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A New Two-Parametric ‘Useful’ Fuzzy Information Measure and its Properties

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A ‘useful’ fuzzy measure of order α and type β is developed. Its validity established with a numerical example.

Keywords: Shannon's entropy, fuzzy set, fuzzy entropy, ‘useful’ information measure

Introduction

Zadeh (1965) presented fuzzy set theory. The degree of fuzziness in a fuzzy set is measured by using the concept of entropy. Ebanks (1983) and Pal and Bezdek (1994) called it fuzzy entropy, which is an important concept for measuring fuzzy information. It has a vital role in fuzzy systems such as neural networks, pattern recognition, decision making, knowledge base, communication, etc. This led to further developments, such as Kaufmann (1975), Pal and Pal (1989), Parkash and Sharma (2002, 2004), Bhat and Baig (2016a, b), Bhat, Baig, and Salam (2016), and Bhat, Bhat, et al. (2017).

Let $X = \{x_1, x_2,\ldots, x_n\}$ be a universal set defined in the universe of discourse. A fuzzy subset ‘A’ in ‘X’ is defined as $A = \{(x_i, \mu_A(x_i)): x_i \in X, \mu_A(x_i) \in [0, 1]\}$ where $\mu_A(x_i)$ is a membership function which is defined as

$$\mu_A(x_i) = \begin{cases} 0 & \text{if } x \notin A \text{ and there is no ambiguity,} \\ 0.5 & \text{if there is maximum ambiguity whether } x \in A \text{ or } x \notin A, \\ 1 & \text{if } x \in A \text{ and there is no ambiguity} \end{cases}$$

Some important concepts related to fuzzy sets are given below:

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- Sum of A and B ($A + B$) is given as

$$\mu_{A+B}(x_i) = \mu_A(x_i) + \mu_B(x_i) - \mu_A(x_i)\mu_B(x_i), \quad \forall x_i \in X \;;$$

- Product of A and B (AB) is given as

$$\mu_{AB}(x_i) = \mu_A(x_i)\mu_B(x_i), \quad \forall x_i \in X \;;$$

- Equality of A and B ($A = B$) is given as

$$\mu_A(x_i) = \mu_B(x_i), \quad \forall x_i \in X \;;$$

- Containment of A and B ($A \subset B$) is given as

$$\mu_A(x_i) \leq \mu_B(x_i), \quad \forall x_i \in X \;;$$

- Complement of A (A') is defined as

$$\mu_{A'}(x_i) = 1 - \mu_A(x_i), \quad \forall x_i \in X \;;$$

- Union of A and B ($A \cup B$) is defined as

$$\mu_{A\cup B}(x_i) = \mathrm{Max}\{\mu_A(x_i), \mu_B(x_i)\}, \quad \forall x_i \in X \;;$$

- Intersection of A and B ($A \cap B$) is defined as:

$$\mu_{A\cap B}(x_i) = \mathrm{Min}\{\mu_A(x_i), \mu_B(x_i)\}, \quad \forall x_i \in X$$

where A and B are two fuzzy subsets of X with membership functions $\mu_A(x_i)$ and $\mu_B(x_i)$, respectively.

Shannon’s Entropy

Let $X = (x_1, x_2, \ldots, x_n)$ be a discrete random variable with probability distribution $P = (p_1, p_2, \ldots, p_n)$ such that $p_i \geq 0\ \forall\ i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} p_i = 1$. Then the Shannon’s information measure, called entropy, is defined as (Shannon, 1948)

$$\mathrm{H}(P) = -\sum_{i=1}^{n} p_i \log_D p_i \,. \tag{1}$$

Corresponding to Shannon’s measure of entropy, De Luca and Termini (1972) gave a measure of fuzzy entropy given as

$$\mathrm{H}(A) = -\sum_{i=1}^{n} \left[\mu_A(x_i) \log \mu_A(x_i) + \left(1 - \mu_A(x_i)\right) \log\left(1 - \mu_A(x_i)\right) \right]. \tag{2}$$

The fuzzy entropy measure should satisfy the following four properties, given by De Luca and Termini (1972):

1. Sharpness: H(A) is minimum if and only if A is a crisp set.
2. Maximality: H(A) is maximum if and only if A is most fuzzy set.
3. Resolution: H(A) ≥ H(A^*), where A^* is sharpened version of A.
4. Symmetry: H(A) = H(A'), where A' is the complement of A.

