1-1-2017

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Recommended Citation

Gheorghe, Bogdan, "The Motivic Cofiber Of T And Exotic Periodicities" (2017). *Wayne State University Dissertations*. 1804.
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THE MOTIVIC COFIBER OF τ AND EXOTIC PERIODICITIES

by

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PH.D. DISSERTATION

Submitted to the Graduate School,

of Wayne State University,

Detroit, Michigan

in partial fulfillment of the requirements

for the degree of

DOCTOR OF PHILOSOPHY

2017

MAJOR: MATHEMATICS

Approved By:

Advisor Date

Committee Member Date

Committee Member Date

Committee Member Date

DEDICATION

For Christa, my extraordinary future wife,

for her unconditional love and support.

For the best parents in the world,

who are always here for me.

ACKNOWLEDGEMENTS

I would like to first and foremost thank my PhD advisor Dan Isaksen for all his help during my five years at Wayne State University. Dan taught me more than just mathematics, he taught me how to be a mathematician, which will undoubtedly have an impact on my entire career. Moreover, I would like to thank him for suggesting the investigation of Theorem 1.1, which started my PhD thesis. This whole project would not have existed without his insight and constant support. Dan, it has been an honor to be your student.

To my beautiful future wife Christa: Thank you for always believing in me and supporting me the way you did. Even if I would not have finished this PhD, coming to Wayne State University was the right move, as I met you. My work hours have sometimes been interfering with our everyday life, thank you for always being understanding of that. Thank you for all your love, our wedding this summer will be awesome.

Of course, I would not have been able to even start this PhD without the constant support and help from my parents, Adrian and Daniela. I would like to thank you here for your dedication to help, you are role models in every aspect of life. I would like to also thank my future parents in law, Chris and June, for welcoming me in their family and giving me a second home where I feel loved. Most of this document was in fact written under your roof, in the conservatory.

Similarly, I would not have been able to start my math career with the help of Kathryn Hess. I would like to thank you for that, as well as for your encouragements and kindness.

I would like to also thank some math faculties from Wayne State. Thank you Andrew Salch for always being so excited to teach math to everybody. I learned a lot from all the seminars you organized, Wayne State is very lucky to have you. Thank you Bob Bruner for always having your door open and willing to teach students everything you know. I would also like to thank Po Hu and John Klein for various classes taught, and helpful conversations.

I would have never been able to finish this PhD thesis without the help from all my math friends. In particular, Nicolas Ricka have not only been my best buddy around, but has answered tons and tons of my questions. Everybody could write a PhD thesis if they have a Nicolas to help. I am also grateful for early conversations with Sean Tilson and Prasit Bhattacharya, which initiated me to some of the necessary mathematics needed to prove Theorem 1.1. I am also grateful for uncountably many discussions with Gabe, Mike and Mike, Hieu and Josh. I learned a lot from these discussions, my experience at Wayne State would not have been the same without you.

I would like to thank Peter May for doubting the first version of the proof of Theorem 1.1 during a visit to Chicago. This lead me to think about a cleaner proof, which slowly evolved until it reached its final version presented here. I would also like to thank Mike Hill for useful conversations during this cleaning process. I am not sure I could have gotten here without you. I would like to also thank Elden Elmanto, Dominic Culver, J.D. Quigley, Mark Behrens, Drew Heard, Achim Krause, Zhouli Xu, Guozhen Wang, Jens Hornbostel, Jeremiah Heller, Paul Goerss, Eric Peterson, Marc Hoyois, Tobias Barthel, Haynes Miller, Kyle Ormsby and Tyler Lawson for various conversations, and/or moral support.

Finally, I would like to thank the rest of my family and friends. I could not have finished my PhD without a healthy environment with you by my side. In particular I would like to acknowledge Alina, Beth, Julius, Matt, Bridgy, and some of the Swiss team Raf, Robin, Denis and Tania. I apologize for the many people that I forgot and illuminate my life.

Thanks.

TABLE OF CONTENTS

CHAPTER 1 INTRODUCTION

1.1 Motivation for this Thesis

Here is the table of contents for this section on motivation for this thesis.

The *chromatic approach* to classical homotopy theory is a very powerful organizational tool to study the homotopy category of (p-local) finite CW-complexes. In particular, given a p-local finite CW-complex X, the chromatic approach provides an algorithm for computing its homotopy groups $\pi_*(X)$. This algorithm relies heavily on the existence of the *chromatic filtration* on $\pi_*(X)$. This is an increasing filtration indexed by non-negative integers $n \in \mathbb{N}_0$, where we say that an element $f \in \pi_*(X)$ in the n^{th} filtration has *height* n. Experience shows that the complexity of the height n part grows exponentially as n increases linearly. For example for the sphere $S_{(p)}$, the height 0 elements are exactly the elements in π_0, the height 1 elements correspond to $\operatorname{im} j$, and the height 2 elements are associated with *tmf*. Following [48], determining the chromatic filtration of an element $f \in \pi_*(X)$ can be done by a recursive algorithm that is based on the following steps.

Step 1: Find a non-nilpotent self-map v on X, not necessarily of degree 0.

Step 2: Since v acts on $\pi_*(X)$ by post-composition, we can use it to break $\pi_*(X)$ into a v-periodic part and a (power of) v-torsion part.

Step 3: If the element f is v-periodic, then it will be detected in some cohomology theory, and we are done. If not, then it lifts to the cofiber of some power of v, and we repeat this process by replacing X with this cofiber. This process increases the height of f by 1.

The execution and good behavior of this algorithm require the following ingredients:

(1) the existence of a non-nilpotent self-map v on every finite complex,

(2) some sort of uniqueness for such a self-map v,

(3) computable cohomology theories detecting v-periodicity.

The first two points are exactly the content of one of the deepest theorems in chromatic homotopy theory called the *Periodicity Theorem* of Devinatz-Hopkins-Smith. Given any finite complex X, this theorem says that there is an essentially unique non-nilpotent self-map on X called a *v_n-self-map.* Finally, the last point is taken care of by the existence of the *Morava K-theories* $K(n)$ [38] which are field spectra[1] and detect exactly v_n-periodicity.

One of the main goals of this thesis is to discuss the story of motivic periodicities. As we will see, the v_n-operators have a motivic analogue, but this is only a fraction of the whole story.

1.1.1 Motivic w_n-periodicity

This section is extracted from the introduction of our paper [15], which we refer to for more details about w_n-periodicity.

Even over very nice base schemes (for example algebraically closed fields of characteristic

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[1] and thus admit a Künneth isomorphism, i.e., are computable.

0), the above *chromatic algorithm* does not apply in the category of motivic spectra. Motivic Morava K-theories detecting v_n-periodicity have been constructed in [8] and [23], and even though they are not quite field spectra[2], they are computable over nice base schemes. The main issue comes from the lack of a Periodicity Theorem, and the fact that there is more periodicity to consider than just v_n-periodicity. One goal of this thesis is to try to explain this phenomenon in the easiest case: for 2-local cellular motivic spectra over $\operatorname{Spec}\mathbb{C}$. Denote by S the 2-local motivic sphere spectrum over $\operatorname{Spec}\mathbb{C}$. The first step in the chromatic approach to compute $\pi_{*,*}(S)$ already fails, as there are two very different non-nilpotent self-maps

$$S \xrightarrow{2} S \qquad \text{and} \qquad S^{1,1} \xrightarrow{\eta} S.$$

This means that the process has to be refined, and that the linear ordering of periodicities has to be replaced by a more complex lattice. It turns out that in the same way that there is a v_n-periodicity story starting with $v_0 = 2$, Haynes Miller suggested that there could be a similar story starting with $w_0 = \eta$. The first evidence is provided by Andrews in [1], where he also pins down the notation of w_n*-periodicity.* Even though no precise definition is given, he shows that in the same way that S/v_0 admits a v_1^4-self-map, the motivic 2-cell complex S/w_0 also admits a w_1^4 self-map. The main goal of Chapter 4 is to give a precise definition of w_n-periodicity, as well as construct motivic field spectra $K(w_n)$ that detect w_n-periodicity. We refer to Theorem 4.26 for more details.

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[2] they are however what could be called *principal ideal domains* spectra.

1.1.2 Towards the Construction of the Fields $K(w_n)$: The Cofiber of τ

The spectra $K(w_n)$ will not be constructed merely as motivic spectra, but as modules over some motivic E_∞ ring spectrum. By the end of this section, we will explain which ring spectrum that is, and why it is easier to construct the $K(w_n)$'s there.

The mod 2 cohomology of the motivic sphere spectrum $S^{0,0}$ over $\operatorname{Spec}\mathbb{C}$ was computed by Voevodsky in [60], and is given by

$$H\mathbb{F}_2{}^{*,*}(S^{0,0}) \cong \mathbb{F}_2[\tau] \qquad \text{where } |\tau| = (0,1).$$

The element τ is sometimes denoted the *Tate twist*. One can run the motivic Adams spectral sequence

$$\operatorname{Ext}_{\mathcal{A}_\mathbb{C}}(\mathbb{F}_2[\tau], \mathbb{F}_2[\tau]) \Longrightarrow \pi_{*,*}\left((S^{0,0})^\wedge_2\right),$$

as constructed in [39], [13], [22], and it is easy to see that the element $\cdot\tau \in \operatorname{Ext}^0$ survives to the E_∞-page. It therefore detects a map

$$S^{0,-1} \xrightarrow{\tau} (S^{0,0})^\wedge_2,$$

i.e., an element in the homotopy groups (of the 2-completed motivic sphere), whose Hurewicz image is the element $\tau \in H\mathbb{F}_{2*,*}((S^{0,0})^\wedge_2)$.

Consider now the first w_n-periodic operator $w_0 = \eta$, and its associated field $K(w_0)$. There is a relation $0 = \tau\eta^4$ in the homotopy groups of $(S^{0,0})^\wedge_2$, which thus holds in the homotopy groups of any motivic 2-completed spectrum. Similarly with the fact that the Morava K-

theories $K(n)$ are naturally 2-completed[3], i.e., are $(S^{0,0})^\wedge_2$-modules, the same occurs for the motivic spectra $K(w_n)$. In particular, since $K(w_0)$ contains $\eta^{\pm 1}$ in its homotopy, this forces τ to act by zero on it. One can quotient by τ on the level of spectra, and the motivic spectrum $S^{0,0}/\tau$ is naturally a highly structured ring spectrum. Moreover, it turns out that τ acts as zero on all the fields $K(w_n)$, and that in this case, this is sufficient to naturally turn them into $S^{0,0}/\tau$-modules. This means that once this category of modules is in place, one can construct $K(w_n)$ in this category, which has the advantage of being much easier than constructing them as only motivic spectra.

1.1.3 The Cofiber of τ as an E_∞ Ring Spectrum

This section is extracted from the introduction of our paper [14], which we refer to for more details about the motivic spectrum $C\tau = S^{0,0}/\tau$.

As we will explain later in diagram (2.4), the map

$$S^{0,-1} \xrightarrow{\tau} (S^{0,0})^\wedge_2,$$

does not exist if we don't 2-complete the target. We will thus work 2-completed, and denote the 2-completed sphere and the smash product in 2-completed motivic spectra simply by $S^{0,0}$ and $- \wedge -$. Consider the cofiber sequence

$$S^{0,-1} \xrightarrow{\tau} S^{0,0} \longrightarrow C\tau \longrightarrow S^{1,-1},$$

where we denote the cofiber of the map τ by $C\tau = S^{0,0}/\tau$. This 2-cell complex already

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[3] except for $K(0) \simeq H\mathbb{Q}$.

appeared in [24], where it is studied via its motivic Adams-Novikov spectral sequence. More precisely, it is proven that its Adams-Novikov spectral sequence collapses at the E_2-page, providing a surprising isomorphism

$$\mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, BP_*) \cong \pi_{*,*}(C\tau). \tag{1.1}$$

The left hand side is the cohomology of the classical (non-motivic) Hopf algebroid (BP_*, BP_*BP) and is very important in chromatic homotopy theory. Notice that since it is the cohomology of a dga, namely the cobar complex associated to (BP_*, BP_*BP), it admits products and higher Massey products. All this algebraic structure gets transferred to the motivic homotopy groups $\pi_{*,*}(C\tau)$, formally endowing it with a (higher) ring structure. One can thus hope that this algebraic ring structure can be lifted to a topological ring structure on $C\tau$. The following is the main result of Chapter 3.

Theorem 1.1. *There exists a unique E_∞ ring structure on $C\tau$.*

This improves the above isomorphism, and we can show that the algebraic structure on $\pi_{*,*}(C\tau)$ does come from $C\tau$.

Proposition 1.2. *The isomorphism* (1.1)

$$\pi_{*,*}(C\tau) \cong \mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, BP_*)$$

is an isomorphism of rings which sends Toda brackets in $\pi_{*,*}$ to Massey products in Ext*, and vice-versa.*

Let's point out that the additive version of this theorem was already exploited by Isaksen in [24] to gain knowledge about the classical Adams-Novikov E_2-page. The idea is to compute $\pi_{*,*}(C\tau)$ in a range using its motivic Adams spectral sequence and the knowledge of $\pi_{*,*}(S^{0,0})$ in this range. Having a multiplicative structure available improves the correspondence in an obvious manner.

1.1.4 The Category of $C\tau$-Modules and Examples

With an E_∞ ring structure on $C\tau$, there is an associated symmetric monoidal ∞-category of $C\tau$-modules, and a free-forget adjunction

$$\begin{tikzcd}\mathbf{Spt}_{\mathbb{C}} \arrow[r, shift left, "-\wedge C\tau"] \arrow[r, phantom, "\perp"] & {}_{C\tau}\mathbf{Mod}, \arrow[l, shift left]\end{tikzcd} \tag{1.2}$$

between motivic spectra and $C\tau$-modules. In Lemma 3.30 we show that the Betti realization of any $C\tau$-module is contractible, which means that the category of $C\tau$-modules lies in the kernel of the Betti realization

$$\mathbf{Spt}_{\mathbb{C}} \xrightarrow{\mathrm{Re}_{\mathbb{C}}} \mathbf{Spt}.$$

This explains in some sense why the motivic spectra $K(w_n)$ should be constructed in $C\tau$-modules, since they do not have a classical analogue.

Given a spectrum X, we call the induced $C\tau$-module $\overline{X} := X \wedge C\tau$ a $C\tau$-induced spectrum. This is in some sense the spectrum X/τ, which is the analogue of X in the category of $C\tau$-modules. It turns out that $C\tau$-induced spectra $\overline{X}$ are easy to understand for many usual motivic spectra X, and admit many interesting properties. We now cite a few results that we show at the end of Chapter 3.

Start with the 2-completed motivic mod 2 Moore spectrum $S^{0,0}/2$. The Toda bracket

$\langle 2, \eta, 2\rangle \ni \tau\eta^2$ is the obstruction to both endowing it with a left unital multiplication, and to a v_1^1-self map. In Theorem 3.37, we will prove the following results about the $C\tau$-induced Moore spectrum, which we denote by $S/(2,\tau)$.

Theorem 1.3. *The $C\tau$-induced motivic mod 2 Moore spectrum $S/(2,\tau)$ admits a unique structure of an E_∞ $C\tau$-algebra.*

Proposition 1.4. *The $C\tau$-induced motivic mod 2 Moore spectrum $S/(2,\tau)$ admits a v_1^1-self map*

$$\Sigma^{2,1} S/(2,\tau) \xrightarrow{v_1} S/(2,\tau).$$

Another interesting spectrum to consider is the 2-completed connective[4] hermitian K-theory spectrum kq.

Proposition 1.5. *The $C\tau$-induced connective hermitian K-theory spectrum $\overline{kq}$ has homotopy groups*

$$\pi_{*,*}(\overline{kq}) \cong \hat{\mathbb{Z}}_2[v_1^2, \eta] \big/ 2\eta.$$

Recall that the homotopy of the motivic spectrum kq contains the 8-fold Bott periodicity element v_1^4, but does not contain v_1^2. In chromatic motivic language, up to the v_0-extensions this can be rewritten as $\mathbb{F}_2[v_0, v_1^2, w_0] \big/ v_0 w_0$. The relation $v_0 w_0 = 0$ is clear as it is already existent in $\pi_{*,*}(S^{0,0})$, but this shows that v_1^2 and w_0 can coexist without any relation between them.

1.1.5 The Structure of the Category of $C\tau$-Modules

The results of this section are joint with Zhouli Xu and Guozhen Wang, and appear both in Chapter 5 and in the first part of the paper [17].

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[4] in the sense of [25, Definiton 4.9 and 4.11].

One strength of the category ${}_{C\tau}\mathbf{Mod}$ is that it is relatively easy to work with $C\tau$-modules. One first observes this phenomenon during the process of proving that $C\tau$ admits an E_∞ ring structure, and for example also trough studying the $C\tau$-induces spectra $S/(2,\tau)$ and $\overline{kq}$. In Chapter 5, we offer an explanation for these phenomena.

Recall the isomorphism of equation (1.1)

$$\pi_{*,*}(C\tau) \cong \mathrm{Ext}_{\overline{MGL}_{*,*}\overline{MGL}}(\overline{MGL}_{*,*}, \overline{MGL}_{*,*}),$$

which is a highly structured isomorphism by Proposition 1.2. This isomorphism hints to the fact that the motivic 2-cell complex $C\tau$ is of algebraic nature, and one can ask how far this comparison can go. In Chapter 5, we will show that some category ${}_{C\tau}\mathbf{Cell}^{\mathrm{comp}}$ of cellular $C\tau$-modules is equivalent the derived bounded category of its heart

$$\mathcal{D}^{\mathrm{b}}({}_{MU_*\widehat{MU}}\mathbf{Comod}) \xrightarrow{\cong} {}_{C\tau}\mathbf{Cell}^{\mathrm{comp}},$$

providing a surprising equivalence between modules over a motivic 2-cell complex, and derived modules over a ring. The spectrum $\widehat{MU}$ is the 2-completion of the complex cobordism (non-motivic) spectrum MU.

This equivalence of categories implies that the homotopy category of cellular (complete) $C\tau$-modules is algebraic in the sense of [54]. This gives a reason as to why it feels easier to manipulate motivic spectra living in ${}_{C\tau}\mathbf{Mod}$, since algebraic categories are usually better behaved than topological categories. For example, algebraic categories admit a $\mathcal{D}(\mathbb{Z})$-enrichment which implies many pleasant properties.

1.1.6 Motivic Thick Subcategories

With the category of $C\tau$-modules in place and reasonably well understood, we construct in Chapter 4 the motivic fields $K(w_n) \in {}_{C\tau}\mathbf{Mod}$. In fact, the spectra $K(w_n)$ are also directly related to the structure of the category of finite 2-local motivic spectra. From the Periodicity Theorem, one can rewind back to one of the deepest and pioneering theorems in chromatic homotopy theory : the *Thick Subcategory Theorem* proved by Devinatz-Hopkins-Smith in [11]. This theorem is equivalent to the Nilpotence Theorem, and in fact implies the Periodicity Theorem. In some sense it is a more global way of understanding the chromatic filtration, without zooming in on a specific object. This theorem describes a filtration

$$* = \mathcal{C}_{-1} \subsetneq \mathcal{C}_0 \subsetneq \mathcal{C}_1 \subsetneq \cdots \subsetneq \mathcal{C}_\infty = \mathbf{FinSpt}_{(p)}$$

by height, on the whole category $\mathbf{FinSpt}_{(p)}$ of finite p-local spectra. The category $\mathcal{C}_n$ is the subcategory of acyclics for the Morava K-theory spectrum $K(n)$ of height n. This filtration is exhaustive, Hausdorff, and admits no refinement by any thick subcategory. In fact, the subcategories $\mathcal{C}_n$ turn out to further be prime ideals, i.e., this filtration gives a complete description of the *Balmer spectrum of* $\mathbf{FinSpt}_{(p)}$. Although it is not true in a general tensor triangulated category that all prime ideals come from field spectra, this happens to be the case for $\mathbf{FinSpt}_{(p)}$.

Motivically, the study of thick subcategories started with the work in [27]. In that paper some thick subcategories of $\mathbf{FinSpt}_{\mathbb{C}}$ were constructed, with the feeling that this is a very hard problem. In fact, even the Balmer spectrum (i.e., just the thick prime ideals) of the category of finite p-local motivic spectra has not been computed over any base scheme. A slightly easier problem that we will consider is the Balmer spectrum of the category of *cellular* finite p-local spectra. Denote this category (over the base $\operatorname{Spec}\mathbb{C}$) by $\mathbf{FinCell}_{\mathbb{C}}$. Since the spectra $K(w_n)$ are new motivic fields, it is now clear that the Balmer spectrum of $\mathbf{FinCell}_{\mathbb{C}}$ is more complicated than the one of **FinSpt**. Morel showed [40] that if the base scheme is a perfect field k of characteristic different than 2, then

$$\pi_{-(*,*)}S^{0,0} \cong K_*^{\mathrm{MW}}(k)$$

is the Milnor-Witt K-theory of k, which contains the Grothendieck-Witt group $K_0^{\mathrm{MW}}(k) = \mathrm{GW}(k)$ in degree 0. In [3], Balmer considers a natural map

$$\rho\colon \operatorname{Spc}(\mathbf{FinCell}_k) \longrightarrow \operatorname{Spec}(\mathrm{GW}(k))$$

from the spectrum of finite motivic cellular spectra. This maps send a thick prime ideal $\mathfrak{p}$ to the prime ideal of elements $f \in \mathrm{GW}(k) = [S^{0,0}, S^{0,0}]$ such that $Cf \notin \mathfrak{p}$, where Cf denotes the 2-cell complex given by the cofiber of the map f. He shows by general methods that in this case, the map ρ is surjective. In the case when k is either $\mathbb{C}$ or a finite field $\mathbb{F}_q$, the only non-trivial prime ideals of $\mathrm{GW}(k)$ are given by (p), and the surjectivity of ρ was already known since the motivic v_n-story covers these ideals. In [28] for the case of finite fields, and more generally in [18], it is shown that Balmer's map ρ factors further through the surjective map $\rho^\bullet$ as in

$$\begin{tikzcd}
\mathrm{Spc}(\mathbf{FinCell}_k) \arrow[r, "\rho^\bullet"] \arrow[dr, "\rho"'] & \mathrm{Spec}^{\mathrm{h}}(K_*^{\mathrm{MW}}(k)) \arrow[d] \\
& \mathrm{Spec}(\mathrm{GW}(k)),
\end{tikzcd}$$

where $\mathrm{Spec}^{\mathrm{h}}$ denotes the space of homogeneous prime ideals. The space $\mathrm{Spec}^{\mathrm{h}}(K_*^{\mathrm{MW}}(k))$ has been computed in [56], the remaining task for understanding the thick prime ideals of finite cellular motivic spectra is thus to identify the fibers of this map. However, this is no easy task, even in what is considered to be the easiest case, i.e., over $\mathrm{Spec}\,\mathbb{C}$. In this case, the lattice of homogeneous prime ideals of $\mathrm{Spec}^{\mathrm{h}}(K_*^{\mathrm{MW}}(\mathbb{C}))$ is given by

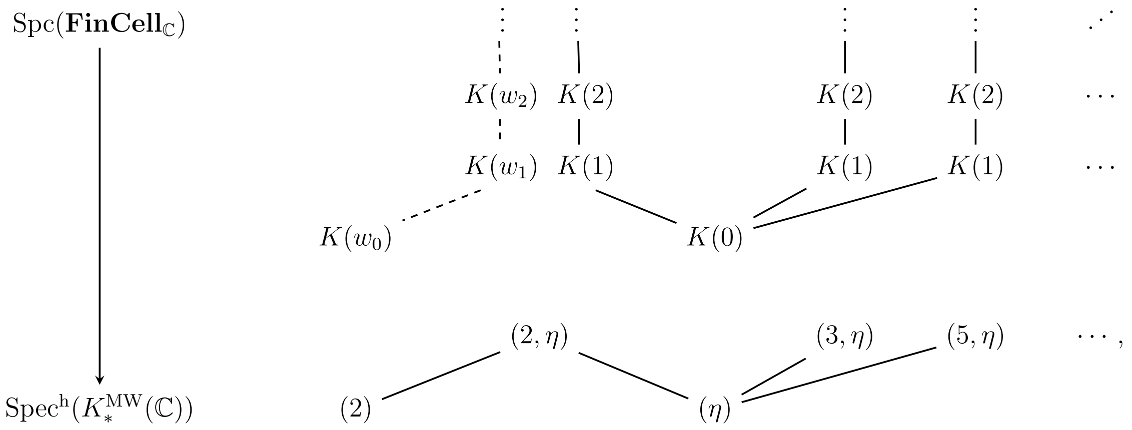
$$\begin{tikzcd}
& & (2,\eta) & & (3,\eta) & (5,\eta) & \cdots . \\
\mathrm{Spec}^{\mathrm{h}}(K_*^{\mathrm{MW}}(\mathbb{C})) & (2) \arrow[ur, dash] & & (\eta) \arrow[ul, dash] \arrow[ur, dash] \arrow[urr, dash] & & &
\end{tikzcd}$$

Some explicit thick prime ideals have been constructed in [18] in the case when the base field k admits an embedding $k \hookrightarrow \mathbb{C}$. For simplicity, let's from now on only work over $\mathrm{Spec}\,\mathbb{C}$. Recall that there is an adjunction

$$\begin{tikzcd}
\mathrm{Real}\colon \mathbf{Spt}_{\mathbb{C}} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathbf{Spt}\colon \mathrm{Sing}, \arrow[l, shift left]
\end{tikzcd}$$

where Real is the *Betti realization* functor induced by taking $\mathbb{C}$-points. In [18], Heller-Ormsby show that the thick prime ideal $\mathrm{Real}^{-1}(\mathcal{C}_n)$ is the subcategory of acyclics for the motivic field spectrum $\mathrm{Sing}(K(n))$. The thick prime ideal generated by $\mathrm{Sing}(K(0))$ sits over (η), while $\mathrm{Sing}(K(n))$ for $n > 0$ sits over (p, η) for the appropriate prime, very much like the classical picture except that (0) becomes (η). In the cellular case, being a motivic field is a weaker condition, since in particular it is implied by the coefficients being a graded field. In this case, the paper [18] constructs a motivic field from the spectrum KT of [20] representing *(higher) Witt groups*. This cellular motivic field generates another thick prime ideal and lives over the ideal (2).

In Theorem 4.26 we construct more cellular motivic fields $K(w_n)$ for every $n \in \mathbb{N}_0$. Since $w_0 = \eta$, the spectrum $K(w_0)$ agrees on homotopy groups with the cellular field of [18]. These new motivic fields sit above $\mathrm{Spec}^{\mathrm{h}}(K_*^{\mathrm{MW}}(\mathbb{C}))$ as is shown in the diagram



where the symbol $K(n)$ in fact stands for the motivic field $\mathrm{Sing}(K(n))$. From this picture, it is tempting to conjecture that the containment of the thick ideals generated by $K(w_n)$ is similar to the $K(n)$ story (the above dotted lines are only conjectural and represent this containment), and that these fields only exist at $p = 2$. In any case, our methods employed to detect w_n-periodicity and to construct $K(w_n)$ in Section 4.2 do not generalize in an obvious way to odd primes. Work in progress of Barthel-Heard-Krause investigate such fields at odd primes.

1.1.7 Detecting Motivic Nilpotence and Periodicity

Having motivic fields $K(w_n)$ that detect w_n-periodicity is a first step towards detecting motivic nilpotence, in the sense explained below (as in [11]). Similarly to the situation in classical homotopy theory, it is desirable to also have a more global spectrum containing all the w_i and detecting them all at once. In classical chromatic homotopy theory, the Brown-Peterson spectrum BP is necessary to enunciate the Nilpotence Theorem. This theorem states that given any p-local finite complex X, a self-map $\Sigma^* X \longrightarrow X$ is nilpotent if and only if it is nilpotent in BP-homology. For example in the case of a single cell $X = S_{(p)}$, this theorem recovers Nishida's Theorem since BP_* is torsion-free. Motivically, it is easy to see that the natural motivic analogue $BPGL$ does not detect nilpotence. We already pointed out that the Hopf map $\eta\colon S^{1,1} \longrightarrow S$ is not nilpotent. Moreover, by construction, the algebraic cobordism spectrum MGL does not detect η, and thus neither does the spectrum $BPGL$. In Section 4.3 we construct an E_∞ ring spectrum wBP with homotopy groups

$$\pi_{*,*}(wBP) \cong \mathbb{F}_2[w_0, w_1, \ldots],$$

with the hope that $BPGL$ together with wBP detect nilpotence. Unfortunately, this turns out to not be the case. More precisely, there is an element $d_1 \in \pi_{32,18}(\widehat{S}_2)$ which is non-nilpotent by work of [24], and which is not detected by either $BPGL$ or wBP. At this point we should mention current work in progress of Barthel-Heard-Krause, which organizes w_n-periodicity in a bigger framework. Their idea is to rewind back to the E_1-page of the May spectral sequence, and consider all the periodicity that can occur from May's elements h_{ij}. In their work, they constructed a cellular motivic ring spectrum that detects the element d_1, as well as more motivic spectra (which are neither ring nor fields) detecting more h_{ij}-periodicity. One can hope that their additional spectra provide all the tools to detect nilpotence in the motivic setting.

1.2 Organization

Here is a brief organization of this thesis.

Chapter 2. This chapter contains some background material (as well as references) to understand this thesis. In particular, it contains a very brief introduction to motivic homotopy theory, as well as the first computational tools over $\operatorname{Spec}\mathbb{C}$, such as motivic Eilenberg-Maclane spectra, the motivic Steenrod algebra and the motivic Adams spectral sequence. Finally, we introduce the motivic element τ and explain how it gives rise to the motivic spectrum $C\tau$.

Chapter 3. In this chapter we show that the motivic spectrum $C\tau$ admits an E_∞ ring structure. We then compute the $C\tau$-linear $\overline{H\mathbb{F}}_2$ Steenrod algebra and its dual, that will be necessary in Chapter 4. We end the chapter with some results about various $C\tau$-induced motivic spectra such as $S/(2,\tau)$, $\overline{kgl}$ and $\overline{kq}$.

Chapter 4. In this chapter we construct the motivic fields $K(w_n)$, as well as associated spectra such as connective versions $k(w_n)$ and an analogue of the Brown-Peterson spectrum wBP.

Chapter 5. This chapter is concerned with the equivalence of categories

$$\mathcal{D}^{\mathrm{b}}({}_{MU_*\widehat{MU}}\mathbf{Comod}) \xrightarrow{\cong} {}_{C\tau}\mathbf{Cell}^{\mathrm{comp}},$$

providing an algebraic model for the category of $C\tau$-modules. Along the way, we will show another equivalence of categories

$$\mathcal{D}^b({}_{\overline{MGL_{*,*}}}\mathbf{Mod}) \xrightarrow{\cong} {}_{\overline{MGL}}\mathbf{Cell}^{\mathrm{b}},$$

which is a required ingredient in showing the first equivalence.

1.3 The Choice of Prime $p = 2$

This thesis is written in a p-completed setting, where we exclusively restrict to the prime $p = 2$. However, some results generalize in a straightforward way to odd primes as well.

Before indicating which of those results generalize to odd primes, let's explain the reason why we restrict to $p = 2$. For odd primes p, the motivic story is somehow easier since it is more closely related to the classical story. In particular, in the case of odd primes, the motivic Steenrod algebra (and its dual) are isomorphic as Hopf algebras to the classical Steenrod algebra (and its dual) adjoined a primitive formal variable τ. This is not the case for $p = 2$, for example because of the relation $\tau_i^2 = \tau\xi_{i+1}$ in the dual motivic Steenrod algebra.

Let now p be an arbitrary prime. The $H\mathbb{F}_p$-based motivic Adams spectral sequence produces the map

$$S^{0,-1} \xrightarrow{\tau} (S^{0,0})^\wedge_p,$$

after p-completing the target for any prime p. Denote again its cofiber by $C\tau$, where the prime p does not appear in the notation. The isomorphism

$$\mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, BP_*) \cong \pi_{*,*}(C\tau)$$

still holds for any prime, producing the same vanishing regions in the homotopy of $C\tau$, and thus endowing $C\tau$ with an E_∞ ring structure. The calculations from the end of Chapter 3 are specific to $p = 2$, so we will skip those results. Similarly, Chapter 4 is specific to $p = 2$, and the story of motivic periodicity for odd primes is somehow more complicated. However, all the results of Chapter 5 apply to odd primes *verbatim*, and will be written in this generality in [17].

CHAPTER 2 BACKGROUND IN MOTIVIC HOMOTOPY THEORY

In this section we give some brief background on the setting of this thesis, which is motivic homotopy theory over $\operatorname{Spec}\mathbb{C}$. For a more detailed introduction to motivic homotopy theory we refer the reader to [41], [42]. Most of our notation agrees with and is taken from [24].

2.1 Motivic Spaces and Spectra over $\operatorname{Spec}\mathbb{C}$

Denote by $\mathrm{Spc}_{\mathbb{C}}$ the category of *(pointed) motivic spaces* over $\operatorname{Spec}\mathbb{C}$ as defined in [42]. This is roughly obtained by starting with $\mathbb{C}$-schemes and

(1) freely adding (homotopy) colimits to allow gluing constructions (attaching cells, suspensions, etc),

(2) restoring some desired geometric colimits by Bousfield localizations,

(3) forcing the affine line $\mathbb{A}^1$ to play the role of the interval I and be contractible.

This category is endowed with a well-behaved $\mathbb{A}^1$-invariant homotopy theory, for example in the form of a closed symmetric monoidal, proper, simplicial and cellular model structure. The paper [46, Chapter 2] is a good source for a careful construction of these model structures. There is a realization functor

$$\mathrm{Spc}_{\mathbb{C}} \xrightarrow{\mathrm{Re}_{\mathbb{C}}} \mathbf{Top},$$

from motivic spaces over $\operatorname{Spec}\mathbb{C}$ to topological spaces called *Betti realization.* This functor is for example constructed in [13, 2.6], [45, Appendix A.7] or [27, Chapter 4], and is induced by taking $\mathbb{C}$-points of the involved $\mathbb{C}$-schemes. It is a strict symmetric monoidal left Quillen functor, whose right adjoint is usually denoted by Sing. In the same spirit as equivariant homotopy theory, motivic homotopy theory has two different types of spheres. Here are 3 of them:

- the *simplicial spheres* that appear during the homotopy cocompletion process, they are constant presheaves that have nothing to do with algebraic geometry,
- the *geometric sphere* that is the (Yoneda image of the) multiplicative group scheme $\mathbb{G}_m$,
- the (Yoneda image of the) *projective line* $\mathbb{P}^1$.

The 1-dimensional simplicial sphere will be denoted by $S^{1,0}$. It Betti realizes to the 1-dimensional sphere S^1 and is the sphere that appears in the triangulated structure with shift functor $\Sigma = S^{1,0} \wedge -$. We will denote the geometric sphere $\mathbb{G}_m$ by $S^{1,1}$, where the first coordinate indicates its *topological dimension* (since $\mathbb{G}_m$ realizes to $\mathbb{G}_m(\mathbb{C}) = \mathbb{C} - \{0\} \simeq S^1$), while the second coordinate indicates its *weight*, or Tate twist. Since $\mathbb{A}^1$ was made contractible, the pushout square

$$\begin{tikzcd}
\mathbb{G}_m \arrow[r, "x"] \arrow[d, "x^{-1}"'] & \mathbb{A}^1 \simeq * \arrow[d] \\
* \simeq \mathbb{A}^1 \arrow[r] & \mathbb{P}^1
\end{tikzcd}$$

exhibits an equivalence of motivic spaces $\mathbb{P}^1 \simeq \Sigma\mathbb{G}_m$. In the above notation we get the equation

$$\mathbb{P}^1 \simeq S^{1,0} \wedge S^{1,1} \simeq S^{2,1}, \tag{2.1}$$

i.e., the homotopy type of the projective line $\mathbb{P}^1$ is obtained as the smash product of a simplicial sphere and a geometric (twisted) sphere.

With spheres at hand, we can now construct the category of *motivic spectra* $\boldsymbol{Spt}_{\mathbb{C}}$ over $\operatorname{Spec}\mathbb{C}$ in the exact same way that is done in topology: by stabilizing with respect to some sphere. Using the above equation (2.1) we observe that inverting the smash product $- \wedge \mathbb{P}^1$ is a good idea as it inverts both smashing with the simplicial $- \wedge S^{1,0}$ and the geometric sphere $- \wedge S^{1,1}$. For any $n, m \in \mathbb{Z}$, there is therefore a *sphere* of *topological dimension* n and *weight* m that breaks uniquely in terms of simplicial and geometric spheres as

$$S^{n,m} = \left(S^{1,1}\right)^{\wedge m} \wedge \left(S^{1,0}\right)^{\wedge n-m}.$$

This provides a bigraded suspension functor that we denote by $\Sigma^{n,m} = - \wedge S^{n,m}$. As noted above, smashing with the simplicial sphere $\Sigma^{1,0} = - \wedge S^{1,0}$ corresponds to the shift functor associated with the triangulated structure on the homotopy category. The category of motivic spectra $\mathbf{Spt}_{\mathbb{C}}$ also supports good model structures which, in particular, are closed symmetric monoidal with respect to the smash product $- \wedge -$, proper, simplicial, cellular, $\ldots$, see [46, Chapter 2] for more details. Moreover, the realization pair stabilizes to a Quillen adjunction[5]

$$\begin{tikzcd}\mathbf{Spt}_{\mathbb{C}} \arrow[r, shift left, "\mathrm{Re}_{\mathbb{C}}"] \arrow[r, phantom, "\perp"] & \mathbf{Spt}, \arrow[l, shift left, "\mathrm{Sing}"]\end{tikzcd}$$

where the Betti functor $\mathrm{Re}_{\mathbb{C}}$ is strict symmetric monoidal, see for example [45, A.45].

Given two spectra $X, Y \in \mathbf{Spt}_{\mathbb{C}}$, the closed symmetric monoidal structure provides a *function motivic spectrum* that we denote by $F(X,Y) \in \mathbf{Spt}_{\mathbb{C}}$. When $X = Y$, we will usually write $\mathrm{End}(X) = F(X,X)$. As usual, we will denote the abelian group of homotopy classes of maps between X and Y by $[X,Y]$. When the source spectrum is a sphere $X = S^{s,w}$, the abelian group

$$\pi_{s,w}(Y) \coloneqq [S^{s,w}, Y]$$

is called the homotopy group of Y in *stem s* and *weight w*. The relation between the two is given by the usual adjunction between the smash product and the function spectrum. After taking homotopy, this becomes the equation

$$\pi_{s,w}(F(X,Y)) \cong [\Sigma^{s,w} X, Y].$$

2.2 The Motivic Steenrod Algebra and the Adams Spectral Sequence

Denote by $H\mathbb{Z}$ Voevodsky's motivic Eilenberg-Maclane spectrum representing integral motivic cohomology on smooth schemes [59, Section 6.1]. Denote by $H\mathbb{F}_2$ the cofiber of multiplication by 2 on $H\mathbb{Z}$, which sits in the cofiber sequence

$$H\mathbb{Z} \xrightarrow{\cdot 2} H\mathbb{Z} \longrightarrow H\mathbb{F}_2.$$

---

[5] Since the Betti realization of $\mathbb{P}^1$ is the topological sphere $\mathbb{P}^1(\mathbb{C}) \simeq S^2$, taking $\mathbb{C}$-points lands in the category of S^2-spectra, i.e., spectra with bonding maps $S^2 \wedge X_n \longrightarrow X_{n+1}$. This is also a model for stable homotopy theory, see [27, Section 4.1] for more details.

The spectrum $H\mathbb{F}_2$ represents mod 2 motivic cohomology on smooth schemes. The coefficients of this spectrum were computed in [60] by Voevodsky, and are given by

$$H\mathbb{F}_{2*,*}(S^{0,0}) \cong \mathbb{F}_2[\tau] \qquad \text{for } |\tau| = (0,-1).$$

Dually, the motivic cohomology of a point is

$$H\mathbb{F}_2{}^{*,*}(S^{0,0}) \cong \mathbb{F}_2[\tau] \qquad \text{for } |\tau| = (0,1),$$

where we abuse notation and use the same symbol τ to denote the Tate twist element in homology and its dual in cohomology. We use the same notation as in [24] for the coefficients

$$\mathbb{M}_2 := H\mathbb{F}_2{}^{*,*}(S^{0,0}) \cong \mathbb{F}_2[\tau] \qquad \text{and} \qquad \overset{\vee}{\mathbb{M}}_2 := H\mathbb{F}_{2*,*}(S^{0,0}) \cong \mathbb{F}_2[\tau].$$

We write $\mathcal{A}_{\mathbb{C}}$ for the mod 2 motivic Steenrod algebra, i.e., the ring of stable cohomology operations on the motivic spectrum $H\mathbb{F}_2$. Its structure has been computed by Voevodsky in [61], [62] : it is the bigraded Hopf algebra over $\mathbb{M}_2$ given by

$$\mathcal{A}_{\mathbb{C}} \cong \mathbb{M}_2\langle \mathrm{Sq}^1, \mathrm{Sq}^2, \ldots \rangle \,/\text{Adem relations}\cdot$$

Observe that as in topology, it is generated by the Steenrod squares Sq^n with the Adem relations between them. The Tate twist $\tau \in \mathbb{M}_2$ has bidegree $|\tau| = (0,1)$, and the Steenrod squares have bidegrees $|\mathrm{Sq}^{2n}| = (2n, n)$ and $|\mathrm{Sq}^{2n+1}| = (2n+1, n)$. Since we work at $p = 2$, the first square $\mathrm{Sq}^1 = \beta$ is again the usual Bockstein operation coming from the short exact sequence of abelian groups

$$0 \longrightarrow \mathbb{Z}/2 \xrightarrow{\cdot 2} \mathbb{Z}/4 \longrightarrow \mathbb{Z}/2 \longrightarrow 0.$$

The dual motivic Steenrod algebra

$$\overset{\vee}{\mathcal{A}}_{\mathbb{C}} \cong \overset{\vee}{\mathbb{M}}_2[\xi_1, \xi_2, \ldots, \tau_0, \tau_1, \ldots] \Big/ \tau\xi_{i+1} = \tau_i^2, \tag{2.2}$$

was also computed by Voevodsky in [61]. Because we are now in homology, the Tate twist $\tau \in \overset{\vee}{\mathbb{M}}_2$ has bidegree $|\tau| = (0, -1)$. The ξ_i's and τ_i's have bidegrees $|\xi_i| = (2^{i+1} - 2, 2^i - 1)$ and $|\tau_i| = (2^{i+1} - 1, 2^i - 1)$. The coproduct is given by the formulas

$$\Delta(\xi_n) = \sum \xi_{n-k}^{2^k} \otimes \xi_k \qquad \text{and} \qquad \Delta(\tau_n) = \tau_n \otimes 1 + \sum \xi_{n-k}^{2^k} \otimes \tau_k.$$

One can now run the motivic Adams spectral sequence

$$\mathrm{Ext}_{\mathcal{A}_{\mathbb{C}}}(\mathbb{M}_2, \mathbb{M}_2) \Longrightarrow \pi_{*,*}((S^{0,0})_2^{\wedge})$$

constructed in [39], [13], [22], that converges to the homotopy groups of the 2-completed motivic sphere $(S^{0,0})_2^{\wedge}$. The $\mathcal{A}_{\mathbb{C}}$-module map

$$\mathbb{M}_2 \xrightarrow{\cdot\tau} \mathbb{M}_2$$

is an element in Hom = Ext^0 of Adams filtration 0 as τ is central in $\mathcal{A}_{\mathbb{C}}$. This element survives to the E_∞-page as it cannot be involved with any differential for degree reasons. Therefore, it detects a map

$$S^{0,-1} \xrightarrow{\tau} (S^{0,0})_2^{\wedge}, \tag{2.3}$$

whose Hurewicz image is the element $\tau \in H\mathbb{F}_{2*,*}((S^{0,0})_2^{\wedge})$.

Unfortunately, the map of equation (2.3) does not lift to a map before 2-completing the target. The situation can be summarized by the following commutative diagram

$$\begin{tikzcd}
S^{0,-1} \arrow[dr, "\tau"] \arrow[drr, dashed, bend left, "\exists\ \tau"] \arrow[ddr, dashed, bend right, "\nexists"'] & & \\
& H\mathbb{F}_2 & (S^{0,0})^{\wedge}_2 \arrow[l] \\
& H\mathbb{Z} \arrow[u] & S^{0,0}. \arrow[l] \arrow[u]
\end{tikzcd} \tag{2.4}$$

The top dotted arrow corresponds to the element (2.3) constructed by the motivic Adams spectral sequence, and the non-existence of the bottom dotted arrow shows that τ does not lift to a map $S^{0,-1} \longrightarrow S^{0,0}$. In fact, a map $S^{0,-1} \longrightarrow H\mathbb{Z}$ corresponds to a cohomology class in the group $H^{0,1}_{\text{mot}}(\operatorname{Spec}\mathbb{C};\mathbb{Z})$, which vanishes. Here is another argument to explain the existence of the map (2.3), that was kindly suggested by the referee during the publication of [14]. This element comes from the Tor spectral sequence for the 2-completed sphere, whenever there is an infinitely 2-divisible element in the Milnor-Witt K-theory $K^{\text{MW}}_1(\operatorname{Spec}\mathbb{C})$, i.e., when the base field has all 2-power roots of unity.

It is crucial for us that this element τ exists in the homotopy groups of the motivic sphere spectrum, and thus acts on the homotopy of any motivic spectrum. Recall that 2-completion is given by the E-Bousfield localization at either the Moore spectrum $S^{0,0}/2$ or the Eilenberg-Maclane spectrum $H\mathbb{F}_2$. In particular, the 2-completed sphere $(S^{0,0})^{\wedge}_2$ is also an E_∞ ring spectrum and admits a good category of (2-completed) modules. Denote temporarily its category of modules by $\widehat{\mathbf{Spt}_{\mathbb{C}}}$. The ring map $S^{0,0} \longrightarrow (S^{0,0})^{\wedge}_2$ induces a forgetful functor

$$\mathbf{Spt}_{\mathbb{C}} \longleftarrow \widehat{\mathbf{Spt}_{\mathbb{C}}}$$

from 2-completed motivic spectra to motivic spectra. As explained in [46, Section 2.8], this forgetful functor creates a symmetric monoidal model structure on $\widehat{\mathbf{Spt}_{\mathbb{C}}}$. Moreover, as indicated in the diagram

$$\begin{tikzcd}
\mathbf{Spt}_{\mathbb{C}} \arrow[r, shift left=2, "- \wedge (S^{0,0})^{\wedge}_{2}"] \arrow[r, shift right=2, "F\left((S^{0,0})^{\wedge}_{2},\, -\right)"'] & \widehat{\mathbf{Spt}_{\mathbb{C}}}, \arrow[l]
\end{tikzcd}$$

it is both a left and right Quillen functor via the usual adjunctions. It follows that the forgetful functor preserves all categorical constructions in $\widehat{\mathbf{Spt}_{\mathbb{C}}}$, i.e., the underlying spectrum of any (co)limit is computed in the underlying category of motivic spectra $\mathbf{Spt}_{\mathbb{C}}$. On finite spectra, 2-completing and smashing with the 2-completed sphere $(S^{0,0})^{\wedge}_{2}$ are equivalent functors. In this thesis, we will only be concerned with finite spectra, and will from now on exclusively work in $\widehat{\mathbf{Spt}_{\mathbb{C}}}$ without further mention, and drop the completion symbol from the notation. For example, we will denote this category by $\mathbf{Spt}_{\mathbb{C}}$, the 2-completed motivic sphere spectrum by $S^{0,0}$, the smash product over the 2-completed sphere by $- \wedge -$, . . . etc. With this notation, the motivic Adams spectral sequence produces a non-trivial map

$$S^{0,-1} \xrightarrow{\tau} S^{0,0},$$

which we can see as being an element in the homotopy groups $\pi_{0,-1}(S^{0,0})$.

