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Estimation of Mean with Two-Parameter Ratio-Product-Ratio Estimator in Double Sampling using Ancillary Information under Non-Response

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Ratio-product-ratio estimators with two parameters in double sampling under non-response are considered along with their properties. Practical conditions are obtained in which the suggested estimators are more proficient than other existing estimators. An example is given.

Keywords: Ancillary variable, main character, double sampling, MSE, comparison

Introduction

Non-response (NR) is an important issue that remains under constant debate amongst statistician for a variety of reasons. Some of which may be (i) refuel to answer the questionnaire, (ii) not available at home, (iii) lack of information, (iv) failure to contact, (v) unable to answer, and (vi) inaccessible. In the case of NR in double sampling, the sampling procedure due to Hansen and Hurwitz (1946) is employed for estimating the universe mean. Cochran (1977), Rao (1986), Khare and Srivastava (1993, 1995), Tabasum and Khan (2004), Singh and Kumar (2008, 2009a, 2009b, 2010a, 2010b), Singh, Kumar, and Kozak (2010), and Pal and Singh (2016, 2017) made their contribution towards the mean estimation of the principal variable y while considering the NR at the next phase. If information (data) on the subsidiary variable x is not readily available, the double sampling method is used, where a large, first-phase sample is drawn from the universe and information is collected over the variable x to achieve a superior estimate of the universe mean $\bar{X}$. A second-phase sample can then be taken, and the main variable y is observed. Wu

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and Luan (2003) discussed that the major benefit of using double sampling is the gain in high precision without significant increase in price.

Methodology

Suppose a finite universe U = (u_1, u_2,…, u_N) of N units. A simple random sample of size n is drawn without replacement from U. Let y_i be the value of the main variable y on the unit u_i (i = 1, 2,…, N). In surveys on human populations, frequently n_1 units 'respond' at first attempt while the remaining n_2 units do not respond. The survey may be conducted through the mail or telephone calls, perhaps computer aided.

If NR occurs at the first attempt, Hansen and Hurwitz (1946) introduced a procedure for estimating the universe mean $\bar{Y}$ containing the subsequent steps: (i) a simple random sample of size n is drawn and the questionnaire is mailed to the sampled units; (ii) a subsample of size $r = n_2k^{-1}$ ($k > 1$) from the n_2 non-responding units in the initial attempt is conducted through personal interviews.

In the Hansen and Hurwitz (1946) procedure, the universe of size N can be assumed to divide into two strata of size N_1 and $N_2 = (N - N_1)$ of "respondents" and "non-respondents".

Let $\bar{Y}$ and S_y^2 be the mean and mean square of the principal character for the finite universe of N units. Let $\bar{Y}_1$ and $S_{y_1}^2$ indicate the mean and mean square of the response group of N_1 units. Similarly, let $\bar{Y}_2$ and $S_{y_2}^2$ indicate the mean and mean square/variance of the NR group N_2.

The universe mean $\bar{Y}$ of the principal variable y is given as

$$\bar{Y} = D_1\bar{Y}_1 + D_2\bar{Y}_2, \tag{1}$$

with $D_1 = (N_1 / N)$ and $D_2 = (N_2 / N)$. For $\bar{Y}$, the unbiased estimator is

$$\bar{y}^* = d_1\bar{y}_1 + d_2\bar{y}_{2r}, \tag{2}$$

with $d_1 = (n_1 / n)$, $d_2 = (n_2 / n)$, and $\bar{y}_1$ and $\bar{y}_2$ are the sample means depend upon n_1 and r units. The variance of $\bar{y}^*$ is

$$\mathrm{V}\left(\bar{y}^*\right) = \bar{Y}^2\left[\lambda C_y^2 + \lambda^* C_{y(2)}^2\right] \tag{3}$$

(Cochran, 1977, p. 371), where $\lambda = (1 - f)n^{-1} = (n^{-1} - N^{-1})$, f is the sampling fraction, and $\lambda^{*} = n^{-1}D_2(k - 1)$. We also define $C_x = S_x / \bar{X}$ and $C_{x(2)} = S_{x(2)} / \bar{X}$ as the coefficients of variation of the whole universe and NR group, respectively.

The Double Sampling Method and Estimators

If the list of units is available but $\bar{X}$ is not known, insert $\bar{x}'$ based on a large sample of size n' in place of $\bar{X}$. The sampling design will be as follows: (1) choose a large sample of size n' in the first-phase via a simple random sampling without replacement (SRSWOR) method and observed x variable. (2) From the selected n' first-phase units, we select a second-phase sample of n via SRSWOR and observe that n_1 and n_2 observations are responding and not-responding, respectively. Collect information on y for n_1 responding units. (3) From the n_2 NR observations, select a sub-sample of size $r = n_2k^{-1}$ ($k > 1$) using SRSWOR by making an extra effort and observe the character y for these r chosen units. There are n' observations on the x variable. Of the n second-phase units there are n_1 observations on the y variable from units who respond, and also r observations on the sub-sample selected from the n_2 NR units of the second-phase sample. Let $\bar{x}'$ be the sample mean of x based on a preliminary large sample n'. Using the information on x when $\bar{X}$ is not known, consider two classes of estimators for $\bar{Y}$ in two unusual situations, which are as follows:

Situation I: The case when $\bar{X}$ is unknown and incomplete information is available on the main variable y and the supplementary variable x. In this situation, we use ($n_1 + r$) responding units for y and x from the sample of size n and $\bar{x}'$ to estimate $\bar{X}$. Khare and Srivastava (1993, 1995) and Tabasum and Khan (2004) suggested the following two-phase sampling ratio and product type estimators for $\bar{Y}$:

$$T_{\mathrm{R}1d} = z^{-1}\bar{y}^{*}, \tag{4}$$

$$T_{\mathrm{P}1d} = z\bar{y}^{*}, \tag{5}$$

where $z = (\bar{x}^{*} / \bar{x}')$.

Up to order n^{-1}, the expression for bias and mean squared error (MSE) of $T_{\mathrm{R}1d}$ and of $T_{\mathrm{P}1d}$ are as follows:

$$\mathrm{B}\left(T_{\mathrm{R}1d}\right)=\bar{Y}\left\{\theta C_x^2+\lambda^* C_{x(2)}^2\right\}\left(1-R_d^*\right), \tag{6}$$

$$\mathrm{MSE}\left(T_{\mathrm{R}1d}\right)=\bar{Y}^2\left[\left(\lambda C_y^2+\lambda^* C_{y(2)}^2\right)+\left\{\theta C_x^2+\lambda^* C_{x(2)}^2\right\}\left(1-2R_d^*\right)\right], \tag{7}$$

$$\mathrm{B}\left(T_{\mathrm{P}1d}\right)=\bar{Y}\left[\theta C_x^2+\lambda^* C_{x(2)}^2\right]R_d^* \tag{8}$$

$$\mathrm{MSE}\left(T_{\mathrm{P}1d}\right)=\bar{Y}^2\left[\left(\lambda C_y^2+\lambda^* C_{y(2)}^2\right)+\left\{\theta C_x^2+\lambda^* C_{x(2)}^2\right\}\left(1+2R_d^*\right)\right], \tag{9}$$

where

$$R_d^*=\left[\theta C C_x^2+\lambda^* C_{(2)} C_{x(2)}^2\right]\Big/\left[\theta C_x^2+\lambda^* C_{x(2)}^2\right]$$

and $\theta = (n^{-1} - n'^{-1})$ for

$$C=\rho_{yx}\left(\frac{C_y}{C_x}\right), C_{(2)}=\rho_{yx(2)}\left(\frac{C_{y(2)}}{C_{x(2)}}\right), \rho_{yx}=\left(\frac{S_{yx}}{S_y S_x}\right), \rho_{yx(2)}=\left(\frac{S_{yx(2)}}{S_{y(2)} S_{x(2)}}\right),$$

where S_{yx} and $S_{yx(2)}$ are the covariance of the entire group and NR group, respectively.

From (3) and (7),

$$\mathrm{V}\left(\bar{y}^*\right)-\mathrm{MSE}\left(T_{\mathrm{R}1d}\right)=\bar{Y}^2\left\{\theta C_x^2+\lambda^* C_{x(2)}^2\right\}\left(2R_d^*-1\right). \tag{10}$$

It follows from (10) that the estimator $T_{\mathrm{R}1d}$ is more accurate than $\bar{y}^*$ if

$$R_d^*>\left(1/2\right). \tag{11}$$

In a similar fashion it can be shown that the estimator $T_{\mathrm{P}1d}$ is more accurate than $\bar{y}^*$ if

$$R_d^*<-\left(1/2\right). \tag{12}$$

Observing conditions (11) and (12), the conventional unbiased estimator $\bar{y}^*$ is to be preferred over the ratio estimator T_{R1d} and product estimator T_{P1d} if

$$-(1/2) \le R_d^* \le (1/2). \tag{13}$$

Situation II: The case when $\bar{X}$ is unknown and incomplete information on y and complete information on x is available. In this situation, using information on the responding units ($n_1 + r$) on y and complete information on x from n, the two-phase sampling estimators for $\bar{Y}$ are

$$T_{R2d} = v^{-1}\bar{y}^*, \tag{14}$$

$$T_{P2d} = v\bar{y}^*, \tag{15}$$

where $v = (\bar{x} / \bar{x}')$.