‘Useful’ Fuzzy Information Measure

Let $U = (u_1, u_2, \ldots, u_n)$ be a set of non-negative numbers such that $u_i > 0$ and u_i represents the utility of the occurrence of element x_i. In general, utility is independent of probability p_i. The information scheme given by

$$\mathbf{U} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ p_1 & p_2 & \cdots & p_n \\ u_1 & u_2 & \cdots & u_n \end{bmatrix}; \quad u_i > 0, p_i \geq 0 \,\&\, \sum_{i=1}^{n} p_i = 1 \tag{3}$$

is called as utility information scheme. Corresponding to the scheme (3), Belis and Guiasu (1968) gave the following measure of information:

$$\mathrm{H}\left(P;\mathbf{U}\right)=-\sum_{i=1}^{n}u_i\,p_i\log_D\,p_i\ . \tag{4}$$

The measure defined in (4) is called ‘useful’ entropy. This measure can be taken as a satisfactory measure for the average quantity of ‘useful’ information provided by the information scheme (3).

For any fuzzy set A, the ‘useful’ fuzzy entropy is defined as

$$\mathrm{H}\left(A;\mathbf{U}\right)=-\sum_{i=1}^{n}u_i\left\{\mu_A\left(x_i\right)\log_D\mu_A\left(x_i\right)+\left(1-\mu_A\left(x_i\right)\right)\log_D\left(1-\mu_A\left(x_i\right)\right)\right\}. \tag{5}$$

Proposed ‘Useful’ Fuzzy Information Measure and Its Properties

The proposed ‘useful’ fuzzy information measure is

$$\mathrm{H}_{\alpha}^{\beta}\left(A;\mathbf{U}\right)=\frac{\beta}{1-\alpha}\log_D\left[\frac{\sum_{i=1}^{n}u_i\left\{\mu_A^{\beta(1-\alpha)}\left(x_i\right)+\left(1-\mu_A\left(x_i\right)\right)^{\beta(1-\alpha)}\right\}}{\sum_{i=1}^{n}u_i}\right]; \tag{6}$$

$$0<\alpha<1, 0<\beta\le 1, \beta>\alpha, u_i>0$$

For (6) to be a valid ‘useful’ fuzzy information measure, it should satisfy the four properties given by De Luca and Termini (1972).

Sharpness. $\mathrm{H}_{\alpha}^{\beta}\left(A;\mathbf{U}\right)$ is minimum if and only if A is a crisp set i.e., $\mathrm{H}_{\alpha}^{\beta}\left(A;\mathbf{U}\right)=0$ iff $\mu_A(x_i)$ = 0 or 1 $\forall$ i = 1, 2,…, n.

Proof. Suppose $\mathrm{H}_{\alpha}^{\beta}\left(A;\mathbf{U}\right)=0$, i.e.,

$$\frac{\beta}{1-\alpha}\log_D\left[\frac{\sum_{i=1}^{n}u_i\left\{\mu_A^{\beta(1-\alpha)}\left(x_i\right)+\left(1-\mu_A\left(x_i\right)\right)^{\beta(1-\alpha)}\right\}}{\sum_{i=1}^{n}u_i}\right]=0$$

$$\Rightarrow \log_D\left[\frac{\sum_{i=1}^n u_i\left\{\mu_A^{\beta(1-\alpha)}(x_i)+(1-\mu_A(x_i))^{\beta(1-\alpha)}\right\}}{\sum_{i=1}^n u_i}\right]=0$$

$$\Rightarrow \sum_{i=1}^n u_i\left\{\mu_A^{\beta(1-\alpha)}(x_i)+(1-\mu_A(x_i))^{\beta(1-\alpha)}\right\}=\sum_{i=1}^n u_i \tag{7}$$

Because $0 < \alpha < 1$, $0 < \beta \leq 1$, and $u_i > 0$, (7) will hold when either $\mu_A(x_i) = 1$ or $\mu_A(x_i) = 0\ \forall\ i = 1, 2,\ldots, n$.

