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Hardy Space Theory And Endpoint Estimates For Multi-Parameter Singular Radon Transforms

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HARDY SPACE THEORY AND ENDPOINT ESTIMATES FOR MULTI-PARAMETER SINGULAR RADON TRANSFORMS

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DEDICATION

To my parents.

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CHAPTER 1 : INTRODUCTION

1.1 Brief Background and Introduction

In [12], Christ, Nagel, Stein and Waigner studied the L^p theories for the singular Radon Transforms. They consider the following form of the operator

$$Tf(x) := \psi(x) \int f(\gamma(x,t))K(t)\,dt, \tag{1.1}$$

where $\gamma_t(x) = \gamma(x,t)$ is a $C^\infty(\mathbb{R}^n \times \mathbb{R}^N)$ function defined in a neighborhood of the origin with $\gamma_0(x) \equiv x$, ψ is a C_0^∞ cut-off function, supported near the origin of $\mathbb{R}^n$ and K is a standard Calderón-Zygmund kernel on $\mathbb{R}^N$ supported for t near 0.

The key assumption on $\gamma_t(x)$ is a certain curvature condition $(\mathcal{C})$, which can be stated in a number of equivalent ways. One of those forms is in terms of a noncommutative version of Taylor's formula. Actually they proved that actually there exists a unique vector fields $\{X_\alpha\}$ with $\alpha = (\alpha_1, \ldots, \alpha_N) \neq 0$, so that asymptotically $\gamma_t(x) \sim \exp\left(\sum_\alpha \frac{t^\alpha}{\alpha!} X_\alpha\right)(x)$ as $t \to 0$. The assumption for $\gamma_t(x)$ that the Lie algebra generated by the X_α should span the tangent space to $\mathbb{R}^n$. Under such curvature condition, they proved

Theorem 1.1 ([12]). *Let the operator T be defined as in* (1.1) *and assume the vector fields X_α in the asymptotic representation of $\gamma_t(x)$ satisfy the curvature condition $(\mathcal{C})$. Then T extends to a bounded operator from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$.*

The methods used to prove the theorem are mainly some "lifting technique" and the Cotlar-lemma estimates, where some techniques appeared in [9] by Christ. The first example of such operators comes from the Hilbert transform along the parabola, $r(x,t) = (x_1 - t, x_2 - t^2)$ (Fabes [17]). And some non-translation invariant cases were studied by Nagel, Stein, Wainger in [54] for L^2 result in the special case of certain plane curves, Geller and Stein in [23] for the Heisenberg group, with the various extensions by Müller in [46–48], and culminated with Christ in [9]. Furthermore, Stein and Street in [64–67], and Street in [68] studied a wider class of singular integral operators in the multi-parameter setting such as the generalized Calderón-Zygmund operators and singular Radon transforms, and established some related theories including the L^p boundedness property.

More specifically, the L^p theory of a multi-parameter version of (1.1) were established by Stein and Street in a series of papers [64–67], where the distribution kernel

$$K(t) = \sum_j \zeta_j^{(2^j)}(t), \quad t = (t_1, \dots, t_v) \in \mathbb{R}^N = \mathbb{R}^{N_1} \times \cdots \times \mathbb{R}^{N_\nu}$$

is a ν-parameter singular kernel with $\{\zeta_j\} \subset C_0^\infty$ supported in a small ball centered at the origin, and $\zeta_j^{(2^j)}(t)$ is a appropriate ν-parameter dilation. Such kernels have equivalent representations in terms of the growth condition and cancellation condition that people are familiar with. The difficulty in the multi-parameter case is that if considering the Taylor's formula for $\gamma_t(x)$, one has to take care of the "non-pure" vector fields X_α, which do not appear in the single parameter case. By giving additional assumptions on these vector fields, they were able to prove the L^p boundedness for such multi-parameter version of (1.1).

The above kernels have some interesting examples. For instance, Stein and Street in [64] considered the case when $K(t)$ is a product kernel, which satisfies some cancellation condition and the growth condition

$$\left|\partial_{t_1}^{\alpha_1} \cdots \partial_{t_\nu}^{\alpha_\nu} K(t)\right| \le C_\alpha |t_1|^{-N_1-|\alpha_1|} \cdots |t_\nu|^{-N_\nu-|\alpha_\nu|}$$

for each multi-index $\alpha = (\alpha_1, \dots, \alpha_\nu)$. Another example of the multi-parameter version of (1.1) is the case when $K(t)$ is a flag kernel. These operators were studied by Nagel, Ricci and Stein [53], and it turns out flag kernels can be applied to a wider class of $\gamma_t(x)$.

Also, a special case for $\gamma_t(x)$ of the above is the following operator

$$f \mapsto \psi(x) \int f(e^{t_1 X_1 + t_2 X_2 + \cdots + t_l X_l} x) K(t_1, \dots, t_l) dt,$$

where $K(t_1, \cdots, t_l)$ is a product kernel relative to the decomposition of $\mathbb{R}^l = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ and $X_1, \ldots, X_l$ are left invariant vector fields on a nilpotent Lie group. In this case, $\gamma_t(x)$ satisfies the required curvature condition obviously, then one can have the L^p boundedness of the operator, see [65] for details.

In [68], Street extended the classical Calderón-Zygmund kernels to the ones in terms of the Carnot-Carathéodory balls in both single and multi-parameter settings when working on smooth, connected, compact manifolds. With these kernels, a more general type of Calderón-Zygmund operators (see Section 2.2 and 3.6 for details), including some Radon transforms and pseudo-differential operators can be defined, and the corresponding L^p theories were established in [68] by giving appropriate assumptions on the involved vector fields. For instance, the following Radon transform were studied

$$Tf(x) = \int_{\mathbb{R}^q} f(\gamma(x,t))\psi(\gamma(x,t))K(x,t)\,dt, \tag{1.2}$$

where $\gamma(x,t) = e^{t\cdot X}x = e^{t_1\cdot X^1+\cdots+t_\nu\cdot X^\nu}x$ with X^μ as the list of vector fields X^μ_1, …, $X^\mu_{q_\mu}$ expanding the $\mathbb{R}^{q_\mu}$ respectively, $\psi \in C_0^\infty(\Omega)$ and K is kernel defined the same as Proposition 3.5.

In this dissertation, we will study the Hardy space H^p and its dual space associated with both the one-parameter and multi-parameter singular Radon transforms, and consider the boundedness of some singular Radon transforms on such Hardy spaces H^p when $0 \le p \le 1$.

The Hardy and BMO spaces play an important role in modern harmonic analysis and applications in partial differential equations. In [19], C. Fefferman and Stein showed that the space of functions of bounded mean oscillation on $\mathbb{R}^n$, BMO$(\mathbb{R}^n)$, is the dual space of the Hardy space $H^1(\mathbb{R}^n)$. They also obtained a characterization of the BMO space in terms of the Carleson measure.

We now begin to give a brief overview on the multi-parameter singular integrals and Hardy spaces theory. Multi-parameter analysis is an important subject in harmonic analysis. The classical Calderón-Zygmund operators theory is an generalization of the well-known Hilbert transform and is closely related to the Hardy-Littlewood maximal operator which commutes with the usual dilations on $\mathbb{R}^n$, $\delta \cdot x = (\delta x_1, ..., \delta x_n)$ for $\delta > 0$. The *multi-parameter* Calderón-Zygmund operators are also singular integral operators that are extension of the double Hilbert transform and are closely associated with the *strong* maximal function which commutes with the multi-parameter dilations on $\mathbb{R}^n$, $\delta \cdot x = (\delta_1 x_1, ..., \delta_n x_n)$ for $\delta = (\delta_1, ..., \delta_n) \in \mathbb{R}^n_+$ [38].

For multi-parameter Calderón-Zygmund operators of the convolution form $Tf = K * f$, where K is homogeneous in the sense of $\delta_1 ... \delta_n K(\delta \cdot x) = K(x)$, or when $K(x)$ satisfies certain differential inequalities and cancellation conditions, such operators and their non-convolution type analogues have been studied extensively in the literature. The L^p $(1 < p < \infty)$ boundedness of such operators of convolution type was established by R. Fefferman and E. Stein [20]. The non-convolution type multi-parameter singular integral operators were first studied by Journé ([39] [40], [41]). More recent work on boundedness on multi-parameter Triebel-Lizorkin and Besov spaces for Fourier multipliers and singular integral operators can also be found in [7,44] and L^p estimates for multi-parameter Fourier integral operators have been established in [35,36] which extend the works of Seeger, Sogge and Stein on the one-parameter Fourier integral operators in [62] and others in [57–60]. L^p estimates for multi-parameter and multi-linear Fourier multipliers were established by Muscalu, Pipher, Tao and Thiele [51, 52] (see also the work of Chen and Lu [6]).

To study the endpoint estimates, the multi-parameter Hardy spaces introduced by Gundy-Stein ([24]) have been extensively investigated by R. Fefferman ([17]), Chang and R. Fefferman ([2], [3], [4]). Motivated by a counterexample of L. Carleson [1], the multi-parameter $BMO(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ and Hardy space $H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ theory was developed by Chang and R. Fefferman in a series of papers ([2], [3], [4]). As has been known, the atoms in multi-parameter Hardy spaces are supported in arbitrary open sets rather than on cubes or rectangles, it was difficult to establish boundedness of singular integral operators from multi-parameter Hardy spaces H^p to H^p or from H^p to L^p. Thanks to R. Fefferman, a boundedness criterion for the H^p to L^p of a multi-parameter operator T was established using atomic decomposition and a geometric lemma of Journé (see Journé [39], [40], [41]) in two-parameter setting. However, this criterion cannot be applied to the case of three or more parameters [39], [40], [41]. The H^p to L^p boundedness for Journé's class of singular integral operators with arbitrary number of parameters was finally obtained by J. Pipher [56]. Subsequently, Ferguson and Lacey gave a new characterization of the product $\mathrm{BMO}(\mathbb{R}\times\mathbb{R})$ by using the Journé covering lemma in [21]. Moreover, Lacey, Petermichl, Pipher and Wick in [42] established a characterization of $\mathrm{BMO}(\mathbb{R}^n\times\mathbb{R}^n)$ using multiparameter commutators of Riesz transforms. More recently, the authors of [27, 28] established the boundedness criterion on multiparameter Hardy spaces for Journé's class of singular integral operators with arbitrary number of parameters. For multi-parameter flag singular integral operators, singular integral operators on the product of Carnot-Carathéodory spaces, the product of homogeneous spaces, the composition of two singular integral operators with different homogeneity, etc., the Hardy space and duality theory have been established in a series of papers [14, 16, 29–32, 43] using discrete Littlewood-Paley-Stein theory. In particular, the multi-parameter flag Hardy spaces theory [32] extend the L^p theory of Muller, Ricci and Stein [49, 50] and Nagel, Ricci and Stein [53].

Inspired by these characterization of the Hardy spaces on product spaces, we will take advantage of the discrete Littlewood-Paley analysis to define the Hardy spaces H^p and the Carleson measure spaces CMO^p associated with the multi-parameter singular Radon transforms. Moreover, we will prove the H^p boundedness of those operators and thus obtain the endpoint estimates for the L^p boundedness of the multi-parameter singular Radon transforms by Stein and Street [68].

CHAPTER 2 : H^p BOUNDEDNESS OF SINGLE-PARAMETER SINGULAR RADON TRANSFORM

2.1 Introduction

In the first place, we introduce some notations. Define $w^\alpha = w_{\alpha_1} \cdots w_{\alpha_L}$, where $\alpha = (\alpha_1, \dots, \alpha_L)$ is a list of elements in $\{1, \dots, r\}$ and $w = (w_1, \dots, w_r)$ contains r non-commuting indeterminates. Also, we denote $|\alpha| = L$, the length of the list. For instance, $\alpha = (5, 1, 2)$, then $|\alpha| = 3$ and $w^\alpha = w_5 w_1 w_2$.

We consider a compact, connected, smooth manifold without boundary, denoted by M and a list of C$^\infty$ vector fields $W_1, \dots, W_r$ on M.

Definition 2.1. Given a list of vector fields, $W_1, \dots, W_r$, if the Lie algebra generated by them,

$$W_1, \dots, W_r, \dots, [W_i, W_j], \dots, [W_i, [W_j, W_l]], \dots,$$
$$\dots, (\text{commutators of order m}), \dots$$

can span the tangent space $T_x M$ for any $x \in M$, then the list of vector fields satisfies Hörmander's condition.

Actually, the commutators of the vector fields satisfying Hörmander's condition can span the tangent space for any $x \in M$ after finite steps m, because of the compactness. In this situation, we also say the list of the vector fields satisfies the Hörmander's condition of order m.

We say $\rho : M \times M \to [0, \infty]$ is the Carnot-Carathéodory distance if

$$\rho(x,y) = \inf \Big\{ \delta > 0 \,\Big|\, \exists \gamma : [0,1] \to M, \gamma(0) = x, \gamma(1) = y, \gamma'(t) = \sum_{j=1}^{q} a_j(t) \delta W_j(\gamma(t)), a_j \in L^\infty([0,1]), \Big\| \sum_{j=1}^{r} |a_j|^2 \Big\|_{L^\infty([0,1])} < 1 \Big\}.$$

where $W_1, \dots, W_r$ are C^∞ vector fields.

It's easy to verify that ρ is an extended metric. Moreover, Chow proved the following theorem for the above distance.

Theorem 2.2 ([8])**.** *If $W_1, \ldots, W_r$ satisfy Hörmander's condition, then ρ is a metric. That is, $\rho(x, y) < \infty$, for every $x, y \in M$.*

Next, let's introduce a class of "strictly positive, smooth measures" mentioned in [68].

Definition 2.3 ([68])**.** A smooth measure, μ on M is a Borel measure on M such that in any local coordinates x, we may write $d\mu = \phi_x dm(x)$, where dm denotes Lebesgue measure, and ϕ_x is a C^∞ function. We say μ is a strictly positive, smooth measure if $\phi_x > 0$ in every local coordinate system.

We write the functional operation as the integration form. For instance, $\forall\ f \in C^\infty(M)$ and $\lambda \in C^\infty(M)'$, $\lambda(f) = \int \lambda f dx$. We deal similarly between $C^\infty(M \times M)$ and $C^\infty(M \times M)'$. Also, we might define the distribution λ by some $L^1_{\text{loc}}(M)$, $C_0^\infty(U)$ function f, if λ is given by integration against f on some open set $U \in M$.

Afterward, we always assume $W_1, \ldots, W_r$ satisfy Hörmander's condition. With the Carnot-Carathéodory distance ρ, we define the Carnot-Carathéodory ball as follows

$$B_W(x, \delta) := \{y \in M \mid \rho(x, y) < \delta\},$$

with the radius δ centered at x. Nagel, Stein, and Wainger deal with some properties of balls on the metric space [55] and obtain the following important estimate for the Carnot-Carathéodory balls.

Theorem 2.4 ([68])**.** *There are constants $Q_2 \geq Q_1 > 0$ such that for any $x \in A$, $\delta > 0$,*

$$2^{Q_1}\text{Vol}(B_W(x, \delta)) \leq \text{Vol}(B_W(x, 2\delta)) \leq 2^{Q_2}\text{Vol}(B_W(x, \delta)).$$

From , the following property follows automatically:

Lemma 2.1 ([68])**.** $\text{Vol}(B_W(x, \rho(x, z))) \approx \text{Vol}(B_W(z, \rho(z, x)))$.

Remark 2.5. The least possible Q_2 in Theorem 2.1 is considered as the homogeneous dimension of (M, ρ, Vol).

2.2 Notations and Preliminaries

To establish the theorems, let's firstly introduce some types of functions that will be frequently used. Some are introduced in [68] and some are defined newly.

Definition 2.6 ([68])**.** We say $\mathcal{B} \subset C^\infty(M) \times M \times (0,1]$ is a bounded set of bump function if:

(i) $\forall(\phi, x, \delta) \in \mathcal{B}$, $\text{supp}(\phi) \subset B_W(x, \delta)$.

(ii) For every ordered multi-index α, there exists C, such that $\forall(\phi, x, \delta) \in \mathcal{B}$,

$$\sup_z |(\delta W)^\alpha \phi(z)| \leq C \text{Vol}(B_W(x, \delta))^{-1}$$

Also, we introduce the space of test functions on M. The similar type of test function is introduced in many articles, such as [30], [31], [32], [34].

Definition 2.7. Let $x_1 \in M$. A function f on M is said to be a test function if there exists a constant $C \geq 0$ such that for every $m \in \mathbb{N}$, and every ordered multi-index α, the following holds

$$\left|W^\alpha f(x)\right| \leq C_{\alpha,m} \frac{(1 + \rho(x, x_1))^{-m}}{\text{Vol}(B_W(x_1, 1 + \rho(x, x_1)))} \tag{2.1}$$

We define the norm of such functions for any $|\alpha| \leq n_0$ and m as

$$\|f\|_{\mathcal{T}(x_1, n_0, m)} = \sup_{|\alpha| \leq n_0} \inf\{C_{\alpha,m}\} < \infty, \text{where } C_{n_0,m} \text{ is as in (2.1)}$$

Note that the definition is made to be invariant by translation. Thus, for another point $x_2 \in M$, $\mathcal{T}(x_1, n_0, m)$ and $\mathcal{T}(x_2, n_0, m)$ are equivalent in the corresponding norm. WLOG, we can denote $\mathcal{T}(x_1, n_0, m)$ by $\mathcal{T}(n_0, m)$ and represent the collection of all test functions by $\mathcal{T}$.

In the history of Hardy space Theory, there are lots of characterizations for Calderón-Zygmund operators. Here, let's refer to the classification by Street in [68].

Definition 2.8 ([68])**.** We say $T : C^\infty(M) \to C^\infty(M)$ is a Calderón-Zygmund operator of order $t \in (-Q_1, \infty)$ if

(i) (Growth Condition) For each ordered multi-indices α, β,

$$|W_x^\alpha W_z^\beta T(x,z)| \leq C_{\alpha,\beta} \frac{\rho(x,z)^{-t-|\alpha|-|\beta|}}{\mathrm{Vol}(B_W(x,\rho(x,z)))},$$

where W_x denotes the list of vector fields $W_1, \ldots, W_r$ thought of as partial differential operators in the x variable and similarly for W_z. In particular, the above implies that the distribution $T(x,z)$ corresponds with a C^∞ function for $x \neq z$.

(ii) (Cancellation Condition) For each bounded set of bump function $\mathcal{B} \subset C^\infty(M) \times M \times (0,1]$ and each ordered multi-index α,

$$\sup_{(\phi,z,\delta)\in\mathcal{B}} \sup_{x\in M} \delta^{t+|\alpha|} \mathrm{Vol}(B_W(z,\delta)) |W^\alpha T\phi(x)| \leq C_{\mathcal{B},\alpha},$$

with the same estimates for T^* in place of T. Here, the formal adjoint T^* is taken in the sense of $L^2(M)$ which is defined in terms of the chosen strictly positive, smooth measure. Namely, we first define the transpose, T^t. The Schwartz kernel of T^t is defined by $T^t(x,y) = T(y,x)$; more precisely,

$$\int T^t(x,t)\phi(x,y)\, dxdy = \int T(x,y)\phi(y,x)\, dxdy,$$

for $\phi \in C_0^\infty(M \times M)$. We define the Schwartz kernel of T^* by $T^* = \overline{T^t}$, where, $\bar{z}$ denotes the complex conjugate of z. Here, for a distribution λ, we are defining the distribution $\overline{\lambda}$ by $\overline{\lambda}(f) = \overline{\lambda(\overline{f})}$.

Now, let's introduce the tool of pre-elementary operators and elementary operators. We write elements of $(0,1]$ as 2^{-j}, where $j \in [0,\infty)$.

Definition 2.9 ([68]). We say $\mathcal{E} \subset C^\infty(M \times M) \times (0,1]$ is a bounded set of pre-elementary operators if: $\forall \alpha, \beta, m$, $\exists C = C(\mathcal{E}, \alpha, \beta, m)$, $\forall (E, 2^{-j}) \in \mathcal{E}$,

$$\left|(2^{-j}W_x)^{\alpha}(2^{-j}W_z)^{\beta}E(x,z)\right| \leq C\frac{(1+2^j\rho(x,z))^{-m}}{\mathrm{Vol}(B_W(x, 2^{-j}(1+2^j\rho(x,z))))}$$

Note by the symmetry and Lemma 2.1, it follows

$$\frac{(1+2^j\rho(x,z))^{-m}}{\mathrm{Vol}(B_W(x, 2^{-j}(1+2^j\rho(x,z))))} \approx \frac{(1+2^j\rho(z,x))^{-m}}{\mathrm{Vol}(B_W(z, 2^{-j}(1+2^j\rho(z,x))))}.$$

Definition 2.10 ([68]). We define the set of bounded sets of elementary operators, $\mathcal{G}$, to be the largest set of subsets of $C^\infty(M \times M) \times (0,1]$ such that for all $\mathcal{E} \in \mathcal{G}$,

(i) $\mathcal{E}$ is a bounded set of pre-elementary operators.

(ii) $\forall\, (E, 2^{-j}) \in \mathcal{E}$,

$$E = \sum_{|\alpha|,|\beta| \leq 1} 2^{-(2-|\alpha|-|\beta|)j}(2^{-j}W)^{\alpha}E_{\alpha,\beta}(2^{-j}W)^{\beta}$$

where $\{(E_{\alpha,\beta}, 2^{-j}) \,|\, (E, 2^{-j}) \in \mathcal{E}\} \in \mathcal{G}$.

We say $\mathcal{E}$ is a bounded set of elementary operators if $\mathcal{E} \in \mathcal{G}$.

In fact, the elementary operators are invariant under some transforms. The details are list in the following properties.

Proposition 2.1 ([68]). *Let $\mathcal{E}$ be a bounded set of elementary operators. Then,*

(a) If $\psi \in C^\infty(M)$, then $\{(\psi E, 2^{-j}), (E\psi, 2^{-j}) \,|\, (E, 2^{-j}) \in \mathcal{E}\}$ is a bounded set of elementary operators. Here, we are identifying ψ with the operator $f \mapsto \psi f$.

(b) $\{(E^*, 2^{-j}) \,|\, (E, 2^{-j}) \in \mathcal{E}\}$ is a bounded set of elementary operators.

(c) Fix an ordered multi-index α. Then

$$\left\{\left((2^{-j}W)^{\alpha}E, 2^{-j}\right), \left(E(2^{-j}W)^{\alpha}, 2^{-j}\right) \,|\, (E, 2^{-j}) \in \mathcal{E}\right\}$$

is a bounded family of elementary operators.

(d) For every $N \in \mathbb{N}$, each $(E, 2^{-j}) \in \mathcal{E}$ can be written as

$$E = \sum_{|\alpha| \le N} 2^{(|\alpha|-N)j} (2^{-j}W)^{\alpha} \widetilde{E}_{\alpha},$$

where $\left\{ \left(\widetilde{E}_{\alpha}, 2^{-j}\right) \,|\, (E, 2^{-j}) \in \mathcal{E} \right\}$ is a bounded set of elementary operators. Similarly, each $(E, 2^{-j}) \in \mathcal{E}$ can be written as

$$E = \sum_{|\alpha| \le N} 2^{(|\alpha|-N)j} \widetilde{E}_{\alpha} (2^{-j}W)^{\alpha},$$

where $\left\{ \left(\widetilde{E}_{\alpha}, 2^{-j}\right) \,|\, (E, 2^{-j}) \in \mathcal{E} \right\}$ is a bounded set of elementary operators.

Lemma 2.2 ([68]). *Fix $t \in \mathbb{R}$ and let $\{(E_j, 2^{-j}) \,|\, j \in \mathbb{N}\}$ be a bounded set of elementary operators. Then the sum $\sum_{j \in \mathbb{N}} 2^{jt} E_j$ converges in the topology of bounded convergence as operators $C^{\infty}(M) \to C^{\infty}(M)$ (and therefore converges in distribution).*

In the other hands, the Calderón-Zygmund operators have several equivalent characterizations.

Theorem 2.11 ([68]). *Let $T : C^{\infty}(M) \to C^{\infty}(M)$, and fix $t \in (-Q_1, \infty)$. The following are equivalent.*

(i) T is a Calderón-Zygmund operator of order t.

(ii) For every bounded set of elementary operator $\mathcal{E}$,

$$\{(2^{-jt} TE, 2^{-j}) \,|\, (E, 2^{-j}) \in \mathcal{E}\}$$

is a bounded set of elementary operators.

(iii) There is a bounded set of elementary operators $\{(E_j, 2^{-j}) \,|\, j \in \mathbb{N}\}$ such that $T = \sum_{j \in \mathbb{N}} 2^{jt} E_j$ in the sense of $C^{\infty}(M)'$.

For the multi-parameter analysis and generalizations later, let's introduce an equivalent family of balls with a slightly different definition. By scaling the vector fields, we can make the balls of any radius equal to the "balls" of unit radius, i.e. $B_W(x, \delta) = B_{\delta W}(x, 1)$. Given a list of vector fields $W = W_1, \ldots, W_r$, we write the unit ball $B_W(x, 1)$ as $B_W(x)$.

Now suppose $W_1, \ldots, W_r$ satisfy Hörmander's condition of order m. We assign to $W_1, \ldots, W_r$ the *formal degree* 1. To vector fields of the form $[W_i, W_j]$ we assign the formal degree 2. Recursively, if Y has formal degree d_0, we assign to $[W_j, Y]$ the formal degree $d_0 + 1$. Let $(X_1, d_1), \ldots, (X_q, d_q)$ be an enumeration of the above collection of vector fields with formal degrees, which have formal degree $\leq m$. Note that, in light of Hörmander's condition $X_1, \ldots X_q$ span T_xM for every x.

The formal degrees encapsulate the above notion of scaling. Indeed, if we replace $W_1, \ldots, W_r$ with $\delta W_1, \ldots, \delta W_r$ in the above, then (X_j, d_j) is replaced by $(\delta^{d_j} X_j, d_j)$. Because this plays a crucial role in the follows, we denote by δX the list of vector fields $\delta X = \delta^{d_1} X_1, \ldots, \delta^{d_q} X_q$.

We define

$$B_{(X,d)}(x, \delta) := B_{(\delta X)}(x).$$

It's clear that $B_W(x, \delta) \subseteq B_{(X,d)}(x, \delta)$. The converse is shown by Nagel, Stein, and Wainger [55]:

Proposition 2.2 ([55]). *There is a constant $c > 0$ such that $B_{(X,d)}(x, c\delta) \subseteq B_W(x, \delta)$, for all $\delta > 0$.*

Then one will be able to replace $B_W(x, \delta)$ with $B_{(X,d)}(x, \delta)$ throughout the previous statements, and obtain equivalent definitions. One can also replace $\rho(x, z)$ with the equivalent metric.

$$\inf\{\delta \,|\, z \in B_{(X,d)}(x, \delta)\}.$$

In this work, we still need the following properties of bounded sets of elementary operators taken from [68]

Proposition 2.3 ([68]). *Let $\mathcal{E}$ be a bounded set of elementary operators. Then, for every N,* *the set*

$$\left\{\left(2^{N|j_1-j_2|}E_1E_2,2^{-j_1}\right),\left(2^{N|j_1-j_2|}E_1E_2,2^{-j_2}\right)|(E_1,2^{-j_1}),(E_2,2^{-j_2})\in\mathcal{E}\right\}$$

is a bounded set of elementary operators.

Definition 2.12. For $j,k\in\mathbb{R}$, we write $j\wedge k$ for the minimum of j and k and $j\vee k$ for the maximum. If, instead, $j=(j_1,\dots j_\nu),k=(k_1,\dots,k_\nu)\in\mathbb{R}^\nu$, then $j\wedge k=(j_1\wedge k_1,\dots,j_\nu\wedge k_\nu)$ and $j\vee k=(j_1\vee k_1,\dots,j_\nu\vee k_\nu)$.

Lemma 2.3 ([68]). *For every $m>Q_1$, and $\forall j_1,j_2\in[0,\infty)$,*

$$\int\frac{(1+2^{j_1}\rho(x,y))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(x,2^{-j_1}+\rho(x,y))\big)}\frac{(1+2^{j_2}\rho(y,z))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(y,2^{-j_2}+\rho(y,z))\big)}\,dy$$
$$\lesssim\frac{(1+2^{j_1\wedge j_2}\rho(x,z))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(x,2^{-j_1\wedge j_2}+\rho(x,z))\big)}$$

where the implicit constant depends on m, but not on $j_1,j_2\in[0,\infty)$.

Lemma 2.4 ([68]). *Let $\mathcal{E}$ be a bounded set of pre-elementary operators. Then, $\forall m$, $\exists C$, $\forall(F_1,2^{-j_1})$, $(F_2,2^{-j_2})\in\mathcal{E}$,*

$$|F_1F_2(x,z)|\le C\frac{(1+2^{j_1\wedge j_2}\rho(x,z))^{-m}}{\mathrm{Vol}(B_{(X,d)}(y,2^{-j_1\wedge j_2}+\rho(x,z)))}.$$

Lemma 2.5 ([68]). $\forall m$, $\exists N$, $\forall j_1,j_2\in[0,\infty)$, $\forall x,y\in M$,

$$2^{-N|j_1-j_2|}\frac{(1+2^{j_1}\rho(x,y))^{-m}}{\mathrm{Vol}(B_{(X,d)}(y,2^{-j_1}+\rho(x,y)))}\le\frac{(1+2^{j_2}\rho(x,y))^{-m}}{\mathrm{Vol}(B_{(X,d)}(y,2^{-j_2}+\rho(x,y)))}$$

Lemma 2.6 ([68]). *Let $\mathcal{E}$ be a bounded set of pre-elementary operators. Then, $\forall m,\alpha$, and β, $\exists N,C$, such that $\forall(F_1,2^{-j_1}),(F_2,2^{-j_2})\in\mathcal{E}$, and letting $k=j_1$ or $k=j_2$, we have*

$$2^{-N|j_1-j_2|}\left|(2^{-k}W_x)^\alpha(2^{-k}W_z)^\beta[F_1F_2](x,z)\right|\le C\frac{(1+2^k\rho(x,z))^{-m}}{\mathrm{Vol}(B_{(X,d)}(y,2^{-k}+\rho(x,z)))}.$$

Lemma 2.7 ([68]). *Suppose $\mathcal{E}_1$ is a bounded set of elementary operators and $\mathcal{E}_2$ is a bounded set of pre-elementary operators. Then, for every N, the set*

$$\Big\{\big(2^{N|j_1-j_2|}E_1E_2, 2^{-j_2}\big), \big(2^{N|j_1-j_2|}E_2E_1, 2^{-j_2}\big) \\ \Big| (E_1, 2^{-j_1}) \in \mathcal{E}_1, (E_2, 2^{-j_2}) \in \mathcal{E}_2, j_1 \geq j_2\Big\}$$

is a bounded set of pre-elementary operators.

Lemma 2.8 ([68]). *Let $\mathcal{B}$ be a bounded set of bump functions and $\mathcal{E}$ be a bounded set of elementary operators. $\forall N$, m, $\exists C$ such that $\forall (E, 2^{-j}) \in \mathcal{E}$, $(\phi, x, 2^{-k}) \in \mathcal{B}$,*

$$|E\phi(z)| \lesssim 2^{-N(j-j\wedge k)} \frac{\big(1 + 2^{j\wedge k}\rho(x,z)\big)^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(x, 2^{-j\wedge k} + \rho(x,z))\big)}.$$

Next, we introduce the continuous version of Littlewood-Paley theory adapted to the geometry $B_{(X,d)}(x,\delta)$, which is obtained by Street [68].

Since $I : C^\infty(M) \to C^\infty(M)$ (the identity operator) is a Caldenón-Zygmund operator of order 0. By the characterization of such operators, there is a bounded set of elementary operators $\{(D_j, 2^{-j}) | (D_j, 2^{-j}) \in \mathcal{D}\}$ with $I = \sum_{j\in\mathbb{N}} D_j$. For $l \in \mathbb{Z}\backslash\mathbb{N}$, define $D_l = 0$. We have

$$I = \Big(\sum_{j\in\mathbb{Z}} D_j\Big)\Big(\sum_{j\in\mathbb{Z}} D_j\Big) = U_N + R_N$$

where $U_N = \sum_{\substack{j\in\mathbb{N}\\|l|\leq N}} D_jD_{j+l} = \sum_{\substack{j\in\mathbb{N}\\|l|\leq N}} D_{j+l}D_j$, $R_N = \sum_{\substack{j\in\mathbb{N}\\|l|>N}} D_jD_{j+l} = \sum_{\substack{j\in\mathbb{N}\\|l|>N}} D_{j+l}D_j$.

By [68], the following properties hold.

Lemma 2.9 ([68]). *Fix p, $1 < p < \infty$. $\lim_{N\to\infty} \|R_N\|_{L^p\to L^p} = 0$.*

Proposition 2.4 ([68]). *Fix p, $1 < p < \infty$, $\exists$ $N = N(p)$, s.t. $U_N : L^p \to L^p$ is an isomorphism. i.e. $\exists$ $V_N : L^p \to L^p$ with $U_NV_N = V_NU_N = I$.*

Consequently, we have

Proposition 2.5 ([68]). *Fix $1 < p < \infty$, then*

$$\|f\|_{L^p} \approx \left\| \left(\sum_{j \in \mathbb{N}} |D_j f|^2 \right)^{\frac{1}{2}} \right\|_{L^p}, \quad f \in C^\infty(M),$$

where the implicit constants depend on p and the particular decomposition of $I = \sum_{j \in \mathbb{N}} D_j$.

Moreover, we can deduce the following continuous version of Calderón reproducing formula.

Theorem 2.13. *Letting $D_j^N = \sum_{|l| \le N} D_{j+l}$ and $\widetilde{D}_j = V_N D_j^N$, thus $f = \sum_{j \in \mathbb{N}} \widetilde{D}_j D_j f$, which converges in the topology of bounded convergence as operators $L^p \to L^p$, and also in the topology of bounded convergence as operators $\mathcal{T}(n_0, m) \to \mathcal{T}(n_0, m)$ (and therefore converges in distribution).*

Theorem 2.14. *Letting $D_j^N = \sum_{|l| \le N} D_{j+l}$ and $\overline{D}_j = D_j^N V_N$, thus $f = \sum_{j \in \mathbb{N}} D_j \overline{D_j} f$ which converges in the topology of bounded convergence as operators $L^p \to L^p$, and also in the topology of bounded convergence as operators $\mathcal{T}(n_0, m) \to \mathcal{T}(n_0, m)$ (and therefore converges in distribution).*

Remark 2.15. B. Street has proved the L^p convergence in [68], and now we will prove the part of $\mathcal{T}(n_0, m)$.

Lemma 2.10. *Fix $N > 0$ and define R_N as above. For any $(\phi, x, \delta) \in \mathcal{B}$, a bounded set of bump function, we have $\forall N_0 > 0$,*

$$\sup_z |(\delta W)^\alpha R_N(\phi)(z)| \le C 2^{-N_0 N} \mathrm{Vol}\big(B_{(X,d)}(x, \delta)\big)^{-1}$$

where C is independent of ϕ.

Proof. WLOG, we consider $\delta = 2^{-k}$ for some $k \in \mathbb{N}$. By the definition, we have

$$|(2^{-k} W)^\alpha R_N(\phi)(z)| \le \sum_{|l| > N} \sum_{j \in \mathbb{N}} 2^{(j-k)|\alpha|} |(2^{-j} W)^\alpha D_j D_{j+l}(\phi)(z)|$$

By using the Proposition 3, we know that $\{(2^{N_0|l|}D_jD_{j+l}, 2^{-j}) \,|\, (D_j, 2^{-j}),\ (D_{j+l}, 2^{-(j+l)}) \in \mathcal{D}\}$ is a bounded set of elementary operators. For the simplicity, we denote the above new set as $\{(E_j, 2^{-j}) \,|\, (E_j, 2^{-j}) \in \mathcal{E}\}$.

Thus, we can rewrite the inequality as

$$|(2^{-k}W)^\alpha R_N(\phi)(z)| \leq \sum_{|l|>N} 2^{-N_0|l|} \sum_{j\in\mathbb{N}} 2^{(j-k)|\alpha|} |(2^{-j}W)^\alpha E_j(\phi)(z)|$$

Next, applying the Lemma 2.8, $\forall N_1, m$,

$$\begin{aligned}|(2^{-k}W)^\alpha R_N(\phi)(z)| &\lesssim \sum_{|l|>N} 2^{-N_0|l|} \sum_{j\in\mathbb{N}} 2^{(j-k)|\alpha|} 2^{-N_1(j-j\wedge k)} \frac{\left(1+2^{j\wedge k}\rho(x,z)\right)^{-m}}{\mathrm{Vol}\left(B_{(X,d)}(x, 2^{-j\wedge k}+\rho(x,z))\right)} \\ &\lesssim \sum_{|l|>N} 2^{-N_0|l|} \sum_{j\in\mathbb{N}} 2^{(j-k)|\alpha|} 2^{-N_1(j-j\wedge k)} \mathrm{Vol}\left(B_{(X,d)}(x, 2^{-j\wedge k})\right)^{-1}.\end{aligned}$$

By the convergence of geometric series, it suffices to verify

$$\sum_{j\in\mathbb{N}} 2^{(j-k)|\alpha|} 2^{-N_1(j-j\wedge k)} \mathrm{Vol}\left(B_{(X,d)}(x, 2^{-j\wedge k})\right)^{-1} \lesssim \mathrm{Vol}\left(B_{(X,d)}(x, 2^{-k})\right)^{-1}$$

We separate the above sum into two parts. The first,

$$\sum_{0\leq j\leq k} 2^{(j-k)|\alpha|} \mathrm{Vol}\left(B_{(X,d)}(x, 2^{-j\wedge k})\right)^{-1}.$$

Using that $\mathrm{Vol}\left(B_{(X,d)}(x, 2^{-j})\right)^{-1} \leq \mathrm{Vol}\left(B_{(X,d)}(x, 2^{-k})\right)^{-1}$, and $j \leq k$, we obtain

$$\sum_{0\leq j\leq k} 2^{(j-k)|\alpha|} \mathrm{Vol}\left(B_{(X,d)}(x, 2^{-j\wedge k})\right)^{-1} \lesssim \mathrm{Vol}\left(B_{(X,d)}(x, 2^{-k})\right)^{-1}$$

The second term is

$$\sum_{k\leq j} 2^{(j-k)|\alpha|} 2^{-N_1(j-k)} \mathrm{Vol}\left(B_{(X,d)}(x, 2^{-k})\right)^{-1}.$$

By taking N_1 large, this is a geometric sum, and therefore bounded by a constant times its largest term. We obtain

$$\sum_{k\leq j} 2^{(j-k)|\alpha|}2^{-N_1(j-k)}\mathrm{Vol}\big(B_{(X,d)}(x,2^{-k})\big)^{-1} \lesssim \mathrm{Vol}\big(B_{(X,d)}(x,2^{-k})\big)^{-1}.$$

Combing all of the above, it follows that

$$|(\delta W)^{\alpha}R_N(\phi)(z)| \leq C2^{-N_0 N}\mathrm{Vol}\big(B_{(X,d)}(x,\delta)\big)^{-1}$$

where z is arbitrary. □

By using the similar proof approach in the above lemma, it's easy to deduce that

Lemma 2.11. *Fix N, N_0, we have $\|R_N\|_{\mathcal{T}(n_0,m)\to\mathcal{T}(n_0,m)} \leq C2^{-NN_0}$. Moreover, R_N is a bounded operator from C^∞ to C^∞ with the norm $C2^{-NN_0}$.*

With the above lemma, we can get

Lemma 2.12. *If N is so large that*

$$C2^{-NN_0} < 1, \tag{2.2}$$

then $V_N = U_N^{-1}$ maps test function space to itself. More precisely, there exists a constant $C > 0$ such that for all $f \in \mathcal{T}(n_0,m)$,

$$\|V_N(f)\|_{\mathcal{T}(n_0,m)} \leq C\|f\|_{\mathcal{T}(n_0,m)}$$

Proof. If we choose $N \in \mathbb{N}$ such that $C2^{-NN_0} < 1$ holds, we have that for all $f \in \mathcal{T}(n_0,m)$,

$$\begin{aligned}\|V_N(f)\|_{\mathcal{T}(n_0,m)} &= \|(I-R_N)^{-1}(f)\|_{\mathcal{T}(n_0,m)}\\ &= \left\|\sum_{h=0}^{\infty}(R_N)^h(f)\right\|_{\mathcal{T}(n_0,m)}\end{aligned}$$

$$\leq \sum_{h=0}^{\infty} (C2^{-NN_0})^h \|f\|_{\mathcal{T}(n_0,m)}$$

$$\lesssim \|f\|_{\mathcal{T}(n_0,m)}$$

which completes the proof the lemma. $\square$

Now we are in the position to prove the part of $\mathcal{T}(n_0, m)$ convergence in Theorem 2.13. Let's restate it:

Lemma 2.13. *Letting $D_j^N = \sum_{|l|\leq N} D_{j+l}$ and $\widetilde{D}_j = V_N D_j^N$, thus $f = \sum_{j\in\mathbb{N}} \widetilde{D}_j D_j f$, which converges in $\mathcal{T}(n_0, m)$ (and therefore converges in distribution).*

Proof. Fix a large integer N such that (2.2) holds. For $L \in \mathbb{N}$, we write

$$\begin{aligned}
\sum_{j\leq L} \widetilde{D}_j D_j f &= V_N\Big(\sum_{j\leq L} D_j^N D_j\Big)(f) \\
&= V_N\Big(U_N - \sum_{j>L} D_j^N D_j\Big)(f) \\
&= V_N U_N f - V_N\Big(\sum_{j>L}\Big)(f) \\
&= f - \lim_{h\to\infty} (R_N)^h(f) - V_N\Big(\sum_{j>L} D_j^N D_j\Big)(f).
\end{aligned}$$

We now verify that

$$\lim_{L\to\infty} \Big\| f - \sum_{j\leq L} \widetilde{D}_j D_j(f) \Big\|_{\mathcal{T}(n_0,m)} = 0.$$

To see this, we write

$$\Big\| f - \sum_{|j|\leq L} \widetilde{D}_j D_j(f) \Big\|_{\mathcal{T}(n_0,m)} \leq \lim_{h\to\infty} \|(R_N)^h(f)\|_{\mathcal{T}(n_0,m)} + \Big\| V_N\Big(\sum_{j>L} D_j^N D_j\Big)(f) \Big\|_{\mathcal{T}(n_0,m)}$$

By the Lemma 2.11, we have

$$\lim_{h\to\infty}\|(R_N)^h(f)\|_{\mathcal{T}(n_0,m)} \leq \lim_{h\to\infty}(C2^{-NN_0})^h\|f\|_{\mathcal{T}(n_0,m)} = 0,$$

We now prove that

$$\lim_{L\to\infty}\Big\|V_N\Big(\sum_{j>L}D_j^N D_j\Big)(f)\Big\|_{\mathcal{T}(n_0,m)} = 0.$$

To this end, by Lemma 2.12, it suffices to verify that there exists some $\theta > 0$ such that for $f \in \mathcal{T}(n_0, m)$,

$$\Big\|\sum_{j>L}D_j^N D_j(f)\Big\|_{\mathcal{T}(n_0,m)} \lesssim 2^{-\theta L}\|f\|_{\mathcal{T}(n_0,m)}$$

By the definition, $\forall x \in M$, we have

$$\Big|W^\alpha\sum_{j>L}D_j^N D_j(f)(x)\Big| \leq \sum_{|l|\leq N}\sum_{j>L}2^{j|\alpha|}\Big|(2^{-j}W)^\alpha D_{j+l}D_j(f)(x)\Big|$$

By using the Proposition 2.3, we know that $\{(2^{N_0|l|}D_{j+l}D_j, 2^{-j})\,|\,(D_j, 2^{-j}) \in \mathcal{D}\}$ is a bounded set of elementary operators. For the simplicity, we denote the above new set as $\{(E_j, 2^{-j})\,|\,(E_j, 2^{-j}) \in \mathcal{E}\}$.

