced by its unbiased estimator. So, the estimator for k is obtained as

$$\hat{k} = k(p)\left(\frac{\hat{\theta}_{\mathrm{MLE}}}{\theta_0}\right)^p\left(\frac{\hat{\theta}_{\mathrm{MLE}}}{\theta_0} - 1\right) = k(p)\left(\frac{\hat{\theta}_{\mathrm{MLE}}}{\theta_0}\right)^p\left(\frac{\hat{\theta}_{\mathrm{MLE}} - \theta_0}{\theta_0}\right). \tag{16}$$

After necessary adjustment, the class of estimators for the scale parameter of the Rayleigh distribution is derived as given in Equation (6).

The MSE of the θ^* estimator is derived by using Equation (16) from Equation (13) as follows:

$$\mathrm{MSE}\left(\theta_p^*\right) = k_{(p)}^2 \frac{\theta^2}{n} + \left(k_{(p)} - 1\right)^2 \left(\theta - \theta_0\right)^2 . \tag{17}$$

Simplification of Equation (17) can be reduced to Equation (7).

The bias of θ_p^* is computed as

$$\mathrm{Bias}\left(\theta_p^*\right) = \mathrm{E}\left(\theta_p^*\right) - \theta . \tag{18}$$

Taking the expectation of Equation (12) and using Equations (10) and (11) in Equation (18), we get Equation (8), which completes the proof.

The Relative Efficiency of the First Estimator

The relative efficiency of the class of estimators θ_p^* with respect to $\hat{\theta}_{\mathrm{MLE}}$ is obtained as

$$\text{Relative Efficiency} = \frac{\mathrm{MSE}\left(\theta_p^*\right)}{\mathrm{Var}\left(\hat{\theta}_{\mathrm{MLE}}\right)} = k_{(p)}^2 + \left(n-2\right) + n\left(k_{(p)-1}\right)^2 \left(1 - \lambda^{-1}\right)^2 . \tag{19}$$

Hence, it is clear that $\mathrm{MSE}\left(\theta_p^*\right) \le \mathrm{Var}\left(\hat{\theta}_{\mathrm{MLE}}\right) < 1$.

Case 1

Consider $p = 1$ in Equation (6). Then an estimator is obtained as

$$\theta_1^* = \theta_0 + \frac{n}{n+1}\left(\hat{\theta}_{\mathrm{MLE}} - \theta_0\right).$$

The MSE of this estimator is calculated as

$$\operatorname{MSE}\left(\theta_1^*\right)=\theta^2\left[\frac{n^2}{(n+1)}+\frac{n}{(n+1)}\left(1-\lambda^{-1}\right)^2\right].$$

The relative efficiency of the estimator θ_1^* with respect to $\hat{\theta}_{\mathrm{MLE}}$ is

$$\frac{\operatorname{MSE}\left(\theta_1^*\right)}{\operatorname{Var}\left(\hat{\theta}_{\mathrm{MLE}}\right)}=n^2\left[\frac{n}{(n+1)}+\frac{1}{(n+1)}\left(1-\lambda^{-1}\right)^2\right].$$

It is clear that θ_1^* is better than $\hat{\theta}_{\mathrm{MLE}}$ if

$$\frac{\operatorname{MSE}\left(\theta_1^*\right)}{\operatorname{Var}\left(\hat{\theta}_{\mathrm{MLE}}\right)}\leq 1 \quad \text{or} \quad \left(1-\lambda^{-1}\right)^2\leq\frac{1-n^2}{n^2}$$

when n tends to ∞, the above inequality reduces to $0 \leq \lambda \leq 1$.

Case 2

Suppose $p = 2$ in Equation (6), then the following estimator is obtained:

$$\theta_2^*=\theta_0+\frac{n^2}{(n+3)}\left(\hat{\theta}_{\mathrm{MLE}}-\theta_0\right)$$

The MSE of the estimator θ_2^* is defined as

$$\operatorname{MSE}\left(\theta_2^*\right)=\theta^2\left[n\left(\frac{n^4}{(n+3)^2(n+2)^2}\right)+\frac{n^2-(n+3)(n+2)}{(n+3)(n+2)}\left(1-\lambda^{-1}\right)^2\right].$$

