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Journal of Modern Applied Statistical Methods

Volume 17 | Issue 2 Article 10

3-12-2019

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Recommended Citation

Jan, U., & Ahmad, S. P. (2018). Bayesian approximation techniques for scale parameter of Laplace distribution. *Journal of Modern Applied Statistical Methods, 17*(2), eP2625. doi: 10.22237/jmasm/1552409925

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Erratum

In the original published version of this article, the caption for Table 4 was incorrectly given as "Posterior estimates of Laplace distribution using normal approximation" instead of "Posterior estimates of Laplace distribution using T-K approximation". This has been corrected.

This regular article is available in Journal of Modern Applied Statistical Methods: https://digitalcommons.wayne.edu/jmasm/vol17/iss2/10

Journal of Modern Applied Statistical Methods
November 2018, Vol. 17, No. 2, eP2625
doi: 10.22237/jmasm/1552409925
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ISSN 1538 − 9472

Bayesian Approximation Techniques for Scale Parameter of Laplace Distribution

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The Bayesian estimation of the scale parameter of a Laplace Distribution is obtained using two approximation techniques, like Normal approximation and Tierney and Kadane (T-K) approximation, under different informative priors.

Keywords: Bayesian estimation, prior distribution, normal approximation, T-K approximation

Introduction

The Laplace distribution is a continuous probability distribution named after Pierre Simon Laplace (1749-1827) who, in 1774, obtained it as the distribution whose likelihood is maximized when the location parameter is set to the median. It is also known as the law of the difference between two independent variables with identical exponential distributions (Abramowitz & Stegun, 1972), as the double exponential distribution because it can be thought of as two exponential distributions spliced together back to back, as well as the two-tailed exponential distribution and the bilateral exponential law (Feller, 1962). Aryal (2006) studied the Laplace and related probability distributions along with their applications to find a probability distribution that could be derived from the Laplace distribution and could be used to frame models for various real-world problems. Nadarajah (2010) obtained two posterior distributions for the mean of the Laplace distribution by deriving the distributions of the product XY and the ratio $X \mid Y$ when X and Y are Student's t and Laplace random variables distributed independently of each other. Abbasi (2011) considered the Bayesian aspect of a discrete Laplace distribution, and compared the Bayes estimator with that of maximum entropy for the discrete Laplace distribution. Ali, Aslam, Abbas, and Ali Kazmi (2012) estimated the scale

doi: 10.22237/jmasm/1552409925 | Accepted: September 9, 2017; Published: March 12, 2019.
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parameter of the Laplace model using different asymmetric loss functions. The estimates were compared using the posterior risks (PRs) under these loss functions. Rasheed and Emad (2015) obtained Bayesian and non-Bayesian estimates for Laplace distribution under different loss functions and made comparison between them through Monte Carlo simulation, depending on mean square errors (MSEs).

The classical Laplace distribution, also known as first law of Laplace, is a probability distribution on $(-\infty < x < \infty)$ given by the density function

$$\mathrm{f}\left(x\right)=\frac{1}{2\lambda}\exp\left[-\frac{\left|x\right|}{\lambda}\right] \quad \lambda>0,-\infty<x<\infty . \tag{1}$$

Here, λ is a scale parameter. It is a symmetric distribution whose tails fall off less sharply than the Gaussian distribution but faster than the Cauchy distribution. Both the normal and Laplace distributions can be used to analyze symmetric data. It is well-known that the normal distribution is used to analyze symmetric data with short tails, whereas the Laplace model is used for data with long tails. Various forms of the skew Laplace distribution have been introduced and applied in several areas including medical science, environmental science, communications, economics and finance, etc. It can be used to model the difference between the waiting times of two events generated by two random processes. It can also be used to describe breaking strength data, modeling the differences in flood stages, etc. (Krishnamoorthy, 2006).

Normal Approximation

When the posterior distribution $\mathrm{P}(\lambda \mid x)$ is unimodal and roughly symmetric, the convenient procedure is to approximate it by a normal distribution centered at the mode, yielding the approximation

$$\mathrm{P}\left(\lambda \mid x\right)\sim \mathrm{N}\left\{\hat{\lambda},\left[\mathrm{I}\left(\hat{\lambda}\right)\right]^{-1}\right\}, \text{where } \mathrm{I}\left(\hat{\lambda}\right)=\frac{-\partial^2}{\partial\lambda^2}\log \mathrm{P}\left(\lambda \mid x\right). \tag{2}$$

If the mode, $\hat{\lambda}$, is in the interior parameter space, then $\mathrm{I}(\lambda)$ is positive; if $\hat{\lambda}$ is a vector parameter, then $\mathrm{I}(\lambda)$ is a matrix.