Up to order n^{-1}, the bias and MSE of T_{R2d} and T_{P2d} are as follows:

$$\mathrm{B}(T_{R2d}) = \bar{Y}\theta C_x^2 (1 - C), \tag{16}$$

$$\mathrm{MSE}(T_{R2d}) = \bar{Y}^2\left[\left(\lambda C_y^2 + \lambda^* C_{y(2)}^2\right) + \theta C_x^2 (1 - 2C)\right], \tag{17}$$

$$\mathrm{B}(T_{P2d}) = \bar{Y}\theta C C_x^2, \tag{18}$$

$$\mathrm{MSE}(T_{R2d}) = \bar{Y}^2\left[\left(\lambda C_y^2 + \lambda^* C_{y(2)}^2\right) + \theta C_x^2 (1 + 2C)\right]. \tag{19}$$

The estimators T_{R2d} and T_{P2d} are respectively better than $\bar{y}^*$ if

$$C > (1/2) \tag{20}$$

and

$$C < -(1/2) \tag{21}$$

However, $\bar{y}^*$ is to be preferred over T_{R1d} and T_{P1d} if

$$-(1/2) \le C \le (1/2) \tag{22}$$

Taking motivation from Chami, Singh, and Thomas (2012), consider a two-parameter ratio-product-ratio (RPR) estimator and its properties in double sampling with non-respondents in two different situations.

The Suggested Two-Parameter RPR Estimator

Consider a two-parameter RPR estimator in two-phase sampling in two situations (i.e. Case I and Case II).

Case I: There Is Non-Response on *y* as Well as on *x*

In this situation, for estimating the $\bar{Y}$ of y, we propose the following two-parameter RPR estimator:

$$T_{d(\alpha,\beta)} = \left[\alpha\left\{\frac{(1-\beta)z+\beta}{\beta z+(1-\beta)}\right\}+(1-\alpha)\left\{\frac{\beta z+(1-\beta)}{(1-\beta)z+\beta}\right\}\right]\bar{y}^*, \tag{23}$$

where α, β are real constants (see Chami et al., 2012). The goal is to derive values for these constants α, β such that the bias and/or the MSE of $T_{d(\alpha,\beta)}$ are minimal. The two parameters α and β may be used to obtain an asymptotically optimum estimator (AOE) $T^{(0)}_{d(\alpha,\beta)}$ that is (up to order n^{-1}) both unbiased and has minimal MSE. The estimator $T^{(0)}_{d(\alpha,\beta)}$ corrects the limitations of the commonly used estimators $\bar{y}^*$, $T_{\mathrm{R}1d}$, and $T_{\mathrm{P}1d}$, which are to be used for a specific range of the parameters (C, $C_{(2)}$, or R^*_d) and, in addition, out-performs the traditional estimators by having the minimum MSE.

$T_{d(\alpha,\beta)} = T_{d(1-\alpha,1-\beta)}$, meaning the estimator $T_{d(\alpha,\beta)}$ is invariant under a point reflection through the point $(\alpha, \beta) = (1/2, 1/2)$. In the point of symmetry $(\alpha, \beta) = (1/2, 1/2)$, the estimator reduces to $\bar{y}^*$ due to Hansen and Hurwitz (1946). In fact, on the entire line $\beta = 1/2$, the suggested estimator reduces to $\bar{y}^*$. For $(\alpha, \beta) = (1, 0)$ or $(\alpha, \beta) = (0, 1)$, the recommended estimator $T_{d(\alpha,\beta)}$ reduces to $T_{d(1,0)} = T_{d(0,1)} = (\bar{x}^*\bar{y}^*) / \bar{x}' = T_{\mathrm{P}1d}$, while for $(\alpha, \beta) = (0, 0)$ or $(\alpha, \beta) = (1, 1)$, it reduces to the ratio estimator $T_{d(0,0)} = T_{d(1,1)} = (\bar{y}^*\bar{x}') / \bar{x}^* = T_{\mathrm{R}1d}$.

All the three estimators $\bar{y}^*$, $T_{\mathrm{R}1d}$, and $T_{\mathrm{P}1d}$ can be obtained from the proposed estimate $T_{d(\alpha,\beta)}$ by using suitable values of the parameters (α, β). Consider estimator

(23) and compare it to the three estimators $\bar{y}^*$, T_{R1d}, and T_{P1d} as follows: In order to derive the bias of $T_{d(\alpha,\beta)}$ up to $O(n^{-1})$, write

$$e_0 = \frac{\bar{y}^* - \bar{Y}}{\bar{Y}}, \quad e_1 = \frac{\bar{x}^* - \bar{X}}{\bar{X}}, \quad \text{and} \quad e_1' = \frac{\bar{x}' - \bar{X}}{\bar{X}}$$

such that $E(e_i) = 0$ for $i = 0, 1$ and $E(e_1') = 0$, with relative variances

$$E(e_0^2) = \left[\lambda C_y^2 + \lambda^* C_{y(2)}^2\right], \quad E(e_1^2) = \left[\lambda C_x^2 + \lambda^* C_{x(2)}^2\right], \quad E(e_1'^2) = \lambda' C_x^2,$$

where $\lambda' = (n'^{-1} - N^{-1})$.

Also,

$$E(e_0 e_1) = \left[\lambda \rho_{yx} C_y C_x + \lambda^* \rho_{yx(2)} C_{y(2)} C_{x(2)}\right], E(e_0 e_1') = \lambda' \rho_{yx} C_y C_x, E(e_1 e_1') = \lambda' C_x^2.$$

Express (23) as

$$T_{d(\alpha,\beta)} = \bar{Y}(1+e_0)\left[\alpha \frac{(1+(1-\beta)e_1 + \beta e_1')}{(1+\beta e_1 + (1-\beta)e_1')} + (1-\alpha)\frac{(1+\beta e_1 + (1-\beta)e_1')}{(1+(1-\beta)e_1 + \beta e_1')}\right]. \quad (24)$$

From (24),

$$T_{d(\alpha,\beta)} \cong \bar{Y}\left[1 + e_0 - (1-2\alpha)(1-2\beta)(e_0 - e_1' + e_0 e_1 - e_0 e_1') + (1-\alpha-\beta)(1-2\beta)e_1^2 + (\alpha-\beta)(1-2\beta)e_1'^2 - (1-4\beta+4\beta^2)e_1 e_1'\right]$$

or

$$(T_{d(\alpha,\beta)} - \bar{Y}) \cong \bar{Y}\left[e_0 - (1-2\alpha)(1-2\beta)(e_0 - e_1' + e_0 e_1 - e_0 e_1') + (1-\alpha-\beta)(1-2\beta)e_1^2 + (\alpha-\beta)(1-2\beta)e_1'^2 - (1-4\beta+4\beta^2)e_1 e_1'\right] \quad (25)$$

Taking expectations together with (25), the expected bias of $T_{d(\alpha,\beta)}$ is obtained as

$$\begin{aligned} \mathrm{B}\left(T_{d(\alpha,\beta)}\right) &= \mathrm{E}\left(T_{d(\alpha,\beta)} - \bar{Y}\right) \\ &\cong \bar{Y}\left(1-2\beta\right)\left\{\theta C_x^2 + \lambda^* C_{x(2)}^2\right\}\left[\left(1-\alpha-\beta\right) - \left(1-2\alpha\right) R_d^*\right] \end{aligned} \tag{26}$$

Equating (26) to zero,

$$\beta = 1/2 \quad \text{or} \quad \beta = \left[1-\alpha-\left(1-2\alpha\right) R_d^*\right]. \tag{27}$$

The proposed RPR estimator $T_{d(\alpha,\beta)}$, substituted with the value of β from (27), becomes an approximately unbiased estimator for $\bar{Y}$. Furthermore, as the sample size n is very large, the bias of $T_{d(\alpha,\beta)}$ will be negligible. If there is response not present on x the result in (27) reduces to

$$\beta = 1/2 \quad \text{or} \quad \beta = \left[1-\alpha-\left(1-2\alpha\right) C\right]. \tag{28}$$

Squaring (25) obtains the approximate expression

$$\begin{aligned} \left(T_{d(\alpha,\beta)} - \bar{Y}\right)^2 \cong \bar{Y}^2 \Big[& e_0^2 + \left(1-2\alpha\right)^2 \left(1-2\beta\right)^2 \left(e_1^2 - e_1'^2 - e_1 e_1'\right) \\ & -2\left(1-2\alpha\right)\left(1-2\beta\right)\left(e_0 e_1 - e_0 e_1'\right)\Big] \end{aligned} \tag{29}$$