The relative efficiency of the estimator θ_2^* with respect to $\hat{\theta}_{\mathrm{MLE}}$ is

$$\frac{\operatorname{MSE}\left(\theta_2^*\right)}{\operatorname{Var}\left(\hat{\theta}_{\mathrm{MLE}}\right)}=\left[n\left(\frac{n^2}{(n+3)(n+2)}\right)^2+\left(\frac{n^2}{(n+3)(n+2)}-1\right)^2\left(1-\lambda^{-1}\right)^2\right].$$

The estimator θ_2^* is more efficient than $\hat{\theta}_{\text{MLE}}$ if and only if

$$\frac{\text{MSE}\left(\theta_2^*\right)}{\text{Var}\left(\hat{\theta}_{\text{MLE}}\right)} < 1$$

$$\frac{1}{1+\sqrt{\dfrac{2n^2+5n+6}{n(5n+6)}}} \leq \lambda \leq \frac{1}{1-\sqrt{\dfrac{2n^2+5n+6}{n(5n+6)}}}$$

Furthermore, when n is very large, i.e., $n \to \infty$, then

$$\left(1-\lambda^{-1}\right)^2 \leq \frac{\left(2+\dfrac{5}{n}+\dfrac{6}{n^2}\right)}{\left(5+\dfrac{6}{n}\right)}$$

reduces to $0.61 \leq \lambda \leq 1.63$.

Theorem 2. Following Jani (1991), the class of shrinkage estimators for the scale parameter of the Rayleigh distribution is defined as

$$\theta_p^{**} = \theta_0 + w_{(p)}\left(\hat{\theta}_{\text{MLE}} - \theta_0\right) \tag{20}$$

where

$$w_{(p)} = n^{-p}\,\frac{(n-p-1)!}{(n-2p-1)!}.$$

The bias of θ_p^{**} is given as

$$\text{Bias}\left(\theta_p^{**}\right) = \text{E}\left(\theta_p^{**}\right) - \theta = \left(1-w_{(p)}\right)\left(\theta-\theta_0\right). \tag{21}$$

The MSE of θ_p^{**} is given below:

$$\text{MSE}\left(\theta_p^{**}\right)=\theta^2\left[\frac{w_{(p)}^2}{n}+\left(w_{(p)}-1\right)^2\left(1-\lambda^{-1}\right)^2\right]. \tag{22}$$

Proof. From Equation (1), the class of shrinkage-type estimators for the scale parameter of the Rayleigh distribution is defined as

$$\theta_p^{**}=\theta_0\left[1+w\left(\frac{\theta_0}{\hat{\theta}_{\text{MLE}}}\right)^p\right]. \tag{23}$$

Here,

$$\text{E}\left(\hat{\theta}_{\text{MLE}}\right)^{-jp}=W_{(jp)}\theta^{-jp} \tag{24}$$

and

$$W_{(jp)}=\frac{(n-jp-1)!n^{jp}}{(n-1)!}. \tag{25}$$

The expression in Equations (24) and (25) are used to calculate the MSE of θ_p^{**}. The MSE of θ_p^{**} is derived as

$$\text{MSE}\left(\theta_p^{**}\right)=\text{E}\left(\theta_p^{**}-\theta\right)^2. \tag{26}$$

Using Equations (23)-(25) in Equation (26), the MSE of θ_p^{**} is obtained as

$$\text{MSE}\left(\theta_p^{**}\right)=\theta^2\left[\left(1+\lambda^{-1}\right)^2+w^2W_{(2p)}\lambda^{-2(p+1)}+2wW_{(1p)}\lambda^{(-p-1)}\left(\lambda^{-1}-1\right)\right]. \tag{27}$$

Differentiating Equation (27) with respect to *w* and equating to zero, we find an optimum value of *w* as given below:

$$w = w_{(p)} \left(\frac{\theta}{\theta_0} \right)^p \left(\frac{\theta}{\theta_0} - 1 \right), \tag{28}$$

where

$$w_{(p)} = \frac{W_{(1p)}}{W_{(2p)}}. \tag{29}$$

On substituting $W_{(1p)}$ and $W_{(2p)}$ in Equation (29), we get

$$w_{(p)} = \frac{(n-p-1)!}{(n-2p-1)!n^p}. \tag{30}$$

Now, using Equation (30) in Equation (28), we get

$$w = \left(\frac{(n-p-1)!}{(n-2p-1)!(n^p)} \right) \frac{\theta^p}{\theta_0^p} \left(\frac{\theta}{\theta_0} - 1 \right). \tag{31}$$