This was reviewed by Ahmed, Khan, and Ahmad (2007) and Ahmad, Ahmed, and Khan (2011), who discussed the Bayesian analysis of exponential and gamma distribution using normal and Laplace approximations. Further, Sultan and Ahmad

(2015a, b) obtained the Bayes' estimates of shape parameters of the Topp-Leone and Kumaraswamy distributions under different priors using Bayesian approximation techniques. Moreover, Jan and Ahmad (2016) studied the behavior of shape parameter of the inverse Lomax distribution using Bayesian approximation techniques.

The likelihood function for a random sample (x_1, x_2,…, x_n), which is taken from the Laplace distribution (1), is given as

$$\mathrm{L}\left(x \mid \lambda\right) = \frac{1}{2^n \lambda^n} \exp\left[-\frac{\sum_{i=1}^{n} \left| x_i \right|}{\lambda} \right]. \tag{3}$$

Using inverted gamma prior

$$\mathrm{f}\left(\lambda\right) = \frac{\beta^\alpha}{\Gamma\left(\alpha\right)} \left(\frac{1}{\lambda}\right)^{\alpha+1} \exp\left[-\frac{\beta}{\lambda} \right], \quad \alpha, \beta, \lambda > 0,$$

where α and β are known hyperparameters, the posterior distribution for λ is

$$\mathrm{P}\left(\lambda \mid x\right) \propto \left(\frac{1}{\lambda}\right)^{n+\alpha+1} \exp\left[-\frac{\left(\beta + T\right)}{\lambda} \right], \quad \text{where } T = \sum_{i=1}^{n} \left| x_i \right|, \tag{4}$$

which is the density similar to the inverted Gamma distribution.

The log posterior is

$$\log \mathrm{P}\left(\lambda \mid x\right) = -\left(n + \alpha + 1\right) \log \lambda - \frac{\left(\beta + T\right)}{\lambda}.$$

The first derivative is

$$\frac{\partial}{\partial \lambda} \log \mathrm{P}\left(\lambda \mid x\right) = \frac{-\left(n + \alpha + 1\right)}{\lambda} + \frac{\left(\beta + T\right)}{\lambda^2},$$

and the posterior mode is obtained as

$$\hat{\lambda} = \frac{\beta + T}{n + \alpha + 1}.$$

The second-order derivative of the log posterior density is given by

$$\frac{\partial^2}{\partial \lambda^2} \log P(\lambda \mid x) = \frac{(n + \alpha + 1)}{\lambda^2} - 2\frac{(\beta + T)}{\lambda^3}.$$

Therefore, the negative Hessian is

$$I(\hat{\lambda}) = \frac{-\partial^2 \log P(\lambda \mid x)}{\partial \lambda^2} = \frac{(n + \alpha + 1)}{\lambda^2} + \frac{2(\beta + T)}{\lambda^3}$$

$$\Rightarrow \left[I(\hat{\lambda}) \right]^{-1} = \frac{(\beta + T)^2}{(n + \alpha + 1)^3}$$

Thus, the posterior distribution can be approximated as

$$P(\lambda \mid x) \sim N\left[\left(\frac{\beta + T}{n + \alpha + 1} \right), \frac{(\beta + T)^2}{(n + \alpha + 1)^3} \right]. \quad (5)$$

Using an inverted chi-square prior,

$$f(\lambda) = \frac{\left(\frac{b}{2}\right)^{\frac{a}{2}}}{2^{\frac{a}{2}}} \left(\frac{1}{\lambda}\right)^{\frac{a}{2}+1} \exp\left[-\frac{b}{2\lambda} \right], \quad a, b, \lambda > 0$$

with hyper parameters (a, b); the posterior distribution of λ for the given data ($x_1, x_2, \ldots, x_n$) is given by

$$P(\lambda \mid x) \propto \frac{1}{\lambda^{\left(\frac{a}{2}+n+1\right)}} \exp\left[-\frac{\left[\frac{b}{2} + T\right]}{\lambda} \right], \quad \lambda > 0 \quad (6)$$

The log posterior is

$$\log \mathrm{P}(\lambda \mid x) = -\left(\frac{a}{2}+n+1\right)\log\lambda - \frac{\left(\frac{b}{2}+T\right)}{\lambda}$$

The first derivative is

$$\frac{\partial}{\partial\lambda}\log \mathrm{P}(\lambda \mid x) = \frac{-\left(\frac{a}{2}+n+1\right)}{\lambda} + \frac{\left(\frac{b}{2}+T\right)}{\lambda^2},$$

and the posterior mode is obtained as

$$\hat{\lambda} = \frac{\left(\frac{b}{2}+T\right)}{\left(\frac{a}{2}+n+1\right)}.$$