CHAPTER 3 THE COFIBER OF τ

This chapter contains the first important result of this thesis, namely the fact that the motivic 2-cell complex $C\tau$ admits an essentially unique E_∞ ring structure. This has many consequences, such as giving more structure to the isomorphism of Proposition 3.1, or more importantly, exhibiting the existence of its category of modules ${}_{C\tau}\mathbf{Mod}$ as a symmetric monoidal ∞-category.

Moreover, this chapter is a requirement for the following two chapters 4 and 5. In Chapter 4 we will construct some motivic spectra detecting a new form of motivic periodicity, called w_n-periodicity. It turns out that these motivic spectra are naturally $C\tau$-modules, and thus it is easier more efficient to construct them in the category ${}_{C\tau}\mathbf{Mod}$, rather than in $\mathbf{Spt}_{\mathbb{C}}$, and further endowing them with the structure of a $C\tau$-module. The computations of Sections 3.3 and 3.4 will be necessary in Chapter 4. In Chapter 5, we will further identify the category ${}_{C\tau}\mathbf{Cell}$ of cellular $C\tau$-modules with the derived category of an abelian category, explain the algebraic nature of the motivic 2-cell complex $C\tau$.

We refer to the Introduction for more motivation about the spectrum $C\tau$. Finally, let's mention that this chapter appears as a separate paper, in [14].

Organization

Here is the organization of this chapter.

Section 3.1. This Section first introduces the spectrum $C\tau$, as well as some vanishing results both in its homotopy groups $\pi_{*,*}(C\tau)$ and in the homotopy classes of self-maps $[C\tau, C\tau]_{*,*}$. These results will be mostly used to endow $C\tau$ with an E_∞ ring structure.

Section 3.2. We first explain the notion of motivic A_∞ and E_∞ ring spectra that we will use in this thesis, and adapt Robinson's obstruction theory [51] to the motivic setting. We then apply this obstruction theory to endow the spectrum $C\tau$ with an E_∞ ring structure.

Section 3.3. In this Section we compute the homotopy types of the E_∞ ring spectrum $C\tau \wedge C\tau$ and of the A_∞ ring spectrum $\mathrm{End}(C\tau)$.

Section 3.4. This Section is about the symmetric monoidal category ${}_{C\tau}\mathbf{Mod}$. We start by showing some generalities on $C\tau$-modules. We then analyze more precisely a few specific $C\tau$-induced spectra:

(1) We compute the Steenrod algebra of operations and co-operations on the $C\tau$-induced mod 2 Eilenberg-Maclane spectrum $H\mathbb{F}_2 \wedge C\tau$.

(2) We show that the $C\tau$-induced mod 2 Moore spectrum $S/(2,\tau)$ admits a unique E_∞ structure as a $C\tau$-algebra, and that it admits a v_1^1-self map.

(3) We compute the homotopy groups of the $C\tau$-induced connective algebraic and hermitian K-theories $kgl \wedge C\tau$ and $kq \wedge C\tau$. In particular, a hidden extension shows that $kq \wedge C\tau$ contains a 4-fold periodicity by the element v_1^2, which is the square root of the usual 8-fold Bott periodicity observed in kq.

3.1 The Spectrum $C\tau$ and its Homotopy

In this Section we will introduce the main object of this thesis, the motivic spectrum $C\tau$. Recall that we work in a 2-completed setting. Define the 2-cell complex $C\tau$ by the cofiber sequence

$$S^{0,-1} \xrightarrow{\tau} S^{0,0} \xrightarrow{i} C\tau \xrightarrow{p} S^{1,-1}, \tag{3.1}$$

where τ denotes the map from equation (2.3), i denotes the inclusion of the bottom cell and p the projection on the top cell. Recall from [13, Section 2.6] that the Betti realization functor $\mathbf{Spt}_{\mathbb{C}} \longrightarrow \mathbf{Spt}$ sends the map τ to the identity id, as shown in the diagram

$$\left(S^{0,-1} \xrightarrow{\tau} S^{0,0}\right) \longmapsto \left(S^{0} \xrightarrow{\mathrm{id}} S^{0}\right).$$

Moreover, it is a left Quillen functor and thus preserves cofiber sequences. This implies that it sends $C\tau$ to a contractible spectrum $* \in \mathbf{Top}$ and thus that $C\tau$ is a purely motivic spectrum living in the kernel of Betti realization. Nonetheless, the motivic spectrum $C\tau$ has very tight connections to classical (non-motivic) homotopy theory. Surprisingly, a computation of Hu-Kriz-Ormsby in [23], allows Isaksen in [24] to express the homotopy groups of this

2-cell complex $\pi_{*,*}(C\tau)$ in terms of the classical Adams-Novikov spectral sequence. Denote by $\mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*)$ the E_2-page of the classical (2-completed) Adams-Novikov spectral sequence for the topological sphere S^0, where as usual s is the Adams filtration and t is the internal degree.

Proposition 3.1 ([24, Proposition 6.2.5]). *The homotopy groups of $C\tau$ are given by*

$$\pi_{s,w}(C\tau) \cong \mathrm{Ext}^{2w-s,2w}_{BP_*BP}(BP_*, BP_*) \qquad \textit{for any } s, w \in \mathbb{Z}.$$

Remark 3.2. Proposition 3.1 is surprising as it is saying that the homotopy groups of a motivic 2-cell complex, which are in principle as complicated to compute as $\pi_{*,*}(S^{0,0})$, are completely algebraic. More precisely, they are given by the cohomology of the Hopf algebroid (BP_*, BP_*BP), which is a very important object in classical chromatic homotopy theory. This bridge allows computations to travel between the classical and the motivic world. See [24, Chapter 5 and 6] for examples where motivic computations of $\pi_{*,*}(C\tau)$ are used to deduce new information about the classical object $\mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, BP_*)$.

Remark 3.3. Since $\mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, BP_*)$ admits a natural ring structure, the isomorphism of Proposition 3.1 induces an artificial ring structure on the motivic homotopy groups $\pi_{*,*}(C\tau)$. The starting point of this project was to ask if this induced ring structure of $\pi_{*,*}(C\tau)$ can be realized by a topological ring structure on the spectrum $C\tau$. Even further, the cohomology groups $\mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, BP_*)$ admit higher structure (Massey products, algebraic squaring operations, ...) and one can hope that this is the shadow of a highly structured ring multiplication on $C\tau$. We will prove in Section 3.2 that $C\tau$ supports an E_∞ ring structure and that the isomorphism

$$\pi_{*,*}(C\tau) \cong \mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, BP_*)$$

preserves higher products (Toda brackets in homotopy and Massey products in algebra). In other words, the E_2-page of the classical Adams-Novikov spectral sequence can be realized with its higher structure as the homotopy of a motivic spectrum.

The ring structure mentioned in Remark 3.3 will be constructed by obstruction theory. To prepare the computations, we will now deduce some Corollaries about $\pi_{*,*}(C\tau)$ and $\pi_{*,*}(\mathrm{End}(C\tau))$.

Corollary 3.4 ([16]). *The group $\pi_{s,w}(C\tau)$ is zero when either $w > s$, or $w \leq \frac{1}{2}s$, or $s < 0$, except that $\pi_{0,0}(C\tau) \cong \hat{\mathbb{Z}}_2$. This is sketched in Figure 1.*

Proof. The vanishing regions in $\pi_{*,*}(C\tau)$ come from the vanishing regions of $\mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, BP_*)$ via the isomorphism

$$\pi_{s,w}(C\tau) \cong \mathrm{Ext}^{2w-s,2w}_{BP_*BP}(BP_*, BP_*)$$

of Proposition 3.1. The region $w > s$ corresponds to the vanishing region above the line $t - s = s$ of slope 1 on the E_2-page of the Adams-Novikov spectral sequence, the region $w \leq \frac{1}{2}s$ corresponds to the E_2-page being 0 in negative Adams filtation $s \leq 0$, and finally $s < 0$ corresponds to E_2-page being zero in negative stems $t - s < 0$. ☐

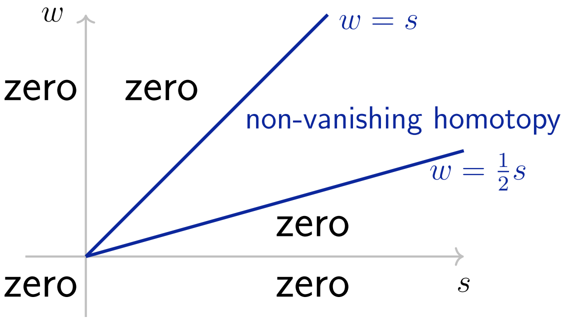


Figure 1: Vanishing regions of the homotopy groups $\pi_{s,w}(C\tau)$.

Corollary 3.5. *The group $[\Sigma^{s,w}C\tau, C\tau]$ is zero if either $w > s + 2$, or $w \leq \frac{1}{2}s$, or $s < -1$, except that $[C\tau, C\tau] \cong \hat{\mathbb{Z}}_2$ in degree $(0, 0)$. This is sketched in Figure 2.*

Proof. Using the cofiber sequence

$$S^{s,w} \xrightarrow{i} \Sigma^{s,w}C\tau \xrightarrow{p} S^{s+1,w-1},$$

we get a long exact sequence

$$\cdots \longleftarrow [S^{s,w}, C\tau] \xleftarrow{i^*} [\Sigma^{s,w} C\tau, C\tau] \xleftarrow{p^*} \left[S^{s+1,w-1}, C\tau\right] \longleftarrow \cdots,$$

after mapping into $C\tau$. The result follows by noticing that the hypothesis of this Corollary force both homotopy groups $\pi_{s,w}(C\tau)$ and $\pi_{s+1,w-1}(C\tau)$ to be 0 by the previous Corollary 3.4.

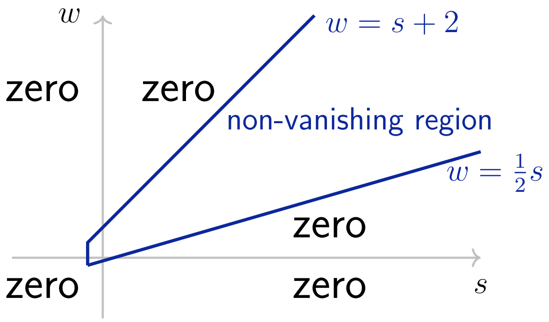


Figure 2: Vanishing regions of the abelian group $[\Sigma^{s,w} C\tau, C\tau]$.

□

Remark 3.6. This result is not sharp and one can slightly improve the non-vanishing region by being careful about choosing which of the 3 conditions of Corollary 3.4 to use. For example, the group $[\Sigma^{-1,0} C\tau, C\tau]$ is zero as it sits in a long exact sequence

$$\cdots \longleftarrow \pi_{-1,0}(C\tau) \xleftarrow{i^*} \left[\Sigma^{-1,0} C\tau, C\tau\right] \xleftarrow{p^*} \pi_{0,-1}(C\tau) \longleftarrow \cdots,$$

and both homotopy groups surrounding it are zero. However, none of the 3 conditions of Corollary 3.5 are satisfied for the pair $(s, w) = (-1, 0)$ and thus we cannot use it to deduce that $[\Sigma^{-1,0} C\tau, C\tau]$ is zero.

The vanishing of the following groups of homotopy classes of maps will often be used in this document.

Corollary 3.7. *The following groups of homotopy classes of maps are zero*

(1) $[\Sigma^{0,-1} C\tau, C\tau] = 0,$

(2) $[\Sigma^{1,0}C\tau, C\tau] = 0$,

(3) $[\Sigma^{1,-1}C\tau, C\tau] = 0$,

(4) $[\Sigma^{n,-n}C\tau, C\tau] = 0$ *for any* $n \geq 1$.

3.2 The E_∞ Ring Structure on $C\tau$

In this Section we construct the E_∞ ring structure on the motivic spectrum $C\tau$. We start by endowing $C\tau$ with a homotopy unital, homotopy associative and homotopy commutative multiplication using elementary techniques with triangulated categories. The E_∞ coherences of such a multiplication cannot be constructed by hand via similar techniques and requires some machinery. We will use a version of Robinson's obstruction theory from [51], that we adapt to the motivic setting in Section 3.2.1.

3.2.1 Motivic A_∞ and E_∞ Operads and Obstruction Theory

Consider a simplicial symmetric monoidal model category presenting $\mathbf{Spt}_\mathbb{C}$, with smash product $- \wedge -$[6], and denote the simplicial mapping space by $\mathrm{Map}(X, Y)$. Given a motivic spectrum X, denote its *endomorphism operad* in simplicial sets by $\mathcal{E}\mathrm{nd}(X)$, where $\mathcal{E}\mathrm{nd}(X)_n$ is the simplicial set $\mathrm{Map}(X^{\wedge n}, X)$. If $F(-,-)$ denotes the internal (motivic) function spectrum, then we recover

$$\pi_n\left(\mathcal{E}\mathrm{nd}(X)_m\right) \cong \pi_{n,0}\left(F(X^{\wedge m}, X)\right), \tag{3.2}$$

only exploiting the weight zero homotopy groups of the function spectrum. Fix an A_∞ or E_∞ operad Θ in simplicial sets. A Θ-*algebra structure* on a motivic spectrum X is a map of operads

$$\Theta \longrightarrow \mathcal{E}\mathrm{nd}(X).$$

Equivalently, one can see Θ as an operad in motivic spaces via the constant functor and define a Θ-algebra via the motivic enrichment, which might seem more natural and internal to motivic homotopy theory. Because of this reason, classical (simplicial) operads transported into the motivic world are sometimes called *constant operads*.

---

[6] For example Jardine's model of motivic symmetric spectra [26].

In this chapter, we will produce A_∞ and E_∞ structures by obstruction theory. The obstruction theory for A_∞ algebras is well-known, for example [2, Theorem 3.1] (itself inspired by [50]) exhibits an obstruction class in a certain abelian group. In all our cases, we will show that all the relevant abelian groups for the obstruction theory are zero. The obstruction theory for E_∞ algebras is less well-known. We will here briefly recap the work done in [51] and adapt it to our motivic situation.

We will consider the simplicial E_∞ operad $\mathcal{T}$ defined in [51, Section 5]. This operad is the product of a combinatorially defined cofibrant simplicial operad with the Barratt-Eccles E_∞ (simplicial) operad $E\Sigma_\bullet$. It inherits both properties and is thus a cofibrant E_∞ operad. The cofibrancy roughly means that the operadic composition maps

$$\mathcal{T}_n \times \mathcal{T}_m \xrightarrow{\circ_i} \mathcal{T}_{m+n-1} \tag{3.3}$$

are injective and that their images intersect in fairly small and regular subcomplexes. We refer to [52, Section 1.5] for more details. The injectivity of these maps is a key property that will be used for inductive arguments, since a map out of $\mathcal{T}_{m+n-1}$ is thus already determined on the image of all these composition maps. The bar filtration on the Barrat-Eccles operad induces a filtration on $\mathcal{T}$, where the n^{th}-filtration space of $\mathcal{T}_m$ is denoted by $\mathcal{T}_m^n \subseteq \mathcal{T}_m$. In particular $\mathcal{T}_m^n = \emptyset$ if $n < 0$. Consider now the *diagonal filtration* $\nabla^\bullet \mathcal{T}$ which is the sum of the bar filtration from the Barratt-Eccles operad and the filtration by operadic subspaces. More precisely, the n^{th}-graded piece $\nabla^n \mathcal{T} \subset \mathcal{T}$ has m^{th}-space given by $\nabla^n \mathcal{T}_m = \mathcal{T}_m^{n-m}$. If $m > n$, then by definition we have $\nabla^n \mathcal{T}_m = \emptyset$. In particular, observe that $\nabla^n \mathcal{T}$ is not a suboperad as it does not contain m-ary operations for $m > n$.

Robinson defines an *n-stage for an E_∞ structure on X* to consist in a map $\nabla^n \mathcal{T} \longrightarrow \mathcal{E}\mathrm{nd}(X)$ satisfying some obvious coherences. More precisely, this is the data of Σ_m-equivariant maps

$$\mathcal{T}_m^{n-m} \longrightarrow \mathcal{E}\mathrm{nd}(X)_m$$

for $0 \leq m \leq n$, which on their restricted domain of definition satisfy the requirements for a morphism of operads. Since the operad $\mathcal{T}$ is non-unital and thus $\mathcal{T}_0 = \mathcal{T}_1 = \emptyset$, we only need to specify these maps for $2 \leq m \leq n$. From the definition of the diagonal filtration one can identify that

- a 2-stage is the data of a map $\mathcal{T}_2^0 \longrightarrow \mathcal{E}\mathrm{nd}(X)_2$, i.e., specifying a map $\mu\colon X \wedge X \longrightarrow X$,
- a 3-stage is the data of a 2-stage with the extra structure of an associative and commutative homotopy for the multiplication μ,
- a 4-stage is the data of a 3-stage with the extra structure of homotopies for the well-known pentagonal and hexagonal axioms [33], as well as a homotopy saying that the commutativity homotopy itself is homotopy commutative,
- an ∞-stage are the coherences of an E_∞ ring structure on X with multiplication μ.

An n-stage determines an $(n-1)$-stage by restriction, and an $(n-1)$-stage determines an n-stage on the boundary $\partial\nabla^n\mathcal{T}$ by injectivity of the composition maps of equation (3.3). We refer to [51, Section 5.2] for more details. Therefore, given an $(n-1)$-stage, the data of an n-stage extending the underlying $(n-1)$-stage consists precisely in the data of extensions

$$\begin{tikzcd}
\partial\nabla^n\mathcal{T}_m \arrow[r] \arrow[dr] & \nabla^n\mathcal{T}_m \arrow[d, dashed] \\
 & \mathcal{E}\mathrm{nd}(X)_m
\end{tikzcd}$$

for every $0 \leq m \leq n$. The cofibrancy of the operad $\mathcal{T}$ is used again to show that for any m, the map

$$\partial\nabla^n\mathcal{T}_m \rightarrowtail \nabla^n\mathcal{T}_m$$

is a principal Σ_m-equivariant cofibration, whose cofiber is a wedge of spheres S^{n+2} indexed over a set with free Σ_m-action. This allows us to formulate the following result.

Proposition 3.8. *Let X be a motivic spectrum with a given $(n-1)$-stage for an E_∞ ring structure.*

(1) If the homotopy groups $\pi_{n-3}(\mathcal{E}\mathrm{nd}(X)_m)$ *are zero for every* $2 \leq m \leq n$*, the given* $(n-1)$*-stage lifts to an* n*-stage.*

(2) If in addition the homotopy groups $\pi_{n-2}(\mathcal{E}\mathrm{nd}(X)_m)$ *are zero for every* $2 \leq m \leq n$*, the extension is (essentially) unique.*

Proof. The fact that $\partial\nabla^n\mathcal{T}_m \rightarrowtail \nabla^n\mathcal{T}_m$ is a principal cofibration allows us to rotate it one step to the left, producing the unstable cofiber sequence of simplicial sets

$$\vee S^{n-3} \longrightarrow \partial\nabla^n\mathcal{T}_m \rightarrowtail \nabla^n\mathcal{T}_m \longrightarrow \vee S^{n-2}.$$

An $(n-1)$-stage produces a map $\partial\nabla^n\mathcal{T}_m \longrightarrow \mathcal{E}\mathrm{nd}(X)_m$, which extends as in the diagram

$$\begin{tikzcd}
\vee S^{n-3} \arrow[r] & \partial\nabla^n\mathcal{T}_m \arrow[r] \arrow[dr] & \nabla^n\mathcal{T}_m \arrow[r] \arrow[d, dashed] & \vee S^{n-2} \\
 & & \mathcal{E}\mathrm{nd}(X)_m &
\end{tikzcd}$$

if and only if the relevant composite is zero in the abelian group

$$\left[\vee S^{n-3}, \mathcal{E}\mathrm{nd}(X)_m\right] \cong \oplus\pi_{n-3}(\mathcal{E}\mathrm{nd}(X)_m).$$

Moreover, if $[S^{n-2}, \mathcal{E}\mathrm{nd}(X)_m] = 0$ then the extension is unique up to homotopy. ☐

By using equation (3.2) and the fact that a 3-stage is equivalent to a unital, associative and commutative monoid in the homotopy category, we get the following Corollary.

Corollary 3.9. *Let* X *be a motivic spectrum with a map* $\mu\colon X \wedge X \longrightarrow X$ *that is homotopy unital, homotopy associative and homotopy commutative.*

(1) If the homotopy groups $\pi_{n-3,0}(F(X^{\wedge m}, X))$ *are zero for every* $n \geq 4$ *and* $2 \leq m \leq n$*, then* μ *can be extended to an* E_∞ *ring structure on* X*.*

(2) If in addition the homotopy groups $\pi_{n-2,0}(F(X^{\wedge m}, X))$ *are zero for every* $n \geq 4$ *and* $2 \leq m \leq n$*, then* μ *can be extended to an* E_∞ *ring structure on* X *in essentially a unique way.*

Remark 3.10. These results are extracted from Robinson's work in [51], even though they do not explicitly appear in this form in his paper. The reason is because this is not a powerful result when applied to the topological setting for the following reason. Fix a (topological) spectrum $X \in \mathbf{Spt}$. To apply this E_∞ obstruction theory to X, its endomorphism operad $\mathcal{E}\mathrm{nd}(X)$ has to satisfy the conditions of Proposition 3.8, which require the homotopy groups $\mathcal{E}\mathrm{nd}(X)_m$ to vanish for all $n \geq 4$ and $2 \leq m \leq n$. In particular, for any fixed m the space $\mathcal{E}\mathrm{nd}(X)_m$ needs to have vanishing homotopy groups in degrees $n \geq m$. The paper [51] proceeds to study what happens during an extension of an $(n-1)$-stage to an n-stage if one allows to perturb underlying stages. This reduces the size of the obstruction groups and gives a constraint between n and m, reducing the number of obstruction groups to check. In our motivic setting the obstructions live in the groups $\pi_{n-3,0}(\mathcal{E}\mathrm{nd}(X)_m)$, which are only a small fraction of all homotopy groups $\pi_{s,w}$. Corollary 3.9 will be sufficient to prove our result.

Remark 3.11. We should point out that, in analogy with the genuine G-equivariant E_∞ operads in [6] (called N_∞ operads), there ought to be a notion of motivic A_∞ and E_∞ operads. An algebra over such a motivic operad would have a lot more structure than an algebra over a constant operad, such as transfers upon changing the base scheme. It is possible that such algebras are exactly the objects corresponding to strict commutative ring spectra. However, for the purpose of this thesis, constant A_∞ and E_∞ operads suffice. We will therefore drop the word "constant" and refer to those just as A_∞ and E_∞ operads.

3.2.2 The Homotopy Ring Structure on $C\tau$

In this Section we construct a ring structure on $C\tau$ up to homotopy. More precisely, we show that $C\tau$ is a unital, associative and commutative monoid in the homotopy category $\mathrm{Ho}(\mathbf{Spt}_{\mathbb{C}})$. Recall that this is a 3-stage in Robinson's obstruction theory, which can be seen as the initial input to start the obstruction theory. In this Section, we will exclusively work in the stable triangulated category $\mathrm{Ho}(\mathbf{Spt}_{\mathbb{C}})$, without further mentioning it.

Lemma 3.12. *There exists a unique left unital multiplication*

$$C\tau \wedge C\tau \xrightarrow{\mu} C\tau.$$

Proof. The equation (3.1) gives an exact triangle

$$S^{0,-1} \xrightarrow{\tau} S^{0,0} \xrightarrow{i} C\tau \xrightarrow{p} S^{1,-1},$$

where i denotes the inclusion of the bottom cell and p denotes the projection on the top cell. By smashing it with $- \wedge C\tau$, we get another triangle

$$S^{0,-1} \wedge C\tau \xrightarrow{\tau} S^{0,0} \wedge C\tau \xrightarrow{i_L} C\tau \wedge C\tau \xrightarrow{p_L} S^{1,-1} \wedge C\tau,$$

where i_L denotes a left unit and p_L the projection on the top cell of the left factor. Since the abelian group of maps $[\Sigma^{0,-1}C\tau, C\tau] = 0$ by Corollary 3.7, the map $\tau \in [\Sigma^{0,-1}C\tau, C\tau]$ is zero on $C\tau$. This produces a left unital multiplication μ on $C\tau$ as shown in the diagram

$$\begin{tikzcd}
S^{0,-1} \wedge C\tau \arrow[r, "\tau"] & S^{0,0} \wedge C\tau \arrow[r, "i_L"] \arrow[dr, "\simeq"] & C\tau \wedge C\tau \arrow[r, "p_L"] \arrow[d, dashed, "\exists\, \mu"] & S^{1,-1} \wedge C\tau \\
 & & C\tau. &
\end{tikzcd}$$

Moreover, since $[\Sigma^{1,-1}C\tau, C\tau] = 0$ by Corollary 3.7, there is no choice for such a map which is unique. ☐

Before studying the properties of this multiplication map μ, we show a fundamental equivalence that will be used throughout the document.

Lemma 3.13. *There is a canonical isomorphism*

$$C\tau \wedge C\tau \cong C\tau \vee \Sigma^{1,-1}C\tau.$$

Proof. Recall that since $[\Sigma^{0,-1}C\tau, C\tau] = 0$, the map τ is zero on $C\tau$. The exact triangle

$$S^{0,-1} \wedge C\tau \xrightarrow{\tau} S^{0,0} \wedge C\tau \xrightarrow{i_L} C\tau \wedge C\tau \xrightarrow{p_L} S^{1,-1} \wedge C\tau,$$

is thus split, giving both a retraction μ and a section s, as in the diagram

$$\begin{tikzcd}
S^{0,-1} \wedge C\tau \arrow[r, "\tau = 0"] & S^{0,0} \wedge C\tau \arrow[r, "i_L"] & C\tau \wedge C\tau \arrow[r, "p_L"] \arrow[l, dotted, bend left, "\exists!\ \mu"] & S^{1,-1} \wedge C\tau \arrow[r, "\tau = 0"] \arrow[l, dotted, bend left, "\exists!\ s"] & \cdots .
\end{tikzcd}$$

As it is the case for μ, the section s is unique since $[\Sigma^{1,-1}C\tau, C\tau] = 0$ by Corollary 3.7. Moreover, the relation $\mu \circ s \cong 0$ is forced since the composite lives in the zero group $[\Sigma^{1,-1}C\tau, C\tau] = 0$. This gives a canonical identification

$$C\tau \wedge C\tau \cong C\tau \vee \Sigma^{1,-1}C\tau,$$

via the inverse maps

$$C\tau \wedge C\tau \xrightarrow{(\mu, p_L)} C\tau \vee \Sigma^{1,-1}C\tau \qquad \text{and} \qquad C\tau \vee \Sigma^{1,-1}C\tau \xrightarrow{i_L + s} C\tau \wedge C\tau.$$

□

Corollary 3.14. *For any $n \geq 2$, there is a canonical isomorphism*

$$C\tau^{\wedge n} \cong \bigvee_{i=0}^{n-1} \binom{n-1}{i} \Sigma^{i,-i} C\tau,$$

where we use $\binom{n-1}{i}\Sigma^{i,-i}C\tau$ to indicate a wedge sum of $\binom{n-1}{i}$ terms of the spectrum $\Sigma^{i,-i}C\tau$.

We will use the identification of Lemma 3.13 to show that μ endows $C\tau$ with a unital, associative and commutative monoid structure in $\mathrm{Ho}(\mathbf{Spt}_{\mathbb{C}})$. We first compute the relevant maps on $C\tau \vee \Sigma^{1,-1}C\tau$ after composing with this identification.

Lemma 3.15. *After the canonical identification $C\tau \wedge C\tau \cong C\tau \vee \Sigma^{1,-1}C\tau$ of Lemma 3.13*

(1) the multiplication map $C\tau \wedge C\tau \xrightarrow{\mu} C\tau$ *is given by the matrix*

$$C\tau \vee \Sigma^{1,-1}C\tau \xrightarrow{[\mathrm{id}\ 0]} C\tau,$$

i.e., by the canonical projection onto the first factor,

(2) the factor swap map $C\tau \wedge C\tau \xrightarrow{\chi} C\tau \wedge C\tau$ *is given by the matrix*

$$C\tau \vee \Sigma^{1,-1}C\tau \xrightarrow{\left[\begin{smallmatrix}\mathrm{id} & 0 \\ i\circ p & -\mathrm{id}\end{smallmatrix}\right]} C\tau \vee \Sigma^{1,-1}C\tau.$$

Proof.

(1) The composite

$$C\tau \vee \Sigma^{1,-1}C\tau \xrightarrow{i_L+s} C\tau \wedge C\tau \xrightarrow{\mu} C\tau$$

restricts to the identity on $C\tau$ since μ is a retraction of i_L, and to zero on $\Sigma^{1,-1}C\tau$ since $s \circ \mu = 0$ by Lemma 3.13.

(2) We claim that the following diagram

$$\begin{tikzcd}
C\tau \wedge C\tau \arrow[r, "\chi"] & C\tau \wedge C\tau \arrow[d, "{(\mu, p_L)}"] \\
C\tau \vee \Sigma^{1,-1}C\tau \arrow[u, "i_L + s"] \arrow[r, dashed, "\left[\begin{smallmatrix}\mathrm{id} & 0 \\ i\circ p & -\mathrm{id}\end{smallmatrix}\right]"] & C\tau \vee \Sigma^{1,-1}C\tau
\end{tikzcd}$$

commutes. First observe that the top right entry is forced to be zero since $[\Sigma^{1,-1}C\tau, C\tau] = 0$ by Corollary 3.7. The bottom left entry can be computed explicitly by a simple diagram chase. It is

$$S^{0,0} \wedge C\tau \xrightarrow{i\wedge \mathrm{id}} C\tau \wedge C\tau \xrightarrow{\chi} C\tau \wedge C\tau \xrightarrow{p\wedge \mathrm{id}} S^{1,-1} \wedge C\tau,$$

which is homotopic to the composite

$$S^{0,0} \wedge C\tau \xrightarrow{\chi} C\tau \wedge S^{0,0} \xrightarrow{\mathrm{id}\wedge i} C\tau \wedge C\tau \xrightarrow{p\wedge \mathrm{id}} S^{1,-1} \wedge C\tau.$$

By commuting $\mathrm{id} \wedge i$ and $p \wedge \mathrm{id}$ and using the canonical equivalences $S^{0,0} \wedge C\tau = C\tau = C\tau \wedge S^{0,0}$ we can rewrite it as

$$C\tau \xrightarrow{p} S^{1,-1} \xrightarrow{i} \Sigma^{1,-1} C\tau.$$

For the diagonal entries, recall that $[C\tau, C\tau] \cong \hat{\mathbb{Z}}_2$ and that the matrix has to be an involution since χ is. This forces the diagonal entries to be $+\mathrm{id}$ and $-\mathrm{id}$. One could conclude by arguing that the top left entry arises by commuting $C\tau$ with $S^{0,0}$, and thus should be $+\mathrm{id}$, while the bottom right entry arises by commuting $C\tau$ with $S^{1,-1}$, and thus should be $-\mathrm{id}$. More precisely, consider the diagram

$$\begin{tikzcd}
S^{0,0} \wedge S^{0,0} \arrow[r, "i \wedge i"] \arrow[d, "\cong"'] & C\tau \wedge C\tau \arrow[d, "\mu"] \\
S^{0,0} \arrow[r, "i"] & C\tau.
\end{tikzcd}$$

By factoring the map $i \wedge i$ as $\mathrm{id} \wedge i$ followed by $i_L = i \wedge \mathrm{id}$, and using that $\mu \circ i_L = \mathrm{id}$, one sees that the diagram commutes up to the usual canonical equivalences of smashing with $S^{0,0}$. By factoring it the other way now, as $i \wedge \mathrm{id}$ followed by $\mathrm{id} \wedge i$, we get that $\mu \circ (\mathrm{id} \wedge i) = \mathrm{id}$. This shows that the top left entry of the matrix is id. The bottom right entry is thus forced to be $-\mathrm{id}$ since the matrix is an involution. ☐

Proposition 3.16. *The unique left unital multiplication map $C\tau \wedge C\tau \xrightarrow{\mu} C\tau$ turns $C\tau$ into a unital, associative and commutative monoid in* $\mathrm{Ho}(\boldsymbol{Spt}_{\mathbb{C}})$.

Proof. Consider the diagram

$$\begin{tikzcd}
 & C\tau \wedge C\tau \arrow[r, "\chi"] \arrow[dd, "{(\mu, p \wedge \mathrm{id})}"] & C\tau \wedge C\tau \arrow[dr, "\mu"] & \\
C\tau \arrow[ur, "i_L"] \arrow[dr, dashed] & & & C\tau, \\
 & C\tau \vee \Sigma^{1,-1} C\tau \arrow[r, "\begin{bmatrix} \mathrm{id} & 0 \\ i \circ p & -\mathrm{id} \end{bmatrix}"] & C\tau \vee \Sigma^{1,-1} C\tau \arrow[uu, "i_L + s"'] \arrow[ur, "\begin{bmatrix} \mathrm{id} & 0 \end{bmatrix}"'] &
\end{tikzcd} \tag{3.4}$$

which is commutative by Lemma 3.15. Since μ is left unital and since $p \circ i = 0$, the dashed arrow is given by the canonical inclusion. It follows that the composite $\mu \circ \chi \circ i_L$ is simply given by the matrix multiplication

$$\begin{bmatrix} \mathrm{id} & 0 \end{bmatrix} \cdot \begin{bmatrix} \mathrm{id} & 0 \\ i \circ p & -\mathrm{id} \end{bmatrix} \cdot \begin{bmatrix} \mathrm{id} \\ 0 \end{bmatrix} = \mathrm{id}\,.$$

Since the right unit is given by $\chi \circ i_L$, this shows that μ is right unital. To show that μ is commutative, we have to compute the composite

$$C\tau \wedge C\tau \xrightarrow{\chi} C\tau \wedge C\tau \xrightarrow{\mu} C\tau.$$

We can again read it from diagram (3.4), where it is given by the matrix multiplication

$$\begin{bmatrix} \mathrm{id} & 0 \end{bmatrix} \cdot \begin{bmatrix} \mathrm{id} & 0 \\ i \circ p & -\mathrm{id} \end{bmatrix} \cdot \begin{bmatrix} \mu \\ p \wedge \mathrm{id} \end{bmatrix} = \mu,$$

showing that μ is commutative. To see that μ is associative, we will show that the map

$$C\tau \wedge C\tau \wedge C\tau \xrightarrow{\mu \circ (1 \wedge \mu - \mu \wedge 1)} C\tau$$

is zero. By left and right unitatlity it restricts to zero on the subspectrum

$$\left(S^{0,0} \wedge C\tau \wedge C\tau\right) \vee \left(C\tau \wedge S^{0,0} \wedge C\tau\right) \vee \left(C\tau \wedge C\tau \wedge S^{0,0}\right) \hookrightarrow C\tau \wedge C\tau \wedge C\tau. \qquad (3.5)$$

By [55, Lemma 3.6], there is a bijection between maps $C\tau \wedge C\tau \wedge C\tau \longrightarrow C\tau$ that restrict to zero on the subspectrum of equation (3.5), and maps

$$S^{3,-3} = S^{1,-1} \wedge S^{1,-1} \wedge S^{1,-1} \longrightarrow C\tau.$$

Here $S^{1,-1}$ appears because it is the cofiber of the unit map $S^{0,0} \longrightarrow C\tau$. By Corollary 3.4, we have that $\pi_{3,-3}(C\tau) = 0$, which shows that there is a unique such map. Since the zero map $C\tau \wedge C\tau \wedge C\tau \longrightarrow C\tau$ restricts to zero on the subspectrum of equation (3.5), it is the unique such map. This shows that $\mu \circ (1 \wedge \mu - \mu \wedge 1)$ is zero, i.e., that μ is associative. $\square$

3.2.3 The E_∞ Ring Structure on $C\tau$

In this Section, we will use Robinson's obstruction theory from Section 3.2.1 to construct the E_∞ ring structure on $C\tau$. In the previous Section 3.2.2 we endowed $C\tau$ with a unital, associative and commutative monoid structure in the the homotopy category $\mathrm{Ho}(\mathbf{Spt}_{\mathbb{C}})$. Recall that this to a 3-stage in Robinson's obstruction theory. We will now use Corollary 3.9 to rigidify this multiplication to an E_∞ ring structure in $\mathbf{Spt}_{\mathbb{C}}$. Although not needed for the E_∞ ring structure, as a warm-up, we first show in Proposition 3.17 that $C\tau$ admits a unique A_∞ ring structure.

Proposition 3.17. *The multiplication μ on $C\tau$ can be uniquely extended to an A_∞ multiplication.*

Proof. An A_2 structure corresponds to unital homotopies (left and right), and an A_3 structure adds an associative homotopy. We constructed both structures in Proposition 3.16. The A_∞ obstruction theory originated in [50] exhibits obstruction classes to extend an A_{n-1} structure to an A_n structure. In more modern language, [2, Theorem 3.1] exhibits the obstruction to go from A_{n-1} structure to an A_n structure as an element in the abelian group

$$\left[\Sigma^{n-3,0} S^{n,-n}, C\tau\right] \cong \left[S^{2n-3,-n}, C\tau\right] = \pi_{2n-3,-n}(C\tau). \tag{3.6}$$

Corollary 3.4 shows that these groups are zero for any n (we really just need $n \geq 4$), which shows that μ can be extended to an A_∞ structure. Furthermore, given that an A_{n-1} structure extends to A_n structure, the possible extensions are in bijection with the abelian group

$$\left[\Sigma^{n-2,0} S^{n,-n}, C\tau\right] \cong \pi_{2n-2,-n}(C\tau).$$

This group is also zero for any n, showing that μ can be uniquely extended to an A_∞

structure. □

Remark 3.18. For the case $n = 3$, i.e., to endow $C\tau$ with an A_3 structure, the obstruction group from equation (3.6) is $\pi_{3,-3}(C\tau)$. Observe that this is the exact same group that appears in Proposition 3.16, where we show with elementary techniques that $C\tau$ admits an A_3 structure.

Remark 3.19. Mahowald conjectured that no non-trivial topological 2-cell complex posses an A_∞ structure. There are 2 trivial cases to exclude which are the cofiber of the zero map and the cofiber of the identity map, as shown in the cofiber sequences

$$S^0 \xrightarrow{0} S^0 \longrightarrow S^1 \vee S^0 \qquad \text{and} \qquad S^0 \xrightarrow{\text{id}} S^0 \longrightarrow *.$$

Since motivic spheres Betti realize to topological spheres, motivic 2-cell complexes Betti realize to topological 2-cell complexes. Moreover, since we are using simplicial (constant) operads, motivic algebras over A_n or E_n operads realize to classical algebras over the same A_n or E_n operads. However, the fact that $C\tau$ admits an A_∞ ring structure does not contradict Mahowald's conjecture, as the map $S^{0,-1} \xrightarrow{\tau} S^{0,0}$ realizes to the identity map $S^0 \xrightarrow{\text{id}} S^0$.

Theorem 3.20. *The multiplication μ on $C\tau$ can be uniquely extended to an E_∞ multiplication.*

Proof. We showed in Proposition 3.16 that $C\tau$ is a unital, associative and commutative monoid in the homotopy category $\mathrm{Ho}(\mathbf{Spt}_{\mathbb{C}})$. This corresponds to a 3-stage in Robinson's obstruction theory. By Corollary 3.9, the obstructions of extending this 3-stage to an E_∞ ring structure live in

$$\pi_{n-3,0}(F(C\tau^{\wedge m}, C\tau)) \cong \left[\Sigma^{n-3,0} C\tau^{\wedge m}, C\tau\right]$$

for $n \geq 4$ and $2 \leq m \leq n$. Recall from Corollary 3.5 that $[\Sigma^{s,w} C\tau, C\tau]$ has in particular a vanishing region for $s \geq 0$ and $2w \leq s$. We now show that all obstruction groups live in this vanishing area. By the equivalence

$$[C\tau^{\wedge m}, C\tau] \cong \bigoplus_{i=0}^{m-1} \binom{m-1}{i} [\Sigma^{i,-i} C\tau, C\tau]$$

of Corollary 3.14, we have

$$\pi_{n-3,0}(F(C\tau^{\wedge m}, C\tau)) \cong \left[\Sigma^{n-3,0} C\tau^{\wedge m}, C\tau\right] \cong \bigoplus_{i=0}^{m-1} \binom{m-1}{i} \left[\Sigma^{n-3+i,-i} C\tau, C\tau\right].$$

In particular, all the obstructions live in groups of the form $[\Sigma^{s,w} C\tau, C\tau]$ where the s-coordinate satisfies

$$s = n - 3 + i \geq 4 - 3 + i \geq 1$$

while the w-coordinate satisfies both

$$w = -i \leq 0 \qquad \text{and} \qquad w = -i = n - s - 3 \geq 1 - s.$$

This corresponds to the region bounded by $s \geq 1$ and $1 - s \leq w \leq s$, which lies entirely in the vanishing area described above. The situation is summarized in Figure 3. Similarly, recall from Corollary 3.9 that the obstructions for uniqueness of such an E_∞ ring structure live in groups of the form

$$\pi_{n-2,0}(F(C\tau^{\wedge m}, C\tau)) \cong \left[\Sigma^{n-2,0} C\tau^{\wedge m}, C\tau\right].$$

A similar analysis shows that all obstruction groups again live in the vanishing region, as described in Figure 3. This shows that $C\tau$ admits a unique E_∞ ring structure. □

Corollary 3.21. *There is an isomorphism of rings*

$$\pi_{*,*}(C\tau) \cong \mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, BP_*),$$

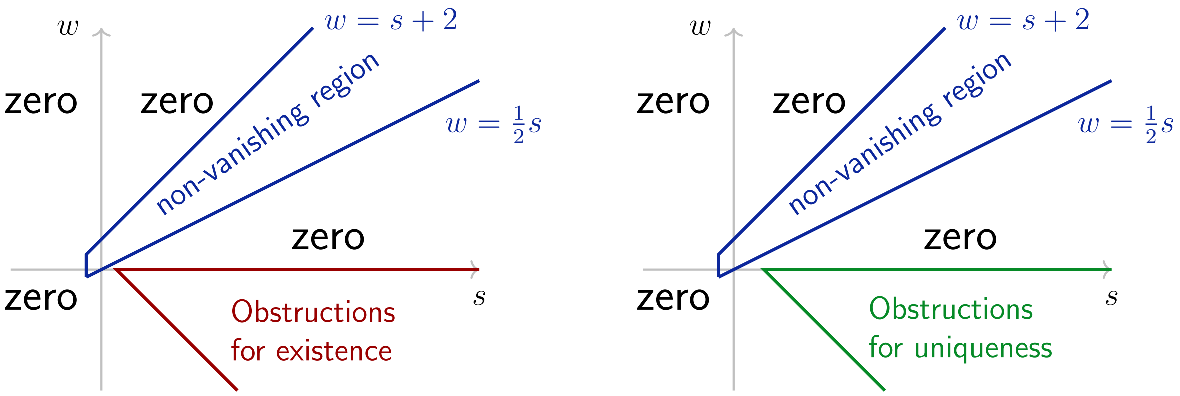


Figure 3: Chart of $[\Sigma^{s,w} C\tau, C\tau]$ where all obstruction groups live in the vanishing region.

which sends Massey products in Ext *to Toda brackets in* $\pi_{*,*}$*, and vice-versa.*

Proof. Since $C\tau$ is an E_∞ ring spectrum, its motivic Adams-Novikov spectral sequence is multiplicative and converges to an associated graded of the ring $\pi_{*,*}(C\tau)$. Recall from Proposition 3.1 that the spectral sequence collapses at E_2 with no possible hidden extensions as a module over the spectral sequence for $S^{0,0}$. For the exact same reason, there are no possible hidden extensions as a multiplicative spectral sequence. By the Moss Convergence Theorem [43], we get a highly structured bigraded isomorphism

$$\mathrm{Ext}_{BPGL_{*,*}BPGL}(BPGL_{*,*}, BPGL_{*,*}/\tau) \cong \pi_{*,*}(C\tau), \tag{3.7}$$

between the E_2-page and the output of the spectral sequence. More precisely, Massey products computed in Ext converge to Toda brackets computed in $\pi_{*,*}(C\tau)$.

Until the end of the proof, denote the motivic Brown-Peterson spectrum $BPGL$ by B. To finish the proof, we have to show that there is a highly structured ring isomorphism

$$\mathrm{Ext}_{B_{*,*}B/\tau}(B_{*,*}/\tau, B_{*,*}/\tau) \cong \mathrm{Ext}_{B_{*,*}B}(B_{*,*}, B_{*,*}/\tau).$$

These are Ext-groups computed in comodules and since the first variable is projective (even free) over the base ring, both of those Ext terms can be computed from their cobar complex [47, Corollary A1.2.12]. Moreover, since the cobar complex also controls the Massey products in the Ext-ring, this will give an isomorphism preserving this structure. The cobar complex of the left Ext-group is given by

$$B_{*,*}/\tau \otimes_{B_{*,*}/\tau} B_{*,*}B/\tau \otimes_{B_{*,*}/\tau} B_{*,*}/\tau \longrightarrow B_{*,*}/\tau \otimes_{B_{*,*}/\tau} B_{*,*}B/\tau \otimes_{B_{*,*}/\tau} B_{*,*}B/\tau \otimes_{B_{*,*}/\tau} B_{*,*}/\tau \longrightarrow \cdots ,$$

while the cobar complex of the right term is given by

$$B_{*,*} \otimes_{B_{*,*}} B_{*,*}B \otimes_{B_{*,*}} B_{*,*}/\tau \longrightarrow B_{*,*} \otimes_{B_{*,*}} B_{*,*}B \otimes_{B_{*,*}} B_{*,*}B \otimes_{B_{*,*}} B_{*,*}/\tau \longrightarrow \cdots .$$

By iterating the ring isomorphism

$$B_{*,*}/\tau \otimes_{B_{*,*}/\tau} B_{*,*}B/\tau \cong B_{*,*} \otimes_{B_{*,*}} B_{*,*}/\tau,$$

these cobar complexes are isomorphic as dga's. By taking cohomology, we get an isomorphism

$$\mathrm{Ext}_{B_{*,*}B}(B_{*,*}, B_{*,*}/\tau) \cong \mathrm{Ext}_{B_{*,*}B/\tau}(B_{*,*}/\tau, B_{*,*}/\tau) \tag{3.8}$$

that preserves Massey products. The trigraded Ext-term $\mathrm{Ext}_{B_{*,*}B/\tau}(B_{*,*}/\tau, B_{*,*}/\tau)$ is really bigraded because of the relation $t = 2w$ between the internal degree t and the weight w. Therefore, when working mod τ, we can regrade everything in sight by keeping the internal degree and forgetting the weight. With this convention, the degree of $v_n \in B_*/\tau$ is the single number $2^{n+1} - 2$ and thus there is an isomorphism of Hopf algebroids $B_*B/\tau \cong BP_*BP$. This provides the (higher) ring isomorphism

$$\mathrm{Ext}_{B_*B/\tau}(B_*/\tau, B_*/\tau) \cong \mathrm{Ext}_{BP_*BP}(BP_*, BP_*). \tag{3.9}$$

By combining the isomorphisms of equation (3.7), (3.8) and (3.9), we get an isomorphism

$$\pi_{*,*}(C\tau) \cong \mathrm{Ext}_{BP_*BP}(BP_*, BP_*)$$

of higher rings, that sends Toda brackets to Massey products and vice-versa. ☐

3.3 (Co-)operations on $C\tau$

In this Section we describe the homotopy types of $C\tau \wedge C\tau$ and $\mathrm{End}(C\tau)$ as ring spectra. Understanding their homotopy types is crucial for the computation of the Steenrod algebra of the spectrum $H\mathbb{F}_2 \wedge C\tau$ in Section 3.4.2. Most proofs are done by diagram chasing and identifying composites of maps.