Thus, we can rewrite the inequality as

$$\Big|W^\alpha\sum_{j>L}D_j^N D_j(f)(x)\Big| \leq \sum_{|l|\leq N}2^{-N_0|l|}\sum_{j>L}2^{j|\alpha|}\Big|(2^{-j}W)^\alpha E_j(f)(x)\Big|$$

Next, applying the similar proof as Lemma 2.8, $\forall N_1, m$,

$$\Big|W^\alpha\sum_{j>L}D_j^N D_j(f)(x)\Big|$$

$$\lesssim \sum_{|l|\leq N} 2^{-N_0|l|} \sum_{j>L} 2^{j|\alpha|} 2^{-N_1 j} \frac{\big(1+\rho(x_1,x)\big)^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(x_1, 1+\rho(x_1,x))\big)}$$

By the convergence of geometric series, it suffices to verify

$$\sum_{j>L} 2^{j(|\alpha|-N_1)} \lesssim 2^{-\theta L}$$

By taking N_1 large, this is a geometric sum, and therefore bounded by a constant times its largest term. Combing all of the above, it follows that

$$\Big|W^\alpha \sum_{j>L} D_j^N D_j(f)(x)\Big| \leq C 2^{-\theta L} \frac{\big(1+\rho(x_1,x)\big)^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(x_1, 1+\rho(x_1,x))\big)}$$

This implies

$$\Big\|\sum_{j>L} D_j^N D_j(f)\Big\|_{\mathcal{T}(n_0,m)} \lesssim 2^{-\theta L} \|f\|_{\mathcal{T}(n_0,m)}$$

The lemma is proved. □

The respective part of Theorem 2.14 can be prove similarly.

We also need the Fefferman-Stein's vector-valued maximal function inequality.

Theorem 2.16 ([18]). *Let $1<p<\infty$, $1<q\leq\infty$, and let $\mathcal{M}$ be the Hardy-Littlewood maximal operator on M. Let $\{f_k\}_{k\in\mathbb{Z}} \subset L^p(M)$ be a sequence of measurable funcitons on M. Then*

$$\Big\|\Big\{\sum_{k=-\infty}^{\infty} |\mathcal{M}(f_k)|^q\Big\}^{1/q}\Big\|_{L^p(M)} \leq C \Big\|\Big\{\sum_{k=-\infty}^{\infty} |f_k|^q\Big\}^{1/q}\Big\|_{L^p(M)}$$

where C is independent of $\{f_k\}_{k\in\mathbb{Z}}$.

2.3 Discrete Calderón Reproducing Formula

We will use the classical decomposition on the homogeneous space by M. Christ in [11] and by Sawyer-Wheeden in [61]. Here, we use the statement in [11].

Lemma 2.14 ([11]). *χ is the space of homogeneous type, $\exists\ \{\mathcal{Q}_\tau^k \subset \chi : k \in \mathbb{Z}, \tau \in I_k\}$ of open subsets, where I_k is some index set, $\delta \in (0,1)$, $C_1, C_2 > 0$, s.t.*

(i) $\mu(\chi \backslash \cup_\tau \mathcal{Q}_\tau^k) = 0$ for each fixed k and $\mathcal{Q}_\alpha^k \cap \mathcal{Q}_\beta^k = \emptyset$ if $\alpha \neq \beta$.

(ii) $\mathcal{Q}_\beta^l \subset \mathcal{Q}_\alpha^k$ or $\mathcal{Q}_\beta^l \cap \mathcal{Q}_\alpha^k = \emptyset$ for $l \geq k$.

(iii) $\exists!\ \beta$, s.t. $\mathcal{Q}_\alpha^k \subset \mathcal{Q}_\beta^l$.

(iv) $\mathrm{diam}(\mathcal{Q}_\tau^k) \leq C_1 \delta^k$;

(v) $\mathcal{Q}_\tau^k$ contains some ball $B(z_\tau^k, C_2\delta^k)$.

Also, we denote by $\mathcal{Q}_\tau^{k,\nu}$, $\nu = 1, 2, \ldots, N(k,\alpha)$, the set of all cubes $\mathcal{Q}_\tau^{k+j} \subset \mathcal{Q}_\tau^k$ and j is a positive large integer such that

$$2^{-j} C_1 < \frac{1}{3}$$

Denote by $z_\tau^{k,\nu}$ the "center" of $\mathcal{Q}_\tau^{k,\tau}$ and by $y_\tau^{k,\nu}$ a point in $\mathcal{Q}_\tau^{k,\nu}$.

Now we're ready to introduce the Discrete Calderón Reproducing Formula. Recall the discrete Riemann sum operator on M,

$$\mathcal{S}(f)(x) = \sum_{k\in\mathbb{N}} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \int_{\mathcal{Q}_\tau^{k,\nu}} D_k^N(y)\, dy D_k(f)(y_\tau^{k,\nu})$$

where $D_k^N = \sum_{|l|\leq N} D_{k+l}$.

We firstly verify $\mathcal{S}$ is well defined and bounded on $L^2(M)$.

Lemma 2.15. *There exists some constant $C > 0$, such that for all $y_\tau^{k,\nu} \in \mathcal{Q}_\tau^{k,\nu}$ and $f \in L^2(M)$,*

$$\sum_{k\in\mathbb{N}} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \mu(\mathcal{Q}_\tau^{k,\nu}) |D_k(f)(y_\tau^{k,\nu})|^2 \leq C \|f\|_{L^2(M)}^2$$

Proof. By Proposition 2.3, we have

$$|D_kD_l(z,x)| \lesssim 2^{-|k-l|}\frac{(1+2^k\rho(z,x))^{-1}}{\mathrm{Vol}(B_{(X,d)}(z,2^{-k}+\rho(z,x)))}$$

where the implicit constant is independent of k,l,z,x. Notice that for all $x\in M$ and any $z,y\in \mathcal{Q}_\tau^{k,\nu}$, we have that $\rho(y,z)\le C_1 2^{-j}2^{-k}\le C_1 2^{-j}(2^{-k}+\rho(x,y))$, where $C_1 2^{-j}<1$. Thus, for all $x\in M$, any $y,z\in\mathcal{Q}_\tau^{k,\nu}$ and all $k,l\in\mathbb{Z}$,

$$|D_kD_l(z,x)|\chi_{\mathcal{Q}_\tau^{k,\nu}}(z)\lesssim 2^{-|k-l|}\frac{(1+2^k\rho(y,x))^{-1}}{\mathrm{Vol}(B_{(X,d)}(x,2^{-k}+\rho(y,x)))}\chi_{\mathcal{Q}_\tau^{k,\nu}}(y)$$

From this, it follows that

$$\begin{aligned}
|D_k(f)(y_\tau^{k,\nu})| &\le \sum_{l\in\mathbb{N}}\int_M |D_k\widetilde{D}_l(y_\tau^{k,\nu},x)|\chi_{\mathcal{Q}_\tau^{k,\nu}}(y_\tau^{k,\nu})|D_l(f)(x)|\,d\mu(x)\\
&\lesssim \sum_{l\in\mathbb{N}}2^{-|k-l|}\int_M\frac{(1+2^k\rho(y,x))^{-1}}{\mathrm{Vol}(B_{(X,d)}(x,2^{-k}+\rho(y,x)))}|D_l(f)(x)|\,d\mu(x)\cdot\chi_{\mathcal{Q}_\tau^{k,\nu}}(y)\\
&\lesssim \sum_{l\in\mathbb{N}}2^{-|k-l|}\int_{B_{(X,d)}(y,2^{-k})}\frac{1}{\mathrm{Vol}\big(B_{(X,d)}(y,2^{-k})\big)}|D_l(f)(x)|\,d\mu(x)\cdot\chi_{\mathcal{Q}_\tau^{k,\nu}}(y)\\
&\quad+\sum_{l\in\mathbb{N}}\sum_{h=1}^{\infty}2^{-|k-l|}\int_{B_{(X,d)}(y,2^{-k+h})\setminus B_{(X,d)}(y,2^{-k+h-1})}\frac{(2^k\rho(x,y))^{-1}}{\mathrm{Vol}\big(B_{(X,d)}(y,\rho(y,x))\big)}\\
&\qquad\times|D_l(f)(x)|\,d\mu(x)\cdot\chi_{\mathcal{Q}_\tau^{k,\nu}}(y)\\
&\lesssim\sum_{l\in\mathbb{N}}2^{-|k-l|}\mathcal{M}(D_l(f))(y)\chi_{\mathcal{Q}_\tau^{k,\nu}}(y)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{k\in\mathbb{N}}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(\tau,\nu)}\mu(\mathcal{Q}_\tau^{k,\nu})|D_k(f)(y_\tau^{k,\nu})|^2 &\lesssim \sum_{k\in\mathbb{N}}\int_M\left[\sum_{l\in\mathbb{N}}2^{-|l-k|}\mathcal{M}(D_l(f))(y)\right]^2dy\\
\lesssim\Big\|\Big\{\sum_{l\in\mathbb{N}}|\mathcal{M}(D_l(f))|^2\Big\}^{1/2}\Big\|_{L^2(M)}^2 &\lesssim \Big\|\Big\{\sum_{l\in\mathbb{N}}|D_l(f)|^2\Big\}^{1/2}\Big\|_{L^2(M)}^2\lesssim\|f\|_{L^2(M)}^2
\end{aligned}$$

□

Also, we have

Lemma 2.16. *Given a sequence* $\{a_\tau^{k,\nu} : k \in \mathbb{Z}, \tau \in I_k, \nu = 1, \ldots, N(k,\tau)\}$ *of numbers with*

$$\sum_{k\in\mathbb{N}} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} |a_\tau^{k,\nu}|^2 < \infty.$$

Then, the function defined by

$$f(x) = \sum_{k\in\mathbb{N}} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} [\mu(\mathcal{Q}_\tau^{k,\nu})]^{-1/2} a_\tau^{k,\nu} \int_{\mathcal{Q}_\tau^{k,\nu}} D_k^N(x,y)\, d\mu(y)$$

is in $L^2(M)$. *Moreover,*

$$\|f\|_{L^2(M)}^2 \le C \sum_{k\in\mathbb{N}} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} |a_\tau^{k,\nu}|^2.$$

Proof. By the definition,

$$\int_M f(x) \cdot \bar{f}(x)\, d\mu(x) = \int_M \Big(\sum_{k\in\mathbb{N}} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} [\mu(\mathcal{Q}_\tau^{k,\nu})]^{-1/2} a_\tau^{k,\nu} \int_{\mathcal{Q}_\tau^{k,\nu}} D_k^N(x,y)\, d\mu(y) \Big)$$
$$\times \overline{\Big(\sum_{k'\in\mathbb{N}} \sum_{\tau'\in I_{k'}} \sum_{\nu'=1}^{N(k',\tau')} [\mu(\mathcal{Q}_{\tau'}^{k',\nu'})]^{-1/2} a_{\tau'}^{k',\nu'} \int_{\mathcal{Q}_{\tau'}^{k',\nu'}} D_{k'}^N(x,y')\, d\mu(y') \Big)}\, d\mu(x)$$

WLOG, we can assume $k \le k'$, thus by using the similar argument in Lemma 2.7, $\forall N_0 \in \mathbb{N}$, we have

$$D_k^N(x,y) D_{k'}^N(x,y') = \sum_{|\alpha|\le N_0} 2^{k'(|\alpha|-N_0)} D_k^N(x,y) (2^{-k'} W_x)^\alpha D_{k',\alpha}^N(x,y')$$
$$= \sum_{|\alpha|\le N_0} 2^{k'(|\alpha|-N_0)} 2^{-|\alpha||k-k'|} D_k^N(x,y) (2^{k'} W_x)^\alpha D_{k',\alpha}^N(x,y')$$
$$= \sum_{|\alpha|\le N_0} 2^{k'(|\alpha|-N_0)} 2^{-|\alpha||k-k'|} 2^{-k(|\alpha|-N_0)} \Big(2^{k(|\alpha|-N_0)} D_k^N(x,y) (2^{k'} W_x)^\alpha \Big) D_{k',\alpha}^N(x,y')$$
$$= \sum_{|\alpha|\le N_0} 2^{-N_0|k-k'|} D_{k,\alpha}^N(x,y) D_{k',\alpha}^N(x,y')$$

where $\{(D^N_{k,\alpha}, 2^{-k}) | (D_k, 2^{-k}) \in \mathcal{D})\}$ and $\{(D^N_{k',\alpha}, 2^{-k'}) | (D_{k'}, 2^{-k'}) \in \mathcal{D}\}$ are both the bounded sets of elementary operators.

Hence, For each $k, \tau, \nu, k', \tau', \nu'$, it follows that

$$
\begin{aligned}
&\int_M \Big[\int_{\mathcal{Q}^{k,\nu}_\tau} D^N_k(x,y)\, d\mu(y) \overline{\int_{\mathcal{Q}^{k',\nu'}_{\tau'}} D^N_{k'}(x,y')\, d\mu(y')} \Big] \, d\mu(x) \\
&= \sum_{|\alpha| \le N_0} 2^{-|k-k'|N_0} \int_M \Big[\int_{\mathcal{Q}^{k,\nu}_\tau} D^N_{k,\alpha}(x,y)\, d\mu(y) \overline{\int_{\mathcal{Q}^{k',\nu'}_{\tau'}} D^N_{k',\alpha}(x,y')\, d\mu(y')} \Big] \, d\mu(x) \\
&\le C_{N_0,N,m,\mathcal{D}} 2^{-|k-k'|N_0} \int_M \Big[\int_{\mathcal{Q}^{k,\nu}_\tau} \frac{(1+2^k \rho(x,y))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(x, 2^{-k} + \rho(x,y))\big)} \, d\mu(y) \\
&\quad \times \overline{\int_{\mathcal{Q}^{k',\nu'}_{\tau'}} \frac{(1+2^{k'} \rho(x,y'))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(x, 2^{-k'} + \rho(x,y'))\big)} \, d\mu(y')} \Big] \, d\mu(x) \\
&\le C_{N_0,N,m,\mathcal{D}} 2^{-|k-k'|N_0} \int_M \Big[\int_{\mathcal{Q}^{k,\nu}_\tau} \frac{(1+2^k \rho(x,y^{k,\nu}_\tau))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(x, 2^{-k} + \rho(x,y^{k,\nu}_\tau))\big)} \, d\mu(y) \\
&\quad \times \overline{\int_{\mathcal{Q}^{k',\nu'}_{\tau'}} \frac{(1+2^{k'} \rho(x,y^{k',\nu'}_{\tau'}))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(x, 2^{-k'} + \rho(x,y^{k',\nu'}_{\tau'}))\big)} \, d\mu(y')} \Big] \, d\mu(x) \\
&\le C_{N_0,N,m,\mathcal{D}} 2^{-|k-k'|N_0} \mu(\mathcal{Q}^{k,\nu}_\tau) \mu(\mathcal{Q}^{k',\nu'}_{\tau'}) \int_M \frac{(1+2^k \rho(y^{k,\nu}_\tau, x))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(y^{k,\nu}_\tau, 2^{-k} + \rho(y^{k,\nu}_\tau, x))\big)} \\
&\quad \times \frac{(1+2^{k'} \rho(x,y^{k',\nu'}_{\tau'}))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(x, 2^{-k'} + \rho(x,y^{k',\nu'}_{\tau'}))\big)} \, d\mu(x)
\end{aligned}
$$

By using Lemma 2.3, the last term can be controlled by

$$
C_{N_0,N,m,\mathcal{D}} 2^{-|k-k'|N_0} \cdot \mu(\mathcal{Q}^{k,\nu}_\tau) \mu(\mathcal{Q}^{k',\nu'}_{\tau'}) \frac{(1+2^{k\wedge k'} \rho(y^{k,\nu}_\tau, y^{k',\nu'}_{\tau'}))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(y^{k,\nu}_\tau, 2^{-k\wedge k'} + \rho(y^{k,\nu}_\tau, y^{k',\nu'}_{\tau'}))\big)}
$$

or

$$
C_{N_0,N,m,\mathcal{D}} 2^{-|k-k'|N_0} \cdot \mu(\mathcal{Q}^{k,\nu}_\tau) \mu(\mathcal{Q}^{k',\nu'}_{\tau'}) \frac{(1+2^{k\wedge k'} \rho(y^{k',\nu'}_{\tau'}, y^{k,\nu}_\tau))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(y^{k',\nu'}_{\tau'}, 2^{-k\wedge k'} + \rho(y^{k',\nu'}_{\tau'}, y^{k,\nu}_\tau))\big)}
$$

Therefore,

$$
\|f\|^2_{L^2(M)} \le C_{N_0,N,m,\mathcal{D}} \sum_{\substack{k,\tau,\nu \\ k',\tau',\nu'}} 2^{-|k-k'|N_0} a^{k,\nu}_\tau a^{k',\nu'}_{\tau'} [\mu(\mathcal{Q}^{k,\nu}_\tau) \mu(\mathcal{Q}^{k',\nu'}_{\tau'})]^{1/2}
$$

$$
\begin{aligned}
&\times \frac{(1+2^{k\wedge k'}\rho(y_\tau^{k,\nu},y_{\tau'}^{k',\nu'}))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(y_\tau^{k,\nu},2^{-k\wedge k'}+\rho(y_\tau^{k,\nu},y_{\tau'}^{k',\nu'}))\big)}\\
\le C_{N_0,N,m,\mathcal{D}} &\sum_{\left\{\substack{k,\tau,\nu\\ k',\tau',\nu':k\le k'}\right\}} 2^{-|k-k'|N_0} a_\tau^{k,\nu} a_{\tau'}^{k',\nu'}[\mu(\mathcal{Q}_\tau^{k,\nu})\mu(\mathcal{Q}_{\tau'}^{k',\nu'})]^{1/2}\\
&\times \frac{(1+2^{k}\rho(y_\tau^{k,\nu},y_{\tau'}^{k',\nu'}))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(y_\tau^{k,\nu},2^{-k}+\rho(y_\tau^{k,\nu},y_{\tau'}^{k',\nu'}))\big)}\\
&\le C_{N_0,N,m,\mathcal{D}} A^{1/2}B^{1/2}
\end{aligned}
$$

where

$$
\begin{aligned}
A &= \sum_{\left\{\substack{k,\tau,\nu\\ k',\tau',\nu':k\le k'}\right\}} |a_\tau^{k,\nu}|^2 2^{-|k-k'|N_0} \frac{\mu(\mathcal{Q}_{\tau'}^{k',\nu'})(1+2^{k}\rho(y_\tau^{k,\nu},y_{\tau'}^{k',\nu'}))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(y_\tau^{k,\nu},2^{-k}+\rho(y_\tau^{k,\nu},y_{\tau'}^{k',\nu'}))\big)},\\
B &= \sum_{\left\{\substack{k,\tau,\nu\\ k',\tau',\nu':k\le k'}\right\}} |a_{\tau'}^{k',\nu'}|^2 2^{-|k-k'|N_0} \frac{\mu(\mathcal{Q}_{\tau}^{k,\nu})(1+2^{k}\rho(y_{\tau'}^{k',\nu'},y_{\tau}^{k,\nu}))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(y_{\tau'}^{k',\nu'},2^{-k}+\rho(y_{\tau'}^{k',\nu'},y_{\tau}^{k,\nu}))\big)}
\end{aligned}
$$

With the Cauchy Schwartz inequality, it's only left to verify,

$$
\sum_{k\le k'}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')} 2^{-2|k-k'|N_0} \frac{\mu(\mathcal{Q}_{\tau'}^{k',\nu'})(1+2^{k}\rho(y_\tau^{k,\nu},y_{\tau'}^{k',\nu'}))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(y_\tau^{k,\nu},2^{-k}+\rho(y_\tau^{k,\nu},y_{\tau'}^{k',\nu'}))\big)} \le C
$$

and

$$
\sum_{k\le k'}\sum_{\tau\in I_{k}}\sum_{\nu=1}^{N(k,\tau)} \frac{\mu(\mathcal{Q}_{\tau}^{k,\nu})(1+2^{k}\rho(y_{\tau'}^{k',\nu'},y_{\tau}^{k,\nu}))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(y_{\tau'}^{k',\nu'},2^{-k}+\rho(y_{\tau'}^{k',\nu'},y_{\tau}^{k,\nu}))\big)} \le C
$$

For A, by Lemma 2.5, $\exists N_1$, s.t.

$$
\begin{aligned}
&\sum_{k\le k'}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\nu')} 2^{-|k-k'|N_0} \frac{\mu(\mathcal{Q}_{\tau'}^{k',\nu'})(1+2^{k}\rho(y_\tau^{k,\nu},y_{\tau'}^{k',\nu'}))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(y_\tau^{k,\nu},2^{-k}+\rho(y_\tau^{k,\nu},y_{\tau'}^{k',\nu'}))\big)}\\
&\le \sum_{k\le k'}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\nu')} 2^{-|k-k'|(N_0-N_1)} \frac{\mu(\mathcal{Q}_{\tau'}^{k',\nu'})(1+2^{k'}\rho(y_\tau^{k,\nu},y_{\tau'}^{k',\nu'}))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(y_\tau^{k,\nu},2^{-k'}+\rho(y_\tau^{k,\nu},y_{\tau'}^{k',\nu'}))\big)}\\
&\le \sum_{k'} 2^{-|k-k'|(N_0-N_1)} \sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\nu')} \int_{\mathcal{Q}_{\tau'}^{k',\nu'}} \frac{(1+2^{k'}\rho(y_\tau^{k,\nu},y_{\tau'}^{k',\nu'}))^{-2m}}{\mathrm{Vol}\big(B_{(X,d)}(y_\tau^{k,\nu},2^{-k'}+\rho(y_\tau^{k,\nu},y_{\tau'}^{k',\nu'}))\big)}\, d\mu(y')
\end{aligned}
$$

$$\leq \sum_{k'} 2^{-|k-k'|(N_0-N_1)} \sum_{\tau' \in I_{k'}} \sum_{\nu'=1}^{N(k',\nu')} \int_{\mathcal{Q}_{\tau'}^{k',\nu'}} \frac{(1+2^{k'}\rho(y_\tau^{k,\nu}, y'))^{-2m}}{\mathrm{Vol}\big(B_{(X,d)}(y_\tau^{k,\nu}, 2^{-k'} + \rho(y_\tau^{k,\nu}, y'))\big)} \, d\mu(y')$$
$$\leq \sum_{k'} 2^{-|k-k'|(N_0-N_1)} \int_M \frac{(1+2^{k'}\rho(y_\tau^{k,\nu}, y'))^{-2m}}{\mathrm{Vol}\big(B_{(X,d)}(y_\tau^{k,\nu}, 2^{-k'} + \rho(y_\tau^{k,\nu}, y'))\big)} \, d\mu(y') \leq C_m$$

and similarly, for B, we have

$$\sum_{k \leq k'} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \frac{\mu(\mathcal{Q}_\tau^{k,\nu})(1+2^k\rho(y_{\tau'}^{k',\nu'}, y_\tau^{k,\nu}))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(y_{\tau'}^{k',\nu'}, 2^{-k} + \rho(y_{\tau'}^{k',\nu'}, y_\tau^{k,\nu}))\big)} \leq \sum_{k \leq k'} 2^{-|k-k'|N_0} C_m \leq C_m$$

□

With the above two lemmas, it follows immediately that

Theorem 2.17. *Let the notation be the same as above with j satisfying $2^{-j}C_1 < \frac{1}{3}$. Then the discrete Riemann sum operator $\mathcal{S}$ is bounded on $L^2(M)$. That is, there is a constant $C > 0$, only depending on N, such that for all $f \in L^2(\mathcal{M})$,*

$$\|\mathcal{S}f\|_{L^2(M)} \leq C\|f\|_{L^2(M)}$$

Proof. From Lemma 2.15 and Lemma 2.16, it follows that

$$\|\mathcal{S}f\|_{L^2(M)}^2 \leq \| \sum_{k \in \mathbb{Z}} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} [\mu(\mathcal{Q}_\tau^{k,\nu})]^{-1/2} |a_\tau^{k,\tau}| \int_{\mathcal{Q}_\tau^{k,\nu}} D_k^N(x,y)\, dy \|_{L^2(M)}$$
$$\leq \sum_{k \in \mathbb{Z}} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} |a_\tau^{k,\nu}|^2 \leq \|f\|_{L^2(M)}^2$$

where $a_\tau^{k,\nu} = [\mu(\mathcal{Q}_\tau^{k,\nu})]^{1/2} |D_k(f)(y_\tau^{k,\nu})|$. □

Next, we prove that $\mathcal{S}$ is invertible. To do this, we define $\mathcal{R} = I - \mathcal{S}$ and denote by $\mathcal{R}(x,y)$ for its kernel. Actually, we can prove: for any $n_0 \in \mathbb{N}$, Claim 1: For $h \in \mathbb{N}$,

$$\lim_{h\to\infty} \|\mathcal{R}^h(f)\|_{L^p(M)} \le \lim_{h\to\infty} (C_{p,N}2^{-\varepsilon_0 N} + C_{p,m,N,\mathcal{D}}2^{-j\varepsilon})^h \|f\|_{L^p(M)} = 0$$

$$\lim_{h\to\infty} \|\mathcal{R}^h(f)\|_{\mathcal{T}(n_0,m)} \le \lim_{h\to\infty} (C_{p,N}2^{-N_0 N} + C_{p,m,N,\mathcal{D}}2^{-j\varepsilon})^h \|f\|_{\mathcal{T}(n_0,m)} = 0$$

Claim 2:

$$\|\mathcal{S}^{-1}\|_{L^p(M)\to L^p(M)} < \infty$$

$$\|\mathcal{S}^{-1}\|_{\mathcal{T}(n_0,m)\to\mathcal{T}(n_0,m)} < \infty$$

Recall that $I = U_N + R_N$, thus

$$\begin{aligned}
\mathcal{R}(f)(x) &= (I-\mathcal{S})(f)(x) \\
&= \sum_{k\in\mathbb{N}}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} \int_{\mathcal{Q}_\tau^{k,\nu}} D_k^N(x,y)[D_k(f)(y) - D_k(f)(y_\tau^{k,\nu})]\,dy + \sum_{k\in\mathbb{N}}\sum_{|l|>M} D_{k+l}D_k(f)(x) \\
&\equiv \sum_{k\in\mathbb{N}} G_k(f)(x) + R_N(f)(x) \\
&\equiv G(f)(x) + R_N(f)(x)
\end{aligned}$$

Let $G_k(x,y)$ be the kernel of G_k. We now verify that $G_k(x,y)$ and hence $G(x,y)$ satisfies all the desired estimates. Clearly,

$$\begin{aligned}
G(x,y) &= \sum_{k\in\mathbb{N}}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} \int_{\mathcal{Q}_\tau^{k,\nu}} D_k^N(x,z)[D_k(z,y) - D_k(y_\tau^{k,\nu},y)]\,dz \\
&= \sum_{k\in\mathbb{N}} G_k(x,y)
\end{aligned}$$

For each G_k, we can prove the following,

Lemma 2.17. $\{(G_k, 2^{-k})\}$ *is a bounded set of the pre-elementary operators.*

Proof. By the construction of dyadic cubes in Lemma 2.14, for any $z \in \mathcal{Q}_\tau^{k,\nu}$,

$$\rho(z, y_\tau^{k,\nu}) \le C_1 2^{-(k+j)} = C_1 2^{-j} 2^{-k} \le C_1 2^{-j}(2^{-k} + \rho(y,z))$$

We recall that j always satisfies $2^{-j}C_1 < \frac{1}{3}$. Thus, we have

$$\frac{1}{2^{-k} + \rho(z, y_\tau^{k,\nu})} \le C \frac{1}{2^{-k} + \rho(y,z)} \tag{2.3}$$

and

$$\frac{1}{\mathrm{Vol}\big(B_{(X,d)}(y, 2^{-k}(1 + 2^k \rho(z, y_\tau^{k,\nu})))\big)} \le C \frac{1}{\mathrm{Vol}\big(B_{(X,d)}(y, 2^{-k}(1 + 2^k \rho(y,z)))\big)} \tag{2.4}$$

Also by the definition of elementary operators, $\forall m$, $\exists C_{m,\mathcal{D}} = C(m, \mathcal{D})$, s.t.

$$\begin{aligned}
|D_k(z,y) - D_k(y_\tau^{k,\nu}, y)| &\le \max_{\substack{z^* \in \mathcal{Q}_\tau^{N(k,\nu)} \\ |\gamma|=1}} \Big|(W_x)^\gamma D_k(z^*, y)\Big| \rho(z, y_\tau^{k,\nu}) \\
&\le C_{m,\mathcal{D}} \max_{z^* \in \mathcal{Q}_\tau^{k,\nu}} 2^k \frac{(1 + 2^k \rho(z^*, y))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(y, 2^{-k}(1 + 2^k \rho(z^*, y)))\big)} \rho(z, y_\tau^{k,\nu}) \\
&\le C_{m,\mathcal{D}} C_1 2^{-j} \frac{(1 + 2^k \rho(y,z))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(y, 2^{-k}(1 + 2^k \rho(y,z)))\big)}
\end{aligned}$$

where the last inequality comes from (2.3), (2.4), $\rho(z, y_\tau^{k,\nu}) \le C_1 2^{-j-k}$ and $|\gamma| = 1$.

Consequently,

$$\begin{aligned}
|G_k(x,y)| &= \Big| \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\nu)} \int_{\mathcal{Q}_\tau^{k,\nu}} D_k^N(x,z)[D_k(z,y) - D_k(y_\tau^{k,\nu}, y)]\, d\mu(z) \Big| \\
&\le C_{m,N,\mathcal{D}} C_1 2^{-j} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\nu)} \int_{\mathcal{Q}_\tau^{k,\nu}} \frac{(1 + 2^k \rho(x,z))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(y, 2^{-k}(1 + 2^k \rho(x,z)))\big)} \\
&\qquad \times \frac{(1 + 2^k \rho(z,y))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(y, 2^{-k}(1 + 2^k \rho(z,y)))\big)}\, d\mu(z) \\
&\le C_{m,N,\mathcal{D}} C_1 2^{-j} \frac{(1 + 2^k \rho(x,y))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(y, 2^{-k}(1 + 2^k \rho(x,y)))\big)}
\end{aligned}$$

where the last inequality comes from Lemma 2.3 and $C_{m,N,\mathcal{D}}$ only depends on m, N and $\mathcal{D}$.

Similarly, $\forall \alpha, \beta, m, N$, $\exists\ C_{\mathcal{D},\alpha,\beta,m,N} = C(\mathcal{D}, \alpha, \beta, m)$, $\forall (G_k, 2^{-k})$,

$$\left|(2^{-k}W_x)^\alpha (2^{-k}W_y)^\beta G_k(x,y)\right|$$

$$= \Big| \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\nu)} \int_{\mathcal{Q}_\tau^{k,\nu}} (2^{-k}W_x)^\alpha D_k^N(x,z) [(2^{-k}W_y)^\beta D_k(z,y) - (2^{-k}W_y)^\beta D_k(y_\tau^{k,\nu}, y)]\, d\mu(z) \Big|$$

$$\leq \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\nu)} \int_{\mathcal{Q}_\tau^{k,\nu}} \left|(2^{-k}W_x)^\alpha D_k^N(x,z)\right| \cdot \max_{\substack{z^* \in \mathcal{Q}_\tau^{N(k,\nu)} \\ |\gamma|=1}} \left|(W_x)^\gamma (2^{-k}W_y)^\beta D_k(z^*, y)\right| \rho(z, y_\tau^{k,\nu})\, d\mu(z)$$

$$\leq C_{\mathcal{D},\alpha,\beta,m,N} C_1 2^{-j} \Bigg| \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\nu)} \int_{\mathcal{Q}_\tau^{k,\nu}} \frac{(1 + 2^k \rho(x,z))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(x, 2^{-k}(1 + 2^k\rho(x,z)))\big)}$$

$$\times \frac{(1 + 2^k \rho(z,y))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(y, 2^{-k}(1 + 2^k\rho(z,y)))\big)}\, d\mu(z) \Bigg|$$

$$\leq C_{\mathcal{D},\alpha,\beta,m,N} C_1 2^{-j} \frac{\big(1 + 2^k\rho(x,z)\big)^{-m}}{\mathrm{Vol}(B_W(x, 2^{-k}(1 + 2^k\rho(x,y))))}$$

Hence, $\{(G_k, 2^{-k}) | k \in \mathbb{N}\}$ is a bounded set of pre-elementary operators. □

In fact, we can furthermore obtain the result as follows:

Lemma 2.18. $\{(G_k, 2^{-k}) | k \in \mathbb{N}\}$ *is a bounded set of the elementary operators.*

Proof. We've verify that $\{(G_k, 2^{-k}) | k \in \mathbb{N}\}$ is a bounded set of pre-elementary operators. The result will follow once we show for $k \in \mathbb{N}$, we have G_k is a sum of derivatives of operators of the same form, as in the definition of elementary operators. But we have, using Proposition 2.1,

$$G_k = \sum_{|\alpha|,|\beta| \leq 1} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\nu)} \int_{\mathcal{Q}_\tau^{k,\nu}} 2^{-(2-|\alpha|-|\beta|)k} (2^{-k}W)^\alpha D_{k,\alpha}^N(x,z)$$

$$\times [D_{k,\beta}(z,y) - D_{k,\beta}(y_\tau^{k,\nu}, y)] (2^{-k}W)^\beta\, d\mu(z)$$

where $\{(D_{k,\alpha}^N, 2^{-k}), (D_{k,\beta}, 2^{-k}) | (D_k, 2^{-k}) \in \mathcal{D}\}$ is a bounded set of elementary operators. And therefore,

$$G_k = \sum_{|\alpha|,|\beta|\le 1} 2^{-(2-|\alpha|-|\beta|)k}(2^{-k}W)^{\alpha}\Big\{\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\nu)}\int_{\mathcal{Q}_\tau^{k,\nu}} D_{k,\alpha}^N(x,z)$$
$$\times[D_{k,\beta}(z,y)-D_{k,\beta}(y_\tau^{k,\nu},y)]\,d\mu(z)\Big\}(2^{-k}W)^{\beta}$$
$$= \sum_{|\alpha|,|\beta|\le 1} 2^{-(2-|\alpha|-|\beta|)k}(2^{-k}W)^{\alpha}G_{k,\alpha,\beta}(2^{-k}W)^{\beta}$$

This completes the proof, since $G_{k,\alpha,\beta}$ is of the same form as G_k. $\square$

Lemma 2.19. *G is a Calderón-Zygmund operator of order 0. Moreover, $\exists\varepsilon>0$, s.t.*

$$\|G\|_{L^p(M)\to L^p(M)} \le C_{m,\mathcal{D},N}2^{-j\varepsilon}$$

and for any $n_0\in\mathbb{N}$,

$$\|G\|_{\mathcal{T}(n_0,m)\to\mathcal{T}(n_0,m)} \le C_{m,\mathcal{D},N}2^{-j\varepsilon}$$

Proof. According to the characterization in Theorem 2.11, it follows that $G=\sum_{k\in\mathbb{N}}G_k$ is a Calderón-Zygmund operator of order 0. Hence, G is $L^p(M)\to L^p(M)$ bounded, i.e. $\forall p>0$, $\exists C_p$, such that

$$\|G\|_{L^p(M)\to L^p(M)} \le C_p$$

Also note that, for all $k,l\in\mathbb{N}$, we have

$$\|G_kG_l^*\|_{L^2(M)\to L^2(M)} \le C_{m,\mathcal{D},N}2^{-j}2^{-|k-l|},$$
$$\|G_lG_k^*\|_{L^2(M)\to L^2(M)} \le C_{m,\mathcal{D},N}2^{-j}2^{-|k-l|}$$

The Colter-Stein Lemma now shows

$$\|G\|_{L^2(M)\to L^2(M)} \leq C_{m,\mathcal{D},N} 2^{-j}$$

Together, by the interpolation, we have

$$\|G\|_{L^p(M)\to L^p(M)} \leq C_{p,m,\mathcal{D},N} 2^{-j\varepsilon}$$

On the other hand, $\forall f \in \mathcal{T}(n_0, m)$ and $x \in M$,

$$|W^\alpha G(f)(x)| \leq \sum_{k\in\mathbb{N}} 2^{k|\alpha|} |(2^{-k}W)^\alpha G_k(f)(x)|$$

By using the Proposition 2.1, we know that $\{((2^{-k}W)^\alpha G_k(f), 2^{-k}) \,|\, (G_k, 2^{-k}), \in \mathcal{D}\}$ is a bounded set of elementary operators. For the simplicity, we denote the above new set as $\{(E_k, 2^{-k}) \,|\, (E_k, 2^{-k})$
$\in \mathcal{E}\}$.

Thus, we can rewrite the inequality as

$$|W^\alpha G(f)(x)| \leq \sum_{k\in\mathbb{N}} 2^{k|\alpha|} |E_k(f)(x)|$$

Again, applying the similar proof as Lemma 2.8, we have $\forall N_1, m$,

$$|W^\alpha G(f)(x)| \lesssim C_{m,\mathcal{D},N} 2^{-j} \sum_{k\in\mathbb{N}} 2^{k|\alpha|} 2^{-N_1 k} \frac{\big(1+\rho(x_1,x)\big)^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(x_1, 1+\rho(x_1,x))\big)} \|f\|_{\mathcal{T}(n_0,m)}$$

By the convergence of geometric series, it suffices to verify

$$\sum_{k\in\mathbb{N}} 2^{k(|\alpha|-N_1)} \lesssim 1$$

which is obviously by taking N_1 large. Therefore , we obtain that

$$|W^\alpha G(f)(x)| \leq C_{m,\mathcal{D},N} 2^{-j} \frac{\big(1+\rho(x_1,x)\big)^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(x_1, 1+\rho(x_1,x))\big)} \|f\|_{\mathcal{T}(n_0,m)}$$

□

Lemma 2.20. *Let $\mathcal{S}$ be the discrete Riemann sum operator on M and $\mathcal{R} = I - \mathcal{S}$. Then $\mathcal{R}$ is $L^p(M)$ bounded, i.e.*

$$\|\mathcal{R}\|_{L^p(M)\to L^p(M)} \leq C_p 2^{-\varepsilon_0 N} + C_{p,m,\mathcal{D},N} 2^{-j\varepsilon}$$

and $\mathcal{R}$ is bounded on $\mathcal{T}(n_0,m)$ for any $n_0 \in \mathbb{N}$, i.e.

$$\|\mathcal{R}\|_{\mathcal{T}(n_0,m)\to\mathcal{T}(n_0,m)} \leq C 2^{-N_0 N} + C_{m,\mathcal{D},N} 2^{-j}$$

Proof. By Lemma 2.9 and Lemma 2.19, it follows that

$$\begin{aligned}\|\mathcal{R}\|_{L^p(M)\to L^p(M)} &\leq \|G\|_{L^p(M)\to L^p(M)} + \|R_N\|_{L^p(M)\to L^p(M)} \\ &\leq C_{p,N} 2^{-\epsilon_0 N} + C_{p,m,\mathcal{D},N} 2^{-j\varepsilon}\end{aligned}$$

Also,

$$\begin{aligned}\|\mathcal{R}\|_{\mathcal{T}(n_0,m)\to\mathcal{T}(n_0,m)} &\leq \|G\|_{\mathcal{T}(n_0,m)\to\mathcal{T}(n_0,m)} + \|R_N\|_{\mathcal{T}(n_0,m)\to\mathcal{T}(n_0,m)} \\ &\leq C_N 2^{-N_0 N} + C_{m,\mathcal{D},N} 2^{-j}\end{aligned}$$

Since N and j are arbitrary, we can choose them large enough such that

$$C_{p,N} 2^{-\epsilon_0 N} + C_{p,m,\mathcal{D},N} 2^{-j\varepsilon} < 1$$

and

$$C_{p,N}2^{-N_0N} + C_{m,\mathcal{D},N}2^{-j} < 1$$

which implies our Claim 1. $\square$

Our Claim 2 follows from the corollary below.

Corollary 2.18. Let $\mathcal{S}$ be the discrete Riemann operator on M. Let $N, j \in \mathbb{N}$ such that

$$C_{p,N}2^{-\epsilon_0 N} + C_{p,m,\mathcal{D},N}2^{-j\varepsilon} < 1$$

and

$$C_N 2^{-N_0N} + C_{m,\mathcal{D},N}2^{-j} < 1$$

hold. Then $\mathcal{S}$ has a bounded inverse in $L^p(M)$ for $p \in (1, \infty)$. Namely, there exists a constant $C > 0$ depending on p such that

$$\|\mathcal{S}^{-1}\|_{L^p(M)\to L^p(M)} \leq C$$

and for any $n_0 \in \mathbb{N}$,

$$\|\mathcal{S}^{-1}\|_{\mathcal{T}(n_0,m)\to\mathcal{T}(n_0,m)} \leq C$$

To establish the discrete Calderón reproducing formula, we still need the following technical lemma.

Lemma 2.21. *Let j satisfy $C_1 2^{-j} < \frac{1}{3}$. For $k \in \mathbb{Z}$, any fixed $y_\tau^{k,\nu} \in Q_\tau^{k,\nu}$ with $\tau \in I_k$ and*

$\nu = \{1, \dots, N(k,\tau)\}$, and any $x \in M$, let

$$H_x(x,y) = \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \int_{\mathcal{Q}_\tau^{k,\nu}} D_k^N(x,z)\, d\mu(z) D_k(y_\tau^{k,\nu}, y).$$

Then $\{(H_k, 2^{-k}) \,|\, (D_k, 2^{-k}) \in \mathcal{D}\}$ is a bouded set of elementary operators.

Proof. $\forall \alpha, \beta, m, N$, $\exists C_{\mathcal{D},\alpha,\beta,m,N} = C(\mathcal{D}, \alpha, \beta, m, N)$, $\forall (H_k, 2^{-k})$,

$$\begin{aligned}
&|(2^{-k}W_x)^\alpha (2^{-k}W_y)^\beta H_k(x,y)| \\
&= \left| \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \int_{\mathcal{Q}_\tau^{k,\nu}} (2^{-k}W_x)^\alpha D_k^N(x,z)\, d\mu(z) (2^{-k}W_y)^\beta D_k(y_\tau^{k,\nu}, y) \right| \\
&\leq C_{\mathcal{D},\alpha,\beta,m,N} \left| \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \int_{\mathcal{Q}_\tau^{k,\nu}} \frac{(1+2^k\rho(x,z))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(x, 2^{-k} + \rho(x,z))\big)}\, d\mu(z) \right. \\
&\qquad \left. \times \frac{(1+2^k\rho(y_\tau^{k,\nu}, y))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(y_\tau^{k,\nu}, 2^{-k} + \rho(y_\tau^{k,\nu}, y))\big)} \right| \\
&= C_{\mathcal{D},\alpha,\beta,m,N} \left| \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \int_{\mathcal{Q}_\tau^{k,\nu}} \frac{(1+2^k\rho(x,z))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(x, 2^{-k} + \rho(x,z))\big)} \right. \\
&\qquad \left. \times \frac{(1+2^k\rho(y_\tau^{k,\nu}, y))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(y_\tau^{k,\nu}, 2^{-k} + \rho(y_\tau^{k,\nu}, y))\big)}\, d\mu(z) \right|
\end{aligned}$$

Note that for any $z, y_\tau^{k,\nu} \in \mathcal{Q}_\tau^{k,\nu}$ and $y \in M$,

$$\frac{(1+2^k\rho(y_\tau^{k,\nu}, y))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(y_\tau^{k,\nu}, 2^{-k} + \rho(y_\tau^{k,\nu}, y))\big)} \approx \frac{(1+2^k\rho(z,y))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(z, 2^{-k} + \rho(z,y))\big)}$$

Hence,

$$\begin{aligned}
&|(2^{-k}W_x)^\alpha (2^{-k}W_y)^\beta H_k(x,y)| \\
&\leq C_{\mathcal{D},\alpha,\beta,m,N} \left| \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \int_{\mathcal{Q}_\tau^{k,\nu}} \frac{(1+2^k\rho(x,z))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(x, 2^{-k} + \rho(x,z))\big)} \right. \\
&\qquad \left. \times \frac{(1+2^k\rho(z,y))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(z, 2^{-k} + \rho(z,y))\big)}\, d\mu(z) \right|
\end{aligned}$$

$$\leq C_{\mathcal{D},\alpha,\beta,m,N}\left|\int_M \frac{(1+2^k\rho(x,z))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(x,2^{-k}+\rho(x,z))\big)}\frac{(1+2^k\rho(z,y))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(z,2^{-k}+\rho(z,y))\big)}\,d\mu(z)\right|$$

$$\leq C_{\mathcal{D},\alpha,\beta,m,N}\frac{(1+2^k\rho(x,y))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(x,2^{-k}+\rho(x,y))\big)}$$

Therefore, $\{(H_k,2^{-k})\,|\,(D_k,2^{-k})\in\mathcal{D}\}$ is a bounded set of pre-elementary operators.