The approximate MSE of $T_{d(\alpha,\beta)}$ is obtained as

$$\begin{aligned} \mathrm{MSE}\left(T_{d(\alpha,\beta)}\right) &= \bar{Y}^2 \Big[\left(\lambda C_y^2 + \lambda^* C_{y(2)}^2\right) + \left(1-2\alpha\right)^2 \left(1-2\beta\right)^2 \left\{\theta C_x^2 + \lambda^* C_{x(2)}^2\right\} \\ & \qquad - 2\left(1-2\alpha\right)\left(1-2\beta\right)\left\{\theta \rho_{yx} C_y C_x + \lambda^* \rho_{yx(2)} C_{y(2)} C_{x(2)}\right\}\Big] \\ &= \bar{Y}^2 \Big[\left(\lambda C_y^2 + \lambda^* C_{y(2)}^2\right) \\ & \qquad + \left(1-2\alpha\right)\left(1-2\beta\right)\left\{\theta C_x^2 + \lambda^* C_{x(2)}^2\right\}\left\{\left(1-2\alpha\right)\left(1-2\beta\right) - 2R_d^*\right\}\Big] \end{aligned} \tag{30}$$

Taking the gradient $\nabla = (\partial/\partial\alpha, \partial/\partial\beta)$ of (30),

$$\begin{aligned} \nabla\, \mathrm{MSE}\left(T_{d(\alpha,\beta)}\right) = 4\bar{Y}^2 & \left\{\theta C_x^2 + \lambda^* C_{x(2)}^2\right\}\left[\left(1-2\alpha\right)\left(1-2\beta\right) - R_d^*\right]; \\ & \left(1-2\alpha, 1-2\beta\right) \end{aligned} \tag{31}$$

Equating (31) to zero to obtain the critical points, we obtain the following solutions:

$$\alpha = 1/2, \beta = 1/2 \tag{32}$$

or

$$(1-2\alpha)(1-2\beta) = R_d^*. \tag{33}$$

The critical point in (32) is a saddle point unless $R_d^* = 0$, in which case a local minimum is obtained. However, the critical points obtained in (33) give the equation of the hyperbola symmetric through $(\alpha, \beta) = (1/2, 1/2)$. The minimum mean squared error (MMSE) of $T_{d(\alpha,\beta)}$ is obtained as

$$\text{MMSE}\left(T_{d(\alpha,\beta)}\right) = \bar{Y}^2\left[\left(\lambda C_y^2 + \lambda^* C_{y(2)}^2\right) - \left\{\theta C_x^2 + \lambda^* C_{x(2)}^2\right\} R_d^{*2}\right], \tag{34}$$

which is independent of α and β.

Theorem 1. Up to $O(n^{-1})$,

$$\text{MSE}\left(T_{d(\alpha,\beta)}\right) \geq \bar{Y}^2\left[\left(\lambda C_y^2 + \lambda^* C_{y(2)}^2\right) - \left\{\theta C_x^2 + \lambda^* C_{x(2)}^2\right\} R_d^{*2}\right]$$

if $(1-2\alpha)(1-2\beta) = R_d^*$.

This is the minimal possible MSE up to order n^{-1} for a wide family of estimators to which the estimator (23) belongs, for instance, for estimators

$$t_{dh} = \bar{y}^* \, \text{h}(z), \tag{35}$$

h(.) being a function of z such that h(1) = 1 and also satisfies certain regularity conditions similar to those given in Srivastava (1971). Whatever value of R_d^* in (30) has, we are always able to select an AOE $T_{d(\alpha,\beta)}^{(0)}$ from the two-parameter family in (23).

Further, putting (32) into $T_{d(\alpha,\beta)}^{(0)}$ yields $\bar{y}^*$ of $\bar{Y}$. Thus, the MSE of $\bar{y}^*$ is

$$\text{MSE}\left(T_{d(1/2,1/2)}\right) = \text{MSE}\left(\bar{y}^*\right) = \bar{Y}^2\left[\lambda C_y^2 + \lambda^* C_{y(2)}^2\right] \tag{36}$$

Remark 1. When NR occurs on both y and x with unknown $\bar{X}$, an alternative to a two-parameter RPR estimator for $\bar{Y}$ is defined by

$$T_{d(\alpha,\beta)}^{(1)} = \left[\alpha z^{(1-2\beta)} + (1-\alpha) z^{(2\beta-1)}\right]\bar{y}^*.$$

The estimator $T_{d(\alpha,\beta)}^{(1)}$ has the same bias and MSE up to order n^{-1} as $T_{d(\alpha,\beta)}^{(0)}$ in (23).

The class of estimators $T_{d(\alpha,\beta)}^{(1)}$ is further generalized along the lines of Singh, Solanki, and Singh (2016) as

$$T_{d(\alpha,\beta)}^{(2)} = \left[\alpha z^{*(1-2\beta)} + (1-\alpha) z^{*(2\beta-1)}\right]\bar{y}^*,$$

where $z^* = (a\bar{x}^* + b) / (a\bar{x}' + b)$, (α, β) are the same defined in Chami et al. (2012, p. 2), and a $(\neq 0)$ and b are either real or the functions of the known parameters associated with x and y or both (x, y).

Efficiency Comparison and Choice of Parameters

Comparing the MSE of $\bar{y}^*$ to $T_{d(\alpha,\beta)}$, observe the following from (3) or (36) and from (30):

$$\begin{aligned} V\left(\bar{y}^*\right) - \text{MSE}\left(T_{d(\alpha,\beta)}\right) = -\bar{Y}^2\Big[&(1-2\alpha)(1-2\beta)\left(\theta C_x^2 + \lambda^* C_{x(2)}^2\right) \\ &+ \left\{(1-2\alpha)(1-2\beta)\right\} - 2R_d^*\Big] \end{aligned} \tag{37}$$

which is positive if

$$(1-2\alpha)(1-2\beta)\left[2R_d^* - (1-2\alpha)(1-2\beta)\right] > 0. \tag{38}$$

Therefore

(i) $\alpha > \dfrac{1}{2}, \beta > \dfrac{1}{2}$ and $R_d^* > \dfrac{1}{2}(1-2\alpha)(1-2\beta)$,

(ii) $\alpha < \frac{1}{2}, \beta > \frac{1}{2}$ and $R_d^* < \frac{1}{2}(1-2\alpha)(1-2\beta)$,

(iii) $\alpha > \frac{1}{2}, \beta < \frac{1}{2}$ and $R_d^* < \frac{1}{2}(1-2\alpha)(1-2\beta)$, or

(iv) $\alpha < \frac{1}{2}, \beta < \frac{1}{2}$ and $R_d^* > \frac{1}{2}(1-2\alpha)(1-2\beta)$.

The conventional unbiased estimator $\bar{y}^*$ is to be preferred if $-1/2 \le R_d^* \le 1/2$. Combining (i) to (iv) with $-1/2 \le R_d^* \le 1/2$, obtain the following explicit range:

(v) If $0 < R_d^* \le \frac{1}{2}$ and $\beta > \frac{1}{2}$, then $\frac{1}{2} < \alpha < \frac{(2\beta + 2R_d^* - 1)}{2(2\beta - 1)}$ from (i).

(vi) If $-\frac{1}{2} \le R_d^* < 0$ and $\beta > \frac{1}{2}$, then $\frac{(2\beta + 2R_d^* - 1)}{2(2\beta - 1)} < \alpha < \frac{1}{2}$ from (ii).

(vii) If $-\frac{1}{2} \le R_d^* < 0$ and $\beta < \frac{1}{2}$, then $\frac{1}{2} < \alpha < \frac{(2\beta + 2R_d^* - 1)}{2(2\beta - 1)}$ from (iii).

(viii) If $0 < R_d^* \le \frac{1}{2}$ and $\beta < \frac{1}{2}$, then $\frac{(2\beta + 2R_d^* - 1)}{2(2\beta - 1)} < \alpha < \frac{1}{2}$ from (iv).