3.3.1 The Spectrum $C\tau \wedge C\tau$

The E_∞ ring structure on $C\tau$ induces an E_∞ ring structure on the smash product $C\tau \wedge C\tau$ via the multiplication

$$\mu_{C\tau\wedge C\tau} \colon (C\tau \wedge C\tau) \wedge (C\tau \wedge C\tau) \xrightarrow{1\wedge\chi\wedge 1} C\tau \wedge C\tau \wedge C\tau \wedge C\tau \xrightarrow{\mu\wedge\mu} C\tau \wedge C\tau.$$

Here μ denotes the multiplication map on $C\tau$ and χ denotes the factor swap map. Recall from Lemma 3.13 that there is a canonical equivalence

$$C\tau \wedge C\tau \simeq C\tau \vee \Sigma^{1,-1} C\tau,$$

describing the additive homotopy type of $C\tau \wedge C\tau$. The next lemma describes its ring structure.

Lemma 3.22. *Under the canonical vertical identifications given by*

$$\begin{tikzcd}
(C\tau \wedge C\tau) \wedge (C\tau \wedge C\tau) \arrow[r, "\mu_{C\tau\wedge C\tau}"] \arrow[d, dash, "\simeq"'] & C\tau \wedge C\tau \arrow[d, dash, "\simeq"] \\
(C\tau \vee \Sigma^{1,-1}C\tau) \wedge (C\tau \vee \Sigma^{1,-1}C\tau) \arrow[r] \arrow[d, dash, "="'] & C\tau \vee \Sigma^{1,-1}C\tau \arrow[d, dash, "="] \\
(C\tau \wedge C\tau) \vee (\Sigma^{1,-1}C\tau \wedge C\tau) \vee (C\tau \wedge \Sigma^{1,-1}C\tau) \vee (\Sigma^{1,-1}C\tau \wedge \Sigma^{1,-1}C\tau) \arrow[r, dashed] & C\tau \vee \Sigma^{1,-1}C\tau,
\end{tikzcd}$$

the multiplication on $C\tau \wedge C\tau$ *is given by the maps*

$$\begin{aligned} C\tau \wedge C\tau &\xrightarrow{(\mu,0)} C\tau \vee \Sigma^{1,-1}C\tau \\ \Sigma^{1,-1}C\tau \wedge C\tau &\xrightarrow{(0,\mu)} C\tau \vee \Sigma^{1,-1}C\tau \\ C\tau \wedge \Sigma^{1,-1}C\tau &\xrightarrow{(0,\mu)} C\tau \vee \Sigma^{1,-1}C\tau \\ \Sigma^{1,-1}C\tau \wedge \Sigma^{1,-1}C\tau &\xrightarrow{(0,0)} C\tau \vee \Sigma^{1,-1}C\tau. \end{aligned}$$

Proof. These four maps are given by a simple diagram chase, where we only have to be careful with the identifications. For simplicity, let's denote the sphere spectrum $S^{0,0}$ by S, and ignore or denote by 1 some identity maps id in the following diagrams. Recall the cofiber sequence

$$S^{0,-1} \xrightarrow{\tau} S^{0,0} \xrightarrow{i} C\tau \xrightarrow{p} S^{1,-1}$$

from equation (3.1). The first map $C\tau \wedge C\tau \longrightarrow C\tau \vee \Sigma^{1,-1}C\tau$ corresponds to the composite

$$(\mu, p \wedge 1) \circ (\mu \wedge \mu) \circ (1 \wedge \chi \wedge 1) \circ (i \wedge i),$$

which is embedded in the commutative diagram

$$\begin{tikzcd}
(C\tau \wedge C\tau) \wedge (C\tau \wedge C\tau) \arrow[r, "1 \wedge \chi \wedge 1"] & C\tau \wedge C\tau \wedge C\tau \wedge C\tau \arrow[r, "\mu \wedge \mu"] & C\tau \wedge C\tau \arrow[r, "{(\mu, p \wedge 1)}"] & C\tau \vee \Sigma^{1,-1}C\tau \\
(S \wedge C\tau) \wedge (S \wedge C\tau) \arrow[u, "i \wedge i"] \arrow[r, dash, "\simeq"] & S \wedge S \wedge C\tau \wedge C\tau \arrow[u, "i \wedge i"] \arrow[r, dash, "\simeq"] & S \wedge C\tau \wedge C\tau. \arrow[u, "i \wedge \mu"] &
\end{tikzcd}$$

We can compute by the other path, where we use that the map

$$S \wedge C\tau \wedge C\tau \xrightarrow{i \wedge \mu} C\tau \wedge C\tau$$

decomposes as

$$S \wedge C\tau \wedge C\tau \xrightarrow{1\wedge\mu} S \wedge C\tau \xrightarrow{i\wedge 1} C\tau \wedge C\tau,$$

and by using that $p \circ i = 0$ and $\mu \circ (i \wedge 1) = \mathrm{id}$. For the second map, the canonical splitting of Lemma 3.13 induces a splitting

$$\Sigma^{1,-1}C\tau \wedge C\tau \simeq \Sigma^{1,-1}C\tau \vee \Sigma^{2,-2}C\tau.$$

By Corollary 3.7 we have $[\Sigma^{1,-1}C\tau, C\tau] = [\Sigma^{2,-2}C\tau, C\tau] = 0$, and thus the second map

$$\Sigma^{1,-1}C\tau \wedge C\tau \longrightarrow C\tau \vee \Sigma^{1,-1}C\tau$$

corestricts to zero on $C\tau$. To compute the other part, recall first from Lemma 3.13 that the map $p \wedge 1$ admits a canonical section s, as shown in the cofiber sequence

$$\begin{tikzcd}
S^{0,-1} \wedge C\tau \arrow[r, "\tau = 0"] & S \wedge C\tau \arrow[r, "i_L"] & C\tau \wedge C\tau \arrow[r, "p_L"] \arrow[l, dotted, bend left, "\exists!\ \mu"] & S^{1,-1} \wedge C\tau \arrow[r, "\tau = 0"] \arrow[l, dotted, bend left, "\exists!\ s"] & \cdots .
\end{tikzcd}$$

The second map is the composite in the commutative diagram

$$\begin{tikzcd}
(C\tau \wedge C\tau) \wedge (C\tau \wedge C\tau) \arrow[r, "1 \wedge \chi \wedge 1"] & C\tau \wedge C\tau \wedge C\tau \wedge C\tau \arrow[r, "\mu \wedge \mu"] & C\tau \wedge C\tau \arrow[r, "p \wedge 1"] & \Sigma^{1,-1}C\tau \\
(S^{1,-1} \wedge C\tau) \wedge (S \wedge C\tau) \arrow[u, "s \wedge (i \wedge 1)"] \arrow[r, "s \wedge (1 \wedge 1)"'] & C\tau \wedge C\tau \wedge S \wedge C\tau \arrow[ul, dashed, "i"'] \arrow[r, dash, "\simeq"] & C\tau \wedge S \wedge C\tau \wedge C\tau. \arrow[u, "1 \wedge \mu"'] \arrow[ul, dashed, "i"'] &
\end{tikzcd}$$

We again compute it by following the other path

$$(p \wedge 1) \circ (1 \wedge \mu) \circ (s \wedge (1 \wedge 1)).$$

The result follows by noticing that the last two maps $p \wedge 1$ and $1 \wedge \mu$ commute with each other, together with the fact that s is a section of $p \wedge 1$. For the third map, we can either do a similar diagram chase, or use the fact that $C\tau \wedge C\tau$ is an E_∞ ring spectrum, and so the third map is homotopic to the second map we just computed. The last map is forced to be nullhomotopic since

$$\Sigma^{1,-1}C\tau \wedge \Sigma^{1,-1}C\tau \simeq \Sigma^{3,-3}C\tau \vee \Sigma^{2,-2}C\tau$$

and there are no non-trivial maps to both $C\tau$ and $\Sigma^{1,-1}C\tau$ by Corollary 3.5. □

The additive splitting $C\tau \wedge C\tau \simeq C\tau \vee \Sigma^{1,-1}C\tau$ gives the isomorphism

$$\pi_{*,*}(C\tau \wedge C\tau) \cong \pi_{*,*}(C\tau) \oplus \beta_\tau \cdot \pi_{*,*}(C\tau).$$

The class β_τ has degree $|\beta_\tau| = (1,-1)$, and is the unit element of the shifted copy given by the composite

$$S^{1,-1} \simeq S^{1,-1} \wedge S^{0,0} \xrightarrow{1\wedge i} S^{1,-1} \wedge C\tau \xrightarrow{s} C\tau \wedge C\tau.$$

We call it β_τ because it induces a τ-Bockstein operations in $H\mathbb{F}_2 \wedge C\tau$-(co)homology, as we show in Propositions 3.33 and 3.34. Lemma 3.22 gives the following multiplicative description of the homotopy groups $\pi_{*,*}(C\tau \wedge C\tau)$.

Corollary 3.23. *The E_∞ ring spectrum $C\tau \wedge C\tau$ has homotopy ring*

$$\pi_{*,*}\left(C\tau \wedge C\tau\right) \cong \pi_{*,*}\left(C\tau\right)\left[\beta_\tau\right] \Big/ \beta_\tau^2,$$

where $|\beta_\tau| = (1,-1)$.

3.3.2 The Endomorphism Spectrum End($C\tau$)

In this Section we explicitly describe the homotopy type of End($C\tau$) as a ring spectrum and give a presentation of its homotopy ring $\pi_{*,*}(\mathrm{End}(C\tau))$, in the same way that we did for $C\tau \wedge C\tau$. However, the endomorphism spectrum End($C\tau$) is a little harder to understand than $C\tau \wedge C\tau$. First, it is only an associative A_∞ spectrum, whereas $C\tau \wedge C\tau$ is E_∞. Second, its multiplication comes from composition of morphisms and has nothing to do with the fact that $C\tau$ is a ring object, whereas the multiplication on $C\tau \wedge C\tau$ is easy to describe in terms of the multiplication of $C\tau$. Finally, it turns out that out of the eight maps that assemble together to give the multiplication on $\mathrm{End}(C\tau)$, only three are forced to be nullhomotopic for degree reasons, whereas five where forced to be nullhomotopic for $C\tau \wedge C\tau$.

An important tool that we use is Spanier-Whitehead duality, adapted to the motivic setting from the categorical treatment in [31, Chapter 3]. We briefly recall some notation and elementary results from both [31, Chapter 3] and [32, Sections 4.6-7]. Consider two motivic spectra X and Y. If X is dualizable, its Spanier-Whitehead dual is defined to be the motivic spectrum

$$DX \coloneqq F(X, S^{0,0}).$$

In particular, finite cell complexes are dualizable. For spheres, there is a canonical identification

$$DS^{m,n} = F(S^{m,n}, S^{0,0}) \simeq F(S^{0,0}, S^{-m,-n}) \simeq S^{-m,-n}. \tag{3.10}$$

Given a map $f\colon X \longrightarrow Y$ between dualizable motivic spectra, denote its Spanier-Whitehead dual by

$$Df \coloneqq F(f, S^{0,0})\colon DY \longrightarrow DX.$$

If X is dualizable, the smashing morphism $F(X, S^{0,0}) \wedge X \xrightarrow{\wedge} F(X, S^{0,0} \wedge X)$ is an equivalence, giving the equivalence

$$DX \wedge X = F(X, S^{0,0}) \wedge X \xrightarrow{\simeq} F(X, S^{0,0} \wedge X) = \mathrm{End}(X). \tag{3.11}$$

Denote the evaluation map that is adjoint to the identity map on $F(X, S^{0,0})$ by

$$DX \wedge X = F(X, S^{0,0}) \wedge X \xrightarrow{\mathrm{ev}} S^{0,0}.$$

The endomorphism spectrum $\mathrm{End}(X)$ is always a motivic A_∞ ring spectrum with multiplication map given by the composite $\mu_{\mathrm{End}(X)}$ in the diagram

$$\begin{tikzcd}
\mathrm{End}(X) \wedge \mathrm{End}(X) \arrow[rr, dashed, "\mu_{\mathrm{End}(X)}"] \arrow[d, equal, "\mathrm{can.}"'] & & \mathrm{End}(X) \arrow[d, equal, "\mathrm{can.}"'] \\
DX \wedge X \wedge DX \wedge X \arrow[r, "1 \wedge \chi \wedge 1"] & DX \wedge DX \wedge X \wedge X \arrow[r, "1 \wedge \mathrm{ev} \wedge 1"] & DX \wedge S^{0,0} \wedge X.
\end{tikzcd} \tag{3.12}$$

The spectrum $C\tau$ is dualizable since it is a 2-cell complex. The A_∞ ring structure on $\mathrm{End}(C\tau)$ can thus be understood in terms of Spanier-Whitehead duality. For this, we have to compute the homotopy type of the Spanier-Whitehead dual $DC\tau$ and identify the evaluation map $DC\tau \wedge C\tau \xrightarrow{\mathrm{ev}} S^{0,0}$.

Proposition 3.24. *We have the following identifications.*

(1) The Spanier-Whitehead dual of $S^{0,-1} \xrightarrow{\tau} S^{0,0}$ *is* $D\tau \simeq \tau \colon S^{0,0} \longrightarrow S^{0,1}$.

(2) The Spanier-Whitehead dual of the cofiber sequence

$$S^{0,-1} \xrightarrow{\tau} S^{0,0} \xrightarrow{i} C\tau \xrightarrow{p} S^{1,-1}$$

is the cofiber sequence

$$S^{0,1} \xleftarrow{\tau} S^{0,0} \xleftarrow{p} \Sigma^{-1,1} C\tau \xleftarrow{i} S^{-1,1}.$$

In particular we have $Di \simeq p$ *and* $Dp \simeq i$, *and a canonical (up to homotopy) identification*

$$DC\tau \simeq \Sigma^{-1,1} C\tau. \tag{3.13}$$

Proof.

(1) Start with the map $S^{0,-1} \xrightarrow{\tau} S^{0,0}$. The functor $D = F(-, S^{0,0})$ and the canonical identification of equation (3.10) gives a map $S^{0,0} \xrightarrow{D\tau} S^{0,1}$, which by definition, sends 1 to τ on $\pi_{0,0}$. Since it lives in the group $[S^{0,0}, S^{0,1}] \cong \hat{\mathbb{Z}}_2$ generated by τ, we get that

$D\tau \simeq \tau$.

(2) Since the dualization functor D preserves cofiber sequences, we get the cofiber sequence

$$DS^{0,-1} \xleftarrow{D\tau} DS^{0,0} \xleftarrow{Di} DC\tau \xleftarrow{Dp} DS^{1,-1}.$$

To understand it, we use the canonical equivalences of equation (3.10) and embed it in the diagram

$$\begin{tikzcd}
DS^{0,-1} \arrow[d, equal, "\text{can.}"'] & DS^{0,0} \arrow[l, "D\tau"'] \arrow[d, equal, "\text{can.}"'] & DC\tau \arrow[l, "Di"'] & DS^{1,-1} \arrow[l, "Dp"'] \arrow[d, equal, "\text{can.}"'] \\
S^{0,1} & S^{0,0} \arrow[l, "\tau"'] & \Sigma^{-1,1}C\tau \arrow[l, "p"'] \arrow[u, dashed] & S^{-1,1}. \arrow[l, "i"']
\end{tikzcd}$$

By the 5-lemma, the map $\Sigma^{-1,1}C\tau \longrightarrow DC\tau$ is an equivalence. Moreover, given two such equivalences, their difference would factor trough the map p and thus trough $S^{0,0}$. It follows that this equivalence is canonical up to homotopy, since by Corollary 3.4 we have

$$\pi_{0,0}(DC\tau) \cong \pi_{0,0}(\Sigma^{-1,1}C\tau) \cong \pi_{1,-1}(C\tau) = 0.$$

□

Lemma 3.25. *Up to sign, the evaluation map $DC\tau \wedge C\tau \xrightarrow{\text{ev}} S^{0,0}$ is given by the commutative diagram*

$$\begin{tikzcd}
DC\tau \wedge C\tau \arrow[r, "ev"] \arrow[d, dash, "\simeq"', "\text{can.}"] & S^{0,0} \\
\Sigma^{-1,1}C\tau \wedge C\tau \arrow[r, "\mu"] & \Sigma^{-1,1}C\tau. \arrow[u, "p"']
\end{tikzcd}$$

Proof. We compute the abelian group of homotopy classes of maps $[DC\tau \wedge C\tau, S^{0,0}]$. We have

$$\begin{aligned}
[DC\tau \wedge C\tau, S^{0,0}] &\cong [\Sigma^{-1,1}C\tau \wedge C\tau, S^{0,0}] && \text{by equation (3.13)}\\
&\cong [\Sigma^{-1,1}C\tau \vee C\tau, S^{0,0}] && \text{by Lemma 3.13}\\
&\cong [\Sigma^{-1,1}C\tau, S^{0,0}] \oplus [C\tau, S^{0,0}] && \\
&\cong [S^{0,0}, S^{0,0}] \oplus 0 && \text{via } \Sigma^{-1,1}C\tau \xrightarrow{p} S^{0,0}\\
&\cong \hat{\mathbb{Z}}_2
\end{aligned}$$

which is generated by the identity. This means that $[DC\tau \wedge C\tau, S^{0,0}]$ is generated by the composite

$$DC\tau \wedge C\tau \simeq \Sigma^{-1,1}C\tau \wedge C\tau \xrightarrow{\mu} \Sigma^{-1,1}C\tau \xrightarrow{p} S^{0,0}.$$

On the other side, by adjunction we have an isomorphism

$$[DC\tau, DC\tau] \cong \left[DC\tau \wedge C\tau, S^{0,0}\right],$$

which sends the identity map to the evaluation map (by definition of the evaluation map). This shows that ev is also one of the two units $\pm 1 \in \hat{\mathbb{Z}}_2$, finishing the proof. $\square$

Lemma 3.26. *Under the vertical identifications given by*

$$\begin{tikzcd}
\mathrm{End}(C\tau) \wedge \mathrm{End}(C\tau) \arrow[r, "\mu_{\mathrm{End}(C\tau)}"] \arrow[d, dash, "\simeq"'] & \mathrm{End}(C\tau) \arrow[d, dash, "\simeq"] \\
(\Sigma^{-1,1}C\tau \vee C\tau) \wedge (\Sigma^{-1,1}C\tau \vee C\tau) \arrow[r] \arrow[d, dash, "="'] & \Sigma^{-1,1}C\tau \vee C\tau \arrow[d, dash, "="] \\
(\Sigma^{-1,1}C\tau \wedge \Sigma^{-1,1}C\tau) \vee (\Sigma^{-1,1}C\tau \wedge C\tau) \vee (C\tau \wedge \Sigma^{-1,1}C\tau) \vee (C\tau \wedge C\tau) \arrow[r, dashed] & \Sigma^{-1,1}C\tau \vee C\tau,
\end{tikzcd}$$

the multiplication on $\mathrm{End}(C\tau)$ *is given by the maps*

$$\Sigma^{-1,1}C\tau \wedge \Sigma^{-1,1}C\tau \xrightarrow{(p\wedge 1,0)} \Sigma^{-1,1}C\tau \vee C\tau$$

$$\Sigma^{-1,1}C\tau \wedge C\tau \xrightarrow{(\mu,0)} \Sigma^{-1,1}C\tau \vee C\tau$$

$$C\tau \wedge \Sigma^{-1,1}C\tau \xrightarrow{(\mu,p\wedge 1)} \Sigma^{-1,1}C\tau \vee C\tau$$

$$C\tau \wedge C\tau \xrightarrow{(0,\mu)} \Sigma^{-1,1}C\tau \vee C\tau.$$

Sketch of proof. This proof is by tedious diagram chases, and is in the spirit as the proof of Lemma 3.22. We will now briefly sketch the steps in the proof. The first part is to break $\mathrm{End}(C\tau)\wedge\mathrm{End}(C\tau)$ in more manageable summands via Spanier-Whitehead duality, and the necessary identifications are done in Proposition 3.24. We then use the definition of the multiplication map on $\mathrm{End}(C\tau)$ from diagram (3.12), as a composite of the factor swap map and the evaluation map. The evaluation map was explicitly computed in Lemma 3.25. The remainder of the proof consists on carefully identifying composites. ☐

The additive splitting $\mathrm{End}(C\tau) \simeq C\tau \vee \Sigma^{-1,1}C\tau$ gives the isomorphism

$$\pi_{*,*}(\mathrm{End}(C\tau)) \cong \pi_{*,*}(C\tau) \oplus \beta_\tau \cdot \pi_{*,*}(C\tau).$$

The class β_τ has degree $|\beta_\tau| = (-1,1)$, and is the unit element of the shifted copy given by the composite given by the composite

$$C\tau \xrightarrow{p} S^{1,-1} \xrightarrow{\Sigma i} \Sigma^{1,-1}C\tau.$$

Lemma 3.26 gives the following multiplicative description of the homotopy groups $\pi_{*,*}(\mathrm{End}(C\tau))$.

Corollary 3.27. *The A_∞ ring spectrum* $\mathrm{End}(C\tau)$ *has homotopy ring*

$$\pi_{*,*}\left(\mathrm{End}(C\tau)\right) \cong \pi_{*,*}\left(C\tau\right)\langle\beta_\tau\rangle \Big/ \begin{array}{l} \alpha\beta_\tau - (-1)^{|\alpha|}\beta_\tau\alpha = i\circ p(\alpha) \\ \beta_\tau^2 = 0 \end{array}$$

where β_τ is a non-commutative variable and α span the elements of $\pi_{*,*}(C\tau)$.

Remark 3.28. The canonical inclusion $C\tau \longrightarrow \mathrm{End}(C\tau)$ is a map of A_∞ ring spectra and on homotopy is the inclusion of $\pi_{*,*}(C\tau)$ onto the non-shifted factor. We can also think of the ring $\pi_{*,*}(\mathrm{End}(C\tau))$ as being the abelian group

$$\pi_{*,*}(\mathrm{End}(C\tau)) \cong \pi_{*,*}(C\tau) \oplus \beta_\tau \cdot \pi_{*,*}(C\tau)$$

with ring structure given by the following multiplication table

$$\begin{aligned}
\alpha \circ \alpha' &= \alpha\alpha' \\
\alpha \circ \beta_\tau \alpha' &= (-1)^{|\alpha|}\beta_\tau \alpha\alpha' + (i \circ p(\alpha))\alpha' \\
\beta_\tau \alpha \circ \alpha' &= \beta_\tau \alpha\alpha' \\
\beta_\tau \alpha \circ \beta_\tau \alpha' &= \beta_\tau (i \circ p(\alpha))\alpha',
\end{aligned}$$

where $\alpha, \alpha' \in \pi_{*,*}(C\tau)$ and $\beta_\tau \alpha, \beta_\tau \alpha' \in \beta_\tau \cdot \pi_{*,*}(\Sigma^{-1,1}C\tau)$.

Remark 3.29. Since $S^{0,0} \xrightarrow{i} C\tau$ is the ring map which induces the $\pi_{*,*}(S^{0,0})$-module structure on $\pi_{*,*}(C\tau)$, we have the compatibility formula

$$i(\alpha)\alpha' = \alpha\alpha' \qquad \text{for } \alpha \in \pi_{*,*}(S^{0,0}), \alpha' \in \pi_{*,*}(C\tau).$$

The first multiplication uses the ring structure of $C\tau$ while the second uses the $S^{0,0}$-module structure on $C\tau$. This simplifies some of the formulas of Corollary 3.27, for example by $\beta_\tau \alpha \circ \beta_\tau \alpha' = \beta_\tau p(\alpha)\alpha'$ since $p(\alpha)$ is in the homotopy groups of the motivic sphere.

3.4 Examples of $C\tau$-Modules

Since the 2-cell complex $C\tau$ is a (cofibrant) commutative ring spectrum, we can use [46, Section 2.8] to endow the category ${}_{C\tau}\mathbf{Mod}$ with a closed symmetric monoidal model structure. The closed monoidal structure is given by the the relative smash product $- \wedge_{C\tau} -$ and the internal function spectrum $F_{C\tau}(-,-)$. Moreover, the model structure is created by the forgetful functor, and is thus part of the Quillen adjunction

$$\mathbf{Spt}_{\mathbb{C}} = {}_{S^{0,0}}\mathbf{Mod} \underset{U}{\overset{-\wedge C\tau}{\rightleftarrows}} {}_{C\tau}\mathbf{Mod}. \tag{3.14}$$

In this section we will first give some elementary lemmas about the category ${}_{C\tau}\mathbf{Mod}$, and then study some important spectra that are induced up from $S^{0,0}$-modules by smashing with $- \wedge C\tau$. We call such a spectrum a *$C\tau$-induced spectrum.*

We start with the $C\tau$-induced Eilenberg-Maclane spectrum $H\mathbb{F}_2 \wedge C\tau$ which has homotopy groups $\pi_{*,*}(H\mathbb{F}_2 \wedge C\tau) \cong \mathbb{F}_2$ in degree $(0,0)$. We will compute its Steenrod algebra of operations (and its dual) as a Hopf algebra, both in $\mathbf{Spt}_{\mathbb{C}}$ and ${}_{C\tau}\mathbf{Mod}$. This computation is used in future work [15] to construct Morava K-theories for the motivic w_i periodic operators. The first operator w_1 was introduced in [1]. We then show that the $C\tau$-induced Moore spectrum $S/(2,\tau)$ admits a unique structure of an E_∞ algebra over $C\tau$. We also observe that it admits a v_1^1-self map, whereas $S^{0,0}/2$ only admits a v_1^4-self map. Finally, we compute the homology and homotopy of the $C\tau$-induced connective algebraic and hermitian K-theory spectra kgl and kq. Here again an interesting phenomenon arises in hermitian K-theory: an obstruction is killed and we can see the element v_1^2 in the homotopy of $kq \wedge C\tau$, whereas we only see its square v_1^4 in kq.

3.4.1 Elementary Results on $C\tau$-Modules

Let X be a (left) $C\tau$-module with action map $\phi_X\colon C\tau \wedge X \longrightarrow X$. The left unitality condition says that the triangle in the diagram

$$\begin{tikzcd}
S^{0,-1} \wedge X \arrow[r, "\tau"] & S^{0,0} \wedge X \arrow[r, "i"] \arrow[dr, "\simeq"] & C\tau \wedge X \arrow[r, "p"] \arrow[d, "\phi_X"] & S^{1,-1} \wedge X \\
 & & X &
\end{tikzcd}$$

commutes, i.e., that ϕ_X is a retraction of the unit. This produces a splitting

$$C\tau \wedge X \xrightarrow{(\phi_X, p)} X \vee \Sigma^{1,-1} X \tag{3.15}$$

up to homotopy, whose inverse map requires a choice of section of p. There is however a canonical choice of section given by the composite

$$S^{1,-1} \wedge X = S^{1,-1} \wedge S^{0,0} \wedge X \xrightarrow{\mathrm{id} \wedge i \wedge \mathrm{id}} S^{1,-1} \wedge C\tau \wedge X \xrightarrow{s \wedge \mathrm{id}} C\tau \wedge C\tau \wedge X \xrightarrow{\mathrm{id} \wedge \phi_X} C\tau \wedge X,$$

by using the canonical section $s\colon \Sigma^{1,-1}C\tau \longrightarrow C\tau \wedge C\tau$ from Lemma 3.13. The Betti realization functor $\mathbf{Spt}_{\mathbb{C}} \longrightarrow \mathbf{Spt}$ naturally extends to ${}_{C\tau}\mathbf{Mod}$ by composing with the forget functor

$$ {}_{C\tau}\mathbf{Mod} \longrightarrow \mathbf{Spt}_{\mathbb{C}} \xrightarrow{\mathrm{Re}_{\mathbb{C}}} \mathbf{Spt}. $$

Lemma 3.30. *Every $C\tau$-module realizes to a contractible spectrum in **Top**.*

Proof. Consider a spectrum $X \in \mathbf{Spt}_{\mathbb{C}}$ endowed with a structure of $C\tau$-module. Since the Betti realization functor is (strict) symmetric monoidal and sends $C\tau$ to a contractible spectrum, we have

$$ \mathrm{Re}_{\mathbb{C}}(C\tau \wedge X) \simeq \mathrm{Re}_{\mathbb{C}}(C\tau) \wedge \mathrm{Re}_{\mathbb{C}}(X) \simeq *. $$

It follows that $\mathrm{Re}_{\mathbb{C}}(X) \simeq *$ as X is a retract of $C\tau \wedge X$ by equation (3.15). ☐

The next two elementary lemmas will often be used for studying $C\tau$-induced spectra.

Lemma 3.31. *Let X be a spectrum with τ-free homotopy (resp. homology) groups, i.e., multiplication by τ is injective on $\pi_{*,*}(X)$ (resp. on $H\mathbb{F}_{2*,*}(X)$). Then the homotopy (resp. homology) groups of the $C\tau$-induced spectrum $X \wedge C\tau$ are given by*

$$ \pi_{*,*}(X \wedge C\tau) \cong {\pi_{*,*}(X)}/{\tau} \qquad \textit{(resp. } H\mathbb{F}_{2*,*}(X \wedge C\tau) \cong {H\mathbb{F}_{2*,*}(X)}/{\tau}\ \textit{).} $$

Moreover if X is an E_∞ ring spectrum, then this isomorphism is a ring isomorphism.

Proof. This follows by the long exact sequence induced from the cofiber sequence

$$ \Sigma^{0,-1}X \xrightarrow{\tau} X \xrightarrow{i} C\tau \wedge X $$

since multiplication by τ is injective. Moreover, if X is an E_∞ ring spectrum, then the map

$$ S^{0,0} \wedge X \xrightarrow{i\wedge \mathrm{id}} C\tau \wedge X $$

is a map of E_∞ ring spectra as well. ☐

Lemma 3.32. *Let X be a spectrum with τ-free $H\mathbb{F}_2$-cohomology groups, i.e., multiplication by τ is injective on ${H\mathbb{F}_2}^{*,*}(X)$. Then the cohomology groups of the $C\tau$-induced spectrum $X \wedge C\tau$ are given by*

$$ {H\mathbb{F}_2}^{*,*}(X \wedge C\tau) \cong {H\mathbb{F}_2}^{*,*}(\Sigma^{1,-1}X)\,/_{\tau}. $$

Proof. Similarly to the proof of Lemma 3.31, this just follows by the long exact sequence induced from the cofiber sequence

$$\begin{tikzcd} C\tau \wedge X \arrow[r] & \Sigma^{1,-1}X \arrow[r, "\tau"] & \Sigma^{1,0}X \end{tikzcd}$$

since multiplication by τ is injective. ☐

3.4.2 The $C\tau$-Induced Eilenberg-Maclane Spectrum

Consider the $C\tau$-induced Eilenberg-Maclane spectrum

$$ \overline{H} \coloneqq H\mathbb{F}_2 \wedge C\tau, $$

which has homotopy $\pi_{*,*}(\overline{H}) \cong \mathbb{F}_2$ concentrated in degree $(0,0)$ by Lemma 3.31. Unlike $H\mathbb{F}_2$, this spectrum detects both cells of $C\tau$ since

$$ {H\mathbb{F}_2}^{*,*}(C\tau) \cong \begin{cases} \mathbb{F}_2 & \text{if } (*,*) = (0,0) \\ \mathbb{F}_2 & \text{if } (*,*) = (1,-1) \\ 0 & \text{otherwise.} \end{cases} $$

This spectrum plays an important role in the theory of motivic periodicities, as it is the building block of the Morava K-theories $K(w_n)$ and of the Brown-Peterson spectrum wBP that we construct in [15]. Denote the $\overline{H}$-Steenrod algebra of operations in $\overline{H}$-cohomology by

$$ \overline{\mathcal{A}}_{\mathbb{C}} \cong \pi_{-*,-*}\left(F(\overline{H}, \overline{H})\right), $$

and its dual algebra of co-operations in $\overline{H}$-homology by

$$\overset{\vee}{\overline{\mathcal{A}}}_{\mathbb{C}} \cong \pi_{*,*}\left(\overline{H} \wedge \overline{H}\right).$$

The two main ingredients for these computations are our previous knowledge of the $H\mathbb{F}_2$-Steenrod algebra $\mathcal{A}_{\mathbb{C}}$, which we recalled in Section 2.2, and the descriptions of $C\tau \wedge C\tau$ and $\mathrm{End}(C\tau)$ from Section 3.3. Since $\tau \in \mathbb{M}_2$ is an element of the base ring, there is an induced Hopf algebra structure over $\mathbb{M}_2/\tau \cong \mathbb{F}_2$ on the quotients $\mathcal{A}_{\mathbb{C}}/_{\tau}$ and $\overset{\vee}{\mathcal{A}}_{\mathbb{C}}/_{\tau}$.

Proposition 3.33. *The dual $\overline{H}$-Steenrod algebra $\overset{\vee}{\overline{\mathcal{A}}}_{\mathbb{C}}$ has the following Hopf algebra structure*

$$\overset{\vee}{\overline{\mathcal{A}}}_{\mathbb{C}} \cong \overset{\vee}{\mathcal{A}}_{\mathbb{C}}/_{\tau} \otimes E(\beta_\tau) \cong \mathbb{F}_2[\xi_1, \xi_2, \ldots] \otimes E(\tau_0, \tau_1, \ldots) \otimes E(\beta_\tau)$$

where β_τ is a τ-Bockstein in degree $(1,-1)$ which is primitive in the coalgebra structure.

Proof. The dual $\overline{H}$-Steenrod algebra is given by the homotopy groups of the E_∞ ring spectrum

$$\overline{H} \wedge \overline{H} = H\mathbb{F}_2 \wedge C\tau \wedge H\mathbb{F}_2 \wedge C\tau \simeq H\mathbb{F}_2 \wedge H\mathbb{F}_2 \wedge C\tau \wedge C\tau.$$

Since $\pi_{*,*}(\overline{H}) \cong \mathbb{F}_2$, the left and right units of the Hopf algebroid $\pi_{*,*}\left(\overline{H} \wedge \overline{H}\right)$ are flat maps and they agree, turning it into a Hopf algebra. If we smash the canonical equivalence $C\tau \wedge C\tau \simeq C\tau \vee \Sigma^{1,-1}C\tau$ of Lemma 3.13 with $H\mathbb{F}_2 \wedge H\mathbb{F}_2$, we get an additive splitting

$$\overline{H} \wedge \overline{H} \simeq (H\mathbb{F}_2 \wedge H\mathbb{F}_2 \wedge C\tau) \vee \left(\Sigma^{1,-1}H\mathbb{F}_2 \wedge H\mathbb{F}_2 \wedge C\tau\right),$$

into two wedge summands that we can understand individually. Since the dual Steenrod algebra $\overset{\vee}{\mathcal{A}}_{\mathbb{C}}$ is τ-free, Lemma 3.31 gives a ring description of the homotopy

$$\pi_{*,*}(H\mathbb{F}_2 \wedge H\mathbb{F}_2 \wedge C\tau) \cong \overset{\vee}{\mathcal{A}}_{\mathbb{C}}/_{\tau},$$

and thus the dual $\overline{H}$-Steenrod algebra is a free module of rank 2 over $\overset{\vee}{\mathcal{A}}_{\mathbb{C}}$. The first generator in degree $(0,0)$ is the unit given by the ring map

$$S^{0,0} \xrightarrow{i} \overline{H} \wedge \overline{H}.$$

The second generator in degree $(1,-1)$ that we call β_τ is given by the map

$$\beta_\tau \colon S^{1,-1} \xrightarrow{i} \Sigma^{1,-1} C\tau \xrightarrow{s} C\tau \wedge C\tau \xrightarrow{i \wedge i} \overline{H} \wedge \overline{H},$$

where i denotes the inclusion of the bottom cell and s denotes the canonical section of μ, as in Lemma 3.13. We choose the name β_τ because its dual element in the $\overline{H}$-Steenrod algebra does behave like a τ-Bockstein in cohomology, as we explain in Proposition 3.34. To finish the description of the ring structure of $\overline{\mathcal{A}}^{\vee}_{\mathbb{C}}$, we have to compute the product $\beta_\tau \cdot \beta_\tau$ which lands in degree $(2,-2)$. This product is the homotopy class of the composite

$$\beta_\tau \cdot \beta_\tau \colon S^{1,-1} \wedge S^{1,-1} \xrightarrow{\beta_\tau \wedge \beta_\tau} \overline{H} \wedge \overline{H} \wedge \overline{H} \wedge \overline{H} \xrightarrow{\mu} \overline{H} \wedge \overline{H}$$

which is nullhomotopic since $\mu_{C\tau} \circ s \simeq 0$. This gives the ring structure as the tensor products

$$\overline{\mathcal{A}}^{\vee}_{\mathbb{C}} \cong \mathcal{A}^{\vee}_{\mathbb{C}} \otimes E(\beta_\tau) \cong \mathbb{F}_2[\xi_1, \xi_2, \ldots] \otimes E(\tau_0, \tau_1, \ldots) \otimes E(\beta_\tau).$$

For the coalgebra structure, the counit is forced as there is only a copy of $\mathbb{F}_2$ in degree $(0,0)$. It thus only remains to compute the coproduct. The ring map

$$H\mathbb{F}_2 \xrightarrow{i} \overline{H}$$

induces the following map of Hopf algebras

$$\mathcal{A}^{\vee}_{\mathbb{C}} \xrightarrow{\psi} \overline{\mathcal{A}}^{\vee}_{\mathbb{C}} \cong \mathcal{A}^{\vee}_{\mathbb{C}}/\tau \otimes E(\beta_\tau) \colon a \longmapsto a \otimes 1,$$

which can be factored as reduction modulo τ and then inclusion into the $- \otimes 1$ factor. It follows that the coproduct $\Delta(a \otimes 1)$ can be computed by choosing a pre-image a of $a \otimes 1$, computing the coproduct in $\overset{\vee}{\mathcal{A}}_{\mathbb{C}}$, and then pushing it back via ψ. Since the coproduct formula on the ξ_i's and τ_i's in $\overset{\vee}{\mathcal{A}}_{\mathbb{C}}$ does not involve any τ-multiples, the exact same formula holds for the coproduct of elements of the form $a \otimes 1 \in \overset{\vee}{\overline{\mathcal{A}}}_{\mathbb{C}}$. It only remains to compute the diagonal on the element $1 \otimes \beta_\tau$. We show in the next Proposition 3.34 that its dual is exterior in the algebra structure of $\overline{\mathcal{A}}_{\mathbb{C}}$, implying that $1 \otimes \beta_\tau$ is primitive. ☐

Proposition 3.34. *The $\overline{H}$-Steenrod algebra $\overline{\mathcal{A}}_{\mathbb{C}}$ has the following Hopf algebra structure*

$$\overline{\mathcal{A}}_{\mathbb{C}} \cong \mathcal{A}_{\mathbb{C}} /_{\tau} \otimes E(\beta_\tau)$$

where β_τ is a τ-Bockstein in degree $(1,-1)$ which is primitive in the coalgebra structure.

Proof. Since $C\tau$ is dualizable we can rewrite

$$F(\overline{H}, \overline{H}) = F(H\mathbb{F}_2 \wedge C\tau, H\mathbb{F}_2 \wedge C\tau) \simeq F(H\mathbb{F}_2, H\mathbb{F}_2) \wedge C\tau \wedge DC\tau.$$

By the identification of Section 3.3.2 we further have

$$F(\overline{H}, \overline{H}) \simeq (F(H\mathbb{F}_2, H\mathbb{F}_2) \wedge C\tau) \vee \left(\Sigma^{-1,1} F(H\mathbb{F}_2, H\mathbb{F}_2) \wedge C\tau\right).$$

By Lemma 3.31 we get that $\overline{\mathcal{A}}_{\mathbb{C}}$ is a free $\mathcal{A}_{\mathbb{C}}/\tau$-module of rank 2 with generators given by the operations

$$\mathrm{id} \colon \overline{H} \longrightarrow \overline{H} \qquad \text{and} \qquad \beta_\tau \colon \overline{H} \xrightarrow{p} \Sigma^{1,-1} H\mathbb{F}_2 \xrightarrow{i} \Sigma^{1,-1}\overline{H},$$

where p denotes the projection of $C\tau$ on its top cell, while i denotes the inclusion of it bottom cell. The definition of β_τ explains why we call it a τ-Bockstein. Since the Steenrod algebra is defined as negative homotopy groups of the endomorphism spectrum, the τ-Bockstein β_τ is in degree $(1, -1)$. This settles the additive structure of $\overline{\mathcal{A}}_{\mathbb{C}}$, and it remains to understand its Hopf algebra structure. Since $\overset{\vee}{\overline{\mathcal{A}}_{\mathbb{C}}}$ is a Hopf algebra of finite type, we can dualize its structure from Proposition 3.33 to get the desired Hopf algebra structure of $\overline{\mathcal{A}}_{\mathbb{C}}$. Recall that we did not yet finish the proof of Proposition 3.33, as we still have to show that $\beta_\tau \in \overset{\vee}{\overline{\mathcal{A}}_{\mathbb{C}}}$ is primitive. This is equivalent to $\beta_\tau \in \overline{\mathcal{A}}_{\mathbb{C}}$ being exterior, which is clear since it is the composite

$$\beta_\tau \circ \beta_\tau \colon \overline{H} \xrightarrow{p} \Sigma^{1,-1} H\mathbb{F}_2 \xrightarrow{i} \Sigma^{1,-1}\overline{H} \xrightarrow{p} \Sigma^{1,-1} H\mathbb{F}_2 \xrightarrow{i} \Sigma^{1,-1}\overline{H},$$

which is nullhomotopic as $p \circ i \simeq 0$. □

Remark 3.35 ($C\tau$-linear $\overline{H}$-homology and cohomology)**.** We can define the $C\tau$-linear homology and cohomology of a $C\tau$-module X to be

$$\overline{H}^{C\tau}_{*,*}(X) \coloneqq \pi_{*,*}(\overline{H} \wedge_{C\tau} X) \qquad \text{and} \qquad \overline{H}^{*,*}_{C\tau}(X) \coloneqq \pi_{-*,-*}\left(F_{C\tau}(X, \overline{H})\right).$$

The relevant $\overline{H}$-Steenrod algebra of $C\tau$-linear operations and co-operations are then

$$\pi_{-*,-*}\left(F_{C\tau}(\overline{H}, \overline{H})\right) \qquad \text{and} \qquad \overset{\vee}{\overline{\mathcal{A}}_{\mathbb{C}}} \cong \pi_{*,*}\left(\overline{H} \wedge_{C\tau} \overline{H}\right).$$

Their computation follows from Lemmas 3.31 and 3.32, and the result is the usual motivic Steenrod algebra and its dual, modulo τ. The only difference with the computations of Propositions 3.33 and 3.34 is that the $C\tau$-linear Steenrod algebras do not contain the τ-Bockstein element β_τ. In particular, the dual $C\tau$-linear $\overline{H}$-Steenrod algebra enjoys the nice formula

$$\mathbb{F}_2[\xi_1, \xi_2, \ldots] \otimes E(\tau_0, \tau_1, \ldots)$$

that is very reminiscent of the odd-primary classical Steenrod algebra.

3.4.3 The $C\tau$-Induced Moore Spectrum

Denote by $S^0/2$ the mod 2 Moore spectrum in the usual category of topological spectra **Spt**. Recall that the classical Toda bracket $\langle 2, \eta, 2\rangle = \eta^2$ implies that $\pi_2(S^0/2) \cong \mathbb{Z}/4$. This shows that multiplication by 2 is not a nullhomotopic map on $S^0/2$, and thus that there is no possible filler in the diagram

$$\begin{tikzcd}
S^0 \wedge S^0/2 \arrow[r, "2"] & S^0 \wedge S^0/2 \arrow[r] \arrow[dr, "\simeq"] & S^0/2 \wedge S^0/2 \arrow[r] \arrow[d, dashed, "\nexists\, \mu"] & \Sigma^1 S^0/2 \\
 & & S^0/2. &
\end{tikzcd}$$

This shows that there exists no left unital multiplication on $S^0/2$.

Denote now the motivic mod 2 Moore spectrum by $S^{0,0}/2$. Similarly, we can compute the motivic homotopy group $\pi_{2,0}(S^{0,0}/2) \cong \mathbb{Z}/4$ via the same argument. More precisely, the analoguous Toda bracket is $\langle 2, \tau\eta, 2\rangle = \tau^2\eta^2$, where $\eta \in \pi_{1,1}(S^{0,0})$ and thus $\tau\eta \in \pi_{1,0}(S^{0,0})$. This again implies that there is no left unital multiplication on the Moore spectrum $S^{0,0}/2$. Observe that this could also have been noticed by the fact that a left unital multiplication on $S^{0,0}/2$ would induce one on $S^0/2$ by Betti realization.

Denote the cofiber of multiplication by τ on $S^{0,0}/2$ by $S/(2,\tau)$. This spectrum does admit a left unital multiplication since

$$\langle 2, \eta, 2\rangle = \tau\eta^2 \equiv 0 \qquad \text{modulo } \tau.$$

This does not imply that there is a ring structure on $S/(2,\tau)$ as this bracket is just one possible obstruction (the obstruction to left unitality). In Theorem 3.37 we show that all obstructions are of this type and that $S/(2,\tau)$ admits the structure of an E_∞ algebra over $C\tau$.

Since cofibers in $C\tau$-modules can be computed in the underlying category of motivic spectra, it follows that the cofiber of 2 on $C\tau$ has underlying spectrum $S/(2,\tau)$. Consider now $S/(2,\tau)$ as a $C\tau$-module, for example as constructed in the category ${}_{C\tau}\mathbf{Mod}$ by the cofiber sequence

$$C\tau \xrightarrow{2} C\tau \xrightarrow{i} S/(2,\tau) \xrightarrow{p} \Sigma^{1,0}C\tau. \tag{3.16}$$

To equip $S/(2,\tau)$ with an E_∞ $C\tau$-algebra structure, we will proceed very similarly as in Section 3.2, which we refer to for more details.