Next, we will show for $k\in\mathbb{N}$, we have H_k is a sum of derivatives of operators of the same form, as in the definition of elementary operators. But we have, using Proposition 2.1,

$$H_k(x,z)=\sum_{|\alpha|,|\beta|\leq 1}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\int_{\mathcal{Q}_\tau^{k,\nu}}2^{-(2-|\alpha|-|\beta|)k}(2^{-k}W)^\alpha D_{k,\alpha}^N(x,z)\,d\mu(z)$$
$$\times D_{k,\beta}(y_\tau^{k,\nu},y)(2^{-k}W)^\beta$$

where $\{(D_{k,\alpha}^N,2^{-k}),(D_{k,\beta},2^{-k})|(D_k,2^{-k})\in\mathcal{D}\}$ is a bounded set of elementary operators. And therefore,

$$H_k(x,y)=\sum_{|\alpha|,|\beta|\leq 1}2^{-(2-|\alpha|-|\beta|)k}(2^{-k}W)^\alpha\Big\{\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\int_{\mathcal{Q}_\tau^{k,\nu}}D_{k,\alpha}^N(x,z)\,d\mu(z)$$
$$\times D_{k,\beta}(y_\tau^{k,\nu},y)\Big\}(2^{-k}W)^\beta$$
$$=\sum_{|\alpha|,|\beta|\leq 1}2^{-(2-|\alpha|-|\beta|)k}(2^{-k}W)^\alpha H_{k,\alpha,\beta}(2^{-k}W)^\beta$$

This completes the proof, since $H_{k,\alpha,\beta}$ is of the same form as H_k. □

Now let's prove the Discrete Calderón Reproducing Formula.

Theorem 2.19 (Discrete Calderón Reproducing Formula)**.** *For any fixed $j\in\mathbb{N}$, there exists a family of linear operators $\{\widetilde{D}_k\}_{k\in\mathbb{N}}$ such that for any fixed $y_\tau^{k,\nu}\in\mathcal{Q}_\tau^{k,\nu}$ with $k\in\mathbb{N}$, $\tau\in I_k$, and $\nu=1,\ldots,N(k,\tau)$, and all $f\in C^\infty(M)$,*

$$f(x)=\sum_{k=0}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\int_{\mathcal{Q}_\tau^{k,\nu}}\widetilde{D}_k(x,y)\,d\mu(y)D_k(f)(y_\tau^{k,\nu})$$

$$= \sum_{k=0}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \widetilde{D}_k(x, y_\tau^{k,\nu}) \int_{\mathcal{Q}_\tau^{k,\nu}} D_k(f)(y)\, d\mu(y)$$
$$= \sum_{k=0}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \mu(\mathcal{Q}_\tau^{k,\nu}) \widetilde{D}_k(x, y_\tau^{k,\nu}) D_k(f)(y_\tau^{k,\nu})$$

where the series converges in both the norm of $\mathcal{T}(n_0, m)$ and the dual space $\mathcal{T}'(n_0, m)$, the the topology of bouned convergence as operators $C^\infty(M) \to C^\infty(M)$ and the norm of $L^p(M)$ with $p \in (1, \infty)$.

Proof. Fix $N, j \in \mathbb{N}$, where $2^{-j}C_1 < \frac{1}{3}$, $C_{p,N}2^{-\epsilon_0 N} + C_{p,m,N,\mathcal{D}}2^{-j\varepsilon} < 1$ and $C_N 2^{-N_0 N} + C_{m,N,\mathcal{D}}2^{-j} < 1$. Let D_k^N for $k \in \mathbb{Z}$ be as above. For $k \in \mathbb{Z}$, let $\widetilde{D}_k(x, y) = \mathcal{S}^{-1}[D_k^N(\cdot, y)](x)$. For $L \in \mathbb{N}$, we write

$$\sum_{k \le L} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \int_{\mathcal{Q}_\tau^{k,\nu}} \widetilde{D}_k(x, y)\, d\mu(y) D_k(y_\tau^{k,\nu})$$
$$= \mathcal{S}^{-1}\Big[\sum_{k \le L} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \int_{\mathcal{Q}_\tau^{k,\nu}} D_k^N(\cdot, y)\, d\mu(y) D_k(f)(y_\tau^{k,\nu})\Big](x)$$
$$= \mathcal{S}^{-1}\Big[\mathcal{S}(f)(\cdot) - \sum_{k > L} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \int_{\mathcal{Q}_\tau^{k,\nu}} D_k^N(\cdot, y)\, d\mu(y) D_k(f)(y_\tau^{k,\nu})\Big](x)$$
$$= f(x) - \lim_{h \to \infty} R^h(f)(x) - \mathcal{S}^{-1}\Big[\sum_{k > L} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \int_{\mathcal{Q}_\tau^{k,\nu}} D_k^N(\cdot, y)\, d\mu(y) D_k(f)(y_\tau^{k,\nu})\Big](x)$$

Claim 1 shows that for all $f \in \mathcal{T}(n_0, m)$ with the center $x_1 \in M$,

$$\lim_{h \to \infty} \|\mathcal{R}^h(f)\|_{\mathcal{T}(n_0,m)} \le \lim_{h \to \infty} (C_{p,N}2^{-N_0 N} + C_{p,m,N,\mathcal{D}}2^{-j})^h \|f\|_{\mathcal{T}(n_0,m)} = 0$$

and for all $f \in L^p(M)$ with $p \in (1, \infty)$,

$$\lim_{h \to \infty} \|\mathcal{R}^h(f)\|_{L^p(M)} \le \lim_{h \to \infty} (C_{p,N}2^{-\epsilon_0 N} + C_{p,m,N,\mathcal{D}}2^{-j\varepsilon})^h \|f\|_{L^p(M)} = 0$$

To finish the proof of the theorem, we still need to verify that for all $f \in \mathcal{T}(n_0, m)$,

$$\lim_{L\to\infty} \left\| \mathcal{S}^{-1}\left\{ \sum_{|k|>L} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} \int_{Q_\tau^{k,\nu}} D_k^N(\cdot, y)\, d\mu(y) D_k(f)(y_\tau^{k,\nu}) \right\} \right\|_{\mathcal{T}(n_0,m)} = 0, \tag{2.5}$$

and for all $f \in L^p(M)$ with $p \in (1, \infty)$,

$$\lim_{L\to\infty} \left\| \mathcal{S}^{-1}\left\{ \sum_{|k|>L} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} \int_{Q_\tau^{k,\nu}} D_k^N(\cdot, y)\, d\mu(y) D_k(f)(y_\tau^{k,\nu}) \right\} \right\|_{L^p(M)} = 0. \tag{2.6}$$

Let's consider (2.5) firstly. $\forall\, x \in M$, we have

$$\left| W^\alpha \sum_{|k|>L} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} \int_{Q_\tau^{k,\nu}} D_k^N(x, y)\, d\mu(y) D_k(f)(y_\tau^{k,\nu}) \right|$$
$$\leq \sum_{|k|>L} 2^{k|\alpha|} |(2^{-k}W)^\alpha H_k(f)(x)|$$

By using the Proposition 2.1, we know that $\{((2^{-k}W)^\alpha H_k(f), 2^{-k}) \,|\, (H_k, 2^{-k}), \in \mathcal{H}\}$ is a bounded set of elementary operators. For the simplicity, we denote the above new set as $\{(E_k, 2^{-k}) \,|$

$(E_k, 2^{-k}) \in \mathcal{E}\}$. Thus, we can rewrite the inequality as

$$\left| W^\alpha \sum_{|k|>L} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} \int_{Q_\tau^{k,\nu}} D_k^N(x, y)\, d\mu(y) D_k(f)(y_\tau^{k,\nu}) \right|$$
$$\leq \sum_{k>L} 2^{k|\alpha|} |E_j(f)(z)|$$

Next, applying the similar proof method as Lemma 2.8, $\forall N_1, m$,

$$\left| W^\alpha \sum_{|k|>L} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} \int_{Q_\tau^{k,\nu}} D_k^N(x, y)\, d\mu(y) D_k(f)(y_\tau^{k,\nu}) \right|$$
$$\lesssim \sum_{k>L} 2^{k|\alpha|} 2^{-N_1 k} \frac{\big(1 + \rho(x_1, x)\big)^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(x_1, 1 + \rho(x_1, x))\big)} \|f\|_{\mathcal{T}(n_0,m)}$$

Hence, it suffices to verify

$$\sum_{k>L} 2^{k|\alpha|}2^{-N_1 k} \lesssim 1$$

By taking N_1 large, this is a geometric sum, and therefore bounded by a constant times its largest term. There exists some $\theta > 0$, such that

$$\sum_{k>L} 2^{k|\alpha|}2^{-N_1 k} \lesssim 2^{-\theta L}.$$

Combing all of the above, it follows that

$$\begin{aligned}&\left| W^\alpha \sum_{|k|>L} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \int_{Q_\tau^{k,\nu}} D_k^N(x,y)\, d\mu(y) D_k(f)(y_\tau^{k,\nu}) \right| \\ &\leq C 2^{-\theta L} \frac{\big(1+\rho(x_1,x)\big)^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(x_1, 1+\rho(x_1,x))\big)} \|f\|_{\mathcal{T}(n_0,m)}\end{aligned}$$

This implies

$$\left\| \sum_{|k|>L} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \int_{Q_\tau^{k,\nu}} D_k^N(\cdot,y)\, d\mu(y) D_k(f)(y_\tau^{k,\nu}) \right\|_{\mathcal{T}(n_0,m)} \lesssim 2^{-\theta L} \|f\|_{\mathcal{T}(n_0,m)}$$

Combing Corollary 2.18, it's easy to obtain (2.5). Next, let's consider (2.6).

For $L \in \mathbb{N}$, let T_L be the operator associated with the kernel

$$K_L(x,y) = \sum_{k>L} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \int_{Q_\tau^{k,\nu}} D_k^N(x,z)\, d\mu(z) D_k(y_\tau^{k,\nu}, y)$$

Hence, with the above claims, it suffices to verify that for all $f \in L^p(M)$ with $p \in (1,\infty)$, $\|T_L(f)\|_{L^p(M)} \to 0$ as $L \to \infty$. By the continuous Calderón reproducing formula, for $f \in L^p(M)$ and $g \in L^{p'}(M)$, $f = \sum_{l\in\mathbb{N}} \widetilde{D}_l D_l(f)$ and $g = \sum_{l\in\mathbb{N}} \widetilde{D}_l D_l(g)$, respectively, in $L^p(M)$ and $L^{p'}(M)$, where $\widetilde{D}_l$ for $l \in \mathbb{N}$ are as in Theorem 2.13. We have the following orthogonality estimate:

$$
\begin{aligned}
|D_k\widetilde{D}_l(y_\tau^{k,\nu}, z)| &\lesssim 2^{-|k-l|}\frac{(1+2^{k\wedge l}\rho(y,z))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(y, 2^{-k\wedge l}(1+2^{k\wedge l}\rho(y,z)))\big)},\\
|(D_k^N)^*\widetilde{D}_l(y, z)| &\lesssim 2^{-|k-l|}\frac{(1+2^{k\wedge l}\rho(y,z))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(y, 2^{-k\wedge l}(1+2^{k\wedge l}\rho(y,z)))\big)}.
\end{aligned}
$$

where the implicit constant is independent of $k, l, y_\tau^{k,\nu}, z$. Besides, we have

$$
\begin{aligned}
&\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\int_{Q_\tau^{k,\nu}}\frac{(1+2^{k\wedge l}\rho(y,z))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(y, 2^{-k\wedge l}(1+2^{k\wedge l}\rho(y,z)))\big)}f(z)\,d\mu(z)\\
&\lesssim \int_M \frac{(1+2^{k\wedge l}\rho(y,z))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(y, 2^{-k\wedge l}(1+2^{k\wedge l}\rho(y,z)))\big)}f(z)\,d\mu(z)\\
&\lesssim \int_{B_{(X,d)}(y,2^{-k\wedge l})}\frac{1}{\mathrm{Vol}\big(B_{(X,d)}(y,2^{-k\wedge l})\big)}|f(z)|\,d\mu(z)\\
&+\sum_{h=1}^{\infty}\int_{B_{(X,d)}(y,2^{-k\wedge l+h})\setminus B_{(X,d)}(y,2^{-k\wedge l+h-1})}\frac{(2^{k\wedge l}\rho(x,y))^{-1}}{\mathrm{Vol}\big(B_{(X,d)}(y,\rho(y,x))\big)}|f(z)|\,d\mu(z)\\
&\lesssim \sum_{h=0}^{\infty}2^{-h}\mathrm{Vol}\big(B_{(X,d)}(x,2^{-k\wedge l+h})\big)^{-1}\int_{B_{(X,d)}(x,2^{-k\wedge l+h})}|f(z)|\,d\mu(z)\lesssim \mathcal{M}(f)(y)
\end{aligned}
$$

Hence, by using the Fefferman-Stein's vector-valued maximal function inequality, and the duality argument, we have

$$
\begin{aligned}
\|T_L(f)\|_{L^p(M)} &= \sup_{\|g\|_{L^{p'}(M)}\le 1}|\langle T_L(f), g\rangle|\\
&= \sup_{\|g\|_{L^{p'}(M)}\le 1}\Big|\sum_{k>L}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\int_{Q_\tau^{k,\nu}}D_k(f)(y_\tau^{k,\nu})(D_k^N)^*(g)(y)\,d\mu(y)\Big|
\end{aligned}
$$

$$
\begin{aligned}
&= \sup_{\|g\|_{L^{p'}(M)}\le 1}\Big|\sum_{k>L}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\int_{Q_\tau^{k,\nu}}D_k\big(\sum_{l=0}^{\infty}\widetilde{D}_lD_l(f)\big)(y_\tau^{k,\nu})(D_k^N)^*\big(\sum_{l=0}^{\infty}\widetilde{D}_lD_l(g)\big)(y)\,d\mu(y)\Big|\\
&\lesssim \sup_{\|g\|_{L^{p'}(M)}\le 1}\Big\|\Big\{\sum_{k>L}\Big[\sum_{l=0}^{\infty}2^{-|k-l|}\mathcal{M}(|D_l(f)|)\Big]^2\Big\}^{1/2}\Big\|_{L^p(M)}\\
&\times\Big\|\Big\{\sum_{k>L}\Big[\sum_{l=0}^{\infty}2^{-|k-l|}\mathcal{M}(|(D_l)^*(g)|)\Big]^2\Big\}^{1/2}\Big\|_{L^{p'}(M)}
\end{aligned}
$$

$$\lesssim 2^{-L/2}\Big\|\Big\{\sum_{l<L/2}[\mathcal{M}(|D_l(f)|)]^2\Big\}^{1/2}\Big\|_{L^p(M)}$$

$$+\Big\|\Big\{\sum_{l\geq L/2}[\mathcal{M}(|D_l(f)|)]^2\Big\}^{1/2}\Big\|_{L^p(M)}\to 0,$$

as $L\to\infty$. □

2.4 Plancherel-Pôlya inequality on M

Using discrete Calderón formula we prove the following Plancherel-Pôlya inequalities on M.

Theorem 2.20. *Let $\{(D_k,2^{-k})|(D_k,2^{-k})\in\mathcal{D}\}$ and $\{(D'_{k'},2^{-k'})|(D'_{k'},2^{-k'})\in\mathcal{D}'\}$ be the bounded sets of elementary operators and both decompose the identity operator I, i.e. $I=\sum_{k\in\mathbb{N}}D_k=\sum_{k'\in\mathbb{N}}D'_{k'}$. For all $f\in C^\infty(M)$,*

$$\Big\|\Big\{\sum_{k\in\mathbb{N}}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\sup_{z\in\mathcal{Q}_\tau^{k,\nu}}|D_k(f)(z)|^2\chi_{\mathcal{Q}_\tau^{k,\nu}}(\cdot)\Big\}^{1/2}\Big\|_{L^p(M)}$$
$$\approx\Big\|\Big\{\sum_{k'\in\mathbb{N}}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}\inf_{z\in\mathcal{Q}_{\tau'}^{k',\nu'}}|D'_{k'}(f)(z)|^2\chi_{\mathcal{Q}_{\tau'}^{k',\nu'}}(\cdot)\Big\}^{1/2}\Big\|_{L^p(M)}$$

To prove Theorem 2.20, we need the following technical lemma:

Lemma 2.22. *If $r<p\leq 1$, then there exists a constant $C>0$ depending only on r such that for all $a_\tau^{k,\nu}\in\mathbb{C}$ and $x\in M$,*

$$\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\mu(\mathcal{Q}_\tau^{k,\nu})\frac{(1+2^{k\wedge k'}\rho(x,y_\tau^{k,\nu}))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(x,2^{-k\wedge k'}+\rho(x,y_\tau^{k,\nu}))\big)}|a_\tau^{k,\nu}|$$
$$\leq C\cdot 2^{[(k\wedge k')-k]Q_2(1-1/r)}\Big\{\mathcal{M}\Big(\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}|a_\tau^{k,\nu}|^r\chi_{\mathcal{Q}_\tau^{k,\nu}}(\cdot)\Big)(x)\Big\}^{1/r}$$

which $\mathcal{M}$ is the Hardy-Littlewood maximal function on M.

Proof. For any positive sequence $\{a_k\}_{k\in\mathbb{N}}$ and the positive number $r\leq 1$, we have the following inequality:

$$\sum_{k\in\mathbb{N}} a_k \leq \Big(\sum_{k\in\mathbb{N}} a_k^r\Big)^{1/r}$$

Thus, the left hand side of the inequality in the lemma is controlled by

$$\begin{aligned}
&\Big\{\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} \mu(\mathcal{Q}_\tau^{k,\tau})^r \frac{(1+2^{k\wedge k'}\rho(x,y_\tau^{k,\nu}))^{-rm}}{\mathrm{Vol}\big(B_{(X,d)}(x,2^{-k\wedge k'}+\rho(x,y_\tau^{k,\nu}))\big)^r}|a_\tau^{k,\nu}|^r\Big\}^{1/r}\\
&=\Big\{\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\int_M \mu(\mathcal{Q}_\tau^{k,\nu})^{r-1}\chi_{\mathcal{Q}_\tau^{k,\nu}}(z)\frac{(1+2^{k\wedge k'}\rho(x,y_\tau^{k,\nu}))^{-rm}}{\mathrm{Vol}\big(B_{(X,d)}(x,2^{-k\wedge k'}+\rho(x,y_\tau^{k,\nu}))\big)^r}|a_\tau^{k,\nu}|^r\,d\mu(z)\Big\}^{1/r}
\end{aligned}$$

By the estimate $\mu(\mathcal{Q}_\tau^{k,\nu})\chi_{\mathcal{Q}_\tau^{k,\nu}}(z)\sim \mathrm{Vol}\big(B_{(X,d)}(z,2^{-k})\big)\chi_{\mathcal{Q}_\tau^{k,\nu}}(z)$ for all $z\in M$, we have

$$\lesssim\Big\{\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\int_M \frac{\mathrm{Vol}\big(B_{(X,d)}(z,2^{-k})\big)^{r-1}(1+2^{k\wedge k'}\rho(x,y_\tau^{k,\nu}))^{-rm}}{\mathrm{Vol}\big(B_{(X,d)}(x,2^{-k\wedge k'}+\rho(x,y_\tau^{k,\nu}))\big)^r}|a_\tau^{k,\nu}|^r\chi_{\mathcal{Q}_\tau^{k,\nu}}(z)\,d\mu(z)\Big\}^{1/r}$$

Moreover, by the following estimate:

$$\mathrm{Vol}\big(B_{(X,d)}(x,2^{-k\wedge k'}+\rho(x,y_\tau^{k,\nu}))\big)\sim \mathrm{Vol}\big(B_{(X,d)}(x,2^{-k\wedge k'}+\rho(x,z))\big),$$

$$1+2^{k\wedge k'}\rho(x,y_\tau^{k,\nu})\sim 1+2^{k\wedge k'}\rho(x,z)$$

for all $z\in\mathcal{Q}_\tau^{k,\nu}$, we can obtain

$$\begin{aligned}
&\lesssim\Big\{\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\int_M \frac{\mathrm{Vol}\big(B_{(X,d)}(z,2^{-k})\big)^{r-1}(1+2^{k\wedge k'}\rho(x,z))^{-rm}}{\mathrm{Vol}\big(B_{(X,d)}(x,2^{-k\wedge k'}+\rho(x,z))\big)^r}|a_\tau^{k,\nu}|^r\chi_{\mathcal{Q}_\tau^{k,\nu}}(z)\,d\mu(z)\Big\}^{1/r}\\
&\lesssim\Big\{\int_M \frac{\mathrm{Vol}\big(B_{(X,d)}(z,2^{-k})\big)^{r-1}(1+2^{k\wedge k'}\rho(x,z))^{-rm}}{\mathrm{Vol}\big(B_{(X,d)}(x,2^{-k\wedge k'}+\rho(x,z))\big)^r}\Big(\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}|a_\tau^{k,\nu}|^r\chi_{\mathcal{Q}_\tau^{k,\nu}}(z)\Big)\,d\mu(z)\Big\}^{1/r}
\end{aligned}$$

Consequently,

$$\lesssim\Big\{\int_{B_{(X,d)}(x,2^{-k\wedge k'})}\frac{\mathrm{Vol}\big(B_{(X,d)}(z,2^{-k})\big)^{r-1}}{\mathrm{Vol}\big(B_{(X,d)}(x,2^{-k\wedge k'})\big)^r}\Big(\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}|a_\tau^{k,\nu}|^r\chi_{\mathcal{Q}_\tau^{k,\nu}}(z)\Big)\,d\mu(z)$$

$$+\sum_{h=1}^{\infty}\int_{B_{(X,d)}(x,2^{-k\wedge k'+h})\setminus B_{(X,d)}(x,2^{-k\wedge k'+h-1})}\frac{\mathrm{Vol}\big(B_{(X,d)}(z,2^{-k})\big)^{r-1}(2^{k\wedge k'}\rho(x,z))^{-rm}}{\mathrm{Vol}\big(B_{(X,d)}(x,\rho(x,z))\big)^{r}}$$
$$\times\Big(\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}|a_\tau^{k,\nu}|^r\chi_{Q_\tau^{k,\nu}}(z)\Big)\,d\mu(z)\Big\}^{1/r}$$

Next, applying the estimate $\mathrm{Vol}\big(B_{(X,d)}(x,r)\big)\sim\mathrm{Vol}\big(B_{(X,d)}(z,r)\big)$ for any x,z satisfying $\rho(x,z)\le\alpha r$, we have

$$\lesssim\Big\{\int_{B_{(X,d)}(x,2^{-k\wedge k'})}\frac{\mathrm{Vol}\big(B_{(X,d)}(x,2^{-k})\big)^{r-1}}{\mathrm{Vol}\big(B_{(X,d)}(x,2^{-k\wedge k'})\big)^{r}}\Big(\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}|a_\tau^{k,\nu}|^r\chi_{Q_\tau^{k,\nu}}(z)\Big)\,d\mu(z)$$
$$+\sum_{h=1}^{\infty}\int_{B_{(X,d)}(x,2^{-k\wedge k'+h})\setminus B_{(X,d)}(x,2^{-k\wedge k'+h-1})}\frac{\mathrm{Vol}\big(B_{(X,d)}(x,2^{-k})\big)^{r-1}(2^{k\wedge k'}\rho(x,z))^{-rm}}{\mathrm{Vol}\big(B_{(X,d)}(x,\rho(x,z))\big)^{r}}$$
$$\times\Big(\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}|a_\tau^{k,\nu}|^r\chi_{Q_\tau^{k,\nu}}(z)\Big)\,d\mu(z)\Big\}^{1/r}$$

Also, by the estimate $\mathrm{Vol}\big(B_{(X,d)}(x,2^{-k\wedge k'})\big)\lesssim 2^{[k-(k\wedge k')]Q_2}\mathrm{Vol}\big(B_{(X,d)}(x,2^{-k})\big)$, we can obtain

$$\lesssim\mathrm{Vol}\big(B_{(X,d)}(x,2^{-k})\big)^{1-1/r}2^{[(k\wedge k')-k]Q_2(1-1/r)}\mathrm{Vol}\big(B_{(X,d)}(x,2^{-k})\big)^{1/r-1}$$
$$\times\Big\{\frac{1}{\mathrm{Vol}\big(B_{(X,d)}(x,2^{-k\wedge k'})\big)}\int_{B_{(X,d)}(x,2^{-k\wedge k'})}\Big(\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}|a_\tau^{k,\nu}|^r\chi_{Q_\tau^{k,\nu}}(z)\Big)\,d\mu(z)$$
$$+\sum_{h=1}^{\infty}\frac{2^{-h[rm-Q_2(1-r)]}}{\mathrm{Vol}\big(B_{(X,d)}(x,\rho(x,z))\big)}\int_{B_{(X,d)}(x,2^{-k\wedge k'+h})\setminus B_{(X,d)}(x,2^{-k\wedge k'+h-1})}$$
$$\times\Big(\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}|a_\tau^{k,\nu}|^r\chi_{Q_\tau^{k,\nu}}(z)\Big)\,d\mu(z)\Big\}^{1/r}$$
$$\lesssim 2^{[(k\wedge k')-k]Q_2(1-1/r)}\Big\{\sum_{h=0}^{\infty}2^{-h[rm-Q_2(1-r)]}\mathcal{M}\Big(\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}|a_\tau^{k,\nu}|^r\chi_{Q_\tau^{k,\nu}}(\cdot)\Big)(x)\Big\}^{1/r}$$

where the last inequality comes from the definition of the definitin of the Hardy-Littlewood maximal funciton.

Finally, we can choose m large enough such that $rm-Q_2(1-r)>0$. Thus, by the convergence of the Geometric series, it follows immediately,

$$\leq 2^{[(k\wedge k')-k]Q_2(1-1/r)}\Big\{\mathcal{M}\Big(\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}|a_\tau^{k,\nu}|^r\chi_{\mathcal{Q}_\tau^{k,\nu}}(\cdot)\Big)(x)\Big\}^{1/r}$$

□

Now let's prove Theorem 2.20.

Proof of Theorem 2.20. For any $f\in C_0^\infty(M)$, by the discrete Calderón reproducing formula, we have

$$f(x)=\sum_{k'\in\mathbb{N}}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}\mu(\mathcal{Q}_{\tau'}^{k',\nu'})\widetilde{D}'_{k'}(x,y_{\tau'}^{k',\nu'})D'_{k'}(f)(y_{\tau'}^{k',\nu'})$$

By the orthogonality argument, $\forall m>0$, $\exists C_{m,\mathcal{D},\mathcal{D}'}$, s.t.

$$|D_k\widetilde{D}'_{k'}(x,y)|\leq C_{m,\mathcal{D},\mathcal{D}'}2^{-N_0|k-k'|}\frac{(1+2^{k\wedge k'}\rho(x,y))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(x,2^{-k\wedge k'}+\rho(x,y))\big)}$$

From the above, for any $k,k'\in\mathbb{N}$, we have Thus,

$$\begin{aligned}&\sum_{k\in\mathbb{N}}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\sup_{z\in\mathcal{Q}_\tau^{k,\nu}}|D_k(f)(z)|^2\chi_{\mathcal{Q}_\tau^{k,\nu}}(x)\\ \leq C_{m,\mathcal{D},\mathcal{D}'}\cdot&\sum_{k\in\mathbb{N}}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\Big(\sum_{k'\in\mathbb{N}}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}\mu(\mathcal{Q}_{\tau'}^{k',\nu'})2^{-N_0|k-k'|}|D'_{k'}(f)(y_{\tau'}^{k',\nu'})|\\ &\times\frac{(1+2^{k\wedge k'}\rho(y_\tau^{k,\nu},y_{\tau'}^{k',\nu'}))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(y_\tau^{k,\nu},2^{-k\wedge k'}+\rho(y_\tau^{k,\nu},y_{\tau'}^{k',\nu'}))\big)}\Big)^2\chi_{\mathcal{Q}_\tau^{k,\nu}}(x)\end{aligned}$$

Equivalently,

$$\begin{aligned}C_{m,\mathcal{D},\mathcal{D}'}\cdot&\sum_{k\in\mathbb{N}}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\Big(\sum_{k'\in\mathbb{N}}2^{-N_0|k-k'|}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}\mu(\mathcal{Q}_{\tau'}^{k',\nu'})|D'_{k'}(f)(y_{\tau'}^{k',\nu'})|\\ &\times\frac{(1+2^{k\wedge k'}\rho(y_\tau^{k,\nu},y_{\tau'}^{k',\nu'}))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(y_\tau^{k,\nu},2^{-k\wedge k'}+\rho(y_\tau^{k,\nu},y_{\tau'}^{k',\nu'}))\big)}\Big)^2\chi_{\mathcal{Q}_\tau^{k,\nu}}(x)\end{aligned}$$

We choose r such that $r < p \le 1$. Now apply Lemma 2.22. The above term is bounded by

$$C_{m,\mathcal{D},\mathcal{D}'} \cdot \sum_{k\in\mathbb{N}} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} \Big(\sum_{k'\in\mathbb{N}} 2^{-N_0|k-k'|} 2^{[(k\wedge k')-k]Q_2(1-\frac{1}{r})}$$
$$\times \big[\mathcal{M}\big(\sum_{\tau'\in I_{k'}} \sum_{\nu'=1}^{N(k',\tau')} \inf_{z\in\mathcal{Q}_{\tau'}^{k',\nu'}} |D'_{k'}(f)(z)|^r \chi_{\mathcal{Q}_{\tau'}^{k',\nu'}}(\cdot)\big)(y_\tau^{k,\nu})\big]^{\frac{1}{r}} \Big)^2 \chi_{\mathcal{Q}_\tau^{k,\nu}}(x)$$

by Cauchy-Schwartz inequality,

$$\le C_{m,\mathcal{D},\mathcal{D}'} \cdot \sum_{k\in\mathbb{N}} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} \Big\{ \Big[\sum_{k'\in\mathbb{N}} 2^{-N_0|k-k'|} 2^{[(k\wedge k')-k]Q_2(1-\frac{1}{r})} \Big]^{1/2}$$
$$\times \Big[\sum_{k'\in\mathbb{N}} 2^{-N_0|k-k'|} 2^{[(k\wedge k')-k]Q_2(1-\frac{1}{r})}$$
$$\times \big[\mathcal{M}\big(\sum_{\tau'\in I_{k'}} \sum_{\nu'=1}^{N(k',\tau')} \inf_{z\in\mathcal{Q}_{\tau'}^{k',\nu'}} |D'_{k'}(f)(z)|^r \chi_{\mathcal{Q}_{\tau'}^{k',\nu'}}(\cdot)\big)(y_\tau^{k,\nu})\big]^{\frac{2}{r}} \Big]^{1/2} \Big\}^2 \chi_{\mathcal{Q}_\tau^{k,\nu}}(x)$$

thus, choosing N_0 large enough such that $N_0 + Q_2(1-\frac{1}{r}) > 0$, we have

$$\le C_{m,\mathcal{D},\mathcal{D}'} \sum_{k\in\mathbb{N}} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} \sum_{k'\in\mathbb{N}} 2^{-N_0|k-k'|} 2^{[(k\wedge k')-k]Q_2(1-\frac{1}{r})}$$
$$\times \big[\mathcal{M}\big(\sum_{\tau'\in I_{k'}} \sum_{\nu'=1}^{N(k',\tau')} \inf_{z\in\mathcal{Q}_{\tau'}^{k',\nu'}} |D'_{k'}(f)(z)|^r \chi_{\mathcal{Q}_{\tau'}^{k',\nu'}}(\cdot)\big)(y_\tau^{k,\nu})\big]^{\frac{2}{r}} \chi_{\mathcal{Q}_\tau^{k,\nu}}(x)$$

Furthermore,

$$\le C_{m,\mathcal{D},\mathcal{D}'} \sum_{k\in\mathbb{N}} \sum_{k'\in\mathbb{N}} 2^{-N_0|k-k'|} 2^{[(k\wedge k')-k]Q_2(1-\frac{1}{r})}$$
$$\times \big[\mathcal{M}\big(\sum_{\tau'\in I_{k'}} \sum_{\nu'=1}^{N(k',\tau')} \inf_{z\in\mathcal{Q}_{\tau'}^{k',\nu'}} |D'_{k'}(f)(z)|^r \chi_{\mathcal{Q}_{\tau'}^{k',\nu'}}(\cdot)\big)(x)\big]^{\frac{2}{r}}$$
$$\le C_{m,\mathcal{D},\mathcal{D}'} \sum_{k'\in\mathbb{N}} \big[\mathcal{M}\big(\sum_{\tau'\in I_{k'}} \sum_{\nu'=1}^{N(k',\tau')} \inf_{z\in\mathcal{Q}_{\tau'}^{k',\nu'}} |D'_{k'}(f)(z)|^r \chi_{\mathcal{Q}_{\tau'}^{k',\nu'}}(\cdot)\big)(x)\big]^{\frac{2}{r}}$$

Since $p/r > 1$ and $2/r > 1$, thus by the Fefferman-Stein vector valued maximal inequality, we have

$$
\Big\|\Big\{\sum_{k'}\Big[\mathcal{M}\big(\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}\inf_{z\in Q_{\tau'}^{k',\nu'}}|D'_{k'}(f)(z)|^{r}\chi_{Q_{\tau'}^{k',\nu'}}(\cdot)\big)(\cdot)\Big]^{2/r}\Big\}^{r/2}\Big\|_{L^{p/r}(M)}^{1/r}
$$

$$
\leq \Big\|\Big\{\sum_{k'}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}\inf_{z\in Q_{\tau'}^{k',\nu'}}|D'_{k'}(f)(z)|^{2}\chi_{Q_{\tau'}^{k',\nu'}}(\cdot)\Big\}^{1/2}\Big\|_{L^{p}(M)}
$$

The result is already proved. □

2.5 The Littlewood-Paley-Stein square function and the Hardy spaces on M

We now introduce the Littlewood-Paley-Stein square function.

Definition 2.21. Let the bounded sets of elementary operators $\{(D_k, 2^{-k})|(D_k, 2^{-k}) \in \mathcal{D}\}$ be an approximation to the identity on M, i.e. $I = \sum_{k\in\mathbb{N}} D_k$. For $f \in \mathcal{T}'$, the Littlewood-Paley-Stein function of f, is defined by

$$
\widetilde{S}(f)(x) = \left\{\sum_{k\in\mathbb{N}} \big|D_k(f)(x)\big|^2\right\}^{1/2}
$$

Street has proved the following result

Theorem 2.22 ([68]). *If $f \in L^p(M)$, $1 < p < \infty$, then* $\|\widetilde{S}(f)\|_{L^p(M)} \approx \|f\|_{L^p(M)}$

We, however, point out that the following discrete Littlewood-Paley-Stein square function is more convenient for the study of the Hardy space H^p when $p \leq 1$.

Definition 2.23. Let the bounded sets of elementary operator $\{(D_k, 2^{-k})|(D_k, 2^{-k}) \in \mathcal{D}\}$ be an approximation to the identity on M, i.e. $I = \sum_{k\in\mathbb{N}} D_k$. For $f \in \mathcal{T}'$, the discrete Littlewood-Paley-Stein square function of f, is defined by

$$
\widetilde{S}_d(f)(x) = \Big\{\sum_{k\in\mathbb{N}}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} |D_k(f)(x)|^2\chi_{Q_\tau^{k,\nu}}(x)\Big\}^{1/2}
$$

By the Plancherel-Pôlya inequalities, it's not difficult to see that the L^p norm of these two kinds of square functions are equivalent. More precisely, we have

Proposition 2.6. *For all $f \in \mathcal{T}'$, $0 < p < \infty$, then* $\|\widetilde{S}(f)\|_{L^p(M)} \approx \|\widetilde{S}_d(f)\|_{L^p(M)}$.

We are ready to introduce the Hardy space on M.

Definition 2.24.

$$H^p(M) = \{f \in \mathcal{T}' : \widetilde{S}_d \in L^p(M)\},$$

and if $f \in H^p(M)$, the norm of f is defined by $\|f\|_{H^p(M)} = \|\widetilde{S}_d(f)\|_{L^p(M)}$.

Obviously, by the Plancherel-Pôlya inequalities inequalities, the Hardy space $H^p(M)$ is well defined. Before ending this section, we prove the following general result which will be used to provide the $H^p - L^p$ boundedness later. We also would like to mention that the proof of this general result does not use atomic decomposition, and thus Journé's covering lemma is not required.

Theorem 2.25. *Let $0 < p \leq 1$. If $f \in L^2(M) \cap H^p(M)$, then $f \in L^p(M)$ and there exists a constant $C_p > 0$ which is independent of the L^2 norm of f such that*

$$\|f\|_{L^p(M)} \leq C_p \|f\|_{H^p(M)}.$$

Proof. Set

$$\Omega_i = \left\{ x \in M : \left[\sum_{k \in \mathbb{N}} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} |\overline{D}_k(f)(x)|^2 \chi_{\mathcal{Q}_\tau^{k,\nu}}(x) \right]^{1/2} > 2^i \right\}.$$

Denote

$$B_i = \left\{ (k, \mathcal{Q}_\tau^{k,\nu}) : \mu\big(\mathcal{Q}_\tau^{k,\nu} \cap \Omega_i\big) > \frac{1}{2}\mu\big(\mathcal{Q}_\tau^{k,\nu}\big), \mu\big(\mathcal{Q}_\tau^{k,\nu} \cap \Omega_{i+1}\big) \leq \frac{1}{2}\mu\big(\mathcal{Q}_\tau^{k,\nu}\big) \right\},$$

where $\mathcal{Q}_\tau^{k,\nu}$ is the dyadic cubes on M. Since $f \in L^2(M)$, by the discrete Calderón formula, we have

$$f(x)=\sum_{k\in\mathbb{N}}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\mu\big(\mathcal{Q}_\tau^{k,\nu}\big)D_k(x,y_\tau^{k,\nu})\overline{D}_k(f)(y_\tau^{k,\nu})$$
$$=\sum_{i\in\mathbb{Z}}\sum_{(k,\mathcal{Q}_\tau^{k,\nu})\in B_i}\mu\big(\mathcal{Q}_\tau^{k,\nu}\big)D_k(x,y_\tau^{k,\nu})\overline{D}_k(f)(y_\tau^{k,\nu}),$$

where the series converges in the L^2 norm, and hence almost everywhere.

We claim

$$\Bigg\|\sum_{(k,\mathcal{Q}_\tau^{k,\nu})\in B_i}\mu\big(\mathcal{Q}_\tau^{k,\nu}\big)D_k(\cdot,y_\tau^{k,\nu})\overline{D}_k(f)(y_\tau^{k,\nu})\Bigg\|_{L^p(M)}^p\le C2^{ip}\mu\big(\Omega_i\big),\tag{2.7}$$

which, together with the fact $0<p\le 1$, yields

$$\|f\|_{L^p(M)}^p\le C\sum_{i\in\mathbb{Z}}2^{ip}\mu\big(\Omega_i\big)\le C\|f\|_{H^p(M)}^p.$$

Thus, it suffices to verify claim (2.7).

