2018

A New Lifetime Distribution For Series System: Model, Properties and Application

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A new lifetime distribution for modeling system lifetime in series setting is proposed that embodies most of the compound lifetime distribution. The reliability analysis of parent and of sub-models has also been discussed. Various mathematical properties that include moment generating function, moments, and order statistics have been obtained. The newly-proposed distribution has a flexible density function; more importantly its hazard rate function can take up different shapes such as bathtub, upside down bathtub, increasing, and decreasing shapes. The unknown parameters of the proposed generalized family have been estimated through MLE technique. The strength and usefulness of the proposed family was tested on a real life data set and it is quite clear from the statistical analysis that proposed family offers a better fit.

Keywords: Generalized Lindley distribution, power series distribution, compounding, hazard function, survival function

Introduction

The modeling of lifetime data has received prime attention from researchers for the last decade. Many continuous probability models such as exponential, gamma, and Weibull were frequently used in statistical literature to analyze the lifetime data, but these probability models cannot be used efficiently to model lifetime data that is bathtub shaped and have unimodal failure rates. To overcome this problem, researchers have focused their attention on compounding mechanisms which are an innovative way to construct suitable, flexible, and alternative models to fit the lifetime data of different types.

Consider a system with N components, where N is a discrete random variable with domain $N = 1, 2,\ldots$. The lifetime of the ith component is a continuous random variable, say X_i, that may follow any one of the lifetime distributions such as exponential, gamma, Weibull, Lindley, etc. The suitable discrete distributions for N may be geometric, zero truncated Poisson, or power series distributions in general. The lifetime of such a system in series and parallel combination is defined and denoted by a non-negative random variable $Y = \min \{ X_i \}_{i=1}^{N}$ or $Y = \max \{ X_i \}_{i=1}^{N}$, respectively.

With this in mind, Adamidis and Loukas (1998) constructed a two parameter lifetime distribution by compounding the exponential distribution with the geometric distribution, called the Exponential Geometric (EG) distribution. Kus (2007) obtained a compound of the exponential distribution with that of Poisson and it was named the Exponential Poisson (EP) distribution. Tahmasbi and Rezaei (2008) obtained the Exponential Logarithmic (EL) distribution by using the same compounding mechanism. The extended EG distribution was considered by Adamidis, Dimitrakopoulou, and Loukas (2005).

Power series distributions contain several classical discrete distributions as special cases, therefore Chahkandi and Ganjali (2009) introduced a compound class of Exponential Power Series (EPS) distributions which contain several compound distributions as special cases. It is known that the Weibull distribution contains the exponential distribution as a special case; in view of this, Morais and Baretto-Souza (2011) replaced the exponential distribution with a Weibull distribution in the compounding mechanism of the EPS distribution and obtained a compound class of Weibull Power Series (WPS) distributions, which contains EPS distribution as a special case. Zakerzadeh and Mahmoudi (2012) obtained a two parameter lifetime distribution by compounding a Lindley distribution with a geometric distribution. Recently, a one parameter Lindley distribution has been used frequently to model lifetime data because it has been observed in several research papers that this distribution performs excellently well when it comes to fit the lifetime data. Adil and Jan (2016) introduced a new family of lifetime distributions which is obtained by compounding Lindley distribution with power series distribution. This new family of continuous lifetime distributions was called the Lindley Power Series (LPS) distribution.

The proposed LPS family of compound distributions contains several lifetime distributions as its special cases that are very flexible and able to accommodate different types of data sets since the probability density function and hazard rate can take on different forms such as increasing, decreasing, and upside down bathtub

shapes which have been shown through graphs for some selected values of parameters, and the potentiality of LPS family has been tested statistically by using it to model some real life data set.

There are many continuous distributions in statistics that can be used to model lifetime data; among them, the most popular are gamma, log normal, and Weibull distributions. The Weibull distribution has been used extensively by researchers to model lifetime data because of its closed form for survival function. However, all these lifetime model suffer from a major drawback, i.e., none of the them exhibit bathtub shapes for their hazard rate functions and hence cannot be used efficiently to model real life data that has bathtub shape for hazard rate function.

Nadarajah, Bakouch, and Tahmasbi (2011) proposed a new lifetime distribution called the generalized Lindley distribution that removes all of these mentioned drawbacks. It was shown that the generalized Lindley distribution has an attractive feature of allowing for monotonically decreasing, monotonically increasing, and bathtub shaped hazard rate functions, while not allowing for constant hazard rate functions. These important features of the generalized Lindley distribution attracted the attention of researchers Adil and Jan (2016), who replaced the Lindley distribution with a generalized Lindley distribution in the compounding mechanism of the LPS distribution.

Construction of the Family

Let $X_1, \ldots, X_n$ be an independent and identically distributed (iid) random variables following the generalized Lindley distribution due to Nadarajah et al. (2011), whose density function is given by

$$g(x;\lambda) = \frac{\alpha\lambda^2}{\lambda+1}\left[1-\left(1+\frac{\lambda x}{\lambda+1}\right)e^{-\lambda x}\right]^{\alpha-1} e^{-\lambda x}(1+x), \quad x>0, \alpha, \lambda>0 \qquad (1)$$

Here, the index N is itself a discrete random variable following a zero truncated power series distribution with probability function given by

$$P(N=n) = \frac{a_n\theta^n}{C(\theta)}, \quad n=1,2,\ldots$$

where a_n depends only on n,

Table 1. Useful quantities of some power series distributions

Distribution	a_n	$C(\theta)$	$C'(\theta)$	$C''(\theta)$	$C^{-1}(\theta)$	θ
Poisson	$n!^{-1}$	$e^{\theta} - 1$	e^{θ}	e^{θ}	$\log(\theta + 1)$	$\theta \in (0, \infty)$
Logarithmic	n^{-1}	$-\log(1 - \theta)$	$(1 - \theta)^{-1}$	$(1 - \theta)^{-2}$	$1 - e^{-\theta}$	$\theta \in (0, 1)$
Geometric	1	$\theta(1 - \theta)^{-1}$	$(1 - \theta)^{-2}$	$2(1 - \theta)^{-3}$	$\theta(\theta + 1)^{-1}$	$\theta \in (0, 1)$
Binomial	$\binom{m}{n}$	$(\theta + 1)^m - 1$	$m(\theta + 1)^{m-1}$	$\dfrac{m(m-1)}{(\theta+1)^{2-m}}$	$(\theta - 1)^{1/m} - 1$	$\theta \in (0, \infty)$

$$C(\theta) = \sum_{n=1}^{\infty} a_n \theta^n$$

and θ is such that $C(\theta)$ is finite. Table 1 shows useful quantities of some zero truncated power series distributions such as Poisson, logarithmic, geometric, and binomial (with m being the number of replicas).

