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For the analysis of two-way contingency tables with ordered categories, Yamamoto, Tahata, Suzuki, and Tomizawa (2011) considered a measure to represent the degree of departure from marginal point-symmetry. The maximum value of the measure cannot distinguish two kinds of marginal complete asymmetry with respect to the midpoint. A measure is proposed which can distinguish two kinds of marginal asymmetry with respect to the midpoint. It also gives large-sample confidence interval for the proposed measure.

Keywords: Asymmetry, marginal proportional point-symmetry, marginal point-symmetry, measure, model, ordered category

Introduction

Consider an $R \times R$ square contingency table with the same row and column classifications. Let p_{ij} denote the probability that an observation will fall in the i^{th} row and j^{th} column of the table ($i = 1,\ldots, R$; $j = 1,\ldots, R$). The symmetry model, which was given by Bowker (1948), is defined by

$$p_{ij} = p_{ji} \quad (i < j).$$

This model indicates that the probability that an observation will fall in row category i and column category j is equal to the probability that the observation falls in row category j and column category i. Namely, this describes a structure of symmetry of the cell probabilities $\{p_{ij}\}$ with respect to the main diagonal of the table. For the symmetry model see also Bishop, Fienberg, and Holland (1975, p.

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282), Caussinus (1965), McCullagh (1978), Goodman (1979), Agresti (2002, p. 424), Tomizawa and Tahata (2007), and Tahata and Tomizawa (2014).

The marginal homogeneity (or marginal symmetry) model is defined by

$$p_{i\cdot} = p_{\cdot i} \quad (i = 1, \ldots, R),$$

where

$$p_{i\cdot} = \sum_{t=1}^{R} p_{it}, \quad p_{\cdot i} = \sum_{s=1}^{R} p_{si};$$

see, e.g., Stuart (1955), Bhapkar (1966), Bishop et al. (1975, p. 282), Tomizawa and Tahata (2007), and Tahata and Tomizawa (2014). The marginal homogeneity model indicates that the row marginal distribution is identical to the column marginal distribution.

Wall and Lienert (1976) considered the point-symmetry model, defined by

$$p_{ij} = p_{i^* j^*} \quad (i = 1, \ldots, R; j = 1, \ldots, R),$$

where the symbol * denotes $i^* = R + 1 - i$.

Consider an $R \times C$ rectangular contingency table with ordered categories. Let p_{ij} denote the probability that an observation will fall in the i^{th} row and j^{th} column of the table ($i = 1, \ldots, R$; $j = 1, \ldots, C$). Tomizawa (1985) extended the point-symmetry model for an $R \times C$ contingency table as follows:

$$p_{ij} = p_{i^* j^{**}} \quad (i = 1, \ldots, R; j = 1, \ldots, C)$$

where $i^* = R + 1 - i$ and $j^{**} = C + 1 - j$. This model indicates that the probability that an observation will fall in row category i and column category j is equal to the probability that the observation falls in row category i^* and column category j^{**}. Namely, this describes a structure of point-symmetry of cell probabilities with respect to the center cell or center point in the table. Also see Tomizawa and Tahata (2007) and Tahata and Tomizawa (2014). Tomizawa (1985) also considered the marginal point-symmetry model, defined by

$$p_{i\cdot} = p_{i^*\cdot} \quad (i = 1, \ldots, R)$$

and

$$p_{\cdot j} = p_{\cdot j^{**}} \quad (j = 1, \ldots, C).$$

This model indicates that the row marginal distribution is point-symmetric with respect to the midpoint and the column marginal distribution is also point-symmetric with respect to the midpoint. Let [ ] denote the Gauss symbol (i.e., for real number x, $[x]$ denotes the maximum integer which is not larger than x). For example, when $R = 4$, $[R/2] = 2$, and when $R = 5$, $[R/2] = 2$. The marginal point-symmetry model is also expressed as essentially

$$p_{i\cdot} = p_{i^*\cdot} \quad \left(i = 1, \ldots, \left[\frac{R}{2}\right]\right)$$

and

$$p_{\cdot j} = p_{\cdot j^{**}} \quad \left(j = 1, \ldots, \left[\frac{C}{2}\right]\right).$$

When the model does not hold, there is also interested in measuring the degree of departure from the model. For the measures to represent the degree of departure from the symmetry, the point-symmetry and the marginal homogeneity models, see, e.g., Tomizawa (1994, 1995), Tomizawa, Seo, and Yamamoto (1998), Tomizawa, Miyamoto, and Hatanaka (2001), and Tomizawa, Yamamoto, and Tahata (2007). For the measure from the marginal point-symmetry model, Yamamoto, Tahata, Suzuki, and Tomizawa (2011) proposed the power-divergence type measure $\psi^{(\lambda)}$; see Appendix 1. However, when the measure $\psi^{(\lambda)}$ equals 1, it is not possible to distinguish two kinds of row (column) complete asymmetry with respect to the midpoint, where row complete asymmetry means (i) $p_{i\cdot} = 0$ (then $p_{i^*\cdot} > 0$) for $i = 1, \ldots, [R/2]$, or (ii) $p_{i^*\cdot} = 0$ (then $p_{i\cdot} > 0$) for $i = 1, \ldots, [R/2]$, and column complete asymmetry means (i) $p_{\cdot j} = 0$ (then $p_{\cdot j^{**}} > 0$) for $j = 1, \ldots, [C/2]$, or (ii) $p_{\cdot j^{**}} = 0$ (then $p_{\cdot j} > 0$) for $j = 1, \ldots, [C/2]$. Because these two kinds of row (column) complete asymmetry indicate the opposite different maximum departures from marginal point-symmetry with respect to the midpoint, the interest is in proposing a measure which can take the different values for them.