Set

$$\widetilde{\Omega}_i=\Big\{x\in M:\mathcal{M}\big(\chi_{\Omega_i}\big)(x)>\frac{1}{100}\Big\}.$$

By the Hölder inequality,

$$\Bigg\|\sum_{(k,\mathcal{Q}_\tau^{k,\nu})\in B_i}\mu\big(\mathcal{Q}_\tau^{k,\nu}\big)D_k(\cdot,y_\tau^{k,\nu})\overline{D}_k(f)(y_\tau^{k,\nu})\Bigg\|_{L^p(M)}^p$$
$$\le\mu\big(\widetilde{\Omega}_i\big)^{1-\frac{p}{2}}\Bigg\|\sum_{(k,\mathcal{Q}_\tau^{k,\nu})\in B_i}\mu\big(\mathcal{Q}_\tau^{k,\nu}\big)D_k(\cdot,y_\tau^{k,\nu})\overline{D}_k(f)(y_\tau^{k,\nu})\Bigg\|_{L^2(M)}^p$$
$$+\Bigg\|\sum_{(k,\mathcal{Q}_\tau^{k,\nu})\in B_i}\mu\big(\mathcal{Q}_\tau^{k,\nu}\big)D_k(\cdot,y_\tau^{k,\nu})\overline{D}_k(f)(y_\tau^{k,\nu})\Bigg\|_{L^p(M\setminus\widetilde{\Omega}_i)}^p=\kappa_i^1+\kappa_i^2.$$

Firstly consider κ_i^1. By the duality argument, for all $g \in L^2(M)$ with $\|g\|_{L^2(M)} \le 1$,

$$\begin{aligned}
&\left|\left\langle \sum_{(k,\mathcal{Q}_\tau^{k,\nu})\in B_i} \mu\big(\mathcal{Q}_\tau^{k,\nu}\big) D_k(\cdot, y_\tau^{k,\nu})\overline{D}_k(f)(y_\tau^{k,\nu}),\, g(\cdot)\right\rangle\right| \\
&\le \left| \sum_{(k,\mathcal{Q}_\tau^{k,\nu})\in B_i} \mu\big(\mathcal{Q}_\tau^{k,\nu}\big) \overline{D}_k(f)(y_\tau^{k,\nu}) D_k(g)(y_\tau^{k,\nu})\right| \\
&\le \left(\sum_{(k,\mathcal{Q}_\tau^{k,\nu})\in B_i} \mu\big(\mathcal{Q}_\tau^{k,\nu}\big) |\overline{D}_k(f)(y_\tau^{k,\nu})|^2\right)^{1/2} \\
&\quad\times \left(\sum_{(k,\mathcal{Q}_\tau^{k,\nu})\in B_i} \mu\big(\mathcal{Q}_\tau^{k,\nu}\big) |D_k(g)(y_\tau^{k,\nu})|^2\right)^{1/2}.
\end{aligned}$$

Note that

$$\begin{aligned}
&\left(\sum_{(k,\mathcal{Q}_\tau^{k,\nu})\in B_i} \mu\big(\mathcal{Q}_\tau^{k,\nu}\big) |D_k(g)(y_\tau^{k,\nu})|^2\right)^{1/2} \\
&\le \left(\sum_{(k,\mathcal{Q}_\tau^{k,\nu})\in B_i} \mu\big(\mathcal{Q}_\tau^{k,\nu}\big) \left| \inf_{z\in\mathcal{Q}_\tau^{k,\nu}} \mathcal{M}\big(D_k(g)\big)(z)\right|^2 \chi_{\mathcal{Q}_\tau^{k,\nu}}(x)\right)^{1/2} \\
&\le \left(\sum_{k\in\mathbb{N}} \int_M \left|\mathcal{M}\big(D_k(g)\big)(x)\right|^2 d\mu(x)\right)^{1/2} \le C\|g\|_{L^2(M)}.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\left\| \sum_{(k,\mathcal{Q}_\tau^{k,\nu})\in B_i} \mu\big(\mathcal{Q}_\tau^{k,\nu}\big) D_k(\cdot, y_\tau^{k,\nu})\overline{D}_k(f)(y_\tau^{k,\nu})\right\|_{L^2(M)} \\
&\le C\left(\sum_{(k,\mathcal{Q}_\tau^{k,\nu})\in B_i} \mu\big(\mathcal{Q}_\tau^{k,\nu}\big) |\overline{D}_k(f)(y_\tau^{k,\nu})|^2\right)^{1/2}.
\end{aligned}$$

Note also that

$$\begin{aligned}
C2^{2i}\mu\big(\Omega_i\big) &\ge \int_{\widetilde{\Omega}_i\backslash\Omega_{i+1}} \left| \left\{ \sum_{k\in\mathbb{N}} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} |\overline{D}_k(f)(x)|^2 \chi_{\mathcal{Q}_\tau^{k,\nu}}(x)\right\}\right| d\mu(x) \\
&\ge \sum_{(k,\mathcal{Q}_\tau^{k,\nu})\in B_i} |\overline{D}_k(f)(y_\tau^{k,\nu})|^2 \mu\big(\mathcal{Q}_\tau^{k,\nu} \cap (\widetilde{\Omega}_i\backslash\Omega_{i+1})\big)
\end{aligned}$$

$$\geq \frac{1}{2} \sum_{(k,\mathcal{Q}_\tau^{k,\nu})\in B_i} \mu\big(\mathcal{Q}_\tau^{k,\nu}\big)|\overline{D}_k(f)(y_\tau^{k,\nu})|^2$$

where the fact that $\mu\big(\mathcal{Q}_\tau^{k,\nu} \cap (\widetilde{\Omega}_i \backslash \Omega_{i+1})\big) > \frac{1}{2A}\mu\big(\mathcal{Q}_\tau^{k,\nu}\big)$ when $(k, \mathcal{Q}_\tau^{k,\nu}) \in B_i$ is used in the last inequality. Also note that $\mu\big(\widetilde{\Omega}_i\big) \leq C\mu\big(\Omega_i\big)$. Hence, we can obtain

$$\kappa_i^1 \leq C\mu\big(\widetilde{\Omega}_i\big)^{1-\frac{p}{2}} \cdot 2^{ip}\mu\big(\widetilde{\Omega}_i\big)^{\frac{p}{2}} \leq C2^{ip}\mu\big(\Omega_i\big)$$

Next, consider κ_i^2. Note that if $(k, \mathcal{Q}_\tau^{k,\nu}) \in B_i$, then $\mathcal{Q}_\tau^{k,\nu} \subset \widetilde{\Omega}_i$. Fix $k \in \mathbb{N}$, we have

$$\begin{aligned}
&\Bigg| \sum_{(k,\mathcal{Q}_\tau^{k,\nu})\in B_i} \mu\big(\mathcal{Q}_\tau^{k,\nu}\big) D_k(x, y_\tau^{k,\nu})\overline{D}_k(f)(y_\tau^{k,\nu})\Bigg| \\
&\leq C\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} \mu\big(\mathcal{Q}_\tau^{k,\nu}\big)\frac{\big(1+2^k\rho(x,y_\tau^{k,\nu})\big)^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(x,2^{-k}+\rho(x,y_\tau^{k,\nu}))\big)}\big|\overline{D}_k(f)(y_\tau^{k,\nu})\big| \\
&\leq C\Bigg\{\mathcal{M}\Bigg(\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} \big|\overline{D}_k(f)(y_\tau^{k,\nu})\big|^r \chi_{\mathcal{Q}_\tau^{k,\nu}\cap\widetilde{\Omega}_i}(\cdot)\Bigg)(x)\Bigg\}^{1/r},
\end{aligned}$$

where C is independent of f. Consequently, by choosing r small enough such that $p/r > 1$, also applying the Fefferman-Stein vector valued maximal inequality, we have

$$\begin{aligned}
&\int_{M\backslash\widetilde{\Omega}_i}\Bigg| \sum_{(k,\mathcal{Q}_\tau^{k,\nu})\in B_i} \mu\big(\mathcal{Q}_\tau^{k,\nu}\big) D_k(x, y_\tau^{k,\nu})\overline{D}_k(f)(y_\tau^{k,\nu})\Bigg|^p d\mu(x) \\
&\leq C\int_{M\backslash\widetilde{\Omega}_i}\Bigg|\sum_{k\in\mathbb{N}}\Bigg\{\mathcal{M}\Bigg(\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} \big|\overline{D}_k(f)(y_\tau^{k,\nu})\big|^r \chi_{\mathcal{Q}_\tau^{k,\nu}\cap\widetilde{\Omega}_i}(\cdot)\Bigg)(x)\Bigg\}^{1/r}\Bigg|^p d\mu(x) \\
&\leq C\int_{M\backslash\widetilde{\Omega}_i}\Bigg|\sum_{k\in\mathbb{N}}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} \big|\overline{D}_k(f)(y_\tau^{k,\nu})\big| \chi_{\mathcal{Q}_\tau^{k,\nu}\cap\widetilde{\Omega}_i}(x)\Bigg|^p d\mu(x) = 0.
\end{aligned}$$

The claim is proved and hence the boundedness follows. $\square$

We would like to point out that the subset $L^2(M) \cap H^p(M)$ is dense in $H^p(M)$. Indeed, we have the following.

Proposition 2.7. *$\mathcal{T}$ is dense in $H^p(M)$.*

The proof of this proposition is similar to the proof of the Plancherel-Pôlya inequalities. More precisely, suppose that $\mathcal{J}$ is any set of indexes of indices of k, τ, ν. Then we have

$$\begin{aligned}
&\Big\| \sum_{\mathcal{J}} \mu\big(Q_\tau^{k,\nu}\big) \widetilde{D}_k(\cdot, y_\tau^{k,\nu}) D_k(f)(y_\tau^{k,\nu}) - f \Big\|_{H^p(M)} \\
&\lesssim \Big\| \sum_{\mathcal{J}^c} \mu\big(Q_\tau^{k,\nu}\big) \widetilde{D}_k(\cdot, y_\tau^{k,\nu}) D_k(f)(y_\tau^{k,\nu}) \Big\|_{H^p(M)} \\
&\lesssim \Big\| \Big\{ \sum_{\mathcal{J}^c} |D_k(f)(\cdot)|^2 \chi_{Q_\tau^{k,\nu}}(\cdot) \Big\}^{1/2} \Big\|_{L^p(M)},
\end{aligned}$$

where $\mathcal{J}^c$ is the complement of $\mathcal{J}$.

2.6 Carleson measure space and duality

Let's define $CMO^p(M)$, $f \in \mathcal{T}'$,

$$\|f\|_{CMO^p(M)} = \sup_{\Omega} \Big\{ \frac{1}{\mu(\Omega)^{2/p-1}} \int_\Omega \sum_{k\in\mathbb{N}} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} |D_k(f)(x)|^2 \chi_I(x)\, d\mu(x) \Big\}^{1/2} < \infty$$

where Ω ranges over all open sets in M with finite measures and where for each k, $Q_\tau^{k,\nu}$ range over dyadic cubes in M.

Theorem 2.26. *Let all the notation be the same as above. Let $\{D_k\}_{k\in\mathbb{N}}$ and $\{D'_{k'}\}_{k'\in\mathbb{N}}$ be two approximations to the identity on M. Then for all $f \in CMO^p(M)$,*

$$\begin{aligned}
&\sup_{\Omega} \Big\{ \frac{1}{\mu(\Omega)^{2/p-1}} \int_\Omega \sum_{k\in\mathbb{N}} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} |D_k(f)(x)|^2 \chi_{Q_\tau^{k,\nu}}(x)\, d\mu(x) \Big\}^{1/2} \\
&\lesssim \sup_{\Omega} \Big\{ \frac{1}{\mu(\Omega)^{2/p-1}} \int_\Omega \sum_{k'\in\mathbb{N}} \sum_{\tau' \in I_{k'}} \sum_{\nu'=1}^{N(k',\tau')} |D'_{k'}(f)(x)|^2 \chi_{Q_{\tau'}^{k',\nu'}}(x)\, d\mu(x) \Big\}^{1/2}
\end{aligned}$$

Proof. For any $f \in CMO^p(M)$, by the discrete reproducing formula,

$$\sup_{z\in \mathcal{Q}_\tau^{k,\nu}} |D_k(f)(z)|^2$$
$$\lesssim \sum_{k'\in\mathbb{N}} 2^{-|k-k'|N_0} \sum_{\tau'\in I_{k'}} \sum_{\nu'=1}^{N(k',\tau')} \mu(\mathcal{Q}_{\tau'}^{k',\nu'}) \frac{(1+2^{k\wedge k'}\rho(y_\tau^{k,\nu}, y_{\tau'}^{k',\nu'}))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(y_\tau^{k,\nu}, 2^{-k\wedge k'}+\rho(y_\tau^{k,\nu}, y_{\tau'}^{k',\nu'})\big)} |D'_{k'}(f)(y_{\tau'}^{k',\nu'})|^2$$

Note that $2^{-|k-k'|} \approx \frac{\mathrm{diam}(\mathcal{Q}_\tau^{k,\nu})}{\mathrm{diam}(\mathcal{Q}_{\tau'}^{k',\nu'})} \wedge \frac{\mathrm{diam}(\mathcal{Q}_{\tau'}^{k',\nu'})}{\mathrm{diam}(\mathcal{Q}_\tau^{k,\nu})}$, $2^{-(k\wedge k')} \approx \mathrm{diam}(\mathcal{Q}_\tau^{k,\nu}) \vee \mathrm{diam}(\mathcal{Q}_{\tau'}^{k',\nu'})$ and $\rho(y_\tau^{k,\nu}, y_{\tau'}^{k',\nu'}) \geq \mathrm{dist}(\mathcal{Q}_\tau^{k,\nu}, \mathcal{Q}_{\tau'}^{k',\nu'})$. Hence,

$$\sup_{z\in \mathcal{Q}_\tau^{k,\nu}} |D_k(f)(z)|^2$$
$$\lesssim \sum_{k'\in\mathbb{N}} \sum_{\tau'\in I_{k'}} \sum_{\nu'=1}^{N(k',\tau')} \mu(\mathcal{Q}_{\tau'}^{k',\nu'}) \Big[\frac{\mathrm{diam}(\mathcal{Q}_\tau^{k,\nu})}{\mathrm{diam}(\mathcal{Q}_{\tau'}^{k',\nu'})} \wedge \frac{\mathrm{diam}(\mathcal{Q}_{\tau'}^{k',\nu'})}{\mathrm{diam}(\mathcal{Q}_\tau^{k,\nu})}\Big]^{N_0}$$
$$\times \frac{[1+(\mathrm{diam}(\mathcal{Q}_\tau^{k,\nu}) \vee \mathrm{diam}(\mathcal{Q}_{\tau'}^{k',\nu'}))^{-1}\mathrm{dist}(\mathcal{Q}_\tau^{k,\nu}, \mathcal{Q}_{\tau'}^{k',\nu'})]^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(y_\tau^{k,\nu}, \mathrm{diam}(\mathcal{Q}_\tau^{k,\nu}) \vee \mathrm{diam}(\mathcal{Q}_{\tau'}^{k',\nu'}) + \mathrm{dist}(\mathcal{Q}_\tau^{k,\nu}, \mathcal{Q}_{\tau'}^{k',\nu'})\big)} |D'_{k'}(f)(y_{\tau'}^{k',\nu'})|^2$$

Applying the above estimate with any arbitrary point $y_{\tau'}^{k',\nu'}$ in $\mathcal{Q}_{\tau'}^{k',\nu'}$, and the fact $ab = (a \vee b)^2\big(\frac{a}{b} \wedge \frac{b}{a}\big)$ for any $a, b > 0$, we obtain that for any open set $\Omega \subset M$ with finite measure,

$$\frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{k\in\mathbb{N}} \sum_{\mathcal{Q}_\tau^{k,\nu}\subset\Omega} \mu(\mathcal{Q}_\tau^{k,\nu}) \sup_{z\in \mathcal{Q}_\tau^{k,\nu}} |D_k(f)(z)|^2$$
$$\lesssim \frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{k\in\mathbb{N}} \sum_{\mathcal{Q}_\tau^{k,\nu}\subset\Omega} \sum_{k'\in\mathbb{N}} \sum_{\mathcal{Q}_{\tau'}^{k',\nu'}} \Big[\frac{\mu(\mathcal{Q}_\tau^{k,\nu})}{\mu(\mathcal{Q}_{\tau'}^{k',\nu'})} \wedge \frac{\mu(\mathcal{Q}_{\tau'}^{k',\nu'})}{\mu(\mathcal{Q}_\tau^{k,\nu})}\Big]\Big[\frac{\mathrm{diam}(\mathcal{Q}_\tau^{k,\nu})}{\mathrm{diam}(\mathcal{Q}_{\tau'}^{k',\nu'})} \wedge \frac{\mathrm{diam}(\mathcal{Q}_{\tau'}^{k',\nu'})}{\mathrm{diam}(\mathcal{Q}_\tau^{k,\nu})}\Big]^{N_0}$$
$$\times\big[\mu(\mathcal{Q}_\tau^{k,\nu}) \vee \mu(\mathcal{Q}_{\tau'}^{k',\nu'})\big] \frac{\mu(\mathcal{Q}_\tau^{k,\nu}) \vee \mu(\mathcal{Q}_{\tau'}^{k',\nu'})}{\mathrm{Vol}\big(B_{(X,d)}(y_\tau^{k,\nu}, \mathrm{diam}(\mathcal{Q}_\tau^{k,\nu}) \vee \mathrm{diam}(\mathcal{Q}_{\tau'}^{k',\nu'}) + \mathrm{dist}(\mathcal{Q}_\tau^{k,\nu}, \mathcal{Q}_{\tau'}^{k',\nu'})\big)}$$
$$\times \frac{\inf_{z\in \mathcal{Q}_{\tau'}^{k',\nu'}} |D'_{k'}(f)(y_{\tau'}^{k',\nu'})|^2}{[1+(\mathrm{diam}(\mathcal{Q}_\tau^{k,\nu}) \vee \mathrm{diam}(\mathcal{Q}_{\tau'}^{k',\nu'}))^{-1}\mathrm{dist}(\mathcal{Q}_\tau^{k,\nu}, \mathcal{Q}_{\tau'}^{k',\nu'})]^m}$$

For convenience, set

$$Q = Q_\tau^{k,\nu}, Q' = Q_{\tau'}^{k',\nu'},$$

$$r(Q,Q') = \Big[\frac{\mu(\mathcal{Q}_\tau^{k,\nu})}{\mu(\mathcal{Q}_{\tau'}^{k',\nu'})} \wedge \frac{\mu(\mathcal{Q}_{\tau'}^{k',\nu'})}{\mu(\mathcal{Q}_\tau^{k,\nu})}\Big]\Big[\frac{\mathrm{diam}(\mathcal{Q}_\tau^{k,\nu})}{\mathrm{diam}(\mathcal{Q}_{\tau'}^{k',\nu'})} \wedge \frac{\mathrm{diam}(\mathcal{Q}_{\tau'}^{k',\nu'})}{\mathrm{diam}(\mathcal{Q}_\tau^{k,\nu})}\Big]^{N_0},$$

$$v(Q,Q') = \mu(\mathcal{Q}_\tau^{k,\nu}) \vee \mu(\mathcal{Q}_{\tau'}^{k',\nu'}),$$

$$P(Q,Q') = \frac{\big[\mu(\mathcal{Q}_\tau^{k,\nu}) \vee \mu(\mathcal{Q}_{\tau'}^{k',\nu'})\big][1 + (\mathrm{diam}(\mathcal{Q}_\tau^{k,\nu}) \vee \mathrm{diam}(\mathcal{Q}_{\tau'}^{k',\nu'}))^{-1}\mathrm{dist}(\mathcal{Q}_\tau^{k,\nu}, \mathcal{Q}_{\tau'}^{k',\nu'})]^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(y_\tau^{k,\nu}, \mathrm{diam}(\mathcal{Q}_\tau^{k,\nu}) \vee \mathrm{diam}(\mathcal{Q}_{\tau'}^{k',\nu'}) + \mathrm{dist}(\mathcal{Q}_\tau^{k,\nu}, \mathcal{Q}_{\tau'}^{k',\nu'})\big)},$$

$$S_Q = \sup_{z \in \mathcal{Q}_\tau^{k,\nu}} |D_k(f)(z)|^2,$$

$$T_{Q'} = \inf_{z \in \mathcal{Q}_{\tau'}^{k',\nu'}} |D'_{k'}(f)(z)|^2.$$

Thus, it can be rewritten as

$$\frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{Q \subset \Omega} \mu(Q) S_Q \lesssim \frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{Q \subset \Omega} \sum_{Q'} r(Q,Q')v(Q,Q')P(Q,Q')P(Q,Q')T_{Q'} \tag{2.8}$$

To complete the proof of the theorem, we need to prove that the right-hand side can be controlled by

$$\sup_{\overline{\Omega}} \frac{1}{\mu(\overline{\Omega})^{\frac{2}{p}-1}} \sum_{Q' \subset \overline{\Omega}} \mu(Q') T_{Q'},$$

where $\overline{\Omega}$ ranges over all open sets in M with finite measures.

Similar as before, we point out that the estimates of $v(Q,Q')$ and $P(Q,Q')$ are based on the geometrical properties between Q and Q'. More precisely, when the difference of the sizes and the distance of Q and Q' get larger, then $v(Q,Q')$ and $P(Q,Q')$ then become smaller, respectively. Therefore, to estimate $v(Q,Q')$ and $P(Q,Q')$, for each $Q \subset \Omega$, we group all subsets Q' in M according to the distances and sizes of Q and Q' as follows:

Define

$$\Omega^0 =: \bigcup_{Q \subset \Omega} 3Q.$$

Then, for any $Q \subset \Omega$, let

$$A_0(Q) = \{Q' : \mathrm{dist}(Q, Q') \le \mathrm{diam}(Q) \vee \mathrm{diam}(Q')\};$$

$$A_j(Q) = \{Q' : 2^{j-1}\big[\mathrm{diam}(Q) \vee \mathrm{diam}(Q')\big] < \mathrm{dist}(Q, Q') \le 2^j\big[\mathrm{diam}(Q) \vee \mathrm{diam}(Q')\big]\}.$$

where $j \ge 1$.

Note that for each subset Q', we have $\lim_{j\to\infty} 2^j Q' = M$. Hence, for any subset $Q \subset \Omega$, there exists some j such that $Q' \in A_j(Q)$.

Consequently, we have

$$\begin{aligned}
&\le \frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{Q\subset\Omega} \sum_{Q'\in A_0} r(Q,Q')v(Q,Q')P(Q,Q')P(Q,Q')T_{Q'} \\
&+ \sum_{j\ge 1} \frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{Q\subset\Omega} \sum_{Q'\in A_j} r(Q,Q')v(Q,Q')P(Q,Q')P(Q,Q')T_{Q'} \\
&=: \mathrm{I} + \mathrm{II}
\end{aligned}$$

We first consider I. Define

$$B_0 = \{Q' : 3Q' \cap \Omega^0 \ne 0\}.$$

Then we claim that

$$\mathrm{I} \le \frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{Q'\in B_0} \sum_{\substack{\{Q:Q\subset\Omega,\\ Q'\in A_0(Q)\}}} r(Q,Q')v(Q,Q')P(Q,Q')P(Q,Q')T_{Q'}$$

In fact, for each $Q' \notin B_0$, according to the definition of B_0, we have $3Q' \cap \Omega^0 = \emptyset$. Hence, for any $Q \subset \Omega$, we have $3Q' \cap 3Q = \emptyset$, which implies that $Q' \notin A_0(Q)$. Therefore, we have $\bigcup_{Q\subset\Omega} A_0(Q) \subset B_0$. As a consequence, we can obtain that claim holds.

We make a further decomposition of B_0. First, for each $h \ge 1$, we define $\mathcal{F}_h^0 = \{Q' :$

$\mu(Q'\cap\Omega^0) > \frac{1}{2^h}\mu(3Q')\}$, $\mathcal{I}_h^0 = \mathcal{F}_h^0\backslash\mathcal{F}_{h-1}^0$, i.e. $\mathcal{I}_h^0 = \{Q' : \frac{1}{2^h}\mu(3Q') \le \mu(Q'\cap\Omega^0) \le \frac{1}{2^{h-1}}\mu(3Q')\}$ and $\mathcal{F}_0^0 = \emptyset$, and $\Omega_h^0 = \bigcup_{Q'\in\mathcal{I}_h^0} Q'$. From the above definitions, we have

$$B_0 = \bigcup_{h\ge1} \mathcal{I}_h^0.$$

Therefore, can be rewritten as

$$\mathrm{I} \le \frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{h\ge1} \sum_{Q'\in\mathcal{I}_h^0} \sum_{\substack{\{Q:Q\subset\Omega,\\ Q'\in A_0(Q)\}}} r(Q,Q')v(Q,Q')P(Q,Q')P(Q,Q')T_{Q'}.$$

From the definition of $\mathcal{I}_h^0$ we can see that for any Q' and any Q satisfying $Q'\in A_0(Q)$, we have $P(Q,Q')\le 1$. Hence, to estimate (2.6), we only need to consider the following:

$$\sum_{\substack{\{Q:Q\subset\Omega,\\ Q'\in A_0(Q)\}}} r(Q,Q')v(Q,Q').$$

In what follows, we use a simple geometrical argument, which is a generalization of Chang and R. Fefferman's idea in [2].

Note that $Q'\in A_0(Q)$ we have $3Q\cap 3Q'\neq\emptyset$. We split into two cases:

Case 1. $\mathrm{diam}(Q')\ge\mathrm{diam}(Q)$. First, it is easy to se that $\mu(Q)\lesssim\mu(3Q\cap3Q')$. So we have

$$\mu(Q)\lesssim\mu(3Q\cap3Q')\lesssim\mu(3Q'\cap\Omega^0)\lesssim\frac{1}{2^{h-1}\mu(3Q')},$$

which yields that $2^{h-1}\lesssim\mu(3Q')/\mu(Q)$, i.e. $2^h\mu(Q)\lesssim\mu(Q')$. Since Q and Q' are all with measure equivalent to 2^{-a} for some $a\in\mathbb{Z}$ and $2^h2^{nQ_2}\mu(Q)\approx\mu(Q')$ for some nonnegative integer n. Also, for each fixed n, the numbers of such Q's must be $\lesssim 2^{nQ_2}$.

Denote by z_Q and $x_{Q'}$ the center of Q and Q', respectively. Since $3Q\cap3Q'\neq\emptyset$, we have that $\rho(z_Q,z_{Q'})\le 6\mathrm{diam}(Q')$, and hence $\mathrm{Vol}\big(B_{(X,d)}(z_Q,6\mathrm{diam}(Q')\big)\approx\mathrm{Vol}\big(B_{(X,d)}(z_{Q'},6\mathrm{diam}(Q')\big)\approx$

$\mu(6Q) \approx \mu(6Q')$. Thus,

$$\frac{\mu(Q')}{\mu(Q)} \approx \frac{\mathrm{Vol}\big(B_{(X,d)}(z_Q, 6\mathrm{diam}(Q')\big)}{\mathrm{Vol}\big(B_{(X,d)}(z_Q, 6\mathrm{diam}(Q)\big)} \lesssim \Big(\frac{\mathrm{diam}(Q')}{\mathrm{diam}(Q)}\Big)^{Q_2}$$

It follows that for each fix $n > 0$,

$$\frac{\mathrm{diam}(Q)}{\mathrm{diam}(Q)} \lesssim \Big(\frac{\mu(Q)}{\mu(Q')}\Big)^{1/Q_2} \lesssim 2^{-h/Q_2 - n}.$$

Thus

$$\begin{aligned}
&\sum_{Q \in \text{Case } 1} r(Q,Q')v(Q,Q') \\
&= \sum_{Q \in \text{Case } 1} \frac{\mu(Q)}{\mu(Q')} \Big(\frac{\mathrm{diam}(Q)}{\mathrm{diam}(Q')}\Big)^{N_0} \mu(Q') \\
&\lesssim \sum_{n \geq 0} 2^{-(h+nQ_2)} 2^{-N_0[n+h/Q_2]} \mu(Q') \\
&\lesssim 2^{-h(1+N_0/Q_2)} \mu(Q')
\end{aligned}$$

Case 2. $\mathrm{diam}(Q') \leq \mathrm{diam}(Q)$. We have

$$\mu(Q') \lesssim \mu(3Q' \cap \Omega_0) \leq \frac{1}{2^{h-1}} \mu(3Q')$$

hence there exists a constant $h_0 > 0$ independent of Q and Q' such that $0 \leq h \leq h_0$. We obtain that $\mu(Q) \approx 2^{h+nQ_2}\mu(Q')$ for some $n > 0$ and that for each fixed n, the number of such Q''s is less than a constant independent of n. Since $3Q \cap 3Q' \neq \emptyset$, we have

$$\frac{\mu(Q)}{\mu(Q')} \lesssim \Big(\frac{\mathrm{diam}(Q)}{\mathrm{diam}(Q')}\Big)^{Q_2}.$$

Thus,

$$\frac{\mathrm{diam}(Q')}{\mathrm{diam}(Q)} \lesssim \Big(\frac{\mu(Q')}{\mu(Q)}\Big)^{\frac{1}{Q_2}} \lesssim 2^{-\frac{h}{Q_2}} 2^{-n}$$

Therefore,

$$\sum_{Q \in Case\ 2} r(Q,Q')v(Q,Q') \lesssim \sum_{n \geq 0} 2^{-\frac{h}{Q_2}} 2^{-nN_0} \mu(Q') \lesssim 2^{-h\frac{N_0}{Q_2}} \mu(Q').$$

Now let us turn to I_1:

$$\mathrm{I} = \frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{h \geq 1} \sum_{Q' \in \mathcal{I}_h^0} \Big(\sum_{Q \in Case\ 1} + \sum_{Q \in Case\ 2} \Big) \times r(Q,Q')v(Q,Q')T_{Q'}$$

$$= I_2 + I_2.$$

Obviously, combing the fact that $\mu(\Omega_h^0) \lesssim h2^h \mu(\Omega)$ for $h > 1$, $\mu(\Omega_0^0) \lesssim \mu(\Omega)$, we have

$$I_1 \lesssim \frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{h \geq 1} 2^{-h(1+\frac{N_0}{Q_2})} \mu(\Omega_h^0)^{\frac{2}{p}-1} \frac{1}{\mu(\Omega_h^0)^{\frac{2}{p}-1}} \sum_{Q' \subset \Omega_h^0} \mu(Q') T_{Q'}$$

$$\lesssim \frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{h \geq 1} 2^{-h(1+\frac{N_0}{Q_2})} (h2^h)^{\frac{2}{p}-1} \mu(\Omega)^{\frac{2}{p}-1} \times \sup_{\overline{\Omega}} \frac{1}{\mu(\overline{\Omega})^{\frac{2}{p}-1}} \sum_{Q' \subset \overline{\Omega}} \mu(Q') T_{Q'}$$

$$\lesssim \sup_{\overline{\Omega}} \frac{1}{\mu(\overline{\Omega})^{\frac{2}{p}-1}} \sum_{Q' \subset \overline{\Omega}} \mu(Q') T_{Q'}.$$

For I_2, observing that

$$I_2 = \frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{h=0}^{h_0} \sum_{Q' \in \mathcal{I}_h^0} \sum_{Q \in Case\ 2} r(Q,Q')v(Q,Q')T_{Q'},$$

we have

$$I_2 \lesssim \frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{h=0}^{h_0} 2^{-h\frac{N_0}{Q_2}} \mu(\Omega_h^0)^{\frac{2}{p}-1} \frac{1}{\mu(\Omega_h^0)^{\frac{2}{p}-1}} \sum_{Q' \subset \Omega_h^0} \mu(Q') T_{Q'}$$

$$\lesssim \frac{1}{\mu(Q)^{\frac{2}{p}-1}} \sum_{h=0}^{h_0} 2^{-h\frac{N_0}{Q_2}} h^{\frac{2}{p}-1} 2^{h(\frac{2}{p}-1)} \mu(\Omega)^{\frac{2}{p}-1} \times \sup_{\overline{\Omega}} \sum_{Q' \subset \overline{\Omega}} \mu(Q') T_{Q'}$$

$$\lesssim \sup_{\overline{\Omega}} \frac{1}{\mu(\overline{\Omega})^{\frac{2}{p}-1}} \sum_{Q' \subset \overline{\Omega}} \mu(Q') T_{Q'}$$

Similarly, we can deal with II. This completes the proof of Theorem 2.26. □

Thus, the space CMP^p is well defined, and moreover, we have the following duality results.

Theorem 2.27. $\big(H^p(M)\big)' = CMO^p(M)$, $\big(H^1(M)\big)' = CMO^1(M) = BMO$.

To show that the dual of $H^p(M)$ is $CMO^p(M)$ for $0 < p \leq 1$, we first introduce sequence spaces s^p and c^p as follows.

Definition 2.28. Let $\widetilde{\chi}_Q(x) = \mu(Q)^{-1/2}\chi_Q(x)$ for any dyadic cube $Q \subset M$. For $0 < p \leq 1$, the sequence space s^p is defined by the collection of all complex-valued sequences $s = \{s_R\}_R$ such that

$$\|s\|_{s^p} = \Big\| \Big\{ \sum_Q \big(|s_Q| \cdot \widetilde{\chi}_Q(x)\big)^2 \Big\}^{1/2} \Big\|_{L^p} \tag{2.9}$$

Similarly, for $0 < p \leq 1$, the sequence space c^p is defined by the collection of all complex-valued sequences $t = \{t_Q\}_Q$ such that

$$\|t\|_{c^p} = \sup_{\Omega} \Big(\frac{1}{\mu(\Omega)^{2/p-1}} \sum_{Q \subseteq \Omega} |t_Q|^2 \Big)^{1/2} \tag{2.10}$$

where the sup is taken over all open sets $\Omega \subset M$ with finite measure and R ranges over all the dyadic rectangles in M.

The duality theorem of these sequence spaces is the following:

Theorem 2.29. $(s^p)' = c^p$.

Proof. First, we prove that for all $t \in c^p$, if

$$L(s) = \sum_Q s_Q \cdot \bar{t}_Q, \quad \forall\, s \in s^p,$$

then $|L(s)| \lesssim \|s\|_{s^p}\|t\|_{c^p}$. To see this, set

$$\Omega_k = \{x \in M \,:\, [\sum_Q (|s_Q|\widetilde{\chi}_Q(x))^2]^{1/2} > 2^k\},$$

$$B_k = \{Q \,:\, \mu(\Omega_k \cap Q) > \frac{1}{2A}\mu(Q), \mu(\Omega_{k+1} \cap Q) \le \frac{1}{2A}\mu(Q)\}$$

and

$$\widetilde{\Omega}_k = \{x \in M \,:\, \mathcal{M}(\chi_{\Omega_k}) > \frac{1}{2A}\},$$

where $\mathcal{M}$ is the Hardy-Littlewood maximal function on M. By the Hölder inequality,

$$\begin{aligned} |L(s)| &\le \Big(\sum_k \big(\sum_{Q \in B_k} |s_Q|^2\big)^{\frac{p}{2}} \big(\sum_{Q \in B_k} |t_Q|^2\big)^{\frac{p}{2}}\Big)^{\frac{1}{p}} \\ &\le \Big(\sum_k \mu(\widetilde{\Omega}_k)^{1-\frac{p}{2}} \big(\sum_{Q \in B_k} |s_Q|^2\big)^{\frac{p}{2}} \big(\frac{1}{\mu(\widetilde{\Omega}_k)^{\frac{2}{p}-1}} \sum_{Q \subset \widetilde{\Omega}_k} |t_Q|^2\big)^{\frac{p}{2}}\Big)^{\frac{1}{p}} \\ &\le \Big(\sum_k \mu(\widetilde{\Omega}_k)^{1-\frac{p}{2}} \big(\sum_{Q \in B_k} |s_Q|^2\big)^{\frac{p}{2}}\Big)^{\frac{1}{p}} \|t\|_{c^p}, \end{aligned} \tag{2.11}$$

where we have used the fact that if $Q \in B_k$, then Q is contained in $\widetilde{\Omega}_k$. Observing that

$$\int_{\widetilde{\Omega}_k \backslash \Omega_{k+1}} \sum_{Q \in B_k} \big(|s_Q|\widetilde{\chi}_Q(x)\big)^2\, dx \le 2^{2(k+1)}\mu(\widetilde{\Omega}_k \backslash \Omega_{k+1}) \le C2^{2k}\mu(\Omega_k)$$

and

$$\int_{\widetilde{\Omega}_k \backslash \Omega_{k+1}} \sum_{Q \in B_k} \big(|s_Q|\widetilde{\chi}_Q(x)\big)^2\, dx \ge \sum_{Q \in B_k} |s_Q|^2 \mu(Q)^{-1}\mu(\widetilde{\Omega}_k \backslash \Omega_{k+1} \cap Q)$$

$$\geq \sum_{Q\in B_k} |s_Q|^2 \mu(Q)^{-1} \frac{1}{2A} \mu(Q) \sum_{Q\in B_k} |s_Q|^2,$$

we obtain $\Big(\sum_{Q\in B_k} |s_Q|^2\Big)^{\frac{p}{2}} \leq 2^{kp} \mu(\Omega_k)^{\frac{p}{2}}$. Substituting this back into (2.11) and noting $\mu(\widetilde{\Omega}_k) \lesssim \mu(\Omega_k)$ yield that $|L(s)| \lesssim \|s\|_{s^p} \|t\|_{c^p}$.

Conversely, we need to verify that for any $L \in (s^p)'$, there exists $t \in c^p$ with $\|t\|_{c^p} \lesssim \|L\|$ such that for all $s \in s^p$, $L(s) = \sum_Q s_Q \bar{t}_Q$.

For any $L \in (s^p)'$, then $L(s) = \sum_Q s_Q \bar{t}_Q$. It suffices to show that $\|t\|_{c^p} \leq \|L\|$. To do this, for any open set $\Omega \subset M$ with finite measure, let $\bar{\mu}$ be a new measure such that $\bar{\mu}(Q) = \frac{\mu(Q)}{\mu(\Omega)^{\frac{p}{2}-1}}$ when $Q \subset \Omega$ and $\bar{\mu}(Q) = 0$ when $Q \not\subseteq \Omega$. Also, let $l^2(\bar{\mu})$ be a sequence space such that when $\{s_Q\} \in l^2(\bar{\mu})$, $\big(\sum_{Q\subset\Omega} |s_Q|^2 \frac{\mu(Q)}{\mu(\Omega)^{\frac{p}{2}-1}}\big)^{1/2} < \infty$. Observe

$$\Big\{\frac{1}{\mu(\Omega)^{\frac{p}{2}-1}} \sum_{Q\subset\Omega} |t_Q|^2\Big\}^{1/2} = \big\|\mu(Q)^{-1/2} \cdot |t_Q|\big\|_{l^2(\bar{\mu})}$$

$$\begin{aligned}
&= \sup_{s:\|s\|_{l^2(\bar{\mu})}\leq 1} \Big| \sum_{Q\subseteq\Omega} (t_Q \mu(Q)^{-1/2}) \cdot \bar{s}_Q \cdot \frac{\mu(Q)}{\mu(\Omega)^{\frac{p}{2}-1}} \Big| \\
&\leq \sup_{s:\|s\|_{l^2(\bar{\mu})}\leq 1} \Big| L\Big(\chi_{Q\subseteq\Omega}(Q) \cdot \frac{\mu(Q)^{1/2}|s_Q|}{\mu(\Omega)^{\frac{p}{2}-1}}\Big) \Big| \\
&\leq \sup_{s:\|s\|_{l^2(\bar{\mu})}\leq 1} \|L\| \cdot \Big\|\chi_{Q\subseteq\Omega}(Q) \cdot \frac{\mu(Q)^{1/2}|s_Q|}{\mu(\Omega)^{\frac{p}{2}-1}}\Big\|_{s^p}.
\end{aligned}$$

By (2.9) and the Hölder, we have

$$\Big\|\chi_{Q\subseteq\Omega}(Q) \cdot \frac{\mu(Q)^{1/2}|s_Q|}{\mu(\Omega)^{\frac{p}{2}-1}}\Big\|_{s^p} \leq \Big(\sum_{Q\subseteq\Omega} |s_Q|^2 \frac{\mu(Q)}{\mu(\Omega)^{\frac{p}{2}-1}}\Big)^{1/2}.$$

Therefore,

$$\|t\|_{c^p} \leq \sup_{s:\|s\|_{l^2(\bar{\mu})}\leq 1} \|L\| \cdot \|s\|_{l^2(\bar{\mu})} \leq \|L\|.$$

□

Definition 2.30. For any $f \in \mathcal{T}$, define the lifting operator S by

$$\{(Sf)_Q\} = \left\{\mu(Q)^{\frac{1}{2}} D_k(f)(z_Q)\right\},$$

where Q is 'dyadic cube' in M, with length $l(Q) = 2^{-k-j}$, and z_Q is the center of Q, respectively.

Definition 2.31. For any complex-valued sequence $\lambda = \{\lambda_Q\}$, where Q are all dyadic cubes in M. Define the projection operator T by

$$T(\lambda_Q)(x) = \sum_{k\in\mathbb{N}} \sum_Q \mu(Q)^{1/2} \widetilde{D}_k(x, z_Q) \cdot \lambda_Q,$$

By discrete Calderón formula, we immediately obtain

$$T \circ S(f)(x) = \sum_{k\in\mathbb{N}} \sum_Q \mu(Q) \widetilde{D}_k(x, z_Q) D_k(f)(z_Q) = f(x).$$

This means that $T \circ S$ is the identity operator. Moreover,

Proposition 2.8. *$\forall f \in H^p(M)$, we have*

$$\|(Sf)_Q\|_{s^p} \lesssim \|f\|_{H^p(M)}.$$

Conversely, for any $s \in c^p$,

$$\|T(s_Q)\|_{H^p(M)} \le \|s_Q\|_{s^p}$$

Also,

Proposition 2.9. *$\forall f \in CMO^p(M)$,*

$$\|(Sf)_Q\|_{c^p} \lesssim \|f\|_{CMO^p(M)}$$

Conversely, for any $t \in c^p$,

$$\|T(t_Q)\|_{CMO^p(M)} \le \|t_Q\|_{c^p}$$

Proof. For $0 < p \le 1$ and any $g \in \mathcal{B}$ and $f \in CMO^p(M)$, by the discrete Calderón reproducing formula, for any $g \in \mathcal{T}$,

$$\begin{aligned}\langle f, g\rangle &= \langle \sum_k \sum_Q \mu(Q)\widetilde{D}_k(\cdot, z_Q)D_k(f)(z_Q), g\rangle \\ &= \sum_Q S_Q(f)\widetilde{S}_Q(g),\end{aligned}$$

where $\widetilde{S}_Q(g) = \{\mu(Q)^{\frac{1}{2}}\widetilde{D}_k(f)(z_Q)\}_{k,Q}$.

By Theorem 2.20 and Definition 2.28, we obtain

$$|\langle f, g\rangle| \le |\langle S(f), \widetilde{S}(g)\rangle| \lesssim \|f\|_{CMO^p(M)}\|g\|_{H^p(M)}.$$

Since $\mathcal{T}(n_0, m)$ is dense in $H^p(M)$, it follows that $CMO^p(M) \subset \big(H^p(M)\big)'$.

Conversely, suppose $l \in \big(H^p(M)\big)'$. Then $l_1 = l \circ T \in (s^p)'$ by Proposition 2.8. So by the duality argument, there exists $t \in c^p$ such that $l_1(s) = \langle t, s\rangle$ for all $s \in s^p$, and $\|t\|_{c^p} \approx \|l_1\| \lesssim \|l\|$, since T is bounded. We have $l_1 \circ S = l \circ T \circ S = l$, hence

$$l(g) = l \circ T(S(g)) = \langle t, S(g)\rangle = \langle T(t), g\rangle.$$

By the definition of s^p and c^p, also applying the min-max comparison theorem, we obtain that

$$\|T(t)\|_{CMO^p(M)} \lesssim \|t\|_{c^p} \lesssim \|l\|.$$

Hence, $\big(H^p(M)\big)' \subseteq CMO^p(M)$. $\square$

As a consequence of the facts that $\left(h^1(M)\right)' = BMO(M)$, $H^1(M) \cap L^2 \subset L^1(M)$ and $H^1(M) \cap L^2$ is dense in $H^1(M)$, we obtain

Proposition 2.10. *$L^\infty(M) \subset BMO(M)$.*

2.7 The Boundedness of singular integral operator on M

An important model case for a manifold endowed with vector fields satisfying Hörmander's condition comes from that of a (stratified) nilpotent Lie group. Let $\mathfrak{g}$ be a Lie algebra. Define $\mathfrak{g}^{(0)} = \mathfrak{g}$ and recursively, $\mathfrak{g}^{(k+1)} = [\mathfrak{g}, \mathfrak{g}^{(k)}]$.

Definition 2.32. We say $\mathfrak{g}$ is nilpotent of step k, if $\mathfrak{g}^{(k+1)} = \{0\}$. We say $\mathfrak{g}$ is nilpotent if it's nilpotent of step k for some k.

We say G is a nilpotent Lie group, whose Lie algebra is nilpotent, if G is Lie group. It is well known that if $\mathfrak{g}$ is a nilpotent Lie algebra and if G is the corresponding connected, simply connected, Lie group, then the exponential map $\exp : \mathfrak{g} \to G$ is a diffeomorphism. In particular, as a manifold $G \cong \mathbb{R}^{\mathrm{diam} G}$.

Definition 2.33. We say a nilpotent Lie algebra $\mathfrak{g}$ is graded if $\mathfrak{g} = \oplus_{\mu=1}^{\nu} V_\mu$ with $[V_{\mu_1}, V_{\mu_2}] = V_{\mu_1+\mu_2}$, where we take $V_\mu = \{0\}$ for $\mu > \nu$. We say a connected, simply connected, nilpotent Lie group is a graded Lie group if its Lie algebra is graded.

Definition 2.34. We say a nilpotent Lie algebra $\mathfrak{g} = \oplus_{\mu=1}^{\nu} V_\mu$ is stratified if $[V_1, V_\mu] = V_{\mu+1}$. We say a connected, simply connected, nilpotent Lie group is a stratified Lie group if its Lie algebra is stratified.

Suppose $\mathfrak{g}$ is stratified, and suppose $W_1, \ldots, W_r$ are a basis for V_1. We may think of $W_1, \ldots, W_r$ are left invariant vector fields on G. By the definition of a stratified Lie group, $W_1, \ldots, W_r$ satisfy Hörmander's condition. This is an important model case for general vector fields which satisfy Hörmander's condition.

Definition 2.35. Let $\mathfrak{g}$ be a nilpotent Lie algebra. A family of dilations $\delta_t : \mathfrak{g} \to \mathfrak{g}$, $t > 0$ is a family of automorphism defined by $\delta_t X_j = t^{d_j} X_j$, when $X_1, \ldots, X_{\mathrm{diam}\,\mathfrak{g}}$ is a basis for $\mathfrak{g}$, and $0 \neq d_j \in (0, \infty)$.