The shrinkage parameter w is obtained as a function of the unknown parameter θ. So, the unknown parameter in Equation (31) is replaced by its unbiased estimator. So, the $\hat{w}$ estimator for w is computed as

$$\hat{w} = \left(\frac{(n-p-1)!}{(n-2p-1)!(n^p)} \right) \frac{\hat{\theta}^p_{\text{MLE}}}{\theta_0^p} \left(\frac{\hat{\theta}^p_{\text{MLE}}}{\theta_0} - 1 \right). \tag{32}$$

Finally, substituting Equation (32) into Equation (23), the class of estimators for the scale parameter of the Rayleigh distribution is obtained as

$$\theta_p^{**} = \theta_0 + w_{(p)} \left(\hat{\theta}_{\text{MLE}} - \theta_0 \right).$$

Similarly, from Equations (25), (27), and (30), the MSE of θ_p^{**} is obtained as

$$\mathrm{MSE}\left(\theta_p^*\right)=w_{(p)}^2\,\frac{\theta^2}{n}+\left(w_{(p)}-1\right)^2\left(\theta-\theta_0\right)^2 \tag{33}$$

After simplification, Equation (33) can be reduced to Equation (22).

The bias of θ_p^{**} is obtained as

$$\mathrm{Bias}\left(\theta_p^{**}\right)=\mathrm{E}\left(\theta_p^{**}\right)-\theta\,.$$

By taking the expectation of Equation (23) and substituting Equations (23) and (24) in Equation (33), Equation (33) reduced to Equation (21).

The Relative Efficiency of the Second Estimator

The relative efficiency of θ_p^{**} with respect to $\hat{\theta}_{\mathrm{MLE}}$ is derived as

$$\text{Relative Efficiency}=\frac{\mathrm{MSE}\left(\theta_p^{**}\right)}{\mathrm{Var}\left(\hat{\theta}_{\mathrm{MLE}}\right)}=\left[w_{(p)}^2+n\left(w_{(p)}-1\right)^2\left(1-\lambda^{-1}\right)^2\right].$$

Case 1

Suppose p = 1 in Equation (20); the MSE of the estimator for the scale parameter can be obtained as

$$\theta_1^{**}=\theta_0\left[1+\frac{n-2}{n}\left(\hat{\theta}-\theta\right)\right]. \tag{34}$$

The MSE of the estimator in Equation (34) is defined as

$$\mathrm{MSE}\left(\theta_1^{**}\right)=\theta^2\left[\frac{\left(n-2\right)^2}{n^3}+\frac{4}{n^2}\left(1-\lambda^{-1}\right)^2\right].$$

The relative efficiency of θ_1^{**} with respect to $\hat{\theta}_{\mathrm{MLE}}$ is given by the following expression:

$$\frac{\operatorname{MSE}\left(\theta_1^{**}\right)}{\operatorname{Var}\left(\hat{\theta}_{\mathrm{MLE}}\right)}=\frac{(n-2)^2}{n^2}+\frac{4}{n}\left(1-\lambda^{-1}\right)^2 .$$

The θ_1^{**} estimator will be more efficient than $\hat{\theta}_{\mathrm{MLE}}$ if

$$\frac{\operatorname{MSE}\left(\theta_1^{**}\right)}{\operatorname{Var}\left(\hat{\theta}_{\mathrm{MLE}}\right)}<1,$$

which gives

$$\frac{(n-2)^2}{n^2}+\frac{4}{n}\left(1-\lambda^{-1}\right)^2 \leq 1 .$$

After simplification, the above inequality becomes

$$\left(1-\lambda^{-1}\right)^2 \leq \frac{2n-3}{4} .$$

For $n \to \infty$, this inequality reduces to $0 < \lambda < 1$.