The second-order derivative of the log posterior density is given by

$$\mathrm{I}\left(\hat{\lambda}\right) = \frac{-\partial^2 \log \mathrm{P}(\lambda \mid x)}{\partial\lambda^2} = \frac{\left(n+\frac{a}{2}+1\right)^3}{\left(\frac{b}{2}+T\right)^2}.$$

$$\Rightarrow \left[\mathrm{I}\left(\hat{\lambda}\right)\right]^{-1} = \frac{\left(\frac{b}{2}+T\right)^2}{\left(n+\frac{a}{2}+1\right)^3}$$

Thus, the posterior distribution can be approximated as

$$\mathrm{P}(\lambda \mid x) \sim \mathrm{N}\left[\left(\frac{\frac{b}{2}+T}{n+\frac{a}{2}+1}\right), \frac{\left(\frac{b}{2}+T\right)^2}{\left(n+\frac{a}{2}+1\right)^3}\right]. \tag{7}$$

Using a Gumbel type II prior, an informative prior with hyperparameter γ,

$$\mathrm{f}(\lambda) = \gamma\left(\frac{1}{\lambda}\right)^2 \exp\left(\frac{-\gamma}{\lambda}\right), \quad \gamma, \lambda > 0,$$

we obtain the posterior distribution of λ for the given data ($x_1, x_2, \ldots, x_n$) as

$$\mathrm{P}(\lambda \mid x) \propto \frac{1}{\lambda^{n+2}} \exp\left[-\frac{(\gamma+T)}{\lambda}\right]. \tag{8}$$

The log posterior is given by

$$\log \mathrm{P}(\lambda \mid x) = -(2+n)\log\lambda - \frac{(\gamma+T)}{\lambda},$$

and the posterior mode is obtained as

$$\hat{\lambda} = \frac{\gamma+T}{n+2}.$$

The second-order derivative of the log posterior density is given as

$$\frac{-\partial^2}{\partial\lambda^2}\log \mathrm{P}(\lambda \mid x) = \frac{n+2}{\lambda^2} - \frac{2(\gamma+T)}{\lambda^3}.$$

The negative Hessian is

$$\mathrm{I}(\hat{\lambda}) = \frac{-\partial^2}{\partial\lambda^2}\log \mathrm{P}(\lambda \mid x) = \frac{(n+2)^3}{(\gamma+T)^2}.$$

Therefore,

$$\left[\mathrm{I}\left(\hat{\lambda}\right)\right]^{-1} = \frac{\left(\gamma + T\right)^2}{\left(n+2\right)^3}$$

Thus, the posterior distribution can be approximated as

$$\mathrm{P}\left(\lambda \mid x\right) \sim \mathrm{N}\left[\left(\frac{\gamma + T}{n+2}\right), \frac{\left(\gamma + T\right)^2}{\left(n+2\right)^3}\right] \tag{9}$$

T-K Approximation

Lindley's approximation method requires the computation of higher-order partial derivatives, which is usually tedious to calculate when the parameter λ is vector-valued. Tierney and Kadane (1986) gave the Laplace method to evaluate $\mathrm{E}[\mathrm{h}(\lambda) \mid x]$ as

$$\mathrm{E}\left[\mathrm{h}\left(\lambda\right) \mid x\right] \cong \frac{\hat{\varphi}^* \exp\left[-n\,\mathrm{h}^*\left(\hat{\lambda}^*\right)\right]}{\varphi \exp\left[-n\,\mathrm{h}\left(\hat{\lambda}\right)\right]} \tag{10}$$

where

$$-n\,\mathrm{h}\left(\hat{\lambda}\right) = \log \mathrm{P}\left(\lambda \mid x\right), \quad -n\,\mathrm{h}^*\left(\hat{\lambda}^*\right) = \log \mathrm{P}\left(\lambda \mid x\right) + \log \mathrm{h}\left(\lambda\right),$$

$$\hat{\varphi}^2 = -\left[-n\,\mathrm{h}''\left(\hat{\lambda}\right)\right]^{-1}, \quad \hat{\varphi}^{*2} = -\left[-n\,\mathrm{h}''^*\left(\hat{\lambda}^*\right)\right]^{-1}$$