Conversely, suppose

$$\log_D\left[\frac{\sum_{i=1}^n u_i\left\{\mu_A^{\beta(1-\alpha)}(x_i)+(1-\mu_A(x_i))^{\beta(1-\alpha)}\right\}}{\sum_{i=1}^n u_i}\right]=0\,. \tag{8}$$

Multiplying both sides of equation (8) by $\beta / (1 - \alpha)$,

$$\frac{\beta}{1-\alpha}\log_D\left[\frac{\sum_{i=1}^n u_i\left\{\mu_A^{\beta(1-\alpha)}(x_i)+(1-\mu_A(x_i))^{\beta(1-\alpha)}\right\}}{\sum_{i=1}^n u_i}\right]=0$$

$$\Rightarrow \mathrm{H}_\alpha^\beta(A;\mathbf{U})=0$$

Hence, $\mathrm{H}_\alpha^\beta(A;\mathbf{U})=0$ if and only if A is a crisp set.

Maximality. $\mathrm{H}_\alpha^\beta(A;\mathbf{U})$ is maximum if and only if A is most fuzzy set.

Proof. We have

$$\mathrm{H}_\alpha^\beta(A;\mathbf{U})=\frac{\beta}{1-\alpha}\log_D\left[\frac{\sum_{i=1}^n u_i\left\{\mu_A^{\beta(1-\alpha)}(x_i)+(1-\mu_A(x_i))^{\beta(1-\alpha)}\right\}}{\sum_{i=1}^n u_i}\right]; \tag{9}$$

$$0<\alpha<1, 0<\beta\leq 1, \beta>\alpha, u_i>0$$

Now, differentiating equation (9) with respect to $\mu_A(x_i)$,

$$\frac{\partial \mathrm{H}_{\alpha}^{\beta}(A;\mathbf{U})}{\partial \mu_A(x_i)} = \beta^2 \left[\frac{u_i \left\{ \mu_A^{\beta(1-\alpha)-1}(x_i) - (1-\mu_A(x_i))^{\beta(1-\alpha)-1} \right\}}{\sum_{i=1}^{n} u_i \left\{ \mu_A^{\beta(1-\alpha)}(x_i) + (1-\mu_A(x_i))^{\beta(1-\alpha)} \right\}} \right].$$

Let $0 \le \mu_A(x_i) < 0.5$; then

$$\frac{\partial \mathrm{H}_{\alpha}^{\beta}(A;\mathbf{U})}{\partial \mu_A(x_i)} > 0; \quad 0 < \alpha < 1, 0 < \beta \le 1, \beta > \alpha, u_i > 0\,.$$

Hence, $\mathrm{H}_{\alpha}^{\beta}(A;\mathbf{U})$ is an increasing function of $\mu_A(x_i)$ whenever $0 \le \mu_A(x_i) < 0.5$.