Case 2

By considering $p = 2$ in Equation (20), an estimator for the scale parameter is obtained as

$$\theta_2^{**}=\theta_0+\frac{(n-3)(n-2)}{n^2}\left(\hat{\theta}_{\mathrm{MLE}}-\theta_0\right).$$

The MSE of the θ_2^{**} estimator is given by

$$\operatorname{MSE}\left(\theta_2^{**}\right)=\theta^2\left[n^{-1}\left((n-3)(n-4)\right)^2+\left(\frac{(n-3)(n-4)}{n^2}-1\right)^2\left(1-\lambda^{-1}\right)^2\right].$$

The relative efficiency of θ_2^{**} with respect to $\hat{\theta}_{\mathrm{MLE}}$ is obtained as

$$\frac{\mathrm{MSE}\left(\theta_2^{**}\right)}{\mathrm{Var}\left(\hat{\theta}_{\mathrm{MLE}}\right)}=\left[n^{-1}\left((n-3)(n-4)\right)^2+\left(\frac{(n-3)(n-4)}{n^2}-1\right)^2\left(1-\lambda^{-1}\right)^2\right].$$

It is clear that θ_2^{**} is better than $\hat{\theta}_{\mathrm{MLE}}$ if

$$\frac{\mathrm{MSE}\left(\theta_2^{**}\right)}{\mathrm{Var}\left(\hat{\theta}_{\mathrm{MLE}}\right)}\leq 1.$$

This gives

$$n\left(1-\frac{(n-3)(n-4)}{n^2}\left(1-\lambda^{-1}\right)^2\right)\leq\left(1+\frac{(n-3)(n-4)}{n^2}\right).$$

After simplification, we get

$$\frac{1}{1+\sqrt{\dfrac{2n^2-7n+12}{n(7n-12)}}}\leq\lambda\leq\frac{1}{1-\sqrt{\dfrac{2n^2-7n+12}{n(7n-12)}}}.$$

When n is very large ($n\rightarrow\infty$), the inequality reduces to $0.65\leq\lambda\leq 2.14$.

Note. It can be identified that the class of estimators proposed by Jani (1991) is directly related to the Singh and Singh (1997) estimator for the scale parameter of the Rayleigh distribution. This relationship is expressed as $k_{(-p)}=w_{(p)}$.

Numerical Study

The percent relative bias and MSE are considered as performance criteria to judge the performance of the proposed estimation. A relative bias can be calculated by dividing Equation (11) into Equation (21). The relative bias is given in Equation (35):

$$\frac{\mathrm{Bias}\left(\theta_p^{*}\right)}{\mathrm{Bias}\left(\theta_p^{**}\right)}=\frac{1-k_{(p)}}{1-w_{(p)}}. \tag{35}$$

The values of relative bias are obtained by means of Equation (35) for n = 10, 15, 20, 25, and 50 and for different values of p. These values are shown in Table 1. It is observed that the biases of θ_p^{**} are smaller (larger) than those of θ_p^{*} when p is negative (positive). Moreover, it is seen that the relative bias decreases (increases) when the sample size increases and p is negative (positive). However, it can be noted that the θ_p^{**} have a smaller bias when the value of p is negative, while they have a larger bias when the value of p is positive.

Table 1. The relative bias of θ_p^{*} relative to θ_p^{**} for different n and p values

	Sample size				
p	**10**	**15**	**20**	**25**	**50**
-1.0	2.2000	2.1333	2.1000	2.0800	2.0400
-0.5	5.1860	5.1214	5.0901	5.0717	5.0354
0.5	0.1928	0.1952	0.1964	0.1971	0.1985
1.0	0.4545	0.4687	0.4761	0.4807	0.4902
1.5	0.5589	0.5805	0.5927	0.6006	0.6176
2.0	0.6189	0.6404	0.6546	0.6644	0.6870
5.0	0.6684	0.6814	0.6942	0.7041	0.7298

Table 2. The percent relative efficiency of $\hat{\theta}_{\text{MLE}}$ with respect to θ_p^{**} estimator

		Sample size				
λ	***p***	**10**	**15**	**20**	**25**	**50**
0.625	-1.0	112.3948	108.2524	106.1851	104.9460	102.4709
	-0.5	102.4065	101.5986	101.1968	100.9564	100.4771
	0.0	100.0000	100.0000	100.0000	100.0000	100.0000
	0.5	110.2408	106.6603	104.9338	103.9179	101.9301
	1.0	105.6803	104.6816	103.8287	103.2098	101.7449
	1.5	66.3761	74.9265	80.0713	83.4819	91.1228
	2.0	36.3377	43.3610	49.2513	54.1038	69.0787
	2.5	22.6909	25.7553	29.2873	32.6959	46.2440
2.500	-1.0	118.0488	111.9126	108.8889	107.0891	103.5223
	-0.5	102.4899	101.6536	101.2378	100.9890	100.4933
	0.0	100.0000	100.0000	100.0000	100.0000	100.0000
	0.5	112.9021	108.2732	106.0883	104.8161	102.3553
	1.0	135.1351	122.2826	116.2791	112.8158	106.2022
	1.5	133.7216	124.6512	119.0925	115.5048	107.9158
	2.0	98.2897	101.5389	102.6273	102.9314	102.4205
	2.5	67.5687	71.7393	75.8728	79.1579	87.8875

Table 2 (continued).