Proposition 3.36. *There is a unique homotopy unital and homotopy commutative $C\tau$-algebra structure on $S/(2,\tau)$.*

Proof. The computation of $[S/(2,\tau), S/(2,\tau)]_{C\tau} \cong \mathbb{Z}/2$ generated by the identity map shows that $\cdot 2$ is nullhomotopic on $S/(2,\tau)$, providing a left unital multiplication μ from the diagram

$$\begin{tikzcd}
C\tau \wedge_{C\tau} S/(2,\tau) \arrow[r, "2"] & C\tau \wedge_{C\tau} S/(2,\tau) \arrow[r, "i_L"] \arrow[dr, "\simeq"] & S/(2,\tau) \wedge_{C\tau} S/(2,\tau) \arrow[r, "p_L"] \arrow[d, dashed, "\exists\, \mu"] & \Sigma^{1,0} C\tau \wedge_{C\tau} S/(2,\tau) \\
 & & S/(2,\tau). &
\end{tikzcd}$$

The computation $[\Sigma^{1,0} C\tau \wedge_{C\tau} S/(2,\tau), S/(2,\tau)]_{C\tau} = 0$ shows that there is a unique left unital multiplication up to homotopy on $S/(2,\tau)$. As in Lemma 3.13, it also implies that there is a unique section s of p_L, giving a canonical additive splitting

$$S/(2,\tau) \wedge_{C\tau} S/(2,\tau) \simeq S/(2,\tau) \vee \Sigma^{1,0} S/(2,\tau). \tag{3.17}$$

The induced multiplication $\widetilde{\mu}$ after this identification is again just projection onto the first factor, and the factor swap map χ is given by the following diagram

$$\begin{tikzcd}
S/(2,\tau) \wedge_{C\tau} S/(2,\tau) \arrow[r, "\chi"] & S/(2,\tau) \wedge_{C\tau} S/(2,\tau) \arrow[d, "{(\mu, p_L)}"] \\
S/(2,\tau) \vee \Sigma^{1,0} S/(2,\tau) \arrow[u, "i_L + s"] \arrow[r, dashed, "\left[\begin{smallmatrix} 1 & 0 \\ i \circ p & 1 \end{smallmatrix}\right]"] & S/(2,\tau) \vee \Sigma^{1,0} S/(2,\tau).
\end{tikzcd}$$

The matrix can be completely determined since $[S/(2,\tau), S/(2,\tau)]_{C\tau} \cong \mathbb{Z}/2$. By an easy matrix multiplication as in Proposition 3.16, this shows that μ is right unital and homotopy commutative. ☐

The next step is to show that this (unique) multiplication map μ on $S/(2,\tau)$ can be extended to an E_∞ multiplication. We proceed in the exact same way as we did in Proposition 3.17 and Theorem 3.20.

Theorem 3.37. *The $C\tau$-algebra structure on $S/(2,\tau)$ can be uniquely extended to an E_∞ structure.*

Proof. We first extend it to an A_∞ structure as in Proposition 3.17, with obstructions living in the abelian group

$$\left[\Sigma^{n-3,0}(\Sigma^{1,0}C\tau)^{\wedge n}, S/(2,\tau)\right]_{C\tau} \cong \left[\Sigma^{2n-3,0}C\tau^{\wedge n}, S/(2,\tau)\right]_{C\tau}$$

for $n \geq 3$. Here we used $\Sigma^{1,0}C\tau$ since it is the cofiber of the unit map $C\tau \xrightarrow{i} S/(2,\tau)$. By using the decomposition formula for $C\tau^{\wedge n}$ from Corollary (3.14), the obstructions live in the group

$$\bigoplus_{i=0}^{n} \binom{n}{i} \left[\Sigma^{2n-3+i,-i}C\tau, S/(2,\tau)\right]_{C\tau}.$$

By the free-forget adjunction these groups are

$$\pi_{2n-3+i,-i}(S/(2,\tau)).$$

For $n \geq 3$ and for any $0 \leq i \leq n$ this homotopy group is zero, making the obstruction group zero and allowing μ to extend to an A_∞ structure. Similarly the obstructions for uniqueness live in zero groups, showing that $S/(2,\tau)$ admits a unique A_∞ algebra structure over $C\tau$.

The A_3 structure gives an associative homotopy, and thus we now have a unital, associative and commutative monoid in the homotopy category. This is a 3-stage in Robinsin's obstruction theory, so we can apply Corollary 3.9 to extend it to an E_∞ ring structure. The obstructions live in

$$\left[\Sigma^{n-3,0}S/(2,\tau)^{\wedge m}, S/(2,\tau)\right]_{C\tau}$$

for $n \geq 4$ and $2 \leq m \leq n$, where the smash product is over $C\tau$. As in the proof of Theorem 3.20, we first break the source in smaller pieces by recursively using equation (3.17). It is then easy to show that all of those groups are zero by using cofiber sequences in the first variable to reduce it to homotopy groups of $S/(2,\tau)$. Similarly, the obstructions for uniqueness live in

$$\left[\Sigma^{n-2,0} S/(2,\tau)^{\wedge m}, S/(2,\tau)\right]_{C\tau}$$

for $n \geq 4$ and $2 \leq m \leq n$. We show by the exact same method that all those groups are zero, finishing the proof. $\square$

Remark 3.38. The fact that multiplication by 2 is nullhomotopic on $S/(2,\tau) \simeq C\tau/2$ is not so surprising, as $C\tau$ is of somehow of algebraic nature. In fact, multiplication by n on X/n is always nullhomotopic in such algebraic categories, as explained in [54, Proposition 1].

Remark 3.39. The Toda bracket $\langle 2,\eta,2\rangle = \eta^2$ is also responsible for the non-existence of a v_1^1-self map on the topological Moore spectrum $S^0/2$. This is illustrated in the diagram

$$\begin{tikzcd}
S^2/2 \arrow[d, dashed, "\nexists"'] & S^2 \arrow[l, "i"'] \arrow[d, "\eta"] \arrow[dl, dashed, "\exists\ \widetilde{\eta}"'] & S^2 \arrow[l, "2"'] \\
S^0/2 \arrow[r, "p"] & S^1 \arrow[r, "2"] & S^1.
\end{tikzcd}$$

The map $\widetilde{\eta}$ exists since $2\eta = 0$, but there is no v_1^1-self map as $2 \cdot \widetilde{\eta} \neq 0$. Motivically, the same diagram has the same problem because of the non-vanishing of the bracket $\langle 2,\eta,2\rangle = \tau\eta^2$. However, in $C\tau$-modules this bracket vanishes and the $C\tau$-induced Moore spectrum admits a v_1^1-self map. The diagram

$$\begin{tikzcd}
\Sigma^{2,1}S/(2,\tau) \arrow[d, dashed, "\exists\ v_1"'] & \Sigma^{2,1}C\tau \arrow[l, "i"'] \arrow[d, "\eta"] \arrow[dl, dashed, "\exists\ \widetilde{\eta}"'] & \Sigma^{2,1}C\tau \arrow[l, "2"'] \\
S/(2,\tau) \arrow[r, "p"] & \Sigma^{1,0}C\tau \arrow[r, "2"] & \Sigma^{1,0}C\tau
\end{tikzcd}$$

exhibits this v_1-self map

$$\Sigma^{2,1}S/(2,\tau) \xrightarrow{v_1} S/(2,\tau).$$

More precisely, this follows since the computation $[\Sigma^{2,1}C\tau, S/(2,\tau)] \cong \mathbb{Z}/2$ forces the relation $2 \cdot \tilde{\eta} \simeq 0$.

3.4.4 The $C\tau$-Induced connective Algebraic and Hermitian K-Theory Spectra

Consider the motivic algebraic K-theory spectrum KGL constructed in [59]. This spectrum represents algebraic K-theory on schemes. More precisely, given any scheme X, the KGL-cohomology of its stabilization $\Sigma^\infty_+ X$ computes the algebraic K-theory of the scheme X. Consider now its connective cover kgl as described in [25] over $\operatorname{Spec}\mathbb{C}$ and in [44] over more general basis. It is shown in [44] that both KGL and kgl admit a unique E_∞ ring structure. Recall that we work in the 2-completed category, and we use kgl to denote the 2-completed connective algebraic K-theory spectrum. Its coefficients and mod 2 homology of kgl over $\operatorname{Spec}\mathbb{C}$ are computed in [25] and given by

$$\pi_{*,*}(kgl) \cong \hat{\mathbb{Z}}_2[\tau, v_1] \qquad \text{and} \qquad H\mathbb{F}_{2*,*}(kgl) \cong \mathbb{F}_2[\tau][\xi_1, \xi_2, \ldots][\tau_2, \tau_3, \ldots] \Big/ \tau_i^2 = \tau\xi_{i+1}\,,$$

where the element v_1 is in degree $(2,1)$ and corresponds to the usual Bott periodicity. Its homology is written as a subalgebra of the mod 2 homology of $H\mathbb{F}_2$ recalled in equation (2.2).

Consider now the hermitian K-theory spectrum KQ defined in [20] and studied in [53]. The paper [25] defines its connective cover kq over $\operatorname{Spec}\mathbb{C}$, by taking appropriate C_2-fixed points (although it is denoted by ko in that paper). It also computes its coefficients and mod 2 homology

$$\pi_{*,*}(kq) \cong \hat{\mathbb{Z}}_2[\tau, \eta, a, b] \big/ 2\eta, \tau\eta^3, a\eta, a^2 = 4b$$
$$H\mathbb{F}_{2*,*}(kq) \cong \mathbb{F}_2[\tau][\xi_1^2, \xi_2, \ldots][\tau_2, \tau_3, \ldots] \Big/ \tau_i^2 = \tau\xi_{i+1}\,.$$

To explain the homotopy ring $\pi_{*,*}(kq)$, Figure 4 displays the E_∞-page of the motivic Adams spectral sequence computing $\pi_{*,*}(kq)$. The horizontal axis represents the stem, i.e., the s in $\pi_{s,w}(kq)$, while the vertical axis represents the Adams filtration. As it is usually done with

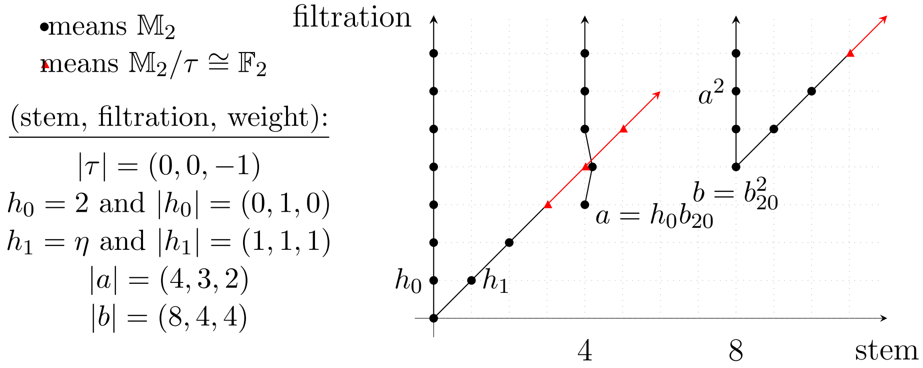


Figure 4: The E_∞-page of the Adams spectral sequence computing $\pi_{*,*}(kq)$.

motivic charts, the weight w in $\pi_{s,w}(kq)$ is suppressed from the chart and one can imagine it on a third axis perpendicular to the page.

In this section we consider the $C\tau$-induced spectra that we denote by

$$\overline{kgl} := kgl \wedge C\tau \qquad \text{and} \qquad \overline{kq} := kq \wedge C\tau.$$

Both of them are $C\tau$-algebras, where kgl is an E_∞ algebra as being the smash product of two E_∞ rings.

The case of algebraic K-theory $\overline{kgl}$

The fact that both its homotopy and homology are τ-free makes the description of $\overline{kgl}$ straightforward. Indeed, by Lemma 3.31 we immediately get

$$\pi_{*,*}(\overline{kgl}) \cong \hat{\mathbb{Z}}_2[v_1] \qquad \text{and} \qquad H\mathbb{F}_{2*,*}(\overline{kgl}) \cong \mathbb{F}_2[\xi_1, \xi_2, \ldots] \otimes E(\tau_2, \tau_3, \ldots).$$

The case of hermitian K-theory $\overline{kq}$

Its homology is τ-free and so again we immediately get

$$H\mathbb{F}_{2*,*}(\overline{kq}) \cong \mathbb{F}_2[\xi_1^2, \xi_2, \ldots] \otimes E(\tau_2, \tau_3, \ldots).$$

Its homotopy is more interesting as it is not τ-free, and we will get contributions both from the cokernel and kernel of multiplication by τ. Moreover, a surprising fact occurs as there is a hidden extension which makes $\overline{kq}$ contain the periodicity element v_1^2 in its homotopy.

Proposition 3.40. *The homotopy ring* $\pi_{*,*}(\overline{kq})$ *has the presentation*

$$\pi_{*,*}(\overline{kq}) \cong \hat{\mathbb{Z}}_2[\eta, v_1^2] \big/ 2\eta.$$

Proof. The usual cofiber sequence (3.1) for $C\tau$, smashed with kq gives the cofiber sequence

$$\Sigma^{0,-1}kq \xrightarrow{\tau} kq \xrightarrow{i} \overline{kq} \xrightarrow{p} \Sigma^{1,-1}kq.$$

Since the homology $H\mathbb{F}_{2*,*}(\Sigma^{0,-1}kq)$ is τ-free, we ge the short exact sequence

$$0 \longrightarrow H\mathbb{F}_{2*,*}\left(\Sigma^{0,-1}kq\right) \xrightarrow{\tau} H\mathbb{F}_{2*,*}(kq) \xrightarrow{i} H\mathbb{F}_{2*,*}\left(\overline{kq}\right) \longrightarrow 0$$

in homology. For any motivic spectrum X, denote by $\mathrm{Ext}^*(X)$ the trigraded term

$$\mathrm{Ext}^{\overset{*}{\vee},*,*}_{\mathcal{A}_{\mathbb{C}}\text{-comod}}(H\mathbb{F}_{2*,*}(S^{0,0}), H\mathbb{F}_{2*,*}(X))$$

that represents the E_2-page of the motivic Adams spectral sequence for X. We use the indicated grading in $\mathrm{Ext}^*(X)$ to denote the homological degree in Ext, i.e., the Adams filtration on the E_2-page. From the above short exact sequence, we get a long exact sequence in Ext-groups

$$\cdots \xrightarrow{\tau} \mathrm{Ext}^*(kq) \xrightarrow{i_*} \mathrm{Ext}^*(\overline{kq}) \xrightarrow{p_*} \mathrm{Ext}^{*+1}(\Sigma^{0,-1}kq) \xrightarrow{\tau} \cdots,$$

i.e., a long exact sequence in E_2-pages. This gives short exact sequences

$$0 \longrightarrow \mathrm{Ext}^*(kq)\big/_{\tau} \xrightarrow{i_*} \mathrm{Ext}^*(\overline{kq}) \xrightarrow{p_*} {}_{\tau}\,\mathrm{Ext}^{*+1}(\Sigma^{0,-1}kq) \longrightarrow 0,$$

where the left term is the cokernel of τ while the right term is the τ-torsion. Since i is a ring map, the term $\mathrm{Ext}(kq)/\tau$ includes as a subring of $\mathrm{Ext}(\overline{kq})$. However, this cokernel can act non-trivially on the τ-torsion part, giving potential extension problems to solve. Since the motivic Adams spectral sequence for kq collapses at the E_2-page with no hidden extensions, the term $\mathrm{Ext}(kq)$ is given by the Figure 4 on page 67. These two pieces assemble to give the additive description of the E_2-page of the motivic Adams spectral sequence for $\overline{kq}$ as described in Figure 5. It still remains to solve the possible extension problems and possible Adams differentials. The only possible extension is whether or not $2 \cdot \widetilde{h_1^3} = \overline{a}$, as indicated in Figure 5. Consider the Toda bracket $\langle \tau, \eta^3, 2\rangle$ as in the diagram

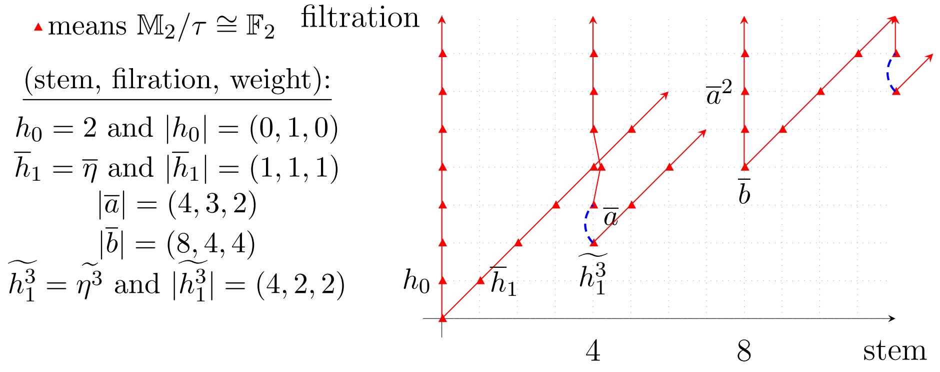


Figure 5: The E_2-page of the motivic Adams spectral sequence for $\overline{kq}$ as an $\mathbb{F}_2$-vector space.

$$\begin{tikzcd}
S^{3,2} \arrow[r, "2"] \arrow[ddrr, dashed] & S^{3,2} \arrow[r, "\eta^3"] \arrow[dr, dashed, "\widetilde{\eta^3}"'] & \Sigma^{0,-1}kq \arrow[r, "\tau"] & kq \\
 & & \Sigma^{-1,0}\overline{kq} \arrow[u, "p"'] & \\
 & & \Sigma^{-1,0}kq, \arrow[u, "i"'] &
\end{tikzcd}$$

where we have that $2 \cdot \widetilde{\eta^3} \in i_*\langle \tau, \eta^3, 2\rangle$ by [24, Section 3.1.1]. We can compute this bracket in the motivic May spectral sequence using May's Convergence Theorem. See [35] for the original reference, and [24, Theorem 2.2.3] for an exposition of the motivic version. More precisely, we can compute it on the motivic May E_3-page via the differential $d_3(b_{20}) = \tau h_1^3$

(since h_0h_1 is already zero). This bracket has no indeterminacy giving

$$\langle \tau, h_1^3, h_0 \rangle = \{b_{20}h_0\}.$$

Recall from Figure 4 that $a = b_{20}h_0$ giving that indeed, in $\pi_{*,*}(\overline{kq})$, there is an extension $2 \cdot \widetilde{h_1^3} = \overline{a}$. This h_0-extension appears as the round dotted line on Figure 5. We now spell out the ring structure of this E_2-page. First observe that

$$4\left(\widetilde{h_1^3}\right)^2 = \left(2\widetilde{h_1^3}\right)^2 = \overline{a}^2 = 4\overline{b}^2,$$

and because there are no possible extensions in that column, we get that $\left(\widetilde{h_1^3}\right)^2 = \overline{b}$. The E_2-page of the motivic Adams spectral sequence for $\overline{kq}$ has therefore the ring presentation

$$E_2 \cong \mathbb{F}\left[h_0, \overline{h}_1, \widetilde{h_1^3}\right] / h_0\overline{h}_1.$$

There are no possible Adams differentials on these 3 generators, and thus Figure 5 also represents the E_∞-page of the Adams spectral sequence for $\overline{kq}$. Except the h_0-towers, there are no possible hidden extensions, giving the multiplicative description

$$\pi_{*,*}(\overline{kq}) \cong \hat{\mathbb{Z}}_2[\eta, \widetilde{h_1^3}] / 2\eta.$$

Finally, we show that $\widetilde{h_1^3}$ detects the element v_1^2. We can smash the cofiber sequence

$$\Sigma^{1,1}kq \xrightarrow{\eta} kq \xrightarrow{i} kgl$$

with $C\tau$ to obtain the cofiber sequence

$$\Sigma^{1,1}\overline{kq} \xrightarrow{\eta} \overline{kq} \xrightarrow{\overline{i}} \overline{kgl}.$$

Since i is a ring map, then so is the induced map $\bar{i}$. The ring map $\bar{i}$ sends the 8-fold Bott periodicity element $\bar{b} = \left(\widetilde{h_1^3}\right)^2$ to the 8-fold Bott periodicity element v_1^4, which forces $\widetilde{h_1^3}$ to be sent to v_1^2. The E_2-page of $\overline{kq}$ has therefore the ring presentation

$$\pi_{*,*}(\overline{kq}) \cong \hat{\mathbb{Z}}_2[\eta, v_1^2] \big/ 2\eta . \square$$

CHAPTER 4 EXOTIC MOTIVIC PERIODICITY

In this chapter, we explain what motivic w_n-periodicity is, and construct the motivic exotic fields $K(w_n)$. These fields are called exotic because our intuition from classical homotopy theory would lead us to believe that the only motivic fields of this form (i.e., associated with periodic operators) are the usual Morava K-theories $K(n)$.

We refer to the Introduction for more motivation about motivic w_n-periodicity. Finally, let's mention that this chapter appears as a separate paper, in [15].

Organization

Here is the organization of this chapter.

Section 4.1. In this section we first describe the setting in which we work, which is the category of cellular $C\tau$-modules. This includes the following : computing the relevant Steenrod algebra and its dual, deriving some important properties, and setting up an appropriate Adams spectral sequence.

Section 4.2. This section contains the construction of the motivic fields $K(w_n)$. This goes through first constructing connective versions $k(w_n)$, endowing them with an E_∞ ring structure, and finally inverting multiplication by w_n.

Section 4.3. This section contains the construction of the spectrum wBP, and its truncations $wBP\langle n\rangle$.

4.1 Recollection on $C\tau$-Modules and its Steenrod Algebra

In this section we will describe the general framework in which all spectra will be constructed. We first start by recalling notation in 4.1.1, as well as set up the category ${}_{C\tau}\mathbf{Cell}$ of cellular (2-completed) $C\tau$-modules. In 4.1.2 we recall from Section 3.4.2 the $C\tau$-induced mod 2 Eilenberg-Maclane spectrum $\overline{H}$, and its Steenrod algebra of operations and co-operations. This spectrum plays in ${}_{C\tau}\mathbf{Cell}$ the role that $H\mathbb{F}_2$ plays in $\mathbf{Spt}_{\mathbb{C}}$, and will serve as a building block for the Postnikov tower constructions of Section 4.2. In 4.1.3 we further study this Steenrod algebra and give the relevant definitions which lead to the definition of w_n periodicity. Finally, in 4.1.4 we briefly describe the $\overline{H}$-based Adams spectral sequence in the category ${}_{C\tau}\mathbf{Cell}$. This spectral sequence will be used in several places in Sections 4.2 and 4.3.

4.1.1 Cellular motivic spectra and $C\tau$-modules

Recall from [12] that a motivic spectrum is called *cellular* if it can be built out of spheres $S^{s,w}$ under filtered colimits. Denote by $\mathbf{Cell}_{\mathbb{C}}$ the category of *cellular motivic spectra over* $\operatorname{Spec}\mathbb{C}$, constructed as the right Bousfield localization at the set of spheres $\{S^{s,w}\}_{s,w\in\mathbb{Z}}$. The weak equivalences in $\mathbf{Cell}_{\mathbb{C}}$ are thus given by $\pi_{*,*}$-isomorphisms. The Bousfield localization is part of an adjunction

$$\begin{tikzcd} \mathbf{Cell}_{\mathbb{C}} \arrow[r, hook, shift left] \arrow[r, phantom, "\perp"] & \mathbf{Spt}_{\mathbb{C}} \arrow[l, shift left, "C"] \end{tikzcd},$$

where the unit is a weak equivalence in $\mathbf{Cell}_{\mathbb{C}}$, and the counit $C(X) \longrightarrow X$ is a $\pi_{*,*}$-isomorphism. This discussion can be carried out both in the world of

- presentable, closed symmetric monoidal ∞-categories, following [49],
- cellular, closed symmetric monoidal model categories, via the motivic symmetric spectra of [26], [46], and the theory of right Bousfield localization in that setting following [4].

In this chapter we will be working in $\mathbf{Cell}_{\mathbb{C}}$ as all spectra constructed will be cellular. This has in particular the advantages that a spectrum $X \in \mathbf{Cell}_{\mathbb{C}}$ is contractible if and only if its homotopy groups $\pi_{*,*}(X)$ vanish, and that our spectral sequences converge.

In Chapter 3, we considered the cofiber of $\tau \in \pi_{0,-1}S^{0,0}$

$$S^{0,-1} \xrightarrow{\tau} S^{0,0} \xrightarrow{i} C\tau,$$

and shown that the motivic 2-cell complex $C\tau$ admits a unique E_∞ ring structure, producing thus a closed symmetric monoidal category $({}_{C\tau}\mathbf{Mod}, - \wedge_{C\tau} -)$ of $C\tau$-modules. The usual adjunction from the ring map $S^{0,0} \longrightarrow C\tau$ restricts to an adjunction

$$\begin{tikzcd} \mathbf{Cell}_{\mathbb{C}} \arrow[r, shift left, "-\wedge C\tau"] \arrow[r, phantom, "\perp"] & {}_{C\tau}\mathbf{Cell} \arrow[l, shift left] \end{tikzcd}$$

on cellular objects. From now on, we will call an object $X \in {}_{C\tau}\mathbf{Cell}$ a $C\tau$-module, omitting the word cellular.

Remark 4.1 (Working in $C\tau$-modules)**.** Similarly with the fact that the Morava K-theories $K(n)$ are 2-completed[7], the spectra $K(w_n)$ are naturally $C\tau$-modules. In the case of $K(w_0)$, this can be seen from the relation $0 = \tau\eta^4 \in \pi_{4,3}(\widehat{S}_2)$. Since $K(w_0)$ contains η^{-1}, this forces τ to act by zero on it, which in this case is sufficient to promote a spectrum to a $C\tau$-module. We will therefore work in the category of ${}_{C\tau}\mathbf{Cell}$ in which we will construct the motivic fields $K(w_n)$.

4.1.2 $C\tau$-linear $\overline{H}$-(co)homology and its (co)operations

We will now exclusively be working in ${}_{C\tau}\mathbf{Cell}$, i.e., with cellular $C\tau$-modules and with $C\tau$-linear maps between them. Most invariants of the underlying spectrum of a $C\tau$-module X can be rewritten in this category. For example, the usual adjunction describes its homotopy groups by

$$\pi_{s,w}(X) \cong [\Sigma^{s,w} C\tau, X]_{C\tau}\,.$$

The analog of the mod 2 Eilenberg-Maclane spectrum in this category is the $C\tau$-induced Eilenberg-Maclane spectrum $\overline{H} := H\mathbb{F}_2 \wedge C\tau$. Recall from Section 3.4.2 that given a $C\tau$-module X, we defined its *$C\tau$-linear $\overline{H}$-homology* to be

$$\overline{H}_{*,*}(X) := \pi_{*,*}(\overline{H} \wedge_{C\tau} X),$$

and its *$C\tau$-linear $\overline{H}$-cohomology* of X to be

$$\overline{H}^{*,*}(X) := \pi_{-*,-*}(F_{C\tau}(X, \Sigma^{-1,1}\overline{H})).$$

As explained in Section 3.4.2, these are isomorphic to the $\overline{H}$-homology and (shifted) cohomology of the underlying spectrum, and are naturally acted upon by the $C\tau$-linear operations

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[7] except for $K(0) \simeq H\mathbb{Q}$.

$\overline{\mathcal{A}}_{\mathbb{C}} := \pi_{-*,-*}F_{C\tau}(\overline{H}, \overline{H})$, and cooperations $\overset{\vee}{\overline{\mathcal{A}}}_{\mathbb{C}} := \pi_{*,*}(\overline{H} \wedge_{C\tau} \overline{H})$. Recall the computation of $\overset{\vee}{\overline{\mathcal{A}}}_{\mathbb{C}}$ from Proposition 3.33.

Proposition 4.2. *The $C\tau$-linear co-operations of $\overline{H}$ are given by the Hopf algebra*

$$\overset{\vee}{\overline{\mathcal{A}}}_{\mathbb{C}} \cong \mathbb{F}_2[\xi_1, \xi_2, \ldots] \otimes E(\tau_0, \tau_1, \ldots),$$

with bidegrees given by $|\xi_n| = (2^{n+1} - 2, 2^n - 1)$ *and* $|\tau_n| = (2^{n+1} - 1, 2^n - 1)$, *and coproduct*

$$\Delta(\xi_n) = \sum_{i=0}^{n} \xi_{n-i}^{2^i} \otimes \xi_i = \xi_n \otimes 1 + \xi_{n-1}^2 \otimes \xi_1 + \cdots + \xi_{n-i}^{2^i} \otimes \xi_i + \cdots + 1 \otimes \xi_n,$$

$$\Delta(\tau_n) = \tau_n \otimes 1 + \sum_{i=0}^{n} \xi_{n-i}^{2^i} \otimes \tau_i = \tau_n \otimes 1 + \xi_n \otimes \tau_0 + \xi_{n-1}^2 \otimes \tau_1 + \cdots + \xi_{n-i}^{2^i} \otimes \tau_i + \cdots + 1 \otimes \tau_n.$$

The advantage of working with the coaction of $\overset{\vee}{\overline{\mathcal{A}}}_{\mathbb{C}}$ instead of $\overset{\vee}{\mathcal{A}}_{\mathbb{C}}$ is now apparent by comparing Proposition 4.2 with Voevodsky's formula (2.2). First, $\overset{\vee}{\overline{\mathcal{A}}}_{\mathbb{C}}$ is smaller and more regular, which will be convenient for computations. Second, since $\tau \in \pi_{0,-1}S^{0,0}$ is nullhomotopic on any $C\tau$-module X, it will act as zero on any algebraic invariant of X. Morally, it is thus natural to expect that the coaction of $\overset{\vee}{\mathcal{A}}_{\mathbb{C}}$ on the homology $H\mathbb{F}_{2*,*}(X)$ should factor through the quotient $\overset{\vee}{\mathcal{A}}_{\mathbb{C}}/\tau \cong \overset{\vee}{\overline{\mathcal{A}}}_{\mathbb{C}}$. Working with $C\tau$-linear $\overline{H}$-homology is a way of making this remark precise. The exact same remark applies to cohomology, where the computation of the $C\tau$-linear Steenrod algebra $\overline{\mathcal{A}}_{\mathbb{C}}$ follows from Proposition 3.34 and is given by the quotient $\overline{\mathcal{A}}_{\mathbb{C}} \cong \mathcal{A}_{\mathbb{C}}/\tau$.

Convention 4.3. Given a $C\tau$-module X, we will always consider its $C\tau$-linear $\overline{H}$-homology (i.e., its $H\mathbb{F}_2$-homology) endowed with the coaction of $\overset{\vee}{\overline{\mathcal{A}}}_{\mathbb{C}}$. Similarly, its $C\tau$-linear $\overline{H}$-cohomology will always be considered as an $\overline{\mathcal{A}}_{\mathbb{C}}$-module.

To state another crucial advantage of $\overline{\mathcal{A}}_{\mathbb{C}}$ over $\mathcal{A}_{\mathbb{C}}$ we need the following definition.

Definition 4.4 (Chow degree)**.** Let $A_{*,*}$ be a bigraded abelian group. The *Chow degree* of an element $x \in A_{s,w}$ is given by the difference $s - 2w$. The bigraded group $A_{*,*}$ splits as a sum of its summands in a fixed Chow degree, which ranges through $\mathbb{Z}$.

Remark 4.5 (The Chow degree on motivic Steenrod algebras)**.** Both $\mathcal{A}_{\mathbb{C}}$ and $\overline{\mathcal{A}}_{\mathbb{C}}$ are generated as algebras (over $\mathbb{F}_2[\tau]$ and $\mathbb{F}_2$ respectively) by the Steenrod squares Sq^n. The even squares Sq^{2n} are in Chow degree 0, while the odd squares Sq^{2n+1} are in Chow degree 1. It follows that the whole Steenrod algebra $\overline{\mathcal{A}}_{\mathbb{C}}$ is concentrated in positive Chow degrees, i.e., is bounded below by 0. This in particular allows recursive arguments on the Chow degree. On the other side, since in this cohomological setting $|\tau| = (0,1)$ is in Chow degree -2, the Steenrod algebra $\mathcal{A}_{\mathbb{C}}$ of Voevodsky is non-vanishing in all Chow degrees and does not allow recursive arguments of this type.

Finally, let's mention that $\overline{H}^{*,*}$-cohomology satisfies a Künneth formula.

Proposition 4.6. *Given two $C\tau$-modules X and Y, there is a Künneth isomorphism*

$$\overline{H}^{*,*}(X \wedge Y) \cong \overline{H}^{*,*}(X) \otimes_{\mathbb{F}_2} \overline{H}^{*,*}(Y)$$

of $\overline{\mathcal{A}}_{\mathbb{C}}$-modules, where the right hand side has the diagonal $\overline{\mathcal{A}}_{\mathbb{C}}$-module structure.

Proof. Consider the motivic Künneth spectral sequence from [12], [58]

$$\mathrm{Tor}^{s,t,w}_{\overline{H}^{*,*}}\left(\overline{H}^{*,*}(X), \overline{H}^{*,*}(Y)\right) \Longrightarrow \overline{H}^{t-s,w}(X \wedge Y),$$

where s is the homological degree, and (t,w) are the usual two internal degrees. Since $\overline{H}^{*,*} \cong \mathbb{F}_2$ is a field, this spectral sequence is concentrated in homological degree $s = 0$ and thus collapses, giving the desired result. ☐

4.1.3 The Motivic Margolis elements P^s_t

In this section we will set-up some notation and formulas in the Steenrod algebra $\overline{\mathcal{A}}_{\mathbb{C}}$. These formulas are the motivic adaptation of classical formulas in $\mathcal{A}_{\mathrm{cl}}$ proven by Milnor, see for example [10, Section 2]. There are two different ways of showing these formulas in the motivic setting

(1) either by brute-force, by adapting the classical proof to the motivic setting, or

(2) by transporting them via a map between the classical and motivic Steenrod algebras.

We chose to use the second option. Consider the injective map

$$\mathcal{A}_{\mathrm{cl}} \hookrightarrow \overline{\mathcal{A}}_{\mathbb{C}} \tag{4.1}$$

of Hopf algebras, defined in [24, Section 2.1.3], where $\mathcal{A}_{\mathrm{cl}}$ denotes the mod 2 classical Steenrod algebra. One can for example define it as the dual to the natural quotient map between dual Steenrod algebras

$$\overset{\vee}{\overline{\mathcal{A}}}_{\mathbb{C}} = \mathbb{F}_2[\xi_1, \xi_2, \dots] \otimes E(\tau_0, \tau_1, \dots) \longrightarrow \overset{\vee}{\mathcal{A}}_{\mathrm{cl}} = \mathbb{F}_2[\xi_1, \xi_2, \dots]. \tag{4.2}$$

Remark 4.7. It is easy to see that both map are graded if the motivic bigraded object is consider as simply graded by the weight. Restricting to the weight in the motivic setting feels artificial, but turns out to be useful for the following reason.. Denote by $c(-)$ the conjugation map on both the classical and motivic Steenrod algebras (and their duals). By analyzing the coproduct on $\overset{\vee}{\overline{\mathcal{A}}}_{\mathbb{C}}$, it is easy to see that the ring map (4.2) is in fact a *graded* map of bialgebras over $\mathbb{F}_2$. Since both are connected Hopf algebras, the conjugation $c(-)$ is uniquely determined, and thus both maps (4.1) and (4.2) are maps of Hopf algebras.

Notation 4.8 (Margolis' P_t^s)**.** Denote by $P_t^s \in \overline{\mathcal{A}}_{\mathbb{C}}$ the element dual to $\xi_t^{2^s} \in \overset{\vee}{\overline{\mathcal{A}}}_{\mathbb{C}}$, by dualizing in the canonical monomial basis.

Example 4.9. Since the motivic Steenrod algebra at $p = 2$ admits a slightly different notation than the classical one, let's look at low dimensional elements. When $s = 0$, these elements are the sequence $P_1 = \mathrm{Sq}^2$, $P_2 = [\mathrm{Sq}^2, \mathrm{Sq}^4]$, etc. Observe that $Q_0 = \mathrm{Sq}^1$ does not appear in this notation. The sequence $\{P_t\}$ is a doubled version of the classical Milnor sequence $\{Q_t\}$.

Lemma 4.10. *The element $P_t^s \in \overline{\mathcal{A}}_{\mathbb{C}}$ is exterior if and only if $s < t$. Moreover, the subalgebra generated by the elements P_t is an exterior commutative algebra.*

Proof. Let's also denote by $P_t^s \in \mathcal{A}_{\mathrm{cl}}$ the classical element dual to $\xi_t^{2^s}$. Since the map (4.2) sends $\xi_t^{2^s}$ to $\xi_t^{2^s}$, its dual map (4.1) sends P_t^s to P_t^s. It is proven in [34, Lemma 15.1.4] that the classical P_t^s's are exterior if and only if $s < t$. The if part follows immediately and the only if part follows by injectivity of the map (4.1).

In the classical setting, recall that the dual element to $\xi_t \in \overset{\vee}{\mathcal{A}}_{\mathrm{cl}}$ is the Milnor primitive $Q_{t-1} \in \mathcal{A}_{\mathrm{cl}}$. It follows that the map (4.1) sends Q_{t-1} to P_t. Since the Q_t's commute, then so do the P_t's, finishing the proof. □

Notation 4.11. Denote by $E(P_t)$ the exterior algebra (in $\overline{\mathcal{A}}_{\mathbb{C}}$) generated by P_t, and by $E(P_1, P_2, \ldots)$ the exterior algebra (in $\overline{\mathcal{A}}_{\mathbb{C}}$) generated by $P_1, P_2, \ldots$.

Since P_t is exterior, one can consider *Margolis homology with respect to P_t*. Recall that given an $\overline{\mathcal{A}}_{\mathbb{C}}$-module M, this is defined as the homology of the complex

$$\begin{tikzcd} M \arrow[r, "\cdot P_t"] & M \arrow[r, "\cdot P_t"] & M, \end{tikzcd}$$

i.e., by the formula $H(M; P_t) = \ker P_t / \operatorname{im} P_t$. If $H(M; P_t) = 0$, one says that P_t is *exact* on M.

Corollary 4.12. *For every t, the element P_t is primitive, exterior and exact on $\overline{\mathcal{A}}_{\mathbb{C}}$.*

Proof. Notice that P_t is primitive since it is dual to the indecomposable element ξ_t, and that it is exterior by Lemma 4.10 applied with $s = 0$.

To show the vanishing of the P_t-Margolis homology on $\overline{\mathcal{A}}_{\mathbb{C}}$, we use the same strategy as in [34, Proposition 19.1.1]. First of all, it is easy to see by inspection that the subalgebra $E(P_1, P_2, \ldots)$ has no Margolis homology for every P_t. By a theorem of Milnor-Moore [37], the Hopf algebra $\overline{\mathcal{A}}_{\mathbb{C}}$ is free over $E(P_1, P_2, \ldots)$, i.e., can be written as a direct sum $\oplus E(P_1, P_2, \ldots)$ as an $E(P_1, P_2, \ldots)$-module. It follows that $\overline{\mathcal{A}}_{\mathbb{C}}$ has no P_t-Margolis homology, i.e., that P_t is exact on $\overline{\mathcal{A}}_{\mathbb{C}}$. □

Remark 4.13 (The Steenrod algebra and w_n-periodicity)**.** There is a tight relation between the periodic operator v_n and the cohomology operation Q_n. This can for example be seen by the interplay between the homotopy and cohomology of the connective Morava K-theory spectrum $k(n)$. In the classical setting, these invariants are

$$\pi_*(k(n)) \cong \mathbb{F}_2[v_n] \qquad \text{and} \qquad H^*(k(n);\mathbb{F}_2) \cong \mathcal{A}_{\mathrm{cl}}//E(Q_n).$$

This can equivalently be seen in the Postnikov tower of $k(n)$, whose layers are given by Eilenberg-Maclane spectra $H\mathbb{F}_2$, which are attached via Q_n. The same relation exists motivically for the motivic Morava K-theories. An intuition for w_n-periodicity is that the relation between w_n and Margolis' P_n is the exact same as the relation between v_n and Q_n.

4.1.4 The $\overline{H}$-based $C\tau$-linear motivic Adams spectral sequence

In what follows, we will construct an $\overline{H}$-based Adams spectral sequence in the category of $C\tau$-modules. One can set it up as in [13, Section 7] by replacing motivic spectra with $C\tau$-modules and $H\mathbb{F}_2$ with $\overline{H}$. In this setting, an Adams resolution of a $C\tau$-module X is given by a diagram of $C\tau$-modules

$$\begin{tikzcd}
X \arrow[d, "g_0"'] & X_1 \arrow[l, "f_0"'] \arrow[d, "g_1"'] & X_2 \arrow[l, "f_1"'] \arrow[d, "g_2"'] & \cdots \arrow[l, "f_2"'] \\
K_0 & K_1 & K_2,
\end{tikzcd}$$

where K_i is a wedge of suspensions of $\overline{H}$ and f_i is zero in $\overline{H}$-(co)homology. Consider the fiber sequence $F \xrightarrow{f} C\tau \xrightarrow{g} \overline{H}$ of $C\tau$-modules, where the map $C\tau \longrightarrow \overline{H}$ comes from the unit $S^{0,0} \longrightarrow H\mathbb{F}_2$. Then we can form a canonical Adams resolution as usual by inductively setting $K_i \simeq X \wedge_{C\tau} F^{\wedge_{C\tau} i} \wedge_{C\tau} \overline{H}$ and $X_i \simeq X \wedge_{C\tau} F^{\wedge_{C\tau} i}$, where f_i and g_i are induced from f and g.

In [13, Section 7] Dugger-Isaksen define a category $\langle S^{0,0} \rangle_{H\mathbb{F}_2}$ for which the motivic Adams spectral sequence converges. This is the full subcategory of $\mathbf{Spt}_{\mathbb{C}}$ containing cellular spectra, which is also closed by smashing with $H\mathbb{F}_2$. This last condition was necessary at that time since it was not known that $H\mathbb{F}_2$ was cellular. It was later proved in [22] that the mod 2 motivic Eilenberg-Maclane spectrum $H\mathbb{F}_2$ is cellular, and thus that $\langle S^{0,0} \rangle_{H\mathbb{F}_2}$ is just the category of motivic cellular spectra.

By copying [13, Section 7], we get a $C\tau$-linear $\overline{H}$-based motivic Adams spectral sequence with E_2-term given by

$$E_2 \cong \mathrm{Ext}_{\overline{\mathcal{A}}_{\mathbb{C}}}(\overline{H}^{*,*}(X), \overline{H}^{*,*}(C\tau)) \cong \mathrm{Ext}_{\overline{\mathcal{A}}_{\mathbb{C}}}(\overline{H}^{*,*}(X), \mathbb{F}_2).$$

It remains to study what the E_∞-page computes. Again by [13, Section 7], the E_∞-page computes the homotopy groups of the homotopy limit of the semi-cosimplicial spectrum

$$\overline{H} \wedge_{C\tau} X \rightrightarrows \overline{H} \wedge_{C\tau} \overline{H} \wedge_{C\tau} X \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \cdots \tag{4.3}$$

where all cofaces are induced from $C\tau \longrightarrow \overline{H}$. To compute this homotopy limit one can compare it with the $H\mathbb{F}_2$-tower of the underlying spectrum of X

$$H\mathbb{F}_2 \wedge X \rightrightarrows H\mathbb{F}_2 \wedge H\mathbb{F}_2 \wedge X \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \cdots \tag{4.4}$$

which we know totalizes to X by [13, Section 7], since X is already 2-complete. Since $\overline{H} \wedge_{C\tau} C\tau \simeq H\mathbb{F}_2 \wedge S^{0,0}$, there are level-wise weak equivalences between these two towers[8] which commute with the coface maps. Since forgetting the $C\tau$-module structure is a right adjoint, it follows that the underlying spectrum of the totalization of the tower (4.3) is also X. There is thus a convergent $C\tau$-linear $\overline{H}$-based motivic Adams spectral sequence

$$\mathrm{Ext}^{s,t,w}_{\overline{\mathcal{A}}_{\mathbb{C}}}(\overline{H}^{*,*}(X), \mathbb{F}_2) \Longrightarrow \pi_{t-s,w}(X). \tag{4.5}$$

Remark 4.14. Even though both towers (4.3) and (4.4) have the same underlying spectrum, they do not live in the same category and thus do not produce the same spectral sequence. They converge to the same object, but the E_2-term of the Adams spectral sequence coming from (4.3) is smaller and more computable.

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[8] more precisely between the underlying tower of (4.3) and the tower (4.4).

4.2 The Motivic Fields $K(w_n)$

The goal of this section is to construct motivic fields $K(w_n)$, that detect w_n periodicity. As explained in Remark 4.1, these spectra will be constructed in the category ${}_{C\tau}\mathbf{Cell}$. We will thus from now on exclusively work in the category ${}_{C\tau}\mathbf{Cell}$ and denote the smash product over $C\tau$ simply by $- \wedge -$, homotopy classes of $C\tau$-linear maps by $[-,-]$, work with $C\tau$-linear $\overline{H}$-homology and cohomology, etc. The spectrum $K(w_n)$ should be a ring spectrum with homotopy groups given by

$$\pi_{*,*}(K(w_n)) \cong \mathbb{F}_2[w_n^{\pm 1}].$$

The strategy is to first construct a connective version $k(w_n)$ with homotopy groups

$$\pi_{*,*}(k(w_n)) \cong \mathbb{F}_2[w_n],$$

endow it with a ring structure, and finally invert multiplication by w_n to get $K(w_n)$. In 4.2.1 we will construct $k(w_n)$ and show that it has the correct homotopy, and appropriate cohomology. The construction is done along an inverse tower, which can be seen as Postnikov tower of $k(w_n)$ (in the stem direction, for example). From this tower one can easily compute the homotopy and cohomology of $k(w_n)$ by the associated spectral sequences. In 4.2.2 we will again use this tower to construct a ring map $k(w_n) \wedge k(w_n) \longrightarrow k(w_n)$. We will then use Robinson's obstruction theory to rigidify it to an E_∞ ring structure, which allows the definition of $K(w_n)$.

4.2.1 The construction of $k(w_n)$

Fix an $n \in \mathbb{N}_0$ until the end of the section. We will now construct $k(w_n)$ via its Postnikov tower, in the category of $C\tau$-modules. Recall that $k(w_n)$ should be a motivic ring spectrum whose homotopy groups are given by the polynomial ring

$$\pi_{*,*}(k(w_n)) \cong \mathbb{F}_2[w_n],$$

where w_n is an element detected by the cohomology operation P_n. This suggests that $k(w_n)$ could be constructed via a tower whose layers are copies of $\overline{H}$, each of which is attached by P_n. We will call this tower the Postnikov tower of $k(w_n)$, as the layers are becoming more and more connected (in both the stem, and the weight). We proceed to explain this construction now.