Definition 2.36. A connected, simply connected, nilpotent Lie group whose Lie algebra is endowed with a family of dilations is called a homogeneous Lie group.

Definition 2.37. Let G be a homogeneous group. A homogeneous norm $|\cdot| : G \to [0, \infty)$ is a continuous function, smooth away from the identity, whose $|x| = 0 \Leftrightarrow x = 0$, and $|\delta_t x| = t|x|$ for $t > 0$.

With a fixed choice of homogeneous norm on a homogeneous group G, there is a natural left invariant metric on G, namely the distance between $x, y \in G$ is given by $\rho(x, y) := |x^{-1}y|$. This metric is also homogeneous: $\rho(\delta_t x, \delta_t y) = t\rho(x, y)$. For $r > 0$, $x \in G$, let $B(x, r) = \{y|\rho(x, y) < r\}$.

Fix a graded group G, $\mathfrak{g} = \oplus_{\mu=1}^{\nu} V_\mu$, and let $q = \dim G$; decompose $\mathbb{R}^q = \mathbb{R}^{\dim V_1} \times \mathbb{R}^{\dim V_\nu}$. For $r > 0$ we define dilations on $\mathbb{R}^q$ by $r(t_1, \ldots, r_\nu) = (rt_1, r^2 t_2, \ldots, r^\nu t_\nu)$. Notice, if we identify $G \cong \mathfrak{g}$ with $\mathbb{R}^q$ (as a manifold) by identifying V_μ with $\mathbb{R}^{\dim V_\mu}$, these are the dilations given by δ_r, though now we have suppressed the δ. With these dilations $d(rt)/dt = r^Q$, where $d(rt)/dt$ denotes the Radon-Nikodym derivative, and $Q = \sum_{\mu=1}^{\nu} \mu \dim V_\mu$ is the so-called "homogeneous dimension". Furthermore, we use this identification with G to define $|t|$ for $t \in \mathbb{R}^q$, where $|\cdot|$ denotes a homogeneous norm. With the above notations, we have $|rx| = r|x|$, for $r > 0$. Finally, in the above identification, Lebesgue measure on $\mathbb{R}^q$ corresponds with the two-sided Haar measure on G. Henceforth, integration on G will always be with respect to this measure.

Using these dilations, we can generalize the Calderón-Zygmund kernels of Definition 2.8. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_\nu) \in \mathbb{N}^q = \mathbb{N}^{\mathrm{diam} V_1} \times \cdots \times \mathbb{N}^{\mathrm{diam} V_\nu}$, we define $\deg(\alpha) = \sum_{\mu=1}^{\nu} \mu |\alpha_\mu|$, where $|\alpha_\mu|$ denotes the usual length of the multi-index, i.e. ℓ^1 norm.

Definition 2.38. $K \in C_0^\infty(\mathbb{R}^q)'$ is a Calderón-Zygmund kernel of order $s \in (-Q, \infty)$ if

(i) (Growth Condition) For every multi-index α, $\left|\partial_t^\alpha K(t)\right| \le C_\alpha |t|^{-Q-s-\deg(\alpha)}$;

(ii) (Cancellation Condition) For every bounded set $\mathcal{B} \subset C_0^\infty(\mathbb{R}^q)$, we assume

$$\sup_{\substack{\phi \in \mathcal{B} \\ R > 0}} R^{-s} \left| \int K(t)\phi(Rt)\, dt \right| < \infty$$

Given $K \in C_0^\infty(\mathbb{R}^q)' = C_0^\infty(G)'$ we may define a left invariant operator $\mathrm{Op}(K) : C_0^\infty(G) \to C^\infty(G)$ by $\mathrm{Op}(K)f(x) = f * K(x) = \int f(xy^{-1})K(y)\,dy$. For a funciton $f \in C^\infty(\mathbb{R}^q)$ and $R > 0$, we define $f^{(R)}(t) = R^Q f(Rt)$, where Rt is defined by the above dilations, and therefore $f^{(R)}$ is defined to preserve the L^1 norm: $\int f^{(R)}(t)\,dt = \int f(t)\,dt$.

Theorem 2.39 ([68]). *Fix $s \in (-Q, \infty)$, and let $K \in \mathcal{S}_0(\mathbb{R}^q)'$. The following are equivalent:*

(i) K is a Calderón-Zygmund kernel of order s.

(ii) $\mathrm{Op}(K) : \mathcal{S}_0 \to \mathcal{S}_0(\mathbb{R}^q)$ and for any bounded set $\mathcal{B} \subset \mathcal{S}_0(\mathbb{R}^q)$, the set

$$\left\{g \in \mathcal{S}_0(\mathbb{R}^q) | \exists R > 0, f \in \mathcal{B}, g^{(R)} = R^{-s}\mathrm{Op}(K)f^{(R)}\right\} \subset \mathcal{S}_0(\mathbb{R}^q)$$

is a bounded set.

(iii) For each $j \in \mathbb{Z}$, there is a function $\varsigma_j \in \mathcal{S}_0(\mathbb{R}^q)$ with $\{\varsigma_j | j \in \mathbb{Z}\} \subset \mathcal{S}_0(\mathbb{R}^q)$ a bounded set and such that

$$K = \sum_{j \in \mathbb{Z}} 2^{js} \varsigma_j^{(2^j)}.$$

The above sum converges in distribution, and the equality is taken in the sense of elements of $\mathcal{S}_0(\mathbb{R}^q)'$.

Furthermore, (ii) and (iii) are equivalent for any $s \in \mathbb{R}$.

Next, we need modify the construction of vector fields (X, d) a little. Suppose $W_1, \ldots, W_r$ satisfy Hörmander's condition of order k, and let $\mathfrak{n}_{k,r}$ be free nilpotent Lie algebra of step k on r generators, and we denote $\widehat{W}_1, \ldots, \widehat{W}_r$ r generators for $\mathfrak{n}_{k,r}$. As in the previous section, let $\widehat{X}_1, \ldots, \widehat{X}_q$ be a basis for $\mathfrak{n}_{k,r}$ with

$$\widehat{X}_j = ad\left(\widehat{W}_{l_1^j}\right) ad\left(\widehat{W}_{l_2^j}\right) \cdots ad\left(\widehat{W}_{l_{d_j-1}^j}\right) \widehat{W}_{l_{d_j}^j}$$

for some choice of $l_1^j, \ldots, l_{d_j}^j$. Note that $\widehat{X}_j$ is homogeneous of degree d_j. On M, we define the corresponding vector fields

$$X_j = ad\Big(W_{l_1^j}\Big)ad\Big(W_{l_2^j}\Big)\cdots ad\Big(W_{l_{d_j-1}^j}\Big)W_{l_{d_j}^j}$$

and we assign to X_j the formal degree d_j. Because $\mathfrak{n}_{k,r}$ is the free nilpotent Lie group of step k on r generators. It follows that every commutator of $W_1, \ldots, W_r$ of order $\leq k$ can be written as a linear combination, with constants coefficient of $X_1, \ldots, X_q$. Thus, $X_1, \ldots, X_q$ span the tangent space at any point of M, since $W_1, \ldots, W_r$ satisfy Hörmander's condition of order k. We let $(X, d) = (X_1, d_1), \ldots, (X_q, d_q)$.

On $\mathbb{R}^q$, we define dilations as in the previous section, for $\delta > 0$, $\delta(t_1, \ldots, t_q) = (\delta^{d_1} t_1, \ldots, \delta^{d_q} t_q)$. Let $K \in C_0^\infty(\mathbb{R}^q)$ be a Calderón-Zygmund kernel of order $s > -Q$ as in Definition 2.38. Consider the operator $T : C^\infty(M) \to C^\infty(M)$. Consider the operator T:

$$Tf(x) = \int f(e^{t\cdot X}x)K(t)\,dt$$

In the proof, we need the following two results. The first one is proved by B.Street [68].

Theorem 2.40 ([68]). *If K is supported on a sufficiently small neighborhood of 0, then T is a Calerón-Zygmund operator of order s, as in Definition 2.38.*

Proposition 2.11. *Given two bounded set of elementary operator $\mathcal{E}_1$ and $\mathcal{E}_2$, $\forall m, N$, $\exists C = C(m, N, \mathcal{E}_1, \mathcal{E}_2)$, s.t. $\forall (D_j.2^{-j}) \in \mathcal{E}_1$ and $(D_k, 2^{-k}) \in \mathcal{E}_2$, we have*

$$|D_j T D_k(x,y)| \leq C 2^{-N|j-k|} \frac{(1 + 2^{j\wedge k}\rho(x,y))^{-m}}{\mathrm{Vol}(B_{(X,d)}(x, 2^{-j\wedge k}(1 + 2^{j\wedge k}\rho(x,y)))}$$

We assume the proposition for the moment and now show the H^p boundedness of T as follows.

Theorem 2.41. *For $0 < p \leq 1$ and $s = 0$, we have*

$$\|Tf\|_{H^p} \leq C\|f\|_{H^p}$$

Proof. For $f \in L^2 \cap H^p$, we have

$$\|Tf\|_{H^p} \lesssim \left\| \Big\{ \sum_{k\in\mathbb{N}} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} |D_k(Tf)(y_\tau^{k,\nu})|^2 \chi_{\mathcal{Q}_\tau^{k,\nu}}(\cdot) \Big\}^{\frac{1}{2}} \right\|_{L^p}$$

Applying the L^2 boundedness of T and the discrete Calderón reproducing formula,

$$\lesssim \left\| \left\{ \sum_{k\in\mathbb{N}} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} \Big| D_k\Big(T \sum_{k'\in\mathbb{N}} \sum_{\tau'\in I_{k'}} \sum_{\nu'=1}^{N(k',\tau')} \mu(\mathcal{Q}_{\tau'}^{k',\tau'}) \right.\right.$$
$$\left.\left. \times D_{k'}(\cdot, y_{\tau'}^{k',\nu'}) \overline{D}_{k'}(f)(y_{\tau'}^{k',\nu'}) \Big)(y_\tau^{k,\nu}) \Big|^2 \chi_{\mathcal{Q}_\tau^{k,\nu}}(\cdot) \right\}^{\frac{1}{2}} \right\|_{L^p}$$

According to the above proposition and the similar procedure while proving the Plancherel-Pôlya inequality, we can obtain

$$\lesssim \left\| \left\{ \sum_{k'\in\mathbb{N}} \sum_{\tau'\in I_{k'}} \sum_{\nu'=1}^{N(k',\tau')} \Big| \overline{D}_{k'}(f)(y_{\tau'}^{k',\nu'}) \Big|^2 \chi_{\mathcal{Q}_{\tau'}^{k',\nu'}}(\cdot) \right\}^{\frac{1}{2}} \right\|_{L^p} \lesssim \|f\|_{H^p}$$

□

We now return to the proof of Proposition 2.11.

Proof. Note that T is the Calderón-Zygmund operator of order 0. Hence, by the characterization in Definition 2.39, $\{(TD_k, 2^{-k}) | D_k \in \mathcal{E}_2\}$ is also a bounded set of elementary operators.

Furthermore, note that for every N, the set

$$\{(2^{N|j-k|} D_j T D_k, 2^{-j\wedge k}), (2^{N|j-k|} T D_k D_j, 2^{-j\wedge k}) | (D_j.2^{-j}) \in \mathcal{E}_1, (D_k, 2^{-k}) \in \mathcal{E}_2\}$$

is a bounded set of pre-elementary operators.

Therefore, by the definition of pre-elementary operators, we have

$$\left|\big(2^{-j\wedge k}W_x\big)^{\alpha}\big(2^{-j\wedge k}W_y\big)^{\beta}\big(2^{N|j-k|}D_jTD_k\big)(x,y)\right| \lesssim \frac{\big(1+2^{j\wedge k}\rho(x,y)\big)^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(x,2^{-j\wedge k}+\rho(x,y))\big)}$$

or equivalently,

$$\left|\big(2^{-j\wedge k}W_x\big)^{\alpha}\big(2^{-j\wedge k}W_y\big)^{\beta}\big(D_jTD_k\big)(x,y)\right| \lesssim 2^{-N|j-k|}\frac{\big(1+2^{j\wedge k}\rho(x,y)\big)^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(x,2^{-j\wedge k}+\rho(x,y))\big)}$$

$\square$

CHAPTER 3 : H^p BOUNDEDNESS OF MULTI-PARAMETER RADON TRANSFORM

3.1 Assumptions for multi-parameter vector fields on product space $M = M_1 \times \cdots \times M_\nu$

Now we assume M takes the form of product spaces. More precisely, for $1 \leq \mu \leq \nu$, let M_μ be a smooth, connected, and compact manifold as in the single parameter setting, and define the product space $M = M_1 \times M_2 \times \cdots \times M_\nu$.

Now we state our assumptions for our vector fields on the product space M. On each piece $M_\mu (1 \leq \mu \leq \nu)$, we assume there are vector fields $W_1^\mu, \dots, W_{r_\mu}^\mu$ satisfying the Hörmander's condition, i.e spanning the tangent space to M_μ at each point. From their iterated commutators, as in the single parameter case, we can create a list of vector fields $(X^\mu, \hat{d}^\mu) = (X_1^\mu, \hat{d}_1^\mu), \dots, (X_{q_\mu}^\mu, \hat{d}_{q_\mu}^\mu)$ that span the tangent space to M_μ at each point. Then, on each M^μ, we have the Carnot-Carathéodory balls $B_{(X^\mu, \hat{d}^\mu)}(x_\mu, \delta_\mu)$ as before, and each $B_{(X^\mu, \hat{d}^\mu)}(x_\mu, \delta_\mu)$ on M_μ induces a Carnot-Carathéodory metric $\rho_\mu(x_\mu, z_\mu) := \inf\{\delta_\mu > 0 : z_\mu \in B_{(X^\mu, \hat{d}^\mu)}(x_\mu, \delta_\mu)\}$ on M_u. Then on M we define the corresponding metric having the vector form

$$\rho((x_1, \dots, x_\nu), (z_1, \dots, z_\nu)) := (\rho(x_1, z_1), \dots, \rho(x_\nu, z_\nu)).$$

We can extend each single-parameter formal degree $\hat{d}_j^\mu$ into a ν-parameter d_j^μ as in the beginning of this section, and then combine these pieces of vector fields $(X^\mu, \hat{d}^\mu)$ for $1 \leq \mu \leq \nu$ together as a list of vector fields $(X, d) = (X_1, d_1), \dots, (X_q, d_q)$ on M. Based on (X, d), we define the ν-parameter balls $B_{(X,d)}(x, \delta)$, and naturally, we hope such balls are "almost" in the product form. Actually, if we denote by $B((x_1, \dots, x_\nu), (\delta_1, \dots, \delta_\nu)) := B_{(X^1, \hat{d}^1)}(x_1, \delta_1) \times \cdots \times B_{(X^\nu, \hat{d}^\nu)}(x_\nu, \delta_\nu)$ and give M the strictly positive smooth density corresponding to the product measure on $M_1 \times \cdots \times M_\mu$, then the following properties hold

$$\mathrm{Vol}(B((x_1, \dots, x_\nu), (\delta_1, \dots, \delta_\nu))) = \prod_{\mu=1}^{\nu} \mathrm{Vol}(B_{(X^\mu, \hat{d}^\mu)}(x_\nu, \delta_\nu)),$$

$$B(x, \delta/C) \subseteq B_{(X,d)}(x, \delta) \subseteq B(x, \delta C), \quad x \in M,\ \delta \in [0, \infty)^{\nu},$$

for some $C > 0$. In other words, the balls $B_{(X,d)}(x, \delta)$ are comparable to the "product balls".

3.2 Discrete Calderón Reproducing Formula on product space M

In this section we will introduce the discrete Calderoón Reproducing formula on the product space M. For simplicity, we just consider the case $\nu = 2$, and the case when $\nu > 2$ follows in the same way. First we introduce the bump functions.

Definition 3.1. We say $\mathcal{B} \subset C^{\infty}(M) \times M \times (0,1]^2$ is a bounded set of bump functions if $\forall m$, $\exists C_m$, $\forall (\phi, x, \delta) \in \mathcal{B}$,

- $\mathrm{supp}(\phi) \subset B(x, \delta)$,
- $\sup_z |(\delta X)^{\alpha} \phi(z)| \leq C_m \mathrm{Vol}(B(x, \delta))^{-1}$.

Also, we introduce the space of test functions on M.

Definition 3.2. Let $(x_1, y_1) \in M$. A function f defined on M is said to be a test function of type (x_1, y_1) if for fixed y, $f(x, y)$ is a test function of type (x_1) and for fixed x, $f(x, y)$ is a test function of type (y_1). More precisely, a function f on M is said to be a test function of type (n_0, m, n_0', m') if for fixed y, $f(x, y)$ is a test function of type (n_0, m) centered at x_1 and satisfies that

$$\left\|X_y^{\alpha'} f(\cdot, y)\right\|_{\mathcal{T}(x_1, n_0, m)} \leq C_{\alpha', m'} \frac{(1 + \rho(y, y_1))^{-m'}}{\mathrm{Vol}(B_{X_y}(y_1, 1 + \rho(y, y_1)))} \tag{3.1}$$

Similarly, for a fixed x, $f(x, y)$ is a test function of type (n_0', m') centered at y_1 and satisfies that

$$\left\|X_x^{\alpha} f(x, \cdot)\right\|_{\mathcal{T}(y_1, n_0', m')} \leq C_{\alpha, m} \frac{(1 + \rho(x, x_1))^{-m}}{\mathrm{Vol}(B_{X_x}(x_1, 1 + \rho(x, x_1)))} \tag{3.2}$$

Moreover, for each (n_0, m, n_0', m'), we denote by $\mathcal{T}(x_1, n_0, m; y_1, n_0', m')$ the set of test functions of type (n_0, m, n_0', m') with the norm

$$\|f\|_{\mathcal{T}(x_1,n_0,m;y_1,n_0',m')} = \sup_{\substack{|\alpha|\le n_0\\ |\alpha'|\le n_0'}} \inf\{C_{\alpha',m'}, C_{\alpha,m}\}, \text{ where } C_{\alpha',m'}, C_{\alpha,m} \text{ is taken from (3.1),(3.2)}$$

It's easy to see that for another point $(x_2, y_2) \in M$, $\mathcal{T}(x_1, n_0, m; y_1, n_0', m')$ and $\mathcal{T}(x_2, n_0, m; y_2, n_0', m')$ are equivalent in the corresponding norm. We can denote $\mathcal{T}(x_1, n_0, m; y_1, n_0', m')$ by $\mathcal{T}(n_0, m; n_0', m')$ and represent the test functions of all types by $\mathcal{T}$.

Theorem 3.3. *Let D_{k_i} and $\widetilde{D}_{k_i}$ be given in Theorem 2.19 on each M_i, $i = 1, 2$, respectively. Then*

$$f(x,y) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{\tau_1\in I_{k_1}}\sum_{\tau_2\in I_{k_2}}\sum_{\nu_1=1}^{N(k_1,\tau_1)}\sum_{\nu_2=1}^{N(k_2,\tau_2)} \mu_1(Q_{\tau_1}^{k_1,\nu_1})\mu_2(Q_{\tau_2}^{k_2,\nu_2})$$
$$\widetilde{D}_{k_1}(x, y_{\tau_1}^{k_1,\nu_1})\widetilde{D}_{k_2}(y, y_{\tau_2}^{k_2,\nu_2})D_{k_1}D_{k_2}(f)(y_{\tau_1}^{k_1,\nu_1}, y_{\tau_2}^{k_2,\nu_2})$$

where the series converges in the norm of $\mathcal{T}(n_0, m; n_0', m')$, the topology of bounded convergence as operators $C^\infty(M) \to C^\infty(M)$ and the norm of $L^p(M_1 \times M_2)$, $1 < p < \infty$.

Proof. The proof of this theorem is based on the method of iteration and some known estimates on one single factor M. We first show the L^p, $1 < p < \infty$, convergence. Denote

$$g(x,y) = \sum_{|k_1|\le L_1}\sum_{|k_2|\le L_2}\sum_{\tau_1\in I_{k_1}}\sum_{\tau_2\in I_{k_2}}\sum_{\nu_1=1}^{N(k_1,\tau_1)}\sum_{\nu_2=1}^{N(k_2,\tau_2)} \mu_1(Q_{\tau_1}^{k_1,\nu_1})\mu_2(Q_{\tau_2}^{k_2,\nu_2})$$
$$\widetilde{D}_{k_1}(x, y_{\tau_1}^{k_1,\nu_1})\widetilde{D}_{k_2}(y, y_{\tau_2}^{k_2,\nu_2})D_{k_1}D_{k_2}(f)(y_{\tau_1}^{k_1,\nu_1}, y_{\tau_2}^{k_2,\nu_2}) - f(x,y)$$
$$=: \quad g_1(x,y) + g_2(x,y)$$

where

$$g_1(x,y) = \sum_{|k_1|\le L_1}\sum_{\tau_1\in I_{k_1}}\sum_{\nu_1=1}^{N(k_1,\tau_1)} \mu_1(Q_{\tau_1}^{k_1,\nu_1})$$

$$\times\widetilde{D}_{k_1}(x,y_{\tau_1}^{k_1,\nu_1})D_{k_1}\Big(\sum_{|k_2|\le L_2}\sum_{\tau_2\in I_{k_2}}\sum_{\nu_2=1}^{N(k_2,\tau_2)}\mu_2(\mathcal{Q}_{\tau_2}^{k_2,\nu_2})\widetilde{D}_{k_2}(y,y_{\tau_2}^{k_2,\nu_2})D_{k_2}(f(\cdot,y_{\tau_2}^{k_2,\nu_2}))\Big)(y_{\tau_1}^{k_1,\nu_1})$$
$$-\sum_{|k_2|\le L_2}\sum_{\tau_2\in I_{k_2}}\sum_{\nu_2=1}^{N(k_2,\tau_2)}\mu_2(\mathcal{Q}_{\tau_2}^{k_2,\nu_2})\widetilde{D}_{k_2}(y,y_{\tau_2}^{k_2,\nu_2})D_{k_2}(f)(x,y_{\tau_2}^{k_2,\nu_2})$$

and

$$g_2(x,y)=\sum_{|k_2|\le L_2}\sum_{\tau_2\in I_{k_2}}\sum_{\nu_2=1}^{N(k_2,\tau_2)}\mu_2(\mathcal{Q}_{\tau_2}^{k_2,\nu_2})\widetilde{D}_{k_2}(y,y_{\tau_2}^{k_2,\nu_2})D_{k_2}(f)(x,y_{\tau_2}^{k_2,\nu_2})-f(x,y).$$

We now need the following estimates from the single parameter setting: There exists a constant C such that for $f\in L^p(M)$, $1<p<\infty$, and any integer L,

$$\|\sum_{|k|\le L}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\mu(\mathcal{Q}_\tau^{k,\nu})\widetilde{D}_k(x,y_\tau^{k,\nu})D_k(f)(y_\tau^{k,\nu})\|_{L^p(M)}\le C\|f\|_{L^p(M)}$$

and

$$\|\sum_{|k|\le L}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\mu(\mathcal{Q}_\tau^{k,\nu})\widetilde{D}_k(x,y_\tau^{k,\nu})D_k(f)(y_\tau^{k,\nu})-f\|_{L^p(M)}\le C\|\{\sum_{|k|\ge L}|D_k(f)|^2\}^{\frac{1}{2}}\|_{L^p(M)}$$

Using the above two estimates, we have

$$\begin{aligned}&\|g_1(x,y)\|_{L^p(M)}\\ &\le C\|\{\sum_{|k_1\ge L_1|}\Big|D_{k_1}\Big(\sum_{|k_2|\le L_2}\sum_{\tau_2\in I_{k_2}}\sum_{\nu_2=1}^{N(k_2,\tau_2)}\mu_2(\mathcal{Q}_{\tau_2}^{k_2,\nu_2})\\ &\times\widetilde{D}_{k_2}(y,y_{\tau_2}^{k_2,\nu_2})D_{k_2}(f(\cdot,y_{\tau_2}^{k_2,\nu_2}))\Big)(y_{\tau_1}^{k_1,\nu_1})\Big|^2\}^{1/2}\|_{L^p(M)}\\ &\le C\|\{\sum_{|k_1|\ge L_1}\sum_{|k_2|\le L_2}|D_{k_1}D_{k_2}(f)|^2\}^{\frac{1}{2}}\|_{L^p(M)},\end{aligned}$$

where the last term goes to zeros as L_1 goes to infinity. $\|g_2(x,y)\|_{L^p(M)}$ can be handled similarly. This implies the convergence in $L^p(M)$, $1<p<\infty$.

To see the convergence in the space of test functions, we need the following estimates on one single factor M: for $f \in \mathcal{T}(n_0, m)$ and any integers L,

$$\| \sum_{|k| \le L} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \mu(\mathcal{Q}_\tau^{k,\nu}) \widetilde{D}_k(x, y_\tau^{k,\nu}) D_k(f)(y_\tau^{k,\nu}) \|_{\mathcal{T}(n_0,m)} \le C \|f\|_{\mathcal{T}(n_0,m)}$$

and

$$\| \sum_{|k| \le L} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \mu(\mathcal{Q}_\tau^{k,\nu}) \widetilde{D}_k(x, y_\tau^{k,\nu}) D_k(f)(y_\tau^{k,\nu}) - f \|_{\mathcal{T}(n_0,m)} \le C 2^{-\theta L} \|f\|_{\mathcal{T}(n_0,m)}$$

where C is a constant.

We observe that if $f \in \mathcal{T}(n_0, m; n_0', m')$, then $\|f(\cdot, y)\|_{\mathcal{T}(n_0,m)}$ as a function of the variable y, is in $\mathcal{T}(n_0', m')$ and

$$\big\| \|f(\cdot, \cdot)\|_{\mathcal{T}(n_0,m)} \big\|_{\mathcal{T}(n_0',m')} \le \|f\|_{\mathcal{T}(n_0,m;n_0',m')}.$$

Similarly, $\big\| \|f(\cdot, \cdot)\|_{\mathcal{T}(n_0',m')} \big\|_{\mathcal{T}(n_0,m)} \le \|f\|_{\mathcal{T}(n_0,m;n_0',m')}$. Therefore, we obtain

$$\begin{aligned}
&\|g_1(\cdot, y)\|_{\mathcal{T}(n_0,m)} \\
&\le C \cdot 2^{-\theta_1 L_1} \| \sum_{|k_2| \le L_2} \sum_{\tau_2 \in I_{k_2}} \sum_{\nu_2=1}^{N(k_2,\tau_2)} \mu_2(\mathcal{Q}_{\tau_2}^{k_2,\nu_2}) \widetilde{D}_{k_2}(y, y_{\tau_2}^{k_2,\nu_2}) D_{k_2}(f(\cdot, y_{\tau_2}^{k_2,\nu_2})) \|_{\mathcal{T}(n_0,m)} \\
&\le C \cdot 2^{-\theta_1 L_1} \Big\| \|f(\cdot, \cdot)\|_{\mathcal{T}(n_0',m')} \frac{(1 + \rho(y, y_1))^{-m'}}{\mathrm{Vol}(B_{X_y}(y_1, 1 + \rho(y, y_1)))} \Big\|_{\mathcal{T}(n_0,m)} \\
&\le C \cdot 2^{-\theta_1 L_1} \|f(\cdot, \cdot)\|_{\mathcal{T}(n_0,m;n_0',m')} \frac{(1 + \rho(y, y_1))^{-m'}}{\mathrm{Vol}(B_{X_y}(y_1, 1 + \rho(y, y_1)))}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\|g_2(\cdot, y)\|_{\mathcal{T}(n_0,m)} \\
&\le C 2^{-\theta_2 L_2} \|f\|_{\mathcal{T}(n_0,m;n_0',m')} \frac{(1 + \rho(y, y_1))^{-m'}}{\mathrm{Vol}(B_{X_y}(y_1, 1 + \rho(y, y_1)))}.
\end{aligned}$$

Note that this implies

$$\|g(x,y)\|_{\mathcal{T}(n_0,m;n_0',m')} \le C(2^{-\theta_1 L_1} + 2^{-\theta_2 L_2})\|f\|_{\mathcal{T}(n_0,m;n_0',m')},$$

which yields the convergence in $\mathcal{T}(n_0,m;n_0',m')$. $\square$

3.3 Plancherel-Pôlya inequality on M

Theorem 3.4. *Let $\{D_{k_i}\}_{k_i\in\mathbb{Z}}$ and $\{D'_{k_i'}\}_{k_i'\in\mathbb{Z}}$ be two bounded sets of elementary operators and decompose the identity map I, i.e. $I=\sum_{k_i\in\mathbb{N}} D_{k_i} = \sum_{k_i'} D'_{k_i'}$ on M_i, $i=1,2$. For all distribution of test function f, $i=1,2$,*

$$\|\{\sum_{k_1\in\mathbb{N}}\sum_{k_2\in\mathbb{N}}\sum_{\tau_1\in I_{k_1}}\sum_{\tau_2\in I_{k_2}}\sum_{\nu_1=1}^{N(k_1,\tau_1)}\sum_{\nu_2=1}^{N(k_2,\tau_2)}$$

$$\sup_{(z_1,z_2)\in\mathcal{Q}_{\tau_1}^{k_1,\nu_1}\times\mathcal{Q}_{\tau_2}^{k_2,\nu_2}} |D_{k_1}D_{k_2}(f)(z_1,z_2)|^2\chi_{\mathcal{Q}_{\tau_1}^{k_1,\nu_1}}(\cdot)\chi_{\mathcal{Q}_{\tau_2}^{k_2,\nu_2}}(\cdot)\}^{\frac{1}{2}}\|_{L^p(M)}$$

$$\approx \|\{\sum_{k_1'\in\mathbb{N}}\sum_{k_2'\in\mathbb{N}}\sum_{\tau_1'\in I_{k_1'}}\sum_{\tau_2'\in I_{k_2'}}\sum_{\nu_1'=1}^{N(k_1',\tau_1')}\sum_{\nu_2'=1}^{N(k_2',\tau_2')}$$

$$\inf_{(z_1,z_2)\in\mathcal{Q}_{\tau_1'}^{k_1',\nu_1'}\times\mathcal{Q}_{\tau_2'}^{k_2',\nu_2'}} |D'_{k_1'}D'_{k_2'}(f)(z_1,z_2)|^2\chi_{\mathcal{Q}_{\tau_1'}^{k_1',\nu_1'}}(\cdot)\chi_{\mathcal{Q}_{\tau_2'}^{k_2',\nu_2'}}(\cdot)\}^{\frac{1}{2}}\|_{L^p(M)}$$

Proof. For any $f\in\mathcal{T}(n_0,m;n_0',m')$, we rewrite Theorem 3.3 by

$$f(x,y)=\sum_{k_1'=0}^{\infty}\sum_{k_2'=0}^{\infty}\sum_{\tau_1'\in I_{k_1'}}\sum_{\tau_2'\in I_{k_2'}}\sum_{\nu_1'=1}^{N(k_1',\tau_1')}\sum_{\nu_2'=1}^{N(k_2',\tau_2')}\mu_1(\mathcal{Q}_{\tau_1'}^{k_1',\nu_1'})\mu_2(\mathcal{Q}_{\tau_2'}^{k_2',\nu_2'})$$

$$\widetilde{D}'_{k_1'}(x,y_{\tau_1'}^{k_1',\nu_1'})\widetilde{D}'_{k_2'}(y,y_{\tau_2'}^{k_2',\nu_2'})D'_{k_1'}D'_{k_2'}(f)(y_{\tau_1'}^{k_1',\nu_1'},y_{\tau_2'}^{k_2',\nu_2'})$$

By the orthogonality argument we obtain

$$|D_{k_i}\widetilde{D}'_{k_i'}(x,y)| \le C_{m,\mathcal{D},\mathcal{D}'}2^{-N_0|k_i-k_i'|}\frac{(1+2^{k_i\wedge k_i'}\rho(x,y))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(x,2^{-k_i\wedge k_i'}+\rho(x,y))\big)}$$

From the above, for any $k_1, k_2 \in \mathbb{N}$, we have

$$
\begin{aligned}
&|D_{k_1} D_{k_2}(f)(x,y)| \\
&= \Big| \sum_{k_1'=0}^{\infty} \sum_{k_2'=0}^{\infty} \sum_{\tau_1' \in I_{k_1'}} \sum_{\tau_2' \in I_{k_2'}} \sum_{\nu_1'=1}^{N(k_1',\tau_1')} \sum_{\nu_2'=1}^{N(k_2',\tau_2')} \mu_1(Q_{\tau_1'}^{k_1',\nu_1'}) \mu_2(Q_{\tau_2'}^{k_2',\nu_2'}) \\
&\quad D_{k_1} \widetilde{D}'_{k_1'}(x, y_{\tau_1'}^{k_1',\nu_1'}) D_{k_2} \widetilde{D}'_{k_2'}(y, y_{\tau_2'}^{k_2',\nu_2'}) D'_{k_1'} D'_{k_2'}(f)(y_{\tau_1'}^{k_1',\nu_1'}, y_{\tau_2'}^{k_2',\nu_2'}) \Big| \\
&\le C_{m,\mathcal{D},\mathcal{D}'} \cdot \sum_{k_1' \in \mathbb{N}} \sum_{k_2' \in \mathbb{N}} \sum_{\tau_1' \in I_{k_1'}} \sum_{\tau_2' \in I_{k_2'}} \sum_{\nu_1'=1}^{N(k_1',\tau_1')} \sum_{\nu_2'=1}^{N(k_2',\tau_2')} \mu_1(Q_{\tau_1'}^{k_1',\nu_1'}) \mu_2(Q_{\tau_2'}^{k_2',\nu_2'}) \\
&\quad \times 2^{-N_{0,1}|k_1-k_1'|} 2^{-N_{0,2}|k_2-k_2'|} |D'_{k_1'} D'_{k_2'}(f)(y_{\tau_1'}^{k_1',\nu_1'}, y_{\tau_2'}^{k_2',\nu_2'})| \\
&\times \frac{(1+2^{k_1\wedge k_1'}\rho(y_{\tau_1}^{k_1,\nu_1}, y_{\tau_1'}^{k_1',\nu_1'}))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(y_{\tau_1}^{k_1,\nu_1}, 2^{-k_1\wedge k_1'} + \rho(y_{\tau_1}^{k_1,\nu_1}, y_{\tau_1'}^{k_1',\nu_1'}))\big)} \cdot \frac{(1+2^{k_2\wedge k_2'}\rho(y_{\tau_2}^{k_2,\nu_2}, y_{\tau_2'}^{k_2',\nu_2'}))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(y_{\tau_2}^{k_2,\nu_2}, 2^{-k_2\wedge k_2'} + \rho(y_{\tau_2}^{k_2,\nu_2}, y_{\tau_2'}^{k_2',\nu_2'}))\big)}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\sum_{k_1 \in \mathbb{N}} \sum_{k_2 \in \mathbb{N}} \sum_{\tau_1 \in I_{k_1}} \sum_{\tau_2 \in I_{k_2}} \sum_{\nu_1=1}^{N(k_1,\tau_1)} \sum_{\nu_2=1}^{N(k_2,\tau_2)} \sup_{(z_1,z_2) \in Q_{\tau_1}^{k_1,\nu_1} \times Q_{\tau_2}^{k_2,\nu_2}} |D_{k_1} D_{k_2}(f)(z_1,z_2)| \\
&\quad \times \chi_{Q_{\tau_1}^{k_1,\nu_1}}(x) \chi_{Q_{\tau_2}^{k_2,\nu_2}}(y) \\
&\le C_{m,\mathcal{D},\mathcal{D}'} \sum_{k_1 \in \mathbb{N}} \sum_{k_2 \in \mathbb{N}} \sum_{\tau_1 \in I_{k_1}} \sum_{\tau_2 \in I_{k_2}} \sum_{\nu_1=1}^{N(k_1,\tau_1)} \sum_{\nu_2=1}^{N(k_2,\tau_2)} \Bigg(\\
&\quad \sum_{k_1'=0}^{\infty} \sum_{k_2'=0}^{\infty} \sum_{\tau_1' \in I_{k_1'}} \sum_{\tau_2' \in I_{k_2'}} \sum_{\nu_1'=1}^{N(k_1',\tau_1')} \sum_{\nu_2'=1}^{N(k_2',\tau_2')} \mu_1(Q_{\tau_1'}^{k_1',\nu_1'}) \mu_2(Q_{\tau_2'}^{k_2',\nu_2'}) \\
&\quad \times 2^{-N_{0,1}|k_1-k_1'|} 2^{-N_{0,2}|k_2-k_2'|} |D'_{k_1'} D'_{k_2'}(f)(y_{\tau_1'}^{k_1',\nu_1'}, y_{\tau_2'}^{k_2',\nu_2'})| \\
&\times \frac{(1+2^{k_1\wedge k_1'}\rho(y_{\tau_1}^{k_1,\nu_1}, y_{\tau_1'}^{k_1',\nu_1'}))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(y_{\tau_1}^{k_1,\nu_1}, 2^{-k_1\wedge k_1'} + \rho(y_{\tau_1}^{k_1,\nu_1}, y_{\tau_1'}^{k_1',\nu_1'}))\big)} \cdot \frac{(1+2^{k_2\wedge k_2'}\rho(y_{\tau_2}^{k_2,\nu_2}, y_{\tau_2'}^{k_2',\nu_2'}))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(y_{\tau_2}^{k_2,\nu_2}, 2^{-k_2\wedge k_2'} + \rho(y_{\tau_2}^{k_2,\nu_2}, y_{\tau_2'}^{k_2',\nu_2'}))\big)} \Bigg)^2 \\
&\quad \times \chi_{Q_{\tau_1}^{k_1,\nu_1}}(x) \chi_{Q_{\tau_2}^{k_2,\nu_2}}(y).
\end{aligned}
$$

Equivalently,

$$
\le C_{m,\mathcal{D},\mathcal{D}'} \sum_{k_1 \in \mathbb{N}} \sum_{k_2 \in \mathbb{N}} \sum_{\tau_1 \in I_{k_1}} \sum_{\tau_2 \in I_{k_2}} \sum_{\nu_1=1}^{N(k_1,\tau_1)} \sum_{\nu_2=1}^{N(k_2,\tau_2)} \Bigg(
$$

$$\sum_{k_1'=0}^{\infty}\sum_{k_2'=0}^{\infty} 2^{-N_{0,1}|k_1-k_1'|}2^{-N_{0,2}|k_2-k_2'|}\sum_{\tau_1'\in I_{k_1'}}\sum_{\tau_2'\in I_{k_2'}}\sum_{\nu_1'=1}^{N(k_1',\tau_1')}\sum_{\nu_2'=1}^{N(k_2',\tau_2')}\mu_1(\mathcal{Q}_{\tau_1'}^{k_1',\nu_1'})\mu_2(\mathcal{Q}_{\tau_2'}^{k_2',\nu_2'})$$

$$\times|D'_{k_1'}D'_{k_2'}(f)(y_{\tau_1'}^{k_1',\nu_1'},y_{\tau_2'}^{k_2',\nu_2'})|$$

$$\times\frac{(1+2^{k_1\wedge k_1'}\rho(y_{\tau_1}^{k_1,\nu_1},y_{\tau_1'}^{k_1',\nu_1'}))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(y_{\tau_1}^{k_1,\nu_1},2^{-k_1\wedge k_1'}+\rho(x,y_{\tau_1'}^{k_1',\nu_1'}))\big)}\cdot\frac{(1+2^{k_2\wedge k_2'}\rho(y_{\tau_2}^{k_2,\nu_2},y_{\tau_2'}^{k_2',\nu_2'}))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(y_{\tau_2}^{k_2,\nu_2},2^{-k_2\wedge k_2'}+\rho(y_{\tau_2}^{k_2,\nu_2},y_{\tau_2'}^{k_2',\nu_2'}))\big)}\Bigg)^2$$

$$\times\chi_{\mathcal{Q}_{\tau_1}^{k_1,\nu_1}}(x)\chi_{\mathcal{Q}_{\tau_2}^{k_2,\nu_2}}(y)$$

We choose r such that $r<p\le 1$. Now apply Lemma 2.22. The above term is bounded by

$$\le C_{m,\mathcal{D},\mathcal{D}'}\sum_{k_1\in\mathbb{N}}\sum_{k_2\in\mathbb{N}}\sum_{\tau_1\in I_{k_1}}\sum_{\tau_2\in I_{k_2}}\sum_{\nu_1=1}^{N(k_1,\tau_1)}\sum_{\nu_2=1}^{N(k_2,\tau_2)}\Bigg($$

$$\sum_{k_1'=0}^{\infty}\sum_{k_2'=0}^{\infty}2^{-N_{0,1}|k_1-k_1'|}2^{[(k_1\wedge k_1')-k_1]Q_2(1-\frac{1}{r})}2^{-N_{0,2}|k_2-k_2'|}2^{[(k_2\wedge k_2')-k_2]Q_2(1-\frac{1}{r})}$$

$$\times\Bigg[\mathcal{M}_1\Bigg(\sum_{\tau_1'\in I_{k_1'}}\sum_{\nu_1'=1}^{N(k_1',\tau_1')}\mathcal{M}_2\Big(\sum_{\tau_2'\in I_{k_2'}}\sum_{\nu_2'=1}^{N(k_2',\tau_2')}$$

$$\inf_{(z_1,z_2)\in\mathcal{Q}_{\tau_1'}^{k_1',\nu_1'}\times\mathcal{Q}_{\tau_2'}^{k_2',\nu_2'}}|D'_{k_1'}D'_{k_2'}(f)(z_1,z_2)|^r\chi_{\mathcal{Q}_{\tau_2'}^{k_2',\nu_2'}}(\cdot)\Big)(y_{\tau_2}^{k_2,\nu_2})\chi_{\mathcal{Q}_{\tau_1'}^{k_1',\nu_1'}}(\cdot)\Bigg)(y_{\tau_1}^{k_1,\nu_1})\Bigg]^{\frac{1}{r}}\Bigg)^2$$

$$\times\chi_{\mathcal{Q}_{\tau_1}^{k_1,\nu_1}}(x)\chi_{\mathcal{Q}_{\tau_2}^{k_2,\nu_2}}(y)$$

By the Cauchy-Schwartz inequality, the last term above is dominated by

$$\le C_{m,\mathcal{D},\mathcal{D}'}\sum_{k_1\in\mathbb{N}}\sum_{k_2\in\mathbb{N}}\sum_{\tau_1\in I_{k_1}}\sum_{\tau_2\in I_{k_2}}\sum_{\nu_1=1}^{N(k_1,\tau_1)}\sum_{\nu_2=1}^{N(k_2,\tau_2)}\Bigg\{$$

$$\Bigg[\sum_{k_1'=0}^{\infty}\sum_{k_2'=0}^{\infty}2^{-N_{0,1}|k_1-k_1'|}2^{[(k_1\wedge k_1')-k_1]Q_2(1-\frac{1}{r})}2^{-N_{0,2}|k_2-k_2'|}2^{[(k_2\wedge k_2')-k_2]Q_2(1-\frac{1}{r})}\Bigg]^{1/2}$$

$$\times\Bigg[\sum_{k_1'=0}^{\infty}\sum_{k_2'=0}^{\infty}2^{-N_{0,1}|k_1-k_1'|}2^{[(k_1\wedge k_1')-k_1]Q_2(1-\frac{1}{r})}2^{-N_{0,2}|k_2-k_2'|}2^{[(k_2\wedge k_2')-k_2]Q_2(1-\frac{1}{r})}$$

$$\times\Big[\mathcal{M}_1\Big(\sum_{\tau_1'\in I_{k_1'}}\sum_{\nu_1'=1}^{N(k_1',\tau_1')}\mathcal{M}_2\Big(\sum_{\tau_2'\in I_{k_2'}}\sum_{\nu_2'=1}^{N(k_2',\tau_2')}$$

$$\inf_{(z_1,z_2)\in\mathcal{Q}^{k'_1,\nu'_1}_{\tau'_1}\times\mathcal{Q}^{k_2,\nu'_2}_{\tau'_2}} |D'_{k'_1}D'_{k'_2}(f)(z_1,z_2)|^r\chi_{\mathcal{Q}^{k'_2,\nu'_2}_{\tau'_2}}(\cdot)\Big)(y^{k_2,\nu_2}_{\tau_2})\chi_{\mathcal{Q}^{k'_1,\nu'_1}_{\tau'_1}}(\cdot)\Big)(y^{k_1,\nu_1}_{\tau_1})\Big]^{\frac{2}{r}}\Bigg]^{1/2}\Bigg\}^2$$

$$\times\chi_{\mathcal{Q}^{k_1,\nu_1}_{\tau_1}}(x)\chi_{\mathcal{Q}^{k_2,\nu_2}_{\tau_2}}(y)$$

