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The Estimation of Missing Values in Rectangular Lattice Designs

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Algebraic expressions for estimating missing data when one or more observation(s) are missing in Rectangular lattice designs with repetition were derived using the method of minimizing the residual sum of squares. Results showed that the estimated value(s) were significantly approximate to that of the actual value(s).

Keywords: Simple Rectangular lattice design, Triple Rectangular lattice design, repetition, missing data, actual value and estimated value

Introduction

In performing designed experiments, there is always a chance that some of the responses are not observed or where some of the recordings for certain plots are missing, Little and Rubin (2002). Such missing responses or recordings from the experiment are usually referred to as missing data or missing values. The existence of missing data has always been a problem of interest, because the occurrence of missing data tends to destroy the orthogonal or balanced structure usually present in a designed experiment.

In experimental work, it frequently happens that one or more experimental units are missing from the data or have to be rejected because of the conditions outside the control of the experimenter. Data values may be absent for various reasons, for example, the inability to measure certain attributes. In such cases, the most popular and simple method of handling missing data is to ignore either the projects or the attributes with missing observations, Azadeh et al. (2013). This technique causes the loss of valuable information and therefore may lead to inaccurate cost estimation models.

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Gad and Ahmed (2006) claimed that ignoring the missing values in this case leads to biased inferences. The problem of handling missing data has been treated adequately in various real world data sets. Several statistical methods for estimating the yield of a missing unit(s) in the field experimental work have been developed and applied by researchers. Allan and Wishart (1930) derived formulas for a single missing plot in a randomized block and in a latin square experiment. Cornish (1943) derived formulae for intra-block estimates of missing values in quasi-factorial design. These estimation methods were expanded by Yates (1933) to cover several missing units in a given experiment which has lead several other researchers to have attempted to obtain formulas for estimating missing data by the least-squares procedures in the analysis of variance (ANOVA) model with no attempt made on the estimation of missing values in rectangular lattice designs.

If more than one observation is missing, the iterative method is used to obtain the missing yields, Kempthorne (1951). With this iterative procedure, initial estimates are assigned all the missing observation but one, which is then estimated by the applicable formula. Using this value, and the initial estimates of all but one of the remaining quantities, the second missing value is found by the formula. This procedure is followed until all estimates show no significant change from one cycle to the next. The number of cycles necessary before convergence is attained depends heavily upon the choice of the initial estimate, John (1966). Using this iterative method, the problem of estimating more than one missing value, reduces to the problem of estimating one such value, which is the disadvantage of the iterative method.

In this study, the aim is to develop a method which eliminates the disadvantage introduced by applying the iterative method. The methods of estimating missing values will be derived when one or more observations are missing in Rectangular lattice designs (Simple and Triple Rectangular lattice designs respectively) with repetitions of the basic designs. A rectangular lattice design is said to be with repetitions when the basic design with n replications is repeated p times to give $r = np$ replications, Cox and Cochran (1957). From the numerical illustrations, an analysis of variance is constructed for the exact values of the simple and triple rectangular lattice designs for one and two observations respectively, from which the exact standard errors, SE, the block errors, E_b and the intra-block errors, E_e were obtained. After the missing yield has been estimated and replaced, the analysis of variance can be performed with the modification that the error degree of freedom will reduce by one (1) and the total degree of freedom will also reduce by one (1), thus we have $(r-1)(k^2-1)-k-1$ and rk^2+rk-2 degrees of freedom respectively, from which the estimated standard errors, the block errors and the intra-block errors were obtained as presented in Table 13.

Methodology

Model for Rectangular lattice designs

Consider an experiment of $t = k(k + 1)$ treatment in $k + 1$ blocks of r replicate, the observations are denoted by Y_{ijk} where i denotes the number of treatments ($i = 1, 2, \ldots, t$); j denotes the number of replicates ($j = 1, 2, \ldots, r$) and $k + 1$ denotes the number of blocks ($k = 1, 2, \ldots, k + 1$).

The model for this design is expressed as;

$$Y_{ijk} = \mu + \tau_i + \theta_j + \rho_{k(j)} + \varepsilon_{ijk} \quad (1)$$

where Y_{ijk} = The response (yield) of treatment, i, for a particular i replicate and a particular block; μ = Grand mean; τ_I = Effect of treatment i; θ_j = Effect of replicate j; $\rho_{k(j)}$ = Effect of block k within replicate j; ε_{ijk} = Random error associated with the response, Y_{ijk}, normally distributed about a mean zero (0) with constant variance σ^2.

This model leads to the ANOVA Table below, where R_j is the sum of the yields of the treatments for j^{th} replicate; T_i is the sum of the yields from all replicates of the treatments; C_{jk} is the total (over all replicates) of all treatments in the block minus (–) rB_{jk}; $r = np$ is the number of replications; n is the number of times the basic design (simple or triple rectangular lattice design) is replicated; p is the number of repetitions; R_c is the sum of the C_{jk} in the j^{th} replicate; A_{hk} is the difference between blocks within group h of block k; $\sum_{i=x}^{y} D_i^2 = D_x^2 + D_y^2$ and $\sum_{i=x}^{z} D_i^2 = D_x^2 + D_y^2 + D_z^2$ if the design is Simple Rectangular Lattice (SRL) and Triple Rectangular Lattice (TRL) designs respectively, with repetitions such that $D_x = R_{x1} - R_{x2}$, $D_y = R_{y1} - R_{y2}$, and $D_z = R_{z1} - R_{z2}$ are the differences between the repetitions of group X, Y for SRL and X, Y, Z for TRL designs respectively; E_b is the block error; E_e is the intra-block error; Sk is the sum of the same peer in the same row; G is the grand total; k is the block size.

Table 1. ANOVA Table for a $k(k+1)$ Rectangular Lattice Design (Complete data set).