Let $X_{(1)} = \min\{X_i\}_{i=1}^{N}$. The conditional cumulative distribution function of $X_{(1)} \mid N = n$ is given by

$$G_{X_{(1)}|N=n} = 1 - \left[\bar{G}(x)\right]^n$$

$$G_{X_{(1)}|N=n} = 1 - \left[1 - \left[1 - \left(1 + \frac{\lambda x}{\lambda + 1}\right)e^{-\lambda x}\right]^{\alpha}\right]^n$$

and

$$p\left(X_{(1)} \leq x, N = n\right) = \left\{1 - \left[1 - \left[1 - \left(1 + \frac{\lambda x}{\lambda + 1}\right)e^{-\lambda x}\right]^{\alpha}\right]^n\right\}\frac{a_n \theta^n}{C(\theta)}, \quad x > 0, n \geq 1$$

The family of generalized Lindley power series distributions is defined by the marginal cumulative distribution function of $X_{(1)}$:

$$F(x) = \sum_{n-1}^{\infty} \left\{ 1 - \left[1 - \left[1 - \left(1 + \frac{\lambda x}{\lambda + 1} \right) e^{-\lambda x} \right]^{\alpha} \right]^{n} \right\} \frac{a_n \theta^n}{C(\theta)}$$

$$= 1 - \frac{C\left(\left[1 - \left[1 - \left(1 + \frac{\lambda x}{\lambda + 1} \right) e^{-\lambda x} \right]^{\alpha} \right] \theta \right)}{C(\theta)} \tag{2}$$

Density, Survival, and Hazard Rate Function

The probability density function of a family of the Lindley power series (GLPS) distribution can be obtained by differentiating (2) both sides with respect to x.

$$f(x) = \frac{\alpha \lambda^2 \theta e^{-\lambda x}}{(\lambda + 1) C(\theta)} \left[1 - \left(1 + \frac{\lambda x}{\lambda + 1} \right) e^{-\lambda x} \right]^{\alpha - 1} (1 + x)$$

$$\times C'\left(\left[1 - \left[1 - \left(1 + \frac{\lambda x}{\lambda + 1} \right) e^{-\lambda x} \right]^{\alpha} \right] \theta \right) \tag{3}$$

If $\alpha = 1$ in (3), it reduces to a Lindley power series distribution. Alternatively, the probability density function of the GLPS distribution can also be obtained using the joint probability function of $X_{(1)}$ and N:

$$g_{X,N}(x,n) = n \frac{\alpha \lambda^2}{\lambda + 1} (1 + x) e^{-\lambda x} \frac{a_n \theta^n}{C(\theta)} \left[1 - \left[1 - \left(1 + \frac{\lambda x}{\lambda + 1} \right) e^{-\lambda x} \right]^{\alpha} \right]^{n-1}$$

$$\times \left[1 - \left(1 + \frac{\lambda x}{\lambda + 1} \right) e^{-\lambda x} \right]^{\alpha + 1}$$

Therefore, density function of a family of the GLPS distribution is defined by the marginal density of $X_{(1)}$:

$$f(x) = \frac{\alpha \lambda^2 \theta e^{-\lambda x}}{(\lambda+1)C(\theta)}\left[1-\left(1+\frac{\lambda x}{\lambda+1}\right)e^{-\lambda x}\right]^{\alpha-1}(1+x)$$

$$\times C'\left(\left[1-\left[1-\left(1+\frac{\lambda x}{\lambda+1}\right)e^{-\lambda x}\right]^{\alpha}\right]\theta\right)$$

which is same as (3). The survival function of the GLPS distribution is given by

$$S(x) = \frac{C\left(\left[1-\left[1-\left(1+\frac{\lambda x}{\lambda+1}\right)e^{-\lambda x}\right]^{\alpha}\right]\theta\right)}{C(\theta)}$$

and the hazard function is

$$h(x) = \frac{\alpha \lambda^2 \theta(1+x)e^{-\lambda x}\left[1-\left(1+\frac{\lambda x}{\lambda+1}\right)e^{-\lambda x}\right]^{\alpha-1}C'\left(\left[1-\left[1-\left(1+\frac{\lambda x}{\lambda+1}\right)e^{-\lambda x}\right]^{\alpha}\right]\theta\right)}{(\lambda+1)C\left(\left[1-\left[1-\left(1+\frac{\lambda x}{\lambda+1}\right)e^{-\lambda x}\right]^{\alpha}\right]\theta\right)}$$

Proposition 1. The generalized Lindley distribution is the limiting case of the proposed distribution when $\theta \to 0^+$.

Proof. From the cumulative distribution function of the GLPS distribution,

$$\lim_{\theta \to 0^+} F(x) = 1 - \lim_{\theta \to 0^+} 1 - \frac{C\left(\left[1-\left[1-\left(1+\frac{\lambda x}{\lambda+1}\right)e^{-\lambda x}\right]^{\alpha}\right]\theta\right)}{C(\theta)}$$

$$= 1 - \lim_{\theta \to 0^+} \frac{\sum_{n=1}^{\infty} a_n \left(\left[1-\left[1-\left(1+\frac{\lambda x}{\lambda+1}\right)e^{-\lambda x}\right]^{\alpha}\right]\theta\right)^n}{\sum_{n=1}^{\infty} a_n \theta^n}$$

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as

$$C(\theta) = \sum_{n=1}^{\infty} a_n \theta^n \tag{4}$$

and, using L'Hospital's rule, it follows that

$$\lim_{\theta \to 0^+} F(x) = 1 - \lim_{\theta \to 0^+} \frac{\sum_{n=1}^{\infty} a_n n \theta^{n-1} \left[1 - \left[1 - \left(1 + \frac{\lambda x}{\lambda+1} \right) e^{-\lambda x} \right]^\alpha \right]^n}{\sum_{n=1}^{\infty} a_n n \theta^{n-1}}$$

$$= \left[1 - \left(1 + \frac{\lambda x}{\lambda+1} \right) e^{-\lambda x} \right]^\alpha$$

which is the distribution function of the generalized Lindley distribution.

Proposition 2. The densities of proposed distribution can be expressed as an infinite linear combination of densities of 1st order statistics of the generalized Lindley distribution

$$f(x) = \sum_{n=1}^{\infty} P(N=n) g_1(x,n)$$

where

$$g_1(x,n) = n \frac{\alpha \lambda^2}{\lambda+1} (1+x) e^{-\lambda x} \left[1 - \left(1 + \frac{\lambda x}{\lambda+1} \right) e^{-\lambda x} \right]^{\alpha-1}$$

$$\times \left(\left[1 - \left[1 - \left(1 + \frac{\lambda x}{\lambda+1} \right) e^{-\lambda x} \right]^\alpha \right] \right)^{n-1}$$

Proof. It is known that

$$C'(\theta) = \sum_{n=1}^{\infty} n a_n \theta^{n-1}$$

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Therefore,

$$f(x) = \frac{\alpha\lambda^2}{\lambda+1}(1+x)e^{-\lambda x}\left[1-\left(1+\frac{\lambda x}{\lambda+1}\right)e^{-\lambda x}\right]^{\alpha-1}$$

$$\times \sum_{n=1}^{\infty}\frac{a_n\theta^n}{C(\theta)}n\left(\left[1-\left[1-\left(1+\frac{\lambda x}{\lambda+1}\right)e^{-\lambda x}\right]^{\alpha}\right]\right)^{n-1} \tag{5}$$

$$= \sum_{n=1}^{\infty}P(N=n)g_1(x,n)$$

where $g_1(x, n)$, defined as in the statement of the proposition, is the pdf of $X_{(1)} = \min(X_1, X_2,\dots, X_n)$. Therefore the densities of the proposed distribution can be expressed as an infinite linear combination of the 1st order statistics of the generalized Lindley distribution.