Similarly, for $0.5 < \mu_A(x_i) \le 1$,

$$\frac{\partial \mathrm{H}_{\alpha}^{\beta}(A;\mathbf{U})}{\partial \mu_A(x_i)} < 0; \quad 0 < \alpha < 1, 0 < \beta \le 1, \beta > \alpha, u_i > 0\,.$$

Hence, $\mathrm{H}_{\alpha}^{\beta}(A;\mathbf{U})$ is a decreasing function of $\mu_A(x_i)$ whenever $0.5 < \mu_A(x_i) \le 1$, and for $\mu_A(x_i) = 0.5$,

$$\frac{\partial \mathrm{H}_{\alpha}^{\beta}(A;\mathbf{U})}{\partial \mu_A(x_i)} = 0; \quad 0 < \alpha < 1, 0 < \beta \le 1, \beta > \alpha, u_i > 0\,.$$

Thus, $\mathrm{H}_{\alpha}^{\beta}(A;\mathbf{U})$ is a concave function which has a global maximum at $\mu_A(x_i) = 0.5$. This implies $\mathrm{H}_{\alpha}^{\beta}(A;\mathbf{U})$ is maximum iff A is most fuzzy set, that is, $\mu_A(x_i) = 0.5$ $\forall\, i = 1, 2, \ldots, n$.

Resolution. $\mathrm{H}_{\alpha}^{\beta}(A;\mathbf{U}) \ge \mathrm{H}_{\alpha}^{\beta}(A^*;\mathbf{U})$, where A^* is sharpened version of A.

Proof. : Because $\mathrm{H}_{\alpha}^{\beta}(A;\mathbf{U})$ is an increasing function of $\mu_A(x_i)$ whenever $0 \le \mu_A(x_i) < 0.5$ and is a decreasing function of $\mu_A(x_i)$ whenever $0.5 < \mu_A(x_i) \le 1$,

$$\mu_{A^*}(x_i) \le \mu_A(x_i) \Rightarrow \mathrm{H}_\alpha^\beta(A;\mathbf{U}) \ge \mathrm{H}_\alpha^\beta(A^*;\mathbf{U}) \text{ in } [0, 0.5) \tag{10}$$

Also,

$$\mu_{A^*}(x_i) \ge \mu_A(x_i) \Rightarrow \mathrm{H}_\alpha^\beta(A;\mathbf{U}) \ge \mathrm{H}_\alpha^\beta(A^*;\mathbf{U}) \text{ in } (0.5, 1] \tag{11}$$

Taking equation (10) and (11) together, $\mathrm{H}_\alpha^\beta(A;\mathbf{U}) \ge \mathrm{H}_\alpha^\beta(A^*;\mathbf{U})$.

Symmetry. $\mathrm{H}_\alpha^\beta(A;\mathbf{U}) = \mathrm{H}_\alpha^\beta(A';\mathbf{U})$, where A' is the compliment of A.

Proof. From the definition of $\mathrm{H}_\alpha^\beta(A;\mathbf{U})$ and $\mu_{A'}(x_i) = 1 - \mu_A(x_i)\ \forall x_i \in X$, we conclude that $\mathrm{H}_\alpha^\beta(A;\mathbf{U}) = \mathrm{H}_\alpha^\beta(A';\mathbf{U})$.

Because the proposed measure $\mathrm{H}_\alpha^\beta(A;\mathbf{U})$ satisfies all the four properties of fuzzy information measure, thus it is a valid measure of 'useful' fuzzy information.

Illustration

Sharpness

From Table 1, conclude A is minimum (i.e., $\mathrm{H}_\alpha^\beta(A;\mathbf{U}) = 0$) iff A is a crisp set (i.e., when $\mu_A(x_i) = 0$ or $\mu_A(x_i) = 1$).

Table 1. Behavior of $\mathrm{H}_\alpha^\beta(A;\mathbf{U})$ when $\mu_A(x_i) = 1$ and $\mu_A(x_i) = 0$ with respect to α and β

α	β	u_i	$\mu_A(x_i)$	$\mathrm{H}_\alpha^\beta(A;\mathbf{U})$	$\mu_A(x_i)$	$\mathrm{H}_\alpha^\beta(A;\mathbf{U})$
0.1	0.2	4	1	0	0	0
		3	1	0	0	0
		2	1	0	0	0
		1	1	0	0	0

Maximality

From Table 2, conclude $H_\alpha^\beta(A;\mathbf{U})$ is an increasing function of $\mu_A(x_i)$ (i.e. $\left(\partial H_\alpha^\beta(A;\mathbf{U})/\partial\mu_A(x_i)\right) > 0$) whenever $0 \le \mu_A(x_i) < 0.5$.