Thus, choosing $N_{0,1}$ and $N_{0,2}$ large enough such that $N_{0,i}+Q_{2,i}(1-\frac{1}{r})>0$, we have

$$\le C_{m,\mathcal{D},\mathcal{D}'}\sum_{k_1\in\mathbb{N}}\sum_{k_2\in\mathbb{N}}\sum_{\tau_1\in I_{k_1}}\sum_{\tau_2\in I_{k_2}}\sum_{\nu_1=1}^{N(k_1,\tau_1)}\sum_{\nu_2=1}^{N(k_2,\tau_2)}$$

$$\sum_{k'_1=0}^{\infty}\sum_{k'_2=0}^{\infty}2^{-N_{0,1}|k_1-k'_1|}2^{[(k_1\wedge k'_1)-k_1]Q_2(1-\frac{1}{r})}2^{-N_{0,2}|k_2-k'_2|}2^{[(k_2\wedge k'_2)-k_2]Q_2(1-\frac{1}{r})}$$

$$\times\Big[\mathcal{M}_1\Big(\sum_{\tau'_1\in I_{k'_1}}\sum_{\nu'_1=1}^{N(k'_1,\tau'_1)}\mathcal{M}_2\Big(\sum_{\tau'_2\in I_{k'_2}}\sum_{\nu'_2=1}^{N(k'_2,\tau'_2)}$$

$$\inf_{(z_1,z_2)\in\mathcal{Q}^{k'_1,\nu'_1}_{\tau'_1}\times\mathcal{Q}^{k_2,\nu'_2}_{\tau'_2}} |D'_{k'_1}D'_{k'_2}(f)(z_1,z_2)|^r\chi_{\mathcal{Q}^{k'_2,\nu'_2}_{\tau'_2}}(\cdot)\Big)(y^{k_2,\nu_2}_{\tau_2})\chi_{\mathcal{Q}^{k'_1,\nu'_1}_{\tau'_1}}(\cdot)\Big)(y^{k_1,\nu_1}_{\tau_1})\Big]^{\frac{2}{r}}$$

$$\times\chi_{\mathcal{Q}^{k_1,\nu_1}_{\tau_1}}(x)\chi_{\mathcal{Q}^{k_2,\nu_2}_{\tau_2}}(y)$$

Furthermore,

$$\le C_{m,\mathcal{D},\mathcal{D}'}\sum_{k_1\in\mathbb{N}}\sum_{k_2\in\mathbb{N}}\sum_{k'_1=0}^{\infty}\sum_{k'_2=0}^{\infty}2^{-N_{0,1}|k_1-k'_1|}2^{[(k_1\wedge k'_1)-k_1]Q_2(1-\frac{1}{r})}2^{-N_{0,2}|k_2-k'_2|}2^{[(k_2\wedge k'_2)-k_2]Q_2(1-\frac{1}{r})}$$

$$\times\Big[\mathcal{M}_1\Big(\sum_{\tau'_1\in I_{k'_1}}\sum_{\nu'_1=1}^{N(k'_1,\tau'_1)}\mathcal{M}_2\Big(\sum_{\tau'_2\in I_{k'_2}}\sum_{\nu'_2=1}^{N(k'_2,\tau'_2)}$$

$$\inf_{(z_1,z_2)\in\mathcal{Q}^{k'_1,\nu'_1}_{\tau'_1}\times\mathcal{Q}^{k_2,\nu'_2}_{\tau'_2}} |D'_{k'_1}D'_{k'_2}(f)(z_1,z_2)|^r\chi_{\mathcal{Q}^{k'_2,\nu'_2}_{\tau'_2}}(\cdot)\Big)(y)\chi_{\mathcal{Q}^{k'_1,\nu'_1}_{\tau'_1}}(\cdot)\Big)(x)\Big]^{\frac{2}{r}}$$

$$\le C_{m,\mathcal{D},\mathcal{D}'}\sum_{k'_1=0}^{\infty}\sum_{k'_2=0}^{\infty}\Big[\mathcal{M}_1\Big(\sum_{\tau'_1\in I_{k'_1}}\sum_{\nu'_1=1}^{N(k'_1,\tau'_1)}\mathcal{M}_2\Big(\sum_{\tau'_2\in I_{k'_2}}\sum_{\nu'_2=1}^{N(k'_2,\tau'_2)}$$

$$\inf_{(z_1,z_2)\in\mathcal{Q}^{k'_1,\nu'_1}_{\tau'_1}\times\mathcal{Q}^{k_2,\nu'_2}_{\tau'_2}} |D'_{k'_1}D'_{k'_2}(f)(z_1,z_2)|^r\chi_{\mathcal{Q}^{k'_2,\nu'_2}_{\tau'_2}}(\cdot)\Big)(y)\chi_{\mathcal{Q}^{k'_1,\nu'_1}_{\tau'_1}}(\cdot)\Big)(x)\Big]^{\frac{2}{r}}$$

Since $p/r > 1$ and $2/r > 1$, thus by the Fefferman-Stein vector valued maximal inequality, we have

$$\|\{\sum_{k_1'=0}^{\infty}\sum_{k_2'=0}^{\infty}\Big[\mathcal{M}_1\Big(\sum_{\tau_1'\in I_{k_1'}}\sum_{\nu_1'=1}^{N(k_1',\tau_1')}\mathcal{M}_2\Big(\sum_{\tau_2'\in I_{k_2'}}\sum_{\nu_2'=1}^{N(k_2',\tau_2')}$$

$$\inf_{(z_1,z_2)\in \mathcal{Q}_{\tau_1'}^{k_1',\nu_1'}\times \mathcal{Q}_{\tau_2'}^{k_2,\nu_2'}}|D_{k_1'}'D_{k_2'}'(f)(z_1,z_2)|^r\chi_{\mathcal{Q}_{\tau_2'}^{k_2',\nu_2'}}(\cdot)\Big)(y)\chi_{\mathcal{Q}_{\tau_1'}^{k_1',\nu_1'}}(\cdot)\Big)(x)\Big]^{\frac{2}{r}}\}^{\frac{r}{2}}\|_{L^{p/r}(M)}^{1/r}$$

$$\leq \|\{\sum_{k_1'\in\mathbb{N}}\sum_{k_2'\in\mathbb{N}}\sum_{\tau_1'\in I_{k_1'}}\sum_{\tau_2'\in I_{k_2'}}\sum_{\nu_1'=1}^{N(k_1',\tau_1')}\sum_{\nu_2'=1}^{N(k_2',\tau_2')}$$

$$\inf_{(z_1,z_2)\in \mathcal{Q}_{\tau_1'}^{k_1',\nu_1'}\times \mathcal{Q}_{\tau_2'}^{k_2',\nu_2'}}|D_{k_1'}'D_{k_2'}'(f)(z_1,z_2)|^2\chi_{\mathcal{Q}_{\tau_1'}^{k_1',\nu_1'}}(\cdot)\chi_{\mathcal{Q}_{\tau_2'}^{k_2',\nu_2'}}(\cdot)\}^{\frac{1}{2}}\|_{L^p(M)}$$

The result is already proved. $\square$

3.4 The Littlewood-Paley-Stein square function and the Hardy spaces on M

We now introduce the Littlewood-Paley-Stein square function.

Definition 3.5. Let the bounded sets of elementary operator $\{(D_{k_i}, 2^{k_i})|(D_{k_i}, 2^{-k_i}) \in \mathcal{D}_i\}$ be an approximation to the identity on M_i, $i = 1, 2$. For $f \in \mathcal{T}'$, $\widetilde{S}(f)$, the Littlewood-Paley-Stein square function of f, is defined by

$$\widetilde{S}(f)(x,y) = \Big\{\sum_{k_1\in\mathbb{N}}\sum_{k_2\in\mathbb{N}}|D_{k_1}D_{k_2}(f)(x,y)|^2\Big\}^{1/2}.$$

By the results on each M_i, $i = 1, 2$, and iteration, we immediately obtain

Theorem 3.6. *If $f \in L^p(M)$, $1 < p < \infty$, then $\|\widetilde{S}(f)\|_p \approx \|f\|_p$.*

We, however, point out that the following discrete Littlewood-Paley-Stein square function is more convenient for the study of the Hardy space H^p when $p \leq 1$.

Definition 3.7. Let the bounded sets of elementary operator $\{(D_{k_i}, 2^{k_i})|(D_{k_i}, 2^{-k_i}) \in \mathcal{D}_i\}$ be an approximation to the identity on M_i, i.e. $I = \sum_{k_i\in\mathbb{N}} D_{k_i}$. For $f \in \mathcal{T}'$, the discrete Littlewood-

Paley-Stein square function of f, is defined by

$$\widetilde{S}_d(f)(x) = \Big\{ \sum_{k_1\in\mathbb{N}} \sum_{k_2\in\mathbb{N}} \sum_{\tau_1\in I_{k_1}} \sum_{\tau_2\in I_{k_2}} \sum_{\nu=1}^{N(k_1,\tau_1)} \sum_{\nu=2}^{N(k_2,\tau_2)} |D_{k_1}D_{k_2}(f)(x,y)|^2 \times \chi_{Q_{\tau_1}^{k_1,\nu_1}}(x)\chi_{Q_{\tau_2}^{k_2,\nu_2}}(y)\Big\}^{1/2}$$

By the Plancherel-Pôlya inequalities, it's not difficult to see that the L^p norm of these two kinds of square functions are equivalent. More precisely, we have

Proposition 3.1. *For all $f \in \mathcal{T}'$, $0 < p < \infty$, then $\|\widetilde{S}(f)\|_{L^p(M)} \approx \|\widetilde{S}_d(f)\|_{L^p(M)}$.*

We are ready to introduce the Hardy space on M.

Definition 3.8.

$$H^p(M) = \{f \in \mathcal{T}' : \widetilde{S}_d(f) \in L^p(M)\}$$

and if $f \in H^p(M)$, the norm of f is defined by $\|f\|_{H^p(M)} = \|\widetilde{S}(f)\|_{L^p(M)}$.

Obviously, by the Plancherel-Pôlya inequalities inequalities, the Hardy space $H^p(M)$ is well defined. Before ending this section, we prove the following general result which will be used to provide the $H^p - L^p$ boundedness later. We also would like to mention that the proof of this general result does not use atomic decomposition, and thus Journé's covering lemma is not required.

Theorem 3.9. *Let $0 < p \le 1$. If $f \in L^2(M) \cap H^p(M)$, then $f \in L^p(M)$ and there exists a constant $C_p > 0$ which is independent of the L^2 norm of f such that*

$$\|f\|_{L^p(M)} \le C_p\|f\|_{H^p(M)}$$

Proof. Set

$$\Omega_i = \Big\{(x,y) \in M : \Big[\sum_{k_1\in\mathbb{N}}\sum_{k_2\in\mathbb{N}}\sum_{\tau_1\in I_{k_1}}\sum_{\tau_2\in I_{k_2}}\sum_{\nu_1=1}^{N(k_1,\tau_1)}\sum_{\nu_2=1}^{N(k_2,\tau_2)} |\overline{D}_{k_1}\overline{D}_{k_2}(f)(x,y)|^2 \times\chi_{\mathcal{Q}_{\tau_1}^{k_1,\nu_1}}(x)\chi_{\mathcal{Q}_{\tau_2}^{k_2,\nu_2}}(y)\Big]^{1/2} > 2^i\Big\}.$$

Denote

$$B_i = \Big\{(k_1, k_2, \mathcal{Q}_{\tau_1}^{k_1,\nu_1}, \mathcal{Q}_{\tau_2}^{k_2,\nu_2}) : \mu_1\times\mu_2\big(\mathcal{Q}_{\tau_1}^{k_1,\nu_1}\times\mathcal{Q}_{\tau_2}^{k_2,\nu_2}\cap\Omega_i\big) > \frac{1}{2}\mu_1\times\mu_2\big(\mathcal{Q}_{\tau_1}^{k_1,\nu_1}\times\mathcal{Q}_{\tau_2}^{k_2,\nu_2}\big), \mu_1\times\mu_2\big(\mathcal{Q}_{\tau_1}^{k_1,\nu_1}\times\mathcal{Q}_{\tau_2}^{k_2,\nu_2}\cap\Omega_{i+1}\big) \le \frac{1}{2}\mu_1\times\mu_2\big(\mathcal{Q}_{\tau_1}^{k_1,\nu_1}\times\mathcal{Q}_{\tau_2}^{k_2,\nu_2}\big)\Big\},$$

where $\mathcal{Q}_{\tau_i}^{k_i,\nu_i}$ is the dyadic cubes on M_i. Since $f \in L^2(M)$, by the discrete Calderón formula, we have

$$\begin{aligned} f(x,y) &= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{\tau_1\in I_{k_1}}\sum_{\tau_2\in I_{k_2}}\sum_{\nu_1=1}^{N(k_1,\tau_1)}\sum_{\nu_2=1}^{N(k_2,\tau_2)} \mu_1(\mathcal{Q}_{\tau_1}^{k_1,\nu_1})\mu_2(\mathcal{Q}_{\tau_2}^{k_2,\nu_2}) \\ &\quad D_{k_1}(x, y_{\tau_1}^{k_1,\nu_1})D_{k_2}(y, y_{\tau_2}^{k_2,\nu_2})\overline{D}_{k_1}\overline{D}_{k_2}(f)(y_{\tau_1}^{k_1,\nu_1}, y_{\tau_2}^{k_2,\nu_2}) \\ &= \sum_{i\in\mathbb{Z}}\sum_{(k_1,k_2,\mathcal{Q}_{\tau_1}^{k_1,\nu_1},\mathcal{Q}_{\tau_2}^{k_2,\nu_2})\in B_i} \mu_1(\mathcal{Q}_{\tau_1}^{k_1,\nu_1})\mu_2(\mathcal{Q}_{\tau_2}^{k_2,\nu_2}) \\ &\quad D_{k_1}(x, y_{\tau_1}^{k_1,\nu_1})D_{k_2}(y, y_{\tau_2}^{k_2,\nu_2})\overline{D}_{k_1}\overline{D}_{k_2}(f)(y_{\tau_1}^{k_1,\nu_1}, y_{\tau_2}^{k_2,\nu_2}) \\ &\triangleq \sum_{i\in\mathbb{Z}} f_i(x,y), \end{aligned}$$

where the series converges in the L^2 norm, and hence almost everywhere.

We claim

$$\|f_i\|_{L^p(M)}^p \le C2^{ip}\cdot\mu_1\times\mu_2\big(\Omega_i\big), \tag{3.3}$$

which, together with the fact $0 < p \le 1$, yields

$$\|f\|^p_{L^p(M)} \le C \sum_{i\in\mathbb{Z}} 2^{ip} \cdot \mu_1 \times \mu_2\big(\Omega_i\big) \le C\|f\|^p_{H^p(M)}.$$

This completes the proof of Theorem. Thus, it suffices to verify claim (3.3).

Set

$$\widetilde{\Omega}_i = \Big\{x \in M : \mathcal{M}\big(\chi_{\Omega_i}\big)(x) > \frac{1}{100}\Big\}.$$

By the Hölder's inequality,

$$\|f_i\|^p_{L^p(M)} \le \mu_1 \times \mu_2\big(\widetilde{\Omega}_i\big)^{1-\frac{p}{2}} \|f_i\|^p_{L^2(M)} + \|f_i\|^p_{L^p(M\setminus\widetilde{\Omega}_i)} \triangleq \kappa_i^1 + \kappa_i^2.$$

Firstly consider κ_i^1. By the duality argument, for all $g \in L^2(M)$ with $\|g\|_{L^2(M)} \le 1$,

$$\begin{aligned}
|\langle f_i,\, g(\cdot)\rangle| \le \Bigg| & \sum_{(k_1,k_2,\mathcal{Q}^{k_1,\nu_1}_{\tau_1},\mathcal{Q}^{k_2,\nu_2}_{\tau_2})\in B_i} \mu_1(\mathcal{Q}^{k_1,\nu_1}_{\tau_1})\mu_2(\mathcal{Q}^{k_2,\nu_2}_{\tau_2}) \\
& \overline{D}_{k_1}\overline{D}_{k_2}(f)(y^{k_1,\nu_1}_{\tau_1}, y^{k_2,\nu_2}_{\tau_2}) D_{k_1}D_{k_2}(g)(y^{k_1,\nu_1}_{\tau_1}, y^{k_2,\nu_2}_{\tau_2})\Bigg| \\
\le \Bigg(& \sum_{(k_1,k_2,\mathcal{Q}^{k_1,\nu_1}_{\tau_1},\mathcal{Q}^{k_2,\nu_2}_{\tau_2})\in B_i} \mu_1(\mathcal{Q}^{k_1,\nu_1}_{\tau_1})\mu_2(\mathcal{Q}^{k_2,\nu_2}_{\tau_2}) |\overline{D}_{k_1}\overline{D}_{k_2}(f)(y^{k_1,\nu_1}_{\tau_1}, y^{k_2,\nu_2}_{\tau_2})|^2\Bigg)^{1/2} \\
\times \Bigg(& \sum_{(k_1,k_2,\mathcal{Q}^{k_1,\nu_1}_{\tau_1},\mathcal{Q}^{k_2,\nu_2}_{\tau_2})\in B_i} \mu_1(\mathcal{Q}^{k_1,\nu_1}_{\tau_1})\mu_2(\mathcal{Q}^{k_2,\nu_2}_{\tau_2}) |D_{k_1}D_{k_2}(g)(y^{k_1,\nu_1}_{\tau_1}, y^{k_2,\nu_2}_{\tau_2})|^2\Bigg)^{1/2}.
\end{aligned}$$

Note that

$$\begin{aligned}
\Bigg(& \sum_{(k_1,k_2,\mathcal{Q}^{k_1,\nu_1}_{\tau_1},\mathcal{Q}^{k_2,\nu_2}_{\tau_2})\in B_i} \mu_1(\mathcal{Q}^{k_1,\nu_1}_{\tau_1})\mu_2(\mathcal{Q}^{k_2,\nu_2}_{\tau_2}) |D_{k_1}D_{k_2}(g)(y^{k_1,\nu_1}_{\tau_1}, y^{k_2,\nu_2}_{\tau_2})|^2\Bigg)^{1/2} \\
& \le \Bigg(\sum_{(k_1,k_2,\mathcal{Q}^{k_1,\nu_1}_{\tau_1},\mathcal{Q}^{k_2,\nu_2}_{\tau_2})\in B_i} \mu_1(\mathcal{Q}^{k_1,\nu_1}_{\tau_1})\mu_2(\mathcal{Q}^{k_2,\nu_2}_{\tau_2})
\end{aligned}$$

$$\times\Big|\inf_{z\in\mathcal{Q}_{\tau_1}^{k_1,\nu_1}\times\mathcal{Q}_{\tau_2}^{k_2,\nu_2}}\mathcal{M}\big(D_{k_1}D_{k_2}(g)\big)(z)\Big|^2\chi_{\mathcal{Q}_{\tau_1}^{k_1,\nu_1}}(x)\chi_{\mathcal{Q}_{\tau_2}^{k_2,\nu_2}}(y)\Bigg)^{1/2}$$

$$\leq\Bigg(\sum_{k_1,k_2\in\mathbb{N}}\int_M\big|\mathcal{M}\big(D_{k_1}D_{k_2}(g)\big)(x,y)\big|^2\,d\mu_1(x)d\mu_2(y)\Bigg)^{1/2}\leq C\|g\|_{L^2(M)}.$$

This implies that

$$\|f_i\|_{L^2(M)}\leq C\Bigg(\sum_{(k_1,k_2,\mathcal{Q}_{\tau_1}^{k_1,\nu_1},\mathcal{Q}_{\tau_2}^{k_2,\nu_2})\in B_i}\mu_1(\mathcal{Q}_{\tau_1}^{k_1,\nu_1})\mu_2(\mathcal{Q}_{\tau_2}^{k_2,\nu_2})|\overline{D}_{k_1}\overline{D}_{k_2}(f)(y_{\tau_1}^{k_1,\nu_1},y_{\tau_2}^{k_2,\nu_2})|^2\Bigg)^{1/2}.$$

Note also that

$$C2^{2i}\mu_1\times\mu_2\big(\Omega_i\big)\geq\int_{\widetilde{\Omega}_i\backslash\Omega_{i+1}}\Bigg|\Bigg\{\sum_{k_1\in\mathbb{N}}\sum_{k_2\in\mathbb{N}}\sum_{\tau_1\in I_{k_1}}\sum_{\tau_2\in I_{k_2}}\sum_{\nu_1=1}^{N(k_1,\tau_1)}\sum_{\nu_2=1}^{N(k_2,\tau_2)}|\overline{D}_{k_1}\overline{D}_{k_2}(f)(x,y)|^2$$

$$\times\chi_{\mathcal{Q}_{\tau_1}^{k_1,\nu_1}}(x)\chi_{\mathcal{Q}_{\tau_2}^{k_2,\nu_2}}(y)\Bigg\}\Bigg|\,d\mu_1(x)d\mu_2(y)$$

$$\geq\sum_{(k_1,k_2,\mathcal{Q}_{\tau_1}^{k_1,\nu_1},\mathcal{Q}_{\tau_2}^{k_2,\nu_2})\in B_i}|\overline{D}_{k_1}\overline{D}_{k_2}(f)(y_{\tau_1}^{k_1,\nu_1},y_{\tau_2}^{k_2,\nu_2})|^2$$

$$\cdot\mu_1\times\mu_2\big((\mathcal{Q}_{\tau_1}^{k_1,\nu_1}\times\mathcal{Q}_{\tau_2}^{k_2,\nu_2})\cap(\widetilde{\Omega}_i\backslash\Omega_{i+1})\big)$$

$$\geq\frac{1}{2}\sum_{(k_1,k_2,\mathcal{Q}_{\tau_1}^{k_1,\nu_1},\mathcal{Q}_{\tau_2}^{k_2,\nu_2})\in B_i}\mu_1(\mathcal{Q}_{\tau_1}^{k_1,\nu_1})\mu_2(\mathcal{Q}_{\tau_2}^{k_2,\nu_2})|\overline{D}_{k_1}\overline{D}_{k_2}(f)(y_{\tau_1}^{k_1,\nu_1},y_{\tau_2}^{k_2,\nu_2})|^2$$

where the fact that $\mu_1\times\mu_2\big((\mathcal{Q}_{\tau_1}^{k_1,\nu_1}\times\mathcal{Q}_{\tau_2}^{k_2,\nu_2})\cap(\widetilde{\Omega}_i\backslash\Omega_{i+1})\big)\leq\frac{1}{2}\mu_1\times\mu_2\big(\mathcal{Q}_{\tau_1}^{k_1,\nu_1}\times\mathcal{Q}_{\tau_2}^{k_2,\nu_2}\big)$ when $(k_1,k_2,\mathcal{Q}_{\tau_1}^{k_1,\nu_1},\mathcal{Q}_{\tau_2}^{k_2,\nu_2})\in B_i$ is used in the last inequality. Also note that $\mu_1\times\mu_2\big(\widetilde{\Omega}_i\big)\leq C\mu_1\times\mu_2\big(\Omega_i\big)$. Hence, we can obtain

$$\kappa_i^1\leq C\mu_1\times\mu_2\big(\widetilde{\Omega}_i\big)^{1-\frac{p}{2}}\cdot2^{ip}\mu_1\times\mu_2\big(\widetilde{\Omega}_i\big)^{\frac{p}{2}}\leq C2^{ip}\mu_1\times\mu_2\big(\Omega_i\big)$$

Next, consider κ_i^2. Note that if $(k_1,k_2,\mathcal{Q}_{\tau_1}^{k_1,\nu_1},\mathcal{Q}_{\tau_2}^{k_2,\nu_2})\in B_i$, then $\mathcal{Q}_{\tau_1}^{k_1,\nu_1}\times\mathcal{Q}_{\tau_2}^{k_2,\nu_2}\subset\widetilde{\Omega}_i$. Fix

$k_1, k_2 \in \mathbb{N}$, we have

$$\Bigg|\sum_{(k_1,k_2,\mathcal{Q}_{\tau_1}^{k_1,\nu_1},\mathcal{Q}_{\tau_2}^{k_2,\nu_2})\in B_i} \mu_1\big(\mathcal{Q}_{\tau_1}^{k_1,\nu_1}\big)\mu_2\big(\mathcal{Q}_{\tau_2}^{k_2,\nu_2}\big)$$

$$D_{k_1}(x, y_{\tau_1}^{k_1,\nu_1})D_{k_2}(y, y_{\tau_2}^{k_2,\nu_2})\overline{D}_{k_1}\overline{D}_{k_2}(f)(y_{\tau_1}^{k_1,\nu_1}, y_{\tau_2}^{k_2,\nu_2})\Bigg|$$

$$\leq C\sum_{\tau_1\in I_{k_1}}\sum_{\nu_1=1}^{N(k_1,\tau_1)}\sum_{\tau_2\in I_{k_2}}\sum_{\nu_2=1}^{N(k_2,\tau_2)} \mu_1\big(\mathcal{Q}_{\tau_1}^{k_1,\nu_1}\big)\mu_2\big(\mathcal{Q}_{\tau_2}^{k_2,\nu_2}\big)$$

$$\times\frac{\big(1+2^{k_1}\rho(x, y_{\tau_1}^{k_1,\nu_1})\big)^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(x, 2^{-k_1}+\rho(x, y_{\tau_1}^{k_1,\nu_1}))\big)}$$

$$\times\frac{\big(1+2^{k_2}\rho(y, y_{\tau_2}^{k_2,\nu_2})\big)^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(x, 2^{-k_2}+\rho(y, y_{\tau_2}^{k_2,\nu_2}))\big)}\big|\overline{D}_{k_1}\overline{D}_{k_2}(f)(y_{\tau_1}^{k_1,\nu_1}, y_{\tau_2}^{k_2,\nu_2})\big|$$

$$\leq C\Bigg\{\mathcal{M}_1\Bigg(\sum_{\tau_1\in I_{k_1}}\sum_{\nu_1=1}^{N(k_1,\tau_1)}\mathcal{M}_2\Big(\sum_{\tau_2\in I_{k_2}}\sum_{\nu_2=1}^{N(k_2,\tau_2)}$$

$$\big|\overline{D}_{k_1}\overline{D}_{k_2}(f)(y_{\tau_1}^{k_1,\nu_1}, y_{\tau_2}^{k_2,\nu_2})\big|^r\chi_{\mathcal{Q}_{\tau_2}^{k_2,\nu_2}}(\cdot)\Big)(y)\chi_{\mathcal{Q}_{\tau_1}^{k_1,\nu_1}}(\cdot)\Bigg)(x)\Bigg\}^{1/r}$$

where C is independent of f. Consequently, by choosing r small enough such that $p/r > 1$, also applying the Fefferman-Stein vector valued maximal inequality, we have

$$\int_{M\backslash\widetilde{\Omega}_i}\Bigg|\sum_{(k_1,k_2,\mathcal{Q}_{\tau_1}^{k_1,\nu_1},\mathcal{Q}_{\tau_2}^{k_2,\nu_2})\in B_i} \mu_1\big(\mathcal{Q}_{\tau_1}^{k_1,\nu_1}\big)\mu_2\big(\mathcal{Q}_{\tau_2}^{k_2,\nu_2}\big)$$

$$D_{k_1}(x, y_{\tau_1}^{k_1,\nu_1})D_{k_2}(y, y_{\tau_2}^{k_2,\nu_2})\overline{D}_{k_1}\overline{D}_{k_2}(f)(y_{\tau_1}^{k_1,\nu_1}, y_{\tau_2}^{k_2,\nu_2})\Bigg|^p d\mu(x)$$

$$\leq C\int_{M\backslash\widetilde{\Omega}_i}\Bigg|\sum_{(k_1,k_2)\in B_i}\Bigg\{\mathcal{M}_1\Bigg(\sum_{\tau_1\in I_{k_1}}\sum_{\nu_1=1}^{N(k_1,\tau_1)}\mathcal{M}_2\Big(\sum_{\tau_2\in I_{k_2}}\sum_{\nu_2=1}^{N(k_2,\tau_2)}$$

$$\big|\overline{D}_{k_1}\overline{D}_{k_2}(f)(y_{\tau_1}^{k_1,\nu_1}, y_{\tau_2}^{k_2,\nu_2})\big|^r\chi_{\mathcal{Q}_{\tau_2}^{k_2,\nu_2}}(\cdot)\Big)(y)\chi_{\mathcal{Q}_{\tau_1}^{k_1,\nu_1}}(\cdot)\Bigg)(x)\Bigg\}^{1/r}\Bigg|^p d\mu(x)$$

$$\leq C\int_{M\backslash\widetilde{\Omega}_i}\Bigg|\sum_{\substack{k_1\in\mathbb{N}\\k_2\in\mathbb{N}}}\sum_{\substack{\tau_1\in I_{k_1}\\\tau_2\in I_{k_2}}}\sum_{\nu_1=1}^{N(k_1,\tau_1)}\sum_{\nu_2=1}^{N(k_2,\tau_2)}\big|\overline{D}_{k_1}\overline{D}_{k_2}(f)(y_{\tau_1}^{k_1,\nu_1}, y_{\tau_2}^{k_2,\nu_2})\big|$$

$$\chi_{(\mathcal{Q}_{\tau_1}^{k_1,\nu_1}\times\mathcal{Q}_{\tau_2}^{k_2,\nu_2})\cap\widetilde{\Omega}_i}(x,y)\Bigg|^p d\mu(x) = 0.$$

The claim is proved and hence the boundedness follows. $\square$

3.5 Product Carleson Measure Space and Duality

To characterize the dual space of $H^p(M)$, we introduce the Carleson measure space CMO^p on M, which is motivated by ideas of Chang and R.Fefferman[2].

Definition 3.10. Let $i = 1, 2$, $\{D_{k_i}\}_{k_i\in\mathbb{Z}}$ be an approximation to the identity. The Carleson measure space $\mathrm{CMO}^p(M)$ is defined to the set of all $f \in (\mathcal{S}_M)'$ such that

$$\|f\|_{\mathrm{CMO}^p(M)} = \sup\Big(\frac{1}{\mu(\Omega)^{\frac{2}{p}-1}}\int_\Omega \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{\tau_1\in I_{k_1}}\sum_{\tau_2\in I_{k_2}}\sum_{\nu_1=1}^{N(k_1,\tau_1)}\sum_{\nu_2=1}^{N(k_2,\tau_2)}$$
$$\times\,\chi_{\{Q_{\tau_1}^{k_1,\nu_1}\times Q_{\tau_2}^{k_2,\nu_2}\subset\Omega\}}(k_1,k_2.\tau_1,\tau_2,\nu_1,\nu_2)|D_{k_1}D_{k_2}f(x_1,x_2)|^2$$
$$\chi_{Q_{\tau_1}^{k_1,\nu_1}}(x_1)\chi_{Q_{\tau_2}^{k_2,\nu_2}}(x_2)d\mu_1(x_1)d\mu_2(x_2)\Big)^{1/2} < \infty,$$

where the sup is taken over all open sets Ω in M with finite measures.

In order to verify that the definition of $\mathrm{CMO}^p(M)$ is independent of the choice of the approximations to identity, we establish the Min-Max comparison principle involving the CMO^p norm. To this end and for the sake of simplicity, we first give some notation as follows.

We write $R = Q_{\tau_1}^{k_1,\nu_1}\times Q_{\tau_2}^{k_2,\nu_2}$, $R' = Q_{\tau_1'}^{k_1',\nu_1'}\times Q_{\tau_2'}^{k_2',\nu_2'}$,

$$\sum_{R\subseteq\Omega} = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{\tau_1\in I_{k_1}}\sum_{\tau_2\in I_{k_2}}\sum_{\nu_1=1}^{N(k_1,\tau_1)}\sum_{\nu_2=1}^{N(k_2,\tau_2)}\chi_{\{Q_{\tau_1}^{k_1,\nu_1}\times Q_{\tau_2}^{k_2,\nu_2}\subset\Omega\}}(k_1,k_2,\tau_1,\tau_2,\nu_1,\nu_2);$$
$$\sum_{R'\subseteq\Omega} = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{\tau_1'\in I_{k_1'}}\sum_{\tau_2'\in I_{k_2'}}\sum_{\nu_1'=1}^{N(k_1',\tau_1')}\sum_{\nu_2'=1}^{N(k_2',\tau_2')}\chi_{\{Q_{\tau_1'}^{k_1',\nu_1'}\times Q_{\tau_2'}^{k_2',\nu_2'}\subset\Omega\}}(k_1',k_2',\tau_1',\tau_2',\nu_1',\nu_2');$$

$$\sum_{R'} = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{\tau_1'\in I_{k_1'}}\sum_{\tau_2'\in I_{k_2'}}\sum_{\nu_1'=1}^{N(k_1',\tau_1')}\sum_{\nu_2'=1}^{N(k_2',\tau_2')};$$

$$\mu(R) = \mu(Q_{\tau_1}^{k_1,\nu_1})\mu(Q_{\tau_2}^{k_2,\nu_2});\quad \mu(R') = \mu(Q_{\tau_1'}^{k_1',\nu_1'})\mu(Q_{\tau_2'}^{k_2',\nu_2'});$$

$$r(R,R') = \prod_{i=1}^{2}\left(\frac{\mu(\mathcal{Q}_{\tau_i}^{k_i,\nu_i})}{\mu(\mathcal{Q}_{\tau_i'}^{k_i',\nu_i'})} \wedge \frac{\mu(\mathcal{Q}_{\tau_i'}^{k_i',\nu_i'})}{\mu(\mathcal{Q}_{\tau_i}^{k_i,\nu_i})}\right)\left(\frac{\mathrm{diam}(\mathcal{Q}_{\tau_i}^{k_i,\nu_i})}{\mathrm{diam}(\mathcal{Q}_{\tau_i'}^{k_i',\nu_i'})} \wedge \frac{\mathrm{diam}(\mathcal{Q}_{\tau_i'}^{k_i',\nu_i'})}{\mathrm{diam}(\mathcal{Q}_{\tau_i}^{k_i,\nu_i})}\right)^{N_0};$$

$$v(R,R') = \left(\mu(\mathcal{Q}_{\tau_1}^{k_1,\nu_1}) \vee \mu(\mathcal{Q}_{\tau_1'}^{k_1',\nu_1'})\right)\left(\mu(\mathcal{Q}_{\tau_2}^{k_2,\nu_2}) \vee \mu(\mathcal{Q}_{\tau_2'}^{k_2',\nu_2'})\right);$$

$$P(R,R') = \prod_{i=1}^{2}\frac{[\mu(\mathcal{Q}_{\tau_i}^{k_i,\nu_i}) \vee \mu(\mathcal{Q}_{\tau_i'}^{k_i',\nu_i'})][1+(\mathrm{diam}(\mathcal{Q}_{\tau_i}^{k_i,\nu_i}) \vee \mathrm{diam}(\mathcal{Q}_{\tau_i'}^{k_i',\nu_i'}))^{-1}\mathrm{dist}(\mathcal{Q}_{\tau_i}^{k_i,\nu_i},\mathcal{Q}_{\tau_i'}^{k_i',\nu_i'})]^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(y_{\tau_i}^{k_i,\nu_i}, \mathrm{diam}(\mathcal{Q}_{\tau_i}^{k_i,\nu_i}) \vee \mathrm{diam}(\mathcal{Q}_{\tau_i'}^{k_i',\nu_i'}) + \mathrm{dist}(\mathcal{Q}_{\tau_i}^{k_i,\nu_i},\mathcal{Q}_{\tau_i'}^{k_i',\nu_i'}))\big)}$$

$$S_R = \sup_{x_1\in\mathcal{Q}_{\tau_1}^{k_1,\nu_1},\, x_2\in\mathcal{Q}_{\tau_2}^{k_2,\nu_2}} |D_{k_1}D_{k_2}(f)(x_1,x_2)|^2;$$

$$T_{R'} = \inf_{x_1'\in\mathcal{Q}_{\tau_1'}^{k_1',\nu_1'},\, x_2'\in\mathcal{Q}_{\tau_2'}^{k_2',\nu_2'}} |D_{k_1'}D_{k_2'}(f)(x_1',x_2')|^2.$$

Now we state the main theorem of this section as follows.

Theorem 3.11. *Let all the notations be the same as above. For $p \le 1$ all $f \in \mathrm{CMO}^p(M)$,*

$$\sup_{\Omega}\Big(\frac{1}{\mu(\Omega)^{\frac{2}{p}-1}}\sum_{R\subseteq\Omega}\mu(R)S_R\Big)^{1/2} \lesssim \sup_{\Omega}\Big(\frac{1}{\mu(\Omega)^{\frac{2}{p}-1}}\sum_{R'\subseteq\Omega}\mu(R')T_{R'}\Big)^{1/2}, \tag{3.4}$$

where Ω ranges over the open sets in M with finite measures.

Proof. First, for each p satisfying $p \le 1$ and any $f \in \mathrm{CMO}^p(M)$, it is easy to see that the right-hand side of (3.4) is finite and can be controlled by $C\|f\|_{\mathrm{CMO}^p(M)}$.

To prove (3.4), we need to show that for any open set $\Omega \in M$ with finite measure, the following inequality holds,

$$\frac{1}{\mu(\Omega)^{\frac{2}{p}-1}}\sum_{R\subseteq\Omega}\mu(R)S_R \lesssim \sup_{\overline{\Omega}}\frac{1}{\mu(\overline{\Omega})^{\frac{2}{p}-1}}\mu(R')T_{R'}, \tag{3.5}$$

where $\bar{\Omega}$ ranges over all open sets in M with finite measures.

To begin with, for each fix Ω, we first consider the estimate of the term S_R in the left-hand side of (3.5) for every $R = \mathcal{Q}_{\tau_1}^{k_1,\nu_1}\times\mathcal{Q}_{\tau_2}^{k_2,\nu_2} \subset \Omega$. To estimate this, we recall the almost orthogonal property of $D_{k_i}\widetilde{D}_{k_i'}$ for $i=1,2$, namely, for any $0<N_0<m$

$$|D_{k_i}\widetilde{D}_{k_i}(x,y)| \le C_{N_0,\mathcal{D},\mathcal{D}'}2^{-|k_i-k_i'|N_0}\frac{(1+2^{k_i\wedge k_i'}\rho(x,y))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(x,2^{-k_i\wedge k_i'}+\rho(x,y))\big)}$$

(see the property of the elementary operators for more details).

Now for any $(x_1,x_2)\in R$, using the discrete Calderón reproducing formula, the above almost orthogonal property and the Hölder's inequality, we can obtain that

$$\begin{aligned}
|D_{k_1}D_{k_2}f(x_1,x_2)|^2 \lesssim& \Bigg|\sum_{k_1'=0}^{\infty}\sum_{k_2'=0}^{\infty}\sum_{\tau_1'\in I_{k_1'}}\sum_{\tau_2'\in I_{k_2'}}\sum_{\nu_1'=1}^{N(k_1',\tau_1')}\sum_{\nu_2'=1}^{N(k_2',\tau_2')}\mu(\mathcal{Q}_{\tau_1'}^{k_1',\nu_1'})\mu(\mathcal{Q}_{\tau_2'}^{k_2',\nu_2'})\\
&\times D_{k_1}D_{k_2}\widetilde{D}_{k_1'}\widetilde{D}_{k_2'}(x_1,x_2,y_1',y_2')D_{k_1'}D_{k_2'}(f)(y_1',y_2')\Bigg|^2\\
\lesssim&\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{\tau_1'\in I_{k_1'}}\sum_{\tau_2'\in I_{k_2'}}\sum_{\nu_1'=1}^{N(k_1',\tau_1')}\\
&\sum_{\nu_2'=1}^{N(k_2',\tau_2')}2^{-|k_1-k_1'|N_0}2^{-|k_2-k_2'|N_0}\mu(\mathcal{Q}_{\tau_1'}^{k_1',\nu_1'})\mu(\mathcal{Q}_{\tau_2'}^{k_2',\nu_2'})\\
&\times\frac{(1+2^{k_i\wedge k_i'}\rho(y_{\tau_1}^{k_1,\nu_1},y_{\tau_1'}^{k_1',\nu_1'}))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(y_{\tau_1}^{k_1,\nu_1},2^{-k_i\wedge k_i'}+\rho(y_{\tau_1}^{k_1,\nu_1},y_{\tau_1'}^{k_1',\nu_1'}))\big)}\\
&\cdot\frac{(1+2^{k_i\wedge k_i'}\rho(y_{\tau_2}^{k_2,\nu_2},y_{\tau_2'}^{k_2',\nu_2'}))^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(y_{\tau_2}^{k_2,\nu_2},2^{-k_i\wedge k_i'}+\rho(y_{\tau_2}^{k_2,\nu_2},y_{\tau_2'}^{k_2',\nu_2'}))\big)}|D_{k_1'}D_{k_2'}(f)(y_1',y_2')|^2.
\end{aligned} \tag{3.6}$$

where N_0 is chosen to satisfy $\varepsilon<N_0<m$, and for $i=1,2$, $y_{\tau_i}^{k_i,\nu_i}$ is the center of $\mathcal{Q}_{\tau_i}^{k_i,v_i}$ and $y_{\tau_i'}^{k_i',\nu_i'}$ is any point in $\mathcal{Q}_{\tau_i'}^{k_i',\nu_i'}$, respectively.