Moment Generating Function

The moment generating function (mgf) of the proposed distribution can be obtained from (5):

$$M_X(t) = \sum_{n=1}^{\infty}P(N=n)M_{X_{(1)}}(t)$$

where $M_{X_{(1)}}(t)$ is the moment generating function of 1st order statistics of the generalized Lindley distribution:

$$M_{X_{(1)}}(t) = \int_0^{\infty}e^{tx}n\frac{\alpha\lambda^2}{\lambda+1}(1+x)e^{-\lambda x}\left[1-\left(1+\frac{\lambda x}{\lambda+1}\right)e^{-\lambda x}\right]^{\alpha-1}$$

$$\times\left(\left[1-\left[1-\left(1+\frac{\lambda x}{\lambda+1}\right)e^{-\lambda x}\right]^{\alpha}\right]\right)^{n-1}dx$$

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$$M_{X_{(1)}}(t) = n\frac{\alpha\lambda^2}{\lambda+1}\sum_{j=0}^{n-1}\binom{n-1}{j}(-1)^j\int_0^\infty e^{tx-\lambda x}(1+x)\left[1-\left(1+\frac{\lambda x}{\lambda+1}\right)e^{-\lambda x}\right]^{\alpha j}$$

$$\times\left[1-\left(1+\frac{\lambda x}{\lambda+1}\right)e^{-\lambda x}\right]^{\alpha-1}dx$$

$$= n\frac{\alpha\lambda^2}{\lambda+1}\sum_{j=0}^{n-1}\binom{n-1}{j}(-1)^j\int_0^\infty e^{tx-\lambda x}(1+x)\left[1-\left(1+\frac{\lambda x}{\lambda+1}\right)e^{-\lambda x}\right]^{\alpha j+\alpha-1}dx$$

$$= n\frac{\alpha\lambda^2}{\lambda+1}\sum_{j=0}^{n-1}\sum_{k=0}^\infty\binom{n-1}{j}\binom{\alpha j+\alpha-1}{k}(-1)^{j+k}\int_0^\infty e^{tx-\lambda x-k\lambda x}(1+x)\left(1+\frac{\lambda x}{\lambda+1}\right)^k dx$$

$$= n\frac{\alpha\lambda^2}{\lambda+1}\sum_{j=0}^{n-1}\sum_{k=0}^\infty\sum_{i=0}^k\binom{n-1}{j}\binom{\alpha j+\alpha-1}{k}\binom{k}{i}\left(\frac{\lambda}{\lambda+1}\right)^{k-i}(-1)^{j+k}$$

$$\times\int_0^\infty e^{-(k\lambda+\lambda-t)x}x^{k-i}(1+x)dx$$

$$= n\frac{\alpha\lambda^2}{\lambda+1}\sum_{j=0}^{n-1}\sum_{k=0}^\infty\sum_{i=0}^k\binom{n-1}{j}\binom{\alpha j+\alpha-1}{k}\binom{k}{i}\left(\frac{\lambda}{\lambda+1}\right)^{k-i}(-1)^{j+k}$$

$$\times\left(\frac{\Gamma(k-i+1)}{(\lambda k+\lambda-t)^{k-i+1}}+\frac{\Gamma(k-i+2)}{(\lambda k+\lambda-t)^{k-i+2}}\right)$$

Hence

$$M_X(t) = \frac{\alpha\lambda^2}{\lambda+1}\sum_{n=1}^\infty\frac{a_n\theta^n}{C(\theta)}n\sum_{j=0}^{n-1}\sum_{k=0}^\infty\sum_{i=0}^k\binom{n-1}{j}\binom{\alpha j+\alpha-1}{k}\binom{k}{i}\left(\frac{\lambda}{\lambda+1}\right)^{k-i}(-1)^{j+k}$$

$$\times\left(\frac{\Gamma(k-i+1)(\lambda k+\lambda-t)^{k-i+1}+\Gamma(k-i+2)}{(\lambda k+\lambda-t)^{k-i+2}}\right)$$

As

$$E(X_{(1)}) = \sum_{j=0}^{n-1}\sum_{k=0}^\infty\sum_{i=0}^k\binom{n-1}{j}\binom{\alpha j+\alpha-1}{k}\binom{k}{i}(-1)^{j+k}\frac{n\alpha\lambda^{k-i+2}}{(\lambda+1)^{k-i+1}}$$

$$\times\left[\frac{\Gamma(m+k-i+1)}{(\lambda k+\lambda)^{m+k-i+1}}+\frac{\Gamma(m+k-i+2)}{(\lambda k+\lambda)^{m+k-i+2}}\right]$$

the mth moment of proposed distribution about origin is

$$
\begin{aligned}
\mathrm{E}\left(X^{m}\right) &= \sum_{n=1}^{\infty} \mathrm{P}(N=n) \int_{0}^{\infty} x^{m} \, \mathrm{g}_{1}(x) dx \\
&= \sum_{n=1}^{\infty} \mathrm{P}(N=n) \mathrm{E}\left(X_{(1)}\right) \\
&= \sum_{n=1}^{\infty} \sum_{j=0}^{n-1} \sum_{k=0}^{\infty} \sum_{i=0}^{k} \frac{a_{n} \theta^{n}}{\mathrm{C}(\theta)} \binom{n-1}{j} \binom{\alpha j + \alpha - 1}{k} \binom{k}{i} (-1)^{j+k} \frac{n \alpha \lambda^{k-i+2}}{(\lambda+1)^{k-i+1}} \\
&\quad \times \left[\frac{\Gamma(m+k-i+1)}{(\lambda k + \lambda)^{m+k-i+1}} + \frac{\Gamma(m+k-i+2)}{(\lambda k + \lambda)^{m+k-i+2}} \right]
\end{aligned}
\tag{6}
$$

Order Statistics and Their Moments

Let $X_1, X_2, \ldots, X_n$ be a random sample from the GLPS distribution, and let $X_{1:n} < X_{2:n} < \ldots < X_{n:n}$ denote the corresponding order statistics. The pdf of the ith order statistic $X_{i:n}$ is given by

$$
\mathrm{f}_{i:n}(x) = \frac{n! \mathrm{f}(x)}{(n-i)!(i-1)!} \left[1 - \frac{\mathrm{C}\left(\left[1 - \left[1 - \left(1 + \frac{\lambda x}{\lambda+1}\right) \mathrm{e}^{-\lambda x} \right]^{\alpha} \right] \theta\right)}{\mathrm{C}(\theta)} \right]^{i-1}
$$

$$
\times \left[\frac{\mathrm{C}\left(\left[1 - \left[1 - \left(1 + \frac{\lambda x}{\lambda+1}\right) \mathrm{e}^{-\lambda x} \right]^{\alpha} \right] \theta\right)}{\mathrm{C}(\theta)} \right]^{n-i}
\tag{7}
$$