From Table 3, conclude $H_\alpha^\beta(A;\mathbf{U})$ is a decreasing function of $\mu_A(x_i)$ (i.e. $\left(\partial H_\alpha^\beta(A;\mathbf{U})/\partial\mu_A(x_i)\right) < 0$) whenever $0.5 < \mu_A(x_i) \le 1$. For $\mu_A(x_i) = 0.5$, $\alpha = 0.1$, and $\beta = 0.2$,

$$\frac{\partial H_\alpha^\beta(A;\mathbf{U})}{\partial\mu_A(x_i)} = 0 \tag{12}$$

Thus, from Tables 2 and 3 and equation (12), conclude $H_\alpha^\beta(A;\mathbf{U})$ is a concave function with global maximum at $\mu_A(x_i) = 0.5$.

Table 2. At $0 \le \mu_A(x_i) < 0.5$ and with respect to α and β

α	β	u_i	$\mu_A(x_i)$	$\frac{\partial H_\alpha^\beta(A;\mathbf{U})}{\partial\mu_A(x_i)}$
0.1	0.2	1	0.00	∞
		2	0.13	0.0202
		3	0.27	0.0118
		4	0.43	0.0039

Table 3. $0.5 < \mu_A(x_i) \le 1$ and with respect to α and β

α	β	u_i	$\mu_A(x_i)$	$\frac{\partial H_\alpha^\beta(A;\mathbf{U})}{\partial\mu_A(x_i)}$
0.1	0.2	1	0.55	–0.000810
		2	0.70	–0.007520
		3	0.85	–0.030181
		4	1.00	–∞

Resolution

From Table 4, conclude $H_\alpha^\beta\left(A^*;\mathbf{U}\right) \le H_\alpha^\beta\left(A;\mathbf{U}\right)$ whenever $\mu_A\left(x_i\right) \ge \mu_{A^*}\left(x_i\right)$ in [0, 0.5).

From Table 5, conclude $H_\alpha^\beta\left(A^*;\mathbf{U}\right) \le H_\alpha^\beta\left(A;\mathbf{U}\right)$ whenever $\mu_A\left(x_i\right) \le \mu_{A^*}\left(x_i\right)$ in (0.5, 1].

Thus, from Tables 4 and 5, conclude $H_\alpha^\beta\left(A^*;\mathbf{U}\right) \le H_\alpha^\beta\left(A;\mathbf{U}\right)$, where A^* is sharpened version of A.

Table 4. At [0, 0.5) and with $\mu_A\left(x_i\right) \ge \mu_{A^*}\left(x_i\right)$

α	β	u_i	$\mu_A(x_i)$	$H_\alpha^\beta\left(A;\mathbf{U}\right)$	$\mu_{A^*}\left(x_i\right)$	$H_\alpha^\beta\left(A^*;\mathbf{U}\right)$
0.2	0.6	1	0.12	0.2539	0.00	0.2306
		2	0.23		0.15	
		3	0.36		0.31	
		4	0.49		0.44	

Table 5. At (0.5, 1] and with $\mu_A\left(x_i\right) \le \mu_{A^*}\left(x_i\right)$

α	β	u_i	$\mu_A(x_i)$	$H_\alpha^\beta\left(A;\mathbf{U}\right)$	$\mu_{A^*}\left(x_i\right)$	$H_\alpha^\beta\left(A^*;\mathbf{U}\right)$
0.2	0.6	1	0.65	0.1823	0.70	0.1226
		2	0.83		0.83	
		3	0.89		0.94	
		4	0.96		1.00	

Symmetry

From Table 6, conclude that $H_\alpha^\beta\left(A;\mathbf{U}\right) = H_\alpha^\beta\left(A';\mathbf{U}\right)$, where A' is the compliment of A.