Thus, for the Laplace model, T-K approximation for the scale parameter λ is obtained, under inverted gamma prior, as

$$\mathrm{f}\left(\lambda\right) \propto \left(\frac{1}{\lambda}\right)^{\alpha+1} \exp\left[-\frac{\beta}{\lambda}\right],$$

the posterior distribution for the scale parameter λ is calculated in (4)

$$-n\,\mathrm{h}(\lambda) = -(n+\alpha+1)\log\lambda - \frac{(\beta+T)}{\lambda}; \quad -n\,\mathrm{h}'(\lambda) = \frac{-(n+\alpha+1)}{\lambda} + \frac{\beta+T}{\lambda^2}$$

$$\Rightarrow \hat{\lambda} = \frac{\beta+T}{n+\alpha+1}$$

Also,

$$-n\,\mathrm{h}''(\hat{\lambda}) = -\frac{(n+\alpha+1)^3}{(\beta+T)^2}.$$

Therefore,

$$\hat{\varphi}^2 = \frac{(\beta+T)^2}{(n+\alpha+1)^3} \quad \text{or} \quad \hat{\varphi} = \frac{\beta+T}{(n+\alpha+1)^{\frac{3}{2}}}.$$

Now,

$$-n\,\mathrm{h}^*(\lambda^*) = -n\,\mathrm{h}(\lambda) + \log\mathrm{h}(\lambda) = (n+\alpha)\log\lambda^* - \frac{(\beta+T)}{\lambda^*}.$$

Also,

$$-n\,\mathrm{h}'^*(\lambda^*) = \frac{-(n+\alpha)}{\lambda^*} + \frac{\beta+T}{\lambda^{*2}} \Rightarrow \hat{\lambda}^* = \frac{\beta+T}{n+\alpha}.$$

Further,

$$-n\,\mathrm{h}''^*(\hat{\lambda}^*) = \frac{(n+\alpha)}{\lambda^2} - \frac{2(\beta+T)}{\lambda^3} = -\frac{(n+\alpha)^3}{(\beta+T)^2} \Rightarrow \hat{\varphi}^* = \frac{\beta+T}{(n+\alpha)^{\frac{3}{2}}}.$$

Using the values in (10),

$$\mathrm{E}(\lambda \mid x) = \left(\frac{n+\alpha+1}{n+\alpha}\right)^{\frac{3}{2}} \frac{\exp\left[-(n+\alpha)\log\hat{\lambda}^{*} - \frac{\beta+T}{\hat{\lambda}^{*}}\right]}{\exp\left[-(n+\alpha+1)\log\hat{\lambda} - \frac{\beta+T}{\hat{\lambda}}\right]} \tag{11}$$

$$= \left(\frac{n+\alpha}{n+\alpha+1}\right)^{n+\alpha-\frac{3}{2}} \left(\frac{\beta+T}{n+\alpha+1}\right) e$$

Similarly,

$$\mathrm{E}(\lambda^2 \mid x) = \frac{\hat{\varphi}^{*} \exp\left[-n\,\mathrm{h}^{*}\left(\hat{\lambda}^{*}\right)\right]}{\hat{\varphi} \exp\left[-n\,\mathrm{h}\left(\hat{\lambda}\right)\right]};$$

here, $-n\,\mathrm{h}^{*}\left(\hat{\lambda}^{*}\right) = \log\lambda^{2} - n\,\mathrm{h}(\lambda)$.

$$-n\,\mathrm{h}^{*}\left(\hat{\lambda}^{*}\right) = -(n+\alpha-1)\log\lambda - \frac{(\beta+T)}{\lambda} \text{ and}$$

$$-n\,\mathrm{h}'^{*}\left(\hat{\lambda}^{*}\right) = \frac{-(n+\alpha-1)}{\lambda} + \frac{\beta+T}{\lambda^{2}}$$

$$\Rightarrow \hat{\lambda}^{*} = \frac{\beta+T}{n+\alpha-1}$$

Now,

$$-n\,\mathrm{h}''^{*}\left(\hat{\lambda}^{*}\right) = \frac{(n+\alpha-1)}{\lambda^{2}} - 2\frac{(\beta+T)}{\lambda^{3}} = -\frac{(n+\alpha-1)^{3}}{(\beta+T)^{2}}.$$

Then,

$$\hat{\varphi}^{*} = \frac{\beta+T}{(n+\alpha-1)^{\frac{3}{2}}}.$$

Therefore,

$$\mathrm{E}\left[\lambda^2 \mid x\right]=\left(\frac{n+\alpha+1}{n+\alpha-1}\right)^{\frac{3}{2}} \frac{\exp\left[-(n+\alpha-1)\log\hat{\lambda}^* - \frac{(\beta+T)}{\hat{\lambda}^*}\right]}{\exp\left[-(n+\alpha+1)\log\lambda - \frac{(\beta+T)}{\hat{\lambda}}\right]}$$

$$=\left(\frac{n+\alpha-1}{n+\alpha+1}\right)^{n+\alpha-\frac{5}{2}}\left(\frac{\beta+T}{n+\alpha+1}\right)^2 e^2$$