From

$$
\mathrm{f}(x)\left[\mathrm{F}(x)\right]^{k+i-1} = \left(\frac{1}{k+i}\right) \frac{d}{dx}\left[\mathrm{F}(x)\right]^{k+i}
$$

the associated CDF is

$$F_{i:n}(x) = \frac{n!}{(n-i)!(i-1)!} \sum_{k=0}^{n-i} \frac{\binom{n-i}{k}(-1)^k}{(k+i)}$$
$$\times \left[1 - \frac{C\left(\left[1-\left[1-\left(1+\frac{\lambda x}{\lambda+1}\right)e^{-\lambda x}\right]^\alpha\right]\theta\right)}{C(\theta)} \right]^{k+i} \quad (8)$$

Alternatively equation (8) can be written as

$$F_{i:n}(x) = 1 - \frac{n!}{(n-i)!(i-1)!} \sum_{k=0}^{i-1} \frac{\binom{i-1}{k}(-1)^k}{(k+i)}$$
$$\times \left[\frac{C\left(\left[1-\left[1-\left(1+\frac{\lambda x}{\lambda+1}\right)e^{-\lambda x}\right]^\alpha\right]\theta\right)}{C(\theta)} \right]^{k+n-i+1}$$

The expression for the rth moment of ith order statistic $X_{i:n}$ with CDF (8) can be obtained, using a well-known result due to Barakat and Abdelkadir (2004), as

$$E(X_{i:n}^r) = r \sum_{k=n-i+1}^{n} \frac{(-1)^{k-n+i-1}}{C(\theta)^k} \binom{k-1}{n-i}\binom{n}{k} \int_0^\infty x^{r-1} C\left(\left[1-\left[1-\left(1+\frac{\lambda x}{\lambda+1}\right)e^{-\lambda x}\right]^\alpha\right]\theta\right)^k dx$$

where $r = 1, 2, 3,\ldots$ and $i = 1, 2,\ldots, n$.

Parameter Estimation

Let $X_1,\ldots, X_N$ be a random sample with observed values $x_1,\ldots, x_n$ from a GLPS distribution, and let $\boldsymbol{\Theta} = (\alpha, \lambda, \theta)^{\mathrm{T}}$ be the unknown parameter vector. The log-likelihood function is given by

$$l_n = l_n(x, \boldsymbol{\Theta})$$

$$= n\log\alpha + 2n\log\lambda + n\log\theta - n\log(\lambda+1) + \sum_{i=1}^{n}\log(1+x_i)$$

$$- n\log\mathrm{C}(\theta) - \lambda\sum_{i=1}^{n}x_i + (\alpha-1)\sum_{i=1}^{n}\log\left[1-\left(1+\frac{\lambda x_i}{\lambda+1}\right)\mathrm{e}^{-\lambda x_i}\right]$$

$$+ \sum_{i=1}^{n}\log\mathrm{C}'\left(\left[1-\left[1-\left(1+\frac{\lambda x_i}{\lambda+1}\right)\mathrm{e}^{-\lambda x_i}\right]^{\alpha}\right]\theta\right)$$

The corresponding score functions are

$$\frac{\partial l_n}{\partial \theta} = \frac{n}{\theta} - \frac{n\mathrm{C}'(\theta)}{\mathrm{C}(\theta)} + \sum_{i=1}^{n}\frac{\mathrm{C}''\left(\theta\mathrm{e}^{-\lambda x}\left(1+\frac{\lambda x}{\lambda+1}\right)\right)}{\mathrm{C}'\left(\theta\mathrm{e}^{-\lambda x}\left(1+\frac{\lambda x}{\lambda+1}\right)\right)}\left[1-\left(1-\left(1+\frac{\lambda x}{\lambda+1}\right)\mathrm{e}^{-\lambda x}\right)^{\alpha}\right]$$

$$\frac{\partial l_n}{\partial \alpha} = \frac{n}{\alpha} - \alpha\theta\sum_{i=1}^{n}\frac{\mathrm{C}''\left(\left[1-\left[1-\left(1+\frac{\lambda x}{\lambda+1}\right)\mathrm{e}^{-\lambda x}\right]^{\alpha}\right]\theta\right)}{\mathrm{C}'\left(\left[1-\left[1-\left(1+\frac{\lambda x}{\lambda+1}\right)\mathrm{e}^{-\lambda x}\right]^{\alpha}\right]\theta\right)}\left(1-\left(1+\frac{\lambda x}{\lambda+1}\right)\mathrm{e}^{-\lambda x}\right)^{\alpha}$$

$$\times\log\left(1-\left(1+\frac{\lambda x}{\lambda+1}\right)\mathrm{e}^{-\lambda x}\right) + \sum_{i=1}^{n}\log\left(1-\left(1+\frac{\lambda x}{\lambda+1}\right)\mathrm{e}^{-\lambda x}\right)$$

$$\frac{\partial l_n}{\partial \lambda} = \frac{2n}{\lambda} - \sum_{i=1}^{n} x_i - \frac{n}{\lambda+1} + \frac{(\alpha-1)}{(\lambda+1)^2} \sum_{i=1}^{n} \left(\frac{\left(\lambda(\lambda+1)^2 + \lambda^2 x(\lambda+1) - x\right)e^{-\lambda x}}{1 - \left(1 + \frac{\lambda x}{\lambda+1}\right)e^{-\lambda x}} \right)$$

$$+ \frac{\alpha\theta}{(\lambda+1)^2} e^{-\lambda x} \left(x - \lambda(\lambda+1)(\lambda x + \lambda + 1) \right)$$

$$\times \sum_{i=1}^{n} \frac{C''\left(\left[1 - \left[1 - \left(1 + \frac{\lambda x}{\lambda+1}\right)e^{-\lambda x}\right]^{\alpha}\right]\theta\right)}{C'\left(\left[1 - \left[1 - \left(1 + \frac{\lambda x}{\lambda+1}\right)e^{-\lambda x}\right]^{\alpha}\right]\theta\right)} \left(1 - \left(1 + \frac{\lambda x}{\lambda+1}\right)e^{-\lambda x}\right)^{\alpha-1}$$

The maximum likelihood estimate of Θ, say $\hat{\Theta}$, is obtained by solving the non-linear system of equations

$$\Theta = \left(\frac{\partial l_n}{\partial \alpha}, \frac{\partial l_n}{\partial \lambda}, \frac{\partial l_n}{\partial \theta} \right)^{\mathrm{T}} = 0$$

The solution of this non-linear system of equation can be found numerically by using R.