Construction 4.15 (The construction of $k(w_n)$)**.** Start the bottom of the tower with the fiber sequence

$$\begin{tikzcd}
\overline{H} \arrow[r, "i_{-1} = \mathrm{id}"] & k(w_n)\langle 0\rangle \arrow[d, "p_{-1}"] & \\
 & * \arrow[r, "k_{-1} = 0"] & \Sigma^{1,0}\overline{H},
\end{tikzcd}$$

where k_{-1} denotes the -1^{st} k-invariant. The Postnikov truncation $k(w_n)\langle 0\rangle$ has thus homotopy groups $\pi_{*,*}(k(w_n)\langle 0\rangle) \cong \pi_{*,*}(\overline{H}) \cong \mathbb{F}_2$. We now want to attach the second copy of $\overline{H}$ via the Steenrod operation P_n. This requires a k-invariant k_0 that restricts to the operation P_n on $\overline{H}$ as shown in the diagram

$$\begin{tikzcd}
\overline{H} \arrow[r, "\mathrm{id}"] \arrow[rr, dashed, bend right, "P_n"'] & k(w_n)\langle 0\rangle \arrow[r, dotted, "\exists ?\, k_0"] \arrow[d] & \Sigma^{\mathbb{r}}\overline{H} \\
 & * \arrow[r] & \Sigma^{1,0}\overline{H},
\end{tikzcd}$$

where we denote the bidegree of P_n by $\mathbb{r} = |P_n|$. There is obviously a unique such filler up to homotopy, which is $k_0 = P_n$. The next step is to take the fiber of k_0 to get the next stage in the tower, which we denote by $k(w_n)\langle 1\rangle$ as shown in

$$\begin{tikzcd}
\Sigma^{\mathbb{r}-(1,0)}\overline{H} \arrow[r, "i_0"] \arrow[rr, dashed, bend right, "P_n"'] & k(w_n)\langle 1\rangle \arrow[r, dotted, "\exists?\ k_1"] \arrow[d, "p_0"] & \Sigma^{2\mathbb{r}-(1,0)}\overline{H} \\
\overline{H} \arrow[r, "\mathrm{id}"] \arrow[rr, dashed, bend right, "P_n"'] & k(w_n)\langle 0\rangle \arrow[r, "k_0"] \arrow[d] & \Sigma^{\mathbb{r}}\overline{H} \\
 & * \arrow[r] & \Sigma^{1,0}\overline{H}.
\end{tikzcd}$$

To continue the process, we need the existence of a k-invariant k_1 that restricts to P_n on $\Sigma^{\mathbb{r}-(1,0)}\overline{H}$. Equivalently, we are trying to extend P_n to $k(w_n)\langle 1\rangle$ as in the diagram

$$\begin{tikzcd}
\Sigma^{-1,0}k(w_n)\langle 0\rangle \arrow[r, "k_0"] & \Sigma^{\mathbb{r}-(1,0)}\overline{H} \arrow[r, "i_0"] \arrow[dr, "P_n"'] & k(w_n)\langle 1\rangle \arrow[r, "p_0"] \arrow[d, dotted, "\exists?\ k_1"] & k(w_n)\langle 0\rangle \\
 & & \Sigma^{2\mathbb{r}-(1,0)}\overline{H}. &
\end{tikzcd}$$

It follows that a k-invariant k_1 exists if and only if the composite

$$\Sigma^{-1,0}k(w_n)\langle 0\rangle \xrightarrow{k_0} \Sigma^{\mathbb{r}-(1,0)}\overline{H} \xrightarrow{P_n} \Sigma^{2\mathbb{r}-(1,0)}\overline{H}$$

is nullhomotopic, which is equivalent to having the relation $P_n k_0 = 0$ in the cohomology $\overline{H}^{2\mathbb{r}}(k(w_n)\langle 0\rangle)$. If such a k-invariant k_1 exists, then the difference of two such extensions would factor through the map p_0, and so one can alter k_1 up to homotopy by the group of maps in $\left[k(w_n)\langle 0\rangle, \Sigma^{2\mathbb{r}-(1,0)}\overline{H}\right]$ modulo k_0-divisibles. In order to keep the outline of the construction clear, we postpone the proof of the relation $P_n k_0 = 0$, and the fact that the moduli of such extensions is trivial to Proposition 4.17. These two facts show that there exists a unique k-invariant k_1 up to homotopy, so the homotopy type $k(w_n)\langle 1\rangle$ is uniquely defined and we can canonically continue to build the tower. We can now repeat the process by taking the fiber of k_1 as shown in the diagram

$$\begin{tikzcd}
\Sigma^{2\mathbb{r}-(2,0)}\overline{H} \arrow[r, "i_1"] \arrow[rr, dashed, bend right, "P_n"'] & k(w_n)\langle 2\rangle \arrow[r, dotted, "\exists ?\ k_2"] \arrow[d, "p_1"] & \Sigma^{3\mathbb{r}-(2,0)}\overline{H} \\
\Sigma^{\mathbb{r}-(1,0)}\overline{H} \arrow[r, "i_0"] \arrow[rr, dashed, bend right, "P_n"'] & k(w_n)\langle 1\rangle \arrow[r, "k_1"] \arrow[d, "p_0"] & \Sigma^{2\mathbb{r}-(1,0)}\overline{H} \\
\overline{H} \arrow[r, "\mathrm{id}"] \arrow[rr, dashed, bend right, "P_n"'] & k(w_n)\langle 0\rangle \arrow[r, "k_0"] \arrow[d] & \Sigma^{\mathbb{r}}\overline{H} \\
 & * \arrow[r] & \Sigma^{1,0}\overline{H}.
\end{tikzcd}$$

Similarly, the k-invariant k_2 exists if and only if the composite $P_n k_1$ is nullhomotopic, and the set of such extensions is given by a subset of homotopy classes of maps in $[k(w_n)\langle 1\rangle, \Sigma^{3\mathbb{r}-(2,0)}\overline{H}]$. As for the previous case, the existence of a unique k-invariant k_2 will be shown in Proposition 4.17.

More generally, if the k-invariants $k_0, \ldots, k_m$ exist, one can define $k(w_n)\langle m+1\rangle$ and ask if we can continue building the tower, i.e., if the next k-invariant k_{m+1} exists. The pattern is clear and a k-invariant k_{m+1} exists if and only if we have the relation

$$P_n k_m = 0 \in \overline{H}^{(m+2)\mathbb{r}-(m,0)}(k(w_n)\langle m\rangle). \tag{4.6}$$

Once k_{m+1} exists, one can alter it by the set of maps

$$\left[k(w_n)\langle m\rangle, \Sigma^{(m+2)\mathbb{r}-(m+1,0)}\overline{H}\right] \cong \overline{H}^{(m+2)\mathbb{r}-(m+1,0)}(k(w_n)\langle m\rangle). \tag{4.7}$$

We will show in Proposition 4.17 that at each step there exists a unique k-invariant k_m, which in addition turns out to satisfy the relation $P_n k_m = 0$, producing k_{m+1}. Having all these k-invariants, we can define a motivic spectrum $k(w_n)$ as the homotopy limit of the tower

$$k(w_n) \coloneqq \operatorname{holim}\left(\ldots \xrightarrow{p_2} k(w_n)\langle 1\rangle \xrightarrow{p_1} k(w_n)\langle 0\rangle \xrightarrow{p_0} *\right). \tag{4.8}$$

The map $k(w_n) \longrightarrow k(w_n)\langle 0\rangle = \overline{H}$ will represent the element 1 in cohomology so we call it the *fundamental class*. Since the choice of k-invariants k_m is canonical, this shows that there is a unique homotopy type $k(w_n)$ whose Postnikov tower has layers $\overline{H}$ that are successively attached via the cohomology operation P_n. □

Construction 4.15 contains the framework for the construction of the motivic spectrum $k(w_n)$. However, as mentioned above, the formula (4.8) does not make sense until we show the existence of the k-invariants $k_1, k_2, \ldots$, which we do in Proposition 4.17. It is folklore that the existence of these k-invariants is equivalent to showing that some specific Toda brackets between Eilenberg-Maclane contain the element zero. Although we don't pursue this direction further, the following remark is meant to explain this folklore result.

Remark 4.16 (Existence of k-invariants from Toda brackets)**.** The first k-invariant k_1 exists if and only if the relation $P_n k_0 = 0$ holds in the $\overline{H}$-cohomology of $k(w_n)\langle 0\rangle = \overline{H}$. The second k-invariant k_2 exists if and only if $P_n k_1 = 0$ in the cohomology of the 2-stage motivic spectrum $k(w_n)\langle 1\rangle$. The diagram

$$\begin{tikzcd}
k(w_n)\langle 0\rangle = \overline{H} \arrow[r, "k_0 = P_n"] & \Sigma^r \overline{H} \arrow[r, "P_n"] \arrow[d, "i_0"'] & \Sigma^{2r}\overline{H} \arrow[r, "P_n"] & \Sigma^{3r}\overline{H} \\
 & \Sigma^{1,0}k(w_n)\langle 1\rangle \arrow[ur, dashed, "\exists\ k_1"'] \arrow[d, "p_0"'] & & \\
 & \Sigma^{1,0}k(w_n)\langle 0\rangle \arrow[uurr, dashed, "\exists\ \langle P_n, P_n, P_n\rangle"'] \arrow[d, "k_0 = P_n"'] & & \\
 & \Sigma^{r+(1,0)}\overline{H} & &
\end{tikzcd}$$

shows that the relation $P_n k_1 = 0$ holds if and only if the Toda bracket $\langle P_n, P_n, P_n\rangle$ contains an element that is P_n-divisible. Since the indeterminacy is also P_n-divisible, this is equivalent to the bracket $\langle P_n, P_n, P_n\rangle$ containing zero. Moreover, the indeterminacy of the Toda bracket corresponds to the choices of such extensions k_2. This generalizes to higher Toda brackets for the higher k_i's, but we do not explore this direction, as we will show by other means that $P_n k_m = 0$.

In the following Proposition 4.17 we now show the existence of these k-invariants, as well as their uniqueness up to homotopy.

Proposition 4.17. *There exist unique k-invariants $k_0, k_1, k_2, \dots$ as described in Construction 4.15, defining a unique homotopy type $k(w_n)$ by equation* (4.8).

Proof. Suppose that the tower has been constructed until the stage

$$\begin{tikzcd}
\Sigma^{(m+1)\mathbb{r}-(m+1,0)}\overline{H} \arrow[r, "i_m"] \arrow[rr, dashed, bend right, "P_n"] & k(w_n)\langle m+1\rangle \arrow[r, dotted, "\exists?\ k_{m+1}"] \arrow[d, "p_m"] & \Sigma^{(m+2)\mathbb{r}-(m+1,0)}\overline{H} \\
\Sigma^{m\mathbb{r}-(m,0)}\overline{H} \arrow[r, "i_{m-1}"] \arrow[rr, dashed, bend right, "P_n"] & k(w_n)\langle m\rangle \arrow[r, "k_m"] \arrow[d, "p_{m-1}"] & \Sigma^{(m+1)\mathbb{r}-(m,0)}\overline{H}, \\
 & \vdots &
\end{tikzcd}$$

and we want to show that there is a unique possible k-invariant k_{m+1}. Note that in Construction 4.15 we uniquely constructed the tower in the case $m = 0$ so we can start the inductive process.

We will show in Lemma 4.18 below that if $k(w_n)\langle m\rangle$ exists (as assumed by the induction hypothesis), then its cohomology $\overline{H}^{*,*}(k(w_n)\langle m\rangle)$ vanishes in Chow degrees less than $-m$. This implies that if a k-invariant k_{m+1} exists then it is unique, since the set of choices from equation (4.7) is a subset of the cohomology of $k(w_n)\langle m\rangle$ that is concentrated in Chow degrees less than $-m-1$, which vanishes since $-m-1 < -m$.

To show existence, by equation (4.6) we have to show that $P_n k_m = 0$. We will show in Lemma 4.20 by induction that despite the group $\overline{H}^{(m+2)\mathbb{r}-(m,0)}(k(w_n)\langle m\rangle)$ not vanishing, we still have the relation $P_n k_m = 0$. By induction, this concludes the existence of unique k-invariants $k_0, k_1, \dots$ and allows us to define a unique homotopy type $k(w_n)$ by equation (4.8). ☐

Lemma 4.18. *Fix an m, and suppose that $k(w_n)\langle 0\rangle, k(w_n)\langle 1\rangle, \dots, k(w_n)\langle m\rangle$ are constructed* *as in Construction 4.15. Then* $\overline{H}^{*,*}(k(w_n)\langle m\rangle)$ *vanishes in Chow degrees less than* $-m$.

Proof. We show it by induction. For the initial case recall that $k(w_n)\langle 0\rangle = \overline{H}$ and thus $\overline{H}^{*,*}(k(w_n)\langle 0\rangle) \cong \overline{\mathcal{A}}_{\mathbb{C}}$ is the Steenrod algebra of operations of $\overline{H}$. Recall from Remark 4.5 that it is concentrated in positive Chow degrees, showing the initial case.

Suppose that the Lemma is shown for $k(w_n)\langle 0\rangle, \ldots, k(w_n)\langle s-1\rangle$. The cofiber sequence

$$\Sigma^{s\mathbb{r}-(s,0)}\overline{H} \longrightarrow k(w_n)\langle s\rangle \longrightarrow k(w_n)\langle s-1\rangle$$

induces a long exact sequence in cohomology

$$\cdots \longleftarrow \overline{\mathcal{A}}_{\mathbb{C}}^{(a,b)-s\mathbb{r}+(s,0)} \longleftarrow \overline{H}^{a,b}k(w_n)\langle s\rangle \longleftarrow \overline{H}^{a,b}k(w_n)\langle s-1\rangle \longleftarrow \cdots .$$

Fix a couple (a,b) in Chow degree less than $-s$, i.e., such that $a-2b < -s$. Then $\overline{\mathcal{A}}_{\mathbb{C}}^{(a,b)-s\mathbb{r}+(s,0)} = 0$ by the initial case since it is in negative Chow degree, and $\overline{H}^{a,b}k(w_n)\langle s-1\rangle = 0$ by the inductive hypothesis since it is in Chow degree less than $-s$, which is less than $-(s-1)$. This implies that $\overline{H}^{a,b}k(w_n)\langle s\rangle = 0$, showing the inductive step and finishing the proof. ☐

Remark 4.19. This bound is sharp for every m, as it can be seen by the long exact sequences in cohomology that the groups $\overline{H}^{*,*}(k(w_n)\langle m\rangle)$ do not vanish in Chow degree $-m$. The composite $P_n k_m \in \overline{H}^{(m+2)\mathbb{r}-(m,0)}(k(w_n)\langle m\rangle)$ lives in this non-zero group in Chow degree $-m$, so we cannot use Lemma 4.18 to show that the product $P_n k_m$ is zero.

Lemma 4.20. *Fix an* m*, and suppose that* $k(w_n)\langle 0\rangle, k(w_n)\langle 1\rangle, \ldots, k(w_n)\langle m\rangle$ *are constructed as in Construction 4.15. Then we have the relation*

$$P_n k_m = 0 \in \overline{H}^{(m+2)\mathbb{r}-(m,0)}(k(w_n)\langle m\rangle).$$

Proof. Consider the cofiber sequence

$$\Sigma^{m\mathbb{r}-(m,0)}\overline{H} \longrightarrow k(w_n)\langle m\rangle \longrightarrow k(w_n)\langle m-1\rangle.$$

The natural transformation $\overline{H}^{*,*} \xrightarrow{\cdot P_n} \overline{H}^{(*,*)+\mathbb{r}}$ induces a map of long exact sequences, which in bidegree $(*,*) = (m+1)\mathbb{r} - (m,0)$ becomes

$$\begin{tikzcd}
\cdots & \overline{\mathcal{A}}_{\mathbb{C}}^{\mathbb{r}} \arrow[l] \arrow[d, "P_n\cdot"] & \overline{H}^{(m+1)\mathbb{r}-(m,0)}(k(w_n)\langle m\rangle) \arrow[l] \arrow[d, "P_n\cdot"] & \overline{H}^{(m+1)\mathbb{r}-(m,0)}(k(w_n)\langle m-1\rangle) \arrow[l] \arrow[d, "P_n\cdot"] & \cdots \arrow[l] \\
\cdots & \overline{\mathcal{A}}_{\mathbb{C}}^{2\mathbb{r}} \arrow[l] & \overline{H}^{(m+2)\mathbb{r}-(m,0)}(k(w_n)\langle m\rangle) \arrow[l] & \overline{H}^{(m+2)\mathbb{r}-(m,0)}(k(w_n)\langle m-1\rangle) \arrow[l] & \cdots . \arrow[l]
\end{tikzcd}$$

Both cohomology groups of $k(w_n)\langle m-1\rangle$ on the right of the above diagram are concentrated in Chow degree $-m$, so they vanish by Lemma 4.18. We thus get the commutative square

$$\begin{tikzcd}
P_n \in \overline{\mathcal{A}}_{\mathbb{C}}^{\mathbb{r}} \arrow[d, "P_n\cdot"'] & \overline{H}^{(m+1)\mathbb{r}-(m,0)}(k(w_n)\langle m\rangle) \ni k_m \arrow[l, hook'] \arrow[d, "P_n\cdot"'] \\
P_nP_n \in \overline{\mathcal{A}}_{\mathbb{C}}^{2\mathbb{r}} & \overline{H}^{(m+2)\mathbb{r}-(m,0)}(k(w_n)\langle m\rangle) \ni P_nk_m, \arrow[l, hook']
\end{tikzcd}$$

with injective horizontal maps. The element k_m lives in the top right corner of the square, and by definition is sent to $P_n \in \overline{\mathcal{A}}_{\mathbb{C}}^{\mathbb{r}}$ along the top horizontal map. Since $P_nP_n = 0 \in \overline{\mathcal{A}}_{\mathbb{C}}^{2\mathbb{r}}$ by Lemma 4.10, the product P_nk_m is also sent to zero in $\overline{\mathcal{A}}_{\mathbb{C}}^{2\mathbb{r}}$ by the bottom horizontal map. Since the bottom horizontal map is injective, it follows that $P_nk_m = 0$. □

This finishes the construction started in 4.15, constructing a homotopy type $k(w_n)$ defined by its Postnikov tower. From this tower, we will now compute its cohomology.

Proposition 4.21. *The cohomology of $k(w_n)$ is given as an $\overline{\mathcal{A}}_{\mathbb{C}}$-module by*

$$\overline{H}^{*,*}(k(w_n)) \cong \overline{\mathcal{A}}_{\mathbb{C}}//E(P_n). \tag{4.9}$$

In particular, it is concentrated in non-negative Chow degrees.

Proof. Applying the contravariant functor $\overline{H}^{*,*}$ to the tower defining $k(w_n)$ gives an unrolled exact couple and thus an associated spectral sequence as shown in Figure 6. The E_∞-page

$$\begin{tikzcd}
 & & & k(w_n) \arrow[d] \\
\vdots \arrow[d, "d_1 = P_n"'] & & & \vdots \arrow[d] \\
\Sigma^{2r-(2,0)}\overline{\mathcal{A}}_{\mathbb{C}} \cdot w_n^2 \arrow[d, "d_1 = P_n"'] & & \Sigma^{2r-(2,0)}\overline{H} \arrow[r, "i_1"] & k(w_n)\langle 2\rangle \arrow[d, "p_1"] \\
\Sigma^{r-(1,0)}\overline{\mathcal{A}}_{\mathbb{C}} \cdot w_n^1 \arrow[d, "d_1 = P_n"'] & & \Sigma^{r-(1,0)}\overline{H} \arrow[r, "i_0"] \arrow[u, dotted, bend left, "P_n \circ"] \arrow[ll, squiggly, "\overline{H}^{*,*}(-)"'] & k(w_n)\langle 1\rangle \arrow[d, "p_0"] \\
\Sigma^{0,0}\overline{\mathcal{A}}_{\mathbb{C}} \cdot w_n^0 & & \overline{H} \arrow[r, "\mathrm{id}"] \arrow[u, dotted, bend left, "P_n \circ"] & k(w_n)\langle 0\rangle \arrow[d] \\
 & & & *
\end{tikzcd}$$

Figure 6: The spectral sequence computing the cohomology of $k(w_n)$ is drawn schematically on the left-hand side.

of this spectral sequence computes the colimit

$$\mathcal{H}^{*,*} := \operatorname{colim}\left(\overline{H}^{*,*}(k(w_n)\langle 0\rangle) \longrightarrow \overline{H}^{*,*}(k(w_n)\langle 1\rangle) \longrightarrow \cdots\right).$$

We will first show that the E_∞-page is isomorphic to the quotient $\overline{\mathcal{A}}_{\mathbb{C}}//E(P_n)$, and then show that in this setting we have

$$\mathcal{H}^{*,*} \cong \overline{H}^{*,*}(k(w_n)),$$

i.e., that the cohomology of this homotopy limit is computed by the colimit of the cohomologies. We index the E_1-page of this spectral sequence as

$$E_1^{s,t,w} = \overline{H}^{t,w}\left(\Sigma^{s\cdot(\mathbb{r}-(1,0))}\overline{H}\right),$$

where s is the homological degree, and (t,w) are the internal degrees. The first differential d_1 is the boundary map in the tower, which is multiplication by the cohomology operation P_n. Since P_n is exact on $\overline{\mathcal{A}}_{\mathbb{C}}$ by Corollary 4.12, the d_1 differential wipes out everything in homological degree $s > 0$, and leaves the quotient $\overline{\mathcal{A}}_{\mathbb{C}}//E(P_n)$ in degree $s = 0$. It thus follows that the spectral sequence collapses at E_2 for degree reasons, with no possible hidden extensions.

It remains to show that this colimit is isomorphic to $\overline{H}^{*,*}(k(w_n))$. This is shown by a standard technique which we explain for the reader's convenience. Fix a bidegree (t,w) until the end of the proof. Consider the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \overline{H}^{t,w}(k(w_n)\langle m\rangle) \arrow[r] \arrow[d] & \overline{H}^{t,w}(k(w_n)\langle m+1\rangle) \arrow[r] \arrow[dl] & \overline{H}^{t,w}(k(w_n)\langle m+2\rangle) \arrow[r] \arrow[dll] & \cdots \\
& \overline{H}^{t,w}(k(w_n)), & & &
\end{tikzcd}$$

where the vertical and diagonal maps are induced by the maps from the diagram defining $k(w_n)$ as a homotopy limit. For m big enough, the maps in the diagram

$$\overline{H}^{t,w}(k(w_n)\langle m\rangle) \xrightarrow{\cong} \overline{H}^{t,w}(k(w_n)\langle m+1\rangle) \xrightarrow{\cong} \cdots \xrightarrow{\cong} \mathcal{H}^{t,w}$$

are all isomorphisms since the successive fibers

$$\Sigma^{m\mathbb{r}-(m,0)}\overline{H},\ \Sigma^{(m+1)\mathbb{r}-(m+1,0)}\overline{H}, \ldots$$

are suspended too much and have no cohomology in degree (t,w). Similarly, for m big enough the fiber of $k(w_n) \longrightarrow k(w_n)\langle m\rangle$ is too connected, and thus also has no cohomology. It follows that for m big enough, the maps in the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \overline{H}^{t,w}(k(w_n)\langle m\rangle) \arrow[r, "\cong"] \arrow[d, "\cong"] & \overline{H}^{t,w}(k(w_n)\langle m+1\rangle) \arrow[r, "\cong"] \arrow[dl, "\cong"] & \overline{H}^{t,w}(k(w_n)\langle m+2\rangle) \arrow[r, "\cong"] \arrow[dll, "\cong"] & \cdots \\
& \overline{H}^{t,w}(k(w_n)), & & &
\end{tikzcd}$$

become isomorphisms, and thus we have the required isomorphism

$$\overline{H}^{t,w}(k(w_n)) \xleftarrow{\cong} \mathcal{H}^{t,w}.$$

Since $\overline{\mathcal{A}}_{\mathbb{C}}$ is concentrated in non-negative Chow degrees by Remark 4.5, then so is $\overline{H}^{*,*}(k(w_n)) \cong \overline{\mathcal{A}}_{\mathbb{C}}//E(P_n)$. ☐

Corollary 4.22. *The cohomology of the smash product $k(w_n) \wedge k(w_n)$ is given by*

$$\overline{H}^{*,*}\left(k(w_n) \wedge k(w_n)\right) \cong \overline{\mathcal{A}}_{\mathbb{C}}//E(P_n) \otimes \overline{\mathcal{A}}_{\mathbb{C}}//E(P_n),$$

with the diagonal $\overline{\mathcal{A}}_{\mathbb{C}}$-module structure. More generally, for any $m \geq 1$ we have

$$\overline{H}^{*,*}\big(k(w_n)^{\wedge m}\big) \cong \big(\overline{\mathcal{A}}_{\mathbb{C}}//E(P_n)\big)^{\otimes m},$$

with the iterated diagonal $\overline{\mathcal{A}}_{\mathbb{C}}$-module structure. In particular, for any m the cohomology $\overline{H}^{*,*}\big(k(w_n)^{\wedge m}\big)$ is concentrated in positive Chow degrees.

Proof. The first part follows from the Künneth isomorphism of Proposition 4.6. An easy induction computes the cohomology of the smash product $k(w_n)^{\wedge m}$. Finally, $\overline{\mathcal{A}}_{\mathbb{C}}//E(P_n)$ is concentrated in positive Chow degrees since $\overline{\mathcal{A}}_{\mathbb{C}}$ is (Remark 4.5), and thus so is $\big(\overline{\mathcal{A}}_{\mathbb{C}}//E(P_n)\big)^{\otimes m}$.
☐

Remark 4.23 (Additive description of $\pi_{*,*}(k(w_n))$)**.** It is easy to see from Construction 4.15

that additively the homotopy groups of $k(w_n)$ are given by sparse copies of $\mathbb{F}_2$'s

$$\pi_{s,w}(k(w_n)) \cong \begin{cases} \mathbb{F}_2 & \text{if } (s,w) = m \cdot \mathbb{r} \text{ for some } m \in \mathbb{N}_0 \\ 0 & \text{otherwise.} \end{cases}$$

For example, one can compute $\pi_{*,*}(k(w_n))$ by Milnor's $\lim^1$ short exact sequence, where the $\lim^1$ term vanishes since p_m is surjective in homotopy groups and thus the inverse sequence satisfies the Mittag-Leffler condition. We will compute the ring structure on $\pi_{*,*}(k(w_n))$ via the motivic $\overline{H}$-based Adams spectral sequence in Theorem 4.26.

4.2.2 The E_∞ ring structure and $K(w_n)$

We will now endow the motivic spectrum $k(w_n)$ with an E_∞ ring structure. The technique is the same as in [14, Section 3] and is done in three steps. The first step is to construct a ring map $\mu\colon k(w_n) \wedge k(w_n) \longrightarrow k(w_n)$, which we do by lifting the fundamental class $1 \in \overline{H}^{0,0}\big(k(w_n)^{\wedge 2}\big)$ along the Postnikov tower of $k(w_n)$. We will then show that this endows $k(w_n)$ with a unital, associative and commutative monoid structure in the homotopy category. The last step is to use Robinson's obstruction theory [51] to extend it to an E_∞ ring structure.

The following Lemma 4.24 provides a homotopy class of maps $\mu\colon k(w_n) \wedge k(w_n) \longrightarrow k(w_n)$, as well as some estimates necessary to apply Robinson's obstruction theory.

Lemma 4.24. *For any $m \geq 1$ the abelian group of homotopy classes of maps satisfies*

$$\big[k(w_n)^{\wedge m}, \Sigma^{t,w} k(w_n)\big] \cong \begin{cases} \mathbb{F}_2 & \textit{if } (t,w) = (0,0) \\ 0 & \textit{if } (t,w) \textit{ is in negative Chow degree, i.e., } t - 2w < 0. \end{cases}$$

Moreover, the non-trivial map $k(w_n)^{\wedge m} \longrightarrow k(w_n)$ preserves the fundamental class in $\overline{H}$-cohomology, i.e., sends 1 to $1^{\otimes m}$.

Proof. Consider the Atiyah-Hirzebruch spectral sequence computing

$$[k(w_n)^{\wedge m}, k(w_n)]_{*,*}$$

given by applying the functor $[k(w_n)^{\wedge m}, -]$ to the tower defining $k(w_n)$ in Construction 4.15. We index the E_1-page by

$$E_1^{s,t,w} = \left[k(w_n)^{\wedge m}, \Sigma^{t,w}\Sigma^{s\mathbb{r}-(s,0)}\overline{H}\right] \cong \overline{H}^{(t,w)+s\mathbb{r}-(s,0)}\big(k(w_n)^{\wedge m}\big),$$

where $s \geq 0$ is the homological degree and (t,w) are the internal degrees. This is a first quadrant spectral sequence (if plotted in the (s,t) plane) and thus converges to

$$E_\infty^{s,t,w} \cong \left[k(w_n)^{\wedge m}, \Sigma^{t,w}k(w_n)\right].$$

Since the cohomology of the smash power $k(w_n)^{\wedge m}$ is concentrated in non-negative Chow degrees by Corollary 4.22, and since $\mathbb{r}$ is in Chow degree zero, the E_1-page is concentrated in degrees

$$t - 2w - s \geq 0.$$

In particular, if $t - 2w < 0$, the spot $E_\infty^{s,t,w}$ is zero for any s and thus also

$$\left[k(w_n)^{\wedge m}, \Sigma^{t,w}k(w_n)\right] = 0.$$

In the case where $(t,w) = (0,0)$, then necessarily $s = 0$. The l^{th} differential d_l goes from (s,t,w) to $(s+l, t+1, w)$, and there is thus no possible differential entering $E_1^{0,0,0}$. Since d_l reduces the quantity $t - 2w - s$ by $l - 1$, the only possible differential exiting $E_1^{0,0,0}$ is a d_1. As in Proposition 4.21, observe that the d_1 differential is multiplication by P_n on $\overline{H}^{*,*}\big(k(w_n)^{\wedge m}\big)$. Recall from Corollary 4.22 that

$$\overline{H}^{*,*}\big(k(w_n)^{\wedge m}\big) \cong \big(\overline{\mathcal{A}}_{\mathbb{C}}//E(P_n)\big)^{\otimes m},$$

with $\overline{H}^{0,0}\big(k(w_n)^{\wedge m}\big) \cong \mathbb{Z}/2^{\otimes m} \cong \mathbb{Z}/2$. Since $P_n \in \overline{\mathcal{A}}_{\mathbb{C}}$ is primitive, it acts as zero on $1 \otimes \cdots \otimes 1$.

This shows that there is no possible differential on $E_1^{0,0,0}$ and thus

$$[k(w_n)^{\wedge m}, k(w_n)] \cong E_1^{0,0,0} = \overline{H}^{0,0}\big(k(w_n)^{\wedge m}\big) \cong \mathbb{Z}/2^{\otimes m} \cong \mathbb{Z}/2.$$

Let's call a representative for the non-trivial class by $\mu\colon k(w_n)^{\wedge m} \longrightarrow k(w_n)$. Unwinding this chain of isomorphisms shows that μ start as the fundamental class $1^{\otimes m} = \mu_0\colon k(w_n)^{\wedge m} \longrightarrow \overline{H}$ and can be uniquely lifted along the Postnikov tower as shown in the diagram

$$\begin{tikzcd}
 & k(w_n) \arrow[d] & \\
 & \vdots \arrow[d] & \\
 & k(w_n)\langle 2\rangle \arrow[r, "k_2"] \arrow[d, "p_1"] & \Sigma^{3r-(2,0)}\overline{H} \\
 & k(w_n)\langle 1\rangle \arrow[r, "k_1"] \arrow[d, "p_0"] & \Sigma^{2r-(1,0)}\overline{H} \\
k(w_n)^{\wedge m} \arrow[r, dashed, "\mu_0 = 1"] \arrow[ru, dashed, "\mu_1"] \arrow[ruu, dashed, "\mu_2"] \arrow[ruuuu, dashed, bend left, "\mu"] & k(w_n)\langle 0\rangle \arrow[r, "k_0"] & \Sigma^{r}\overline{H}.
\end{tikzcd}$$

This shows that μ sends the fundamental class 1 to the fundamental class $1^{\otimes m}$ in cohomology since the vertical composite is non-trivial and thus 1 in cohomology. $\square$

Proposition 4.25. *The map $k(w_n) \wedge k(w_n) \xrightarrow{\mu} k(w_n)$ is homotopy unital, associative and commutative.*

Proof. Since $\pi_{0,0}(k(w_n)) \cong \mathbb{F}_2$ from Remark 4.23, the non-zero element

$$S^{0,0} \xrightarrow{i} k(w_n)$$

will be the unit of the ring structure on $k(w_n)$. The multiplication μ is homotopy left unital if and only if the composite

$$S^{0,0} \wedge k(w_n) \xrightarrow{i \wedge \mathrm{id}} k(w_n) \wedge k(w_n) \xrightarrow{\mu} k(w_n)$$

is homotopy equivalent to the identity map on $k(w_n)$. By Corollary 4.22 the group of homotopy classes of self-maps of degree $(0,0)$ on $k(w_n)$ is $\mathbb{Z}/2$, so it suffices to show that the above composite is not nullhomotopic. We can do so by embedding it in the following commutative diagram

$$\begin{tikzcd}
S^{0,0} \wedge k(w_n) \arrow[r, "i \wedge \mathrm{id}"] & k(w_n) \wedge k(w_n) \arrow[r, "\mu"] \arrow[d, "1 \wedge 1"] & k(w_n) \arrow[d, "1"] \\
S^{0,0} \wedge S^{0,0} \arrow[u, "\mathrm{id} \wedge i"] \arrow[r, "i \wedge i"] & \overline{H} \wedge \overline{H} \arrow[r, "\mu"] & \overline{H},
\end{tikzcd}$$

where both squares are seen to commute by Lemma 4.24 since the map 1 represents 1 in the cohomology of $k(w_n)$. The top horizontal composite cannot be nullhomotopic since the bottom horizontal composite is the unit of $\overline{H}$ and thus not nullhomotopic. This shows that μ is left unital, and by a similar argument that μ is also right unital.

Recall from Corollary 4.22 that μ lives in the group $[k(w_n) \wedge k(w_n), k(w_n)] \cong \mathbb{Z}/2 \cdot \{\mu\}$. Precomposing with the factor swap map $\chi \colon k(w_n) \wedge k(w_n) \longrightarrow k(w_n) \wedge k(w_n)$ is an involution on this group, which forces $\mu \circ \chi \cong \mu$, i.e., showing that μ is homotopy commutative.

For associativity, we need to compare the two maps $\mu \circ (\mu \wedge \mathrm{id})$ and $\mu \circ (\mathrm{id} \wedge \mu)$ in the group of homotopy classes of maps $[k(w_n) \wedge k(w_n) \wedge k(w_n), k(w_n)]$. Since μ is unital, precomposing both maps with the units

$$S^{0,0} \wedge S^{0,0} \wedge S^{0,0} \xrightarrow{i \wedge i \wedge i} k(w_n) \wedge k(w_n) \wedge k(w_n)$$

gives the non-zero map $S^{0,0} \xrightarrow{i} k(w_n)$. This means that both maps $\mu \circ (\mu \wedge \mathrm{id})$ and $\mu \circ (\mathrm{id} \wedge \mu)$ are not nullhomotopic, and since $[k(w_n) \wedge k(w_n) \wedge k(w_n), k(w_n)] \cong \mathbb{Z}/2$ by Corollary 4.22, they are homotopic. □

Theorem 4.26. *For any n, the motivic spectrum $k(w_n) \in {}_{C\tau}\boldsymbol{Cell}$ admits an essentially unique E_∞ ring structure and satisfies*

- $\overline{H}^{*,*}(k(w_n)) \cong \overline{\mathcal{A}}_{\mathbb{C}}//E(P_n)$ *as an $\overline{\mathcal{A}}_{\mathbb{C}}$-module,*
- $\pi_{*,*}(k(w_n)) \cong \mathbb{F}_2[w_n]$ *as a ring.*

Proof. We start by rigidifying the homotopy ring structure on $k(w_n)$ to an E_∞ ring structure by using Robinson's obstruction theory. This obstruction theory has been adapted to the motivic setting in [14, Corollary 3.2]. More precisely, the multiplication μ extends to an E_∞ ring structure if the groups

$$\left[k(w_n)^{\wedge m}, \Sigma^{3-m',0}k(w_n)\right]$$

are all zero for $m' \geq 4$ and $2 \leq m \leq m'$. For a fixed $m' \geq 4$, observe that these groups are in negative Chow degree for any m, and thus vanish by Lemma 4.24. This shows that the multiplication map μ can be extended to an E_∞ ring structure on $k(w_n)$. Furthermore, [14, Corollary 3.2] shows that this E_∞ ring structure is unique if

$$\left[k(w_n)^{\wedge m}, \Sigma^{2-m',0}k(w_n)\right]$$

are all zero for $m' \geq 4$ and $2 \leq m \leq m'$. Another application of Lemma 4.24 shows these are zero and thus that $k(w_n)$ admits a unique E_∞ ring structure.

Its cohomology has been computed in Proposition 4.21. For its homotopy, we already know from remark 4.23 that $\pi_{*,*}(k(w_n))$ is given by a copy of $\mathbb{F}_2$ in every degree of the form $m \cdot (\mathbb{r} - (1,0))$ where $m \geq 0$ and $\mathbb{r} - (1,0)$ is the bidegree of the class w_n. To show that the ring structure is polynomial, one can for example consider the $C\tau$-linear $\overline{H}$-based motivic Adams spectral sequence

$$E_2 = \mathrm{Ext}_{\overline{\mathcal{A}}_\mathbb{C}}\left(\overline{H}^{*,*}(k(w_n)), \mathbb{F}_2\right) \Longrightarrow \pi_{*,*}(k(w_n))$$

which is now multiplicative since $k(w_n)$ is a motivic ring spectrum. Since $\overline{H}^{*,*}(k(w_n)) \cong \overline{\mathcal{A}}_\mathbb{C}//E(P_n)$ we can apply the usual change of rings to the E_2-page

$$E_2 \cong \mathrm{Ext}_{E(P_n)}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2[w_n].$$

The spectral sequence collapses now at E_2 with no possible hidden extensions. $\square$

Another relation between the homotopy element w_n and the cohomology operation P_n is given in the following proposition.

Proposition 4.27. *For any n, there is a cofiber sequence*

$$\Sigma^{\mathrm{r}-(1,0)}k(w_n) \xrightarrow{w_n} k(w_n) \xrightarrow{1} \overline{H} \xrightarrow{\beta_n} \Sigma^{\mathrm{r}}k(w_n),$$

where the boundary map β_n is such that the composite

$$\overline{H} \xrightarrow{\beta_n} \Sigma^{\mathrm{r}}k(w_n) \xrightarrow{1} \Sigma^{\mathrm{r}}\overline{H}$$

gives the cohomology operation $P_n \in \overline{\mathcal{A}}_{\mathbb{C}}$.

Proof. Consider the cofiber sequence

$$\Sigma^{\mathrm{r}-(1,0)}k(w_n) \xrightarrow{w_n} k(w_n) \longrightarrow C \longrightarrow \Sigma^{\mathrm{r}}k(w_n),$$

where we denote by C the cofiber of multiplication by w_n. Comparing it with the cofiber sequence coming from the beginning of the tower of $k(w_n)$ gives a diagram

$$\begin{tikzcd}
\Sigma^{\mathrm{r}-(1,0)}k(w_n) \arrow[r, "w_n"] & k(w_n) \arrow[r] \arrow[d] \arrow[dr, "1"] & C \arrow[r, "q"] & \Sigma^{\mathrm{r}}k(w_n) \\
\Sigma^{\mathrm{r}-(1,0)}\overline{H} \arrow[r] & k(w_n)\langle 1\rangle \arrow[r, "1"] & k(w_n)\langle 0\rangle = \overline{H} \arrow[r, "P_n"] & \Sigma^{\mathrm{r}}\overline{H},
\end{tikzcd}$$

where $k(w_n) \longrightarrow k(w_n)\langle 1\rangle$ is the natural map and 1 denotes the fundamental class. The composite $1 \circ w_n$ lives in the Chow degree -1 part of the cohomology of $k(w_n)$, which is zero by Proposition 4.21. This implies that there exists a filler ψ

$$\begin{tikzcd}
\Sigma^{\mathrm{r}-(1,0)}k(w_n) \arrow[r, "w_n"] & k(w_n) \arrow[r] \arrow[d] \arrow[dr, "1"] & C \arrow[r, "q"] \arrow[d, dashed, "\exists\psi"] & \Sigma^{\mathrm{r}}k(w_n) \arrow[r, "w_n"] \arrow[d, dashed, "\exists!\ \phi"] & \Sigma^{1,0}k(w_n) \\
\Sigma^{\mathrm{r}-(1,0)}\overline{H} \arrow[r] & k(w_n)\langle 1\rangle \arrow[r, "1"] & k(w_n)\langle 0\rangle = \overline{H} \arrow[r, "P_n"] & \Sigma^{\mathrm{r}}\overline{H} \arrow[r] & \Sigma^{1,0}k(w_n)\langle 1\rangle
\end{tikzcd}$$

which itself implies that there is another filler ϕ making all squares commute. By the long exact sequence in cohomology, it is easy to see that $\left[C, \overline{H}\right] \cong \mathbb{Z}/2$. Observe that ψ is non-zero since $k(w_n) \longrightarrow k(w_n)\langle 0\rangle$ is non-zero, which implies that ψ is unique up to homotopy and induces an isomorphism on homotopy groups. It follows that its cofiber is contractible and thus that it is an equivalence. Observe that ϕ is unique up to homotopy since $\left[\Sigma^{1,0}k(w_n), \Sigma^{\mathbb{r}}\overline{H}\right] = 0$ by Proposition 4.21. Since $\left[\Sigma^{1,0}k(w_n), \Sigma^{\mathbb{r}}\overline{H}\right] \cong \mathbb{Z}/2$ and $P_n \neq 0$, then ϕ is also non-zero, forcing it to be the fundamental class $\phi = 1$. This gives the desired cofiber sequence

$$\Sigma^{\mathbb{r}-(1,0)}k(w_n) \xrightarrow{w_n} k(w_n) \xrightarrow{1} \overline{H} \xrightarrow{\beta_n} \Sigma^{\mathbb{r}}k(w_n),$$

where we denote the composite $q \circ \psi^{-1}$ by β_n. The composite $\overline{H} \xrightarrow{\beta_n} \Sigma^{\mathbb{r}}k(w_n) \xrightarrow{1} \Sigma^{\mathbb{r}}\overline{H}$ is the cohomology operation P_n by the above comparison of cofiber sequences. ☐

Corollary 4.28. *For any n, there is a motivic E_∞ graded field $K(w_n)$ with*

$$\pi_{*,*}(K(w_n)) \cong \mathbb{F}_2[w_n^{\pm 1}].$$

Proof. As a module, define $K(w_n)$ as the homotopy colimit

$$K(w_n) := \operatorname{hocolim}\left(k(w_n) \xrightarrow{w_n} \Sigma^{-|w_n|}k(w_n) \xrightarrow{w_n} \cdots\right).$$

By compactness of $S^{s,w}$, its homotopy groups are given by

$$\pi_{*,*}(K(w_n)) \cong \pi_{*,*}(k(w_n))[w_n^{-1}] \cong \mathbb{F}_2[w_n^{\pm 1}].$$

It remains to show that this localization can be performed in motivic E_∞ rings. For this, one can apply the methods of [19]. We will now give a minimal argument to explain how this applies to the motivic setting, and refer to [19, Section 3.1] for more details. Recall that the motivic E_∞ operad that we consider is the simplicial operad where $E\Sigma_n$ is a constant motivic space. In particular, this space admits a cellular filtration where the layers are spheres $S^{*,0}$ in weight zero. There is thus a spectral sequence computing the homotopy groups of $(E\Sigma_n)_+ \wedge_{\Sigma_n} Z$, which has as input the homotopy groups of various suspensions $\Sigma^{*,0}Z$. One can now apply [19] since the acyclics form a localizing subcategory that is closed under suspensions of the form $\Sigma^{*,0}$. □

4.3 The Motivic Spectrum wBP

In this section we will construct E_∞ ring spectra wBP and $wBP\langle n\rangle$ with homotopy groups given by

$$\pi_{*,*}(wBP) \cong \mathbb{F}_2[w_0, w_1, \ldots] \qquad \text{and} \qquad \pi_{*,*}(wBP\langle n\rangle) \cong \mathbb{F}_2[w_0, w_1, \ldots, w_n].$$

These spectra will be constructed for the property that

$$\overline{H}^{*,*}(wBP) \cong \overline{\mathcal{A}}_{\mathbb{C}}//E(P_1, P_2, \ldots) \qquad \text{and} \qquad \overline{H}^{*,*}(wBP\langle n\rangle) \cong \overline{\mathcal{A}}_{\mathbb{C}}//E(P_1, \ldots, P_n)$$

with the natural $\overline{\mathcal{A}}_{\mathbb{C}}$-module structure. As we did in Section 4.1.3 for $k(w_n)$, in Section 4.3.1 we derive some formulas in the Steenrod algebra $\overline{\mathcal{A}}_{\mathbb{C}}$ and its dual $\overset{\vee}{\overline{\mathcal{A}}}_{\mathbb{C}}$. In Section 4.3.2 we proceed to construct wBP, by using a version of Toda's Realization Theorem [57, Lemma 3.1]. We finally endow it with an E_∞ ring structure in Section 4.3.3.

4.3.1 More formulas in the $\overline{H}$-Steenrod algebra

Recall from Proposition 4.2 that the $C\tau$-linear dual $\overline{H}$-Steenrod algebra, i.e., the Hopf algebra of $C\tau$-linear co-operations on $\overline{H}$ is given by

$$\pi_{*,*}(\overline{H} \wedge_{C\tau} \overline{H}) = \overset{\vee}{\overline{\mathcal{A}}}_{\mathbb{C}} \cong \mathbb{F}_2[\xi_1, \xi_2, \ldots] \otimes E(\tau_0, \tau_1, \ldots).$$

In this section we will need to work in Milnor's basis, whose notation we recall.

Notation 4.29 (Milnor's basis and the P^R notation)**.** Given a sequence $R = (r_1, r_2, \ldots)$ of non-negative integers with only finitely many non-zero entries, denote by $P^R \in \overline{\mathcal{A}}_{\mathbb{C}}$ the dual element to $\xi_1^{r_1}\xi_2^{r_2}\cdots$. The *length* of a sequence R is the non-negative number

$$l(R) = r_1 + r_2 + \cdots .$$

Denote by Δ_j the sequence of length 1 containing a 1 in position j, and thus we recover $P^{\Delta_j} = P_j$. Given two sequences R and R', denote by

$$P^{R-R'} = \begin{cases} \text{dual to } \xi_1^{r_1-r_1'}\xi_2^{r_2-r_2'}\cdots & \text{if } r_j \geq r_j' \text{ for all } j \\ 0 & \text{if not.} \end{cases}$$

A sequence R is called *even* if every r_i is even. Given a sequence R, denote by P^{2R} the dual to $\xi_1^{2r_1}\xi_2^{2r_2}\cdots$, and thus $P^{2\Delta_j}$ is dual to ξ_j^2.