Source of Variation	***Degrees of Freedom***	***Sum of Squares***	***Mean Squares***
Replicates	$r-1$	$\frac{\sum R_j^2}{k(k+1)} - \frac{G^2}{rk(k+1)}$	
Treatments (unadj.)	k^2+k-1	$\frac{\sum \tau i}{r} - \frac{G^2}{rk(k+1)}$	
Component (a)	$(r-n)k$	$\frac{\sum_{h=x}^{y}\sum_{k=1}^{k+1} A_{hk}^2}{nk} - \frac{\sum_{i=x}^{y,z} D_i^2}{nk(k+1)}$	
Component (b)	nk	$\frac{\sum C_{jk}^2}{r(nk-k-1)} - \frac{\sum R_c^2}{r(k+1)(nk-k-1)} - \frac{\sum S_k^2}{r(n-1)(k+1)(nk-k-1)}$	E_b(*error*)
Intra-block error	$(r-1)(k^2-1)-k$	*By Subtraction*	E_e(*error*)
Total	rk^2+rk-1		

Estimation of Missing Data when One Observation is missing.

R_j is the sum of the yields of the treatments for j^{th} replicate; τ_i is the sum of the yields from all replicates of the treatments; C_{jk} is the total (over all replicates) of all treatments in the block minus (–) rB_{jk} : $r = np$ is the number of replications; n is the number of times the basic design (simple or triple rectangular lattice design) is replicated; p is the number of repetitions; R_c is the sum of the C_{jk} in the j^{th} replicate; A_{hk} is the

Suppose the yield on the plot in replicate j, block k, and receiving treatment i denoted by u is missing. Let R_j, τ_i, C_{jk}, R_c, S and G retain their usual meanings so that R'_j is the sum of the observations of all the treatments in replicate j; τ'_i is the sum of the observations from all replicates of treatment i; C'_{jk} is the total (over replicate j) of all treatments in block k minus (–) rB'_{jk}; R'_c is the sum of the C_{jk} in replicate j; S'_k is the sum of the observations in the same row and G' is the unknown grand total. When estimating u for one observation missing, u and v for two observations missing in SRL and TRL designs respectively, the F' in all the cases is $D_x + D_y$ for SRL and $D_x + D_y + D_z$ for TRL designs.

Minimizing the intra-block error sum of squares (2) with respect to the missing plot, u, say Q, we have

$$Q = SS_E = \left(u^2 - \frac{(G'+u)^2}{B}\right) - \left(\frac{(R'_j+u)^2}{A} - \frac{(G'+u)^2}{B}\right) - \left(\frac{(\tau'_j+u)^2}{r} - \frac{(G'+u)^2}{B}\right)$$
$$-\left(\frac{(A_{hk}+u)^2}{nk} - \frac{\left(\sum_{i=x}^{y,z} D_i + u\right)^2}{nA}\right) - \left(\frac{(C'_{jk}+u)^2}{C} - \frac{(R'_c+u)^2}{D} - \frac{(S'_k+u)^2}{E}\right)$$

(2)

$$= u^2 + \frac{(G'+u)^2}{B} - \frac{(R'_j+u)^2}{A} - \frac{(\tau'_j+u)^2}{r} - \frac{(A_{hk}+u)^2}{nk} + \frac{\left(\sum_{i=x}^{y,z} D_i + u\right)^2}{nA} - \frac{(C'_{jk}+u)^2}{C} + \frac{(R'_c+u)^2}{D} + \frac{(S'_k+u)^2}{E} \tag{3}$$

Differentiating Q with respect to u, and equating to zero,

$$\frac{\partial Q}{\partial u} = 2u + \frac{2(G'+u)}{B} - \frac{2(R'_j+u)}{A} - \frac{2(\tau'_j+u)}{r} - \frac{2(A_{hk}+u)}{nk} + \frac{2\left(\sum_{i=x}^{y,z} D_i + u\right)}{nA} - \frac{2(C'_{jk}+u)}{C} + \frac{2(R'_c+u)}{D} + \frac{2(S'_k+u)}{E} = 0 \tag{4}$$

Then solving for u,

$$u = \frac{F(n-1)\begin{bmatrix} nrR'_j + nA\tau'_i - nG' \\ +r(k+1)A_{hk} \\ -rF' \end{bmatrix} + nk\begin{bmatrix} (n-1)(k+1)C'_{jk} \\ -(n-1)R'_c \\ -S'_k \end{bmatrix}}{F\left[n(n-1)(r-1)(k^2+k-1) - k(rn-r-n)\right]} \tag{5}$$

Where $A = k(k + 1)$, $B = rk(k + 1)$, $C = r(nk - k - 1)$, $D = r(k + 1)(nk - k - 1)$, $E = r(n - 1)(k + 1)(nk - k - 1)$, $F = (nk - k - 1)$, and $F' = \sum_{i=x}^{y,z} D_i^2$ depends on whether the design is a SRL or TRL design, respectively. Equation (5) holds true for simple rectangular lattice designs ($n = 2$) and triple rectangular lattice designs ($n = 3$).

Estimation of missing data when two (2) observations are missing.

There are various cases in which two data can be missing in rectangular lattice designs with repetitions. Supposing u and v are the two values missing in group X and Y respectively, given any of the cases below, we differentiate the sum of squares errors with respect to u and v and equating to zero to give a system of simultaneous equations for estimating the two missing values. The missing of u and v can occur either in X and Y or X and Z or Y and Z respectively and the algebraic expression for each case is the same for both simple ($n = 2$) and triple ($n = 3$) rectangular lattice designs. Thus the derived formulas for each basic design can be extended to (for) any replication depending on where the values are missing.