		Sample size				
λ	***p***	**10**	**15**	**20**	**25**	**50**
5.000	-1.0	121.0000	113.7778	110.2500	108.1600	104.0400
	-0.5	102.5305	101.6803	101.2577	101.0049	100.5012
	0.0	100.0000	100.0000	100.0000	100.0000	100.0000
	0.5	114.2392	109.0729	106.6572	105.2573	102.5627
	1.0	156.2500	133.1361	123.4568	118.1474	108.5069
	1.5	263.1945	183.7882	155.9376	141.8949	118.5045
	2.0	566.8934	290.5475	216.2630	183.0105	133.7114
	2.5	1644.9660	533.9921	331.6380	254.5150	155.9988
7.500	-1.0	118.0488	111.9126	108.8889	107.0891	103.5224
	-0.5	102.4899	101.6536	101.2378	100.9890	100.4933
	0.0	100.0000	100.0000	100.0000	100.0000	100.0000
	0.5	112.9021	108.2732	106.0883	104.8161	102.3553
	1.0	135.1351	122.2826	116.2791	112.8158	106.2022
	1.5	133.7216	124.6512	119.0925	115.5048	107.9158
	2.0	98.2897	101.5389	102.6273	102.9314	102.4205
	2.5	67.5687	71.7393	75.8728	79.1579	87.8875
12.500	-1.0	98.7755	98.9372	99.1011	99.2293	99.5598
	-0.5	102.1671	101.4407	101.0790	100.8624	100.4305
	0.0	100.0000	100.0000	100.0000	100.0000	100.0000
	0.5	103.2356	102.2745	101.7465	101.4158	100.7256
	1.0	64.9350	74.0131	79.3650	82.8912	90.7770
	1.5	27.0942	34.8759	41.2049	46.4272	62.9312
	2.0	12.9108	16.3661	19.7223	22.8709	35.6598
	2.5	7.7921	9.0520	10.5823	12.1559	19.5613

Here the relative efficiencies of the defined estimator classes with respect to the unbiased estimator of the scale parameter of the Rayleigh distribution is calculated using the different values of n, p, and λ. The values of λ are selected by considering the efficiency range for large sample size values in numerical studies. The percent relative efficiency of θ_p^{**} estimator with respect to $\hat{\theta}_{\mathrm{MLE}}$ is calculated for the different values of n, p, and λ with the help of Equation (19). These calculated values are summarized in Table 2. The θ_p^{**} estimator performed better as compared to the unbiased estimator whenever the value of p was between –1 and 1, for each value of λ and n, while increased p values cause a decrease in the efficiency of the proposed biased estimator class with respect to the unbiased estimator.

Table 3. The percent relative efficiency of $\hat{\theta}_{\mathrm{MLE}}$ with respect to θ_p^{*}