Comparing the MSE of $T_{\mathrm{R1}d}$ and $T_{d(\alpha,\beta)}$, from (7) and (36),

$$\begin{aligned} &\mathrm{MSE}(T_{\mathrm{R1}d}) - \mathrm{MSE}\left(T_{d(\alpha,\beta)}\right) \\ &\quad = 4\bar{Y}^2\left\{\theta C_x^2 + \lambda^* C_{x(2)}^2\right\}\left[(2\alpha\beta - \alpha - \beta)\left\{R_d^* - 1 - (2\alpha\beta - \alpha - \beta)\right\}\right] \end{aligned} \tag{39}$$

which is positive if

$$(2\alpha\beta - \alpha - \beta)\left\{R_d^* - 1 - (2\alpha\beta - \alpha - \beta)\right\} > 0. \tag{40}$$

Therefore,

$$R_d^* - 1 > (2\alpha\beta - \alpha - \beta) > 0 \tag{41}$$

or

$$R_d^* - 1 < \left(2\alpha\beta - \alpha - \beta\right) < 0 . \tag{42}$$

Hence, from (42), when $R_d^* > 1$,

$$\text{If } \beta < \frac{1}{2}, \text{ then } \frac{\left(\beta + R_d^* - 1\right)}{\left(2\beta - 1\right)} < \alpha < \frac{\beta}{\left(2\beta - 1\right)} . \tag{43}$$

$$\text{If } \beta > \frac{1}{2}, \text{ then } \frac{\beta}{\left(2\beta - 1\right)} < \alpha < \frac{\left(\beta + R_d^* - 1\right)}{\left(2\beta - 1\right)} . \tag{44}$$

Further, from (42), when $1/2 < R_d^* < 1$ we have

$$\text{If } \beta < \frac{1}{2}, \text{ then } \frac{\beta}{\left(2\beta - 1\right)} < \alpha < \frac{\left(\beta + R_d^* - 1\right)}{\left(2\beta - 1\right)} . \tag{45}$$

$$\text{If } \beta > \frac{1}{2}, \text{ then } \frac{\left(\beta + R_d^* - 1\right)}{\left(2\beta - 1\right)} < \alpha < \frac{\beta}{\left(2\beta - 1\right)} . \tag{46}$$

Comparing the MSE of $T_{\mathrm{P1}d}$ to $T_{d(\alpha,\beta)}$, from (9) and (36),

$$\begin{aligned} &\mathrm{MSE}\left(T_{\mathrm{P1}d}\right) - \mathrm{MSE}\left(T_{d(\alpha,\beta)}\right) \\ &\quad = 4\bar{Y}^2 \left\{\theta C_x^2 + \lambda^* C_{x(2)}^2\right\}\left[\left(1 + 2\alpha\beta - \alpha - \beta\right)\left\{R_d^* - \left(2\alpha\beta - \alpha - \beta\right)\right\}\right] \end{aligned} \tag{47}$$

The expression (47) is positive if

$$\left(1 + 2\alpha\beta - \alpha - \beta\right)\left[R_d^* - \left(2\alpha\beta - \alpha - \beta\right)\right] > 0 . \tag{48}$$

Obtain the following two cases:

$$R_d^* > \left(2\alpha\beta - \alpha - \beta\right) > -1 \text{ if both factors in (47) are positive or} \tag{49}$$

$$-1, R_d^* < \left(2\alpha\beta - \alpha - \beta\right) \text{ if both factors in (47) are negative.} \tag{50}$$

Unbiased Asymptotically Optimum Estimators

Solving the two equations (27) and (33), calculate the parameters α and β, where the proposed class of estimators $T_{d(\alpha,\beta)}$ turns out to be, at least up to first degree of approximation, an unbiased AOE. Obtain a line

$$\beta = \frac{1}{2}, \quad \left(C, C_{(2)}\right) = (0,0) \text{ or } R_d^* = 0 \tag{51}$$

(recall that on this line the recommended family $T_{d(\alpha,\beta)}$ always reduces to $\bar{y}^*$) and a curve $\left(\alpha^*\left(R_d^*\right), \beta^*\left(R_d^*\right), R_d^*\right) \in \mathrm{R}^3$ in the parameter space with

$$\alpha^*\left(R_d^*\right) = \frac{1}{2}\left(1 \pm \sqrt{\frac{R_d^*}{2R_d^* - 1}}\right), \quad \beta^*\left(R_d^*\right) = \frac{1}{2}\left(1 \pm \sqrt{R_d^*\left(2R_d^* - 1\right)}\right). \tag{52}$$

Inserting the values of $\alpha^*\left(R_d^*\right)$ and $\beta^*\left(R_d^*\right)$ given by (52) in (23), obtain the estimator of $\bar{Y}$ as

$$T_{d\left(R^*\right)} = T_{d\left(\alpha^*\left(R_d^*\right), \beta^*\left(R_d^*\right)\right)}$$
$$= \bar{y}^*\left[\frac{2\left(R_d^* + 1\right)\bar{x}'^2 - 2\left(R_d^* - 1\right)\bar{x}^{*2} + \left(2R_d^{*2} - R_d^* - 1\right)\left(\bar{x}' - \bar{x}^*\right)^2}{4\bar{x}'\bar{x}^* - \left(2R_d^{*2} - R_d^* - 1\right)\left(\bar{x}' - \bar{x}^*\right)^2}\right] \tag{53}$$

The denominator vanishes if

$$R_d^* = 0.25\left[1 \pm \sqrt{9 + \frac{32\bar{x}'\bar{x}^*}{u^2}}\right], \quad u = \left(\bar{x}^* - \bar{x}'\right).$$

Thus,

$$\mathrm{B}\left(T_{d\left(R^*\right)}\right) = 0$$

and

$$\mathrm{MSE}\left(T_{d(R^*)}\right)=\bar{Y}^2\left[\left(\lambda C_y^2+\lambda^* C_{y(2)}^2\right)-\frac{\left\{\theta C C_x^2+\lambda^* C_{(2)} C_{x(2)}^2\right\}^2}{\theta C_x^2+\lambda^* C_{x(2)}^2}\right] \tag{54}$$
$$=\mathrm{V}\left(\bar{y}^*\right)\left(1-\rho'^{*2}\right)$$

where ρ'^* is the correlation coefficient between $\bar{y}^*$ and u.

The estimator $T_{d(R^*)}$ in (53) is an unbiased AOE. One might be interested to know whether inside $0<R_d^*\leq 1/2$ there is a choice of real parameters $(\alpha, \beta) \in \mathrm{R}^2$ such that an AOE with small bias is obtained. Putting (33) in (27) yields the first-degree approximation of the bias of an AOE:

$$\mathrm{B}\left(T^*_{d(\alpha,\beta)}\right)=\bar{Y}^2\left[\theta C_x^2+\lambda^* C_{x(2)}^2\right]\left[R_d^*\left(1-2R_d^*\right)+\left(1-2\beta\right)^2\right]. \tag{55}$$

It follows from (33) and (55) the bias can only be made zero if $R_d^*\leq 0$ or $R_d^*\geq 1/2$. Otherwise, there is always a positive contribution coming from the term $R_d^*\left(1-2R_d^*\right)$ that does not vanish regardless of what is chosen for (see Chami et al., 2012, p. 10).

Case II: There is Non-Response on *y* Only, Complete Information is Available for a Sample of Size *n* on the Subsidiary Variable *x*

If NR occurs only on y and information lacks about $\bar{X}$, a two-parameter RPR estimator suggested for $\bar{Y}$ is

$$P_{d(\eta,\delta)}=\left[\eta\left\{\frac{(1-\delta)v+\delta}{\delta v+(1-\delta)}\right\}+(1-\eta)\left\{\frac{\bar{x}v+(1-\delta)}{(1-\delta)v+\delta}\right\}\right]\bar{y}^* \tag{56}$$

where (η, δ) are real constants.

The objective is to obtain values for these scalars (η, δ) such that bias or the MSE of $P_{d(\eta,\delta)}$ are minimal. Note that $P_{d(\eta,\delta)} = P_{d(1-\eta,1-\delta)}$; that is, the estimator $P_{d(\eta,\delta)}$ is invariant under a point reflection through the point $(\eta, \delta) = (1/2, 1/2)$. In the point of symmetry $(\eta, \delta) = (1/2, 1/2)$, the proposed class estimators reduces to the conventional unbiased estimator $\bar{y}^*$; that is, we have $P_{d(1/2,1/2)} = \bar{y}^*$.

The recommended family $P_{d(\eta,\delta)}$ reduces to $T_{R2d} = \bar{y}^*(\bar{x} / \bar{x}')$ for $(\eta, \delta) = (0, 0)$ or $(1, 1)$, and to $T_{R2d} = \bar{y}^*(\bar{x} / \bar{x}')$ for $(\eta, \delta) = (1, 0)$ or $(0, 1)$. Write $e_2 = (\bar{x} - \bar{X}) / \bar{X}$ such that $\mathrm{E}(e_2) = 0$, $\mathrm{E}\left(e_2^2\right) = \lambda C_x^2$, $\mathrm{E}(e_0 e_2) = \lambda \rho_{yx} C_y C_x$, and $\mathrm{E}\left(e_2 e_1'\right) = \lambda' C_x^2$. Expressing (56),

$$P_{d(\eta,\delta)} = \bar{Y}\left(1+e_0\right)\left[\eta \frac{1+\left(1-\delta\right)e_2+\delta e_1'}{1+\delta e_2+\left(1-\delta\right)e_1'} + \left(1-\eta\right)\frac{1+\delta e_2+\left(1-\delta\right)e_1'}{1+\left(1-\delta\right)e_2+\delta e'}\right]. \quad (57)$$

The expression (57) can be approximated as

$$\begin{aligned} P_{d(\eta,\delta)} \cong \bar{Y}\Big[&1+e_0-\left(1-2\eta\right)\left(1-2\delta\right)\left(e_2-e_1'+e_0e_2-e_0e_1'\right) \\ &+\left(1-\eta-\delta\right)\left(1-2\delta\right)e_2^2-\left(1-2\delta\right)^2 e_2e_1'+\left(\eta-\delta\right)e_1'^2\Big] \end{aligned} \quad (58)$$

or

$$\begin{aligned} \left(P_{d(\eta,\delta)}-\bar{Y}\right) \cong \bar{Y}\Big[&e_0-\left(1-2\eta\right)\left(1-2\delta\right)\left(e_2-e_1'+e_0e_2-e_0e_1'\right) \\ &+\left(1-\eta-\delta\right)\left(1-2\delta\right)e_2^2-\left(1-2\delta\right)^2 e_2e_1'+\left(\eta-\delta\right)e_1'^2\Big] \end{aligned} \quad (59)$$