Case 1: Different treatments missing in different replicates, but same block. Let u and v be missing observations in simple and triple rectangular lattice designs given this case, the sum of squares error, where all symbols in $\Delta i (i = 1,2,3,4)$ have the same meaning as in the case of one observation missing, is given as

$$SS_E = u^2 + v^2 + \Delta_1 - \frac{\left(A_{hk} + u\right)^2 + \left(A_{hk} + v\right)^2}{nk} + \frac{\left(D_x + u\right)^2 + \left(D_y + v\right)^2}{nk\left(k + 1\right)} \tag{6}$$

where

$$\Delta_1 = \frac{\left(G' + u + v\right)^2}{B} - \frac{\left(R'_j + u\right)^2 + \left(R'_j + v\right)^2}{A} - \frac{\left(\tau'_i + u\right)^2 + \left(\tau'_i + v\right)^2}{r} - \frac{\left(C'_{jk} + u\right)^2 + \left(C'_{jk} + v\right)^2}{C}$$
$$+ \frac{\left(R'_c + u\right)^2 + \left(R'_c + v\right)^2}{D} + \frac{\left(S'_k + u + v\right)^2}{E}$$

Minimizing the intra-block error sum of squares (6) with respect to the missing plots, u and v; solving simultaneously to obtain estimates of the missing data, we have

$$u = v = \frac{\begin{array}{l}(r-1)\left[nAC'_{jk} - nFG'\right] \\ -F(r-1)\left[rF' - nrR'_j - r(k+1)A_{hk} - nA\tau'_i\right] \\ -nk\left[(r-1)R'_c - S'_k\right]\end{array}}{\begin{array}{l} k(r-1)\left[\begin{array}{l} n^2(2-r) + r(k+1) \\ -nk(3r-1) \\ +nk(r-1)(n(k+1)-k)\end{array}\right] \\ -n\left[(r-3)(k-r)+2\right]\end{array}} \tag{7}$$

Expression (7) remains true for u and v for both simple ($n = 2$) and triple ($n = 3$) rectangular lattice designs.

Case 2: Different treatments missing in same replicates, but same block. Let u and v be missing observations in simple and triple rectangular lattice designs given this case, the sum of squares error is given as;

$$SS_E = u^2 + v^2 + \Delta_2 - \frac{(A_{hk} + u + v)^2}{nk} + \frac{(D_x + D_y + u + v)^2}{nk(k+1)} \tag{8}$$

where

$$\begin{array}{l}\Delta_2 = \dfrac{(G' + u + v)^2}{B} - \dfrac{(R'_j + u + v)^2}{A} - \dfrac{(\tau'_i + u)^2 + (\tau'_i + v)^2}{r} - \dfrac{(C'_{jk} + u + v)^2}{C} \\ \quad + \dfrac{(R'_c + u + v)^2}{D} + \dfrac{(S'_k + u + v)^2}{E}\end{array} \tag{9}$$

By minimizing the intra-block error sum of squares (9) with respect to the missing plots u and v and solving simultaneously for SRL and TRL designs respectively, we have for the simple ($n = 2$) and triple ($n = 3$) rectangular lattice designs

$$u=v=\frac{\begin{array}{l}(r-1)\left[nAC'_{jk}-nFG'\right]\\ -F(r-1)\left[rF'-nrR'_j-r(k+1)A_{hk}-nA\tau'_i\right]\\ -nk\left[(r-1)R'_c-S'_k\right]\end{array}}{2rkF(r-1)+n\left[\begin{array}{l}k^2(r-1)\left(2r-n(r-1)\right)\\ +(r-1)^2\left(k\left(k^2+2n\right)-2\right)\\ -k\left(nk(r-1)^2+r(r-2)+3\right)\end{array}\right]} \tag{10}$$

Case 3: Different treatments missing in same replicates, but different blocks. Let *u* and *v* be missing observations in simple and triple rectangular lattice designs given this case, the sum of squares error is given as

$$SS_E=u^2+v^2+\Delta_3-\frac{\left(A_{hk}+u\right)^2+\left(A_{hk}+v\right)^2}{nk}+\frac{\left(D_x+D_y+u+v\right)^2}{nk(k+1)} \tag{11}$$

where

$$\begin{aligned}\Delta_3=&\frac{\left(G'+u+v\right)^2}{B}-\frac{\left(R'_j+u+v\right)^2}{A}-\frac{\left(\tau'_i+u\right)^2+\left(\tau'_i+v\right)^2}{r}\\ &-\frac{\left(C'_{jk}+u\right)^2+\left(C'_{jk}+v\right)^2}{C}+\frac{\left(R'_c+u+v\right)^2}{D}+\frac{\left(S'_k+u\right)^2+\left(S'_k+v\right)^2}{E}\end{aligned} \tag{12}$$

Then for the missing observations *u* and *v* as in Case 2, for the simple rectangular lattice design (n = 2)

$$u = v = \frac{\begin{array}{l}(r-1)\left[nAC'_{jk} - nFG'\right] \\ -F(r-1)\left[rF' - nrR'_j - r(k+1)A_{hk} - nA\tau'_i\right] \\ -nk\left[(r-1)R'_c - S'_k\right]\end{array}}{2rkF(r-1) - n\begin{bmatrix} k^2(r-1)\left(2r - nk(r-1) - 1\right) \\ +(r-1)^2\left(k\left(k^2+2n\right)-2\right) \\ -k\left(nk(r-1)^2 + r(r+1)+1\right)\end{bmatrix}} \tag{13}$$

and for the triple rectangular lattice design ($n = 3$)

$$u = v = \frac{\begin{array}{l}(r-1)\left[nAC'_{jk} - nFG'\right] \\ -F(r-1)\left[rF' - nrR'_j - r(k+1)A_{hk} - nA\tau'_i\right] \\ -nk\left[(r-1)R'_c - S'_k\right]\end{array}}{rF(r-1)(k-1) - n\begin{bmatrix} k^2(r-1)\left(2r - nk(r-1) - 1\right) \\ +(r-1)^2\left(k\left(k^2+2n\right)-2\right) \\ -k\left(nk(r-1)^2 + r(r+1)+1\right)\end{bmatrix}} \tag{14}$$

Case 4: Same treatments missing in different replicates, but different blocks. Let u and v be missing observations in simple and triple rectangular lattice designs given this case, then the sum of squares error is given as

$$SS_E = u^2 + v^2 + \Delta_4 - \frac{(A_{hk}+u)^2 + (A_{hk}+v)^2}{nk} + \frac{(D_x+u)^2 + (D_y+v)^2}{nk(k+1)} \tag{15}$$

where

$$\begin{aligned}\Delta_4 = {} & \frac{(G'+u+v)^2}{B} - \frac{(R'_j+u)^2 + (R'_j+v)^2}{A} - \frac{(\tau'_i+u+v)^2}{r} \\ & - \frac{(C'_{jk}+u)^2 + (C'_{jk}+v)^2}{C} + \frac{(R'_c+u)^2 + (R'_c+v)^2}{D} + \frac{(S'_k+u)^2 + (S'_k+v)^2}{E}\end{aligned}$$