Recall the ungraded injective map of Hopf algebras $\mathcal{A}_{\text{cl}} \hookrightarrow \overline{\mathcal{A}}_{\mathbb{C}}$ from equation (4.1). By using the same P^R notation in the classical setting (and so in $\mathcal{A}_{\text{cl}}$ we have $P_j = Q_{j-1}$), this map sends P^R to P^R and $P_j = Q_{j-1}$ to P_j. Moreover, the classical formula $c(Q_j) = Q_j \in \mathcal{A}_{\text{cl}}$ implies that motivically $c(P_j) = P_j \in \overline{\mathcal{A}}_{\mathbb{C}}$.

We will construct wBP by assembling $k(w_0)$'s, and since its cohomology is given by $\overline{H}^{*,*}(k(w_0)) \cong \overline{\mathcal{A}}_{\mathbb{C}}//E(P_1)$, we need to derive some formulas in the Hopf algebra quotient $\overline{\mathcal{A}}_{\mathbb{C}}//E(P_1)$.

Lemma 4.30. *In $\overline{\mathcal{A}}_{\mathbb{C}}//E(P_1)$, the following relations hold*

(1) $P_{j+1} = P_1 \cdot c(P^{2\Delta_j})$, for any $j \geq 2$.

(2) $P_1 \cdot c(P^{2R}) = \sum_{j\geq 1} c(P^{2R-2\Delta_j}) \cdot P_1 \cdot c(P^{2\Delta_j})$, for any sequence R.

Proof. In [10, Section 2], it is shown that the following formula

$$P^{2R}Q_0 + Q_0P^{2R} = \sum_{j\geq 1} Q_jP^{2R-2\Delta_j} \in \mathcal{A}_{\text{cl}}$$

holds in the classical Steenrod algebra for any sequence R. We warn the reader that there is a switch in notation between this formula and [10, Formula 2.5], as Brown and Peterson adopt a different notation in the case $p = 2$, where they let P^R be the dual of $\xi_1^{2r_1}\xi_2^{2r_2}\cdots$. Through the map $\mathcal{A}_{\text{cl}} \hookrightarrow \overline{\mathcal{A}}_{\mathbb{C}}$, this relation gives the motivic formula

$$P_1 P^{2R} + P^{2R} P_1 = \sum_{j\geq 1} P_{j+1} P^{2R-2\Delta_j}. \tag{4.10}$$

When $R = \Delta_j$, this formula becomes

$$P_1 P^{2\Delta_j} + P^{2\Delta_j} P_1 = P_{j+1}. \tag{4.11}$$

Applying the anti-morphism $c(-)$ and considering it in the quotient $\overline{\mathcal{A}}_{\mathbb{C}}//E(P_1) = \overline{\mathcal{A}}_{\mathbb{C}} \otimes_{E(P_1)} \mathbb{F}_2$ gives the desired first formula $P_{j+1} = P_1 c(P^{2\Delta_j})$. By plugging equation (4.11) in (4.10) we get

$$P_1 P^{2R} + P^{2R} P_1 = \sum_{j\geq 1} P_1 P^{2\Delta_j} P^{2R-2\Delta_j} + P^{2\Delta_j} P_1 P^{2R-2\Delta_j}.$$

Applying the anti-morphism $c(-)$ and considering it in the quotient $\overline{\mathcal{A}}_{\mathbb{C}}//E(P_1) = \overline{\mathcal{A}}_{\mathbb{C}} \otimes_{E(P_1)} \mathbb{F}_2$ gives the desired second formula

$$P_1 \cdot c(P^{2R}) = \sum_{j\geq 1} c(P^{2R-2\Delta_j}) \cdot P_1 \cdot c(P^{2\Delta_j}).$$

$\square$

Remark 4.31. The formulas of Lemma 4.30 live in the quotient $\overline{\mathcal{A}}_{\mathbb{C}}//E(P_1) \cong \overline{\mathcal{A}}_{\mathbb{C}} \otimes_{E(P_1)} \mathbb{F}_2$, where the action of P_1 on the *right* is reduced to zero. The first formula (4.11) is symmetric, and the conjugation morphism $c(-)$ is unnecessary for this formula alone. However, the point of applying the conjugation $c(-)$ is because the element P_{j+1} is multiplied on the left in equation (4.10). Without applying the anit-morphism $c(-)$ there would be no simplification after plugging-in (4.10) in (4.11) and the formulas would not be as nice. In fact, as we will see in Proposition 4.32, topologically realizing the differential of the chain complex (4.16) would not be possible without applying $c(-)$.

4.3.2 The construction of wBP

We will construct wBP via a certain tower, in the category of $C\tau$-modules. Recall that we are trying to construct a motivic ring spectrum wBP whose homotopy groups are given by the polynomial ring

$$\pi_{*,*}(wBP) \cong \mathbb{F}_2[w_0, w_1, \ldots],$$

where w_n is an element detected by the cohomology operation P_n. Unlike the previous section, the Postnikov tower approach is not tractable for wBP as it is hard to isolate the monomials $w_0^{n_0} w_1^{n_1} \cdots$ of a given bidegree, and thus hard to describe the layers.

Observe however that as in the case of $k(w_n)$, it suffices to construct a motivic spectrum with cohomology given by the Hopf algebra quotient $\overline{\mathcal{A}}_{\mathbb{C}}//E(P_1, P_2, \ldots)$. In fact, by a change of rings theorem and a careful analysis of degrees, the $C\tau$-linear $\overline{H}$-based Adams spectral sequence computing the homotopy of such a motivic spectrum collapses at $E_2 \cong \mathbb{F}_2[w_0, w_1, \ldots]$.

We will construct such a motivic spectrum by following an idea of Toda from [57, Lemma 3.1], where Toda constructs classical spectra with given cohomology by attaching copies of $H\mathbb{F}_2$ together. We also need to adapt and incorporate a trick which was used in [10] to construct the classical Brown-Peterson spectrum BP. This trick is to construct wBP by attaching together wedges of $k(w_0)$'s, instead of wedges of $\overline{H}$'s. This has the effect of reducing the number of wedge summands in the layers, and also of reducing the complexity of some computations, since these are included in the construction of $k(w_0)$ that was already done in Theorem 4.26.

More precisely, we will construct an inverse tower of motivic spectra, whose associated graded will be a topological realization of a resolution of $\overline{\mathcal{A}}_{\mathbb{C}}//E(P_1, P_2, \ldots)$. The following proposition is the first step in doing so, by constructing this associated graded, i.e., the layers of the desired tower.

Proposition 4.32. *There exists a complex of motivic spectra* $kV_0 \xrightarrow{\delta_0} kV_1 \xrightarrow{\delta_1} \cdots$*, where*

(1) the composite of two consecutive maps is nullhomotopic,

(2) each X_i is a locally finite wedge[9] *of suspensions of $k(w_0)$,*

(3) the $\overline{H}$-cohomology of this cochain complex is an $\overline{\mathcal{A}}_{\mathbb{C}}//E(P_1)$-free resolution of $\overline{\mathcal{A}}_{\mathbb{C}}//E(P_1, P_2, \ldots)$.

Proof. For every $j \geq 2$, consider the periodic bigraded $E(P_j)$-free resolution of $\mathbb{Z}/2$ given by

$$0 \longleftarrow \mathbb{Z}/2 \longleftarrow E(P_j) \xleftarrow{P_j} \Sigma^{|P_j|}E(P_j) \xleftarrow{P_j} \Sigma^{2|P_j|}E(P_j) \longleftarrow \cdots . \tag{4.12}$$

For simplicity, we will denote the exterior subalgebra generated by $P_2, P_3, \ldots$ by

$$E \coloneqq E(P_1, P_2, \ldots)//E(P_1) \cong E(P_2, P_3, \ldots).$$

By tensoring together these resolutions[10] for every $j \geq 2$, we get a bigraded E-free resolution

$$0 \longleftarrow \mathbb{Z}/2 \longleftarrow E \otimes V_0 \xleftarrow{d_0} E \otimes V_1 \xleftarrow{d_1} E \otimes V_2 \longleftarrow \cdots , \tag{4.13}$$

for some bigraded $\mathbb{F}_2$-vector space V_i. A preferred $\mathbb{F}_2$-basis of V_i is given by the set of sequences

$$\{e_R \mid R = (r_2, r_3, \ldots) \text{ satisfies } l(R) = r_2 + r_3 + \cdots = i\} \tag{4.14}$$

of length i. It follows that the bigrading on V_i is given by

$$|e_R| = \sum_{j\geq 2} r_j \cdot |P_j| = \sum_{j\geq 2} r_j \cdot |\xi_j| = \sum_{j \geq 2} r_j \cdot (2^{j+1} - 2, 2^j - 1). \tag{4.15}$$

In the notation $E \otimes V_i$, the E-linear differential is given on this basis by

$$d_i(1 \otimes e_R) = \sum_{j\geq 2} P_j \otimes (e_{R-\Delta_j}).$$

We can now tensor up the resolution of equation (4.13) via $\overline{\mathcal{A}}_{\mathbb{C}}//E(P_1) \otimes_E -$ to obtain the

---

[9] i.e., a possibly infinite wedge $\coprod_{\alpha\in A} \Sigma^{r_\alpha} X_\alpha$ with a finite number of wedge summands X_α's in any given (bi)-degree $\mathbb{r}$.

[10] where the term $\mathbb{Z}/2$ is not part of the resolution and $E(P_j)$ is in homological degree 0.

algebraic $\overline{\mathcal{A}}_{\mathbb{C}}//E(P_1)$-free resolution

$$0 \longleftarrow \overline{\mathcal{A}}_{\mathbb{C}}//E(P_1, P_2, \ldots) \longleftarrow \overline{\mathcal{A}}_{\mathbb{C}}//E(P_1) \otimes V_0 \xleftarrow{d_0} \overline{\mathcal{A}}_{\mathbb{C}}//E(P_1) \otimes V_1 \xleftarrow{d_1} \cdots \tag{4.16}$$

of $\overline{\mathcal{A}}_{\mathbb{C}}//E(P_1, P_2, \ldots)$. The goal is to now realize this resolution topologically. Since the V_i are finite dimensional and the terms in (4.16) are free $\overline{\mathcal{A}}_{\mathbb{C}}//E(P_1)$-modules, they are realized by locally finite wedges of suspensions of $k(w_0)$ indexed over the same basis. For simplicity, denote by kV_i the bigraded wedge of suspensions of $k(w_0)$ indexed by the chosen basis of V_i given by equation (4.14), which thus has the prescribed cohomology $\overline{H}^{*,*}(kV_i) \cong \overline{\mathcal{A}}_{\mathbb{C}}//E(P_1)\otimes V_i$. Similarly, denote by $\overline{H}V_i$ the bigraded wedge of suspensions of $\overline{H}$ indexed by the same basis of V_i. To realize the differentials, observe that by Lemma 4.30, the differential d_i in (4.16) can be simplified to the formula

$$d_i(1 \otimes e_R) = \sum_{j \geq 2} P_j \otimes (e_{R-\Delta_j}) = P_1 \cdot \sum_{j \geq 2} c(P^{2\Delta_{j-1}}) \otimes (e_{R-\Delta_j}). \tag{4.17}$$

The differential d_i can thus be realized by the composite

$$\delta_i \colon kV_i \xrightarrow{1} \overline{H}V_i \xrightarrow{\coprod c(P^{2\Delta_j})} \Sigma^{-(2,1)}\overline{H}V_{i+1} \xrightarrow{\beta_0} kV_{i+1},$$

where β_0 is the Bockstein from Proposition 4.27. The middle map is a locally finite matrix with entries in $\overline{\mathcal{A}}_{\mathbb{C}}$, where for a given sequence R of length i, it is assembled from the maps

$$c(P^{2\Delta_j}) \colon \overline{H}\{e_R\} \longrightarrow \overline{H}\{e_{R+\Delta_j}\} \hookrightarrow \overline{H}V_{i+1}.$$

The composite δ_i realizes the differential d_i since for a given sequence R of length $i+1$ we have

$$\begin{tikzcd}
kV_i \arrow[r, "1"] & \overline{H}V_i \arrow[r, "\coprod c(P^{2\Delta_j})"] & \Sigma^{-(2,1)}\overline{H}V_{i+1} \arrow[r, "\beta_0"] \arrow[dr, dashed, "P_1"] & kV_{i+1} \arrow[d, "1"] \\
 & & & \overline{H}\{e_R\},
\end{tikzcd}$$

which recovers exactly formula (4.17) since $1 \circ \beta_0 = P_1$ by Proposition 4.27. We have thus defined a sequence of motivic spectra

$$kV_0 \xrightarrow{\delta_0} kV_1 \xrightarrow{\delta_1} kV_2 \xrightarrow{\delta_2} \cdots,$$

in which each term is a locally finite wedge of suspensions of $k(w_0)$'s, and which produces an $\overline{\mathcal{A}}_{\mathbb{C}}//E(P_1)$-free resolution of $\overline{\mathcal{A}}_{\mathbb{C}}//E(P_1, P_2, \ldots)$ after applying $\overline{H}$-cohomology.

It remains to show that the composites $\delta_{i+1} \circ \delta_i$ are nullhomotopic. This is accomplished by the following commutative diagram

$$\begin{tikzcd}
kV_i \arrow[r, "1"] \arrow[rrr, dashed, bend left, "\delta_i"] \arrow[rrrrr, dashed, bend right, "P_1 \circ \coprod c(P^{2R}) \circ 1"'] & \overline{H}V_i \arrow[r, "\coprod c(P^{2\Delta_j})"] & \Sigma^{-(2,1)}\overline{H}V_{i+1} \arrow[r, "\beta_0"] \arrow[rr, dashed, bend right, "P_1"'] & kV_{i+1} \arrow[r, "1"] \arrow[rrr, dashed, bend left, "\delta_{i+1}"] & \overline{H}V_{i+1} \arrow[r, "\coprod c(P^{2\Delta_j})"] & \overline{H}V_{i+2} \arrow[r, "\beta_0"] & kV_{i+2},
\end{tikzcd}$$

where the wedges $\coprod c(P^{2R})$ are taken over sequences R of length 2, and the composite $kV_i \longrightarrow \overline{H}V_{i+2}$ is identified from a sum over all such R's of the equation

$$P_1 c(P^{2R}) = \sum_{j \geq 1} c(P^{2R-2\Delta_j}) P_1 c(P^{2\Delta_j}) \in \overline{\mathcal{A}}_{\mathbb{C}}//E(P_1)$$

from lemma 4.30. The total composite

$$\beta_0 \circ P_1 \circ \coprod c(P^{2R}) \circ 1 = 0$$

is zero since $\beta_0 \circ P_1 = \beta_0 \circ 1 \circ \beta_0$, and $\beta_0 \circ 1 = 0$ since they are consecutive maps in the cofiber sequence of Proposition 4.27. ☐

The next step is to construct an inverse tower of motivic spectra, whose layers and induced d_1-differential are exactly the cochain complex of motivic spectra

$$\begin{tikzcd} kV_0 \arrow[r, "\delta_0"] & kV_1 \arrow[r, "\delta_1"] & kV_2 \arrow[r, "\delta_2"] & \cdots \end{tikzcd}$$

from Proposition 4.32. The idea is that once we construct this tower, we can compute the cohomology of its inverse limit by the spectral sequence emerging from applying the cohomological functor $\overline{H}^{*,*}(-)$. We will define wBP to be the inverse limit of this tower. The E_1-page of the associated spectral sequence is the cohomology of the layers, i.e., the cohomology of the above cochain complex. We just showed in Proposition 4.32 that this cohomology forms a resolution of $\overline{\mathcal{A}}_{\mathbb{C}}//E(P_1, P_2, \ldots)$, and thus the spectral sequence collapses at $E_2 = E_\infty$ with output $\overline{\mathcal{A}}_{\mathbb{C}}//E(P_1, P_2, \ldots)$.

Construction 4.33 (Construction of wBP via Toda's realization method)**.** Recall that the goal is now to construct a tower of motivic spectra

$$\begin{tikzcd} X_{-1} = * & X_0 \arrow[l] & X_1 \arrow[l] & \cdots \arrow[l] \end{tikzcd} ,$$

with layers and induced d_1-differential given by the cochain complex

$$\begin{tikzcd} kV_0 \arrow[r, "\delta_0"] & kV_1 \arrow[r, "\delta_1"] & kV_2 \arrow[r, "\delta_2"] & \cdots \end{tikzcd} .$$

The beginning of the tower is given by

$$\begin{tikzcd}
 & kV_0 \arrow[d, "i_{-1} = \mathrm{id}"] \arrow[dd, dashed, bend right, "\delta_0"'] \\
X_{-1} = * \arrow[d, "k_{-1}"] & X_0 = kV_0 \arrow[l, "p_{-1}"'] \arrow[d, dotted, "\exists?\ k_0"] \\
\Sigma^{1,0} kV_0 & kV_1,
\end{tikzcd}$$

where the suspension by $\Sigma^{1,0}$ of the first layer is a small adjustment to get the correct output. Evidently, since $i_{-1} = \mathrm{id}$, there is a unique filler k_0 up to homotopy. We can thus set $k_0 = \delta_0$, denote its fiber by X_1, and ask if the following filler k_1 exists in the diagram

$$\begin{tikzcd}
 & kV_0 \arrow[d, "i_{-1} = \mathrm{id}"] \arrow[dd, dashed, bend right, "\delta_0"'] & \Sigma^{-1,0} kV_1 \arrow[d, "i_0"] \arrow[dd, dashed, bend right, "\delta_1"'] \\
X_{-1} = * \arrow[d, "k_{-1}"] & X_0 = kV_0 \arrow[l, "p_{-1}"'] \arrow[d, "k_0 = \delta_0"] & X_1 \arrow[l, "p_0"'] \arrow[d, dotted, "\exists?\ k_1"] \\
\Sigma^{1,0} kV_0 & kV_1 & \Sigma^{-1,0} kV_2.
\end{tikzcd}$$

By taking the fiber of i_0, this problem becomes an extension problem in the cofiber sequence

$$\begin{tikzcd}
\Sigma^{-1,0} X_0 \arrow[r, "k_0 = \delta_0"] & \Sigma^{-1,0} kV_1 \arrow[r, "i_0"] \arrow[dr, "\delta_1"'] & X_1 \arrow[r, "p_0"] \arrow[d, dotted, "\exists?\ k_1"] & X_0 \\
 & & \Sigma^{-1,0} kV_2, &
\end{tikzcd}$$

where a filler k_1 exists since the composite $\delta_1 \delta_0$ is nullhomotopic by Propositon 4.32. The choices of such extensions are parametrized by the quotient

$$[X_0, \Sigma^{-1,0} kV_2] \big/ \delta_0\text{-divisible elements.}$$

Recall that both $X_0 = kV_0$ and kV_2 are locally finite wedges of suspension of $k(w_0)$'s, and we can read from equation (4.15) that the bidegrees of all suspensions are in Chow degree 0. The set of homotopy classes of maps $[X_0, \Sigma^{-1,0}kV_2]$ is thus built out of self-maps of $k(w_0)$ in Chow degree -1. This is zero by Lemma 4.24, and thus there exists a unique filler k_1.

Getting this far was the base case for the inductive process. Suppose now that the tower

$$\begin{tikzcd}
 & & \Sigma^{-(n-1),0}kV_{n-1} \arrow[d, "i_{n-2}"] \arrow[dd, dashed, bend right, "\delta_{n-1}"'] & \Sigma^{-n,0}kV_n \arrow[d, "i_{n-1}"] \arrow[dd, dashed, bend right, "\delta_n"'] \\
\cdots & X_{n-2} \arrow[l] \arrow[d, "k_{n-2}"] & X_{n-1} \arrow[l, "p_{n-2}"'] \arrow[d, "k_{n-1}"] & X_n \arrow[l, "p_{n-1}"'] \arrow[d, dotted, "\exists ?\ k_n"] \\
 & \Sigma^{-(n-2),0}kV_{n-1} & \Sigma^{-(n-1),0}kV_n & \Sigma^{-n,0}kV_{n+1},
\end{tikzcd}$$

has been constructed for some $n \geq 2$. As above, we can desuspend one step, and rewrite this extension problem as

$$\begin{tikzcd}
\Sigma^{-1,0}X_{n-1} \arrow[r, "k_{n-1}"] & \Sigma^{-n,0}kV_n \arrow[r, "i_{n-1}"] \arrow[dr, "\delta_n"'] & X_n \arrow[r, "p_{n-1}"] \arrow[d, dotted, "\exists ?\ k_n"] & X_{n-1} \\
 & & \Sigma^{-n,0}kV_{n+1}. &
\end{tikzcd}$$

We will now show that $\delta_n k_{n-1} = 0$ and thus that a filler k_n exists, and that $[X_{n-1}, \Sigma^{-n,0}kV_{n+1}] = 0$ and thus that such a filler is unique.

Let's first deal with the uniqueness part, as the statement we show will come up in the existence part as well. A slightly more general result is true, namely that the $k(w_0)$-cohomology of any X_m vanishes in Chow degrees strictly smaller than $-m$. This is easy to show by an induction on m (for all m less than n, so that X_m is already constructed), and is completely analogous to Lemma 4.18. The base case is $X_0 = kV_0$, whose $k(w_0)$-cohomology vanishes in negative Chow degrees by Lemma 4.24. The induction is done by inspecting the long exact sequence in cohomology of the cofiber sequence $\Sigma^{-m,0}kV_m \longrightarrow X_m \longrightarrow X_{m-1}$. We refer to Lemma 4.18 for more details. We can now use this statement to show that $[X_{n-1}, \Sigma^{-n,0}kV_{n+1}] = 0$. In fact, this set of maps is made out of the part in Chow degree $-n$ of the $k(w_0)$-cohomology of X_{n-1}, which vanishes since $-n < -(n-1)$.

For the existence part, consider the $\Sigma^{-1,0}$-desuspension of the cofiber sequence

$$\Sigma^{-(n-1),0}kV_{n-1} \xrightarrow{i_{n-2}} X_{n-1} \xrightarrow{p_{n-2}} X_{n-2}.$$

By applying the cohomological functor $[-, \Sigma^{-n,0}kV_{n+1}]$, we get a long exact sequence

$$\cdots \longleftarrow \left[\Sigma^{-n,0}kV_{n-1}, \Sigma^{-n,0}kV_{n+1}\right] \xleftarrow{i_{n-2}^*} \left[\Sigma^{-1,0}X_{n-1}, \Sigma^{-n,0}kV_{n+1}\right] \xleftarrow{p_{n-2}^*} \left[\Sigma^{-1,0}X_{n-2}, \Sigma^{-n,0}kV_{n+1}\right] \longleftarrow \cdots,$$

which contains the composite $\delta_n k_{n-1}$ in its middle term. The right term $\left[X_{n-2}, \Sigma^{-(n-1),0}kV_{n+1}\right]$ vanishes since it is concentrated in the Chow degree $-(n-1)$ part of the $k(w_0)$-cohomology of X_{n-2}. This simplifies the long exact sequence to

$$\left[\Sigma^{-n,0}kV_{n-1}, \Sigma^{-n,0}kV_{n+1}\right] \xleftarrow{i_{n-2}^*} \left[\Sigma^{-1,0}X_{n-1}, \Sigma^{-n,0}kV_{n+1}\right] \longleftarrow 0.$$

The image of $\delta_n k_{n-1}$ under the precomposition map i_{n-2}^* is

$$i_{n-2}^*(\delta_n k_{n-1}) = \delta_n k_{n-1} i_{n-2} = \delta_n \delta_{n-1},$$

which is zero by Proposition 4.32. Since i_{n-2}^* is injective, it follows that $\delta_n k_{n-1} = 0$ and thus that k_n exists. □

Definition 4.34. Define a motivic spectrum $wBP \in {}_{C\tau}\mathbf{Cell}$ as the inverse limit of the tower

$$wBP \coloneqq \operatorname{holim}\left(* = X_{-1} \longleftarrow X_0 \longleftarrow X_1 \longleftarrow \cdots\right) \tag{4.18}$$

from Construction 4.33.

It remains to show that this spectrum has the correct cohomology. The situation is very similar to the case of $k(w_n)$ in Section 4.2.

Proposition 4.35. *The cohomology of* wBP *is given as an* $\overline{\mathcal{A}}_{\mathbb{C}}$*-module by the Hopf algebra quotient*

$$\overline{H}^{*,*}(wBP) \cong \overline{\mathcal{A}}_{\mathbb{C}}//E(P_1, P_2, \ldots).$$

Proof. This is analogous to Proposition 4.21, which we refer to for more details. Applying the contravariant functor $\overline{H}^{*,*}$ to the tower defining wBP gives a spectral sequence computing the algebraic colimit $\operatorname{colim} \overline{H}^{*,*}(X_i)$. The E_1-page is given by the cohomology of the layers kV_i, which form a resolution of $\overline{\mathcal{A}}_{\mathbb{C}}//E(P_1, P_2, \ldots)$ by Proposition 4.32. The E_2-page is thus given by $\overline{\mathcal{A}}_{\mathbb{C}}//E(P_1, P_2, \ldots)$ concentrated in homological degree $s = 0$, and the spectral sequence collapses with no possible hidden extensions. Observe that the layers $\Sigma^{-(i-1),0}kV_i$ are more and more connected. In fact, the element of lowest bidegree corresponds to the sequence $R = (r_2 = i, 0, 0, \ldots)$, and so $\Sigma^{-(i-1),0}kV_i$ has no cohomology in degrees lower than $(5i + 1, 3i)$. By using this bound, the exact same proof as in Proposition 4.21 shows that

$$\overline{H}^{*,*}(wBP) \xleftarrow{\cong} \operatorname{colim} \overline{H}^{*,*}(X_i) \cong \overline{\mathcal{A}}_{\mathbb{C}}//E(P_1, P_2, \ldots).$$

$\square$

Corollary 4.36. *For any* $m \geq 1$ *we have an isomorphism of* $\overline{\mathcal{A}}_{\mathbb{C}}$*-modules*

$$\overline{H}^{*,*}\big(wBP^{\wedge m}\big) \cong \big(\overline{\mathcal{A}}_{\mathbb{C}}//E(P_1, P_2, \ldots)\big)^{\otimes m},$$

where the right hand side has the iterated diagonal $\overline{\mathcal{A}}_{\mathbb{C}}$*-module structure. In particular, for any* m *the cohomology* $\overline{H}^{*,*}\big(wBP^{\wedge m}\big)$ *is concentrated in positive Chow degrees.*

Proof. This proof is similar to the proof of Corollary 4.22. $\square$

4.3.3 The E_∞ ring structure on wBP, and $wBP\langle n\rangle$

Having constructed a motivic spectrum $wBP \in {}_{C\tau}\mathbf{Cell}$ with correct cohomology, we will now follow the same methodology as we did for $k(w_n)$ in Section 4.2.2. We first construct a ring map $\mu\colon wBP^{\wedge 2} \longrightarrow wBP$ by understanding some parts of various groups of homotopy classes of maps $[wBP^{\wedge m}, wBP]_{*,*}$. We then show that μ turns wBP into a unital, associative and commutative monoid in the homotopy category. Finally, the vanishing of $[wBP^{\wedge m}, wBP]_{*,*}$ in some particular degrees feeds Robinson's obstruction theory [51] which rigidifies μ to an E_∞ ring structure on wBP. The $C\tau$-linear $\overline{H}$-based motivic Adams spectral sequence is now multiplicative and collapses at E_2, showing that wBP has polynomial homotopy in the periodicity elements $w_0, w_1, \ldots$.

Most of these steps are very similar to the case of $k(w_n)$. We will only sketch the arguments in these cases and refer to the appropriate proof in Section 4.2 for more details. Since wBP is built out of a tower whose layers are wedges of suspensions of $k(w_0)$'s, there is a spectral sequence computing $[wBP^{\wedge m}, wBP]_{*,*}$ whose layers are made out of $[wBP^{\wedge m}, k(w_0)]_{*,*}$. We first say something about these layers.

Lemma 4.37. *For any $m \geq 1$ we have*

$$\left[wBP^{\wedge m}, \Sigma^{t,w} k(w_n)\right] \cong \begin{cases} \mathbb{F}_2 & \textit{if } (t,w) = (0,0) \\ 0 & \textit{if } (t,w) \textit{ is in negative Chow degree, i.e., } t-2w<0. \end{cases}$$

Proof. The exact same proof as in Lemma 4.24 applies by changing $k(w_0)$ to wBP, since the cohomology of the smash powers $wBP^{\wedge m}$ is also concentrated in non-negative Chow degrees, with a copy of $\mathbb{Z}/2$ in degree $(0,0)$. □

Lemma 4.38. *For any $m \geq 1$ we have*

$$\left[wBP^{\wedge m}, \Sigma^{t,w} wBP\right] \cong \begin{cases} \mathbb{F}_2 & \textit{if } (t,w) = (0,0) \\ 0 & \textit{if } (t,w) \textit{ is in negative Chow degree, i.e., } t-2w<0. \end{cases}$$

Moreover, the non-trivial map $wBP^{\wedge m} \longrightarrow wBP$ *sends* 1 *to* $1^{\otimes m}$ *in cohomology.*

Proof. As in Lemma 4.24 and Lemma 4.37, this is another Atiyah-Hirzebruch spectral sequence, by applying the functor $[wBP^{\wedge m}, -]$ to the inverse limit tower defining wBP. With the indexing

$$E_1^{s,t,w} = \left[wBP^{\wedge m}, \Sigma^{t,w}\Sigma^{-s,0}kV_s\right] \cong \left[wBP^{\wedge m}, \Sigma^{t-s,w}kV_s\right],$$

this is a first quadrant spectral sequence in the (s,t) plane which converges to

$$E_\infty^{s,t,w} \cong \left[wBP^{\wedge m}, \Sigma^{t,w}wBP\right].$$

Recall that each kV_s is a locally finite wedge of suspensions of $k(w_0)$, where the suspensions are in Chow degree 0. By Lemma 4.37, the $k(w_0)$-cohomology of wBP is concentrated in non-negative Chow degree, with only a $\mathbb{Z}/2$ in degree $(0,0)$. The proof of Lemma 4.24 applies by changing $\overline{H}$ to $k(w_0)$, and $k(w_0)$ to wBP. □

In the case $m = 2$, call a representative of the non-trivial class of maps by $\mu\colon wBP \wedge wBP \longrightarrow wBP$. We are now ready to show that this map can be extended to an E_∞ ring structure on wBP.

Theorem 4.39. *The motivic spectrum* $wBP \in {}_{C\tau}\textbf{\textit{Cell}}$ *admits an essentially unique* E_∞ *ring structure and satisfies*

- $\overline{H}^{*,*}(wBP) \cong \overline{\mathcal{A}}_{\mathbb{C}}//E(P_1, P_2, \ldots)$ *as an* $\overline{\mathcal{A}}_{\mathbb{C}}$*-module,*
- $\pi_{*,*}(wBP) \cong \mathbb{F}_2[w_0, w_1, \ldots]$ *as a ring.*

Proof. First use the proof of Proposition 4.25 by changing $k(w_0)$ to wBP to show that μ turns wBP into a unital, associative and commutative monoid in the homotopy category. Similarly, by using the vanishing results of Lemma 4.38, the proof of Theorem 4.26 shows that μ can be uniquely extended to an E_∞ ring structure. Finally, the $C\tau$-linear $\overline{H}$-based motivic Adams spectral sequence

$$E_2 = \operatorname{Ext}_{\overline{\mathcal{A}}_{\mathbb{C}}}\left(\overline{H}^{*,*}(wBP), \mathbb{F}_2\right) \Longrightarrow \pi_{*,*}(wBP)$$

is multiplicative since wBP is a ring spectrum. Since $\overline{H}^{*,*}(wBP) \cong \overline{\mathcal{A}}_{\mathbb{C}}//E(P_1, P_2, \ldots)$ we can apply the usual change of rings to the E_2-page

$$E_2 \cong \mathrm{Ext}_{E(P_1,P_2,\ldots)}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2[w_0, w_1, \ldots].$$

The spectral sequence collapses now at E_2 with no possible hidden extensions. ☐

Corollary 4.40. *For any $n \geq 0$ there exists a motivic spectrum $wBP\langle n\rangle \in {}_{C\tau}\textbf{\textit{Cell}}$, which admits an essentially unique E_∞ ring structure and satisfies*

- $\overline{H}^{*,*}(wBP\langle n\rangle) \cong \overline{\mathcal{A}}_{\mathbb{C}}//E(P_1, P_2, \ldots, P_{n+1})$ *as an $\overline{\mathcal{A}}_{\mathbb{C}}$-module,*
- $\pi_{*,*}(wBP\langle n\rangle) \cong \mathbb{F}_2[w_0, w_1, \ldots, w_n]$ *as a ring.*

Proof. The whole proof is very similar to the case of wBP, so we will only indicate what needs to be changed. Tensoring together for $2 \leq j \leq n+1$ the resolutions of equation (4.12) gives a resolution

$$0 \longleftarrow \mathbb{Z}/2 \longleftarrow E \otimes V_0 \xleftarrow{d_0} E \otimes V_1 \xleftarrow{d_1} E \otimes V_2 \longleftarrow \cdots,$$

where $E = E(P_2, P_3, \ldots, P_{n+1})$ and where V_i is a finite dimensional bigraded $\mathbb{F}_2$-vector space with basis given by the set of sequences

$$\{R = (r_2, r_3, \ldots, r_{n+1}) \mid l(R) = r_2 + r_3 + \cdots + r_{n+1} = i\}.$$

Tensoring up to $\overline{\mathcal{A}}_{\mathbb{C}}//E(P_1) \otimes_E -$ gives the $\overline{\mathcal{A}}_{\mathbb{C}}//E(P_1)$-free resolution

$$0 \longleftarrow \overline{\mathcal{A}}_{\mathbb{C}}//E(P_2, P_3, \ldots, P_{n+1}) \longleftarrow \overline{\mathcal{A}}_{\mathbb{C}}//E(P_1) \otimes V_0 \xleftarrow{d_0} \overline{\mathcal{A}}_{\mathbb{C}}//E(P_1) \otimes V_1 \xleftarrow{d_1} \longleftarrow \cdots.$$

The differential is still given by

$$d_i(1 \otimes R) = \sum_{j\geq 2} P_j \otimes (R - \Delta_j) = P_1 \cdot \sum_{j \geq 2} c(P^{2\Delta_{j-1}}) \otimes (R - \Delta_j),$$

since the formula we use still lives in the quotient $\overline{\mathcal{A}}_{\mathbb{C}}//E(P_1)$. From here on, the rest of the proof is a copy of the proof for wBP, with the advantage that the terms kV_i are now finite wedges of suspensions of copies of $k(w_0)$'s. □

Remark 4.41. One can also construct the underlying spectrum of $wBP\langle n\rangle$ by taking the quotient

$$wBP \Big/ w_{n+1}, w_{n+2}, \ldots .$$

The fact that these quotients are E_∞ rings requires some extra work from this point of view though.

CHAPTER 5 AN ALGEBRAIC MODEL FOR $C\tau$-MODULES

DISCLAIMER: We warn the reader that this chapter is currently wrong, and MGL should be replaced by $BPGL$. The original version of this work was done with $BPGL$, because this is how the traditional Adams-Novikov spectral sequence is considered. The idea to switch to MGL was to bypass the issue that BP is potentially not E_∞. However, some lemmas true for $BPGL$ are here wrong for MGL. This is however not crucial, and in the end the main result is true as stated (because MGL and $BPGL$ should have equivalent categories of modules, as their associated stack of cooperations have the same cohomology).

In this chapter, we will show that some category ${}_{C\tau}\mathbf{Cell}^{\mathrm{comp}}$ of cellular $C\tau$-modules is equivalent the derived bounded category of its heart

$$\mathcal{D}^{\mathrm{b}}({}_{MU_*\widehat{MU}}\mathbf{Comod}) \xrightarrow{\cong} {}_{C\tau}\mathbf{Cell}^{\mathrm{comp}},$$

where $\widehat{MU}$ denotes the 2-completion of the complex cobordism spectrum MU. We will in fact show that

$$\mathcal{D}^{b}({}_{\overline{MGL}_{*,*}\overline{MGL}}\mathbf{Comod}^{\mathrm{ev}}) \xrightarrow{\cong} {}_{C\tau}\mathbf{Cell}^{\mathrm{comp}},$$

where $\overline{MGL}$ is a version of the motivic spectrum of algebraic cobordism. This result is hinted by the isomorphism

$$\pi_{*,*}(C\tau) \cong \mathrm{Ext}_{\overline{MGL}_{*,*}\overline{MGL}}(\overline{MGL}_{*,*}, \overline{MGL}_{*,*})$$

that is easily deduced from Proposition 3.1, which shows that the motivic 2-cell complex $C\tau$ is of algebraic nature. In the joint paper [17], we show an important application of this result, by identifying two spectral sequences: the motivic Adams spectral sequence for $C\tau$, and the algebraic Novikov-spectral sequence for Ext_{BP_*BP}. Another application would be that the previous Chapter 4 could be entirely rewritten in algebraic language, via the bridge of equation (5). Finally, work in progress of Barthel-Drew-Krause about further motivic periodicity is also relying on the equivalence of equation (5). We believe that the main theorem of this chapter (corresponding to equation (5)) has many more future applications.

We refer to the Introduction for more application of this equivalence of categories. The contents of this chapter will appear in joint work in preparation with Zhouli Xu and Guozhen Wang in [17]. This chapter represents the part of the collaboration that was contributed by the author of this thesis.

Organization

Here is the organization of this chapter.

Section 5.1. In this section we show a general result about t-structure on stable ∞-categories in two different versions, one using injective objects, and one using projective objects. Each version will be used exactly once in the remaining of the chapter, and provides a strategy for proving the main Theorem.

Section 5.2. This section contains an easier version of our main Theorem, where we replace $C\tau$-modules with $\overline{MGL}$-modules. The main result of this section is interesting on its own, but also necessary for proving our main Theorem.

Section 5.3. This section contains the main Theorem, i.e., the equivalence of categories of equation (5). Along the way, we set-up a very general motivic Adams-Novikov spectral sequence, which takes most of the section. The strategy of the proof of our main Theorem mimics the proof of the main Theorem of Section 5.2, even though it uses results from there.

5.1 A Theorem on t-structures

In this section, we state a result about t-structures that will be used twice in this chapter, once stated in its projective version in Proposition 5.1 and once it is injective version in Proposition 5.2. Both versions follow rather directly from [32, Proposition 1.3.3.7], which itself is inspired from a triangulated version in [5]. Before stating the Theorem, we will very briefly recall some terminology and results about t-structures, and refer to [32, Chapter 1] for more details.

Let $\mathcal{C}$ be a stable ∞-category. Denote by $h\mathcal{C}$ its homotopy category, and by $[-,-]$ the group of homotopy classes of maps in $\mathcal{C}$, i.e., the group of maps in $h\mathcal{C}$. Recall that a *t-structure* on $\mathcal{C}$ is the data of two full subcategories $\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0} \subset \mathcal{C}$ which are closed under isomorphisms, and which satisfy the following three axioms

(1) for $X \in \mathcal{C}_{\geq 0}$ and $Y \in \Sigma^{-1}\mathcal{C}_{\leq 0}$, we have $[X, Y] = 0$,

(2) there is an inclusion $\Sigma\mathcal{C}_{\geq 0} \subseteq \mathcal{C}_{\geq 0}$,

(3) for any $X \in \mathcal{C}$, there exists a fiber sequence

$$X_{>0} \longrightarrow X \longrightarrow X_{\leq 0},$$

with $X_{\leq 0} \in \mathcal{C}_{\leq 0}$ and $X_{>0} \in \Sigma\mathcal{C}_{\geq 0}$.

Denote by $\mathcal{C}_{\geq n}$ and $\mathcal{C}_{\leq n}$ the categories $\Sigma^n\mathcal{C}_{\geq 0}$ and $\Sigma^n\mathcal{C}_{\leq 0}$ respectively. For every $n \in \mathbb{Z}$, these subcategories sit in adjunctions

$$\begin{tikzcd} \mathcal{C}_{\geq n} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, shift left, "\tau_{\geq n}"] \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} \mathcal{C} \arrow[r, shift left, "\tau_{\leq n}"] \arrow[r, phantom, "\perp"] & \mathcal{C}_{\leq n}, \arrow[l, shift left] \end{tikzcd}$$

where $\tau_{\geq n}$ and $\tau_{\leq n}$ are called the n^{th}*-truncation functors*.

Denote the subcategories of *left-bounded* and *right-bounded* objects in $\mathcal{C}$ by

$$\mathcal{C}^{+} := \operatorname{hocolim}\Big(\mathcal{C}_{\leq 0} \hookrightarrow \mathcal{C}_{\leq 1} \hookrightarrow \dots\Big) \qquad \text{and} \qquad \mathcal{C}^{-} := \operatorname{hocolim}\Big(\mathcal{C}_{\geq 0} \hookrightarrow \mathcal{C}_{\geq -1} \hookrightarrow \dots\Big),$$

and let $\mathcal{C}^{\mathrm{b}} := \mathcal{C}^{+} \cap \mathcal{C}^{-}$ be the subcategory of *bounded objects*. We say that the t-structure is *left-bounded*, *right-bounded*, or *bounded*, if the inclusion of $\mathcal{C}^{+}$, $\mathcal{C}^{-}$ or $\mathcal{C}^{\mathrm{b}}$ respectively, in $\mathcal{C}$, is an equivalence.

At the other extreme, define the *left and right completions* of the t-structure by

$$\widehat{\mathcal{C}_l} := \operatorname{holim}\big(\cdots \xrightarrow{\tau_{\leq 1}} \mathcal{C}_{\leq 1} \xrightarrow{\tau_{\leq 0}} \mathcal{C}_{\leq 0}\big) \qquad \text{and} \qquad \widehat{\mathcal{C}_r} := \operatorname{holim}\big(\cdots \xrightarrow{\tau_{\geq -1}} \mathcal{C}_{\geq -1} \xrightarrow{\tau_{\geq 0}} \mathcal{C}_{\geq 0}\big).$$

We say that the t-structure is *left-complete* if the functor $\mathcal{C} \longrightarrow \widehat{\mathcal{C}_l}$ is an equivalence, and *right-complete* if $\mathcal{C} \longrightarrow \widehat{\mathcal{C}_r}$ is an equivalence. The left and right completions are again stable ∞-categories and inherit a t-structure from $\mathcal{C}$.

Finally, denote the intersection by $\mathcal{C}^{\heartsuit} = \mathcal{C}_{\geq 0} \cap \mathcal{C}_{\leq 0}$ the *heart* of the t-structure, whose homotopy category $h\mathcal{C}^{\heartsuit}$ is always an abelian category. The goal of this section is to determine under what hypotheses we can reconstruct the whole category $\mathcal{C}$ from its heart.

Let's thus start with an abelian category $\mathcal{A}$ with enough projective objects. There is an associated *right-bounded derived category* $\mathcal{D}^-(\mathcal{A})$ with objects classes of chain complexes M_* of projective objects in $\mathcal{A}$ such that $M_n \simeq 0$ for n small enough[11]. This is a stable ∞-category that admits a natural t-structure defined by

- $\mathcal{D}^-(\mathcal{A})_{\geq 0}$ is the full ∞-subcategory spanned by those complexes $\{M_n\}_{n\in\mathbb{Z}}$ with vanishing homology in negative degrees $n < 0$,
- $\mathcal{D}^-(\mathcal{A})_{\leq 0}$ is the full ∞-subcategory spanned by those complexes $\{M_n\}_{n\in\mathbb{Z}}$ with vanishing homology in positive degrees $n > 0$.

Moreover, this t-structure is left-complete and right-bounded.

Given a stable ∞-category $\mathcal{C}$ with t-structure, one can ask what are necessary conditions for $\mathcal{C}$ to be equivalent to some versions of the derived category of its heart $h\mathcal{C}^{\heartsuit}$. The derived category is universal in the sense that there will always be a functor from it to $\mathcal{C}$. Moreover, the following result states that under mild conditions on the t-structure on $\mathcal{C}$, this functor is fullyfaithful with explicit image. The following proposition stated in its projective version will be used below in Section 5.2.

Proposition 5.1. *Let $\mathcal{C}$ be a stable ∞-category with a t-structure. Suppose that*

(1) the abelian category $h\mathcal{C}^{\heartsuit}$ has enough projectives,

(2) for any objects $X, Y \in h\mathcal{C}^{\heartsuit}$ with X projective we have $[X, \Sigma^i Y] = 0$ for all $i > 0$.

---

[11] The condition that $\mathcal{A}$ has enough projectives is necessary in order to define the derived category.

Then there exists an essentially unique t-exact functor F making the diagram

$$\begin{tikzcd}
\mathcal{C}^{\heartsuit} \arrow[r, hook] \arrow[d] & \widehat{\mathcal{C}_l} \\
\mathcal{D}^{-}(h\mathcal{C}^{\heartsuit}), \arrow[ur, dashed, "\exists!\ F"'] &
\end{tikzcd}$$

commute. Moreover, its essential image is $(\widehat{\mathcal{C}_l})^{-}$. If in addition the t-structure on $\mathcal{C}$ is right-bounded, then F induces a t-exact equivalence of ∞-categories

$$F\colon \mathcal{D}^{b}(h\mathcal{C}^{\heartsuit}) \xrightarrow{\cong} \mathcal{C}^{b}.$$

Proof. The existence, essential uniqueness, and essential image of F are provided by [32, Proposition 1.3.3.7] applied to the left-completion $\widehat{\mathcal{C}_l}$. This gives a t-exact equivalence

$$F\colon \mathcal{D}^{-}(h\mathcal{C}^{\heartsuit}) \xrightarrow{\cong} \left(\widehat{\mathcal{C}_l}\right)^{-}.$$

By taking left-bounded objects on both sides one obtains another equivalence

$$F\colon \mathcal{D}^{\mathrm{b}}(h\mathcal{C}^{\heartsuit}) \xrightarrow{\cong} \left(\left(\widehat{\mathcal{C}_l}\right)^{-}\right)^{+}.$$

The result follows by the zig-zag of equivalences

$$\left(\left(\widehat{\mathcal{C}_l}\right)^{-}\right)^{+} \xrightarrow{\cong} \left(\widehat{\mathcal{C}_l}\right)^{+} \xleftarrow{\cong} \mathcal{C}^{+} \xrightarrow{\cong} \mathcal{C}^{\mathrm{b}},$$

where the first equivalence is true since the t-structure is right-bounded, the second equivalence is from [32, Remark 1.2.1.18] and the last one again uses that the t-structure is right-bounded. ☐

In Section 5.3, we will be dealing with comodule categories. These categories tend to not always have enough projective objectives, but rather enough injectives [47, Appendix 1]. We thus need a version of this result when the heart of the t-structure only has enough injectives.

Let thus $\mathcal{A}$ be an abelian category with enough injective objects. In this case, there is a *left-bounded derived category* $\mathcal{D}^+(\mathcal{A})$ with objects classes of chain complexes M_* of injective objects in $\mathcal{A}$ such that $M_n \simeq 0$ for n large enough. This is again a stable ∞-category that admits a natural t-structure defined by

- $\mathcal{D}^+(\mathcal{A})_{\geq 0}$ is the full ∞-subcategory spanned by those complexes $\{M_n\}_{n\in\mathbb{Z}}$ with vanishing homology in negative degrees $n < 0$,
- $\mathcal{D}^+(\mathcal{A})_{\leq 0}$ is the full ∞-subcategory spanned by those complexes $\{M_n\}_{n\in\mathbb{Z}}$ with vanishing homology in positive degrees $n > 0$.