		Sample size				
λ	***p***	**10**	**15**	**20**	**25**	**50**
0.625	-1.0	100.1629	100.1627	100.1624	100.1622	100.1609
	-0.5	100.0502	100.0502	100.0501	100.0501	100.0500
	0.0	100.0000	100.0000	100.0000	100.0000	100.0000
	0.5	100.0098	100.0098	100.0098	100.0098	100.0098
	1.0	100.0768	100.0768	100.0767	100.0767	100.0764
	1.5	100.1982	100.1978	100.1975	100.1971	100.1952
	2.0	100.3710	100.3697	100.3683	100.3670	100.3603
	2.5	100.5922	100.5888	100.5853	100.5819	100.5647
2.500	-1.0	100.1633	100.1632	100.1631	100.1630	100.1626
	-0.5	100.0502	100.0502	100.0502	100.0502	100.0502
	0.0	100.0000	100.0000	100.0000	100.0000	100.0000
	0.5	100.0098	100.0098	100.0098	100.0098	100.0098
	1.0	100.0769	100.0769	100.0769	100.0768	100.0768
	1.5	100.1987	100.1986	100.1985	100.1984	100.1977
	2.0	100.3728	100.3724	100.3719	100.3715	100.3693
	2.5	100.5969	100.5957	100.5946	100.5935	100.5879
5.000	-1.0	100.1635	100.1635	100.1635	100.1635	100.1635
	-0.5	100.0502	100.0502	100.0502	100.0502	100.0502
	0.0	100.0000	100.0000	100.0000	100.0000	100.0000
	0.5	100.0098	100.0098	100.0098	100.0098	100.0098
	1.0	100.0769	100.0769	100.0769	100.0769	100.0769
	1.5	100.1990	100.1990	100.1990	100.1991	100.1990
	2.0	100.3737	100.3737	100.3737	100.3737	100.3737
	2.5	100.5991	100.5991	100.5991	100.5991	100.5991
7.500	-1.0	100.1633	100.1632	100.1631	100.1630	100.1626
	-0.5	100.0502	100.0502	100.0502	100.0502	100.0502
	0.0	100.0000	100.0000	100.0000	100.0000	100.0000
	0.5	100.0098	100.0098	100.0098	100.0098	100.0098
	1.0	100.0769	100.0769	100.0769	100.0768	100.0768
	1.5	100.1987	100.1986	100.1985	100.1984	100.1977
	2.0	100.3728	100.3724	100.3719	100.3715	100.3693
	2.5	100.5969	100.5957	100.5946	100.5935	100.5879
12.500	-1.0	100.1620	100.1612	100.1605	100.1597	100.1559
	-0.5	100.0501	100.0500	100.0499	100.0499	100.0495
	0.0	100.0000	100.0000	100.0000	100.0000	100.0000
	0.5	100.0098	100.0098	100.0098	100.0098	100.0098
	1.0	100.0766	100.0764	100.0763	100.0761	100.0753
	1.5	100.1967	100.1956	100.1945	100.1934	100.1878
	2.0	100.3658	100.3619	100.3580	100.3540	100.3344
	2.5	100.5789	100.5687	100.5586	100.5485	100.4980

Similarly, the relative efficiency of the proposed class estimator θ_p^* with respect to $\hat{\theta}_{\mathrm{MLE}}$ is computed for different values of *n*, *p*, and *λ*. These computed values are given in Table 3. It can also see that the proposed class estimator θ_p^* performs better for each value of *n*, *p*, and *λ* as compared to the unbiased estimator. Moreover, when the value of *λ* is 7 the proposed class of estimator θ_p^* performs similar to $\hat{\theta}_{\mathrm{MLE}}$. Furthermore, when the estimator θ_p^{**} given in Table 2 is compared to the estimator given in Table 3, it is detected that the efficiency range of the estimator class θ_p^{**} with respect to the estimator class θ_p^* is larger than that of $\hat{\theta}_{\mathrm{MLE}}$. It is also observed that the proposed class of shrinkage estimators given in Equation (6) performs better than the traditional MLE for each value of *n* and *p*. However, the percent relative efficiency of the proposed class of biased estimators is symmetric around $\lambda = 5$ for a given *p* and *n*. It shows that the proposed class of estimators performs well even when the prior guess θ_0 is away from the true but unknown value of the parameter *θ*. Also, the relative efficiency of this class of estimators is not much higher. Meanwhile, the proposed class of estimators given in Equation (20) performs well compared to the usual MLE but its performance depends upon the values of *n*, *p*, and *λ*. For smaller values of *n* and *p*, it performs better at a given *λ*. As the sample size increases, *λ* tends to away from 5, and its performance deteriorates and becomes very poor when sample size, *p* and *λ* are larger. It is observed that the proposed class of estimators can be made highly efficient by taking $p = 2, 2.5$ when sample size is around 20 and *λ* is expected to be approximately 5. Moreover, it is observed that, in general, *p* should be in the interval (–1, 1) so that the proposed class of estimators may be fruitfully used. Above discussion establishes the superiority of the proposed classes of biased estimators.