Table 6. Verification of symmetry property

α	β	u_i	$\mu_A(x_i)$	$H_\alpha^\beta\left(A;\mathbf{U}\right)$	$1-\mu_A(x_i)$	$H_\alpha^\beta\left(A';\mathbf{U}\right)$
0.2	0.6	1	0.65	0.1858	0.35	0.1858
		2	0.78		0.22	
		3	0.89		0.11	
		4	0.96		0.04	

Behavior of Proposed 'Useful' Fuzzy Information Measure of Order *α* and Type *β*

In order to study the behavior of the proposed 'useful' fuzzy information measure, fix β and observe the behavior of $H_\alpha^\beta(A;\mathbf{U})$ at different values of α and vice-versa. Consider the membership function $\mu_A(x_i)$ = {0.11, 0.45, 0.23, 0.65, 0.82, 0.31, 0.72, 0.56, 0.92} with the utilities u_i = {1, 2, 3, 4, 5, 6, 7, 8, 9}.

Table 7. Behavior $H_\alpha^\beta(A;\mathbf{U})$ of at different values of α and $\beta = 1$

α	0.15	0.29	0.36	0.40	0.53	0.61	0.70	0.85	0.90
$H_\alpha^1(A;\mathbf{U})$	0.0978	0.2340	0.3278	0.3923	0.6858	0.9714	1.4846	3.7597	6.0577

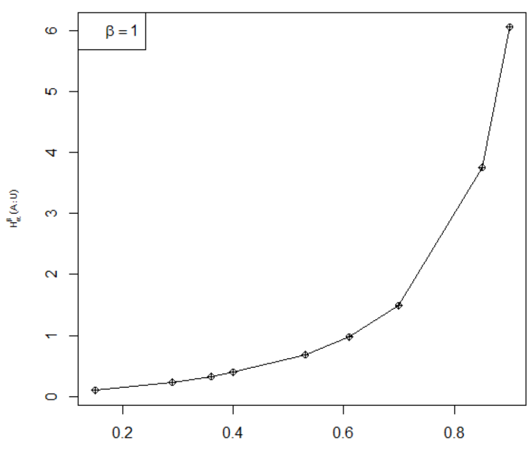

Figure 1. Behavior $H_\alpha^\beta(A;\mathbf{U})$ of at different values of α and $\beta = 1$

Table 8. Behavior $H_\alpha^\beta(A;\mathbf{U})$ of at different values of β and $\alpha = 0.2$

β	0.29	0.36	0.40	0.53	0.61	0.70	0.85	0.90	0.92
$H_{0.2}^\beta(A;\mathbf{U})$	0.1804	0.2045	0.2151	0.2348	0.2363	0.2288	0.1962	0.1800	0.1729

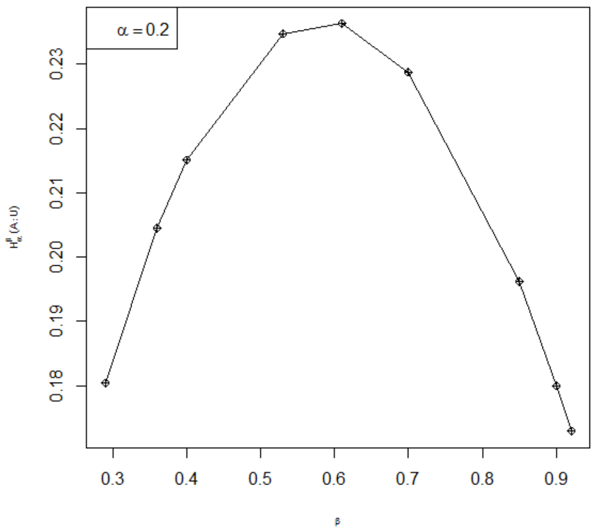

Figure 2. Behavior $H_\alpha^\beta(A;\mathbf{U})$ of at different values of β and α = 0.2

On observing the behavior of $H_\alpha^\beta(A;\mathbf{U})$ at different values of β and fixed α, $H_\alpha^\beta(A;\mathbf{U})$ increases up to $\alpha = 0.59$ and after this value $H_\alpha^\beta(A;\mathbf{U})$ starts decreasing.