The purpose of present study is to propose a measure which can distinguish two kinds of the marginal complete asymmetry with respect to the midpoint for rectangular contingency tables with ordered categories. The measure lies between −1 and 1 although the measure $\psi^{(\lambda)}$ lies between 0 and 1, and it may be useful for comparing the degree of departure from marginal point-symmetry in several tables.

Measure

Consider the $R \times C$ contingency table. Let

$$\delta_1 = \sum_{i=1}^{[R/2]} \left(p_{i\cdot} + p_{i^*\cdot} \right), \quad \delta_2 = \sum_{j=1}^{[C/2]} \left(p_{\cdot j} + p_{\cdot j^{**}} \right),$$

$$q_{i\cdot} = \frac{p_{i\cdot}}{\delta_1}, \quad q_{i^*\cdot} = \frac{p_{i^*\cdot}}{\delta_1} \quad \left(i = 1, \ldots, \left[\frac{R}{2} \right] \right),$$

and

$$q_{\cdot j} = \frac{p_{\cdot j}}{\delta_2}, \quad q_{\cdot j^{**}} = \frac{p_{\cdot j^{**}}}{\delta_2}, \quad \left(j = 1, \ldots, \left[\frac{C}{2} \right] \right).$$

Assuming $\{p_{i\cdot} + p_{i^*\cdot} \neq 0\}$ and $\{p_{\cdot j} + p_{\cdot j^{**}} \neq 0\}$, a measure is proposed to represent the degree of departure from marginal point-symmetry, defined by

$$\varphi_{\mathrm{MPS}} = \frac{\delta_1 \varphi_1 + \delta_2 \varphi_2}{\delta_1 + \delta_2},$$

where

$$\varphi_1 = \frac{4}{\pi} \sum_{i=1}^{[R/2]} \left(q_{i\cdot} + q_{i^*\cdot} \right) \left(\theta_{1(i)} - \frac{\pi}{4} \right),$$

$$\theta_{1(i)} = \cos^{-1} \left(\frac{q_{i\cdot}}{\sqrt{q_{i\cdot}^2 + q_{i^*\cdot}^2}} \right),$$

$$\varphi_2 = \frac{4}{\pi} \sum_{j=1}^{[C/2]} \left(q_{\cdot j} + q_{\cdot j^{**}} \right) \left(\theta_{2(j)} - \frac{\pi}{4} \right),$$

$$\theta_{2(j)} = \cos^{-1}\left(\frac{q_{\cdot j}}{\sqrt{q_{\cdot j}^2 + q_{\cdot j^{**}}^2}}\right).$$

The submeasure φ_1 represents the degree of departure from point-symmetry of row marginal distribution and the submeasure φ_2 represents the degree of departure from point-symmetry of column marginal distribution. The measure φ_{MPS}, being the weighted sum of the submeasures φ_1 and φ_2, represents the degree of departure from marginal point-symmetry.

The ranges of $\{\theta_{1(i)}\}$ and $\{\theta_{2(j)}\}$ are $0 \le \theta_{1(i)} \le \pi/2$ and $0 \le \theta_{2(j)} \le \pi/2$. The submeasures φ_1 and φ_2 lie between -1 and 1. Therefore, the measure φ_{MPS} lies between -1 and 1. The submeasure φ_1 has characteristics that (i) $\varphi_1 = 1$ if and only if $p_{i\cdot} = 0$ (then $p_{i^*\cdot} > 0$) for $i = 1,\ldots, [R/2]$, say, row upper complete asymmetry with respect to the midpoint, and (ii) $\varphi_1 = -1$ if and only if $p_{i^*\cdot} = 0$ (then $p_{i\cdot} > 0$) for $i = 1,\ldots, [R/2]$, say, row lower complete asymmetry with respect to the midpoint. Similarly, the submeasure φ_2 has characteristics that (i) $\varphi_2 = 1$ if and only if $p_{\cdot j} = 0$ (then $p_{\cdot j^{**}} > 0$) for $j = 1,\ldots, [C/2]$, say, column upper complete asymmetry with respect to the midpoint, and (ii) $\varphi_2 = -1$ if and only if $p_{\cdot j^{**}} = 0$ (then $p_{\cdot j} > 0$) for $j = 1,\ldots, [C/2]$, say, column lower complete asymmetry with respect to the midpoint. The measure φ_{MPS} has characteristics that (i) $\varphi_{\text{MPS}} = 1$ if and only if $\varphi_1 = \varphi_2 = 1$, and (ii) $\varphi_{\text{MPS}} = -1$ if and only if $\varphi_1 = \varphi_2 = -1$.