Some Special Sub-Models

Generalized Lindley Poisson Distribution

The Poisson distribution is a special case of the power series distribution for $C(\theta) = e^{\theta} - 1$; therefore the cdf of a compound of the generalized Lindley Poisson (GLP) distribution is obtained by using the same arguments in (2):

$$F(x) = 1 - \frac{e^{\left(1 - \left[1 - \left(1 + \frac{\lambda x}{\lambda+1}\right)e^{-\lambda x}\right]^{\alpha}\right)\theta} - 1}{e^{\theta} - 1}, \quad x > 0$$

The associated pdf, hazard, and survival functions are given by, respectively,

$$f(x) = \frac{\alpha\lambda^2\theta e^{-\lambda x}}{(\lambda+1)e^{\theta-1}}\left[1-\left(1+\frac{\lambda x}{\lambda+1}\right)e^{-\lambda x}\right]^{\alpha-1}(1+x)e^{\left(1-\left[1-\left(1+\frac{\lambda x}{\lambda+1}\right)e^{-\lambda x}\right]^{\alpha}\right)\theta}$$

$$h(x) = \frac{\alpha\lambda^2\theta e^{-\lambda x}\left[1-\left(1+\frac{\lambda x}{\lambda+1}\right)e^{-\lambda x}\right]^{\alpha-1}(1+x)e^{\left(1-\left[1-\left(1+\frac{\lambda x}{\lambda+1}\right)e^{-\lambda x}\right]^{\alpha}\right)\theta}}{(\lambda+1)\left(e^{\left(1-\left[1-\left(1+\frac{\lambda x}{\lambda+1}\right)e^{-\lambda x}\right]^{\alpha}\right)\theta}-1\right)}$$

$$S(x) = \frac{e^{\left(1-\left[1-\left(1+\frac{\lambda x}{\lambda+1}\right)e^{-\lambda x}\right]^{\alpha}\right)\theta}-1}{e^{\theta}-1}$$

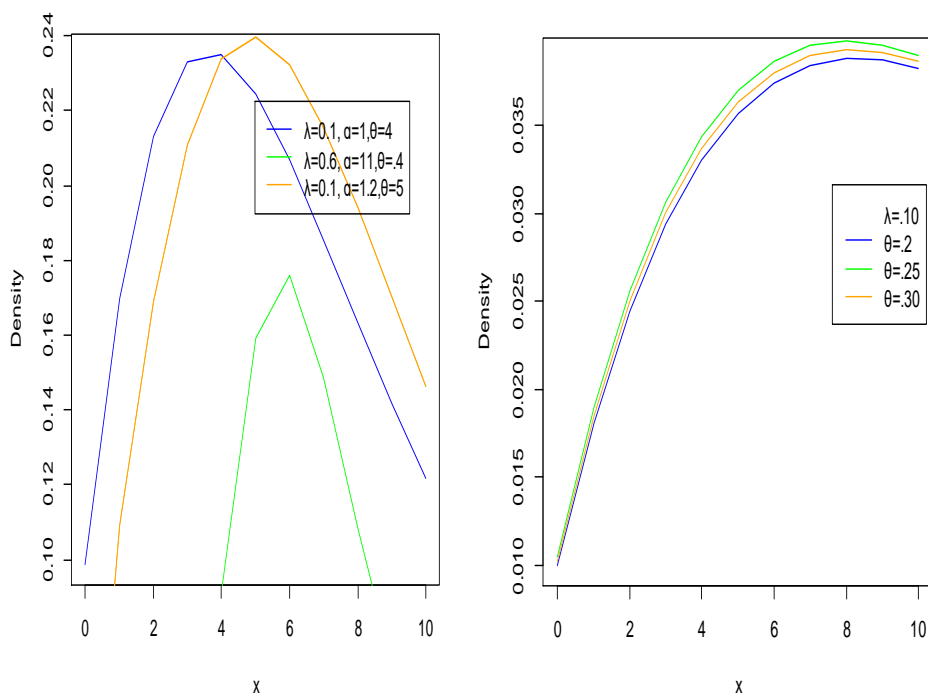**Figure 1.** The flexibility of density of the GLP and LP distributions for some selected values of parameters
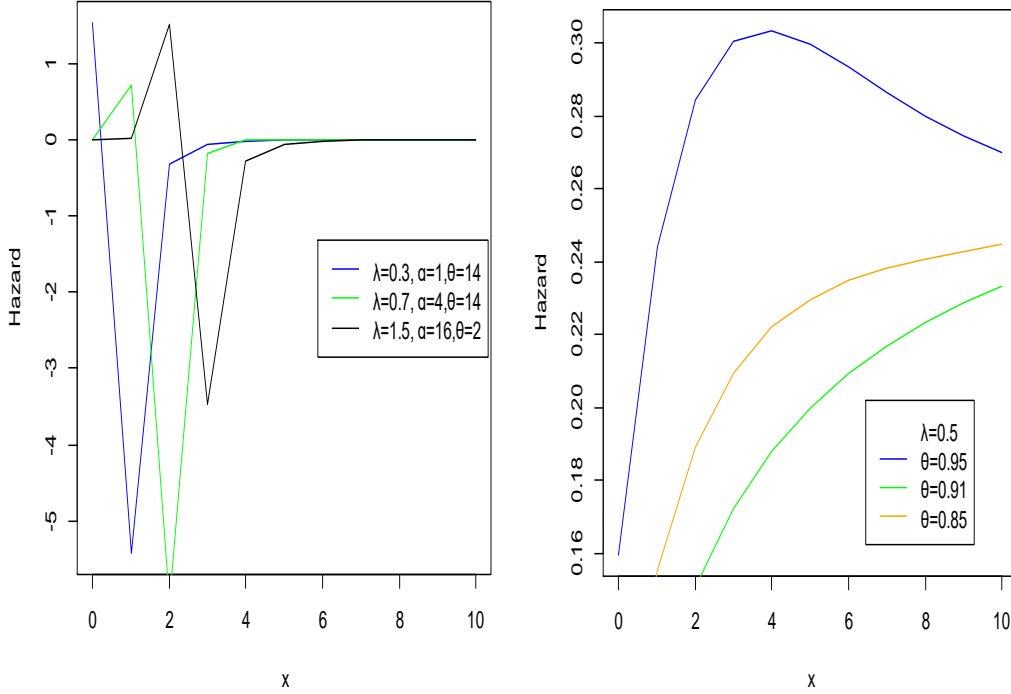
Figure 2. The flexibility of the hazard rate functions of the GLP and LP distributions for some selected values of parameters

for x, α, $\lambda > 0$ and $0 < \theta < \infty$, respectively. The GLP distribution is very flexible in terms of its density and hazard rate function, which is also corroborated by Figure 1 and Figure 2. The expression for the mth moment of a random variable following the generalized Lindley Poisson distribution becomes, by taking $a_n = n!^{-1}$ and $C(\theta) = e^{\theta} - 1$ in (6),

$$\mathrm{E}\left(X^m\right) = \left(e^{\theta} - 1\right)^{-1} \sum_{n=1}^{\infty} \sum_{j=0}^{n-1} \sum_{k=0}^{\infty} \sum_{i=0}^{k} \frac{\theta^n}{n!} \binom{n-1}{j} \binom{\alpha j + \alpha - 1}{k} \binom{k}{i} (-1)^{j+k} \frac{n\alpha\lambda^{k-i+2}}{(\lambda+1)^{k-i+1}}$$
$$\times \left[\frac{\Gamma(m+k-i+1)}{(\lambda k + \lambda)^{m+k-i+1}} + \frac{\Gamma(m+k-i+2)}{(\lambda k + \lambda)^{m+k-i+2}} \right]$$