Hence,

$$\text{Variance} = \mathrm{E}\left[\lambda^2 \mid x\right]-\left\{\mathrm{E}\left[\lambda \mid x\right]\right\}^2$$

$$=\left(\frac{n+\alpha-1}{n+\alpha+1}\right)^{n+\alpha-\frac{5}{2}} \frac{(\beta+T)^2}{(n+\alpha+1)^2} e^2-\left[\left(\frac{n+\alpha}{n+\alpha+1}\right)^{n+\alpha-\frac{3}{2}}\left(\frac{\beta+T}{n+\alpha+1}\right)e\right]^2$$

Using an inverted chi square prior,

$$\mathrm{f}(\lambda) \propto \left(\frac{1}{\lambda}\right)^{\frac{a}{2}+1} \exp\left[\frac{-b}{2\lambda}\right],$$

the posterior distribution for λ is given by equation (6):

$$-n\,\mathrm{h}(\lambda)=-\left(n+\frac{a}{2}+1\right)\log\lambda-\frac{\left(\frac{b}{2}+T\right)}{\lambda}; \quad -n\,\mathrm{h}'(\lambda)=\frac{-\left(n+\frac{a}{2}+1\right)}{\lambda}+\frac{\frac{b}{2}+T}{\lambda^2}$$

$$\Rightarrow \hat{\lambda}=\frac{\frac{b}{2}+T}{n+\frac{a}{2}+1}$$

Now,

$$-n\,\mathrm{h}''\left(\hat{\lambda}\right) = -\frac{-\left(n+\frac{a}{2}+1\right)^{3}}{\left(\frac{b}{2}+T\right)^{2}} \Rightarrow \hat{\varphi} = \frac{\frac{b}{2}+T}{\left(n+\frac{a}{2}+1\right)^{\frac{3}{2}}}.$$

Also,

$$-n\,\mathrm{h}^{*}\left(\lambda^{*}\right) = -n\,\mathrm{h}\left(\lambda\right) + \log \mathrm{h}\left(\lambda\right) = -\left(n+\frac{a}{2}\right)\log\lambda - \frac{\left(\frac{b}{2}+T\right)}{\lambda}$$

$$-n\,\mathrm{h}'^{*}\left(\lambda^{*}\right) = -\frac{\left(n+\frac{a}{2}\right)}{\lambda} + \frac{\frac{b}{2}+T}{\lambda^{2}} \Rightarrow \hat{\lambda}^{*} = \frac{\frac{b}{2}+T}{n+\frac{a}{2}}$$

Then,

$$-n\,\mathrm{h}''\left(\hat{\lambda}^{*}\right) = -\frac{\left(n+\frac{a}{2}\right)^{3}}{\left(\frac{b}{2}+T\right)^{2}} \Rightarrow \hat{\varphi}^{*} = \frac{\frac{b}{2}+T}{\left(n+\frac{a}{2}\right)^{\frac{3}{2}}}.$$

Therefore,

$$\mathrm{E}\left[\lambda \mid x\right] = \frac{\hat{\varphi}^{*}\exp\left[-n\,\mathrm{h}^{*}\left(\hat{\lambda}^{*}\right)\right]}{\hat{\varphi}\exp\left[-nh\left(\hat{\lambda}\right)\right]} = \left(\frac{n+\frac{a}{2}}{n+\frac{a}{2}+1}\right)^{n+\frac{a}{2}-\frac{3}{2}}\left(\frac{\frac{b}{2}+T}{n+\frac{a}{2}+1}\right)e. \qquad (12)$$

Now,

$$\mathrm{E}\left(\lambda^{2} \mid x\right) = \frac{\hat{\varphi}^{*}\exp\left[-n\,\mathrm{h}^{*}\left(\hat{\lambda}^{*}\right)\right]}{\hat{\varphi}\exp\left[-nh\left(\hat{\lambda}\right)\right]}.$$

Here $-n\,\mathrm{h}^*\left(\hat{\lambda}^*\right)=\log\lambda^2-n\,\mathrm{h}(\lambda)$,

$$\Rightarrow \mathrm{E}\left[\lambda^2\mid x\right]=\left(\frac{n+\frac{a}{2}-1}{n+\frac{a}{2}+1}\right)^{n+\frac{a}{2}-\frac{5}{2}}\left(\frac{\frac{b}{2}+T}{n+\frac{a}{2}+1}\right)e^2,$$

$$\mathrm{V}\left[\lambda\mid x\right]=\left(\frac{n+\frac{a}{2}-1}{n+\frac{a}{2}+1}\right)^{n+\frac{a}{2}-\frac{5}{2}}\left(\frac{\frac{b}{2}+T}{n+\frac{a}{2}+1}\right)^2 e^2-\left\{\left(\frac{n+\frac{a}{2}}{n+\frac{a}{2}+1}\right)^{n+\frac{a}{2}-\frac{3}{2}}\left(\frac{\frac{b}{2}+T}{n+\frac{a}{2}+1}\right)e\right\}^2$$