The submeasure $\varphi_1 = 0$ indicates the average of $\{\theta_{1(i)} - (\pi/4)\}$, $i = 1,\ldots, [R/2]$, equals zero. When $\varphi_1 = 0$, this structure is referred to as the row average point-symmetry. Similarly, when $\varphi_2 = 0$, this structure is referred to as the column average point-symmetry. If the marginal point-symmetry model holds then the row (column) average point-symmetry holds; but the converse does not. Using the submeasure φ_1 (φ_2), it can be deteremined whether the row (column) average point-symmetry departs toward the row (column) upper complete asymmetry with respect to the midpoint or toward the row (column) lower complete asymmetry with respect to the midpoint. When $\varphi_{\text{MPS}} = 0$, this structure will be referred to as the marginal average point-symmetry. We note that if the marginal point-symmetry model holds then the marginal average point-symmetry holds; but the converse does not. However, it is difficult to consider the exact interpretation of the marginal average point-symmetry. If the submeasures φ_1 and φ_2 equal 0 then φ_{MPS} equals 0, but the converse does not hold.

For example, consider the artificial probabilities in Table 1. For Table 1a, since there are the structures of row upper complete asymmetry with respect to the midpoint (i.e., $\varphi_1 = 1$) and column upper complete asymmetry with respect to the midpoint (i.e., $\varphi_2 = 1$), we see that the measure $\varphi_{MPS} = 1$. Also, for Table 1b, because there are the structures of row lower complete asymmetry (i.e., $\varphi_1 = -1$) and column lower complete asymmetry (i.e., $\varphi_2 = -1$), we see that the measure $\varphi_{MPS} = -1$. For Table 1c, because there are the structures of row lower complete asymmetry (i.e., $\varphi_1 = -1$) and column upper complete asymmetry (i.e., $\varphi_2 = 1$), we see that the measure $\varphi_{MPS} = 0$.

Relationship between Measure and Model

Consider the relationship between the measure φ_{MPS} and a model. Define the model by

$$p_{i\cdot} = \Delta_1 p_{i^*\cdot} \quad \left(i = 1, \ldots, \left[\frac{R}{2} \right] \right)$$

Table 1. Artificial probabilities

(a)

X \ *Y*	(1)	(2)	(3)	(4)	Total
(1)	0.0	0.0	0.0	0.0	0.0
(2)	0.0	0.0	0.0	0.0	0.0
(3)	0.0	0.0	0.3	0.2	0.5
(4)	0.0	0.0	0.2	0.3	0.5
Total	0.0	0.0	0.5	0.5	1.0

(b)

X \ *Y*	(1)	(2)	(3)	(4)	Total
(1)	0.3	0.2	0.0	0.0	0.5
(2)	0.2	0.3	0.0	0.0	0.5
(3)	0.0	0.0	0.0	0.0	0.0
(4)	0.0	0.0	0.0	0.0	0.0
Total	0.5	0.5	0.0	0.0	1.0

(c)

X \ *Y*	(1)	(2)	(3)	(4)	Total
(1)	0.0	0.0	0.3	0.2	0.5
(2)	0.0	0.0	0.2	0.3	0.5
(3)	0.0	0.0	0.0	0.0	0.0
(4)	0.0	0.0	0.0	0.0	0.0
Total	0.0	0.0	0.5	0.5	1.0

and

$$p_{\cdot j} = \Delta_2 p_{\cdot j^{**}} \quad \left(j = 1, \ldots, \left[\frac{C}{2} \right] \right),$$

where Δ_1 and Δ_2 are unspecified. This structure is referred to as the marginal proportional point-symmetry model. A special case of this model obtained by setting $\Delta_1 = \Delta_2 = 1$ is the marginal point-symmetry model. If there is a structure of the marginal proportional point-symmetry model in the table, then the submeasures φ_1 and φ_2 can be simply (as a function of parameters Δ_1 and Δ_2, respectively) expressed as

$$\varphi_1 = \frac{4}{\pi} \cos^{-1}\left(\frac{\Delta_1}{\sqrt{\Delta_1^2 + 1}} \right) - 1$$

and

$$\varphi_2 = \frac{4}{\pi} \cos^{-1}\left(\frac{\Delta_2}{\sqrt{\Delta_2^2 + 1}} \right) - 1 .$$