Simulation

A random sample was generated with different sample sizes from the Rayleigh distribution with $\theta = 5$ and then calculated the unbiased and biased estimators from the generated samples. The procedure is iterated 10000 times. The MSE of both estimators is calculated to judge the performance of the proposed estimators. We considered $n = 10$, 20, and 50 and $p = -1.00$ to 2.50 with an interval of 0.50. The simulated results are shown in Table 4. It can be seen that the proposed estimator θ_p^* performs well compared to the estimator θ_p^{**} when the value of *p* lies in the interval $(-1 \leq p \leq 0)$ for each value of λ. Furthermore, the percent relative efficiency of both estimators decreased when sample size increases. It is evident from Table 4 that the estimator θ_p^* performs much better than the estimator θ_p^{**} and these results are in agreement with the theoretical results.

Table 4. The percent relative efficiency of the estimators θ_p^* and θ_p^{**} with respect to $\hat{\theta}_{MLE}$

		Sample size					
		10		20		50	
λ	p	θ_p^*	θ_p^{**}	θ_p^*	θ_p^{**}	θ_p^*	θ_p^{**}
0.5	-1.0	134.8489	117.9759	116.1170	108.8182	105.9455	103.3986
	-0.5	113.0125	102.5095	106.0271	101.2272	102.4421	100.5107
	0.0	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000
	0.5	102.4252	112.4855	101.2368	106.0785	100.4898	102.3402
	1.0	117.9180	134.8382	108.9946	116.5100	103.5303	106.2363
	1.5	135.0329	131.6477	116.9538	117.2962	107.4184	108.4199
	2.0	137.1994	99.3177	118.5453	102.8011	107.1770	102.0707
	2.5	113.9579	67.8402	104.8259	76.1585	100.7715	88.3023
1.0	-1.0	156.2500	121.0000	123.4568	110.2500	108.5069	104.0400
	-0.5	114.2392	102.5304	106.6572	101.2577	102.5628	100.5012
	0.0	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000
	0.5	102.5304	114.2392	101.2577	106.6572	100.5012	102.5628
	1.0	121.0000	156.2500	110.2500	123.4568	104.0400	108.5069
	1.5	162.0400	263.1945	128.5714	155.9376	110.8653	118.5045
	2.0	243.3600	566.8934	160.0225	216.2630	121.5286	133.7115
	2.5	405.8534	1644.9660	211.8779	331.6380	136.9558	155.9989
1.5	-1.0	134.3012	117.7642	116.3531	108.9283	106.7376	103.7642
	-0.5	113.0051	102.5049	105.9939	101.2203	102.4261	100.5078
	0.0	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000
	0.5	102.4557	112.7095	101.1636	105.6888	100.4870	102.3230
	1.0	118.2849	135.4221	109.0479	116.6330	103.5407	106.2374
	1.5	137.1121	134.3306	117.6372	118.6150	107.5710	108.5778
	2.0	138.1145	99.2876	119.0729	103.3666	108.3324	103.5204
	2.5	115.3105	67.9699	102.1367	74.0475	100.9496	88.2497
2.0	-1.0	96.4793	110.2760	98.2127	104.5576	99.9544	102.0701
	-0.5	109.0627	102.3512	104.8835	101.2702	101.6787	100.4558
	0.0	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000
	0.5	102.4054	109.3374	101.1150	104.0505	100.4903	101.8299
	1.0	110.0865	96.2502	105.3312	99.6157	101.3720	98.5737
	1.5	93.0756	53.9501	95.9266	70.9762	97.6059	85.5818
	2.0	58.9238	28.1040	65.6110	39.6435	81.1719	61.2328
	2.5	36.4582	17.6169	40.9252	22.8336	56.2884	38.2200

Conclusion

Sometimes, it may be preferable to use biased estimators given that they have smaller MSE than the variance of the unbiased estimators. Such biased estimators are generally developed using shrinkage estimation techniques. Taking into consideration such case, in this study, two different classes of shrinkage estimators are proposed. The proposed estimators are defined after minimizing the MSEs. To judge the relative performance of the proposed estimators, a numerical and simulation study are conducted where percent relative bias and MSE are considered as performance criteria. As a conclusion, it is stated that in order to have a better estimator when there is a little information about the likely value of the unknown parameter of interest, one should use the proposed class of estimators given in Equation (6). However, if some information is available as a prior guess about the likely values of the parameter, one should use the proposed class of estimators given in Equation (20) and p should be chosen in the vicinity of 0.

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