Using a Gumbel type II prior,

$$\mathrm{f}(\lambda)\propto\left(\frac{1}{\lambda}\right)^2\exp\left(-\frac{\gamma}{\lambda}\right),\quad \gamma,\lambda>0,$$

the posterior distribution for the parameter λ is given by (9):

$$-n\,\mathrm{h}(\lambda)=\log\mathrm{P}(\lambda\mid x)=-(n+2)\log\lambda-\frac{(\gamma+T)}{\lambda},$$

and

$$-n\,\mathrm{h}'(\lambda)=\frac{-(n+2)}{\lambda}+\frac{(\gamma+T)}{\lambda^2}\Rightarrow\hat{\lambda}=\frac{\gamma+T}{n+2}$$

Also,

$$-n\,\mathrm{h}''\left(\hat{\lambda}\right)=-\frac{-(n+2)^3}{(\gamma+T)^2}\Rightarrow\hat{\varphi}=\frac{\gamma+T}{(n+2)^{\frac{3}{2}}}.$$

Now,

$$-n\,\mathrm{h}^{*}\left(\lambda^{*}\right)=-n\,\mathrm{h}\left(\lambda\right)+\log\mathrm{h}\left(\lambda\right)=-\left(n+1\right)\log\lambda-\frac{\left(\gamma+T\right)}{\lambda}$$

Then,

$$-n\,\mathrm{h}'^{*}\left(\lambda^{*}\right)=-\frac{-\left(n+1\right)}{\lambda}+\frac{\left(\gamma+T\right)}{\lambda^{2}}\Rightarrow\hat{\lambda}^{*}=\frac{\gamma+T}{n+1}$$

Further,

$$-n\,\mathrm{h}''\left(\hat{\lambda}^{*}\right)=-\frac{-\left(n+1\right)^{3}}{\left(\gamma+T\right)^{2}}\Rightarrow\hat{\varphi}^{*}=\frac{\gamma+T}{\left(n+1\right)^{\frac{3}{2}}}\,.$$

Therefore,

$$\mathrm{E}\left(\lambda\,|\,x\right)=\frac{\hat{\varphi}^{*}\exp\left[-n\,\mathrm{h}^{*}\left(\hat{\lambda}^{*}\right)\right]}{\hat{\varphi}\exp\left[-nh\left(\hat{\lambda}\right)\right]}$$

$$=\left(\frac{n+1}{n+2}\right)^{n-\frac{1}{2}}\left(\frac{\gamma+T}{n+2}\right)e \qquad (13)$$

Now,

$$\mathrm{E}\left(\lambda^{2}\,|\,x\right)=\frac{\hat{\varphi}^{*}\exp\left[-n\,\mathrm{h}^{*}\left(\lambda^{*}\right)\right]}{\hat{\varphi}\exp\left[-n\,\mathrm{h}\left(\lambda\right)\right]},$$

where $-n\,\mathrm{h}^{*}\left(\lambda^{*}\right)=\log\lambda^{2}-n\,\mathrm{h}\left(\lambda\right)$.

$$\Rightarrow\mathrm{E}\left(\lambda^{2}\,|\,x\right)=\left(\frac{n}{n+2}\right)^{n-\frac{3}{2}}\left(\frac{\gamma+T}{n+2}\right)^{2}e^{2}$$

$$\therefore\mathrm{V}\left[\lambda\,|\,x\right]=\left(\frac{n}{n+2}\right)^{n-\frac{3}{2}}\left(\frac{\gamma+T}{n+2}\right)^{2}e^{2}-\left[\left(\frac{n+1}{n+2}\right)^{n-\frac{1}{2}}\left(\frac{\gamma+T}{n+2}\right)e\right]^{2}$$