The approximate bias of $P_{d(\eta,\delta)}$ is

$$\begin{aligned} \mathrm{B}\left(P_{d(\eta,\delta)}\right) &= \mathrm{E}\left(P_{d(\eta,\delta)}-\bar{Y}\right) \\ &= \bar{Y}C_x^2\left(1-2\delta\right)\theta\left[\left(1-\eta-\delta\right)-\left(1-2\eta\right)C\right] \end{aligned} \quad (60)$$

The suggested class of estimators $P_{d(\eta,\delta)}$ would be almost unbiased if

$$\begin{aligned} \mathrm{B}\left(P_{d(\eta,\delta)}\right) = 0 \quad &\Rightarrow \left(1-2\delta\right)\left[\left(1-\eta-\delta\right)-\left(1-2\eta\right)C\right] = 0 \\ &\Rightarrow \delta = 1/2 \text{ or } \delta = 1-\eta-C+2\eta C \end{aligned} \quad (61)$$

If $\delta = 1/2$ in (56), $P_{d(\eta,1/2)} = \bar{y}^*$ (the conventional unbiased estimator), and for $\delta = 1 - \eta - C + 2\eta C$ in (56), $P_{d(\eta,\delta)}$ yields an almost unbiased estimator for $\bar{Y}$ as

$$P_{d(\eta)} = \bar{y}^* \left[\eta \frac{(\eta + C - 2\eta C)\bar{x} + (1 - \eta - C + 2\eta C)\bar{\bar{X}}}{(1 - \eta - C + 2\eta C)\bar{x} + (\eta + C - 2\eta C)\bar{\bar{X}}} + (1-\eta)\frac{(1 - \eta - C + 2\eta C)\bar{x} + (\eta + C - 2\eta C)\bar{\bar{X}}}{(\eta + C - 2\eta C)\bar{x} + (1 - \eta - C + 2\eta C)\bar{\bar{X}}} \right] \tag{62}$$

The estimator (62) depends on the parameter C, which can be determined through a pilot sample survey. The bias of $P_{d(\eta,\delta)}$ is ignorable if the sample sizes (n, n') approach the universe size N because the factors λ and λ' tend to zero. Squaring both sides of (57), the approximated expressions is

$$\begin{aligned} \left(P_{d(\eta,\delta)} - \bar{Y}\right)^2 &\cong \bar{Y}^2 \left[e_0 - (1-2\eta)(1-2\delta)(e_1 - e_1')\right]^2 \\ &= \bar{Y}^2 \Big[e_0^2 (1-2\eta)^2 (1-2\delta)^2 \left(e_2^2 - 2e_2 e_1' + e_1'^2\right) \\ &\qquad - 2(1-2\eta)(1-2\delta)(e_0 e_2 - e_0 e_1')\Big] \end{aligned} \tag{63}$$

The approximate MSE of $P_{d(\eta,\delta)}$ is

$$\begin{aligned} \mathrm{MSE}\left(P_{d(\eta,\delta)}\right) = \bar{Y}^2 \Big[&\lambda C_y^2 + \lambda^* C_{y(2)}^2 \\ &+ \theta(1-2\eta)(1-2\delta)C_x^2 \left\{(1-2\eta)(1-2\delta) - 2C\right\}\Big] \end{aligned} \tag{64}$$

which is minimum when

$$(1-2\eta)(1-2\delta) = C \tag{65}$$

The MMSE of $P_{d(\eta,\delta)}$ is given by

$$\mathrm{MMSE}\left(P_{d(\eta,\delta)}\right) = \left[\lambda S_y^2 \left(1 - \rho_{yx}^2\right) + \lambda' \rho_{yx}^2 S_y^2 + \lambda^* S_{y(2)}^2\right] \tag{66}$$

Theorem 2. Up to $O(n^{-1})$,

$$\mathrm{MSE}\left(P_{d(\eta,\delta)}\right) \geq \left[\lambda S_y^2 \left(1 - \rho_{yx}^2\right) + \lambda' \rho_{yx}^2 S_y^2 + \lambda^* S_{y(2)}^2\right]$$

if $(1 - 2\eta)(1 - 2\delta) = C$.

Singh and Kumar (2009b) showed the quantity $\lambda S_y^2\left(1-\rho_{yx}^2\right)+\lambda'\rho_{yx}^2 S_y^2+\lambda^* S_{y(2)}^2$ is the minimal possible MSE, up to order n^{-1}, for a wide family of estimators to which the estimator (56) also belongs. For instance, for estimators of the form $P_{dh} = \bar{y}^* h(v)$ where h(.) is a function of v such that h(.) = 1. Singh and Kumar (2009a) showed incorporating the sample and universe variances of x might yield an estimator that has a lower MSE than $\lambda S_y^2\left(1-\rho_{yx}^2\right)+\lambda'\rho_{yx}^2 S_y^2+\lambda^* S_{y(2)}^2$, especially when relationship between y and x is markedly nonlinear. For every value of C, it is possible to select an AOE $P_{d(\eta,\delta)}^{(0)}$ from the two-parameter family in (56) (Chami et al., 2012, p. 6).

Remark 2. An alternative to the two-parameter RPR estimator defined in (56) is given by

$$P_{d(\eta,\delta)}^{(1)}=\left[\eta v^{(1-2\delta)}+(1-\eta)v^{(2\delta-1)}\right]\bar{y}^*.$$

Up to order n^{-1}, the bias and MSE of $P_{d(\eta,\delta)}^{(1)}$ are same as defined for the family of $P_{d(\eta,\delta)}$. The class of estimators $P_{d(\eta,\delta)}^{(1)}$ is further generalized as

$$P_{d(\eta,\delta)}^{(2)}=\left[\eta v'^{(1-2\delta)}+(1-\eta)v'^{(2\delta-1)}\right]\bar{y}^*,$$

where $v' = (a\bar{x} + b) / (a\bar{x}' + b)$ and η, δ, a, and b are the same as defined earlier.

Efficiency Comparison and Choice of Parameter

From (3) and (64),

$$\begin{aligned}\mathrm{V}\left(\bar{y}^*\right)-\mathrm{MSE}\left(P_{d(\eta,\delta)}\right)\\ =(\lambda-\lambda')\bar{Y}^2C_x^2(1-2\eta)(1-2\delta)\left\{2C-(1-2\alpha)(1-2\beta)\right\}\end{aligned} \tag{67}$$

which is positive if

$$(1-2\eta)(1-2\delta)\left\{2C-(1-2\alpha)(1-2\beta)\right\}>0. \tag{68}$$

Therefore, either

(i) $\eta > \frac{1}{2}, \delta > \frac{1}{2}$ and $C > \frac{1}{2}(1-2\eta)(1-2\delta)$,

(ii) $\eta < \frac{1}{2}, \delta < \frac{1}{2}$ and $C > \frac{1}{2}(1-2\eta)(1-2\delta)$,

(iii) $\eta < \frac{1}{2}, \delta > \frac{1}{2}$ and $C < \frac{1}{2}(1-2\eta)(1-2\delta)$, or

(iv) $\eta > \frac{1}{2}, \delta > \frac{1}{2}$ and $C < \frac{1}{2}(1-2\eta)(1-2\delta)$.

The family $P_{d(\eta,\delta)}$ is better than $\bar{y}^*$ as long as the above conditions hold true. It can be easily shown that $P_{d(\eta,\delta)}$ is more precise than

(i) T_{R2d} (in the presence of non-response) if
either $C - 1 > (2\eta\delta - \eta - \delta) > 0$
or $C - 1 < (2\eta\delta - \eta - \delta) < 0$

(ii) T_{P2d} (in the presence of non-response) if
either $C > (2\eta\delta - \eta - \delta) > -1$
or $C < (2\eta\delta - \eta - \delta) < -1$

Remark 3. For a more explicit range of η, δ, and C, the reader is referred to Chami et al. (2012).