Similarly, solving for the missing observations u and v as in Case 1,

$$u = v = \frac{\begin{array}{l}(r-1)\left[nAC'_{jk} - nFG'\right] \\ -F(r-1)\left[rF' - nrR'_j - r(k+1)A_{hk} - nA\tau'_i\right] \\ -nk\left[(r-1)R'_c - S'_k\right]\end{array}}{rkF(r-1) - n\left[\begin{array}{l}k^2(r-1)\left(2r - nk(r-2) - 3\right) \\ +(r-1)(r-2)\left(k^3 + nk - 1\right) \\ -k\left(nk(r-1)(r-2) + 1\right)\end{array}\right]} \tag{16}$$

Expression (16) remains true for u and v for both simple ($n = 2$) and triple ($n = 3$) rectangular lattice designs.

Standard Errors of the Differences Between Treatment Means

The standard error (SE) is a measure of statistical accuracy of an estimate observed by taking the square root of the error variance of the difference between the treatment means. In simple and triple rectangular lattice designs with repetition, there are two cases in which the treatment means can be adjusted given missing data. The expressions of the standard error for these cases are the same for SRL and TRL designs respectively, except for the values of r where r is four (4) for SRL and six (6) for TRL.

The standard error of the difference between the means of two treatments occurring together in the same block given missing data is

$$SE_1 = \sqrt{E_e\left(\frac{k+1}{k}\right)\left\{\frac{2}{r} + \frac{k(k+1)}{r\left((r-1)\left(k^2-1\right)-k\right)}\right\}} \tag{17}$$

For the treatments not occurring together in the same block given missing data, the standard error of the difference between the means is

$$SE_2 = \sqrt{E_e\left(\frac{k+2}{k}\right)\left\{\frac{2}{r}+\frac{k^3}{r(k+2)\left((r-1)(k^2-1)-k\right)}\right\}} \tag{18}$$

Results

For numerical illustrations, a simple and triple rectangular lattice example is considered from Harshbarger (1949). The data were on 12 treatments which are to be tested so a 3×4 rectangular lattice with repetition was used. The experiment was set up as in Table 2, with $n = 2$, $r = 4$ for SRL design and $n = 3$, $r = 6$ for TRL design, $k = 3$ and $k + 1 = 4$ are the same for both the SRL and TRL designs.

Table 2. Yield of Alfalfa for a 3×4 Rectangular Lattice Design.

Group X

Blocks	***Repetition 1***			***Block Totals***	***Blocks***	***Repetition 2***			***Block Totals***
1	(1)13.06	(2)5.68	(3)6.28	25.02	1	(1)10.70	(2)4.36	(3)5.66	20.72
2	(4)8.24	(5)8.32	(6)7.84	24.40	2	(4)10.34	(5)6.44	(6)10.06	26.84
3	(7)7.32	(8)6.86	(9)5.04	19.22	3	(7)5.62	(8)7.90	(9)7.70	21.22
4	(10)8.88	(11)11.42	(12)10.38	30.68	4	(10)6.46	(11)8.36	(12)6.74	21.56
			Total R_{X1}	99.32				***Total R_{X2}***	90.34

Group Y

Blocks	***Repetition 1***			***Block Totals***	***Blocks***	***Repetition 2***			***Block Totals***
1	(4)8.55	(7)9.72	(10)4.00	22.27	1	(4)10.74	(7)8.18	(10)8.92	27.84
2	(1)10.56	(8)6.60	(11)9.64	26.80	2	(1)12.62	(8)8.52	(11)11.92	33.06
3	(2)6.76	(5)8.60	(12)8.06	23.42	3	(2)9.18	(5)9.76	(12)8.34	27.28
4	(3)7.60	(6)7.82	(9)7.98	23.40	4	(3)7.76	(6)10.38	(9)10.70	28.84
			Total R_{Y1}	95.89				***Total R_{Y2}***	117.02

Table 2, Continued.

Group Z

Blocks	***Repetition 1***			***Block Totals***	***Blocks***	***Repetition 2***			***Block Totals***
1	(5)9.86	(9)9.28	(11)13.04	32.18	1	(5)5.68	(9)9.40	(11)9.98	28.06
2	(3)8.74	(7)9.34	(12)10.68	28.76	2	(3)5.46	(7)9.41	(12)10.52	25.39
3	(1)11.36	(6)8.52	(10)6.32	26.20	3	(1)14.02	(6)11.76	(10)8.84	34.62
4	(2)5.54	(4)10.58	(8)8.88	25.00	4	(2)8.96	(4)12.00	(8)9.64	30.60
			Total R_{Z1}	112.14				***Total R_{Z2}***	118.67

(Treatment numbers are enclosed in parentheses)

Simple Rectangular Lattice Design

A single observation missing: Suppose that the observation 13.06 for treatment (1) in Group X, Block 1, Replicate 1 of the presented simple rectangular lattice design had been missing in Table 2. From the block, replicate, treatment and grand totals, just as in the ordinary analysis, it is helpful to insert a *u* for the missing observation and to include it in all totals where it should appear. When the value of *u* has been found, it can then be inserted in all the appropriate plots and the data are ready for computing the analysis of variance.

Table 3. Computation procedure for one missing observation.