Under the marginal proportional point-symmetry model, $\varphi_1 = \varphi_2 = 0$ if and only if the marginal point-symmetry model holds, i.e., $\Delta_1 = \Delta_2 = 1$. As the value of Δ_1 (Δ_2) approaches zero, the submeasure φ_1 (φ_2) approaches 1, and as the value of Δ_1 (Δ_2) approaches infinity, the submeasure φ_1 (φ_2) approaches −1. Denote φ_1 and φ_2 by $\varphi_1(\Delta_1)$ and $\varphi_2(\Delta_2)$, respectively. Let

$$\theta_1 = \cos^{-1}\left(\frac{\Delta_1}{\sqrt{\Delta_1^2 + 1}} \right).$$

Noting $\varphi_1(\Delta_1) = 4\theta_1/\pi - 1$,

$$\varphi_1\left(\frac{1}{\Delta_1}\right)=\frac{4}{\pi}\cos^{-1}\left(\frac{\frac{1}{\Delta_1}}{\sqrt{\frac{1}{\Delta_1}+1}}\right)-1$$
$$=\frac{4}{\pi}\cos^{-1}\left(\frac{1}{\sqrt{1+\Delta_1^2}}\right)-1$$
$$=\frac{4}{\pi}\left(\frac{\pi}{2}-\theta_1\right)-1$$
$$=-\varphi_1\left(\Delta_1\right)$$

Thus, $|\varphi_1(\Delta_1)|$ equals $|\varphi_1(1/\Delta_1)|$. Similarly, $|\varphi_2(\Delta_2)|$ equals $|\varphi_2(1/\Delta_2)|$. For comparisons between several tables, if it can be estimated that there is a structure of the marginal proportional point-symmetry model in each table, then the measure φ_{MPS} would be adequate for representing and comparing the degree of departure from the marginal point-symmetry model.

Approximate Confidence Interval for Measures

Let n_{ij} denote the observed frequency in the i^{th} row and j^{th} column of the table ($i = 1,\ldots, R$; $j = 1,\ldots, C$). Assuming that a multinomial distribution applies to the $R \times C$ table, consider an approximate standard error and large-sample confidence interval for the measure φ_{MPS} and the submeasures φ_1 and φ_2 using the delta method, descriptions of which are given by, for example, Bishop et al. (1975, Sec. 14.6). The sample version of φ_{MPS}, $\hat{\varphi}_{\text{MPS}}$, is given by φ_{MPS} with $\{p_{ij}\}$ replaced by $\{\hat{p}_{ij}\}$, where $\hat{p}_{ij} = n_{ij}/n$ and $n = \Sigma\Sigma n_{ij}$. Using the delta method, $\sqrt{n}\left(\hat{\varphi}_{\text{MPS}}-\varphi_{\text{MPS}}\right)$ has asymptotically (as $n \to \infty$) a normal distribution with mean zero and variance

$$\sigma^2\left[\varphi_{\text{MPS}}\right]=\sum_{i=1}^{R}\sum_{j=1}^{C}p_{ij}\left(\frac{\partial\varphi_{\text{MPS}}}{\partial p_{ij}}\right)^2,$$

$$\frac{\partial\varphi_{\text{MPS}}}{\partial p_{ij}}=\frac{1}{\left(\delta_1+\delta_2\right)^2}\left\{\left(\delta_1+\delta_2\right)\left(\frac{\partial\varphi_1}{\partial p_{ij}}\delta_1+\frac{\partial\varphi_2}{\partial p_{ij}}\delta_2\right)+\left(\varphi_1-\varphi_2\right)\left(\frac{\partial\delta_1}{\partial p_{ij}}\delta_2-\frac{\partial\delta_2}{\partial p_{ij}}\delta_1\right)\right\}$$

$$\frac{\partial \delta_1}{\partial p_{ij}} = \begin{cases} 1 & i = 1, \ldots, \left[\frac{R}{2}\right], \left[\frac{R+1}{2}\right] + 1, \ldots, R;\ j = 1, \ldots, C, \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{\partial \delta_2}{\partial p_{ij}} = \begin{cases} 1 & i = 1, \ldots, R;\ j = 1, \ldots, \left[\frac{C}{2}\right], \left[\frac{C+1}{2}\right] + 1, \ldots, C, \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{\partial \varphi_1}{\partial p_{ij}} = \begin{cases} \frac{4}{\pi \delta_1} \left\{ \cos^{-1}\left(\frac{p_{i\cdot}}{\sqrt{p_{i\cdot}^2 + p_{i^*\cdot}^2}} \right) - \frac{p_{i^*\cdot}\left(p_{i\cdot} + p_{i^*\cdot}\right)}{p_{i\cdot}^2 + p_{i^*\cdot}^2} \right\} - \frac{\varphi_1 + 1}{\delta_1} & i = 1, \ldots, \left[\frac{R}{2}\right], \\ \frac{4}{\pi \delta_1} \left\{ \cos^{-1}\left(\frac{p_{i^*\cdot}}{\sqrt{p_{i\cdot}^2 + p_{i^*\cdot}^2}} \right) + \frac{p_{i^*\cdot}\left(p_{i\cdot} + p_{i^*\cdot}\right)}{p_{i\cdot}^2 + p_{i^*\cdot}^2} \right\} - \frac{\varphi_1 + 1}{\delta_1} & i = \left[\frac{R+1}{2}\right] + 1, \ldots, R, \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{\partial \varphi_2}{\partial p_{ij}} = \begin{cases} \frac{4}{\pi \delta_2} \left\{ \cos^{-1}\left(\frac{p_{\cdot j}}{\sqrt{p_{\cdot j}^2 + p_{\cdot j^{**}}^2}} \right) - \frac{p_{\cdot j^{**}}\left(p_{\cdot j} + p_{\cdot j^{**}}\right)}{p_{\cdot j}^2 + p_{\cdot j^{**}}^2} \right\} - \frac{\varphi_2 + 1}{\delta_2} & j = 1, \ldots, \left[\frac{C}{2}\right], \\ \frac{4}{\pi \delta_2} \left\{ \cos^{-1}\left(\frac{p_{\cdot j^{**}}}{\sqrt{p_{\cdot j}^2 + p_{\cdot j^{**}}^2}} \right) + \frac{p_{\cdot j^{**}}\left(p_{\cdot j} + p_{\cdot j^{**}}\right)}{p_{\cdot j}^2 + p_{\cdot j^{**}}^2} \right\} - \frac{\varphi_2 + 1}{\delta_2} & j = \left[\frac{C+1}{2}\right] + 1, \ldots, C, \\ 0 & \text{otherwise} \end{cases}$$