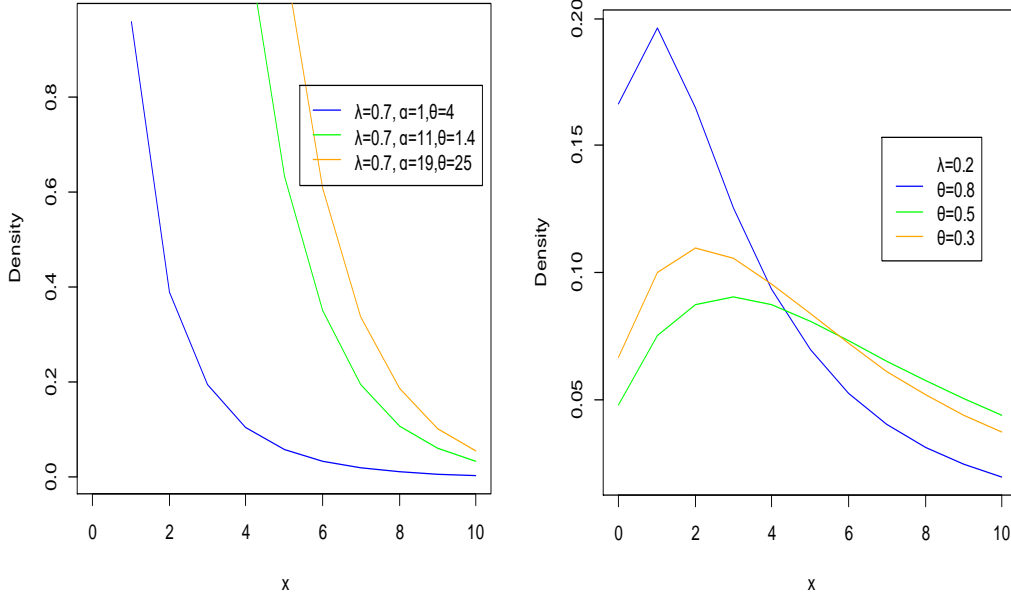for α, $\lambda > 0$ and $0 < \theta < \infty$.

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Figure 3. The flexibility of the density function of the GLL and LL distributions for some selected values of parameters

Generalized Lindley Logarithmic Distribution

The logarithmic distribution is a special case of the PSD when $C(\theta) = -\log(1 - \theta)$, and a compound of the generalized Lindley logarithmic (GLL) distribution follows from (2) from the above $C(\theta)$:

$$F(x) = 1 - \frac{\log\left(\left(1 - \left[1 - \left(1 + \frac{\lambda x}{\lambda + 1}\right)e^{-\lambda x}\right]^{\alpha}\right)\theta\right)}{\log(1 - \theta)}$$

The associated pdf, hazard rate, and survival functions are

$$f(x) = \frac{\alpha \lambda^2 \theta e^{-\lambda x} \left[1 - \left(1 + \frac{\lambda x}{\lambda + 1}\right) e^{-\lambda x}\right]^{\alpha - 1} (1 + x)}{(\lambda + 1)\left(\left[1 - \left[1 - \left(1 + \frac{\lambda x}{\lambda + 1}\right) e^{-\lambda x}\right]^{\alpha}\right] \theta - 1\right) \log(1 - \theta)}$$

$$h(x) = \frac{\alpha \lambda^2 \theta e^{-\lambda x} \left[1 - \left(1 + \frac{\lambda x}{\lambda + 1}\right) e^{-\lambda x}\right]^{\alpha - 1} (1 + x) \left(\left[1 - \left[1 - \left(1 + \frac{\lambda x}{\lambda + 1}\right) e^{-\lambda x}\right]^{\alpha}\right] \theta - 1\right)^{-1}}{(\lambda + 1) \log\left(1 - \left[1 - \left[1 - \left(1 + \frac{\lambda x}{\lambda + 1}\right) e^{-\lambda x}\right]^{\alpha}\right] \theta\right)}$$

$$S(x) = \frac{\log\left(1 - \left[1 - \left[1 - \left(1 + \frac{\lambda x}{\lambda + 1}\right) e^{-\lambda x}\right]^{\alpha}\right] \theta\right)}{\log(1 - \theta)}$$

The density and hazard rate functions of the GLL distribution take different shapes for different values of parameters, which is evident from Figures 3 and 4. The expression for the mth moment of a random variable following a generalized Lindley logarithmic distribution becomes, by taking $a_n = n^{-1}$ and $C(\theta) = -\log(1 - \theta)$ in (6),

$$E(X^m) = \frac{1}{\log(1 - \theta)}^{-1} \sum_{n=1}^{\infty} \sum_{j=0}^{n-1} \sum_{k=0}^{\infty} \sum_{i=0}^{k} \theta^n \binom{n-1}{j} \binom{\alpha j + \alpha - 1}{k} \binom{k}{i} (-1)^{j+k} \frac{\alpha \lambda^{k-i+2}}{(\lambda + 1)^{k-i+1}}$$

$$\times \left[\frac{\Gamma(m+k-i+1)}{(\lambda k + \lambda)^{m+k-i+1}} + \frac{\Gamma(m+k-i+2)}{(\lambda k + \lambda)^{m+k-i+2}}\right]$$
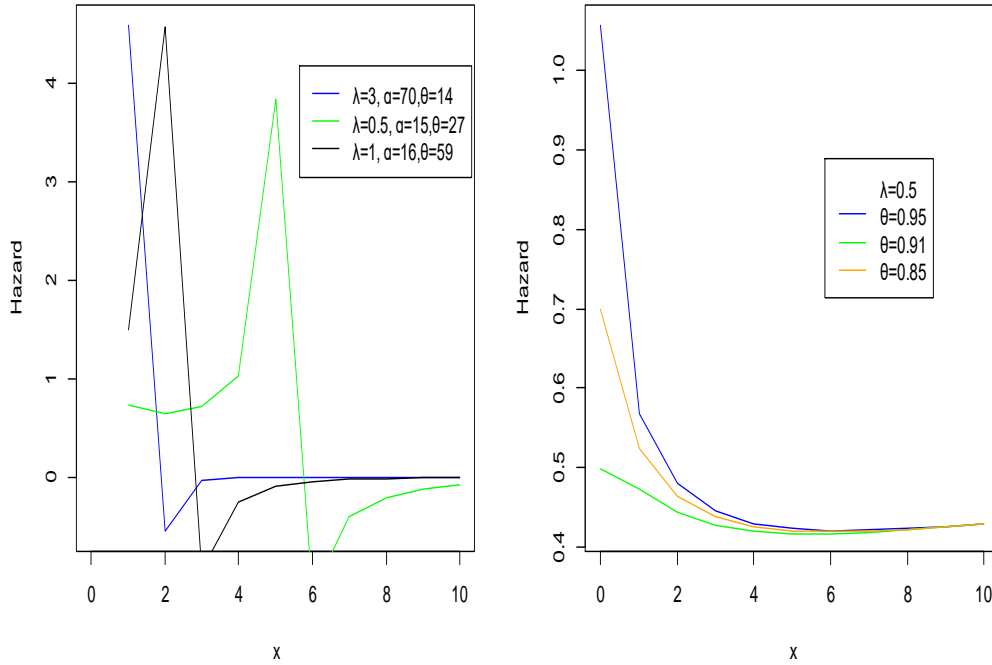
for $\alpha, \lambda > 0$ and $0 < \theta < 1$.