From Lemma 2.14, we know that each dyadic cube $\mathcal{Q}_\alpha^k$ satisfies that

$$\mathrm{diam}(\mathcal{Q}_\tau^{k,\nu})\sim 2^{-k},\text{ which yields } 2^{-|k_i-k_i'|}\sim\frac{\mathrm{diam}(\mathcal{Q}_{\tau_i}^{k_i,\nu_i})}{\mathrm{diam}(\mathcal{Q}_{\tau_i'}^{k_i',\nu_i'})}\wedge\frac{\mathrm{diam}(\mathcal{Q}_{\tau_i'}^{k_i',\nu_i'})}{\mathrm{diam}(\mathcal{Q}_{\tau_i}^{k_i,\nu_i})}$$

$$\text{and } 2^{-(k_i\wedge k_i')}\sim\mathrm{diam}(\mathcal{Q}_{\tau_i}^{k_i,\nu_i})\vee\mathrm{diam}(\mathcal{Q}_{\tau_i'}^{k_i',\nu_i'})\text{ for } i=1,2.$$

Also note that $\rho(y_{\tau_i}^{k_i,\nu_i}, y_{\tau_i'}^{k_i',\nu_i'}) \geq \mathrm{dist}(\mathcal{Q}_{\tau_i}^{k_i,\nu_i}, \mathcal{Q}_{\tau_i'}^{k_i',\nu_i'})$. Since the last inequality of (3.6) is independent of (x_1, x_2), then combining the above estimates, it follows that

$$\begin{aligned} S_R \lesssim & \sum_{k_1'=0}^{\infty} \sum_{k_2'=0}^{\infty} \sum_{\tau_1' \in I_{k_1'}} \sum_{\tau_2' \in I_{k_2'}} \sum_{\nu_1'=1}^{N(k_1',\tau_1')} \sum_{\nu_2'=1}^{N(k_2',\tau_2')} \mu(\mathcal{Q}_{\tau_1'}^{k_1',\nu_1'}) \mu(\mathcal{Q}_{\tau_2'}^{k_2',\nu_2'}) \\ & \times \left(\frac{\mathrm{diam}(\mathcal{Q}_{\tau_1}^{k_1,\nu_1})}{\mathrm{diam}(\mathcal{Q}_{\tau_1'}^{k_1',\nu_1'})} \wedge \frac{\mathrm{diam}(\mathcal{Q}_{\tau_1'}^{k_1',\nu_1'})}{\mathrm{diam}(\mathcal{Q}_{\tau_1}^{k_1,\nu_1})} \right)^{N_0} \\ & \cdot \frac{[1 + (\mathrm{diam}(\mathcal{Q}_{\tau_1}^{k_1,\nu_1}) \vee \mathrm{diam}(\mathcal{Q}_{\tau_1'}^{k_1',\nu_1'}))^{-1} \mathrm{dist}(\mathcal{Q}_{\tau_1}^{k_1,\nu_1}, \mathcal{Q}_{\tau_1'}^{k_1',\nu_1'})]^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(y_{\tau_1}^{k_1,\nu_1}, \mathrm{diam}(\mathcal{Q}_{\tau_1}^{k_1,\nu_1}) \vee \mathrm{diam}(\mathcal{Q}_{\tau_1'}^{k_1',\nu_1'}) + \mathrm{dist}(\mathcal{Q}_{\tau_1}^{k_1,\nu_1}, \mathcal{Q}_{\tau_1'}^{k_1',\nu_1'}))\big)} \\ & \times \left(\frac{\mathrm{diam}(\mathcal{Q}_{\tau_2}^{k_2,\nu_2})}{\mathrm{diam}(\mathcal{Q}_{\tau_2'}^{k_2',\nu_2'})} \wedge \frac{\mathrm{diam}(\mathcal{Q}_{\tau_2'}^{k_2',\nu_2'})}{\mathrm{diam}(\mathcal{Q}_{\tau_2}^{k_2,\nu_2})} \right)^{N_0} \\ & \cdot \frac{[1 + (\mathrm{diam}(\mathcal{Q}_{\tau_2}^{k_2,\nu_2}) \vee \mathrm{diam}(\mathcal{Q}_{\tau_2'}^{k_2',\nu_2'}))^{-1} \mathrm{dist}(\mathcal{Q}_{\tau_2}^{k_2,\nu_2}, \mathcal{Q}_{\tau_2'}^{k_2',\nu_2'})]^{-m}}{\mathrm{Vol}\big(B_{(X,d)}(y_{\tau_2}^{k_2,\nu_2}, \mathrm{diam}(\mathcal{Q}_{\tau_2}^{k_2,\nu_2}) \vee \mathrm{diam}(\mathcal{Q}_{\tau_2'}^{k_2',\nu_2'}) + \mathrm{dist}(\mathcal{Q}_{\tau_2}^{k_2,\nu_2}, \mathcal{Q}_{\tau_2'}^{k_2',\nu_2'}))\big)} \cdot T_{R'}. \end{aligned} \tag{3.7}$$

Now combining (3.7) and the following equality

$$\prod_{i=1}^{2} \mu(\mathcal{Q}_{\tau_i}^{k_i,\nu_i}) \mu(\mathcal{Q}_{\tau_i'}^{k_i',\nu_i'}) = \prod_{i=1}^{2} \Big(\mu(\mathcal{Q}_{\tau_i}^{k_i,\nu_i}) \vee \mu(\mathcal{Q}_{\tau_i'}^{k_i',\nu_i'}) \Big)^2 \left(\frac{\mu(\mathcal{Q}_{\tau_i}^{k_i,\nu_i})}{\mu(\mathcal{Q}_{\tau_i'}^{k_i',\nu_i'})} \wedge \frac{\mu(\mathcal{Q}_{\tau_i'}^{k_i',\nu_i'})}{\mu(\mathcal{Q}_{\tau_i}^{k_i,\nu_i})} \right),$$

we obtain the left-hand side (3.5), namely, $\frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{R \subset \Omega} \mu(R) S_R$, is bounded by

$$\frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{R \subset \Omega} \sum_{R'} v(R, R') r(R, R') P(R, R') T_{R'}. \tag{3.8}$$

Thus, to finish the proof the theorem, we need to prove that (3.8) can be controlled by

$$\sup_{\bar{\Omega}} \frac{1}{\mu(\bar{\Omega})^{\frac{2}{p}-1}} \sum_{R' \subset \bar{\Omega}} \mu(R') T_{R'}, \tag{3.9}$$

where $\bar{\Omega}$ ranges over the open sets in M with finite measures.

We first point out that the terms $r(R, R')$ and $P(R, R')$ characterize the geometrical properties between R and R'. Namely, when the difference of the sizes of R and R' grows bigger,

$r(R,R')$ becomes smaller; when the distance between R and R' grows bigger, $P(R,R')$ becomes smaller. Hence, what we should do next is to that, for each R, decompose the set of all dyadic rectangles $\{R'\}$ into annuli according to the distance between R and R'. Next, for each annuli, we give a precise estimate by considering the difference of the sizes of R and R'. Finally, we add up all the estimates on each annuli and then finish our proof.

Now let's go into the details. For the sake of simplicity, we denote $\mathcal{Q}_{\tau_i}^{k_i,\nu_i}$, $\mathcal{Q}_{\tau_i'}^{k_i',\nu_i'}$ by $\mathcal{Q}_i$, $\mathcal{Q}$, respectively, for $i=1,2$. Define

$$\Omega^0 =: \bigcup_{R=\mathcal{Q}_1\times\mathcal{Q}_2\subset\Omega} 3(\mathcal{Q}_1\times\mathcal{Q}_2).$$

And for each R, let

$$\begin{aligned}
A_{0,0}(R) =&\{R' : \operatorname{dist}(\mathcal{Q}_1,\mathcal{Q}_1') \le \operatorname{diam}(\mathcal{Q}_1)\vee\operatorname{diam}(\mathcal{Q}_1'),\\
&\operatorname{dist}(\mathcal{Q}_2,\mathcal{Q}_2') \le \operatorname{diam}(\mathcal{Q}_2)\vee\operatorname{diam}(\mathcal{Q}_2')\};\\
A_{j,0}(R) =&\{R' : 2^{j-1}\big(\operatorname{diam}(\mathcal{Q}_1)\vee\operatorname{diam}(\mathcal{Q}_1')\big) < \operatorname{dist}(\mathcal{Q}_1,\mathcal{Q}_1') \le 2^j\big(\operatorname{diam}(\mathcal{Q}_1)\vee\operatorname{diam}(\mathcal{Q}_1')\big),\\
&\operatorname{dist}(\mathcal{Q}_2,\mathcal{Q}_2') \le \operatorname{diam}(\mathcal{Q}_2)\vee\operatorname{diam}(\mathcal{Q}_2')\};\\
A_{0,k}(R) =&\{R' : \operatorname{dist}(\mathcal{Q}_1,\mathcal{Q}_1') \le \operatorname{diam}(\mathcal{Q}_1)\vee\operatorname{diam}(\mathcal{Q}_1')\\
&2^{k-1}\big(\operatorname{diam}(\mathcal{Q}_2)\vee\operatorname{diam}(\mathcal{Q}_2')\big) < \operatorname{dist}(\mathcal{Q}_2,\mathcal{Q}_2') \le 2^k\big(\operatorname{diam}(\mathcal{Q}_2)\vee\operatorname{diam}(\mathcal{Q}_2')\big)\};\\
A_{j,k}(R) =&\{R' : 2^{j-1}\big(\operatorname{diam}(\mathcal{Q}_1)\vee\operatorname{diam}(\mathcal{Q}_1')\big) < \operatorname{dist}(\mathcal{Q}_1,\mathcal{Q}_1') \le 2^j\big(\operatorname{diam}(\mathcal{Q}_1)\vee\operatorname{diam}(\mathcal{Q}_1')\big),\\
&2^{k-1}\big(\operatorname{diam}(\mathcal{Q}_2)\vee\operatorname{diam}(\mathcal{Q}_2')\big) < \operatorname{dist}(\mathcal{Q}_2,\mathcal{Q}_2') \le 2^k\big(\operatorname{diam}(\mathcal{Q}_2)\vee\operatorname{diam}(\mathcal{Q}_2')\big)\},
\end{aligned}$$

where $j,k\ge 1$.

Since for each $R'=\mathcal{Q}_1'\times\mathcal{Q}_2'$, $\lim_{j,k\to\infty} 3(2^j\mathcal{Q}_1'\times 2^k\mathcal{Q}_2') = M$, we can see that for any $R\subset\Omega$, there must be some j and k such that $R'\in A_{j,k}(R)$. This implies that for each $R\subset\Omega$, $\{R'\}\subset\cup_{j,k\ge0}A_{j,k}(R)$.

Then, we have

$$
\begin{aligned}
(3.8) \leq & \frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{R\subset\Omega} \sum_{R'\in A_{0,0}(R)} v(R,R')r(R,R')P(R,R')T_{R'} \\
& + \sum_{j\geq 1} \frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{R\subset\Omega} \sum_{R'\in A_{j,0}(R)} v(R,R')r(R,R')P(R,R')T_{R'} \\
& + \sum_{k\geq 1} \frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{R\subset\Omega} \sum_{R'\in A_{0,k}(R)} v(R,R')r(R,R')P(R,R')T_{R'}
\end{aligned}
$$

$$
\begin{aligned}
& + \sum_{j,k\geq 1} \frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{R\subset\Omega} \sum_{R'\in A_{j,k}(R)} v(R,R')r(R,R')P(R,R')T_{R'} \\
=: & I + II + III + IV.
\end{aligned}
$$

We first estimate term I. Define

$$B_{0,0} = \{R' : 3R' \cap \Omega^0 \neq \emptyset\}.$$

Then we claim that

$$I \leq \frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{R'\in B_{0,0}} \sum_{\{R: R\subset\Omega, R'\in A_{0,0}(R)\}} v(R,R')r(R,R')P(R,R')T_{R'} \tag{3.10}$$

To show this claim, we only need to point out that for any $R' \notin B_{0,0}$, we have $3R' \cap \Omega^0 = \emptyset$. Thus, for any $R \subset \Omega$, we can see that $3R' \cap 3R = \emptyset$, which implies that $R' \notin A_{0,0}(R)$. Hence, we can obtain that $\cup_{R\subset\Omega} A_{0,0}(R) \subset B_{0,0}$. This yields that the claim (3.10) holds.

Now we continue to decompose $B_{0,0}$. Let $\mathcal{F}_h^{0,0} = \{R' : \mu(3R' \cap \Omega^0) > \frac{1}{2^h}(3R')\}$, $\mathcal{D}_h^{0,0} = \mathcal{F}_h^{0,0} \backslash \mathcal{F}_{h-1}^{0,0}$, $h \geq 1$, $\mathcal{F}_0^{0,0} = \emptyset$, and $\Omega_h^{0,0} = \cup_{R'\in\mathcal{D}_h^{0,0}} R'$, $h \geq 1$. From these definitions, we can see that

$$B_{0,0} = \bigcup_{h \geq 1} \mathcal{D}_h^{0,0}.$$

Then (3.10) can be rewritten as

$$I \leq \frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{h \geq 1} \sum_{R' \in \mathcal{D}_h^{0,0}} \sum_{\{R: R \subset \Omega, R' \in A_{0,0}(R)\}} v(R,R') r(R,R') P(R,R') T_{R'} \tag{3.11}$$

To estimate the right-hand side of (3.11) we only need to consider

$$\sum_{\{R: R \subset \Omega, R' \in A_{0,0}(R)\}} v(R,R') r(R,R') \tag{3.12}$$

since $P(R,R') \leq 1$ for any $R' \in \mathcal{D}_h^{0,0}$ and R satisfying $R' \in A_{0,0}(R)$. In what follows, we use a simple geometrical argument, which is a generalization of Chang and R.Fefferman's idea, see more details in [2].

Since $3R \cap 3R' \notin \emptyset$, we can split (3.12) into four cases:

Case 1: $\mathrm{diam}(\mathcal{Q}_1') \geq \mathrm{diam}(\mathcal{Q}_1)$, $\mathrm{diam}(\mathcal{Q}_2') \leq \mathrm{diam}(\mathcal{Q}_2)$.

First, it is easy to see that $\mu(\mathcal{Q}_1 \times 3\mathcal{Q}_2') \lesssim \mu(3R \cap 3R')$. So we have

$$\frac{\mu(\mathcal{Q}_1)}{\mu(3\mathcal{Q}_1')} \mu(3R') \lesssim \mu(3R \cap 3R') \leq \mu(3R' \cap \Omega^0) \leq \frac{1}{2^{h-1}\mu(3R')},$$

which yield that $2^{h-1}\mu(\mathcal{Q}_1) \lesssim \mu(3\mathcal{Q}_1') \lesssim \mu(\mathcal{Q}_1')$. Since all the $\mathcal{Q}_i$ and $\mathcal{Q}_i'$ $(i = 1, 2)$ are dyadic cubes with measures equivalent to $2^{aQ_{1,2}}$ for some $a \in \mathbb{Z}$, then we have $\mu(\mathcal{Q}_1') \sim 2^{h+n_1 Q_{1,2}} \mu(\mathcal{Q}_1)$, for some $n_1 \geq 0$. For each fixed n_1, the numbers of such $\mathcal{Q}_1$'s must be $\lesssim \frac{C_1}{C_2} \cdot 2^{n_1 Q_{1,2}}$, where C_1, C_2 are the constants in Christ's construction Lemma.

Denote by $x_{\mathcal{Q}_1}$ and $x_{\mathcal{Q}_1'}$ the centers of $\mathcal{Q}_1$ and $\mathcal{Q}_1'$, respectively. Since $3R \cap 3R' \neq \emptyset$, we have $3\mathcal{Q}_1 \cap 3\mathcal{Q}_1' \neq \emptyset$, which implies that $d(x_{\mathcal{Q}_1}, x_{\mathcal{Q}_1'}) \leq 6\mathrm{diam}(\mathcal{Q}_1')$, and hence $\mathrm{Vol}\big(B_{(X,d)}(x_{\mathcal{Q}_1}, 6\mathrm{diam}(\mathcal{Q}_1'))\big) \approx$

$\mathrm{Vol}\big(B_{(X,d)}(x_{\mathcal{Q}_1'}, 6\mathrm{diam}(\mathcal{Q}_1')\big) \approx \mu(6\mathcal{Q}_1') \approx \mu(\mathcal{Q}_1')$. Thus,

$$\frac{\mu(\mathcal{Q}_1')}{\mu(\mathcal{Q}_1)} \approx \frac{\mathrm{Vol}\big(B_{(X,d)}(x_{\mathcal{Q}_1}, 6\mathrm{diam}(\mathcal{Q}_1'))\big)}{\mathrm{Vol}\big(B_{(X,d)}(x_{\mathcal{Q}_1}, \mathrm{diam}(\mathcal{Q}_1))\big)} \lesssim \Big(\frac{\mathrm{diam}(\mathcal{Q}_1')}{\mathrm{diam}(\mathcal{Q}_1)}\Big)^{Q_{1,2}}.$$

It follows that for each fix $n_1 > 0$,

$$\frac{\mathrm{diam}(\mathcal{Q}_1)}{\mathrm{diam}(\mathcal{Q}_1')} \lesssim \Big(\frac{\mu(\mathcal{Q}_1)}{\mu(\mathcal{Q}_1')}\Big)^{\frac{1}{Q_{1,2}}} \lesssim 2^{-\frac{h}{Q_{1,2}} - n_1}.$$

As for $\mathcal{Q}_2$, $\mu(\mathcal{Q}_2) \sim 2^{n_2 Q_{2,2}} \mu(\mathcal{Q}_2')$ for some $n_2 \geq 0$. For each fixed n_2, the number of such $\mathcal{Q}_2$'s is less than a constant independent of n_2, since $3\mathcal{Q}_2 \cap 3\mathcal{Q}_2' \neq \emptyset$ and $\mu(\mathcal{Q}_2) \gtrsim \mu(\mathcal{Q}_2')$. Moreover, we have

$$\frac{\mathrm{diam}(\mathcal{Q}_2')}{\mathrm{diam}(\mathcal{Q}_2)} \lesssim 2^{-n_2}.$$

Thus,

$$\begin{aligned}
&\sum_{\text{Case 1}} r(R,R')v(R,R') \\
&\quad = \sum_{\text{Case 1}} \frac{\mu(\mathcal{Q}_1)}{\mu(\mathcal{Q}_1')} \cdot \frac{\mu(\mathcal{Q}_2')}{\mu(\mathcal{Q}_2)} \cdot \left(\frac{\mathrm{diam}(\mathcal{Q}_1)}{\mathrm{diam}(\mathcal{Q}_1')}\right)^{N_0} \left(\frac{\mathrm{diam}(\mathcal{Q}_2')}{\mathrm{diam}(\mathcal{Q}_2)}\right)^{N_0} \mu(\mathcal{Q}_1')\mu(\mathcal{Q}_2) \\
&\quad \lesssim \sum_{n_1, n_2 \geq 0} 2^{-(h + n_1 Q_{1,2})} 2^{-(\frac{h}{Q_{1,2}} + n_1)N_0} 2^{-n_2 N_0} 2^{n_1 Q_{1,2}} \mu(\mathcal{Q}_1')\mu(\mathcal{Q}_2') \\
&\quad \lesssim 2^{-h(1 + \frac{N_0}{Q_{1,2}})} \mu(R').
\end{aligned}$$

Case 2: $\mathrm{diam}(\mathcal{Q}_1') \leq \mathrm{diam}(\mathcal{Q}_1)$, $\mathrm{diam}(\mathcal{Q}_2') \geq \mathrm{diam}(\mathcal{Q}_2)$.

This can be handled in a similar way to that of Case 1. We have

$$\sum_{\text{Case 2}} r(R,R')v(R,R') \lesssim 2^{-h(1 + \frac{N_0}{Q_{2,2}})} \mu(R').$$

Case 3: $\mathrm{diam}(\mathcal{Q}_1') \geq \mathrm{diam}(\mathcal{Q}_1)$, $\mathrm{diam}(\mathcal{Q}_2') \geq \mathrm{diam}(\mathcal{Q}_2)$.

Since

$$\mu(R) \lesssim \mu(3R' \cap 3R) \leq \mu(3R' \cap \Omega^0) \leq \frac{1}{2^{h-1}}\mu(3R'),$$

we have $2^{h-1}\mu(R) \lesssim \mu(R')$. Using the same idea as in Case 1, we can obtain that $\mu(R') \sim 2^{h+n_1Q_{1,2}+n_2Q_{2,2}}\mu(R)$ for some $n_1, n_2 \geq 0$. For each fixed n_1 and n_2, the number of such R's is $\lesssim 2^{n_1Q_{1,2}}2^{n_2Q_{2,2}}$.

Similar to Case 1, since $3R \cap 3R' \neq \emptyset$, we have $\frac{\mu(\mathcal{Q}_1')}{\mu(\mathcal{Q}_1)} \lesssim \left(\frac{\mathrm{diam}(\mathcal{Q}_1')}{\mathrm{diam}(\mathcal{Q}_1)}\right)^{Q_{1,2}}$ and $\frac{\mu(\mathcal{Q}_2')}{\mu(\mathcal{Q}_2)} \lesssim \left(\frac{\mathrm{diam}(\mathcal{Q}_2')}{\mathrm{diam}(\mathcal{Q}_2)}\right)^{Q_{2,2}}$. Hence, $\frac{\mathrm{diam}(\mathcal{Q}_1)\mathrm{diam}(\mathcal{Q}_2)}{\mathrm{diam}(\mathcal{Q}_1')\mathrm{diam}(\mathcal{Q}_2')} \lesssim \left(\frac{\mu(R)}{\mu(R')}\right)^{\frac{1}{Q_{1,2}\vee Q_{2,2}}} \lesssim 2^{-\frac{h}{Q_{1,2}\vee Q_{2,2}}}2^{-\frac{n_1Q_{1,2}+n_2Q_{2,2}}{Q_{1,2}\vee Q_{2,2}}}$. Combining these results, we can get

$$\begin{aligned}
&\sum_{\text{Case 3}} r(R,R')v(R,R') \\
&\quad = \sum_{\text{Case 3}} \frac{\mu(R)}{\mu(R')} \cdot \left(\frac{\mathrm{diam}(\mathcal{Q}_1)\mathrm{diam}(\mathcal{Q}_2)}{\mathrm{diam}(\mathcal{Q}_1')\mathrm{diam}(\mathcal{Q}_2')}\right)^{N_0} \mu(\mathcal{Q}_1')\mu(\mathcal{Q}_2') \\
&\quad \lesssim \sum_{n_1,n_2\geq 0} 2^{-(h+n_1Q_{1,2}+n_2Q_{2,2})}2^{-\frac{h}{Q_{1,2}\vee Q_{2,2}}N_0}2^{-\frac{n_1Q_{1,2}+n_2Q_{2,2}}{Q_{1,2}\vee Q_{2,2}}N_0}2^{n_1Q_{1,2}+n_2Q_{2,2}}\mu(R') \\
&\quad \lesssim 2^{-h(1+\frac{N_0}{Q_{1,2}\vee Q_{2,2}})}\mu(R').
\end{aligned}$$

Case 4: $\mathrm{diam}(\mathcal{Q}_1') \leq \mathrm{diam}(\mathcal{Q}_1)$, $\mathrm{diam}(\mathcal{Q}_2') \leq \mathrm{diam}(\mathcal{Q}_2)$.

From

$$\mu(R') \lesssim \mu(3R' \cap 3R) \leq \mu(3R' \cap \Omega^0) \leq \frac{1}{2^{h-1}}\mu(3R'),$$

we have that $\mu(R') \leq C\frac{1}{2^{h-1}}\mu(R')$, where C is a constant depending on only $Q_{1,2}$, $Q_{2,2}$, C_1 and C_2. This yields that $h \leq h_0 = [\log_2(2C)] + 1$. Thus we can see that in this case, there are at most h_0 terms in (3.11) in nonzero.

Since $\mu(R) \geq \mu(R')$, we obtain that $\mu(R) \sim 2^{n_1Q_{1,2}+n_2Q_{2,2}}\mu(R')$ for some $n_1, n_2 \geq 0$. For each fixed n, the number of such R's is less than a constant independent of n_1 and n_2. Also, by using the same skills as in Case 3, we have $\frac{\operatorname{diam}(\mathcal{Q}_1')\operatorname{diam}(\mathcal{Q}_2')}{\operatorname{diam}(\mathcal{Q}_1)\operatorname{diam}(\mathcal{Q}_2)} \lesssim 2^{-\frac{n_1 Q_{1,2}+n_2 Q_{2,2}}{Q_{1,2}\vee Q_{2,2}}}$. Therefore

$$
\begin{aligned}
\sum_{\text{Case 4}} r(R,R')v(R,R') &= \sum_{\text{Case 4}} \frac{\mu(R')}{\mu(R)} \cdot \left(\frac{\operatorname{diam}(\mathcal{Q}_1')\operatorname{diam}(\mathcal{Q}_2')}{\operatorname{diam}(\mathcal{Q}_1)\operatorname{diam}(\mathcal{Q}_2)}\right)^{N_0} \mu(\mathcal{Q}_1)\mu(\mathcal{Q}_2) \\
&\lesssim \sum_{n\geq 0} 2^{-\frac{n_1 Q_{1,2}+n_2 Q_{2,2}}{Q_{1,2}\vee Q_{2,2}} N_0} \mu(R') \\
&\lesssim \mu(R').
\end{aligned}
$$

Now we have finished the estimate of (3.12). Then from (3.11), we have

$$
\begin{aligned}
I \leq \;& \frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{h\geq 1} \sum_{R'\in \mathcal{D}_h^{0,0}} \Bigg(\sum_{R\in \text{Case 1}} + \sum_{R\in\text{Case 2}} + \\
&+ \sum_{R\in\text{Case 3}} + \sum_{R\in\text{Case 4}} v(R,R')r(R,R')T_{R'} \\
=: \;& I_1 + I_2 + I_3 + I_4.
\end{aligned}
$$

We first consider the terms I_1, I_2 and I_3. Noting that we have chosen ϵ and N_0 satisfying that $\frac{N_0}{Q_{1,2}\vee Q_{2,2}} > \frac{2}{p}$ and combining with the fact that $\mu(\Omega_h^{0,0}) \lesssim h2^h\mu(\Omega)$ for $h \geq 1$, we have

$$
\begin{aligned}
I_1, I_2, I_3 \lesssim \;& \frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{h\geq 1} 2^{-h(1+\frac{N_0}{Q_{1,2}\vee Q_{2,2}})} \mu(\Omega_h^{0,0})^{\frac{2}{p}-1} \frac{1}{\mu(\Omega_h^{0,0})^{\frac{2}{p}-1}} \sum_{R'\subset \Omega_h^{0,0}} \mu(R')T_{R'} \\
\lesssim \;& \frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{h\geq 1} 2^{-h(1+\frac{N_0}{Q_{1,2}\vee Q_{2,2}})} (h2^h)^{\frac{2}{p}-1} \mu(\Omega)^{\frac{2}{p}-1} \\
& \times \sup_{\overline{\Omega}} \frac{1}{\mu(\overline{\Omega})^{\frac{2}{p}-1}} \sum_{R'\subset\overline{\Omega}} \mu(R')T_{R'} \\
\lesssim \;& \sup_{\overline{\Omega}} \frac{1}{\mu(\overline{\Omega})^{\frac{2}{p}-1}} \sum_{R'\subset\overline{\Omega}} \mu(R')T_{R'}.
\end{aligned}
$$

As for I_4, from the estimate in Case 4 we can see that

$$
I_4 \leq \frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{h=1}^{h_0} \sum_{R'\in\mathcal{D}_h^{0,0}} \sum_{R\in\text{Case 4}} r(R,R')v(R,R')T_{R'}.
$$

Thus, we have

$$
\begin{aligned}
I_4 \lesssim\ & \frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{h=1}^{h_0} \mu(\Omega_h^{0,0})^{\frac{2}{p}-1} \frac{1}{\mu(\Omega_h^{0,0})^{\frac{2}{p}-1}} \sum_{R' \subset \Omega_h^{0,0}} \mu(R') T_{R'} \\
\lesssim\ & \frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{h=1}^{h_0} (h2^h)^{\frac{2}{p}-1} \mu(\Omega)^{\frac{2}{p}-1} \times \sup_{\overline{\Omega}} \frac{1}{\mu(\overline{\Omega})^{\frac{2}{p}-1}} \sum_{R' \subset \overline{\Omega}} \mu(R') T_{R'} \\
\lesssim & \sup_{\overline{\Omega}} \frac{1}{\mu(\overline{\Omega})^{\frac{2}{p}-1}} \sum_{R' \subset \overline{\Omega}} \mu(R') T_{R'}.
\end{aligned}
$$

Combining the estimates from I_1 to I_4, we can get

$$
I \lesssim \sup_{\overline{\Omega}} \frac{1}{\mu(\overline{\Omega})^{\frac{2}{p}-1}} \sum_{R' \subset \Omega} \mu(R') T_{R'}.
$$

Similarly, we can dealt with *II*, *III* and *IV* with only minor modifications. The proof of the Min-Max comparison principle for $\mathrm{CMO}^p(M)$ is complete. $\square$

We will establish the following duality result in the multi-parameter setting for $\widetilde{M}$.

Theorem 3.12. $\big(H^p(\widetilde{M})\big)' = CMO^p(\widetilde{M})$, $\big(H^1(\widetilde{M})\big)' = CMO^1(\widetilde{M}) = BMO$.

We introduce the product sequence spaces s^p and c^p as follows.

Definition 3.13. Let $\widetilde{\chi}_{\mathcal{Q}} = \mu(\mathcal{Q})^{-\frac{1}{2}} \chi_{\mathcal{Q}}(x)$. The product sequence space s^p, $0 < p \le 1$, is defined as the collection of all complex-value sequences

$$
\lambda = \Big\{ \lambda_{\mathcal{Q}_{\tau_1}^{k_1,\nu_1} \times \mathcal{Q}_{\tau_2}^{k_2,\nu_2}} \Big\}_{k_1,k_2 \in \mathbb{N}; \tau_1 \in I_{k_1}, \tau_2 \in I_{k_2}; \nu_1 = 1,\dots,N(k_1,\tau_1), \nu_2 = 1,\dots,N(k_2.\tau_2)}
$$

such that $\|\lambda\|_{s^p}$

$$
\begin{aligned}
= \Bigg\| \Bigg\{ & \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{\tau_1 \in I_{k_1}} \sum_{\tau_2 \in I_{k_2}} \sum_{\nu_1=1}^{N(k_1,\tau_1)} \\
& \cdot \sum_{\nu_2=1}^{N(k_2,\tau_2)} \Big(|\lambda_{\mathcal{Q}_{\tau_1}^{k_1,\nu_1} \times \mathcal{Q}_{\tau_2}^{k_2,\nu_2}}| \cdot \widetilde{\chi}_{\mathcal{Q}_{\tau_1}^{k_1,\nu_1}}(\cdot) \widetilde{\chi}_{\mathcal{Q}_{\tau_2}^{k_2,\nu_2}}(\cdot) \Big)^2 \Bigg\}^{1/2} \Bigg\|_{L^p} < \infty.
\end{aligned}
$$

Similarly, c^p, $0 < p \le 1$, is defined as the collection of all complex-value sequences

$$t = \left\{ t_{\mathcal{Q}_{\tau_1}^{k_1,\nu_1} \times \mathcal{Q}_{\tau_2}^{k_2,\nu_2}} \right\}_{k_1,k_2 \in \mathbb{N}; \tau_1 \in I_{k_1}, \tau_2 \in I_{k_2}; \nu_1 = 1,\ldots,N(k_1,\tau_1), \nu_2 = 1,\ldots,N(k_2.\tau_2)}$$

such that $\|t\|_{c^p}$

$$\begin{aligned}
= \sup_{\Omega} \Bigg(& \frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \int_{\Omega} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{\tau_1 \in I_{k_1}} \sum_{\tau_2 \in I_{k_2}} \\
& \cdot \sum_{\nu_1=1}^{N(k_1,\tau_1)} \sum_{\nu_2=1}^{N(k_2,\tau_2)} \chi_{\{\mathcal{Q}_{\tau_1}^{k_1,\nu_1} \times \mathcal{Q}_{\tau_2}^{k_2,\nu_2} \subset \Omega\}}(k_1,k_2,\tau_1,\tau_2,\nu_1,\nu_2) \\
& \times \left(|t_{\mathcal{Q}_{\tau_1}^{k_1,\nu_1} \times \mathcal{Q}_{\tau_2}^{k_2,\nu_2}}| \cdot \widetilde{\chi}_{\mathcal{Q}_{\tau_1}^{k_1,\nu_1}}(x_1) \widetilde{\chi}_{\mathcal{Q}_{\tau_2}^{k_2,\nu_2}}(x_2) \right)^2 d\mu(x_1) d\mu(x_2) \Bigg)^{1/2} < \infty.
\end{aligned}$$

For simplicity, $\forall s \in s^p$, we rewrite $s = \{s_R\}_R$, and

$$\|s\|_{s^p} = \left\| \left\{ \sum_R |s_R \widetilde{\chi}_R(x_1,x_2)|^2 \right\}^{1/2} \right\|_{L^p}, \tag{3.13}$$

similarly, $\forall t \in c^p$, rewrite $t = \{t_R\}$, and

$$\|t\|_{c^p} = \sup_{\Omega} \left(\frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{R \subseteq \Omega} |t_R|^2 \right)^{1/2}, \tag{3.14}$$

where R runs over all the dyadic rectangles in M. The main result in this section is the following duality theorem.

Theorem 3.14. *$(s^p)' = c^p$ for $0 < p \le 1$.*

Proof. First, we prove that for all $t \in c^p$, let

$$L(s) = \sum_R s_R \cdot \bar{t}_R, \quad \forall s \in s^p, \tag{3.15}$$

then $|L(s)| \lesssim \|s\|_{s^p} \|t\|_{c^p}$.

To see this, let

$$\Omega_k = \Big\{(x_1,x_2)\in M : \big\{\sum_R (|s_R|\widetilde{\chi}_R(x_1,x_2))^2\big\}^{1/2} > 2^k\Big\}.$$

And define

$$\begin{aligned}
B_k =& \Big\{R : \mu(\Omega_k\cap R) > \frac{1}{2}\mu(R), \mu(\Omega_{k+1}\cap R)\le \frac{1}{2}\mu(R)\Big\},\\
\tilde{\Omega}_k =& \Big\{(x_1,x_2)\in M : \mathcal{M}_s(\chi_{\Omega_k}) > \frac{1}{2}\Big\},
\end{aligned}$$

where $\mathcal{M}_s$ is the strong maximal function on M. By (3.15) and the Hölder's inequality,

$$\begin{aligned}
|L(s)| \le& \left(\sum_k \big(\sum_{R\in B_k} |s_R|^2\big)^{\frac{p}{2}} \big(\sum_{R\in B_k} |t_R|^2\big)^{\frac{p}{2}}\right)^{\frac{1}{p}}\\
\le& \left(\sum_k \mu(\tilde{\Omega}_k)^{1-\frac{p}{2}} \big(\sum_{R\in B_k} |s_R|^2\big)^{\frac{p}{2}} \Big(\frac{1}{\mu(\tilde{\Omega})^{\frac{2}{p}-1}} \sum_{R\subset\tilde{\Omega}_k} |t_R|^2\Big)^{\frac{p}{2}}\right) \qquad (3.16)\\
\le& \left(\sum_k \mu(\tilde{\Omega}_k)^{1-\frac{p}{2}} \Big(\sum_{R\in B_k} |s_R|^2\Big)^{\frac{p}{2}}\right)^{\frac{1}{p}} \|t\|_{c^p}.
\end{aligned}$$

Combining the fact that $\int\limits_{\tilde{\Omega}_k\backslash\Omega_{k+1}} \sum_{R\in B_k}(|s_R|\widetilde{\chi}_R(x))^2 d\mu(x) \le 2^{2(k+1)}\mu(\tilde{\Omega}\backslash\Omega_{k+1}) \le C2^{2k}\mu(\Omega_k)$ and that

$$\int_{\tilde{\Omega}\backslash\Omega_{k+1}} \sum_{R\in B_k} (|s_R|\widetilde{\chi}_R(x))^2 d\mu(x) \ge \sum_{R\in B_k} |s_R|^2\mu(R)^{-1}\mu(\tilde{\Omega}_k\backslash\Omega_{k+1}\cap R)$$

Since $R\in B_k$, then R is contained in $\tilde{\Omega}_k$. It follows that

$$\begin{aligned}
\int_{\tilde{\Omega}\backslash\Omega_{k+1}} \sum_{R\in B_k} (|s_R|\widetilde{\chi}_R(x))^2 d\mu(x) \ge& \sum_{R\in B_k} |s_R|^2\mu(R)^{-1}\frac{1}{2}\mu(R)\\
\ge& \frac{1}{2}\sum_{R\in B_k}|s_R|^2,
\end{aligned}$$

we obtain $(\sum_{R\in B_k}|s_R|^2)^{\frac{p}{2}} \lesssim 2^{kp}\mu(\Omega_k)^{\frac{p}{2}}$. Substituting this back into the last term of (3.16) and noting $\mu(\tilde{\Omega}_k)\lesssim\mu(\Omega)$ yields that $|L(s)|\lesssim \|s\|_{s^p}\|t\|_{t^p}$.

We point out that an idea similar to the one used in the above proof was used earlier to get an atomic decomposition from a wavelet expansion by Meyer in [45].

Conversely, we need to verify that for any $L \in (s^p)'$, there exists $t \in c^p$ with $\|t\|_{c^p} \leq \|L\|$ such that for all $s \in s^p$, $L(s) = \sum_R s_R \bar{t}_R$. Here we adapt a similar idea in one-parameter case of Frazier and Jawerth in [9] to our multi-parameter situation.

Now define $s^i_R = 1$ when $R = R_i$ and $s^i_R = 0$ for all other R. Then is is easy to see that $\|s^i_R\|_{s^p} = 1$. Now for all $s \in s^p$, $s = \{s_R\} = \sum_i s_{R_i} s^i_{R_i}$, the limit holds in the norm of s^p, here we index all dyadic rectangles in M by $\{R_i\}_{i\in\mathbb{Z}}$. For any $L \in (s^p)'$, let $\bar{t}_{R_i} = L(s^i)$, then $L(s) = L(\sum_i s_{R_i} s^i) = \sum_i s_{R_i} \bar{t}_{R_i} = \sum_R s_R \bar{t}_R$. Let $t = \{t_R\}$. Then we only need to check that $\|t\|_{c^p} \leq \|L\|$.

For any open set $\Omega \subset M$ with finite measure, let $\bar{\mu}$ be a new measure such that $\bar{\mu}(R) = \frac{\mu(R)}{\mu(\Omega)}$ when $R \subset \Omega$, $\bar{R} = 0$ when $R \not\subseteq \Omega$. And let $l^2(\mu)$ be a sequence space such that when $s \in l^2(\bar{\mu})$, $(\sum_{R\subset\Omega} |s_R|^2 \frac{\mu(R)}{\mu(\Omega)^{\frac{p}{2}-1}})^{1/2} < \infty$. It is easy to see that $(l^2(\bar{\mu}))' = l^2(\bar{\mu})$. Then,

$$
\begin{aligned}
\Big\{\frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{R\subset\Omega} |t_R|^2\Big\}^{1/2} &= \big\|\mu(R)^{-1/2}|t_R|\big\|_{l^2(\bar{\mu})} \\
&= \sup_{s:\|s\|_{l^2(\bar{\mu})}\leq 1} \Big| \sum_{R\subseteq\Omega} (|t_R|\mu(R)^{-1/2}) \cdot s_R \cdot \frac{\mu(R)}{\mu(\Omega)^{\frac{1}{p}-1}} \Big| \\
&\leq \sup_{s:\|s\|_{l^2(\bar{\mu})}\leq 1} \Big| L\Big(\chi_{R\subseteq\Omega}(R) \frac{\mu(R)^{1/2}|s_R|}{\mu(\Omega)^{\frac{p}{2}-1}}\Big)\Big| \\
&\leq \sup_{s:\|s\|_{l^2(\bar{\mu})}\leq 1} \|L\| \cdot \Big\|\chi_{R\subseteq\Omega}(R) \frac{\mu(R)^{1/2}|s_R|}{\mu(\Omega)^{\frac{p}{2}-1}}\Big\|_{s^p}.
\end{aligned}
$$

By (3.13) and the Hölder inequality, we have

$$
\Big\|\chi_{R\subseteq\Omega}(R) \frac{\mu(R)^{1/2}|s_R|}{\mu(\Omega)^{\frac{p}{2}-1}}\Big\|_{s^p} \leq \Big(\sum_{R\subseteq\Omega} |s_R|^2 \frac{\mu(R)}{\mu(\Omega)^{\frac{p}{2}-1}}\Big)^{1/2}.
$$

Hence,

$$
\|t\|_{c^p} \leq \sup_{s:\|s\|_{l^2(\bar{\mu})}\leq 1} \|L\| \cdot \|s\|_{l^2(\bar{\mu})} \leq \|L\|.
$$

□

In this section, we prove Theorem 3.12. First, we define the lifting and projection operators as follows.

Definition 3.15. Let $\{D_{k_i}\}_{k_i\in\mathbb{N}}$ be an approximation to the identity, for $i=1,2$. For any $f\in\mathcal{T}'$ define the lifting operator S_D by

$$S_D(f)=\left\{\mu(\mathcal{Q}_{\tau_1}^{k_1,\nu_1})^{1/2}\mu(\mathcal{Q}_{\tau_2}^{k_2,\nu_2})^{1/2}D_{k_1}D_{k_2}(f)(y_{\tau_1}^{k_1,\nu_1},y_{\tau_2}^{k_2,\nu_2})\right\}_{\mathcal{Q}_{\tau_1}^{k_1,\nu_1},\mathcal{Q}_{\tau_2}^{k_2,\nu_2}}, \tag{3.17}$$

where $y_{\tau_i}^{k_i,\nu_i}$ is the center of $\mathcal{Q}_{\tau_i}^{k_i,\nu_i}$, $k_i\in\mathbb{N}$, $\tau_i\in I_{k_i}$, $\nu_i=1,\dots,N(k_i,\tau_i)$ for $i=1,2$.

Definition 3.16. Let all the notation be the same as above. For any sequence s, define the projection operator $T_{\tilde{D}}$ by

$$\begin{aligned}T_{\tilde{D}}(s)(x_1,x_2)=&\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{\tau_1\in I_{k_1}}\sum_{\tau_2\in I_{k_2}}\sum_{\nu_1=1}^{N(k_1,\tau_1)}\sum_{\nu_2=1}^{N(k_2,\tau_2)}s_{\mathcal{Q}_{\tau_1}^{k_1,\nu_1}\times\mathcal{Q}_{\tau_2}^{k_2,\nu_2}}\\&\times\mu(\mathcal{Q}_{\tau_1}^{k_1,\nu_1})^{1/2}\mu(\mathcal{Q}_{\tau_2}^{k_2,\nu_2})^{1/2}\tilde{D}_{k_1}\tilde{D}_{k_2}(x_1,x_2,y_{\tau_1}^{k_1,\nu_1},y_{\tau_2}^{k_2,\nu_2}),\end{aligned} \tag{3.18}$$

where $y_{\tau_i}^{k_i,\nu_i}$ is the center of $\mathcal{Q}_{\tau_i}^{k_i,\nu_i}$ and $\tilde{D}_{k_i}$ is the same operator as in the Calderón reproducing formula associated with D_{k_i} for $i=1,2$.

To work at the level of product sequence spaces, we still need the following two propositions.

Proposition 3.2. *Let all the notation be the same as above. Then for any $f\in H^p(M)$,*

$$\|S_D(f)\|_{s^p}\lesssim\|f\|_{H^p(M)}. \tag{3.19}$$

Conversely, for any $s\in s^p$,

$$\|T_{\tilde{D}}(s)\|_{H^p(M)}\lesssim\|s\|_{s^p}$$

Moreover, $T_{\tilde{D}}\circ S_D$ is the identity on $H^p(M)$.

Proposition 3.3. *Let all the notation be the same as above. Then for any* $f \in \mathrm{CMO}^p(M)$,

$$\|S_D(f)\|_{c^p} \lesssim \|f\|_{\mathrm{CMO}^p(M)}. \tag{3.20}$$

Conversely, for any $t \in c^p$,

$$\|T_{\tilde{D}}(t)\|_{\mathrm{CMO}^p(M)} \lesssim \|t\|_{c^p}. \tag{3.21}$$

Moreover, $T_{\tilde{D}} \circ S_D$ *is the identity on* $\mathrm{CMO}^p(M)$.

Assume the above two propositions first, then we give the proof of Theorem 3.12 with p_0.