Let $\hat{\sigma}^2[\varphi_{\text{MPS}}]$ denote $\sigma^2[\varphi_{\text{MPS}}]$ with $\{p_{ij}\}$ replaced by $\{\hat{p}_{ij}\}$. Then $\hat{\sigma}[\varphi_{\text{MPS}}]/\sqrt{n}$ is an estimated approximate standard error of $\hat{\varphi}_{\text{MPS}}$, and

$$\hat{\varphi}_{\text{MPS}} \pm z_{\alpha/2} \hat{\sigma}[\varphi_{\text{MPS}}]/\sqrt{n}$$

is an approximate $100(1-\alpha)\%$ confidence interval for φ_{MPS}, where $z_{\alpha/2}$ is the percentage point from the standard normal distribution corresponding to a two-tail probability of α.

Similarly, for $k = 1, 2$, $\sqrt{n}(\hat{\varphi}_k - \varphi_k)$ has asymptotically (as as $n \to \infty$) a normal distribution with mean zero and variance

$$\sigma^2[\varphi_k] = \sum_{i=1}^{R}\sum_{j=1}^{C} p_{ij}\left(\frac{\partial \varphi_k}{\partial p_{ij}}\right)^2,$$

and

$$\hat{\varphi}_k \pm z_{\alpha/2}\hat{\sigma}[\varphi_k]\big/\sqrt{n}$$

is an approximate $100(1-\alpha)\%$ confidence interval for φ_k.

Example

Consider the data in Table 2 taken from Agresti (2002, p. 462). They are the results of a randomized double-blind clinical trial comparing an active hypnotic drug with a placebo in patients with insomnia. The outcome variable is patient's reported time to fall asleep, measured using 4 categories (< 20 minutes, 20-30 minutes, 30-60 minutes, and > 60 minutes).

From Table 3a, for the data in Table 2a, the estimated value of the submeasure φ_1 is 0.545 (> 0), and the confidence interval for φ_1 does not include zero. These would indicate patient's reported time before treatment in the Active treatment group is estimated to be 54.5 percent of the maximum departure toward the row upper complete asymmetry with respect to the midpoint.

Table 2. Insomniac patient's reported time (in minutes) to fall asleep after going to bed, from from Agresti (2002, p. 462)

(a)

	Follow-up				
Initial	**< 20**	**20-30**	**30-60**	**> 60**	**Total**
< 20	7	4	1	0	12
20-30	11	5	2	2	20
30-60	13	23	3	1	40
> 60	9	17	13	8	47
Total	40	49	19	11	119

(b)

	Follow-up				
Initial	**< 20**	**20-30**	**30-60**	**> 60**	**Total**
< 20	7	4	2	1	14
20-30	14	5	1	0	20
30-60	6	9	18	2	35
> 60	4	11	14	22	51
Total	31	29	35	25	120

Table 3. The estimated measures, their estimated approximate standard errors, and approximate 95% confidence intervals for measures, applied to (a) Table 2a and (b) Table 2b

(a)

Measure	Estimated measure	Standard error	Confidence interval
φ_1	0.545	0.087	(0.375, 0.714)
φ_2	−0.584	0.082	(−0.745, −0.424)
φ_{MPS}	−0.020	0.063	(−0.143, 0.103)
$\psi_1^{(0)}$	0.176	0.060	(0.058, 0.294)
$\psi_2^{(0)}$	0.189	0.063	(0.662, 0.312)
$\psi^{(0)}$	0.182	0.038	(0.107, 0.258)

(b)

Measure	Estimated measure	Standard error	Confidence interval
φ_1	0.512	0.089	(0.337, 0.688)
φ_2	0.000	0.115	(−0.226, 0.226)
φ_{MPS}	0.256	0.088	(0.083, 0.429)
$\psi_1^{(0)}$	0.159	0.058	(0.047, 0.272)
$\psi_2^{(0)}$	0.007	0.013	(−0.019, 0.033)
$\psi^{(0)}$	0.083	0.028	(0.029, 0.138)

Also, from Table 3a, the estimated value of the submeasure φ_2 is −0.584 (< 0), and the confidence interval for φ_2 does not include zero. These would indicate patients' reported time after treatment in the Active treatment group is estimated to be 58.4% of the maximum departure toward the column lower complete asymmetry with respect to the midpoint. Because the absolute values of submeasures φ_1 and φ_2 are almost the same, the measure φ_{MPS} is estimated to be close to zero, and the confidence interval for φ_{MPS} includes zero. However, the estimated values of $\psi_1^{(0)}$ and $\psi_2^{(0)}$ are 0.176 and 0.189, respectively. Because these values are almost the same, the estimated value of the measure $\psi^{(0)}$ is also close to $\psi_1^{(0)}$ and $\psi_2^{(0)}$.