Dual to the case of $\mathcal{D}^-$, this t-structure is now right-complete and left-bounded. Moreover, one has a t-exact equivalence

$$\mathcal{D}^+(\mathcal{A})^{\mathrm{op}} \cong \mathcal{D}^-(\mathcal{A}^{\mathrm{op}}) \tag{5.1}$$

of ∞-categories, induced by the identity functor. We can now state the main result in its injective version, that will be used below in Section 5.3.

Proposition 5.2. *Let $\mathcal{C}$ be a stable ∞-category with a t-structure. Suppose that*

(1) the abelian category $h\mathcal{C}^\heartsuit$ has enough injectives,

(2) for any objects $X, Y \in h\mathcal{C}^\heartsuit$ with Y injecitive we have $[X, \Sigma^i Y] = 0$ for all $i > 0$,

(3) the t-structure is left-bounded.

Then there exists an essentially unique t-exact functor that induces a t-exact equivalence

$$\mathcal{D}^b(h\mathcal{C}^\heartsuit) \xrightarrow{\cong} \mathcal{C}^b$$

of ∞-categories.

Proof. By applying Proposition 5.1 with the opposite t-structure on $\mathcal{C}^{\mathrm{op}}$, we get a t-exact equivalence

$$F\colon \mathcal{D}^{\mathrm{b}}\left(h(\mathcal{C}^{\mathrm{op}})^{\heartsuit}\right) \xrightarrow{\cong} \left(\mathcal{C}^{\mathrm{op}}\right)^{\mathrm{b}}.$$

Since the identity functor induces the equivalence $h(\mathcal{C}^{\mathrm{op}})^{\heartsuit} \cong \left(h\mathcal{C}^{\heartsuit}\right)^{\mathrm{op}}$ as abelian categories, it follows from equation (5.1) that the identity functor induces t-exact equivalences

$$\mathcal{D}^{\mathrm{b}}\left(h(\mathcal{C}^{\mathrm{op}})^{\heartsuit}\right) \cong \mathcal{D}^{\mathrm{b}}\left(\left(h\mathcal{C}^{\heartsuit}\right)^{\mathrm{op}}\right) \cong \mathcal{D}^{\mathrm{b}}\left(h\mathcal{C}^{\heartsuit}\right)^{\mathrm{op}}.$$

The identity functor induces another t-exact equivalence $\left(\mathcal{C}^{\mathrm{op}}\right)^{\mathrm{b}} \cong \left(\mathcal{C}^{\mathrm{b}}\right)^{\mathrm{op}}$, and thus F induces the t-exact equivalence

$$\mathcal{D}^{\mathrm{b}}\left(h\mathcal{C}^{\heartsuit}\right)^{\mathrm{op}} \xrightarrow{\cong} \left(\mathcal{C}^{\mathrm{b}}\right)^{\mathrm{op}}.$$

The required equivalence follows by applying $(-)^{\mathrm{op}}$. □

5.2 Warm-up : An Algebraic Model for $\overline{MGL}$-Modules

As its name indicates, this section is a warm-up for the equivalence of categories to be proved in Section 5.3. As explained in more details underneath, in this section we will provide an algebraic model for $\overline{MGL}$-modules, which will be used in order to prove the main result in Section 5.3.

5.2.1 The categories ${}_{\overline{MGL}}\mathbf{Cell}$ and ${}_{\overline{MGL}_{*,*}}\mathbf{Mod}$

Since $C\tau$ is a cell complex, the usual adjunction from the ring map $S^{0,0} \longrightarrow C\tau$ between the categories of modules restricts to an adjunction

$$\begin{tikzcd} \mathbf{Cell}_{\mathbb{C}} \arrow[r, shift left, "C\tau\wedge-"] \arrow[r, phantom, "\perp"] & {}_{C\tau}\mathbf{Cell}, \arrow[l, shift left, "U"] \end{tikzcd} \tag{5.2}$$

between cellular motivic spectra and cellular (left) $C\tau$-modules. The analogue of the algebraic cobordism spectrum in $C\tau$-modules is the tensored up spectrum $\overline{MGL} \coloneqq C\tau \wedge MGL$. This is a cellular E_∞ $C\tau$-algebra since MGL is E_∞ [45], and tensoring up with an E_∞ ring spectrum (here with $C\tau$) is symmetric monoidal [32, Remark 4.5.3.2], and thus preserves E_∞-algebras. We will sometimes see it as a right $C\tau$-module, by again using the fact that

$C\tau$ is commutative. Denote by ${}_{\overline{MGL}}\mathbf{Cell}$ the category of cellular left $\overline{MGL}$-modules, which sits in the adjunction [12]

$$\begin{tikzcd}
\mathbf{Cell}_{\mathbb{C}} \arrow[r, shift left, "-\wedge C\tau"] \arrow[r, phantom, "\perp"] & {}_{C\tau}\mathbf{Cell} \arrow[l, shift left, "U"] \arrow[r, shift left, "-\wedge_{C\tau}\overline{MGL}"] \arrow[r, phantom, "\perp"] & {}_{\overline{MGL}}\mathbf{Cell}. \arrow[l, shift left, "U"]
\end{tikzcd} \tag{5.3}$$

Although the end goal of Chapter 5 is to study the category of ${}_{C\tau}\mathbf{Cell}$ by using the $\overline{MGL}$-based Adams-Novikov spectral sequence, we will now focus on the category ${}_{\overline{MGL}}\mathbf{Cell}$ of cellular $\overline{MGL}$-modules. In fact, given a $C\tau$-module, the terms in its $\overline{MGL}$-based Adams resolution will be some injective $\overline{MGL}$-modules with certain good properties. Similarly to the situation in homological algebra, to construct such a resolution we need enough of those injective objects. We will show the existence of those objects by better understanding the category of $\overline{MGL}$-modules, and in fact showing that it is equivalent to an algebraic category, in which we know how to find injective objects.

Denote by $[-,-]_{\overline{MGL}}$ the abelian group of homotopy classes of $\overline{MGL}$-linear maps, and by $- \wedge_{\overline{MGL}} -$ the relative smash product. We will compare the topological category of $\overline{MGL}$-modules with an algebraic category via the functor of homotopy groups

$$\begin{tikzcd}
{}_{\overline{MGL}}\mathbf{Cell} \arrow[r, "\pi_{*,*}"] & {}_{\overline{MGL}_{*,*}}\mathbf{Mod}.
\end{tikzcd}$$

Before computing the homotopy groups of $\overline{MGL}$, recall from Definition 4.4 that given a bigraded abelian group $M_{*,*}$, the *Chow degree* of an element $x \in M_{s,w}$ is the integer $s - 2w$.

Lemma 5.3. *The homotopy groups of $\overline{MGL}$ are given as a ring by*

$$\overline{MGL}_{*,*} \cong \widehat{\mathbb{Z}}_2[x_1, x_2, \ldots] \cong \pi_{*,*}(\widehat{MGL}_2)\,/_{\tau},$$

where $|x_i| = (2i, i)$ are in Chow degree 0.

Proof. This follows immediately from Lemma 3.31. ☐

---

[12] the usual adjunction on modules restricts to one on cellular modules since $\overline{MGL}$ is a cellular $C\tau$-module.

We now give the central definition that will lead to the t-structure on the topological category ${}_{\overline{MGL}}\mathbf{Cell}^{\mathrm{b}}$.

Definition 5.4 (Chow degree in ${}_{\overline{MGL}}\mathbf{Cell}$)**.** We say that an object $X \in {}_{\overline{MGL}}\mathbf{Cell}$ is concentrated in *Chow degrees* $[a, b]$ if its homotopy groups $\pi_{*,*}(X)$ are concentrated in Chow degrees $[a, b]$, i.e., if they satisfy

$$\pi_{2*+k,*}(X) = 0 \qquad \text{for any } k \notin [a, b].$$

We say that X has *bounded Chow degree* if it is concentrated in Chow degrees $[a, b]$ for some finite $a, b \neq \pm\infty$.

Notation 5.5. Denote by ${}_{\overline{MGL}}\mathbf{Cell}^{\mathrm{b}}$ the full ∞-subcategory spanned by objects with bounded Chow degree. Denote by ${}_{\overline{MGL}}\mathbf{Cell}_{\geq 0}$, ${}_{\overline{MGL}}\mathbf{Cell}_{\leq 0}$ and ${}_{\overline{MGL}}\mathbf{Cell}^{\heartsuit}$, the full ∞-subcategory of ${}_{\overline{MGL}}\mathbf{Cell}^{\mathrm{b}}$ spanned by objects concentrated in respectively non-negative, non-positive and zero Chow degrees. Similarly, there are obvious generalizations to ${}_{\overline{MGL}}\mathbf{Cell}_{\geq n}$, ${}_{\overline{MGL}}\mathbf{Cell}_{>n}$, ${}_{\overline{MGL}}\mathbf{Cell}_{\leq n}$, ${}_{\overline{MGL}}\mathbf{Cell}_{<n}$. We emphasize that all objects in these subcategories have in particular bounded Chow degree.

Remark 5.6. Even if we use the notation ${}_{\overline{MGL}}\mathbf{Cell}^{\heartsuit}$ and call it the *heart* of the category ${}_{\overline{MGL}}\mathbf{Cell}^{\mathrm{b}}$, we do not claim yet that the Chow degree on ${}_{\overline{MGL}}\mathbf{Cell}^{\mathrm{b}}$ defines a t-structure. In fact, showing that this forms a t-structure is the hardest part of the argument, as the main result in this section, Theorem 5.13, will then follow by a straightforward application of Proposition 5.2.

Observe that by definition of the Chow degree, the homotopy groups of objects in the heart ${}_{\overline{MGL}}\mathbf{Cell}^{\heartsuit}$ land in a smaller category than ${}_{\overline{MGL}_{*,*}}\mathbf{Mod}$. This target category will be important, so let's give it a name.

Definition 5.7. Denote by ${}_{\overline{MGL}_{*,*}}\mathbf{Mod}^{\mathrm{ev}}$ the full subcategory of ${}_{\overline{MGL}_{*,*}}\mathbf{Mod}$ spanned by all modules M concentrated in Chow degree 0, i.e., satisfying $M_{s,w} = 0$ whenever $s \neq 2w$.

We thus have a commutative diagram

$$\begin{tikzcd}
{}_{\overline{MGL}}\mathbf{Cell} \arrow[r, "\pi_{*,*}"] & {}_{\overline{MGL}_{*,*}}\mathbf{Mod} \\
{}_{\overline{MGL}}\mathbf{Cell}^{\heartsuit} \arrow[u, hook, "\text{inc.}"] \arrow[r, dashed, "\pi_{*,*}"] & {}_{\overline{MGL}_{*,*}}\mathbf{Mod}^{\mathrm{ev}}, \arrow[u, hook, "\text{inc.}"']
\end{tikzcd}$$

and we will show that the restriction of $\pi_{*,*}$ to the heart ${}_{\overline{MGL}}\mathbf{Cell}^{\heartsuit}$ induces an equivalence

$$\pi_{*,*} \colon h\left({}_{\overline{MGL}}\mathbf{Cell}^{\heartsuit}\right) \xrightarrow{\cong} {}_{\overline{MGL}_{*,*}}\mathbf{Mod}^{\mathrm{ev}}.$$

It is easy to see that

$${}_{\overline{MGL}_{*,*}}\mathbf{Mod}^{\mathrm{ev}} \cong {}_{\widehat{MU}_*}\mathbf{Mod}^{\mathrm{ev}} \cong {}_{\widehat{MU}_*}\mathbf{Mod},$$

where $\widehat{MU}_* \cong \widehat{\mathbb{Z}}_2[x_1, x_2, \ldots]$ for $|x_i| = 2i$, are the homotopy groups of the 2-completed spectrum MU, and where ${}_{\widehat{MU}_*}\mathbf{Mod}^{\mathrm{ev}}$ is the subcategory of ${}_{\widehat{MU}_*}\mathbf{Mod}$ spanned by modules concentrated in even degrees.

Our main tool to compute homotopy classes of maps in the ∞-category ${}_{\overline{MGL}}\mathbf{Cell}^{\mathrm{b}}$ will be the Universal Coefficient spectral sequence constructed in [12].

Theorem 5.8 (Universal Coefficient spectral sequence)**.** *For any $X, Y \in {}_{\overline{MGL}}\boldsymbol{Cell}$, there is a conditionally convergent spectral sequence*

$$\mathrm{Ext}^{s,t,w}_{\overline{MGL}_{*,*}}(\pi_{*,*}(X), \pi_{*,*}(Y)) \Longrightarrow \left[\Sigma^{-t-s,-w}X, Y\right]_{\overline{MGL}}.$$

If X, Y have bounded Chow degree, then the spectral sequence is strongly convergent and collapses at a finite page.

Proof. We refer to [12, Proposition 7.7] for the precise construction of the spectral sequence and the statement about conditional convergence. However, to study the convergence in the case where X and Y have bounded Chow degree, we need to briefly recall the construction and pay careful attention to degrees. Recall that this spectral sequence arises from an $\overline{MGL}_{*,*}$-free resolution

$$0 \longleftarrow \pi_{*,*}(X) \twoheadleftarrow \pi_{*,*}(F_0) \longleftarrow \pi_{*,*}(F_1) \longleftarrow \cdots .$$

Since $\overline{MGL}_{*,*}$ is concentrated in Chow degree 0, if $\pi_{*,*}(X)$ is concentrated in Chow degrees $[a, b]$, then so is every $\pi_{*,*}(F_s)$, independently of the homological degree s. The E_1-page is given by

$$E_1^{s,t,w} := \mathrm{Hom}_{\overline{MGL}_{*,*}}(\pi_{*,*}(\Sigma^{-t,-w}F_s), \pi_{*,*}(Y)),$$

and the E_2-page is the cohomology of this chain complex, giving the claimed Ext groups. Suppose that $\pi_{*,*}(Y)$ is concentrated in Chow degrees $[c, d]$. Since $\pi_{*,*}(\Sigma^{-t,-w}F_s)$ is concentrated in Chow degrees $[a - (t - 2w), b - (t - 2w)]$, it follows that for a fixed weight w, the group E_1 is possibly non-zero only for

$$t \in [a - d + 2w, b - c + 2w].$$

Since $E_r^{s,t,w}$ is an iterated subquotient of $E_1^{s,t,w}$, it is also possibly non-zero only when $t \in [a - d + 2w, b - c + 2w]$. Recall that the d_r-differential has the form

$$E_r^{s,t,w} \xrightarrow{d_r} E_r^{s+r,t-r+1,w}.$$

In particular, if $r - 1 > (b - c) - (a - d)$, the $d_r = 0$ for degree reasons, that is, the spectral sequence collapses at the $E_{b-a+d-c+2}$-page. □

5.2.2 Proof of the t-structure, the heart and the equivalence

As mentioned before, the main tool for computing homotopy classes of maps in the category ${}_{\overline{MGL}}\mathbf{Cell}$ will be the Universal Coefficient spectral sequence of Theorem 5.8. An immediate Corollary of its proof is the following result.

Corollary 5.9. *The functor*

$$\pi_{*,*}\colon h\left({}_{\overline{MGL}}\boldsymbol{Cell}^{\heartsuit}\right) \longrightarrow {}_{\overline{MGL}_{*,*}}\boldsymbol{Mod}^{ev}$$

is fully faithful.

Proof. Pick two objects $X, Y \in {}_{\overline{MGL}}\mathbf{Cell}^{\heartsuit}$. Since they are both concentrated in Chow degree 0, the end of the proof of Theorem 5.8 shows that the spectral sequence

$$\mathrm{Ext}^{s,t,w}_{\overline{MGL}_{*,*}}(\pi_{*,*}(X), \pi_{*,*}(Y)) \Longrightarrow \left[\Sigma^{-t-s,-w}X, Y\right]_{\overline{MGL}}$$

is concentrated in $t = 2w$ and collapses at the E_2-page. Since there are also no possible hidden extensions for degree reasons, it thus take the form of isomorphisms

$$\mathrm{Ext}^{s,2w,w}_{\overline{MGL}_{*,*}}(\pi_{*,*}(X), \pi_{*,*}(Y)) \cong \left[\Sigma^{-2w-s,-w}X, Y\right]_{\overline{MGL}}.$$

In particular, in the case $2w + s = w = 0$, we get $s = 0$ and the edge homomorphism

$$[X, Y]_{\overline{MGL}} \xrightarrow{\pi_{*,*}} \mathrm{Hom}_{\overline{MGL}_{*,*}}(\pi_{*,*}(X), \pi_{*,*}(Y))$$

is an isomorphism, showing the faithfullness of $\pi_{*,*}$. □

We will now prove the essential surjectivity of $\pi_{*,*}$ on ${}_{\overline{MGL}}\mathbf{Cell}^{\heartsuit}$, identifying the heart of the category ${}_{\overline{MGL}}\mathbf{Cell}^{\mathrm{b}}$.

Proposition 5.10. *The functor*

$$\pi_{*,*}\colon h\left({}_{\overline{MGL}}\boldsymbol{Cell}^{\heartsuit}\right) \xrightarrow{\cong} {}_{\overline{MGL}_{*,*}}\boldsymbol{Mod}^{ev}$$

is an equivalence of categories.

Proof. Pick a module $M \in {}_{\overline{MGL}_{*,*}}\mathbf{Mod}^{\mathrm{ev}}$ and let's realize it topologically in ${}_{\overline{MGL}}\mathbf{Cell}^{\heartsuit}$. If M is a free $\overline{MGL}_{*,*}$-module, say $M \cong \oplus\overline{MGL}_{*,*}$, then M is the homotopy groups of the wedge $\vee\overline{MGL}$ indexed by the same set in Chow degree zero.

For an arbitrary M, pick a free resolution

$$0 \longleftarrow M \longleftarrow F_0 \xleftarrow{f_1} F_1 \xleftarrow{f_2} F_2 \longleftarrow \cdots \tag{5.4}$$

in ${}_{\overline{MGL}_{*,*}}\mathbf{Mod}^{\text{ev}}$. By the above step, each F_i can be realized topologically by an $\overline{MGL}$-module $Z_i \in {}_{\overline{MGL}}\mathbf{Cell}^{\heartsuit}$, and by the previous Corollary 5.9, one can also realize the maps f_i by a tower

$$Z_0 \longleftarrow Z_1 \longleftarrow Z_2 \longleftarrow \cdots .$$

We will now construct a tower

$$X_1 \longrightarrow X_2 \longrightarrow \cdots ,$$

with the property that the homotopy groups of X_i are given by

$$\pi_{2*+k,*}(X_i) = \begin{cases} M = \operatorname{coker} f_1 & \text{if } k = 0 \\ \ker f_i & \text{if } k = i \\ 0 & \text{if } k \neq 0, i. \end{cases}$$

Let $X_1 = \operatorname{cof}(Z_1 \longrightarrow Z_0)$, and it is easy to verify that it satisfies the required properties from the long exact sequence in homotopy groups. Suppose now that the tower has been constructed until X_{i-1}. Since $\overline{MGL}_{*,*}$ is concentrated in Chow degree 0, there is a splitting

$$\pi_{*,*}(X_{i-1}) \cong M \oplus \ker f_{i-1}$$

as $\overline{MGL}_{*,*}$-modules. The homomorphism

$$\pi_{*,*}(Z_i) \cong F_i \twoheadrightarrow \operatorname{im}(f_{i+1}) \cong \ker(f_{i-1}) \hookrightarrow \pi_{*,*}(\Sigma^{1-i,0}X_{i-1})$$

of $\overline{MGL}_{*,*}$-modules corresponds to a unique homotopy class of maps $Z_i \longrightarrow \Sigma^{1-i,0}X_{i-1}$ by Corollary 5.9. Define X_i to be the cofiber of its $\Sigma^{i-1,0}$ suspension, as

$$\Sigma^{i-1,0}Z_i \longrightarrow X_{i-1} \longrightarrow X_i.$$

By the associated long exact sequence in homotopy groups, it is easy to show that X_i satisfies the required properties. The homotopy colimit

$$X \coloneqq \operatorname{hocolim}(X_1 \longrightarrow X_2 \longrightarrow \cdots)$$

realizes M since its homotopy groups are given by the colimit

$$\pi_{*,*}(X) \cong \operatorname{colim}(\pi_{*,*}(X_1) \longrightarrow \pi_{*,*}(X_2) \longrightarrow \cdots) = \begin{cases} M & \text{if } k = 0 \\ 0 & \text{if } k \neq 0. \end{cases}$$

□

We can now start showing the required axioms for the t-structure. We will use the following Lemma several times in what follows.

Lemma 5.11. *Given $X \in {}_{\overline{MGL}}\boldsymbol{Cell}_{\geq 0}$ and $Y \in {}_{\overline{MGL}}\boldsymbol{Cell}_{\leq 0}$, the group of homotopy classes of maps of degree $(0,0)$ can be computed algebraically by*

$$\operatorname{Hom}_{\overline{MGL}_{*,*}}(\pi_{*,*}(X), \pi_{*,*}(Y)) \cong [X,Y]_{\overline{MGL}}\,.$$

Proof. It is clear from the proof of Theorem 5.8 that the E_2-page of the Universal Coefficient spectral sequence

$$E_2^{s,t,w} = \operatorname{Ext}^{s,t,w}_{\overline{MGL}_{*,*}}(\pi_{*,*}(X), \pi_{*,*}(Y))$$

is concentrated in $s \geq 0$ and $t - 2w \geq 0$. We are interested in understanding $[X,Y]_{\overline{MGL}}$, which is assembled from $E_\infty^{s,t,w}$ with $s + t = 0$ and $w = 0$. These conditions imply that

$s = t = w = 0$, and thus that $[X, Y]_{\overline{MGL}}$ is a (possibly iterated) subquotient of

$$E_2^{0,0,0} \cong \mathrm{Hom}_{\overline{MGL}_{*,*}}(\pi_{*,*}(X), \pi_{*,*}(Y)).$$

Recall that the d_r-differential has the form $E_r^{s,t,w} \xrightarrow{d_r} E_r^{s+r,t-r+1,w}$. There are thus no possible differentials entering $E_r^{0,0,0}$ since the homological degree has to be non-negative, and the only possible differential exiting $E_r^{0,0,0}$ is a d_1 since $t - 2w \geq 0$ implies that we must have $-r+1 \geq 0$. However, we already are at the E_2-page and thus $E_2^{0,0,0}$ survives to the E_∞-page giving the isomorphism

$$\mathrm{Hom}_{\overline{MGL}_{*,*}}(\pi_{*,*}(X), \pi_{*,*}(Y)) \cong [X, Y]_{\overline{MGL}}.$$

□

Proposition 5.12. *The pair of subcategories* $({}_{\overline{MGL}}\boldsymbol{Cell}_{\geq 0}, {}_{\overline{MGL}}\boldsymbol{Cell}_{\leq 0})$ *defines a* t*-structure on* ${}_{\overline{MGL}}\boldsymbol{Cell}^b$*, whose heart is given by the equivalence*

$$\pi_{*,*} \colon h\left({}_{\overline{MGL}}\boldsymbol{Cell}^{\heartsuit}\right) \xrightarrow{\cong} {}_{\overline{MGL}_{*,*}}\boldsymbol{Mod}^{ev}.$$

Proof. The heart has been identified in Proposition 5.10. There are now three axioms to show for the t-structure.

Given objects $X \in {}_{\overline{MGL}}\mathbf{Cell}_{\geq 0}$ and $Y \in {}_{\overline{MGL}}\mathbf{Cell}_{<0}$, the first axioms requires that $[X, Y]_{\overline{MGL}} = 0$. In particular observe that $Y \in {}_{\overline{MGL}}\mathbf{Cell}_{\leq 0}$ and thus we have

$$[X, Y]_{\overline{MGL}} \cong \mathrm{Hom}_{\overline{MGL}_{*,*}}(\pi_{*,*}(X), \pi_{*,*}(Y))$$

by Lemma 5.11. The right hand side is zero because this is a graded Hom-set, and $\pi_{*,*}(X)$ is concentrated in non-negative Chow degrees while $\pi_{*,*}(Y)$ is concentrated in negative Chow degrees.

The second axiom requires that if $X \in {}_{\overline{MGL}}\mathbf{Cell}_{\geq 0}$, then $\Sigma^{1,0}X \in {}_{\overline{MGL}}\mathbf{Cell}_{\geq 1}$. This is clear from the definition of these subcategories.

Finally, for any spectrum $X \in {}_{\overline{MGL}}\mathbf{Cell}^{\mathrm{b}}$, the last axiom asks for a fiber sequence

$$\begin{tikzcd} X_{\geq 0} \arrow[r] & X \arrow[r] & X_{<0} \end{tikzcd}$$

with $X_{\geq 0} \in {}_{\overline{MGL}}\mathbf{Cell}_{\geq 0}$ and $X_{<0} \in {}_{\overline{MGL}}\mathbf{Cell}_{<0}$. Since X, or more precisely $\pi_{*,*}(X)$ has bounded Chow degree, denote by n the largest integer that bounds it below. If n is non-negative then

$$\begin{tikzcd} X \arrow[r] & X \arrow[r] & * \end{tikzcd}$$

is the desired fiber sequence since $X \in {}_{\overline{MGL}}\mathbf{Cell}_{\geq n} \subset {}_{\overline{MGL}}\mathbf{Cell}_{\geq 0}$. If not, consider the $\overline{MGL}_{*,*}$-module $\pi_{2*,*}(\Sigma^{-n,0}X)$ that is concentrated in Chow degree 0. By Proposition 5.10 there is a spectrum $X_n \in {}_{\overline{MGL}}\mathbf{Cell}^{\heartsuit}$ with

$$\pi_{2*,*}(X_n) \cong \pi_{2*,*}(\Sigma^{-n,0}X).$$

Moreover $\Sigma^{-n,0}X \in {}_{\overline{MGL}}\mathbf{Cell}_{\geq 0}$ since n is maximal, and so the algebraic map

$$\begin{tikzcd} \pi_{*,*}(\Sigma^{-n,0}X) \arrow[r, two heads] & \pi_{2*,*}(\Sigma^{-n,0}X) \cong \pi_{2*,*}(X_n) \arrow[r, hook, "\cong"] & \pi_{*,*}(X_n) \end{tikzcd}$$

can be topologically realized by a map $\Sigma^{-n,0}X \longrightarrow X_n$ by Lemma 5.11. Denote by $X_{[n+1,m+1]}$ the fiber of its $\Sigma^{n,0}$-suspension as in

$$\begin{tikzcd} X_{[n+1,m+1]} \arrow[r] & X \arrow[r] & \Sigma^{n,0}X_n. \end{tikzcd}$$

It is easy to see by the long exact sequence in homotopy groups that $X_{[n+1,m+1]}$ is concentrated in Chow degrees $[n+1, m+1]$ as its name suggests, and that the map $X_{[n+1,m+1]} \longrightarrow X$ induces an isomorphism in Chow degrees $[n+1, m]$. By re-iterating this process we construct a finite sequence of spectra

$$X_{[0,m-n]} \longrightarrow X_{[-1,m-n-1]} \longrightarrow \cdots \longrightarrow X_{[n+1,m+1]} \longrightarrow X,$$

where $X_{[0,m-n]} \in {}_{\overline{MGL}}\mathbf{Cell}_{\geq 0}$. By letting $X_{\geq 0} \coloneqq X_{[0,m-n]}$, the cofiber of this composite gives the desired cofiber sequence

$$X_{\geq 0} \longrightarrow X \longrightarrow X_{<0},$$

where $X_{<0} \in {}_{\overline{MGL}}\mathbf{Cell}_{<0}$ since $X_{\geq 0} \longrightarrow X$ induces an isomorphism in Chow degrees $[0, m]$. □

Having this t-structure on ${}_{\overline{MGL}}\mathbf{Cell}^{\mathrm{b}}$, the main result of this section follows from an easy application of Proposition 5.1.

Theorem 5.13. *There is a t-exact equivalence of ∞-categories*

$$\mathcal{D}^b({}_{\overline{MGL}_{*,*}}\boldsymbol{Mod}) \xrightarrow{\cong} {}_{\overline{MGL}}\boldsymbol{Cell}^b.$$

Proof. We need to check the conditions of Proposition 5.1 for the t-structure on ${}_{\overline{MGL}}\mathbf{Cell}^{\mathrm{b}}$. The t-structure is clearly right-bounded (it is even bounded), and its heart has enough projectives by Proposition 5.10 since module categories have enough projectives. It remains to show that for any two motivic spectra $X, Y \in {}_{\overline{MGL}}\mathbf{Cell}^{\heartsuit}$ with $\pi_{*,*}(X)$ a projective $\overline{MGL}_{*,*}$-module, there are no non-trivial maps in $[X, \Sigma^{i,0}Y]_{\overline{MGL}}$ for any $i > 0$. We apply the Universal Coefficient spectral sequence

$$\mathrm{Ext}^{s,t,w}_{\overline{MGL}_{*,*}}(\pi_{*,*}(X), \pi_{*,*}(Y)) \Longrightarrow [\Sigma^{-s-t,-w}X, Y]_{\overline{MGL}}$$

of Theorem 5.8, and we are interested in the case $-s-t = -i$ and $w = 0$. Since $\pi_{*,*}(X)$ is a projective $\overline{MGL}_{*,*}$-module, the spectral sequence is concentrated on the line $s = 0$, and thus $t = i$, and collapses at E_2. Moreover, since both X and Y are concentrated in Chow degree 0, it is clear from the proof of Theorem 5.8 that the E_2-page is concentrated in degrees

$t - 2w = 0$, and thus $t = 0$. It follows that in weight 0, there are only maps of bidegree $(0,0)$, and thus $[X, \Sigma^{i,0} Y]_{\overline{MGL}} = 0$ for $i > 0$. □

5.3 Main Result: An Algebraic Model for $C\tau$-Modules

In this section we will prove the main theorem of the chapter, by constructing an algebraic model for the category of cellular $C\tau$-modules. The strategy is very similar to what we did in Section 5.2. In Definition 5.14, we first define the main functor of interest from the topological category ${}_{C\tau}\mathbf{Cell}$ to an algebraic category. The goal will then to put a t-structure on ${}_{C\tau}\mathbf{Cell}$ (or rather ${}_{C\tau}\mathbf{Cell}^{\mathrm{comp}}$), whose heart will be isomorphic to that algebraic category. As in Section 5.2, we will then need a tool which plays well with the t-structure to compute homotopy classes of maps in ${}_{C\tau}\mathbf{Cell}$. Unfortunately, as we explain more at the beginning of Section 5.3.2, the Universal Coefficient spectral sequence in ${}_{C\tau}\mathbf{Cell}$ does not play well with the notion of Chow degree in that category. We will thus have to construct a generalized motivic Adams-Novikov spectral sequence in Section 5.3.2. This is the most difficult part of the section, and the main difference with Section 5.2. However, the proof of the main result in Section 5.3.3 is very similar to that of Section 5.2, by replacing the Universal Coefficient spectral sequence with the motivic Adams-Novikov spectral sequence.

5.3.1 The categories ${}_{C\tau}\mathbf{Cell}$ and ${}_{\overline{MGL}_{*,*}\overline{MGL}}\mathbf{Comod}$

Recall the E_∞ $C\tau$-algebra $\overline{MGL} = C\tau \wedge MGL$, with homotopy groups $\overline{MGL}_{*,*} \cong \widehat{\mathbb{Z}}_2[x_i]$. In this section we will use $\overline{MGL}$-homology to study the category of $C\tau$-modules. As done in Chapter 4, it is more efficient to consider the $C\tau$-linear $\overline{MGL}$-homology for $C\tau$-modules, which is isomorphic but admits more structure that the MGL-homology of the underlying spectrum. We refer to section 4.1.2 for more details.

Definition 5.14 ($C\tau$-linear $\overline{MGL}$-homology)**.** Given a $C\tau$-module X, define its *$C\tau$-linear $\overline{MGL}$-homology* to be

$$\overline{MGL}_{*,*}(X) \coloneqq \pi_{*,*}(\overline{MGL} \wedge_{C\tau} X).$$

As it was the case for its homotopy groups, the $C\tau$-linear cooperations of $\overline{MGL}$ are easy to deduce from the cooperations of MGL.

Lemma 5.15. *The $C\tau$-linear cooperations of $\overline{MGL}$ are given as a Hopf algebroid by*

$$\overline{MGL}_{*,*}(\overline{MGL}) \cong \overline{MGL}_{*,*}[t_1, t_2, \ldots] \cong MGL_{*,*}\widehat{MGL}_2 /_{\tau}$$

where $|t_i| = (2i, i)$ are in Chow degree 0.

Proof. The proof is similar to the proof of Proposition 3.33 and Remark 3.35, by replacing $\overline{H}$ with $\overline{MGL}$. ☐

Remark 5.16. Given a $C\tau$-module X, its $C\tau$-linear $\overline{MGL}$-homology is isomorphic as an abelian group to the MGL-homology of the underlying spectrum of X since

$$U(\overline{MGL} \wedge_{C\tau} X) \simeq MGL \wedge C\tau \wedge_{C\tau} X \simeq MGL \wedge X,$$

where U denotes the underlying spectrum from the adjunction (5.2). Moreover, the action of $MGL_{*,*}$ factors trough the action of $\overline{MGL}_{*,*}$, and the coaction of the Hopf algebroid of cooperations of MGL factors trough the coaction of $C\tau$-linear cooperations of $\overline{MGL}$. The only difference is the simple fact that τ acts and coacts as zero on $\overline{MGL}_{*,*}(X)$. This has important consequences, for example since $\overline{MGL}_{*,*}\overline{MGL}$ is concentrated in Chow degree 0, coacting by an element in $\overline{MGL}_{*,*}\overline{MGL}$ preserves the Chow degree, which is not true for every element in $MGL_{*,*}MGL$.

Definition 5.17. Denote by ${}_{\overline{MGL}_{*,*}\overline{MGL}}\mathbf{Comod}$ the category of comodules over the Hopf algebroid $\overline{MGL}_{*,*}\overline{MGL}$. Denote by ${}_{\overline{MGL}_{*,*}\overline{MGL}}\mathbf{Comod}^{\mathrm{ev}}$ the full subcategory of ${}_{\overline{MGL}_{*,*}\overline{MGL}}\mathbf{Comod}$ spanned by all comodules M whose underlying $\overline{MGL}$-module is in ${}_{\overline{MGL}_{*,*}}\mathbf{Mod}^{\mathrm{ev}}$, i.e., is concentrated in Chow degree 0.

Recall that the adjunction between modules and comodules

$$\begin{tikzcd} U\colon {}_{\overline{MGL}_{*,*}\overline{MGL}}\mathbf{Comod} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & {}_{\overline{MGL}_{*,*}}\mathbf{Mod}\colon - \otimes_{\overline{MGL}_{*,*}} \overline{MGL}_{*,*}\overline{MGL}, \arrow[l, shift left] \end{tikzcd} \tag{5.5}$$

is reverse from the usual situation for just module categories. In particular, the forgetful functor is only a left adjoint, while the tensor-up functor itself admits another right adjoint. We refer to [21, Section 1.1] for more details.

By using the ring map $C\tau \longrightarrow \overline{MGL}$, we can form the commutative diagram

$$\begin{tikzcd}
{}_{\overline{MGL}}\mathbf{Cell} \arrow[r, "\pi_{*,*}"] & {}_{\overline{MGL}_{*,*}}\mathbf{Mod}^{\mathrm{ev}} \\
{}_{C\tau}\mathbf{Cell} \arrow[u, "\overline{MGL} \wedge_{C\tau} -"] \arrow[r, "\overline{MGL}_{*,*}"] & {}_{\overline{MGL}_{*,*}\overline{MGL}}\mathbf{Comod}^{\mathrm{ev}}. \arrow[u, "\text{forget}"']
\end{tikzcd}$$

Recall that the algebraic identification of ${}_{\overline{MGL}}\mathbf{Cell}^{\mathrm{b}}$ of Theorem 5.13 relies on a t-structure on ${}_{\overline{MGL}}\mathbf{Cell}^{\mathrm{b}}$, such that the restriction of $\pi_{*,*}$ on the heart is an equivalence. The proof of the main Theorem 5.37 follows the exact same strategy, where the goal is to endow ${}_{C\tau}\mathbf{Cell}^{\mathrm{b}}$ with a t-structure whose heart is isomorphic to the category ${}_{\overline{MGL}_{*,*}\overline{MGL}}\mathbf{Comod}^{\mathrm{ev}}$. Proving the axioms for this t-structure is significantly harder, and will actually use Theorem 5.13. We can now make the fundamental definition that will lead to the t-structure, which is very similar to that of Definition 5.4.

Definition 5.18 (Chow degree in ${}_{C\tau}\mathbf{Cell}$)**.** We say that an object $X \in {}_{C\tau}\mathbf{Cell}$ is concentrated in *Chow degrees* $[a, b]$ if its $\overline{MGL}$-homology $\overline{MGL}_{*,*}(X)$ is concentrated in Chow degrees $[a, b]$, i.e., if it satisfies

$$\overline{MGL}_{2*+k,*}(X) = 0 \qquad \text{for any } k \notin [a, b].$$

We say that X has *bounded Chow degree* if it is concentrated in Chow degrees $[a, b]$ for some finite numbers $a, b \neq \pm\infty$.

Notation 5.19. Denote by ${}_{C\tau}\mathbf{Cell}^{\mathrm{b}}$ the full subcategory of ${}_{C\tau}\mathbf{Cell}$ spanned by objects of bounded Chow degree. We define ${}_{C\tau}\mathbf{Cell}^{\heartsuit}$, ${}_{C\tau}\mathbf{Cell}_{\geq n}$, ${}_{C\tau}\mathbf{Cell}_{>n}$, ${}_{C\tau}\mathbf{Cell}_{\leq n}$ and ${}_{C\tau}\mathbf{Cell}_{<n}$ as the obvious subcategories of ${}_{C\tau}\mathbf{Cell}^{\mathrm{b}}$, exactly as in Notation 5.5.

The plan is now to show that this defines a t-structure on ${}_{C\tau}\mathbf{Cell}^{\mathrm{b}}$, and that the bottom horizontal arrow in the commutative diagram

$$\begin{tikzcd}
{}_{C\tau}\mathbf{Cell} \arrow[r, "\overline{MGL}_{*,*}"] & {}_{\overline{MGL}_{*,*}\overline{MGL}}\mathbf{Comod} \\
{}_{C\tau}\mathbf{Cell}^{\heartsuit} \arrow[u, hook, "\text{inc.}"] \arrow[r, dashed, "\overline{MGL}_{*,*}"] & {}_{\overline{MGL}_{*,*}\overline{MGL}}\mathbf{Comod}^{\mathrm{ev}}, \arrow[u, hook', "\text{inc.}"']
\end{tikzcd}$$

induces the equivalence

$$\pi_{*,*}\colon h\left({}_{C\tau}\mathbf{Cell}^{\heartsuit}\right) \xrightarrow{\cong} {}_{\overline{MGL}_{*,*}\overline{MGL}}\mathbf{Comod}^{\mathrm{ev}}$$

of abelian categories. The main theorem will then follow from an obvious application of Proposition 5.2.

5.3.2 The $C\tau$-linear Adams-Novikov spectral sequence

In the previous Section 5.2, the tool that we used to compute maps $[X,Y]_{\overline{MGL}}$ was the Universal Coefficient spectral sequence of Theorem 5.8. This was a very convenient tool to use since both the E_2-page $\mathrm{Ext}_{\overline{MGL}}(\pi_{*,*}(X),\pi_{*,*}(Y))$ and the t-structure were defined in terms of homotopy groups. More precisely, bounds in the t-structure transferred to vanishing areas in the spectral sequence, allowing us to prove what we need. In the category of $C\tau$-modules, the t-structure is now defined in terms of $\overline{MGL}$-homology, and so the Universal Coefficient spectral sequence will not be of any use since being bounded for this t-structure does not imply bounds in homotopy groups. We thus need a spectral sequence with $\overline{MGL}$-homology as input and $C\tau$-linear maps as output.

Recall the usual MGL-based Adams-Novikov spectral sequence

$$\mathrm{Ext}_{MGL_{*,*}MGL}(MGL_{*,*}(S^{0,0}), MGL_{*,*}(Y)) \Longrightarrow \pi_{*,*}\left((Y)^{\wedge}_{MGL}\right)$$

constructed in [13, Section 8] or [23]. We need a spectral sequence of the form

$$\mathrm{Ext}_{\overline{MGL}_{*,*}\overline{MGL}}(\overline{MGL}_{*,*}(X), \overline{MGL}_{*,*}(Y)) \Longrightarrow [X, (Y)^{\wedge}_{\overline{MGL}}]_{C\tau},$$

which is more general in two aspects. First, we do not want to compute maps of motivic spectra, but $C\tau$-linear maps. This is easy to fix by working internally in the category of $C\tau$-modules by using the relative smash product, $C\tau$-linear mapping spaces, etc. Second, we need to allow more general spectra than the sphere (or $C\tau$ in the $C\tau$-linear setting) for the first variable X, as we do not only compute homotopy groups, but general maps. This means that we cannot use a standard Adams-Novikov resolution for the second variable Y as is done traditionally [47, Chapter 2] (or [13, Section 8], [23] in the motivic setting) to set up this spectral sequence. In fact, such a resolution induces a resolution of $\overline{MGL}_{*,*}(Y)$ by relative injective comodules, which allows to compute the E_2 term as Ext-groups only when the first variable $\overline{MGL}_{*,*}(X)$ is projective as a module [47, Corollary A1.2.12]. Since our first variable X is arbitrary, this will generally not be the case and we have no (useful) ways of computing the E_2-term.

The solution that we choose is to construct another Adams tower that produces a resolution of $\overline{MGL}_{*,*}(Y)$ by absolute injectives, rather than merely relative injectives. Since absolute injective resolutions are not very popular[13], we need some preparation before constructing the tower. The first step in Lemma 5.21 is to even produce enough $C\tau$-modules whose $\overline{MGL}_{*,*}$-homology will be an injective comodule of a special form. This is done below by using Theorem 5.8. The second step in Lemma 5.22 is to show that algebraically one can resolve comodules in ${}_{\overline{MGL}_{*,*}\overline{MGL}}\mathbf{Comod}^{\mathrm{ev}}$ by the injective comodules arising from topology. Before we show these two Lemmas, let's point out a technical detail about injectives in ${}_{\overline{MGL}_{*,*}\overline{MGL}}\mathbf{Comod}$.

Remark 5.20 (Monomorphisms in comodule categories). Since injective objects are defined

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[13] in the case of Adams spectral sequences, it is more common to use either projective, or relative injective resolutions.

via (extensions along) monomorphisms, let's briefly recall what those are in comodule categories. More generally, given such a category with a forget functor F to **Set**, injective maps are monomorphisms if and only if F is fully faithfull, and monomorphisms are injective if and only if F preserves pullbacks. In this case, the forget functor is faithfull and exact (but does not preserve infinite products), and thus monomorphisms and injective maps agree. We will thus use those two notions interchangeably.

Lemma 5.21. *For any injective module* $I \in {}_{\overline{MGL}_{*,*}}\boldsymbol{Mod}^{ev}$ *concentrated in Chow degree 0, there is a* $C\tau$*-module* $Y \in {}_{C\tau}\boldsymbol{Cell}^{\heartsuit}$ *satisfying*

(1) $\pi_{*,*}(Y) \cong I$ *is an injective* $\overline{MGL}_{*,*}$*-module,*

(2) $\overline{MGL}_{*,*}(Y) \cong \overline{MGL}_{*,*}\overline{MGL} \otimes_{\overline{MGL}_{*,*}} I$ *is an injective* $\overline{MGL}_{*,*}\overline{MGL}$*-comodule,*

(3) for any $X \in {}_{C\tau}\boldsymbol{Cell}^{b}$ *and any bidegree* (t, w)*, we have isomorphisms*

$$[X, \Sigma^{t,w}Y]_{C\tau} \cong \mathrm{Hom}_{\overline{MGL}_{*,*}}(\overline{MGL}_{*,*}(X), \pi_{*,*}(\Sigma^{t,w}Y)) \cong \mathrm{Hom}_{\overline{MGL}_{*,*}\overline{MGL}}(\overline{MGL}_{*,*}(X), \overline{MGL}_{*,*}(\Sigma^{t,w}Y)).$$

Proof. From the equivalence of Proposition, 5.10 there exists an essentially unique $\overline{MGL}$-module Z with the property that $\pi_{*,*}(Z) \cong I$ as an $\overline{MGL}_{*,*}$-module. By using the free-forget adjunction of equation (5.3), the underlying $C\tau$-module UZ will be the spectrum Y that we look for. We now verify the conditions that UZ has to satisfy. First of all, its homotopy groups are given by

$$\pi_{*,*}(UZ) \cong [\Sigma^{*,*}C\tau, UZ]_{C\tau} \cong [\Sigma^{*,*}\overline{MGL}, Z]_{\overline{MGL}} \cong \pi_{*,*}(Z) \cong I.$$

By using the equivalence $UZ \simeq \overline{MGL} \wedge_{\overline{MGL}} Z$ as (left) $C\tau$-modules, we have the equivalence

$$\overline{MGL} \wedge_{C\tau} UZ \simeq \overline{MGL} \wedge_{C\tau} \left(\overline{MGL} \wedge_{\overline{MGL}} Z\right) \simeq \left(\overline{MGL} \wedge_{C\tau} \overline{MGL}\right) \wedge_{\overline{MGL}} Z.$$

The homotopy groups of the right term can be computed via the Tor-spectral sequence of

[12, Proposition 7.7]

$$\mathrm{Tor}^{\overline{MGL}_{*,*}}_{s,t,w}\left(\overline{MGL}_{*,*}\overline{MGL}, \pi_{*,*}(Z)\right) \Longrightarrow \pi_{t+s,w}\left(\overline{MGL} \wedge_{C\tau} \overline{MGL} \wedge_{\overline{MGL}} Z\right).$$

Since $\overline{MGL}_{*,*}\overline{MGL}$ is $\overline{MGL}_{*,*}$-free by Lemma 5.15, the spectral sequence is concentrated on the line $s = 0$ and collapses at E_2, giving the isomorphism of comodules

$$\overline{MGL}_{*,*}(UZ) \cong \overline{MGL}_{*,*}\overline{MGL} \otimes_{\overline{MGL}} \pi_{*,*}(Z).$$

It is well known (for example from [47, Lemma A1.2.2]) that comodules induced from injective modules are injective as comodules, which shows that $\overline{MGL}_{*,*}(UZ)$ is injective.