Unbiased Asymptotically Optimum Estimator

Combining (61) and (64), calculate the parameters η and δ, where the suggested estimator becomes, at least up least up to $O(n^{-1})$, an unbiased AOE. Obtain a line with

$$\delta = 1/2, C = 0 \tag{69}$$

(recall that on this line our estimator always reduces to $\bar{y}^*$) or a curve $(\eta^*(C), \delta^*(C), C) \in R^3$ in the parameter space with

$$\eta^*(C) = \frac{1}{2}\left(1 \pm \sqrt{C/(2C-1)}\right), \delta^*(C) = \frac{1}{2}\left(1 \pm \sqrt{C(2C-1)}\right). \tag{70}$$

The parametric curve in (69) is only defined for $C \leq 0$ or $C > 1/2$. It is three hyperbolas. In the region $0 < C \leq 1/2$ of the parameter space, minimizing MSE comes with a tradeoff in bias. Putting (70) in (56), the unbiased estimator for $\bar{Y}$ is

$$P_{d\left(\eta^*(C),\delta^*(C)\right)} = \left[\frac{2(C+1)\bar{x}' - 2(C-1)\bar{x}^2 + \left(2C^2 - C - 1\right)\left(\bar{x}' - \bar{x}\right)}{4\bar{x}'\bar{x} - \left(2C^2 - C - 1\right)\left(\bar{x}' - \bar{x}\right)^2}\right]\bar{y}^*.$$

The denominator vanishes if

$$C = 0.25\left(1 \pm \sqrt{9 + \left(32\bar{x}'\bar{x}\right)/\left(\bar{x}' - \bar{x}\right)^2}\right) \tag{71}$$

It can be shown up to order n^{-1} that the bias and MSE of the suggested estimators is

$$\mathrm{B}\left(P_{d\left(\eta^*(C),\delta^*(C)\right)}\right) = 0, \mathrm{MSE}\left(P_{d\left(\eta^*(C),\delta^*(C)\right)}\right) = \left[\lambda S_y^2\left(1 - \rho_{yx}^2\right) + \lambda^* S_{y(2)}^2\right].$$

Thus, the estimator $P_{d\left(\eta^*(C),\delta^*(C)\right)}$ is a biased AOE.

Price Aspects on Both Cases

Derivation of Optimum Values of *n*', *n*, and *k* for Fixed Price *C*' ≤ *C*₀

Denote the total (fixed) price of the surveys, apart from overhead, by C_0. The expected total price of the survey apart from overhead is given by

$$C' = c_1'n' + n\left(c_1 + c_2W_1 + \frac{c_3W_2}{k}\right), \tag{72}$$

where c_1' is the price per unit of identifying and observing the supplementary character, c_1 is the price per unit of mailing a questionnaire/visiting the unit in the second-phase, c_2 is the price per unit of collecting or processing data obtained from the n_1 responding units, and c_3 is the price per unit of obtaining data for the sub-sampled units. For the sake of convenience of determination of n', n, and k for (i) fixed price and (ii) specified MSE, retaining the terms of order n^{-1}, we write the approximate MSEs of the estimators $Q_1 = T_{d(\alpha,\beta)}$ and $Q_2 = P_{d(\eta,\delta)}$ as

$$\text{MSE}\left(Q_i\right) = \frac{1}{n}\bar{Y}^2 V_{0i} + \frac{1}{n'}\bar{Y}^2 V_{1i} + \frac{k}{n}\bar{Y}^2 V_{2i} - \frac{1}{N}\bar{Y}^2 C_y^2, \quad i = 1, 2, \tag{73}$$

where $\bar{Y}^2 V_{0i}$, $\bar{Y}^2 V_{1i}$, and $\bar{Y}^2 V_{2i}$ are coefficients of the terms $1/n$, $1/n'$, and k/n in $\text{MSE}(Q_i)$, $i = 1, 2$.

Consider a function φ:

$$\varphi = \text{MSE}\left(Q_i\right) + \lambda_i \left\{ c_1' n' + n\left(c_1 + c_2 W_1 + \frac{c_3 W_2}{k} \right) \right\}. \tag{74}$$

Using the calculus of obtaining the optimality,

$$n' = \bar{Y}\sqrt{\frac{V_{1i}}{\lambda_i c_1'}} = \frac{1}{\lambda_i^*}\sqrt{\frac{V_{1i}}{c_1'}}, \tag{75}$$

$$n = \bar{Y}\sqrt{\frac{V_{0i} + kV_{1i}}{\lambda_i\left(c_1 + c_2 W_1 + \frac{c_3 W_2}{k} \right)}} = \frac{1}{\lambda_i^*}\sqrt{\frac{V_{0i} + kV_{1i}}{c_1 + c_2 W_1 + \frac{c_3 W_2}{k}}}, \tag{76}$$

$$\frac{n}{k} = \bar{Y}\sqrt{\frac{V_{2i}}{\lambda_i c_3 W_2}} = \frac{1}{\lambda_i^*}\sqrt{\frac{V_{2i}}{c_3 W_2}}. \tag{77}$$

where

$$\lambda_i^* = \frac{1}{C_0}\left[\sqrt{V_{1i} c_1'} + \sqrt{\left(V_{0i} + k_{\text{opt}}^{(i)} V_{2i} \right)\left(c_1 + c_2 W_1 + \frac{c_3 W_2}{k_{\text{opt}}^{(i)}} \right)} \right]$$

The optimum value of k is

$$k_{\text{opt}}^{(i)} = \sqrt{\frac{V_{0i} c_3 W_2}{\left(c_1 + c_2 W_1 \right) V_{2i}}}. \tag{78}$$

With the help of (72), (75), (76), and (78),

$$\bar{Y}\lambda_i^* = \sqrt{\lambda_i} \tag{79}$$

The resulting minimum MSE(Q_i), i = 1, 2, are

$$\text{MMSE}(Q_i) = \bar{Y}^2 \left[\frac{1}{C_0} \left\{ \sqrt{V_{1i} c_1'} + \sqrt{\left(V_{0i} + k_{\text{opt}}^{(i)} V_{2i}\right)\left(c_1 + c_2 W_1 + \frac{c_3 W_2}{k_{\text{opt}}^{(i)}}\right)} \right\} - \frac{1}{N} C_y^2 \right] \tag{80}$$

Derivation of *n*', *n*, and *k* for Specified MSE, $V \le V_0^*$

Observing (78), the optimum value of k is independent of the total price or specified precision. Let V_0^* be the fixed MSE of Q_i, i = 1, 2:

$$V_0^* = \bar{Y}^2 \left[\frac{1}{n} V_{0i} + \frac{1}{n'} V_{1i} + \frac{k}{n} V_{2i} - \frac{1}{N} C_y^2 \right], \quad i = 1, 2\,. \tag{81}$$

Thus,

$$\sqrt{\frac{1}{\lambda_i}} = \bar{Y}^2 \left[\frac{1}{C_0} \left\{ \sqrt{V_{1i} c_1'} + \sqrt{\left(V_{0i} + k_{\text{opt}}^{(i)} V_{2i}\right)\left(c_1 + c_2 W_1 + \frac{c_3 W_2}{k_{\text{opt}}^{(i)}}\right)} \right\} \right] \left[V_0^* + \frac{\bar{Y}^2 C_y^2}{N} \right]^{-1} \tag{82}$$

Using the optimum value of the expected price, obtain the MSE

$$\text{D}(Q_i) = \bar{Y}^2 \left[\frac{1}{C_0} \left\{ \sqrt{V_{1i} c_1'} + \sqrt{\left(V_{0i} + k_{\text{opt}}^{(i)} V_{2i}\right)\left(c_1 + c_2 W_1 + \frac{c_3 W_2}{k_{\text{opt}}^{(i)}}\right)} \right\} \right]^2 \left[V_0^* + \frac{\bar{Y}^2 C_y^2}{N} \right]^{-1} \tag{83}$$

Numerical Example

Population source: Khare and Sinha (2007). The values of required parameters are

$$N=95, n=35, n'=70, D_1=0.75, D_2=0.25, C_y=0.15613, C_{(y)}=0.12075,$$
$$C_x=0.03006, C_{x(2)}=0.02478, \rho_{yx}=0.328, C_{yx(2)}=0.477$$

Case I: There Is Non-Response on *y* as Well as on *x*

Compute the optimum values of α for given k and β by using the formula

$$\alpha_{\text{opt}} = 1/2\left[1-\left\{R_d^*/(1-2\beta)\right\}\right] \tag{84}$$

for k = 5 (−1) 2 and β (> 1/2) = 0.51, 0.75, 1.00 (0.25) 2.50, β (< 1/2) = 0.49, 0.25, 0.00 (0.25) −1.50. Findings are shown in Table 1. However, the optimum values of β for given k and α may also be computed by using the formula

$$\beta_{\text{opt}} = 1/2\left[1-\left\{R_d^*/(1-2\alpha)\right\}\right]. \tag{85}$$

Table 1. Optimum values of α for selected values of β and for k = 5 (−1) 2

	β								
k	**0.51**	**0.75**	**1.00**	**1.25**	**1.50**	**1.75**	**2.00**	**2.25**	**2.50**
5	52.0309	2.5612	1.5306	1.1871	1.0153	0.9122	0.8435	0.7945	0.7577
4	50.9240	2.5170	1.5085	1.1723	1.0042	0.9034	0.8362	0.7881	0.7521
3	49.3693	2.4548	1.4774	1.1516	0.9887	0.8910	0.8258	0.7793	0.7443
2	47.0260	2.3610	1.4305	1.1203	0.9653	0.8722	0.8102	0.7659	0.7326
k	**0.49**	**0.25**	**0.00**	**−0.25**	**−0.50**	**−0.75**	**−1.00**	**−1.25**	**−1.50**
5	−51.0309	−1.5612	−0.5306	−0.1871	−0.0153	0.0878	0.1565	0.2055	0.2423
4	−49.9240	−1.5170	−0.5085	−0.1723	−0.0042	0.0966	0.1638	0.2119	0.2479
3	−48.3693	−1.4548	−0.4774	−0.1516	0.0113	0.1090	0.1742	0.2207	0.2557
2	−46.0260	−1.3610	−0.4305	−0.1203	0.0347	0.1278	0.1898	0.2341	0.2674