From Table 3b, for the data in Table 2b the estimated value of the submeasure φ_1 is 0.512 (> 0), and the confidence interval for φ_1 does not include zero. These indicates patients' reported time before treatment in the Placebo treatment group is estimated to be 51.2% of the maximum departure toward the row upper complete asymmetry with respect to the midpoint. From Table 3b, the estimated value of the submeasure φ_2 is 0.000, and the confidence interval for φ_2 includes zero. These indicates there is a structure of the column average point-symmetry for patient's reported time after treatment in the Placebo treatment group. The estimated values of the measures $\psi_1^{(0)}$, $\psi_2^{(0)}$, and $\psi^{(0)}$ show a similar trend with the estimated values of the measures φ_1, φ_2, and φ_{MPS}, respectively.

In addition, when the data in Tables 2a and 2b are compared using the estimated submeasure φ_1, the degree of departure toward the row upper complete asymmetry with respect to the midpoint is almost same for the data in Tables 2a and 2b. However, when comparing using the estimated submeasure φ_2, for patients' reported time after treatment, the degree of departure toward the column lower complete asymmetry with respect to the midpoint is greater in the Active treatment than in the Placebo treatment. Therefore, patients' reported time after treatment in the Active treatment would tend to be shorter than that in the Placebo treatment. These interpretation cannot be obtained by using the measures $\psi_1^{(0)}$, $\psi_2^{(0)}$, and $\psi^{(0)}$.

Conclusion

The measure φ_{MPS} and submeasures φ_1 and φ_2 can distinguish two kinds of complete asymmetry with respect to the midpoint although the measure $\psi^{(\lambda)}$ in Yamamoto et al. (2011) cannot distinguish them. Because the measure φ_{MPS} lies between −1 and 1 (although the measure $\psi^{(\lambda)}$ lies between 0 and 1) without the dimension and the sample size, φ_{MPS} is useful for comparing the degrees of departure from marginal average point-symmetry in several tables. When the marginal proportional point-symmetry model holds, the measure φ_{MPS} are represented by two parameters: Δ_1 and Δ_2. $|\varphi_{MPS}|$ increases as the value of Δ_1 (Δ_2) approaches zero or infinity from 1. Therefore, φ_{MPS} would be adequate for representing the degrees of departure from the marginal point-symmetry model distinguishing two kinds of marginal asymmetry with respect to the midpoint.

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Appendix 1

For an $R \times C$ contingency table with ordered categories, the measure of departure from the marginal point-symmetry model considered by Yamamoto et al. (2011) is given as follows: assuming that $\{p_{i\cdot} + p_{i^*\cdot} \neq 0\}$ and $\{p_{\cdot j} + p_{\cdot j^{**}} \neq 0\}$, for $\lambda > -1$,

$$\psi^{(\lambda)} = \frac{\delta_1 \psi_1^{(\lambda)} + \delta_2 \psi_2^{(\lambda)}}{\delta_1 + \delta_2},$$

where

$$\delta_1 = \sum_{i=1}^{[R/2]} \left(p_{i\cdot} + p_{i^*\cdot} \right), \quad \delta_2 = \sum_{j=1}^{[C/2]} \left(p_{\cdot j} + p_{\cdot j^{**}} \right),$$
$$q_{i\cdot} = \frac{p_{i\cdot}}{\delta_1}, \quad q_{i^*\cdot} = \frac{p_{i^*\cdot}}{\delta_1}, \quad q_{i\cdot}^{\mathrm{MPS}} = \frac{q_{i\cdot} + q_{i^*\cdot}}{2} \quad \left(i = 1, \ldots, \left[\frac{R}{2} \right] \right),$$
$$q_{\cdot j} = \frac{p_{\cdot j}}{\delta_2}, \quad q_{\cdot j^{**}} = \frac{p_{\cdot j^{**}}}{\delta_2}, \quad q_{\cdot j}^{\mathrm{MPS}} = \frac{q_{\cdot j} + q_{\cdot j^{**}}}{2} \quad \left(j = 1, \ldots, \left[\frac{C}{2} \right] \right),$$