Figure 4. The flexibility of the hazard rate function of the GLL and LL distributions for some selected values of parameters

Generalized Lindley Geometric Distribution

The geometric distribution is a particular case of the PSD when $C(\theta) = \theta(1-\theta)^{-1}$, and a compound of the generalized Lindley geometric distribution followed from (2):

$$F(x) = 1 - (1-\theta)\frac{\left(1 - \left[1 - \left(1 + \frac{\lambda x}{\lambda+1}\right)e^{-\lambda x}\right]^{\alpha}\right)}{\left(1 - \left[1 - \left[1 - \left(1 + \frac{\lambda x}{\lambda+1}\right)e^{-\lambda x}\right]^{\alpha}\right]\theta\right)}, \quad x > 0$$

The associated pdf, hazard rate, and survival functions are

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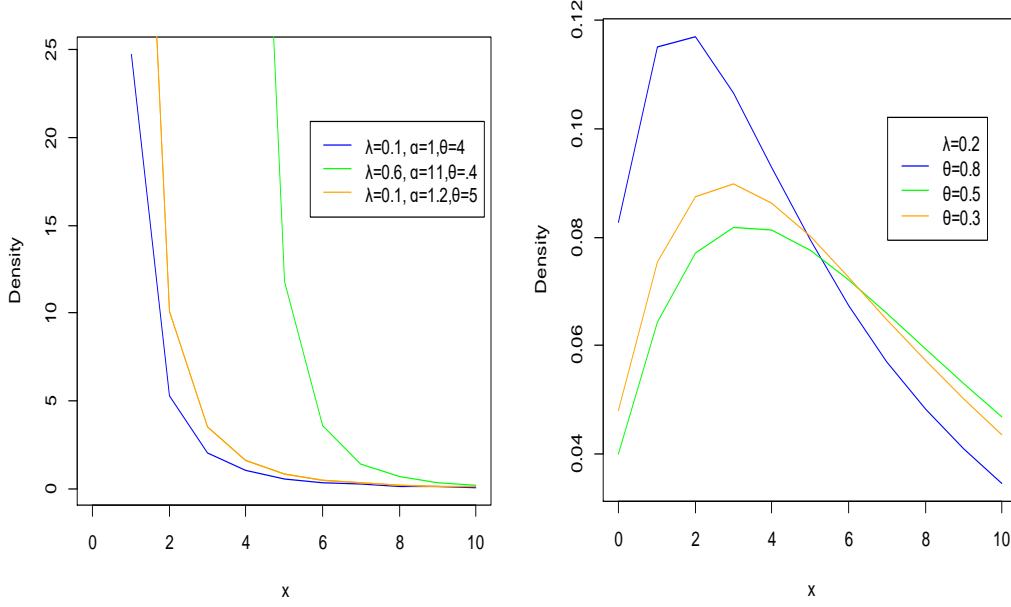**Figure 5.** The flexibility of the density of the GLG and LG distributions for some selected values of parameters

$$f(x) = \frac{\alpha\lambda^2(1-\theta)(1+x)e^{-\lambda x}\left[1-\left(1+\frac{\lambda x}{\lambda+1}\right)e^{-\lambda x}\right]^{\alpha-1}}{(\lambda+1)\left(\left[1-\left[1-\left(1+\frac{\lambda x}{\lambda+1}\right)e^{-\lambda x}\right]^{\alpha}\right]\theta\right)^2}$$

$$h(x) = \frac{\alpha\lambda^2(1+x)e^{-\lambda x}\left[1-\left(1+\frac{\lambda x}{\lambda+1}\right)e^{-\lambda x}\right]^{\alpha-1}}{(\lambda+1)\left(1-\left[1-\left(1+\frac{\lambda x}{\lambda+1}\right)e^{-\lambda x}\right]^{\alpha}\right)\left(1-\left[1-\left[1-\left(1+\frac{\lambda x}{\lambda+1}\right)e^{-\lambda x}\right]^{\alpha}\right]\theta\right)} \quad (9)$$

$$S(x) = \frac{(1-\theta)\left(1-\left[1-\left(1+\frac{\lambda x}{\lambda+1}\right)e^{-\lambda x}\right]^{\alpha}\right)}{\left(1-\left[1-\left[1-\left(1+\frac{\lambda x}{\lambda+1}\right)e^{-\lambda x}\right]^{\alpha}\right]\theta\right)}$$
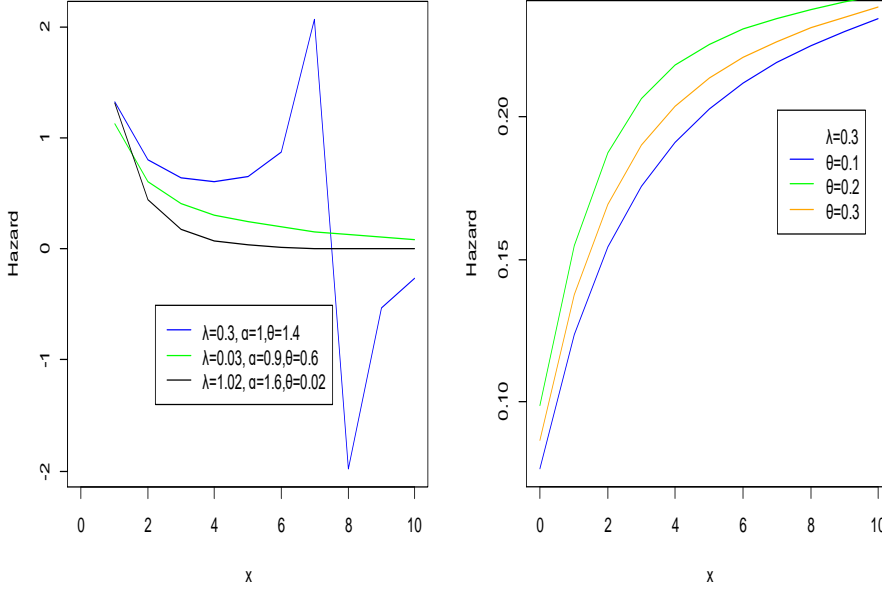
Figure 6. The flexibility of the hazard rate function of the GLG and LG distributions for some selected values of parameters

for x, α, $\lambda > 0$ and $0 < \theta < 1$, respectively. Figure 5 and Figure 6 show that the density and hazard rate functions of the GLG distribution take different shapes for different values of parameters. The expression for the mth moment of a random variable following a generalized Lindley geometric (GLG) distribution becomes, by taking $a_n = 1$ and $C(\theta) = \theta(1 - \theta)^{-1}$ in (6),

$$E\left(X^m\right) = \sum_{n=1}^{\infty}\sum_{j=0}^{n-1}\sum_{k=0}^{\infty}\sum_{i=0}^{k}\frac{\theta^{n-1}}{(1-\theta)}\binom{n-1}{j}\binom{\alpha j + \alpha - 1}{k}\binom{k}{i}(-1)^{j+k}\frac{n\alpha\lambda^{k-i+2}}{(\lambda+1)^{k-i+1}}$$
$$\times\left[\frac{\Gamma\left(m+k-i+1\right)}{\left(\lambda k+\lambda\right)^{m+k-i+1}}+\frac{\Gamma\left(m+k-i+2\right)}{\left(\lambda k+\lambda\right)^{m+k-i+2}}\right]$$