Block Symbols	C_X	C_Y	S_k
1	*u* – 4.32	-3.25	*u* – 7.57
2	4.61	-1.56	3.05
3	11.26	-8.78	2.48
4	-1.36	-9.66	-11.02
Total Rc	*u* + 10.19	-23.25	

The quantities needed for SRL in (5) are R'_1 = 86.26, τ'_1 = 33.88, F' = –25.21, C'_{11} = –4.32, R'_c = 10.19, A_{hx} = –8.76, S'_1 = –7.47, and G' = 389.51. Hence,

$$u = \frac{2\begin{bmatrix} 8(86.26)+24(33.88) \\ -2(389.51)+16(-8.76) \\ -4(-25.21) \end{bmatrix} + 6\begin{bmatrix} 4(-4.32)-10.19 \\ -(-7.57) \end{bmatrix}}{2[66-18]} = 13.02$$

Two Observations Missing in a Simple Rectangular Lattice Design:

Case 1: Different treatments missing in different replicates but same block. Suppose that the observation 13.06 for treatment (1) in Group X, Block 1, Replicate 1 of the presented simple rectangular lattice design and the observation 8.55 for treatment (4) in Group Y, Block 1, Replicate 1 had been missing in Table 2. Let *u* and *v* be the missing treatments which will be obtained using the derived methods of estimation for this case.

Table 4. Computation procedure for estimating u and v given Case 1.

Block Symbols	$C_{X(u)}$	$C_{Y(u)}$	$S_{k(u)}$	$C_{X(v)}$	$C_{Y(v)}$	$S_{k(v)}$
1	u – 4.32	-3.25	u – 7.57	v + 0.19	-3.25	v – 3.06
2	4.61	-1.56	3.05	4.61	-1.56	3.05
3	11.26	-8.78	2.48	11.26	-8.78	2.48
4	-1.36	-9.66	-11.02	-1.36	-9.66	-11.02
Total Rc	u + 10.19	-23.25		v + 14.7	-23.25	

The quantities needed for SRL in (7) for u are $R'_1 = 86.26$, $\tau'_1 = 33.88$, $F' = -25.21$, $C'_{11} = -4.32$, $R'_c = 10.19$, $A_{hx} = -8.76$, $S'_1 = -7.57$, $G' = 389.51$, and for v are $R'_1 = 87.34$, $\tau'_1 = 29.32$, $F' = -20.7$, $C'_{11} = 0.19$, $R'_c = 14.7$, $A_{hx} = -14.12$, $S'_1 = -3.06$, and $G' = 398.02$. Hence

$$u = \frac{3\begin{bmatrix} 24(-4.32) \\ -4(389.51) \end{bmatrix} - 6\begin{bmatrix} 4(-25.21) - 8(86.26) \\ -16(-8.76) - 24(33.88) \end{bmatrix} - 6\begin{bmatrix} 3(10.19) \\ +(-7.57) \end{bmatrix}}{9(32) - 2(1)} = 12.80$$

and

$$v = \frac{3\begin{bmatrix} 24(0.19) \\ -4(394.02) \end{bmatrix} - 6\begin{bmatrix} 4(-20.7) - 8(87.34) \\ -16(-14.12) - 24(29.32) \end{bmatrix} - 6\begin{bmatrix} 3(14.7) \\ +(-3.06) \end{bmatrix}}{9(32) - 2(1)} = 9.07$$

Case 2: Different treatments missing in same replicates but same block. Suppose the observation 6.28 for treatment (3) in Group X, Block 1, Replicate 1 of the presented simple rectangular lattice design, and the observation 5.68 for treatment (2) in Group X, Block 1, Replicate 1 had been missing in Table 2.

Table 5. Computation procedure for estimating u and v given Case 2.

Block Symbols	$C_{X(u)}$	$C_{Y(u)}$	$S_{k(u)}$	$C_{X(v)}$	$C_{Y(v)}$	$S_{k(v)}$
1	u + 2.46	-3.25	u – 0.79	v + 1.54	-3.25	v – 0.19
2	4.61	-1.56	3.05	4.61	-1.56	3.05
3	11.26	-8.78	2.48	11.26	-8.78	2.48
4	-1.36	-9.66	-11.02	-1.36	-9.66	-11.02
Total Rc	u + 16.97	-23.25		v + 16.05	-23.25	

The quantities needed for SRL in (10) for u are $R'_1 = 78.48$, $\tau'_1 = 20.52$, $F' = -18.93$, $C'_{11} = 2.46$, $R'_c = 16.97$, $A_{hx} = -2.48$, $S'_1 = -0.79$, $G' = 396.29$, and for v are $R'_1 = 93.64$, $\tau'_1 = 20.3$, $F' = -25.21$, $C'_{11} = 1.54$, $R'_c = 16.05$, $A_{hx} = -1.38$, $S'_1 = -0.19$, and $G' = 369.89$. Hence

$$u = \frac{3\begin{bmatrix} 24(2.46) \\ -4(396.29) \end{bmatrix} - 6\begin{bmatrix} 4(-18.93) - 8(97.48) \\ -16(-2.48) - 24(20.52) \end{bmatrix} - 6\begin{bmatrix} 3(16.97) \\ +(-0.79) \end{bmatrix}}{522} = 4.04$$

and

$$v = \frac{3\begin{bmatrix} 24(1.54) \\ -4(396.89) \end{bmatrix} - 6\begin{bmatrix} 4(-17.83) - 8(93.64) \\ -16(-1.38) - 24(20.3) \end{bmatrix} - 6\begin{bmatrix} 3(16.05) \\ +(-0.19) \end{bmatrix}}{522} = 5.31$$

Case 3. Different treatments missing in same replicates but different blocks. Suppose the observation 11.42 for treatment (11) in Group X, Block 1, Replicate 1 of the presented simple rectangular lattice design and the observation 8.24 for treatment (4) in Group X, Block 2, Replicate 1 had been missing in Table 2.

Table 6. Computation procedure for estimating u and v given Case 3.