with

$$\psi_1^{(\lambda)} = \frac{1}{2^{\lambda} - 1} \sum_{i=1}^{[R/2]} \left[q_{i\cdot} \left\{ \left(\frac{q_{i\cdot}}{q_{i\cdot}^{\mathrm{MPS}}} \right)^{\lambda} - 1 \right\} + q_{i^*\cdot} \left\{ \left(\frac{q_{i^*\cdot}}{q_{i\cdot}^{\mathrm{MPS}}} \right)^{\lambda} - 1 \right\} \right],$$
$$\psi_2^{(\lambda)} = \frac{1}{2^{\lambda} - 1} \sum_{j=1}^{[C/2]} \left[q_{\cdot j} \left\{ \left(\frac{q_{\cdot j}}{q_{\cdot j}^{MPS}} \right)^{\lambda} - 1 \right\} + q_{\cdot j^{**}} \left\{ \left(\frac{q_{\cdot j^{**}}}{q_{\cdot j}^{MPS}} \right)^{\lambda} - 1 \right\} \right],$$

and the value at $\lambda = 0$ is taken to be the limit as $\lambda \to 0$. Note that (i) $0 \leq \psi^{(\lambda)} \leq 1$, (ii) $\psi^{(\lambda)} = 0$ if and only if the marginal point-symmetry model holds, and (iii) $\psi^{(\lambda)} = 1$ if and only if $p_{i\cdot} = 0$ (then $p_{i^*\cdot} > 0$) or $p_{i^*\cdot} = 0$ (then $p_{i\cdot} > 0$) for $i = 1, \ldots, [R/2]$ and $p_{\cdot j} = 0$ (then $p_{\cdot j^{**}} > 0$) or $p_{\cdot j^{**}} = 0$ (then $p_{\cdot j} > 0$) for $j = 1, \ldots, [C/2]$.

The sample version of $\psi^{(\lambda)}$, $\hat{\psi}^{(\lambda)}$, is given by $\psi^{(\lambda)}$ with $\{p_{ij}\}$ replaced by $\{\hat{p}_{ij}\}$. Yamamoto et al. (2011) also gave that $\sqrt{n}\left(\hat{\psi}^{(\lambda)} - \psi^{(\lambda)}\right)$ has asymptotically (as $n \to \infty$) a normal distribution with mean zero and variance

$$\sigma^2\left[\psi^{(\lambda)}\right]=\frac{1}{\left(\delta_1+\delta_2\right)^2}\sum_{i=1}^{R}\sum_{j=1}^{C}\left(w_{ij}^{(\lambda)}-d_{ij}\psi^{(\lambda)}\right)^2 p_{ij}\ ,$$

where $\sigma^2[\psi^{(0)}] = \lim_{\lambda\to 0} \sigma^2[\psi^{(\lambda)}]$, and

$$\begin{aligned} w_{ij}^{(\lambda)} &= d_{1(i)}\psi_1^{(\lambda)}+\delta_1\Delta_{1(i)}^{(\lambda)}+d_{2(j)}\psi_2^{(\lambda)}+\delta_2\Delta_{2(j)}^{(\lambda)}, \\ d_{ij} &= d_{1(i)}+d_{2(j)}, \end{aligned}$$

with

$$d_{1(i)}=\begin{cases}0 & R \text{ odd and } i=\dfrac{R+1}{2}\\ 1 & \text{otherwise,}\end{cases}$$
$$d_{2(j)}=\begin{cases}0 & C \text{ odd and } j=\dfrac{C+1}{2}\\ 1 & \text{otherwise,}\end{cases}$$

and, for $\lambda > -1$, $\lambda \neq 0$,

$$\Delta_{1(i)}^{(\lambda)}=\begin{cases}0 & R \text{ odd, } i=\dfrac{R+1}{2},\\ \dfrac{1}{\delta_1}\left[1-\psi_1^{(\lambda)}-\dfrac{2^\lambda}{2^\lambda-1}\left\{1-\left(q_{i\cdot}^c\right)^\lambda-\lambda q_{i^*\cdot}^c\left(\left(q_{i\cdot}^c\right)^\lambda-\left(q_{i^*\cdot}^c\right)^\lambda\right)\right\}\right] & \text{otherwise,}\end{cases}$$
$$\Delta_{2(j)}^{(\lambda)}=\begin{cases}0 & C \text{ odd, } j=\dfrac{C+1}{2}\\ \dfrac{1}{\delta_2}\left[1-\psi_2^{(\lambda)}-\dfrac{2^\lambda}{2^\lambda-1}\left\{1-\left(q_{\cdot j}^c\right)^\lambda-\lambda q_{\cdot j^{**}}^c\left(\left(q_{\cdot j}^c\right)^\lambda-\left(q_{\cdot j^{**}}^c\right)^\lambda\right)\right\}\right] & \text{otherwise,}\end{cases}$$

$$q_{i\cdot}^c=\frac{q_{i\cdot}}{q_{i\cdot}+q_{i^*\cdot}},\quad q_{i^*\cdot}^c=\frac{q_{i^*\cdot}}{q_{i\cdot}+q_{i^*\cdot}}\quad \left(i=1,\ldots,\left[\frac{R}{2}\right]\right),$$
$$q_{\cdot j}^c=\frac{q_{\cdot j}}{q_{\cdot j}+q_{\cdot j^{**}}},\quad q_{\cdot j^{**}}^c=\frac{q_{\cdot j^{**}}}{q_{\cdot j}+q_{\cdot j^{**}}}\quad \left(j=1,\ldots,\left[\frac{C}{2}\right]\right).$$