Finally, let $X \in {}_{C\tau}\mathbf{Cell}^{\mathrm{b}}$. We want to understand the set of homotopy classes of maps

$$[X, \Sigma^{t,w}UZ]_{C\tau} \cong [\overline{MGL} \wedge_{C\tau} X, \Sigma^{t,w}Z]_{\overline{MGL}},$$

which can be computed by the Universal Coefficient spectral sequence of Theorem 5.8

$$\mathrm{Ext}^{s,t,w}_{\overline{MGL}_{*,*}}(\overline{MGL}_{*,*}(X), \pi_{*,*}(Z)) \Longrightarrow [\Sigma^{-s-t,-w}\overline{MGL} \wedge_{C\tau} X, Z]_{\overline{MGL}},$$

since both Z and $\overline{MGL} \wedge_{C\tau} X$ live in ${}_{\overline{MGL}}\mathbf{Cell}^{\mathrm{b}}$. Since $\pi_{*,*}(Z) \cong I$ is an injective $\overline{MGL}_{*,*}$-module, the spectral sequence is concentrated on the line $s = 0$ and collapses at E_2. This gives the graded isomorphism

$$\mathrm{Hom}_{\overline{MGL}_{*,*}}(\overline{MGL}_{*,*}(X), \pi_{*,*}(\Sigma^{t,w}Z)) \cong [\overline{MGL} \wedge_{C\tau} X, \Sigma^{t,w}Z]_{\overline{MGL}} \cong [X, \Sigma^{t,w}UZ]_{C\tau}.$$

Finally, the last isomorphism

$$\mathrm{Hom}_{\overline{MGL}_{*,*}}(\overline{MGL}_{*,*}(X), \pi_{*,*}(\Sigma^{t,w}Z)) \cong \mathrm{Hom}_{\overline{MGL}_{*,*}\overline{MGL}}(\overline{MGL}_{*,*}(X), \overline{MGL}_{*,*}(\Sigma^{t,w}Z))$$

follows by adjunction since $\overline{MGL}_{*,*}(\Sigma^{t,w}Z) \cong \overline{MGL}_{*,*}\overline{MGL} \otimes_{\overline{MGL}_{*,*}} \pi_{*,*}(\Sigma^{t,w}Z)$. □

This will be our source of motivic $C\tau$-modules whose $\overline{MGL}$-homology is injective as a comodule. We now show that these suffice to resolve any comodule in Chow degree 0.

Lemma 5.22. *For any comodule* $M \in {}_{\overline{MGL}_{*,*}\overline{MGL}}\boldsymbol{Comod}^{ev}$ *concentrated in Chow degree 0, there exists a monomorphism into a comodule which*

(1) is concentrated in Chow degree 0,

(2) is of the form $\overline{MGL}_{*,*}\overline{MGL} \otimes_{\overline{MGL}_{*,*}} I$ *for some injective module* $I \in {}_{\overline{MGL}_{*,*}}\boldsymbol{Mod}^{ev}$.

Proof. This proof is the standard way of showing that an abelian category has enough injectives, by inducing them from $\mathbb{Z}$-modules (or $\widehat{\mathbb{Z}}_2$ in our case). Start with the monomorphism

$$M \hookrightarrow \prod_{x \in M \setminus 0} \Sigma^{|x|}\widehat{\mathbb{Q}}_2/\widehat{\mathbb{Z}}_2 \tag{5.6}$$

of bigraded $\widehat{\mathbb{Z}}_2$-modules, where the target is an injective $\widehat{\mathbb{Z}}_2$-module that is concentrated in Chow degree 0. By adjointing the above map trough the two adjunctions

$$\begin{tikzcd} {}_{\overline{MGL}_{*,*}\overline{MGL}}\mathbf{Comod} \arrow[r, shift left, "\text{res.}"] \arrow[r, phantom, "\perp"] & {}_{\overline{MGL}}\mathbf{Mod} \arrow[l, shift left, "\text{ext.}"] \arrow[r, shift left, "\text{res.}"] \arrow[r, phantom, "\perp"] & {}_{\widehat{\mathbb{Z}}_2}\mathbf{Mod}, \arrow[l, shift left, "\text{coext.}"] \end{tikzcd}$$

we get the monomorphism

$$M \hookrightarrow \Sigma^{|x|}\overline{MGL}_{*,*}\overline{MGL} \otimes_{\overline{MGL}_{*,*}} \mathrm{Hom}_{\widehat{\mathbb{Z}}_2}(\overline{MGL}_{*,*}, \prod_{x \in M \setminus 0} \widehat{\mathbb{Q}}_2/\widehat{\mathbb{Z}}_2) \tag{5.7}$$

of comodules. The target is injective since both forget functors (which are the left adjoints) preserve injective maps, i.e., monomorphisms. To see that the map (5.7) is still injective, observe that postcomposing it with the two counits recovers the injective map (5.6). □

Remark 5.23. We warn the reader that the underlying $\overline{MGL}_{*,*}$-module of

$$\Sigma^{|x|}\overline{MGL}_{*,*}\overline{MGL} \otimes_{\overline{MGL}_{*,*}} \mathrm{Hom}_{\widehat{\mathbb{Z}}_2}(\overline{MGL}_{*,*}, \prod_{x \in M \setminus 0} \widehat{\mathbb{Q}}_2/\widehat{\mathbb{Z}}_2)$$

is isomorphic to

$$\overline{MGL}_{*,*}\overline{MGL} \otimes_{\overline{MGL}_{*,*}} \prod_{x \in M\backslash 0} \Sigma^{|x|} \operatorname{Hom}_{\widehat{\mathbb{Z}}_2}(\overline{MGL}_{*,*}, \widehat{\mathbb{Q}}_2/\widehat{\mathbb{Z}}_2)$$

but not to

$$\prod_{x \in M\backslash 0} \Sigma^{|x|}\overline{MGL}_{*,*}\overline{MGL} \otimes_{\overline{MGL}_{*,*}} \operatorname{Hom}_{\widehat{\mathbb{Z}}_2}(\overline{MGL}_{*,*}, \widehat{\mathbb{Q}}_2/\widehat{\mathbb{Z}}_2),$$

since the forgetful functor from comodules to modules is not a right adjoint and does not commute with infinite products.

Having enough $C\tau$-modules at hand to handcraft injective resolutions in the category of $\overline{MGL}_{*,*}\overline{MGL}$-comodules, we can now construct an $\overline{MGL}$-based Adams-Novikov tower.

Proposition 5.24. *Any $Y \in {}_{C\tau}\boldsymbol{Cell}^b$ admits a tower*

$$\begin{tikzcd}
Y = Y_0 \arrow[d] & Y_1 \arrow[l] \arrow[d] & Y_2 \arrow[l] & \cdots \arrow[l] ,\\
I_0 & I_1 & &
\end{tikzcd}$$

in ${}_{C\tau}\boldsymbol{Cell}^b$ that we call an absolute Adams-Novikov tower for Y*, where*

(1) each map $Y_s \longrightarrow Y_{s-1}$ is zero in $\overline{MGL}$-homology,

(2) each cofiber $I_s \in {}_{C\tau}\boldsymbol{Cell}^{\heartsuit}$ is in the heart and satisfies the conditions of Lemma 5.21.

Moreover, any map $f\colon X \longrightarrow Y \in {}_{C\tau}\boldsymbol{Cell}^b$ can be lifted to a map of such towers.

Proof. Since $Y \in {}_{C\tau}\mathbf{Cell}^{\mathrm{b}}$, suppose that its homology $\overline{MGL}_{*,*}(Y)$ is concentrated in Chow degree $[a, b]$, that is

$$\overline{MGL}_{*,*}(Y) \cong \bigoplus_{k=a}^{b} \overline{MGL}_{2*+k,*}(Y).$$

For every $k \in [a, b]$, pick a monomorphism

$$\overline{MGL}_{2*+k,*}(Y) \cong \overline{MGL}_{2*,*}(\Sigma^{-k,0}Y) \hookrightarrow \overline{MGL}_{*,*}\overline{MGL} \otimes_{\overline{MGL}_{*,*}} N_{0,k}$$

by Lemma 5.22, for some injective module $N_{0,k}$. By Lemma 5.21, one can realize $N_{0,k}$ by some spectrum $I_{0,k} \in {}_{C\tau}\mathbf{Cell}^{\heartsuit}$ satisfying all the properties of Lemma 5.21. In particular, the algebraic map

$$\overline{MGL}_{2*,*}(\Sigma^{-k,0}Y) \hookrightarrow \overline{MGL}_{*,*}\overline{MGL} \otimes_{\overline{MGL}_{*,*}} N_{0,k} \cong \overline{MGL}_{*,*}(I_{0,k})$$

of comodules can be realized to a $C\tau$-linear map $\Sigma^{-k,0}Y \longrightarrow I_{0,k}$. Denote the product $I_0 \coloneqq \prod \Sigma^{k,0} I_{0,k}$, combine these maps into a unique map $Y \longrightarrow I_0$, and denote its fiber by Y_1 as in the diagram

$$\begin{tikzcd}
Y \arrow[d] & Y_1. \arrow[l] \\
I_0 &
\end{tikzcd}$$

Since the vertical map is an injection in $\overline{MGL}$-homology, the long exact sequence in $\overline{MGL}$-homology shows that $Y_1 \longrightarrow Y$ induces the zero map. This implies that Y_1 is concentrated in Chow degrees $[a-1, b-1]$, so one can just repeat the procedure, producing a tower

$$\begin{tikzcd}
Y \arrow[d] & Y_1 \arrow[l] \arrow[d] & Y_2 \arrow[l] & \cdots \arrow[l] \\
I_0 & I_1 & &
\end{tikzcd}$$

satisfying the desired properties.

To show the second claim, pick a map $f\colon X \longrightarrow Y \in {}_{C\tau}\mathbf{Cell}^{\mathrm{b}}$. We may assume that X and Y have $\overline{MGL}$-homology bounded above and below by the same bounds. Denote the first step of their tower by

$$\begin{tikzcd}
X \arrow[r, "f = f_0"] \arrow[d] & Y \arrow[d] \\
I_0 & J_0,
\end{tikzcd}$$

where I_0 and J_0 are the products constructed above. Applying $\overline{MGL}$-homology produces a diagram of comodules

$$\begin{tikzcd}
\overline{MGL}_{*,*}(X) \arrow[r, "f_0"] \arrow[d, hook] & \overline{MGL}_{*,*}(Y) \arrow[d] \\
\overline{MGL}_{*,*}(I_0) \arrow[r, dashed, "\exists\ \phi_0"] & \overline{MGL}_{*,*}(J_0),
\end{tikzcd}$$

which admits a filler ϕ_0 by the universal property of injective objects. Since J_0 satisfies the properties of Lemma 5.21, the map ϕ_0 is actually realized by a map $g_0 \colon I_0 \longrightarrow J_0$, such that the square

$$\begin{tikzcd}
X \arrow[r, "f = f_0"] \arrow[d] & Y \arrow[d] \\
I_0 \arrow[r, "g_0"] & J_0
\end{tikzcd}$$

commutes up to homotopy. This induces a filler f_1 as in

$$\begin{tikzcd}
X \arrow[d] \arrow[dd, bend right=60, "f_0"'] & X_1 \arrow[l] \arrow[d] \arrow[dd, dashed, bend right=40, "\exists f_1"'] & X_2 \arrow[l] & \cdots \arrow[l] \\
I_0 \arrow[dd, bend right=60, "g_0"'] & I_1 & & \\
Y \arrow[d] & Y_1 \arrow[l] \arrow[d] & Y_2 \arrow[l] & \cdots \arrow[l] \\
J_0 & J_1. & &
\end{tikzcd}$$

Iterating this process produces the desired map of towers. ☐

Every absolute injective tower gives raise to a $C\tau$-linear Adams-Novikov spectral sequence. In the following theorem, we show in addition that the spectral sequence does not depend on the resolution, converges strongly, and we identify its E_2 term as well as what it converges to.

Theorem 5.25. *For any* $X, Y \in {}_{C\tau}\boldsymbol{Cell}^b$ *there is a strongly convergent* $\overline{MGL}$*-based Adams-Novikov spectral sequence*

$$\mathrm{Ext}^{s,t,w}_{\overline{MGL}_{*,*}\overline{MGL}}\left(\overline{MGL}_{*,*}(X), \overline{MGL}_{*,*}(Y)\right) \Longrightarrow \left[\Sigma^{t-s,w}X, Y^{\wedge}_{\overline{MGL}}\right]_{C\tau},$$

with differential

$$d_r \colon E_r^{s,t,w} \longrightarrow E_r^{s+r,t+(r-1),w},$$

where $Y^{\wedge}_{\overline{MGL}}$ *is the* $\overline{MGL}$*-completion of* Y*. Moreover, this spectral sequence collapses at a finite page.*

Proof. Pick an absolute injective resolution of Y by Proposition 5.24

$$\begin{tikzcd}
Y \arrow[d] & Y_1 \arrow[l] \arrow[d] & Y_2 \arrow[l] \arrow[d] & Y_3 \arrow[l] & \cdots \arrow[l] \\
I_0 \arrow[r, dotted, "\circ" marking, "d_1"'] & I_1 \arrow[r, dotted, "\circ" marking, "d_1"'] & I_2.
\end{tikzcd} \tag{5.8}$$

By the construction of the tower, the cochain complex

$$\overline{MGL}_{*,*}(I_0) \longrightarrow \overline{MGL}_{*,*}(\Sigma^{1,0}I_1) \longrightarrow \cdots \tag{5.9}$$

is an (absolute) injective resolution of the comodule $\overline{MGL}_{*,*}(Y)$. Applying the functor $[X, -]_{C\tau}$ to the tower (5.8) gives an exact couple that gives the desired Adams-Novikov spectral sequence. We grade the E_1-page as

$$E_1^{s,t,w} := [\Sigma^{t,w} X, \Sigma^{s,0} I_s],$$

and the d_1 differential in the tower (5.8) gives the cochain complex

$$[X, I_0]_{C\tau} \xrightarrow{d_1} [X, \Sigma^{1,0} I_1]_{C\tau} \xrightarrow{d_1} [X, \Sigma^{2,0} I_2]_{C\tau} \longrightarrow \cdots .$$

The cohomology of this cochain complex gives the E_2-page, which we now identify. Since I_i satisfy the properties of Lemma 5.21, the terms in this cochain complex can be identified with the terms in the complex obtained from applying $\mathrm{Hom}_{\overline{MGL_{*,*}MGL}}(\overline{MGL}_{*,*}(X), -)$ to the injective resolution of (5.9). The differentials agree as well by standard methods [47, Chapter 2], so the E_2-page is given by

$$E_2^{s,t,w} \cong \mathrm{Ext}^{s,t,w}_{\overline{MGL_{*,*}MGL}}(\overline{MGL}_{*,*}(X), \overline{MGL}_{*,*}(Y)).$$

We will now show that the spectral sequence is strongly convergent under the hypotheses that X and Y have bounded Chow degree. This is very similar to the argument given in Theorem 5.8. Suppose that X has Chow degree bounded by $[a, b]$, and Y has Chow degree bounded by $[c, d]$. Recall that the E_1-page is given by

$$E_1^{s,t,w} \cong [\Sigma^{t,w} X, \Sigma^{s,0} I_s]_{C\tau} \cong \mathrm{Hom}_{\overline{MGL_{*,*}MGL}}(\overline{MGL}_{*,*}(\Sigma^{t,w} X), \overline{MGL}_{*,*}(\Sigma^{s,0} I_s)),$$

where $\Sigma^{t,w} X$ is concentrated in Chow degree $[a + t - 2w, b + t - 2w]$. From the construction of the tower (5.8) in Proposition 5.24, it follows that $\Sigma^{s,0} I_s$ has Chow degree bounded by $[c, d]$. In particular, if $E_r^{s,t,w} \neq 0$, then we must have

$$t \in [c - b + 2w, d - a + 2w].$$

By chasing trough the tower, observe that the d_r differential has the form

$$d_r \colon E_r^{s,t,w} \longrightarrow E_r^{s+r,t+(r-1),w}.$$

In particular it increases t by $r-1$, and thus it is not possible for both $E_r^{s,t,w}$ and $E_r^{s+r,t+(r-1),w}$ to be non-zero when $r-1 > d-a+2w-(c-b+2w)$. This implies that the spectral sequence collapses when $r > d-c+b-a+1$, i.e., at the $E_{d-c+b-a+2}$-page.

We will now show that the spectral sequence does not depend (from E_2 onward) on the absolute injective resolution. Consider two such resolutions $\{Y_s, I_s\}$ and $\{\tilde{Y}_s, \tilde{I}_s\}$ for Y. The identity map $\mathrm{id}\colon Y \longrightarrow Y$ produces by Proposition 5.24 a map of towers, and in particular compatible maps $g_s\colon I_s \longrightarrow \tilde{I}_s$. These maps induce a lift of the identity map between the two injective resolutions as illustrated in the diagram

$$\begin{tikzcd}
0 \arrow[r] & \overline{MGL}_{*,*}(Y) \arrow[r] \arrow[d, "\mathrm{id}"'] & \overline{MGL}_{*,*}(I_0) \arrow[r] \arrow[d, dashed, "\overline{MGL}_{*,*}(g_1)"] & \overline{MGL}_{*,*}(I_1) \arrow[r] \arrow[d, dashed, "\overline{MGL}_{*,*}(g_2)"] & \cdots \\
0 \arrow[r] & \overline{MGL}_{*,*}(Y) \arrow[r] & \overline{MGL}_{*,*}(\tilde{I}_0) \arrow[r] & \overline{MGL}_{*,*}(\tilde{I}_1) \arrow[r] & \cdots .
\end{tikzcd}$$

The maps $\overline{MGL}_{*,*}(g_s)$ are thus unique up to homotopy by the Fundamental Theorem of Homological Algebra. It follows that they induce an isomorphism on the E_2-page, and thus an isomorphism of spectral sequences by, for example, [7, Theorem 5.3]. We refer to [47, Section 2.2] for more details.

It remains to identify what is computed by the E_∞-page. By abstract non-sense, the E_∞-page is the associated graded on some filtration on the set of homotopy classes of maps $[X, \widehat{Y}]_{C\tau}$, for some $C\tau$-module $\widehat{Y}$. We will show that $\widehat{Y}$ has the same homotopy groups as the $\overline{MGL}$-completion of Y (in the sense of Bousfield's [9]), and this will be enough to identify $[X, \widehat{Y}]_{C\tau} \cong [X, Y^{\wedge}_{\overline{MGL}}]_{C\tau}$ since X is cellular. To show this, set $X = C\tau$. Since $\overline{MGL}_{*,*}(C\tau) \cong \overline{MGL}_{*,*}$ is free as an $\overline{MGL}_{*,*}$-module, we can use the canonical cobar resolution [47, Definition 2.2.10] for Y in this case. One can apply the same procedure as we did above to get a map from the canonical cobar resolution to any tower of Y, which extends the identity map on Y. This gives a homomorphism from the usual ($C\tau$-linear) motivic Adams-Novikov spectral sequence for Y to our Adams-Novikov spectral sequence. Since this map extends the identity map on Y, one can argue as above to show that this map induces an isomorphism of spectral sequences and thus a weak equivalence $\widehat{Y} \xrightarrow{\simeq} Y^{\wedge}_{\overline{MGL}}$. Since any cellular $C\tau$-module X can be written in terms of filtered colimits and cofibers of cells $C\tau$, there is an isomorphism

$$[X, Y^{\wedge}_{\overline{MGL}}]_{C\tau} \xrightarrow{\cong} [X, \widehat{Y}]_{C\tau},$$

showing that our Adams-Novikov spectral sequence computes $[X, Y^{\wedge}_{\overline{MGL}}]_{C\tau}$. □

Recall that we needed this motivic Adams-Novikov spectral sequence in order to compute maps $[X, Y]_{C\tau}$ and not merely maps $[X, Y^{\wedge}_{\overline{MGL}}]_{C\tau}$ into a completion of Y. We thus have to restrict to those $C\tau$-modules that are already complete with respect to $\overline{MGL}$. In fact, since X is cellular, it is sufficient to restrict to those $C\tau$-modules whose completion map induces an isomorphism on homotopy groups.

Definition 5.26 ($\pi_{*,*}$-complete)**.** A $C\tau$-module Y is called *$\pi_{*,*}$-complete (with respect to $\overline{MGL}$)* if the natural map $Y \longrightarrow Y^{\wedge}_{\overline{MGL}}$ is a weak equivalence, i.e., a $\pi_{*,*}$-isomorphism. Denote by ${}_{C\tau}\mathbf{Cell}^{\mathrm{comp}}$ the full ∞-subcategory of ${}_{C\tau}\mathbf{Cell}^{\mathrm{b}}$ spanned by $\pi_{*,*}$-complete objects.

Theorem 5.25 immediately implies that the homotopy groups of $Y \in {}_{C\tau}\mathbf{Cell}^{\mathrm{comp}}$ can be computed via the motivic Adams-Novikov spectral sequence. By passing to filtered colimits in the first variable we get the following Corollary.

Corollary 5.27. *For any $X, Y \in {}_{C\tau}\mathbf{Cell}^{comp}$, there is a strongly convergent motivic Adams-Novikov spectral sequence*

$$\mathrm{Ext}^{s,t,w}_{\overline{MGL}_{*,*}\overline{MGL}}\left(\overline{MGL}_{*,*}(X), \overline{MGL}_{*,*}(Y)\right) \Longrightarrow \left[\Sigma^{t-s,w} X, Y\right]_{C\tau}.$$

Moreover, this spectral sequence collapses at a finite page.

The following Corollary is easily deduced from the proof of Theorem 5.25 and from Corollary 5.27, and will be of great use in what follows.

Corollary 5.28. *If* $X, Y \in {}_{C\tau}\boldsymbol{Cell}$ *are each concentrated in a single Chow degree, and* Y *is* $\pi_{*,*}$*-complete, then there is an isomorphism*

$$[\Sigma^{t,w} X, Y]_{C\tau} \cong \mathrm{Ext}^{2w-t,t,w}_{\overline{MGL}_{*,*}\overline{MGL}}(\overline{MGL}_{*,*}(\Sigma^{t,w} X), \overline{MGL}_{*,*}(Y))$$

for any bidegree (t, w)*.*

Proof. We will show that the motivic Adams-Novikov spectral sequence collapses, by using similar methods to those of Corollary 5.9, with minor changes in indexing. Since both X and Y are concentrated in a single Chow degree, the end of the proof of Theorem 5.25, together with the fact that Y is $\pi_{*,*}$-complete shows that the spectral sequence

$$\mathrm{Ext}^{s,t,w}_{\overline{MGL}_{*,*}\overline{MGL}}\left(\overline{MGL}_{*,*}(X), \overline{MGL}_{*,*}(Y)\right) \Longrightarrow \left[\Sigma^{t-s,w} X, Y\right]_{C\tau}$$

is concentrated in $t = 2w$ and collapses at the E_2-page. By re-indexing, and since no hidden extensions are possible for degree reasons, we get the desired isomorphisms

$$\mathrm{Ext}^{2w-t,t,w}_{\overline{MGL}_{*,*}\overline{MGL}}(\overline{MGL}_{*,*}(X), \overline{MGL}_{*,*}(Y)) \cong \left[\Sigma^{t,w} X, Y\right]_{C\tau}.$$

. ☐

Before moving on to endow ${}_{C\tau}\mathbf{Cell}^{\mathrm{comp}}$ with its Chow t-structure, we prove a Lemma about $\pi_{*,*}$-complete $C\tau$-modules that we will need for identifying the heart.

Lemma 5.29. *Let* Y_α *be a filtered system in* ${}_{C\tau}\boldsymbol{Cell}^{\heartsuit}$ *such that each* Y_α *is* $\pi_{*,*}$*-complete. Then* $\operatorname{hocolim} Y_\alpha$ *is also* $\pi_{*,*}$*-complete.*

Proof. Consider the motivic Adams-Novikov of Theorem 5.25

$$\operatorname{Ext}^{s,t,w}_{\overline{MGL}_{*,*}\overline{MGL}}\left(\overline{MGL}_{*,*}(C\tau), \overline{MGL}_{*,*}(Y)\right) \Longrightarrow \left[\Sigma^{t-s,w}C\tau, Y^{\wedge}_{\overline{MGL}}\right]_{C\tau} \cong \pi_{t-s,w}(Y^{\wedge}_{\overline{MGL}})$$

for $Y :=$ hocolim Y_α and $X = C\tau$. Since both $C\tau$ and Y are in the heart, the spectral sequence is concentrated in $t = 2w$ and collapses at E_2 with no possible hidden extensions by Corollary 5.28. This gives isomorphisms

$$\operatorname{Ext}^{s,2w,w}_{\overline{MGL}_{*,*}\overline{MGL}}\left(\overline{MGL}_{*,*}(C\tau), \overline{MGL}_{*,*}(Y)\right) \cong \pi_{2w-s,w}(Y^{\wedge}_{\overline{MGL}}) \tag{5.10}$$

for any bidegrees. Since $\overline{MGL}_{*,*}(C\tau) \cong \overline{MGL}_{*,*}$ is free over $\overline{MGL}_{*,*}$, one can use the cobar complex [47, Corollary A1.2.12] (or [13], [23]) to set-up this motivic Adams-Novikov spectral sequence. Since the cobar complex is functorial, the Fundamental Theorem of Homological Algebra shows that the isomorphism

$$\operatorname{colim} \overline{MGL}_{*,*}(Y_\alpha) \xrightarrow{\cong} \overline{MGL}_{*,*}(Y)$$

induces an isomorphism

$$\operatorname{colim} \operatorname{Ext}_{\overline{MGL}_{*,*}\overline{MGL}}\left(\overline{MGL}_{*,*}, \overline{MGL}_{*,*}(Y_\alpha)\right) \xrightarrow{\cong} \operatorname{Ext}_{\overline{MGL}_{*,*}\overline{MGL}}\left(\overline{MGL}_{*,*}, \overline{MGL}_{*,*}(Y)\right). \tag{5.11}$$

Since each Y_α is concentrated in Chow degree 0 and is $\pi_{*,*}$-complete, Corollary 5.28 gives isomorphisms

$$\operatorname{Ext}^{s,2w,w}_{\overline{MGL}_{*,*}\overline{MGL}}\left(\overline{MGL}_{*,*}(C\tau), \overline{MGL}_{*,*}(Y_\alpha)\right) \cong \pi_{2w-s,w}(Y_\alpha) \tag{5.12}$$

for any bidegree and any α. By combining equations (5.10), (5.11), (5.12) with the fact that homotopy groups commute with filtered colimits, we get the desired isomorphism

$$\pi_{2w-s,w}(Y) \xrightarrow{\cong} \pi_{2w-s,w}(Y^{\wedge}_{\overline{MGL}})$$

for every bidegree. □

Remark 5.30. Lemma 5.29 can be generalized to the case where there is a uniform bound $[a, b]$ independent of α, such that each Y_α is concentrated in Chow degree $[a, b]$. In this case, more care has to be taken for ruling out hidden extensions.

5.3.3 Proof of the t-structure, the heart and the equivalence

This section mimics Section 5.2.2. Some proofs however require different techniques, such as Landweber's Filtration Theorem [29] [30]. The goal is to endow ${}_{C\tau}\mathbf{Cell}^{\mathrm{comp}}$ with a t-structure, identify its heart, and conclude by identifying the subcategory of t-bounded objects with an algebraic category. As in Section 5.2.2, once the t-structure is in place, the main result follows by a straightforward application of Proposition 5.2.

Definition 5.31 (Chow degree in ${}_{C\tau}\mathbf{Cell}^{\mathrm{comp}}$)**.** We say that an object $X \in {}_{C\tau}\mathbf{Cell}^{\mathrm{comp}}$ is concentrated in *Chow degrees* $[a, b]$, if it is so when seen in ${}_{C\tau}\mathbf{Cell}^{\mathrm{b}}$, i.e., if its homology $\overline{MGL}_{*,*}(X)$ is concentrated in Chow degrees $[a, b]$. As in Notation 5.5, consider the subcategories ${}_{C\tau}\mathbf{Cell}^{\mathrm{comp},\heartsuit}$, ${}_{C\tau}\mathbf{Cell}^{\mathrm{comp}}_{\geq 0}$, ${}_{C\tau}\mathbf{Cell}^{\mathrm{comp}}_{\leq 0}$, etc.

These categories sit in a commutative diagram

$$\begin{tikzcd}
{}_{C\tau}\mathbf{Cell}^{\mathrm{comp}} \arrow[r, "\overline{MGL}_{*,*}"] & {}_{\overline{MGL}_{*,*}\overline{MGL}}\mathbf{Comod} \\
{}_{C\tau}\mathbf{Cell}^{\mathrm{comp},\heartsuit} \arrow[r, "\overline{MGL}_{*,*}"] \arrow[u, hook, "\text{inc.}"] & {}_{\overline{MGL}_{*,*}\overline{MGL}}\mathbf{Comod}^{\mathrm{ev}}. \arrow[u, hook, "\text{inc.}"']
\end{tikzcd}$$

We will first show that the bottom horizontal arrow induces an equivalence

$$\overline{MGL}_{*,*} \colon h\left({}_{C\tau}\mathbf{Cell}^{\mathrm{comp},\heartsuit}\right) \xrightarrow{\cong} {}_{\overline{MGL}_{*,*}\overline{MGL}}\mathbf{Comod}^{\mathrm{ev}}.$$

As in Section 5.2.2, faithfullness is an easy Corollary of the construction of the spectral sequence.

Corollary 5.32. *The functor*

$$\pi_{*,*}\colon h\left({}_{C\tau}\boldsymbol{Cell}^{comp,\heartsuit}\right) \longrightarrow {}_{\overline{MGL}_{*,*}\overline{MGL}}\boldsymbol{Comod}^{ev}$$

is fully faithful.

Proof. Pick two objects $X, Y \in {}_{C\tau}\mathbf{Cell}^{\mathrm{comp},\heartsuit}$. Since they are both concentrated in Chow degree 0 and Y is $\pi_{*,*}$-complete, Corollary 5.28 gives isomorphisms

$$\mathrm{Ext}^{2w-t,t,w}_{\overline{MGL}_{*,*}\overline{MGL}}(\overline{MGL}_{*,*}(X), \overline{MGL}_{*,*}(Y)) \cong \left[\Sigma^{t,w}X, Y\right]_{C\tau}$$

for any bidegree (t, w). The desired isomorphism follows by letting $t = w = 0$, which moreover is realized by the edge homomorphism

$$\overline{MGL}_{*,*}\colon\ [X, Y]_{C\tau} \xrightarrow{\cong} \mathrm{Hom}_{\overline{MGL}_{*,*}\overline{MGL}}\left(\overline{MGL}_{*,*}(X), \overline{MGL}_{*,*}(Y)\right).$$

□

We will now show the essential surjectivity of this functor, proving the equivalence of categories. This result is proven in a different way than its analogous Proposition 5.10, since we do not have free resolutions available. We will instead use Landweber's Filtration Theorem to realize all finitely presented comodules, and extend the result by filtered colimits. We start with the following Lemma.

Lemma 5.33. *Consider a short exact sequence*

$$0 \longrightarrow M' \xrightarrow{f'} M \xrightarrow{f''} M'' \longrightarrow 0 \tag{5.13}$$

in ${}_{\overline{MGL}_{*,*}\overline{MGL}}\boldsymbol{Comod}^{ev}$*. If two of the three comodules are realizable in* ${}_{C\tau}\boldsymbol{Cell}^{comp,\heartsuit}$*, then so* *is the third.*

Proof. Suppose first that $M' \cong \overline{MGL}_{*,*}(X')$ and $M \cong \overline{MGL}_{*,*}(X)$ are realizable. The algebraic map f' is also realizable to a homotopy class of maps by Corollary 5.32. Since ${}_{C\tau}\mathbf{Cell}^{\mathrm{comp},\heartsuit}$ is closed under fibers and cofibers, the cofiber of any representative realizes the comodule M''. The case where M and M'' is similar by taking the fiber of any representative.

Finally suppose that $M' \cong \overline{MGL}_{*,*}(X')$ and $M'' \cong \overline{MGL}_{*,*}(X'')$ are realizable. The short exact sequence (5.13) corresponds to an element in $\mathrm{Ext}^{1,0,0}_{\overline{MGL}_{*,*}\overline{MGL}}(M'', M')$, which can be realized by a homotopy class of maps $\Sigma^{-1,0}X'' \longrightarrow X'$ by Corollary 5.28. The cofiber of this map realizes M. □

Proposition 5.34. *The functor*

$$\overline{MGL}_{*,*}\colon h\left({}_{C\tau}\boldsymbol{Cell}^{comp,\heartsuit}\right) \xrightarrow{\cong} {}_{\overline{MGL}_{*,*}\overline{MGL}}\boldsymbol{Comod}^{ev}$$

is an equivalence of categories.

Proof. Recall from [29] [30] that there are elements $v_n \in \widehat{MU}_*$ with $v_0 = 2$, giving the invariant prime ideals $I_n = (v_0, \ldots, v_n) \trianglelefteq \widehat{MU}_*$. Moreover, these elements satisfy the formula

$$\eta_R(v_n) \equiv v_n \mod I_{n-1},$$

and so $\widehat{MU}_*/I_n$ is canonically an $MU_*\widehat{MU}$-comodule. This gives short exact sequences

$$0 \longrightarrow \widehat{MU}_*/I_n \xrightarrow{\cdot v_n} \widehat{MU}_*/I_n \longrightarrow \widehat{MU}_*/I_{n+1} \longrightarrow 0$$

of comodules for every $n \in \mathbb{N}_0$. Landweber's Filtration Theorem [29] [30] states that any comodule $M \in {}_{MU_*\widehat{MU}}\mathbf{Comod}$ whose underlying $\widehat{MU}_*$-module is finitely presented, can be reconstructed by finitely many extensions of suspensions of MU/I_{n_i}.

All these results transfer to the category ${}_{\overline{MGL}_{*,*}\overline{MGL}}\mathbf{Comod}^{\mathrm{ev}}$ via the equivalences

$$ {}_{\overline{MGL}_{*,*}\overline{MGL}}\mathbf{Comod}^{\mathrm{ev}} \cong {}_{MU_*\widehat{MU}}\mathbf{Comod}^{\mathrm{ev}} \cong {}_{MU_*\widehat{MU}}\mathbf{Comod}. $$

In particular, there are invariant prime ideals $I_n \trianglelefteq \overline{MGL}_{*,*}$ and short exact sequences

$$ 0 \longrightarrow \overline{MGL}_{*,*}/I_n \xrightarrow{\cdot v_n} \overline{MGL}_{*,*}/I_n \longrightarrow \overline{MGL}_{*,*}/I_{n+1} \longrightarrow 0 $$

of comodules, where the quotients $\overline{MGL}_{*,*}/I_n$ are naturally $\overline{MGL}_{*,*}\overline{MGL}$-comodules. Moreover, any comodule $M \in {}_{\overline{MGL}_{*,*}\overline{MGL}}\mathbf{Comod}^{\mathrm{ev}}$ in Chow degree 0 and whose underlying $\overline{MGL}_{*,*}$-module is finitely presented, can be constructed in finitely many extensions from suspensions of various $\overline{MGL}/I_{n_i}$. Finally, by mimicking the proof of [36, Lemma 2.11], or by [21, Chapter 1], any $\overline{MGL}_{*,*}\overline{MGL}$-comodule in Chow degree 0 can be written as a filtered colimit of finitely presented ones.

Pick now an arbitrary $M \in {}_{\overline{MGL}_{*,*}\overline{MGL}}\mathbf{Comod}^{\mathrm{ev}}$. By the above discussion, this can be written as a filtered colimit $M \cong \operatorname{colim} M_\alpha$, where each $M_\alpha \in {}_{\overline{MGL}_{*,*}\overline{MGL}}\mathbf{Comod}^{\mathrm{ev}}$ is finitely presented as an $\overline{MGL}_{*,*}$-module. Since $\overline{MGL}_{*,*}(C\tau) \cong \overline{MGL}_{*,*}$ is realizable, Lemma 5.33 shows that every quotient $\overline{MGL}/I_n$ can be realized in ${}_{C\tau}\mathbf{Cell}^{\mathrm{comp},\heartsuit}$. Another application of Lemma 5.33 shows that one can realize $M_\alpha \cong \overline{MGL}_{*,*}(X_\alpha)$ by $X_\alpha \in {}_{C\tau}\mathbf{Cell}^{\mathrm{comp},\heartsuit}$. By Corollary 5.32, one can realize the whole filtered system $\{M_\alpha\}$ to a filtered system $\{X_\alpha\}$. The colimit $X := \operatorname{hocolim} X_\alpha$ realizes M since $\overline{MGL}_{*,*}$ commutes with filtered colimits. Moreover, since the X_α are all concentrated in Chow degree 0, Lemma 5.29 shows that $X \in {}_{C\tau}\mathbf{Cell}^{\mathrm{comp},\heartsuit}$. This shows that the functor $\overline{MGL}_{*,*}(-)$ is essentially surjective, and that it is an equivalence of categories by combining this with Corollary 5.32. □

We will now show that the Chow degree induces a t-structure on the category ${}_{C\tau}\mathbf{Cell}^{\mathrm{comp}}$. The following Lemma is the analogue of Lemma 5.11.

Lemma 5.35. *Given $X \in {}_{C\tau}\boldsymbol{Cell}^{comp}_{\geq 0}$ and $Y \in {}_{C\tau}\boldsymbol{Cell}^{comp}_{\leq 0}$, the group of homotopy classes* *of degree* $(0,0)$ *can be computed algebraically by*

$$\mathrm{Hom}_{\overline{MGL}_{*,*}\overline{MGL}}(\overline{MGL}_{*,*}(X), \overline{MGL}_{*,*}(Y)) \cong [X,Y]_{C\tau}.$$

Proof. The proof of Theorem 5.25 shows that for X in non-negative degrees and Y in non-positive degrees, the E_2-page of the motivic Adams-Novikov spectral sequence

$$E_2^{s,t,w} = \mathrm{Ext}^{s,t,w}_{\overline{MGL}_{*,*}\overline{MGL}}(\overline{MGL}_{*,*}(X), \overline{MGL}_{*,*}(Y))$$

is concentrated in degrees $2t-w \leq 0$ and $s \geq 0$. We are interested in understanding $[X,Y]_{C\tau}$, which is assembled from $E_\infty^{s,t,w}$ with $t-s=0$ and $w=0$. These conditions imply that $w=0$ and that $s=t\geq 0$, and thus that $[X,Y]_{C\tau}$ is a (possibly iterated) subquotient of

$$E_2^{0,0,0} \cong \mathrm{Hom}_{\overline{MGL}_{*,*}\overline{MGL}}(\overline{MGL}_{*,*}(X), \overline{MGL}_{*,*}(Y)).$$

Recall that the d_r-differential has the form $E_r^{s,t,w} \xrightarrow{d_r} E_r^{s+r,t+r-1,w}$. There are thus no possible differentials entering $E_r^{0,0,0}$ since the homological degree has to be non-negative, and the only possible differential exiting $E_r^{0,0,0}$ is a d_1 since $t-2w \leq 0$ implies that we must have $r-1 \leq 0$. However, we are already at the E_2-page and thus $E_2^{0,0,0}$ survives to the E_∞-page giving the isomorphism

$$\mathrm{Hom}_{\overline{MGL}_{*,*}\overline{MGL}}(\overline{MGL}_{*,*}(X), \overline{MGL}_{*,*}(Y)) \cong [X,Y]_{C\tau}.$$

□

We can finally show that the category ${}_{C\tau}\mathbf{Cell}^{\mathrm{comp}}$ admits a t-structure.

Theorem 5.36 (t-structure on ${}_{C\tau}\mathbf{Cell}^{\mathrm{comp}}$)**.** *The pair of subcategories* $({}_{C\tau}\boldsymbol{Cell}^{comp}_{\geq 0}, {}_{C\tau}\boldsymbol{Cell}^{comp}_{\leq 0})$ *defines a* t*-structure on* ${}_{C\tau}\boldsymbol{Cell}^{comp}$*, whose heart is given by the equivalence*

$$\overline{MGL}_{*,*}\colon h\left({}_{C\tau}\boldsymbol{Cell}^{comp,\heartsuit}\right) \xrightarrow{\cong} {}_{\overline{MGL}_{*,*}\overline{MGL}}\boldsymbol{Comod}^{ev}.$$

Proof. There are three axioms to show for the t-structure, and we have already identified the heart in Proposition 5.34. The proof is very similar to that of Proposition 5.12, which we refer to for more details.

For the first axiom, consider objects $X \in {}_{C\tau}\mathbf{Cell}^{\text{comp}}_{\geq 0}$ and $Y \in {}_{C\tau}\mathbf{Cell}^{\text{comp}}_{<0}$. One shows that $[X, Y]_{C\tau} = 0$ as in Proposition 5.12 by using Lemma 5.35 instead of Lemma 5.11.

The second axiom requires that if $X \in {}_{C\tau}\mathbf{Cell}^{\text{comp}}_{\geq 0}$, then $\Sigma^{1,0}X \in {}_{C\tau}\mathbf{Cell}^{\text{comp}}_{\geq 1}$. This is clear from the definition of these subcategories.

Finally, for any spectrum $X \in {}_{C\tau}\mathbf{Cell}^{\text{comp}}$, the last axiom asks for a fiber sequence

$$X_{\geq 0} \longrightarrow X \longrightarrow X_{<0}$$

with $X_{\geq 0} \in {}_{C\tau}\mathbf{Cell}^{\text{comp}}_{\geq 0}$ and $X_{<0} \in {}_{C\tau}\mathbf{Cell}^{\text{comp}}_{<0}$. This is again shown as in Proposition 5.12, by using Proposition 5.34 and Lemma 5.35 instead of Proposition 5.10 and Lemma 5.11. □

We can now state and prove the main result of this chapter.

Theorem 5.37. *There is a t-exact equivalence of ∞-categories*

$$\mathcal{D}^b({}_{\overline{MGL}_{*,*}\overline{MGL}}\boldsymbol{Comod}^{ev}) \xrightarrow{\cong} {}_{C\tau}\boldsymbol{Cell}^{comp}.$$

Proof. The proof is very similar to the proof Theorem 5.13. We need to check the conditions of Proposition 5.2 for the t-structure on ${}_{C\tau}\mathbf{Cell}^{\text{comp}}$. It is clearly right-bounded (even bounded), and its heart has enough injectives by Proposition 5.34 since comodule categories have enough injectives. It remains to show that for objects $X, Y \in {}_{C\tau}\mathbf{Cell}^{\text{comp},\heartsuit}$ with $\overline{MGL}_{*,*}(Y)$ an injective $\overline{MGL}_{*,*}\overline{MGL}$-comodule, there are no non-trivial maps in $[X, \Sigma^{i,0}Y]_{C\tau}$ for any $i > 0$. We will use the motivic Adams-Novikov spectral sequence of Corollary 5.27

$$\mathrm{Ext}^{s,t,w}_{\overline{MGL}_{*,*}\overline{MGL}}\left(\overline{MGL}_{*,*}(X), \overline{MGL}_{*,*}(Y)\right) \Longrightarrow \left[\Sigma^{t-s,w}X, Y\right]_{C\tau},$$

and we are interested in the case $t - s = -i$ and $w = 0$. Since $\overline{MGL}_{*,*}(Y)$ is an injective $\overline{MGL}_{*,*}\overline{MGL}$-comodule, the spectral sequence collapses at E_2 and is concentrated on the line $s = 0$, and thus $t = -i$. Moreover, since both X and Y are concentrated in Chow degree 0, it is clear from the proof of Theorem 5.25 that the E_2-page is concentrated in degrees $t - 2w = 0$, and thus $t = 0$. It follows that in weight 0, there are only maps of bidegree $(0, 0)$, and thus $[X, \Sigma^{i,0}Y]_{C\tau} = 0$ for $i > 0$. □

Corollary 5.38. *There is a t-exact equivalence of ∞-categories*

$$\mathcal{D}^b({}_{MU_*\widehat{MU}}\boldsymbol{Comod}) \xrightarrow{\cong} {}_{C\tau}\boldsymbol{Cell}^{comp},$$

where $\widehat{MU}$ denotes the 2-completion of the complex cobordism spectrum MU.

Proof. This follows from the equivalence of abelian categories

$$ {}_{\overline{MGL}_{*,*}\overline{MGL}}\mathbf{Comod}^{\mathrm{ev}} \cong {}_{MU_*\widehat{MU}}\mathbf{Comod}^{\mathrm{ev}} \cong {}_{MU_*\widehat{MU}}\mathbf{Comod}.$$

□

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ABSTRACT

THE MOTIVIC COFIBER OF τ AND EXOTIC PERIODICITIES

by

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August 2017

Advisor: Dr. Daniel C. Isaksen
Major: Mathematics
Degree: Doctor of Philosophy

Consider the Tate twist $\tau \in H^{0,1}(S^{0,0})$ in the mod 2 cohomology of the motivic sphere. After 2-completion, the motivic Adams spectral sequence realizes this element as a map $\tau \colon S^{0,-1} \longrightarrow S^{0,0}$. This thesis begins with the study of its cofiber, that we denote by $C\tau$.

We first show that this motivic 2-cell complex can be endowed with a unique E_∞ ring structure. This promotes the known isomorphism $\pi_{*,*}C\tau \cong \mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, BP_*)$ to an isomorphism of rings which also preserves higher products.

This structure allows us to consider its closed symmetric monoidal category of modules $({}_{C\tau}\mathbf{Mod}, - \wedge_{C\tau} -)$, which happens to live in the kernel of Betti realization. This category has surprising applications, and moreover contains many interesting motivic spectra. In particular, we construct exotic motivic fields $K(w_n)$, detecting motivic w_n-periodicity. This theory of motivic w_n-periodicity can be roughly seen as perpendicular to the v_n-periodicity story, detected by the motivic Morava K-theories $K(n)$.

Finally, we also explain why the category ${}_{C\tau}\mathbf{Mod}$ is so computable. The above isomorphism comes in a more structured version. In work that is joint with Zhouli Xu and Guozhen Wang, we show that there is an equivalence of ∞-categories

$$\mathcal{D}^b({}_{\overline{MGL}_{*,*}\overline{MGL}}\mathbf{Comod}^{\mathrm{ev}}) \xrightarrow{\cong} {}_{C\tau}\mathbf{Cell}^{\mathrm{comp}}$$

between an algebraic category, and the subcategory ${}_{C\tau}\mathbf{Cell}^{\mathrm{comp}}$ of cellular $C\tau$-modules that are complete with respect to a version of the algebraic cobordism spectrum MGL.

AUTOBIOGRAPHICAL STATEMENT

BOGDAN GHEORGHE

Bogdan Gheorghe was born in Bucharest, Romania in 1989. His parents both graduated from the Polytechnical School in Bucharest, and his dad continued to work on a Ph.D. in electronical studies. In 1994, the whole family moved to Switzerland where Bogdan grew up. He obtained his B.S. in Mathematics in 2011 from the Ecole Polytechnique Federale de Lausanne (EPFL), and his M.A. in 2012. Under the recommendation of his master's advisor Kathryn Hess Bellwald, he applied to Wayne State University (WSU) to work on a Ph.D. with Dan Isaksen, and got accepted to start in 2012. His time at WSU was incredible, as the topology department is very friendly and energetic, with excellent professors. After his thesis defense in July 2017, he married his beautiful wife Christa the 5th of August in the north of Michigan. Bogdan accepted a Research Assistant position at the Max Planck Institute for Mathematics (MPIM) in Bonn, Germany, where he will move with his wife at the end of August. Outside of academia he enjoys many sports, brewing and drinking beer, and day-trading.