Conclusion

The present communication introduces a new 'useful' fuzzy information measure i.e., $H_\alpha^\beta(A;\mathbf{U})$, of order α and type β. The properties of $H_\alpha^\beta(A;\mathbf{U})$ were considered via hypothetical data. Further, the behavior of $H_\alpha^\beta(A;\mathbf{U})$ at different values of α and β were studied.

References

Belis, M., & Guiasu, S. (1968). A quantitative-qualitative measure of information in cybernetic system. *IEEE Transactions on Information Theory, 14*(4), 593-594. doi: 10.1109/TIT.1968.1054185

Bhat, A. H., & Baig, M. A. K. (2016a). Coding theorems on generalized useful fuzzy inaccuracy measure. *International Journal of Modern Mathematical Science, 14*(1), 54-62.

Bhat, A. H., & Baig, M. A. K. (2016b). Generalized useful fuzzy inaccuracy measures and their bounds. *International Journal of Advanced Research in Engineering Technology and Sciences, 3*(6), 28-33. Retrieved from http://ijarets.org/publication/24/IJARETS%20V-3-6-6.pdf

Bhat, A. H., Baig, M. A. K., & Salam, A. (2016). Bounds on two parametric new generalized fuzzy entropy. *Mathematical Theory and Modeling*, *6*(7), 7-17. Retrieved from https://iiste.org/Journals/index.php/MTM/article/view/31542

Bhat, A. H., Bhat, M. A., Baig, M. A. K., & Sofi, S. M. (2017). Noiseless coding theorems of generalized useful fuzzy inaccuracy measure of order α and type β. *International Journal of Fuzzy Mathematical Archive*, *13*(2), 135-143. Retrieved from http://www.researchmathsci.org/IJFMAart/IJFMA-v13n2-4.pdf

De Luca, A., & Termini, S. (1972). A definition of non-probabilistic entropy in the setting of fuzzy set theory. *Information and Control, 20*(4), 301-312. doi: 10.1016/S0019-9958(72)90199-4

Ebanks, B. R. (1983). On measures of fuzziness and their representations. *Journal of Mathematical Analysis and Applications, 94*(1), 24-37. doi: 10.1016/0022-247X(83)90003-3

Kaufmann, A. (1975). *Introduction to theory of fuzzy subsets: Fundamental theoretical elements* (Vol. 1). New York: Academic Press.

Pal, N. R., & Bezdek, J. C. (1994). Measuring fuzzy uncertainty. *IEEE Transaction on Fuzzy Systems*, *2*(2), 107-118. doi: 10.1109/91.277960

Pal, N. R., & Pal, S. K. (1989). Object-background segmentation using new definition of entropy. *IEE Proceedings E - Computers and Digital Techniques, 136*(4), 136-284. doi: 10.1049/ip-e.1989.0039

Parkash, O., & Sharma, P. K. (2002). A new class of fuzzy coding theorems. *Caribbean Journal of Mathematical and Computing Sciences, 12*, 1-10.

Parkash, O., & Sharma, P. K. (2004). Noiseless coding theorems corresponding to fuzzy entropies. *Southeast Asian Bulletin of Mathematics, 27*(6), 1073-1080. Retrieved from http://www.seams-bull-math.ynu.edu.cn/downloadfile.jsp?filemenu=_200406&filename=14.pdf

Shannon, C. E. (1948). A mathematical theory of communication. *Bell System Technical Journal, 27*(3), 379-423. doi: 10.1002/j.1538-7305.1948.tb01338.x

Zadeh, L. A. (1965). Fuzzy sets. *Information and Control*, *8*(3), 338-353. doi: 10.1016/S0019-9958(65)90241-X