Block Symbols	$C_{X(u)}$	$C_{Y(u)}$	$S_{k(u)}$	$C_{X(v)}$	$C_{Y(v)}$	$S_{k(v)}$
1	$u - 12.78$	-3.25	$u - 22.44$	$v + 0.5$	-3.25	$v - 2.75$
2	4.61	-1.56	3.05	4.61	-1.56	3.05
3	11.26	-8.78	2.48	11.26	-8.78	2.48
4	-1.36	-9.66	-11.02	-1.36	-9.66	-11.02
Total Rc	$u + 11.83$	-23.25		$v + 15.01$	-23.25	

The quantities needed for SRL in (13) for u are $R'_1 = 84.62$, $\tau'_1 = 29.92$, $F' = -23.57$, $C'_{11} = -12.78$, $R'_c = 11.83$, $A_{hx} = -2.3$, $S'_1 = -22.44$, $G' = 391.15$, and for v are $R'_1 = 91.08$, $\tau'_1 = 29.63$, $F' = -20.39$, $C'_{11} = 0.5$, $R'_c = 15.01$, $A_{hx} = -10.68$, $S'_1 = -2.75$, and $G' = 394.33$. Hence

$$u = \frac{3\begin{bmatrix} 24(-12.78) \\ -4(391.15) \end{bmatrix} - 6\begin{bmatrix} 4(-23.57) - 8(84.62) \\ -16(-2.3) - 24(29.92) \end{bmatrix} - 6\begin{bmatrix} 3(111.83) \\ +(-22.44) \end{bmatrix}}{282} = 10.72$$

and

$$v = \frac{3\begin{bmatrix} 24(0.5) \\ -4(394.33) \end{bmatrix} - 6\begin{bmatrix} 4(-20.39) - 8(91.08) \\ -16(-10.68) - 24(29.63) \end{bmatrix} - 6\begin{bmatrix} 3(15.01) \\ +(-2.75) \end{bmatrix}}{282} = 11.18$$

Case 4. Same treatments missing in different replicates but different blocks. Suppose the observation 13.06 for treatment (1) in Group X, Block 1, Replicate 1 of the presented simple rectangular lattice design and the observation 10.56 for treatment (1) in Group Y, Block 2, Replicate 1 had been missing in Table 2.

Table 7. Computation procedure for estimating *u* and *v* given Case 4.

Block Symbols	$C_{X(u)}$	$C_{Y(u)}$	$S_{k(u)}$	$C_{X(v)}$	$C_{Y(v)}$	$S_{k(v)}$
1	*u* – 4.32	-3.25	*u* – 7.57	*v* – 1.82	-3.25	*v* – 5.07
2	4.61	-1.56	3.05	4.61	-1.56	3.05
3	11.26	-8.78	2.48	11.26	-8.78	2.48
4	-1.36	-9.66	-11.02	-1.36	-9.66	-11.02
Total Rc	*u* + 10.19	-23.25		*v* + 12.69	-23.25	

The quantities needed for SRL in (16) for u are $R'_1 = 86.26$, $\tau'_1 = 33.88$, $F' = -25.21$, $C'_{11} = -4.32$, $R'_c = 10.19$, $A_{hx} = -8.76$, $S'_1 = -7.57$, $G' = 389.51$, and for v are $R'_1 = 85.33$, $\tau'_1 = 36.38$, $F' = -22.71$, $C'_{11} = -1.82$, $R'_c = 12.69$, $A_{hx} = -16.82$, $S'_1 = -5.07$, and $G' = 392.01$. Hence

$$u = \frac{3\begin{bmatrix} 24(-4.32) \\ -4(389.51) \end{bmatrix} - 6\begin{bmatrix} 4(-25.21) - 8(86.26) \\ -16(-8.76) - 24(33.88) \end{bmatrix} - 6\begin{bmatrix} 3(10.19) \\ +(-7.57) \end{bmatrix}}{288} = 12.71$$

and

$$v=\frac{3\begin{bmatrix}24(-1.82)\\-4(392.01)\end{bmatrix}-6\begin{bmatrix}4(-22.71)-8(85.33)\\-16(-16.82)-24(36.38)\end{bmatrix}-6\begin{bmatrix}3(12.69)\\+(-5.07)\end{bmatrix}}{288}=11.22$$

Triple Rectangular Lattice Design

A single plot missing: Suppose that the observation 13.06 for treatment (1) in Group X, Block 1, Replicate 1 of the presented triple rectangular lattice design had been missing in Table 2. From the block, replicate, treatment and grand totals, just as in the ordinary analysis, it is helpful to insert a *u* for the missing observation and to include it in all totals where it should appear. When the value of *u* has been found, it can then be inserted in all the appropriate places and the data are ready for computing the analysis of variance.

Table 8. Computation procedure for one missing observation.

Block Symbols	C_X	C_Y	C_Z	S_k
1	*u* – 4.02	3.13	-14.6	*u* – 7.45
2	14.77	5.50	-16.64	3.63
3	26.77	-5.24	-10.34	11.19
4	5.78	-8.74	-17.47	-20.43
Total Rc	*u* + 51.34	-5.35	-59.05	

The quantities needed for TRL in (5) are $R'_1 = 86.26$, $\tau'_1 = 59.26$, $F' = -31.74$, $C'_{11} = 4.02$, $R'_c = 51.34$, $A_{hx} = -8.76$, $S'_1 = -7.45$, and $G' = 620.32$. Hence,

$$u=\frac{10\begin{bmatrix}18(86.26)+36(59.26)\\-3(620.32)+24(-8.76)\\-6(-31.74)\end{bmatrix}+9\begin{bmatrix}8(4.02)-2(51.34)\\-(-7.57)\end{bmatrix}}{5[330-27]}=11.54$$

Two Plots Missing in a Triple Rectangular Lattice Design:

Case 1: Different treatments missing in different replicates but same block. Suppose the observation 13.06 for treatment (1) in Group X, Block 1, Replicate 1 of the presented triple rectangular lattice design and the observation 8.55 for treatment (4) in Group Y, Block 1, Replicate 1 had been missing in Table 2. The missing treatments u and v will be obtained using the derived methods of estimation for this case.

Table 9. Computation procedure for estimating u and v given Case 1.