Table 1. Posterior estimates of Laplace distribution using normal approximation

		Inverse gamma prior			Inverse chi square prior			Gumbel type II prior		
n	***λ***	***α*=*β*=0.5**	***α*=*β*=1**	***α*=*β*=2**	***a*=*b*=0.5**	***a*=*b*=1**	***a*=*b*=2**	***γ*=0.5**	***γ*=1**	***γ*=2**
25	0.5	0.63052	0.63736	0.65031	0.62700	0.63052	0.63736	0.61884	0.63736	0.67440
		(0.01499)	(0.01503)	(0.01509)	**(0.01496)**	(0.01499)	(0.01503)	(0.01417)	(0.01503)	(0.01683)
	1.0	0.88203	0.88422	0.88835	0.88091	0.88203	0.88422	0.86570	0.92126	0.92126
		(0.02934)	(0.02895)	(0.02817)	(0.02955)	(0.02934)	(0.02894)	**(0.02774)**	(0.02894)	(0.03142)
	2.0	2.22916	2.20640	2.16332	2.24087	2.22916	2.20640	2.18788	2.20640	2.24344
		(0.18750)	(0.18029)	**(0.16713)**	(0.19128)	(0.18750)	(0.18029)	(0.03250)	(0.18029)	(0.18639)
50	0.5	0.49012	0.49502	0.50455	0.48763	0.49012	0.49502	0.48541	0.49502	0.51425
		(0.00465)	(0.00470)	(0.00479)	(0.00462)	(0.00465)	(0.00470)	**(0.00452)**	(0.00470)	(0.00507)
	1.0	1.09274	1.09185	1.09011	1.09319	1.09274	1.09185	1.08223	1.09185	1.11108
		(0.02317)	(0.02291)	**(0.02241)**	(0.02330)	(0.02317)	(0.02291)	(0.02251)	(0.02291)	(0.02373)
	2.0	2.32566	2.31292	2.28814	2.33213	2.32566	2.31292	2.30330	2.31292	2.33215
		(0.10501)	(0.10286)	**(0.09877)**	(0.10611)	(0.10501)	(0.10286)	(0.10201)	(0.10286)	(0.10458)
100	0.5	0.45510	0.45777	0.46304	0.45376	0.45510	0.45777	0.45287	0.45777	0.46758
		(0.00203)	(0.00204)	(0.00207)	(0.00202)	(0.00203)	(0.00204)	**(0.00200)**	(0.00204)	(0.00213)
	1.0	1.13627	1.13560	1.13428	1.13660	1.13627	1.13560	1.13070	1.13560	1.14540
		(0.01271)	(0.01263)	**(0.01248)**	(0.01274)	(0.01271)	(0.01263)	(0.01252)	(0.01263)	(0.01285)
	2.0	2.21633	2.21036	2.19861	2.21933	2.21633	2.21036	2.20546	2.21036	2.22017
		(0.04838)	(0.04838)	**(0.04692)**	(0.04863)	(0.04838)	(0.04788)	(0.04767)	(0.04788)	(0.04831)

Table 2. Posterior estimates of Laplace distribution using T-K approximation

		Inverse gamma prior			**Inverse chi square prior**			**Gumbel type II prior**		
n	***λ***	***α*=*β*=0.5**	***α*=*β*=1**	***α*=*β*=2**	***a*=*b*=0.5**	***a*=*b*=1**	***a*=*b*=2**	***γ*=0.5**	***γ*=1**	***γ*=2**
25	0.5	0.35256	0.36549	0.38988	0.34589	0.35256	0.36549	0.34553	0.36549	0.40543
		(0.00526)	(0.00553)	(0.00605)	**(0.00511)**	(0.00526)	(0.00553)	(0.00494)	(0.00553)	(0.00681)
	1.0	1.39540	1.38754	1.37273	1.39944	1.39540	1.38754	1.36757	1.38754	1.42748
		(0.08257)	(0.07994)	**(0.07513)**	(0.08393)	(0.08257)	(0.07994)	(0.07766)	(0.07994)	(0.08461)
	2.0	2.32739	2.30095	2.25111	2.34101	2.32739	2.30095	2.28098	2.30095	2.34089
		(0.22970)	(0.21987)	**(0.20207)**	(0.23488)	(0.22970)	(0.21987)	(0.21607)	(0.21987)	(0.22756)
50	0.5	0.56176	0.56614	0.57465	0.55954	0.56176	0.56614	0.55615	0.56614	0.58613
		(0.00649)	(0.00652)	(0.00658)	(0.00647)	(0.00649)	(0.00652)	**(0.00629)**	(0.00652)	(0.00699)
	1.0	1.11816	1.11699	1.11470	1.11876	1.11816	1.11699	1.10699	1.11699	1.13698
		(0.02574)	(0.02543)	**(0.02482)**	(0.02590)	(0.02574)	(0.02543)	(0.02497)	(0.02543)	(0.02635)
	2.0	1.76988	1.76219	1.74726	1.77378	1.76988	1.76219	1.75219	1.76219	1.78218
		(0.06452)	(0.06332)	**(0.06099)**	(0.06514)	(0.06452)	(0.06331)	(0.06259)	(0.06331)	(0.06475)
100	0.5	0.49407	0.49660	0.50158	0.49280	0.49407	0.49660	0.49160	0.49660	0.50660
		(0.00246)	(0.00248)	(0.00250)	(0.00246)	(0.00246)	(0.00248)	**(0.00243)**	(0.00248)	(0.00258)
	1.0	1.03193	1.03177	1.03145	1.03201	1.03193	1.03177	1.02677	1.03177	1.04177
		(0.01079)	(0.01074)	**(0.01062)**	(0.01082)	(0.01079)	(0.01074)	(0.01063)	(0.01074)	(0.01096)
	2.0	1.90119	1.89668	1.88781	1.90345	1.90119	1.89668	1.89168	1.89668	1.90668
		(0.03667)	(0.03631)	(0.03631)	(0.03685)	(0.03667)	(0.03631)	**(0.03612)**	(0.03631)	(0.03670)