Proof. First, let $\{D_{k_i}\}_{k_i \in \mathbb{N}}$ be an approximation to the identity, for $i = 1, 2$. For any $g \in \mathcal{T}(n_0, m; n_0',$

$m')$ and $f \in \mathrm{CMO}^p(M)$, from the two propositions above, we have

$$\langle f, g\rangle = \langle T_{\tilde{D}} \circ S_D(f), g\rangle = \langle S_D(f), S_{\tilde{D}}(g)\rangle,$$

where $S_{\tilde{D}}(g) = \{\mu(\mathcal{Q}_{\tau_1}^{k_1,\nu_1})^{1/2}\mu(\mathcal{Q}_{\tau_2}^{k_2,\nu_2})^{1/2}\tilde{D}_{k_1,k_2}(g)(y_{\tau_1}^{k_1,\nu_1}, y_{\tau_2}^{k_2,\nu_2})\}_{\mathcal{Q}_{\tau_1}^{k_1,\nu_1}, \mathcal{Q}_{\tau_2}^{k_2,\nu_2}}$.

By Definition 3.13 and the Min-Max comparison principle in Lemma 3.4, we obtain $\|S_{\tilde{D}}(g)\|_{s^p} \lesssim \|g\|_{H^p(M)}$. Hence $|\langle f, g\rangle| \le |\langle S_D(f), S_{\tilde{D}}(g)\rangle| \lesssim \|f\|_{\mathrm{CMO}^p(M)}\|g\|_{H^p(M)}$, where the last inequality follows from the two propositions above. Since $\mathcal{T}$ is dense in $H^p(M)$, it follows from a standard density argument that $\mathrm{CMO}^p(M) \subseteq (H^p(M))'$.

Conversely, suppose $l \in (H^p(M))'$. Then $l_1 \equiv l \circ T_{\tilde{D}} \in (s^p)'$ by Proposition 3.2. So by Theorem 3.14, there exists $t \in c^p$ such that $l_1(s) = \langle t, s\rangle$ for all $s \in s^p$, and $\|t\|_{c^p} \approx \|l_1\| \lesssim \|l\|$, since $T_{\tilde{D}}$ is bounded. We have $l_1 \circ S_D = l \circ T_{\tilde{D}} \circ S_D = l$, hence

$$l(g) = l \circ T_{\tilde{D}}(S_D(g)) = \langle t, S_D(g)\rangle = \langle T_D(t), g\rangle,$$

where

$$T_D(t)=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{\tau_1\in I_{k_1}}\sum_{\tau_2\in I_{k_2}}\sum_{\nu_1=1}^{N(k_1,\tau_1)}\sum_{\nu_2=1}^{N(k_2,\tau_2)} t_{\mathcal{Q}_{\tau_1}^{k_1,\nu_1}\times\mathcal{Q}_{\tau_2}^{k_2,\nu_2}}\mu(\mathcal{Q}_{\tau_1}^{k_1,\nu_1})^{\frac{1}{2}}\mu(\mathcal{Q}_{\tau_2}^{k_2,\nu_2})^{\frac{1}{2}}$$

$$\times D_{k_1}D_{k_2}(x_1,x_2,y_{\tau_1}^{k_1,\nu_1},y_{\tau_2}^{k_2,\nu_2}).$$

By Definition 3.13 and the Min-Max comparison principle in Theorem 3.4, we obtain that $\|T_D(t)\|_{\mathrm{CMO}^p(M)}\le\|t\|_{c^p}\le\|l\|$. Hence $(H^p(M))'\subseteq\mathrm{CMO}^p(M)$. □

Now we give brief proofs to the above propositions.

Proof of Proposition 3.2. To show this proposition, we first point out that the proof is closely related to the Min-Max comparison principle for $H^p(M)$, namely, Lemma 3.4. (3.19) is a direct consequence of Lemma 3.4 and the proof of (3.2) follows the same routine as the proof of Lemma 3.4.

Now let us go into the details. We first prove (3.19). By Definition 3.13 and 3.15, we can see that for any $f\in H^p(M)$,

$$\|S_D(f)\|_{s^p}$$

$$\le\Bigg\|\Big\{\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{\tau_1\in I_{k_1}}\sum_{\tau_2\in I_{k_2}}\sum_{\nu_1=1}^{N(k_1,\tau_1)}\sum_{\nu=2}^{N(k_2,\tau_2)}\sup_{u\in\mathcal{Q}_{\tau_1}^{k_1,\nu_1},v\in\mathcal{Q}_{\tau_2}^{k_2,\nu_2}}|D_{k_1}D_{k_2}(f)(u,v)|^2$$

$$\times\chi_{\mathcal{Q}_{\tau_1}^{k_1,\nu_1}}\chi_{\mathcal{Q}_{\tau_2}^{k_2,\nu_2}}(\cdot)\Big\}^{\frac{1}{2}}\Bigg\|_{L^p}$$

$$\lesssim\Bigg\|\Big\{\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{\tau_1\in I_{k_1}}\sum_{\tau_2\in I_{k_2}}\sum_{\nu_1=1}^{N(k_1,\tau_1)}\sum_{\nu=2}^{N(k_2,\tau_2)}\inf_{u\in\mathcal{Q}_{\tau_1}^{k_1,\nu_1},v\in\mathcal{Q}_{\tau_2}^{k_2,\nu_2}}|D_{k_1}D_{k_2}(f)(u,v)|^2$$

$$\times\chi_{\mathcal{Q}_{\tau_1}^{k_1,\nu_1}}(\cdot)\chi_{\mathcal{Q}_{\tau_2}^{k_2,\nu_2}}(\cdot)\Big\}^{\frac{1}{2}}\Bigg\|_{L^p}$$

$$\le\Bigg\|\Big\{\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}|D_{k_1}D_{k_2}(f)(\cdot,\cdot)|^2\Big\}^{\frac{1}{2}}\Bigg\|_{L^p}$$

$$\lesssim \|f\|_{H^p}.$$

Now let us turn to (3.2). For simplicity, we only need to work with the dyadic cubes of form $\{\mathcal{Q}^{k_i}_{\tau_i} : k_i \in \mathbb{N}, \tau_i \in I_{k_i+J}\}$ for $i = 1, 2$.

To simplify our notation, let $m_{\mathcal{Q}^{k_1}_{\tau_1}\times\mathcal{Q}^{k_2}_{\tau_2}}(x_1,x_2) = \mu(\mathcal{Q}^{k_2}_{\tau_2})^{1/2}\mu(\mathcal{Q}^{k_1}_{\tau_1})^{1/2}\tilde{D}_{k_1}(x_1,y^{k_1}_{\tau_1})$ $\cdot\,\tilde{D}_{k_2}(x_2,y^{k_2}_{\tau_2})$. Now we first estimate $D_{j_1}D_{j_2}(m_{\mathcal{Q}^{k_1}_{\tau_1}\times\mathcal{Q}^{k_2}_{\tau_2}})(x_1,x_2)$.

From Definition 3.8 and 3.16, we have

$$
\begin{aligned}
&\|T_{\tilde{D}}(s)(x_1,x_2)\|^p_{H^p(M)} = \|g(T_{\tilde{D}}(s))\|^p_{L^p(M)}\\
&\lesssim \Bigg\|\Bigg\{\sum_{j_1,j_2}\Bigg(\Bigg[\sum_{k_1>j_1,k_2>j_2} + \sum_{k_1>j_1,k_2\le j_2} + \sum_{k_1\le j_1,k_2>j_2} + \sum_{k_1\le j_1,k_2\le j_2}\Bigg]\\
&\quad\times\sum_{\tau_1\in I_{k_1+J_1}}\sum_{\tau_2\in I_{k_2+J_2}}|s_{\mathcal{Q}^{k_1}_{\tau_1}\times\mathcal{Q}^{k_2}_{\tau_2}}||D_{j_1}D_{j_2}(m_{\mathcal{Q}^{k_1}_{\tau_1}\times\mathcal{Q}^{k_2}_{\tau_2}})(x_1,x_2)|\Bigg)^2\Bigg\}^{1/2}\Bigg\|^p_{L^p(M)}\\
&\lesssim I + II + III + IV.
\end{aligned}
$$

We now first estimate I. Note that

$$
\begin{aligned}
&\sum_{k_1>j_1,k_2>j_2}\sum_{\tau_1\in I_{k_1+J_1}}\sum_{\tau_2\in I_{k_2+J_2}}|s_{\mathcal{Q}^{k_1}_{\tau_1}\times\mathcal{Q}^{k_2}_{\tau_2}}||D_{j_1}D_{j_2}(m_{\mathcal{Q}^{k_1}_{\tau_1}\times\mathcal{Q}^{k_2}_{\tau_2}})(x_1,x_2)|\\
&\lesssim \sum_{k_1>j_1,k_2>j_2}\sum_{\tau_1\in I_{k_1+J_1}}\\
&\quad\cdot\sum_{\tau_2\in I_{k_2+J_2}}2^{(j_1-k_1)(1+\varepsilon')}2^{(j_2-k_2)(1+\varepsilon')}|s_{\mathcal{Q}^{k_1}_{\tau_1}\times\mathcal{Q}^{k_2}_{\tau_2}}|\mu(\mathcal{Q}^{k_1}_{\tau_1})^{-1/2}\mu(\mathcal{Q}^{k_2}_{\tau_2})^{-1/2}
\end{aligned}
$$

$$
\begin{aligned}
&\times\frac{1}{(1+2^{j_1}\rho(x_1,y^{k_1}_{\tau_1}))^{1+\varepsilon'}}\frac{1}{(1+2^{j_2}\rho(x_2,y^{k_2}_{\tau_2}))^{1+\varepsilon'}}\\
&\lesssim \sum_{k_1>j_1,k_2>j_2}2^{(j_1-k_1)(1+\varepsilon'-\frac{1}{r})}2^{(j_2-k_2)(1+\varepsilon'-\frac{1}{r})}\Bigg(\mathcal{M}_1\Bigg[\sum_{\tau_1\in I_{k_1+J}}
\end{aligned}
$$

$$\times \mathcal{M}_2\Bigg(\sum_{\tau_2\in I_{k_2+J}} |s_{\mathcal{Q}^{k_1}_{\tau_1}\times\mathcal{Q}^{k_2}_{\tau_2}}\mu(\mathcal{Q}^{k_1}_{\tau_1})^{-1/2}\mu(\mathcal{Q}^{k_2}_{\tau_2})^{-1/2}|^r\chi_{\mathcal{Q}^{k_1}_{\tau_1}(\cdot)}\Bigg](x_1)\Bigg)^{\frac{1}{r}},$$

where $\frac{2}{2+\theta} < r < p$ and $\mathcal{M}_i$, $i = 1, 2$, is the Hardy-Littlewood Maximal function with respect to the first and the second variable, respectively. The last inequality follows from an iteration of the result which can be found in [22], for $\mathbb{R}^n$ and [34], for spaces of homogeneous type.

Let $k = (k_1, k_2)$, $j = (j_1, j_2)$, $x = (x_1, x_2)$ and

$$\begin{aligned}
&a(x) = \{a_k(x)\}_k \\
&= \Bigg(\mathcal{M}_1\Bigg[\sum_{\tau_1\in I_{k_1+J}} \mathcal{M}_2\Bigg(\sum_{\tau_2\in I_{k_2+J}} |s_{\mathcal{Q}^{k_1}_{\tau_1}\times\mathcal{Q}^{k_2}_{\tau_2}}\mu(\mathcal{Q}^{k_1}_{\tau_1})^{-1/2}\mu(\mathcal{Q}^{k_2}_{\tau_2})^{-1/2}|^r\chi_{\mathcal{Q}^{k_2}_{\tau_2}(\cdot)}\Bigg) \\
&\quad \cdot (x_2)\chi_{\mathcal{Q}^{k_1}_{\tau_1}}(\cdot)\Bigg](x_1)\Bigg)^{\frac{1}{r}}; \\
&b = \{b_k\}_k = \Big\{2^{k_1(1+\varepsilon'-\frac{1}{r})}2^{k_2(1+\varepsilon'-\frac{1}{r})}\chi_{k_1<0}(k_1)\chi_{k_2<0}(k_2)(k_2)\Big\}_k; \\
&(a*b)_j = \sum_k a_k b_{j-k}.
\end{aligned}$$

By the Young inequality and an iterative application of the Fefferman and Stein vector-valued maximal function inequality in [18] on $L^{\frac{p}{r}}(M)$, we have

$$\begin{aligned}
IV &\lesssim \Big\|\Big\{\sum_j |(a*b)_j|^2\Big\}^{1/2}\Big\| \lesssim \|\|a*b\|_{l^2}\|^p_{L^p(M)} \\
&\lesssim \|\|a\|_{l^2}\|b\|_{l^1}\|^p_{L^p(M)} \\
&\lesssim \|\|a\|_{l^2}\|^p_{L^p(M)} \\
&\lesssim \|s\|^p_{s^p}
\end{aligned}$$

Using the same skill, we can get that *II, III, IV* $\lesssim \|s\|^p_{s^p}$. Thus

$$\|T_{\tilde{D}}(s)(x_1, x_2)\|_{H^p(M)} \lesssim \|s\|_{s^p}.$$

Finally, it is easy to check that from the Calderón reproducing formula, $T_{\tilde{D}} \circ S_D$ equals identity on $H^p(M)$. The proof of proposition is complete. $\square$

Proof of Proposition 3.3. This proposition is similar as the above one since its proof is closely related to the Min-Max comparison principle for $\mathrm{CMO}^p(M)$, namely, Theorem 3.4. (3.20) is a direct consequence of Theorem 3.4 and the proof of (3.21) follows the same routine as the proof of Theorem 3.4.

Now we give the details of the proof. We first prove (3.20). According to Definition 3.13 and 3.15, for any $f \in \mathrm{CMO}^p(M)$, we have

$$
\begin{aligned}
&\|S_D(f)\|_{c^p} \\
&\lesssim \sup_{\Omega}\Bigg(\frac{1}{\mu(\Omega)^{\frac{2}{p}-1}}\int_{\Omega}\sum_{k_1,k_2=0}^{\infty}\sum_{\tau_1\in I_{k_1}}\sum_{\tau_2\in I_{k_2}}\sum_{\nu=1}^{N(k_1,\tau_1)} \\
&\quad\cdot\sum_{\nu_2=1}^{N(k_2,\tau_2)}\chi_{Q_{\tau_1}^{k_1,\nu_1}\times Q_{\tau_2}^{k_2,\nu_2}}(k_1,k_2,\tau_1,\tau_2,\nu_1,\nu_2) \\
&\quad\times\sup_{u\in Q_{\tau_1}^{k_1,\nu_1},v\in Q_{\tau_2}^{k_2,\nu_2}}|D_{k_1}D_{k_2}(f)(u,v)|^2\chi_{Q_{\tau_1}^{k_1,\nu_1}}(x_1)\chi_{Q_{\tau_2}^{k_2,\nu_2}}(x_2)d\mu(x_1)\mu(x_2)\Bigg)^{1/2} \\
&\lesssim \sup_{\Omega}\Bigg(\frac{1}{\mu(\Omega)^{\frac{2}{p}-1}}\int_{\Omega}\sum_{k_1,k_2=0}^{\infty}\sum_{\tau_1\in I_{k_1}}\sum_{\tau_2\in I_{k_2}}\sum_{\nu_1=1}^{N(k_1,\tau_1)} \\
&\quad\cdot\sum_{\nu_2=1}^{N(k_2,\tau_2)}\chi_{Q_{\tau_1}^{k_1,\nu_1}\times Q_{\tau_2}^{k_2,\nu_2}\subset\Omega}(k_1,k_2,\tau_1,\tau_2,\nu_1,\nu_2) \\
&\quad\times\inf_{u\in Q_{\tau_1}^{k_1,\nu_1},v\in Q_{\tau_2}^{k_2,\nu_2}}|D_{k_1}D_{k_2}(f)(u,v)|^2\chi_{Q_{\tau_1}^{k_1,\nu_1}}(x_1)\chi_{Q_{\tau_2}^{k_2,\nu_2}}(x_2)d\mu(x_1)\mu(x_2)\Bigg)^{1/2} \\
&\lesssim \sup_{\Omega}\Bigg(\frac{1}{\mu(\Omega)^{\frac{2}{p}-1}}\int_{\Omega}\sum_{k_1,k_2=0}^{\infty}\sum_{\tau_1\in I_{k_1}}\sum_{\tau_2\in I_{k_2}} \\
&\quad\cdot\sum_{\nu_1=1}^{N(k_1,\tau_1)}\sum_{\nu_2=1}^{N(k_2,\tau_2)}\chi_{Q_{\tau_1}^{k_1,\nu_1}\times Q_{\tau_2}^{k_2,\nu_2}\subset\Omega}(k_1,k_2,\tau_1,\tau_2,\nu_1,\nu_2) \\
&\quad\times|D_{k_1}D_{k_2}(f)(x_1,x_2)|^2\chi_{Q_{\tau_1}^{k_1,\nu_1}}(x_1)\chi_{Q_{\tau_2}^{k_2,\nu_2}}(x_2)d\mu(x_1)\mu(x_2)\Bigg)^{1/2} \\
&\leq \|f\|_{\mathrm{CMO}^p}.
\end{aligned}
$$

Now let us prove (3.21). For any $t \in c^p$, by the definition of norm of CMO^p, we have

$$\|T_{\tilde{D}}(t)\|_{\mathrm{CMO}^p(M)} \lesssim \sup_{\Omega}\Big(\frac{1}{\mu(\Omega)^{\frac{2}{p}-1}}\int_{\Omega}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{\tau_1\in I_{k_1}}\sum_{\tau_2\in I_{k_2}}\sum_{\nu_1=1}^{N(k_1,\tau_1)} \cdot \sum_{\nu_2=1}^{N(k_2,\tau_2)} \chi_{\mathcal{Q}_{\tau_1}^{k_1,\nu_1}}(x_1)\chi_{\mathcal{Q}_{\tau_2}^{k_2,\nu_2}}(x_2)$$

$$\times \chi_{\mathcal{Q}_{\tau_1}^{k_1,\nu_1}\times\mathcal{Q}_{\tau_2}^{k_2,\nu_2}\subset\Omega}(k_1,k_2,\tau_1,\tau_2,\nu_1,\nu_2) \cdot \big|D_{k_1}D_{k_2}\big(T_{\tilde{D}}(t)\big)(x_1,x_2)\big|^2 d\mu(x_1)d\mu(x_2)\Big)^{\frac{1}{2}}.$$

From the definition of $T_{\tilde{D}}(t)$ and the same skill as in the estimate of (3.7), we can obtain that

$$\begin{aligned}
&\sup_{x_1\in\mathcal{Q}_{\tau_1}^{k_1,\nu_1},x_2\in\mathcal{Q}_{\tau_2}^{k_2,\nu_2}} |D_{k_1}D_{k_2}\big(T_{\tilde{D}}(t)\big)(x_1,x_2)|^2 \\
&\lesssim \sum_{k_1'=0}^{\infty}\sum_{k_2'=0}^{\infty}\sum_{\tau_1'\in I_{k_1'}}\sum_{\tau_2'\in I_{k_2'}}\sum_{\nu_1'=1}^{N(k_1',\tau_1')} \\
&\cdot \sum_{\nu_2'=1}^{N(k_2',\tau_2')} 2^{-|k_1-k_1'|\varepsilon'}2^{-|k_2-k_2'|\varepsilon'}\mu(\mathcal{Q}_{\tau_1'}^{k_1',\nu_1'})\mu(\mathcal{Q}_{\tau_2'}^{k_2',\nu_2'}) \\
&\times \frac{2^{-(k_1\wedge k_1')\varepsilon'}}{(2^{-(k_1\wedge k_1')}+\rho(y_1,y_1'))^{1+\varepsilon'}}\frac{2^{-(k_2\wedge k_2')\varepsilon'}}{(2^{-(k_2\wedge k_2')}+\rho(y_2,y_2'))^{1+\varepsilon'}} \\
&\times \big|t_{\mathcal{Q}_{\tau_1'}^{k_1',\nu_1'}\times\mathcal{Q}_{\tau_2'}^{k_2',\nu_2'}}\mu(\mathcal{Q}_{\tau_1'}^{k_1',\nu_1'})^{-1/2}\mu(\mathcal{Q}_{\tau_2'}^{k_2',\nu_2'})^{-1/2}\big|^2,
\end{aligned} \tag{3.22}$$

where y_i is the center of $\mathcal{Q}_{\tau_i}^{k_i,\nu_i}$ and y_i' is the center of $\mathcal{Q}_{\tau_i'}^{k_i',\nu_i'}$ for $i = 1, 2$.

Comparing (3.22) with (3.7), we can that the only thing different is that the last term in the right-hand side of (3.7) is $T_{R'}$, while the last term in the right-hand side of (3.22) is $\big|t_{\mathcal{Q}_{\tau_1'}^{k_1',\nu_1'}\times\mathcal{Q}_{\tau_2'}^{k_2',\nu_2'}} \cdot \mu(\mathcal{Q}_{\tau_1'}^{k_1',\nu_1'})^{-1/2}\mu(\mathcal{Q}_{\tau_2'}^{k_2',\nu_2'})^{-1/2}\big|^2$. However, when proving the Theorem 3.4, we can see the term $T_{R'}$ is fixed throughout the whole proof. This implies that we can prove this proposition just following the proof of Theorem 3.4 without any changes.

Thus, we can obtain that

$$\|T_{\tilde{D}}(t)\|_{\mathrm{CMO}^p(M)} \lesssim \sup_{\Omega} \Bigg(\frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{\tau_1 \in I_{k_1}} \sum_{\tau_2 \in I_{k_2}} \sum_{\nu_1=1}^{N(k_1,\tau_1)} \cdot \sum_{\nu_2=1}^{N(k_2,\tau_2)} \chi_{\mathcal{Q}_{\tau_1}^{k_1,\nu_1} \times \mathcal{Q}_{\tau_2}^{k_2,\nu_2} \subset \Omega}(k_1,k_2,\tau_1,\tau_2,\nu_1,\nu_2)$$

$$\times \mu(\mathcal{Q}_{\tau_1}^{k_1,\nu_1})\mu(\mathcal{Q}_{\tau_1}^{k_1,\nu_1}) \Big| t_{\mathcal{Q}_{\tau_1'}^{k_1',\nu_1'} \times \mathcal{Q}_{\tau_2'}^{k_2',\nu_2'}} \mu(\mathcal{Q}_{\tau_1'}^{k_1',\nu_1'})^{-1/2} \mu(\mathcal{Q}_{\tau_2'}^{k_2',\nu_2'})^{-1/2} \Big|^2 \Bigg)^{1/2}$$

$$\lesssim \|t\|_{c^p}.$$

Finally, we can easily get that form the Calderón reproducing formula $T_{\tilde{D}} \circ S_D$ is the identity operator on CMO^p. We finish the proof of the proposition. □

3.6 The Boundedness of singular integral operator on M

Before we give the proof of the theorem, let's review some known results stated in [68].

Definition 3.17. We say $T : C^\infty(M) \to C^\infty(M)$ is a product singular integral operator of order $t = (t_1, \dots, t_\nu) \in (-Q_1^1, \infty) \times \cdots \times (-Q_1^\nu, \infty) \subseteq \mathbb{R}^\nu$ if

(i) (Growth Condition) For each ordered multi-indices α, β,

$$|X_x^\alpha X_z^\beta T(x,z)| \le C_{\alpha,\beta} \frac{\rho(x,z)^{-t-\deg(\alpha)-\deg(\beta)}}{\mathrm{Vol}(B(x,\rho(x,z)))},$$

where X_x denotes the list of vector fields $X_1, \dots, X_q$ thought of as partial differential operators in the x variable and similarly for X_z. In particular, the above implies that the distribution $T(x,z)$ corresponds with a C^∞ function on the set $x_1 \ne z_1, \dots, x_\nu \ne z_\nu$.

(ii) (Cancellation Condition) For each bounded set of bump function $\mathcal{B} \subset C^\infty(M) \times M \times$

$(0,1]^\nu$ and each ordered multi-index α,

$$\sup_{(\phi,z,\delta)\in\mathcal{B}} \sup_{x\in M} \delta^{t+\deg(\alpha)} \mathrm{Vol}(B(z,\delta))|X^\alpha T\phi(x)| \le C_{\mathcal{B},\alpha},$$

with the same estimates for T^* in place of T. Here, the formal adjoint T^* is taken in the sense of $L^2(M)$ which is defined in terms of the chosen strictly positive, smooth measure.

If we consider topology on the space of product singular integral operators, we can have the following equivalent definition of these operators.

Definition 3.18. When $\nu = 0$, we define the space of product singular integral operators to be C with the usual topology. For $\nu \ge 1$ the space of product singular integral operators of order $t = (t_1, \dots, t_\nu) \in (-Q_1^1, \infty) \times \cdots \times (-Q_1^\nu, \infty) \subseteq \mathbb{R}^\nu$ if

(i) (Growth Condition) For each ordered multi-indices α, β,

$$|X_x^\alpha X_z^\beta T(x,z)| \le C_{\alpha,\beta} \frac{\rho(x,z)^{-t-\deg(\alpha)-\deg(\beta)}}{\mathrm{Vol}(B(x,\rho(x,z)))},$$

In particular we assume $T((x_1,\dots,x_\nu),(z_1,\dots,z_\nu))$ agree with a C^∞ function on the set $x_1 \neq z_1, \dots, x_\nu \neq z_\nu$.

(ii) (Cancellation Condition) For each ν, $1 \le \mu \le \nu$, we assume that following holds. For every bounded set of bump functions $\mathcal{B}_\mu$ and M_μ, we have the following. For every $x_\mu \in M_\mu$, $(\phi_\mu, z_\mu, \delta_\mu) \in \mathcal{B}_\mu$, we define the function $x_\mu \mapsto T^{\phi_\mu, x_\mu}$, $M_\mu \to C^\infty(M_1 \times \cdots \times M_{\mu-1} \times M_{\mu+1} \times \cdots \times M_\nu)'$ by

$$\begin{aligned} &\langle T(\phi_1 \otimes \cdots \otimes \phi_\nu), \psi_1 \otimes \cdots \otimes \psi_\nu \rangle \\ = &\int_{M_\mu} \langle T^{\phi_\mu, x_\mu}(\otimes_{\mu' \neq \mu} \phi_{\mu'}), \otimes_{\mu' \neq \mu} \psi_{\mu'} \rangle \psi_\mu(x_\mu) dx_\mu \end{aligned}$$

T^{ϕ_μ, x_μ} is a priori only defined as a distribution in the x_ν variable, but we assume it to agree with a C^∞ function in that variable. Furthermore, we assume that for every ordered multi-index

α, the operator

$$\mathrm{Vol}\big(B_{(X^\mu,\hat{d}^\mu)}(x_\mu,\delta_\mu)\big)\delta_\mu^{t_\mu}(\delta_\mu X^\mu_{x_\mu})^\alpha T^{\phi_\mu,x_\mu}:\ C^\infty\big(\prod_{\mu'\neq\mu} M_{\mu'}\big)\to C^\infty\big(\prod_{\mu'\neq\mu} M_{\mu'}\big)$$

is a product singular integral operator of order $(t_1,\dots t_{\mu-1},t_{\mu+1},\dots,t_\nu)$ on the $\nu-1$ factor space $M_1\times\cdots\times M_{\mu-1}\times M_{\mu+1}\times\cdots\times M_\nu$. Finally for every continuous semi-norm, $\|\cdot\|$, for product kernels of order $(t_1,\dots t_{\mu-1},t_{\mu+1},\dots,t_\nu)$ on $M_1\times\cdots\times M_{\mu-1}\times M_{\mu+1}\times\cdots\times M_\nu$, every ordered multi-index α, and every bounded set of bump functions $\mathcal{B}_\mu$ on M_μ, we define a semi-norm $\|\cdot\|_{\alpha,\mathcal{B}_\mu}$, on product singular integrals of order t by

$$\|T\|_{\alpha,\mathcal{B}_\mu}:=\sup_{(\phi_\mu,z_\mu,x_\mu)\in\mathcal{B}_\mu}\left|\mathrm{Vol}\big(B_{(X^\mu,\hat{d}^\mu)}(x_\mu,\delta_\mu)\big)\delta_\mu^{t_\mu}(\delta_\mu X^\mu_{x_\mu})^\alpha T^{\phi_\mu,x_\mu}\right|$$

which we assume to be finite. We do the same for the transpose of T in the μ variable, where we define $z_\mu\mapsto T^{z_\mu,\psi_\mu}$ reversing the roles of x_μ,z_μ and ϕ_μ,ψ_μ; thereby obtaining another semi-norm.

Definition 3.19. We say $\mathcal{E}\subset C^\infty(M\times M)\times(0,1]^\nu$ is a bounded set of pre-elementary operators if: $\forall\alpha,\beta,m$, $\exists C=C(\mathcal{E},\alpha,\beta,m)$, $\forall(E,2^{-j})\in\mathcal{E}$,

$$\left|(2^{-j}X_x)^\alpha(2^{-j}X_z)^\beta E(x,z)\right|\le C\frac{\left(1+2^j\rho(x,z)\right)^{-m}}{\mathrm{Vol}(B_X(x,2^{-j}+\rho(x,z)))}.$$

Definition 3.20. We define the set of bounded sets of elementary operators, $\mathcal{G}$, to be the largest set of subsets of $C^\infty(M\times M)\times(0,1)^\nu$ such that for all $\mathcal{E}\in\mathcal{G}$,

(i) $\mathcal{E}$ is a bounded set of pre-elementary operators.

(ii) Let $e=(1,\dots,1)\in\mathbb{N}^\nu$. We write $\deg(\alpha)\le e$ to denote the inequality holding coordinate-wise. We assume $\forall\,(E,2^{-j})\in\mathcal{E}$,

$$E=\sum_{\deg(\alpha),\deg(\beta)\le e}2^{-(2e-\deg(\alpha)-\deg(\beta))j}\left(2^{-j}X\right)^\alpha E_{\alpha,\beta}\left(2^{-j}X\right)^\beta,$$

where $\{(E_{\alpha,\beta}, 2^{-j}) \,|\, (E, 2^{-j}) \in \mathcal{E}\} \in \mathcal{G}$.

We call elements $\mathcal{E} \in \mathcal{G}$ bounded set of elementary operators.

Theorem 3.21 ([68]). *Fix $t \in (-Q_1^1, \infty) \times \cdots \times (-Q_1^\nu, \infty)$, and let $T : C^\infty(M) \to C^\infty(M)$. The following are equivalent:*

(i) T is a product singular integral operator of order t as in Definition 3.17

(ii) T is a product singular integral operator of order t as in Definition 3.18

(iii) For every bounded set of elementary operator $\mathcal{E}$,

$$\{(2^{-j\cdot t}TE, 2^{-j})|(E, 2^{-j}) \in \mathcal{E}\}$$

is a bounded set of elementary operators.

(iv) There is a bounded set of elementary operators $\{(E_j, 2^{-j})|j \in \mathbb{N}^\nu\}$ such that $T = \sum_{j\in\mathbb{N}^\nu} 2^{j\cdot t}E_j$. (Every such sum converges in the topology of bounded convergence as operators $C^\infty(M) \to C^\infty(M)$; this can be seen just as in Lemma 2.0.28.)

Furthermore, (iii) and (iv) are equivalent for any $t \in \mathbb{R}^\nu$.

Lemma 3.1 ([68]). *For each μ, let $\mathcal{E}_\mu \subset C^\infty(M_\mu \times M_\mu) \times (0, 1]^\nu$ be a bounded set of elementary operators as in the single parameter case. Then, the set*

$$\{(E_1 \otimes \cdots \otimes E_\nu, (2^{-j_1}, \ldots, 2^{-j_\nu}))|(E_1, 2^{j_1}) \in \mathcal{E}_1, \ldots, (E_\nu, 2^{j_\mu}) \in \mathcal{E}_\nu\}$$

is a bounded set of elementary operators as in Definition 3.20

Corollary 3.22 ([68]). There is a bounded set of elementary operators

$$\{(E_j, 2^{-j})|j \in \mathbb{N}^\nu\}$$

such that $I = \sum_{j\in\mathbb{N}^\nu} E_j$, where $I : C^\infty(M) \to C^\infty(M)$ is the identity operators.

Proposition 3.4. *Let $\mathcal{E}$ be a bounded set of elementary operators. Then, for every N, the set*

$$\Big\{\big(2^{N|j_1-j_2|}E_1E_2, 2^{-j_1}\big), \big(2^{N|j_1-j_2|}E_1E_2, 2^{-j_2}\big), \big(2^{N|j_1-j_2|}E_1E_2, 2^{-j_1\wedge j_2}\big)|$$
$$(E_1, 2^{-j_1}), (E_2, 2^{-j_2}) \in \mathcal{E}\Big\}$$

is a bounded set of elementary operators.

The following results in [68] are related to the multi-parameter singular operators and the pseudo-differential operators. Now for each point in M, we need to work on a small neighborhood of this point, so that one can apply the Frobenius Theorem (see [68] for details).

Definition 3.23. Fix $s \in \mathbb{R}^\nu$ and $T : C^\infty(M) \to C^\infty(M)$. We say $T \in \mathcal{A}^s$ if for each $j \in \mathbb{N}^\nu$ there is $E_j \in C_0^\infty(M \times M)$ such that $\{(E_j, 2^{-j})|j \in \mathbb{N}^\nu\}$ is a bounded set of elementary operators and

$$T = \sum_{j\in\mathbb{N}^\nu} 2^{j\cdot s}E_j,$$

where the sum taken in the sense of distribution. We will (Remark 5.3.3) that every such sum converges in the sense of distribution. In fact, every such sum converges in the topology of bounded convergence as operators $C^\infty(M) \to C^\infty(M)$.

Definition 3.24. For a distribution $K \in C_0^\infty(\mathbb{R}^q)'$, s $\in \mathbb{R}^\nu$, and $a > 0$, we say $K \in \mathrm{PK}(s, a)$ if there is $\eta \in C_0^\infty(B^q(a))$ and a bounded set $\{\varsigma_j|j \in \mathbb{N}^\nu\} \subset (\mathbb{S}^q)$ with $\varsigma_j \in \mathcal{S}_0^{\mu|j_\mu\neq 0}$ such that

$$K = \eta\sum_{j\in\mathbb{N}^\nu} \varsigma_j^{(2^j)}.$$

Proposition 3.5. *Let $K \in C^\infty(M \times \mathbb{R}^q)'$ be supported in $M \times B^q(a)$ and let $m \in (-Q_1, \infty) \times \cdots \times (-Q_\nu, \infty)$. If K is a product kernel of order m, then $K \in C^\infty(M)\widehat{\otimes}\mathrm{PK}(m, a)$.*

Definition 3.25. Let $a > 0$ be a small number to be chosen later. We say $T : C^\infty(M) \to C^\infty(M)$ is a pseudo-differential operator of order $m \in \mathbb{R}^\nu$ if there is $K \in C^\infty(M)\widehat{\otimes}\mathrm{PK}(m, a)$

such that

$$Tf(x) = \int f(\gamma(x,t))K(x,t)\,dt,$$

γ is given by either of the following formulas:

$$\gamma(x, t_1, \ldots, t_\nu) = e^{t_1 \cdot X^1} \cdots e^{t_\nu \cdot X^\nu} x,$$

$$\gamma(x,t) = e^{t\cdot X}x = e^{t_1 \cdot X^1 + \cdots + t_\nu \cdot X^\nu} x.$$

where X^ν denotes the list of vector fields $X_1^\mu, \cdots, X_{q_\mu}^\mu$ and X denotes the list of vector fields $X_1, \ldots, X_q$.

Theorem 3.26 ([68]). *If $a > 0$ is sufficiently small, and if T is a pseudo-differential operator of order $s \in \mathbb{R}^\nu$, then $T \in \mathcal{A}^s$.*

Theorem 3.27 ([68]). *If $T \in \mathcal{A}^s$, then T is a product singular integral operator of order s in the sense of Definition 3.21.*

And from the above, if we can prove the product singular operator T satisfies the H^p boundedness, then the boundedness of $T \in \mathcal{A}^s$ follows immediately. In other words, the multi-parameter pseudo-differential operator defined above also the H^p boundedness. Hence, let's prove the H^p boundedness of the product singular operator T right now. And for the simplicity, we still consider the two parameter cases. The multi-parameter cases proof follows the similar steps. To achieve this target, we also need the next proposition.

Proposition 3.6. *Given two bounded set of elementary operator $\mathcal{E}_1$ and $\mathcal{E}_2$ on M, $\forall m, N$, $\exists C = C(m, N, \mathcal{E}_1, \mathcal{E}_2)$, s.t. $\forall (D_j.2^{-j}) \in \mathcal{E}_1$ and $(D_k, 2^{-k}) \in \mathcal{E}_2$, we have*

$$|D_1^j D_2^j T D_1^k D_2^k(x,y)| \leq C 2^{-N|j-k|} \frac{(1 + 2^{j\wedge k}\rho(x,y))^{-m}}{\mathrm{Vol}(B_{(X,d)}(x, 2^{-j\wedge k}(1 + 2^{j\wedge k}\rho(x,y)))}$$

where $D_j = D_1^j \otimes D_2^j$ and $D_k = D_1^k \otimes D_2^k$.

We assume the proposition for the moment and now show the H^p boundedness of T as follows.

Theorem 3.28. *For $0 < p \leq 1$ and $s = 0$, we have*

$$\|Tf\|_{H^p(M)} \leq \|f\|_{H^p(M)}$$

where T is defined as Theorem 3.21.

Proof. For $f \in L^2 \cap H^p$, we have

$$\|Tf\|_{H^p} \lesssim \Bigg\| \{ \sum_{k_1\in\mathbb{N}} \sum_{k_2\in\mathbb{N}} \sum_{\tau_1\in I_{k_1}} \sum_{\tau_2\in I_{k_2}} \sum_{\nu_1=1}^{N(k_1,\tau_1)} \sum_{\nu_2=1}^{N(k_2,\tau_2)}$$
$$|D_{k_1}D_{k_2}(Tf)(y_{\tau_1}^{k_1,\nu_1}, y_{\tau_2}^{k_2,\nu_2})|^2 \chi_{\mathcal{Q}_{\tau_1}^{k_1,\nu_1}}(\cdot)\chi_{\mathcal{Q}_{\tau_2}^{k_2,\nu_2}}(\cdot)\}^{\frac{1}{2}} \Bigg\|_{L^p}$$

Applying the L^2 boundedness of T and the discrete Calderón reproducing formula,

$$\lesssim \Bigg\| \{ \sum_{k_1\in\mathbb{N}} \sum_{k_2\in\mathbb{N}} \sum_{\tau_1\in I_{k_1}} \sum_{\tau_2\in I_{k_2}} \sum_{\nu_1=1}^{N(k_1,\tau_1)} \sum_{\nu_2=1}^{N(k_2,\tau_2)} |D_{k_1}D_{k_2}\big(T \sum_{k_1'\in\mathbb{N}} \sum_{k_2'\in\mathbb{N}}$$
$$\sum_{\tau_1'\in I_{k_1'}} \sum_{\tau_2'\in I_{k_2'}} \sum_{\nu_1'=1}^{N(k_1',\tau_1')} \sum_{\nu_2'=1}^{N(k_2',\tau_2')} \mu_1(\mathcal{Q}_{\tau_1'}^{k_1',\nu_1'})\mu_2(\mathcal{Q}_{\tau_2'}^{k_2',\nu_2'})D_{k_1'}D_{k_2'}$$
$$\overline{D}_{k_1'}\overline{D}_{k_2'}(f)(y_{\tau_1'}^{k_1',\nu_1'}, y_{\tau_2'}^{k_2',\nu_2'})\big)(y_{\tau_1}^{k_1,\nu_1}, y_{\tau_2}^{k_2,\nu_2})|^2 \chi_{\mathcal{Q}_{\tau_1}^{k_1,\nu_1}}(\cdot)\chi_{\mathcal{Q}_{\tau_2}^{k_2,\nu_2}}(\cdot)\}^{\frac{1}{2}} \Bigg\|_{L^p(M)}$$

According to the above proposition and the similar procedure while proving the Plancherel-Pôlya inequality, we can obtain

$$\lesssim \Bigg\| \{ \sum_{k_1'\in\mathbb{N}} \sum_{k_2'\in\mathbb{N}} \sum_{\tau_1'\in I_{k_1'}} \sum_{\tau_2'\in I_{k_2'}} \sum_{\nu_1'=1}^{N(k_1',\tau_1')} \sum_{\nu_2'=1}^{N(k_2',\tau_2')} |\overline{D}_{k_1'}\overline{D}_{k_2'}(f)(y_{\tau_1'}^{k_1',\nu_1'}, y_{\tau_2'}^{k_2',\nu_2'})|^2$$
$$\chi_{\mathcal{Q}_{\tau_1'}^{k_1',\nu_1'}}(\cdot)\chi_{\mathcal{Q}_{\tau_2'}^{k_2',\nu_2'}}(\cdot)\}^{\frac{1}{2}} \Bigg\|_{L^p} \lesssim \|f\|_{H^p(M)}$$

$\square$

We now return to the proof of Proposition.

Proof of Proposition. Note that T is the product singular operator of order 0. Hence, by the second equivalent definition, $\{(TD_k, 2^{-k})|D_k \in \mathcal{E}_2\}$ is also a bounded set of elementary operators. Furthermore, note that for every N, the set

$$\{(2^{N|j-k|}D_jTD_k, 2^{-j\wedge k}), (2^{N|j-k|}TD_kD_j, 2^{-j\wedge k})|(D_j.2^{-j}) \in \mathcal{E}_1, (D_k, 2^{-k}) \in \mathcal{E}_2\}$$

is a bounded set of pre-elementary operators.

Therefore, by the definition of pre-elementary operators, we have

$$\left|(2^{-j\wedge k}X_x)^\alpha (2^{-j\wedge k}X_y)^\beta (2^{N|j-k|}D_jTD_k)(x,y)\right| \lesssim \frac{\left(1+2^{j\wedge k}\rho(x,y)\right)^{-m}}{\mathrm{Vol}\left(B_{(X,d)}(x, 2^{-j\wedge k}+\rho(x,y))\right)}$$

or equivalently,

$$\left|(2^{-j\wedge k}X_x)^\alpha (2^{-j\wedge k}X_y)^\beta (D_jTD_k)(x,y)\right| \lesssim 2^{-N|j-k|}\frac{\left(1+2^{j\wedge k}\rho(x,y)\right)^{-m}}{\mathrm{Vol}\left(B_{(X,d)}(x, 2^{-j\wedge k}+\rho(x,y))\right)}$$

$\square$

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ABSTRACT

HARDY SPACE THEORY AND ENDPOINT ESTIMATES FOR MULTI-PARAMETER SINGULAR RADON TRANSFORMS

by

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Advisor: Dr. Guozhen Lu

Major: Mathematics

Degree: Doctor of Philosophy

In [12], Christ, Nagel, Stein and Waigner studied the L^p theories for the singular Radon Transforms. Furthermore, B. Street in [68], and Stein and Street in [64–67] extended the theories of the L^p boundedness for multi-parameter singular integral operators, such as the Calderón Zygmund operators and singular Radon transforms. In this dissertation, we will study the Hardy space H^p and its dual space associated with both the one-parameter and multi-parameter singular Radon transforms, and consider the boundedness of the singular Radon transforms on such Hardy spaces H^p when $0 \leq p \leq 1$.

Inspired by recent characterization of the Hardy spaces on product spaces, we will take advantage of the discrete Littlewood-Paley analysis [14, 32, 43] to define the Hardy spaces H^p and the Carleson measure spaces CMO^p associated with the multi-parameter singular Radon transforms. Moreover, we will prove the H^p boundedness of those operators and thus obtain the endpoint estimates for the L^p boundedness of the singular Radon transforms by Christ, Nagel, Wainger and Stein [12] and for multi-parameter singular Radon transforms by Street and Stein [65–68].

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3. *The Characterization of Sobolev spaces in the metric spaces,* preprint (with G. Lu).