Block Symbols	$C_{X(u)}$	$C_{Y(u)}$	$C_{Z(u)}$	$S_{k(u)}$	$C_{X(v)}$	$C_{Y(v)}$	$C_{Z(v)}$	$S_{k(v)}$
1	u + 4.02	3.13	-14.60	u – 7.45	v + 0.19	3.13	-14.60	v – 2.94
2	14.77	5.50	-16.64	3.63	14.77	5.50	-16.64	3.63
3	26.77	-5.24	-10.34	11.19	26.77	-5.24	-10.34	11.19
4	5.78	-8.74	-17.47	-20.43	5.78	-8.74	-17.47	-20.43
Total Rc	u + 51.34	-5.35	-59.05		v + 14.70	-5.35	-59.05	

The quantities needed for TRL in (7) for u are $R'_1 = 86.26$, $\tau'_1 = 59.26$, $F' = -31.74$, $C'_{11} = -4.02$, $R'_c = 51.34$, $A_{hx} = -8.76$, $S'_1 = -7.45$, $G' = 620.32$, and for v are $R'_1 = 87.34$, $\tau'_1 = 51.9$, $F' = -27.23$, $C'_{11} = 8.53$, $R'_c = 55.85$, $A_{hx} = -4.43$, $S'_1 = -2.94$, and $G' = 624.83$. Hence

$$u = \frac{5\begin{bmatrix} 36(4.02) \\ -15(620.32) \end{bmatrix} - 25\begin{bmatrix} 6(-31.74) - 18(86.26) \\ -24(-8.76) - 36(59.26) \end{bmatrix} - 9\begin{bmatrix} 5(51.34) \\ +(-7.54) \end{bmatrix}}{15(240) - 3(-7)} = 12.04$$

and

$$v = \frac{5\begin{bmatrix} 36(8.53) \\ -15(624.83) \end{bmatrix} - 25\begin{bmatrix} 6(-27.23) - 18(87.34) \\ -24(-4.43) - 36(51.9) \end{bmatrix} - 9\begin{bmatrix} 5(55.85) \\ +(-2.94) \end{bmatrix}}{15(240) - 3(-7)} = 10.94$$

Case 2: Different treatments missing in same replicates but same block. Suppose the observation 6.28 for treatment (3) in Group X, Block 1, Replicate 1 of the presented triple rectangular lattice design and the observation 5.68 for treatment (2) in Group X, Block 1, Replicate 1 had been missing in Table 2.

Table 10. Computation procedure for estimating *u* and *v* given Case 2.

Block Symbols	$C_{X(u)}$	$C_{Y(u)}$	$C_{Z(u)}$	$S_{k(u)}$	$C_{X(v)}$	$C_{Y(v)}$	$C_{Z(v)}$	$S_{k(v)}$
1	*u* + 10.8	3.13	-14.60	*u* – 0.67	*v* + 11.4	3.13	-14.60	*v* – 0.07
2	14.77	5.50	-16.64	3.63	14.77	5.50	-16.64	3.63
3	26.77	-5.24	-10.34	11.19	26.77	-5.24	-10.34	11.19
4	5.78	-8.74	-17.47	-20.43	5.78	-8.74	-17.47	-20.43
Total Rc	*u* + 58.12	-5.35	-59.05		*v* + 58.72	-5.35	-59.05	

The quantities needed for TRL in (10) for u are $R'_1 = 93.04$, $\tau'_1 = 35.22$, $F' = -24.96$, $C'_{11} = 10.8$, $R'_c = 58.12$, $A_{hx} = -1.98$, $S'_1 = -0.67$, $G' = 627.1$, and for v are $R'_1 = 93.64$, $\tau'_1 = 34.8$, $F' = -31.74$, $C'_{11} = 11.4$, $R'_c = 58.72$, $A_{hx} = -1.38$, $S'_1 = -0.07$, and $G' = 627.7$. Hence

$$u = \frac{5\begin{bmatrix} 36(10.8) \\ -15(627.1) \end{bmatrix} - 25\begin{bmatrix} 6(-24.96) - 18(93.04) \\ -24(-1.98) - 36(36.22) \end{bmatrix} - 9\begin{bmatrix} 5(58.12) \\ +(-0.67) \end{bmatrix}}{4212} = 6.75$$

and

$$v = \frac{5\begin{bmatrix} 36(11.4) \\ -15(627.7) \end{bmatrix} - 25\begin{bmatrix} 6(-24.36) - 18(93.64) \\ -24(-1.38) - 36(34.8) \end{bmatrix} - 9\begin{bmatrix} 5(58.72) \\ +(-0.07) \end{bmatrix}}{4212} = 6.79$$

Case 3: Different treatments missing in same replicates but different blocks. Suppose the observation 11.42 for treatment (11) in Group X, Block 1, Replicate 1 of the presented triple rectangular lattice design and the observation 8.24 for treatment (4) in Group X, Block 2, Replicate 1 had been missing in Table 2.

Table 11. Computation procedure for estimating *u* and *v* given Case 3.

Block Symbols	$C_{X(u)}$	$C_{Y(u)}$	$C_{Z(u)}$	$S_{k(u)}$	$C_{X(v)}$	$C_{Y(v)}$	$C_{Z(v)}$	$S_{k(v)}$
1	*u* + 5.66	3.13	-14.60	*u* – 5.81	*v* + 8.84	3.13	-14.60	*v* – 2.63
2	14.77	5.50	-16.64	3.63	14.77	5.50	-16.64	3.63
3	26.77	-5.24	-10.34	11.19	26.77	-5.24	-10.34	11.19
4	5.78	-8.74	-17.47	-20.43	5.78	-8.74	-17.47	-20.43
Total Rc	*u* + 52.98	-5.35	-59.05		*v* + 56.16	-5.35	-59.05	