Table 3. Posterior estimates of Laplace distribution using normal approximation

	Inverse gamma prior			**Inverse chi square prior**			**Gumbel type II prior**		
	***α*=*β*=0.5**	***α*=*β*=1**	***α*=*β*=2**	***a*=*b*=0.5**	***a*=*b*=1**	***a*=*b*=2**	***γ*=0.5**	***γ*=1**	***γ*=2**
Posterior mean	84.05195	82.98718	80.93750	84.59477	84.05195	82.98718	82.97436	82.98718	83.01282
Posterior variance	183.4995	176.5865	**163.7720**	187.0922	183.4995	176.5865	176.5319	176.5865	176.6956

Table 4. Posterior estimates of Laplace distribution using T-K approximation

	Inverse gamma prior			**Inverse chi square prior**			**Gumbel type II prior**		
	***α*=*β*=0.5**	***α*=*β*=1**	***α*=*β*=2**	***a*=*b*=0.5**	***a*=*b*=1**	***a*=*b*=2**	***γ*=0.5**	***γ*=1**	***γ*=2**
Posterior mean	88.58938	87.40748	85.13681	89.19252	88.58938	87.40748	87.39398	87.40748	87.43449
Posterior variance	220.7281	211.9021	**195.6173**	225.3260	220.7281	211.9021	211.8367	211.9021	212.0331

Simulation Study

In a simulation study, samples were generated of sizes $n = 25$, 50, and 100 to examine the effect of small, medium, and large samples of posterior estimates using R. The performance of Bayes estimates of the scale parameter of the Laplace distribution were compared under different informative priors using the two approximation techniques. The value of the scale parameter λ has been chosen as 0.5, 1, and 2, and the value of the hyperparameters have also been taken as 0.5, 1, and 2. These results have been replicated 5000 times and the estimates have been obtained. The results are presented in the tables below with the posterior variances enclosed in the parentheses.

Results

From Tables 1 and 2, it is clearly evident that the posterior variance of the scale parameter of the Laplace distribution in most of the cases is minimum under inverse gamma prior, especially when the value of the hyperparameters α and β is taken to be 2.

Real-Life Example

The data below were discussed by Schmee and Nelson (1977) and show the number, in thousands of miles, at which different locomotive controls failed in a life test involving 96 controls. The test was terminated after 135,000 miles by which time 37 failures had occurred. The failure times for the 37 failed units are:

> 22.5, 37.5, 46.0, 48.5, 51.5, 53.0, 54.5, 57.5, 66.5, 68.0, 69.5, 76.5, 77.0, 78.5, 80.0, 81.5, 82.0, 83.0, 84.0, 91.5, 93.5, 102.5, 107.0, 108.5, 112.5, 113.5, 116.0, 117.0, 118.5, 119.0, 120.0, 122.5, 123.0, 127.5, 131.0, 132.5, 134.0. (p. 3)

Conclusion

The aim was to focus on the efficiency of the various informative priors used for the scale parameter λ of the Laplace distribution under the two Bayesian approximation techniques. It is observed from the simulation study presented in Tables 1 and 2 that, among all the informative priors, the inverse gamma prior proves to be more efficient with minimum posterior variance when the value of hyperparameters is taken to be 2. Further, the results are validated by considering the estimates obtained from a real-life data set as shown in Tables 3 and 4, thereby justifying the results of our simulation study. In addition to this, it can be seen that the values of posterior variance in Tables 1 and 2 decreases with the increase in sample size. Also, it can be seen that normal approximation gives lesser values of posterior variance of shape parameter of the inverse Lomax distribution than T-K approximation in both the generated and real-life data sets.

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