Appendix 2

The code is available in R. The data matrix formed by observed frequencies should be an $r \times c$ matrix named **m**. The function mod.MPS gives a result of a 3×4 matrix, including estimated measure, standard error, and 95% Confidence interval for three kinds of measures.

```
file <- "table"
m <- read.table(paste(file,".txt",sep=""))
N <- sum(m)
r <- nrow(m)
c <- ncol(m)
p <- m/N

mod.MPS <- function(){

  rend <- ifelse((r%%2)==0,r/2,(r-1)/2)
  cend <- ifelse((c%%2)==0,c/2,(c-1)/2)
  p1 <- apply(p,1,sum)
  p2 <- apply(p,2,sum)
  delta1 <- sum(p1[1:rend]) + sum(p1[(r-rend+1):r])
  delta2 <- sum(p2[1:cend]) + sum(p2[(c-cend+1):c])
  q1 <- p1/delta1
  q2 <- p2/delta2

  phi1 <- 0
  phi2 <- 0
  for(i in 1:rend){
    phi1 <- phi1 + (q1[i]+q1[r-i+1])*(acos(p1[i]/sqrt(p1[i]^2+p1[r-
    i+1]^2))-pi/4)*4/pi
  }
  for(j in 1:cend){
    phi2 <- phi2 + (q2[j]+q2[c-j+1])*(acos(p2[j]/sqrt(p2[j]^2+p2[c-
    j+1]^2))-pi/4)*4/pi
  }
  phi.MPS <- (delta1*phi1+delta2*phi2)/(delta1+delta2)

```

```
phi1p <- matrix(c(0),r,c)
phi2p <- matrix(c(0),r,c)
for(k in 1:rend){
  for(l in 1:c){
    phi1p[k,l] <- 4*(acos(p1[k]/sqrt(p1[k]^2+p1[r-k+1]^2)) -
    (p1[k]+p1[r-k+1])*p1[r-k+1]/(p1[k]^2+p1[r-k+1]^2))/(pi*delta1) -
    (phi1+1)/delta1
    phi1p[r-k+1,l] <- 4*(acos(p1[k]/sqrt(p1[k]^2+p1[r-k+1]^2)) +
    (p1[k]+p1[r-k+1])*p1[k]/(p1[k]^2+p1[r-k+1]^2))/(pi*delta1) -
    (phi1+1)/delta1
  }
}
for(k in 1:r){
  for(l in 1:cend){
    phi2p[k,l] <- 4*(acos(p2[l]/sqrt(p2[l]^2+p2[c-l+1]^2)) -
    (p2[l]+p2[c-l+1])*p2[c-l+1]/(p2[l]^2+p2[c-l+1]^2))/(pi*delta2) -
    (phi2+1)/delta2
    phi2p[k,c-l+1] <- 4*(acos(p2[l]/sqrt(p2[l]^2+p2[c-l+1]^2)) +
    (p2[l]+p2[c-l+1])*p2[l]/(p2[l]^2+p2[c-l+1]^2))/(pi*delta2) -
    (phi2+1)/delta2
  }
}
delta1p <- matrix(c(0),r,c)
delta2p <- matrix(c(0),r,c)
for(k in 1:rend){
  for(l in 1:c){
    delta1p[k,l] <- 1
    delta1p[r-k+1,l] <- 1
  }
}
for(k in 1:r){
  for(l in 1:cend){
    delta2p[k,l] <- 1
    delta2p[k,c-l+1] <- 1
  }
}

sigma1 <- sum(p*phi1p^2) - sum(p*phi1p)^2
```

```
  min1 <- phi1 - 1.96*sqrt(sigma1/N)
  max1 <- phi1 + 1.96*sqrt(sigma1/N)
  phi1set <- c(phi1,sqrt(sigma1/N),min1,max1)

  sigma2 <- sum(p*phi2p^2) - sum(p*phi2p)^2
  min2 <- phi2 - 1.96*sqrt(sigma2/N)
  max2 <- phi2 + 1.96*sqrt(sigma2/N)
  phi2set <- c(phi2,sqrt(sigma2/N),min2,max2)

  phiMPSp <- ((delta1+delta2)*(delta1*phi1p+delta2*phi2p) + (phi1-
  phi2)*(delta2*delta1p-delta1*delta2p))/(delta1+delta2)^2
  sigmaMPS <- sum(p*phiMPSp^2) - sum(p*phiMPSp)^2
  minMPS <- phi.MPS - 1.96*sqrt(sigmaMPS/N)
  maxMPS <- phi.MPS + 1.96*sqrt(sigmaMPS/N)
  phiMPSset <- c(phi.MPS,sqrt(sigmaMPS/N),minMPS,maxMPS)

  return(rbind(phi1set,phi2set,phiMPSset))

}
```