Table 2. The PRE of $T_{d(\alpha,\beta)}$ with respect to $\bar{y}^{*}$

		α								
k	***β***	**0.51 or 0.49**	**0.75 or 0.25**	**1.00 or 0.00**	**1.25 or −0.25**	**1.50 or −0.50**	**1.75 or −0.75**	**2.00 or −1.00**	**2.25 or −1.25**	**2.50 or −1.50**
5	0.51 or 0.49	100.0059	100.1464	100.2926	100.4385	100.5840	100.7293	100.8743	101.0189	101.1633
	0.75 or 0.25	100.1464	103.5650	106.8834	109.8822	112.4890	114.6353	116.2613	117.3191	117.7766
	1.00 or 0.00	100.2926	106.8834	112.4890	116.2613	117.7766	116.8529	113.6019	108.3970	101.7727
	1.25 or −0.25	100.4385	109.8822	116.2613	117.6196	113.6019	105.2259	*	*	*
	1.50 or −0.50	100.5840	112.4890	117.7766	113.6019	101.7727	*	*	*	*
	1.75 or −0.75	100.7293	114.6353	116.8529	105.2259	*	*	*	*	*
	2.00 or −1.00	100.8743	116.2613	113.6019	*	*	*	*	*	*
	2.25 or −1.25	101.0189	117.3191	108.3970	*	*	*	*	*	*
	2.50 or −1.50	101.1633	117.7766	101.7727	*	*	*	*	*	*
4	0.51 or 0.49	100.0056	100.1396	100.2789	100.4179	100.5566	100.6950	100.8330	100.9707	101.1081
	0.75 or 0.25	100.1396	103.3888	106.5215	109.3299	111.7470	113.7105	115.1669	116.0747	116.4070
	1.00 or 0.00	100.2789	106.5215	111.7470	115.1669	116.4070	115.3230	112.0413	106.9231	100.4724
	1.25 or −0.25	100.4179	109.3299	115.1669	116.1539	112.0413	103.8304	*	*	*
	1.50 or −0.50	100.5566	111.7470	116.4070	112.0413	100.4724	*	*	*	*
	1.75 or −0.75	100.6950	113.7105	115.3230	103.8304	*	*	*	*	*
	2.00 or −1.00	100.8330	115.1669	112.0413	*	*	*	*	*	*
	2.25 or −1.25	100.9707	116.0747	106.9231	*	*	*	*	*	*
	2.50 or −1.50	101.1081	116.4070	100.4724	*	*	*	*	*	*

Note: * indicates the PRE was less than 100

Table 2 (continued).

		α								
k	***β***	**0.51 or 0.49**	**0.75 or 0.25**	**1.00 or 0.00**	**1.25 or −0.25**	**1.50 or −0.50**	**1.75 or −0.75**	**2.00 or −1.00**	**2.25 or −1.25**	**2.50 or −1.50**
3	0.51 or 0.49	100.0056	100.1396	100.2789	100.4179	100.5566	100.6950	100.8330	100.9707	101.1081
	0.75 or 0.25	100.1396	103.3888	106.5215	109.3299	111.7470	113.7105	115.1669	116.0747	116.4070
	1.00 or 0.00	100.2789	106.5215	111.7470	115.1669	116.4070	115.3230	112.0413	106.9231	100.4724
	1.25 or −0.25	100.4179	109.3299	115.1669	116.1539	112.0413	103.8304	*	*	*
	1.50 or −0.50	100.5566	111.7470	116.4070	112.0413	100.4724	*	*	*	*
	1.75 or −0.75	100.6950	113.7105	115.3230	103.8304	*	*	*	*	*
	2.00 or −1.00	100.8330	115.1669	112.0413	*	*	*	*	*	*
	2.25 or −1.25	100.9707	116.0747	106.9231	*	*	*	*	*	*
	2.50 or −1.50	101.1081	116.4070	100.4724	*	*	*	*	*	*
2	0.51 or 0.49	100.0047	100.1181	100.2359	100.3534	100.4704	100.5871	100.7034	100.8194	100.9349
	0.75 or 0.25	100.1181	102.8378	105.3980	107.6267	109.4733	110.8936	111.8519	112.3233	112.2953
	1.00 or 0.00	100.2359	105.3980	109.4733	111.8519	112.2953	110.7574	107.3959	102.5336	*
	1.25 or −0.25	100.3534	107.6267	111.8519	111.7687	107.3959	*	*	*	*
	1.50 or −0.50	100.4704	109.4733	112.2953	107.3959	*	*	*	*	*
	1.75 or −0.75	100.5871	110.8936	110.7574	*	*	*	*	*	*
	2.00 or −1.00	100.7034	111.8519	107.3959	*	*	*	*	*	*
	2.25 or −1.25	100.8194	112.3233	102.5336	*	*	*	*	*	*
	2.50 or −1.50	100.9349	112.2953	*	*	*	*	*	*	*

Note: * indicates the PRE was less than 100

Table 3. PREs of AOE $T^{(0)}_{d(\alpha,\beta)}$ and of T_{R1d} with respect to $\bar{y}^*$

k	5	4	3	2
$\text{PRE}\left(T^{(0)}_{d(\alpha,\beta)}, \bar{y}^*\right)$	117.7951	116.4084	114.6544	112.3728
$\text{PRE}(T_{R1d}, \bar{y}^*)$	112.4890	111.7470	110.7814	109.4733

The percent relative efficiency (PRE) of $T_{d(\alpha,\beta)}$ with respect to $\bar{y}^*$ is obtained using the formula, the results of which are given in Table 2.

$$\begin{aligned}\text{PRE}\left(T_{d(\alpha,\beta)}, \bar{y}^*\right) &= \left(\lambda C_y^2 + \lambda^* C_{y(2)}^2\right)\left(\lambda C_y^2 + \lambda^* C_{y(2)}^2\right. \\ &\quad \left. + (1-2\alpha)(1-2\beta)\left(\theta C_x^2 + \lambda^* C_{x(2)}^2\right)\left[(1-2\alpha)(1-2\beta) - 2R_d^*\right]\right)^{-1} \times 100\end{aligned} \tag{86}$$

The PREs of the AOE $T^{(0)}_{d(\alpha,\beta)}$ and of T_{R1d} with respect to $\bar{y}^*$ are obtained using the formulae

$$\text{PRE}\left(T^{(0)}_{d(\alpha,\beta)}, \bar{y}^*\right) = \frac{\lambda C_y^2 + \lambda^* C_{y(2)}^2}{\left(\lambda C_y^2 + \lambda^* C_{y(2)}^2\right) - \left(\theta C_x^2 + \lambda^* C_{x(2)}^2\right) R_d^{*2}} \times 100 \tag{87}$$

$$\text{PRE}\left(T_{R1d}, \bar{y}^*\right) = \frac{\lambda C_y^2 + \lambda^* C_{y(2)}^2}{\left(\lambda C_y^2 + \lambda^* C_{y(2)}^2\right) - \left(\theta C_x^2 + \lambda^* C_{x(2)}^2\right)\left(1 - 2R_d^{*2}\right)} \times 100 \tag{88}$$

Findings are given Table 3.

Observe from Table 1 that (i) when k is fixed, α_{opt} decreases when β (> 1/2) increases up to 2.50; (ii) for fixed values of k, the magnitude of α_{opt} (i.e., absolute optimum value of α) decreases when β (< 1/2) decreases to −1.50. Table 2 shows that the PRE of $T_{d(\alpha,\beta)}$ with respect to $\bar{y}^*$ is larger than 100 for $(\alpha, \beta) \in (0.51, 2.50)$, $(\alpha, \beta) \in (-1.50, 0.49)$, and all values of $k = 5\ (-1)\ 2$. Thus it follows that, in said range of (α, β) and all the values of $k = 5\ (-1)\ 2$, the suggested estimator $T_{d(\alpha,\beta)}$ is more accurate than $\bar{y}^*$. A large number of flexible values of (α, β) exist for which the suggested estimator is superior to $\bar{y}^*$. From Table 3, observe the AOE $T^{(0)}_{d(\alpha,\beta)}$ is more accurate than $\bar{y}^*$ and T_{R1d} with a substantial gain in efficiency.

Table 4. Optimum values of η for given δ

δ	0.51	0.75	1.00	1.25	1.50	1.75	2.00	2.25	2.50
η_{opt}	43.09040	2.20360	1.35180	1.06790	0.92591	0.84070	0.78390	0.74340	0.71300
δ	0.49	0.25	0.00	−0.25	−0.50	−0.75	−1.00	−1.25	−1.50
η_{opt}	−42.09040	−1.20360	−0.35180	−0.06790	0.07410	0.15930	0.21610	0.25660	0.28700

Case II: When NR Occurs Only on *y* and Information on *x* is Available

Compute the optimum values of η for a given δ using the following formula:

$$\eta_{\text{opt}} = 1/2\left[1 - \left\{C/(1-2\delta)\right\}\right] \tag{89}$$

The results are given in Table 4.