The quantities needed for TRL in (14) for u are $R'_1 = 87.9$, $\tau'_1 = 52.94$, $F' = -30.1$, $C'_{11} = -5.66$, $R'_c = 52.98$, $A_{hx} = -7.12$, $S'_1 = -5.81$, $G' = 621.96$, and for v are $R'_1 = 91.08$, $\tau'_1 = 52.21$, $F' = -26.92$, $C'_{11} = 8.84$, $R'_c = 56.16$, $A_{hx} = -10.68$, $S'_1 = -2.63$, and $G' = 625.14$. Hence

$$u = \frac{5\begin{bmatrix}36(5.66)\\ -15(621.96)\end{bmatrix} - 25\begin{bmatrix}6(-30.1)-18(87.9)\\ -24(-7.12)-36(54.94)\end{bmatrix} - 9\begin{bmatrix}5(52.98)\\ +(-5.81)\end{bmatrix}}{3321} = 11.89$$

and

$$v = \frac{5\begin{bmatrix}36(8.84)\\ -15(625.14)\end{bmatrix} - 25\begin{bmatrix}6(-26.92)-18(91.08)\\ -24(-10.68)-36(52.21)\end{bmatrix} - 9\begin{bmatrix}5(56.16)\\ +(-2.63)\end{bmatrix}}{3321} = 11.38$$

Case 4: Same treatments missing in different replicates but different blocks. Suppose the observation 13.06 for treatment (1) in Group X, Block 1, Replicate 1 of the presented triple rectangular lattice design and the observation 10.56 for treatment (1) in Group Y, Block 2, Replicate 1 had been missing in Table 2.

Table 12. Computation procedure for estimating *u* and *v* given Case 4.

Block Symbols	$C_{X(u)}$	$C_{Y(u)}$	$C_{Z(u)}$	$S_{k(u)}$	$C_{X(v)}$	$C_{Y(v)}$	$C_{Z(v)}$	$S_{k(v)}$
1	u + 4.02	3.13	-14.60	u – 7.45	v + 6.52	3.13	-14.60	v – 4.95
2	14.77	5.50	-16.64	3.63	14.77	5.50	-16.64	3.63
3	26.77	-5.24	-10.34	11.19	26.77	-5.24	-10.34	11.19
4	5.78	-8.74	-17.47	-20.43	5.78	-8.74	-17.47	-20.43
Total Rc	u + 51.34	-5.35	-59.05		v + 53.84	-5.35	-59.05	

The quantities needed for TRL in (16) for u are $R'_1 = 86.26$, $\tau'_1 = 59.26$, $F' = -31.74$, $C'_{11} = -4.02$, $R'_c = 51.34$, $A_{hx} = -8.76$, $S'_1 = -7.45$, $G' = 620.32$, and for v are $R'_1 = 85.33$, $\tau'_1 = 61.76$, $F' = -29.24$, $C'_{11} = 6.52$, $R'_c = 53.84$, $A_{hx} = -16.82$, $S'_1 = -4.95$, and $G' = 622.82$. Hence

$$u=\frac{5\begin{bmatrix}36(4.02)\\-15(620.32)\end{bmatrix}-25\begin{bmatrix}6(-31.74)-18(86.26)\\-24(-8.76)-36(59.26)\end{bmatrix}-9\begin{bmatrix}5(51.34)\\+(-7.45)\end{bmatrix}}{3624}=12.03$$

and

$$v=\frac{5\begin{bmatrix}36(6.52)\\-15(622.82)\end{bmatrix}-25\begin{bmatrix}6(-29.24)-18(85.33)\\-24(-16.82)-36(61.76)\end{bmatrix}-9\begin{bmatrix}5(53.84)\\+(-4.95)\end{bmatrix}}{3624}=11.14$$

Table 13. Summary Table of estimates, standard errors, block and intra-block errors for simple and triple rectangular lattice designs.

Missing Values		***One Observation***	***Two Observations***			
			Case 1	***Case 2***	***Case 3***	***Case 4***
Actual	***SRL***	13.06	13.06	6.28	11.42	13.06
			8.55	5.68	8.24	10.56
	SE	1.3337	1.3337	1.3337	1.3337	1.3337
		1.3835	1.3835	1.3835	1.3835	1.3835
	E_b	2.3726	2.3726	2.3726	2.3726	2.3726
	E_e	1.2522	1.2522	1.2522	1.2522	1.2522
Estimated	***SRL***	13.02	12.80	4.04	10.72	12.71
			9.07	5.31	11.18	11.22
	SE	1.4830	1.9427	1.7811	1.5610	1.5141
		1.5981	1.9875	1.8294	1.7063	1.6014
	E_b	2.5326	2.4018	2.5815	2.5637	2.5364
	E_e	1.2822	1.3680	1.3562	1.3014	1.2898
Actual	***TRL***	13.06	13.06	6.28	11.42	13.06
			8.55	5.68	8.24	10.56
	SE	0.6713	0.6713	0.6713	0.6713	0.6713
		0.6946	0.6946	0.6946	0.6946	0.6946
	E_b	1.0243	1.0243	1.0243	1.0243	1.0243
	E_e	0.2481	0.2481	0.2481	0.2481	0.2481
Estimated	***TRL***	11.54	12.04	6.75	11.89	12.03
			10.94	6.79	11.38	11.14
	SE	0.7137	0.8033	0.8087	0.7198	0.9147
		0.9371	0.9118	0.9465	0.8510	0.9380
	E_b	1.0526	1.2236	1.5138	1.3687	1.1558
	E_e	0.2905	0.3845	0.5011	0.3412	0.2917

Conclusion

Least squares method for estimating missing data was used to derive formulas for when one or more observations are missing in simple rectangular lattice and triple rectangular lattice designs using the intra-block information. This was done by suggesting algebraic values for the missing cases and solving simultaneously instead of the iteration approach. Groups *X* and *Y* were used for the derivation of the algebraic expressions for more than one observation missing, these expressions can be used also for group *X* and *Z* or *Y* and *Z* given a triple rectangular lattice design.

Comparing the actual value(s) with the estimated value(s) for one and two observations missing in simple and triple rectangular lattice designs respectively as in Table 13, it could be deduced the estimated value(s) are significantly approximate to the actual value(s) which indicates the effectiveness of the derived algebraic procedures. Comparing also the exact standard errors, block errors and intra-block errors with the estimated standard errors, block errors and intra-block errors respectively, it was observed that the estimated errors appear more than the exact errors as could be anticipated in designed experiments due to the missing value(s) in the rectangular lattice designs under consideration. This shows that missing values increase experimental errors.

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