for α, $\lambda > 0$ and $0 < \theta < 1$.
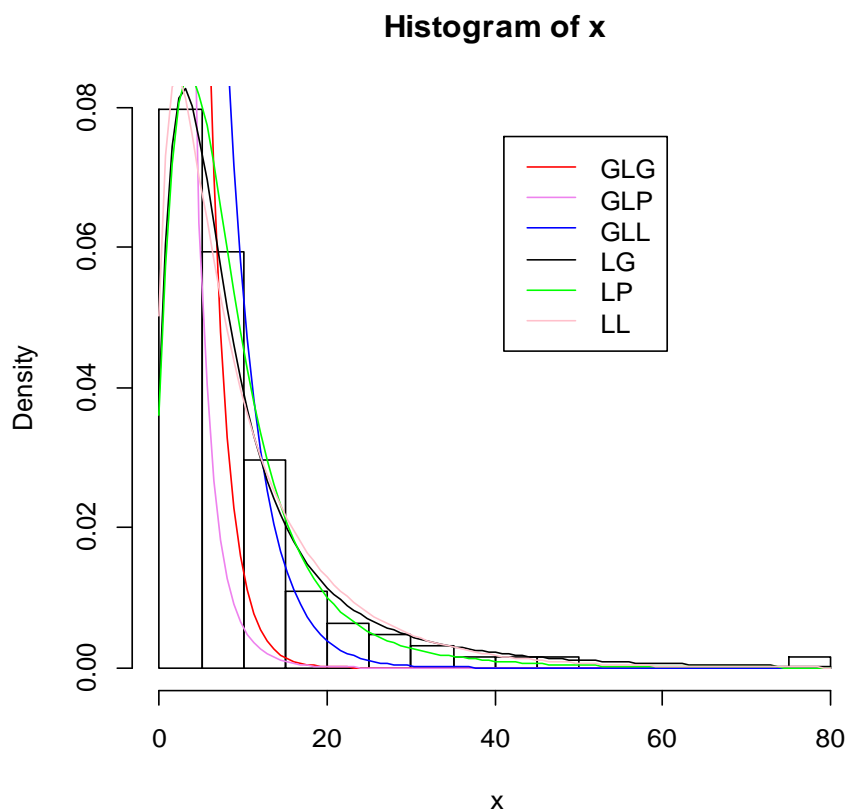
Histogram of x

Figure 7. Fitting of GLP, GLG, GLL, LG, LP, and LL distributions to the cancer data

Generalized Lindley Binomial Distribution

The binomial distribution is a particular case of the PSD for $C(\theta) = (\theta + 1)^m + 1$, with m a positive integer, and a compound of generalized Lindley binomial (GLB) distribution follows from (2):

$$F(x) = 1 - \frac{\left(\left(1 - \left[1 - \left(1 + \frac{\lambda x}{\lambda + 1}\right)e^{-\lambda x}\right]^{\alpha}\right)\theta + 1\right)^m - 1}{(\theta + 1)^m - 1}, \quad x > 0$$

The associated pdf, hazard rate, and survival functions are

$$f(x) = \frac{m\alpha\lambda^2\theta e^{-\lambda x}(1+x)}{(\lambda+1)((\theta+1)^m-1)}\left[1-\left(1+\frac{\lambda x}{\lambda+1}\right)e^{-\lambda x}\right]^{\alpha-1}$$

$$\times\left(\left(1-\left[1-\left(1+\frac{\lambda x}{\lambda+1}\right)e^{-\lambda x}\right]^\alpha\right)\theta+1\right)^{m-1}$$

$$h(x) = m\alpha\lambda^2\theta(1+x)e^{-\lambda x}\left[1-\left(1+\frac{\lambda x}{\lambda+1}\right)e^{-\lambda x}\right]^{\alpha-1}$$

$$\times\left(\left(1-\left[1-\left(1+\frac{\lambda x}{\lambda+1}\right)e^{-\lambda x}\right]^\alpha\right)\theta+1\right)^{m-1}$$

$$\times\left[(\lambda+1)\left(\left(\left(1-\left[1-\left(1+\frac{\lambda x}{\lambda+1}\right)e^{-\lambda x}\right]^\alpha\right)\theta+1\right)^m-1\right)\right]^{-1}$$

$$S(x) = \frac{\left(\left(1-\left[1-\left(1+\frac{\lambda x}{\lambda+1}\right)e^{-\lambda x}\right]^\alpha\right)\theta+1\right)^m-1}{(\theta+1)^m-1}$$

respectively, for x, λ, $\alpha > 0$ and $0 < \theta < \infty$. The expression for the rth moment of a random variable following generalized Lindley binomial distribution becomes, by taking $a_n = \binom{m}{n}$ and $C(\theta) = (\theta+1)^m + 1$ in (6),

$$E(X^r) = \sum_{n=1}^{\infty}\sum_{j=0}^{n-1}\sum_{k=0}^{\infty}\sum_{i=0}^{k}\binom{m}{n}\theta^n\binom{n-1}{j}\binom{\alpha j+\alpha-1}{k}\binom{k}{i}(-1)^{j+k}\frac{n\alpha\lambda^{k-i+2}}{(\lambda+1)^{k-i+1}}$$

$$\times\left[\frac{\Gamma(m+k-i+1)}{(\lambda k+\lambda)^{m+k-i+1}}+\frac{\Gamma(m+k-i+2)}{(\lambda k+\lambda)^{m+k-i+2}}\right]$$

for $n \le m$, α, $\lambda > 0$, and $0 < \theta < \infty$. The new family of LPS distributions due to Adil and Jan (2016) is obtained by taking $\alpha = 1$ in (1). Sub-models GLP, GLG, GLL, and GLB are new compound distributions. The graphical representation of the density function and hazard rate function of all the sub-models of the proposed family have also been presented.

Table 2. Parameter estimates, log likelihood, and Akaike Information Criterion (AIC)

Model	Maximum Likelihood Estimates	Log-likelihood	AIC
GLG	$\hat{\lambda} = 0.06, \hat{\theta} = 0.92, \hat{\alpha} = 1.07$	-409.31	824.62
GLP	$\hat{\lambda} = 0.08, \hat{\theta} = 3.65, \hat{\alpha} = 0.86$	-410.42	826.84
GLL	$\hat{\lambda} = 0.11, \hat{\theta} = 0.96, \hat{\alpha} = 1.22$	-411.20	828.41
LG	$\hat{\lambda} = 0.07, \hat{\theta} = 0.88$	-409.59	823.18
LP	$\hat{\lambda} = 0.11, \hat{\theta} = 3.17$	-411.38	826.76
LL	$\hat{\lambda} = 0.12, \hat{\theta} = 0.90$	-411.77	827.54

Application

The applicability of the proposed model will be explored further. A comparison among its sub-classes will also be undertaken using a real life data set based on the remission times (in months) of a random sample of 128 bladder cancer patients as reported in Lee and Wang (2003). The parameter estimates, log likelihood, and Akaike Information Criterion (AIC) are shown in Table 2.

Conclusion

A new lifetime distribution was proposed for series setting that not only generalizes the Lindley power series family of distribution, but has a flexible density function; more importantly, its hazard function can take up different shapes such as bathtub, upside down bathtub, increasing, and decreasing shapes. The potential of the proposed family has been shown by fitting it to the real life data set, and it is clear from the statistical analysis that the proposed family offers a better fit.

Acknowledgements

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