The optimum values of δ for a given η may also be computed by using the following formula:

$$\delta_{\text{opt}} = 1/2\left[1 - \left\{C/(1-2\eta)\right\}\right] \tag{90}$$

The PRE of $P_{d(\eta,\delta)}$ with respect to $\bar{y}^*$ is obtained using the formula

$$\begin{aligned}&\text{PRE}\left(P_{d(\eta,\delta)}, \bar{y}^*\right)\\ &= \left(\lambda C_y^2 + \lambda^* C_{y(2)}^2\right)\Big(\lambda C_y^2 + \lambda^* C_{y(2)}^2\\ &\quad + (1-2\eta)(1-2\delta)\left(\theta C_x^2 + \lambda^* C_{x(2)}^2\right)\left[(1-2\eta)(1-2\delta) - 2C\right]\Big)^{-1} \times 100\end{aligned} \tag{91}$$

The results are given in Table 5. The PREs of the AOE $P_{d(\eta,\delta)}^{(0)}$ and of $T_{\text{R}2d}$ with respect to $\bar{y}^*$ are obtained using the formulae below, the results are given Table 6.

$$\text{PRE}\left(P_{d(\eta,\delta)}^{(0)}, \bar{y}^*\right) = \frac{\lambda C_y^2 + \lambda^* C_{y(2)}^2}{\left(\lambda C_y^2 + \lambda^* C_{y(2)}^2\right) - \theta C^2 C_x^2} \times 100 \tag{92}$$

$$\text{PRE}\left(P_{\text{R}2d}, \bar{y}^*\right) = \frac{\lambda C_y^2 + \lambda^* C_{y(2)}^2}{\left(\lambda C_y^2 + \lambda^* C_{y(2)}^2\right) - \theta C_x^2\left(1 - 2C^2\right)} \times 100 \tag{93}$$

Table 5. The PRE of $P_{d(\eta,\delta)}$ with respect to $\bar{y}^*$

		η								
k	***δ***	**0.51 or 0.49**	**0.75 or 0.25**	**1.00 or 0.00**	**1.25 or −0.25**	**1.50 or −0.50**	**1.75 or −0.75**	**2.00 or −1.00**	**2.25 or −1.25**	**2.50 or −1.50**
5	0.51 or 0.49	100.0021	100.0512	100.1022	100.1529	100.2034	100.2536	100.3036	100.3533	100.4028
	0.75 or 0.25	100.0512	101.2040	102.2400	103.0967	103.7648	104.2364	104.5062	104.5709	104.4299
	1.00 or 0.00	100.1022	102.2400	103.7648	104.5062	104.4299	103.5394	101.8759	*	*
	1.25 or −0.25	100.1529	103.0967	104.5062	104.0847	101.8759	*	*	*	*
	1.50 or −0.50	100.2034	103.7648	104.4299	101.8759	*	*	*	*	*
	1.75 or −0.75	100.2536	104.2364	103.5394	*	*	*	*	*	*
	2.00 or −1.00	100.3036	104.5062	101.8759	*	*	*	*	*	*
	2.25 or −1.25	100.3533	104.5709	*	*	*	*	*	*	*
	2.50 or −1.50	100.4028	104.4299	*	*	*	*	*	*	*
4	0.51 or 0.49	100.0023	100.0583	100.1164	100.1741	100.2316	100.2889	100.3458	100.4024	100.4588
	0.75 or 0.25	100.0583	101.3729	102.5580	103.5406	104.3084	104.8513	105.1622	105.2368	105.0742
	1.00 or 0.00	100.1164	102.5580	104.3084	105.1622	105.0742	104.0492	102.1411	*	*
	1.25 or −0.25	100.1741	103.5406	105.1622	104.6766	102.1411	*	*	*	*
	1.50 or −0.50	100.2316	104.3084	105.0742	102.1411	*	*	*	*	*
	1.75 or −0.75	100.2889	104.8513	104.0492	*	*	*	*	*	*
	2.00 or −1.00	100.3458	105.1622	102.1411	*	*	*	*	*	*
	2.25 or −1.25	100.4024	105.2368	*	*	*	*	*	*	*
	2.50 or −1.50	100.4588	105.0742	*	*	*	*	*	*	*

Note: * indicates the PRE was less than 100

Table 5 (continued).

		η								
k	***δ***	**0.51 or 0.49**	**0.75 or 0.25**	**1.00 or 0.00**	**1.25 or −0.25**	**1.50 or −0.50**	**1.75 or −0.75**	**2.00 or −1.00**	**2.25 or −1.25**	**2.50 or −1.50**
3	0.51 or 0.49	100.0027	100.0677	100.1351	100.2022	100.2690	100.3354	100.4016	100.4674	100.5329
	0.75 or 0.25	100.0677	101.5970	102.9812	104.1330	105.0355	105.6750	106.0417	106.1298	105.9379
	1.00 or 0.00	100.1351	102.9812	105.0355	106.0417	105.9379	104.7306	102.4937	*	*
	1.25 or −0.25	100.2022	104.1330	106.0417	105.4691	102.4937	*	*	*	*
	1.50 or −0.50	100.2690	105.0355	105.9379	102.4937	*	*	*	*	*
	1.75 or −0.75	100.3354	105.6750	104.7306	*	*	*	*	*	*
	2.00 or −1.00	100.4016	106.0417	102.4937	*	*	*	*	*	*
	2.25 or −1.25	100.4674	106.1298	*	*	*	*	*	*	*
	2.50 or −1.50	100.5329	105.9379	*	*	*	*	*	*	*
2	0.51 or 0.49	100.0032	100.0807	100.1610	100.2410	100.3206	100.3999	100.4788	100.5574	100.6356
	0.75 or 0.25	100.0807	101.9086	103.5724	104.9635	106.0579	106.8357	107.2826	107.3900	107.1560
	1.00 or 0.00	100.1610	103.5724	106.0579	107.2826	107.1560	105.6878	102.9854	*	*
	1.25 or −0.25	100.2410	104.9635	107.2826	106.5850	102.9854	*	*	*	*
	1.50 or −0.50	100.3206	106.0579	107.1560	102.9854	*	*	*	*	*
	1.75 or −0.75	100.3999	106.8357	105.6878	*	*	*	*	*	*
	2.00 or −1.00	100.4788	107.2826	102.9854	*	*	*	*	*	*
	2.25 or −1.25	100.5574	107.3900	*	*	*	*	*	*	*
	2.50 or −1.50	100.6356	107.1560	*	*	*	*	*	*	*

Note: * indicates the PRE was less than 100

Table 6. PREs of AOE $P_{d(\eta,\delta)}^{(0)}$ and of T_{R2d} with respect to $\bar{y}^{*}$

k	5	4	3	2
$\text{PRE}\left(P_{d(\eta,\delta)}^{(0)}, \bar{y}^{*}\right)$	104.5744	105.2409	106.1346	107.3959
$\text{PRE}(T_{R2d}, \bar{y}^{*})$	103.7648	104.3084	105.0355	106.0529

Table 4 exhibits that (i) the optimum value of η decreases as $\delta\,(> 1/2)$ increases up to 2.50; (ii) the absolute of optimum value of η also decreases when $\delta\,(< 1/2)$ decreases to −1.50. Observe from Table 5 that (i) for $-1.50 \le \eta \le 2.50$, $0.51 \le \delta \le 1.00$, and $k = 5\,(-1)\,2$, the proposed class of estimators $P_{d(\eta,\delta)}$ are always better than $\bar{y}^{*}$; (ii) for $-0.50 \le \eta, \delta \le 1.50$ and $k = 5\,(-1)\,2$, the envisaged estimator $P_{d(\eta,\delta)}$ is more efficient than $\bar{y}^{*}$ with a considerable gain in efficiency. Table 6 shows that the envisaged AOE $P_{d(\eta,\delta)}^{(0)}$ is more efficient than $\bar{y}^{*}$ and T_{R2d} for $k = 5\,(-1)\,2$.

From Table 2, note there is enough flexibility in choosing the values of the scalars η, δ in order to get estimators η, δ for $P_{d(\eta,\delta)}$ and better than $\bar{y}^{*}$, T_{R2d}. It is also observed from Table 5 that, for fixed values of η, δ, the values of $\text{PRE}(P_{d(\eta,\delta)}, \bar{y}^{*})$ increase as the values of k decrease. Comparing the results shown in Tables 2 and 3 to Tables 5 and 6, the proposed family of estimators $T_{d(\alpha,\beta)}$ (where NR occurs on both the variables y, x) performs better than the corresponding estimator $P_{d(\eta,\delta)}$ (where NR occurs only on the main variable y). The recommendation favors that the suggested estimators $T_{d(\alpha,\beta)}$ and $P_{d(\eta,\delta)}$ be used in practice.

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