1-1-2014

Can Animal Spirits Solve The Forward Premium Puzzle?

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CAN ANIMAL SPIRITS SOLVE THE FORWARD PREMIUM PUZZLE?

by

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DISSERTATION

Submitted to the Graduate School

of Wayne State University,

Detroit, Michigan

in partial fulfillment of the requirements

for the degree of

DOCTOR OF PHILOSOPHY

2014

MAJOR: ECONOMICS

Approved by:

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Advisor Date

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DEDICATION

To my wife, Victoria.

ACKNOWLEDGMENTS

I would like to acknowledge and thank the following individuals that led to the successful completion of this dissertation,

- My mother and father for their belief in me from the beginning and ability to turn "I want to quit" into this dissertation.
- Dr. Addington Coppin for guiding my undergraduate experience in the directon of Economics.
- Dr. Allen Goodman for providing the opportunity, guidance, and mentorship throughout my graduate excursion. It is an honor to close the door you opened for me, over four years ago.
- Dr. Tatsuma Wada for advisorship, direction, rigorous discipline in the art of Macroeconomics, and for pushing me to go the extra mile.
- Dr. Robert J. Rossana, Dr. Kevin Cotter, Dr. Li Way Lee, Dr.Nivedita Mukherji, Dr. Anandi Sahu, and Dr. Ronald Tracy for their guidance and impact on my experience in economics.
- My cousins Mark Shkreli, Nicholas Sinishtaj, Joe Gojcaj, Agron Shkreli, Robert Shkreli, Nick Shkreli, Agron Martini, Alex Elezaj, and Tom Shkreli for their family support and optimistic conversations.

- My in-laws for their generous support and encouragement.

- Last, but most certainly not least, my wife Victoria. Your endurance and commitment to this dissertation, through its high's and low's, has shown me the depths of your love, patience, and spirit. This is for us and those which will come from us.

I sincerely appreciate each of you.

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CHAPTER 1 AN OVERVIEW OF THE FORWARD PREMIUM PUZZLE, ANIMAL SPIRITS, AND EVERYTHING IN BETWEEN

1 Introduction

There lies a long tradition in analyzing the relationship between two country's spot and forward exchange rates. Early empirical works such as Hansen and Hodrick (1980) and Bilson (1981) utilize the forward exchange rate as a predictor for future spot exchange rates, establishing the consensus that the former is a poor indicator of future movements of the latter. Other early important works like Frankel (1982), Hsieh (1982), Hodrick and Srivastave (1984) focus on the question of whether forward exchange rates explained variation contained within premiums. However, it wasn't until Fama (1984) testing the variational relationship between the forward exchange rate premium and expected future spot exchange rate elements of forward exchange rates that the conanical form of the field began to take shape. In his paper, Fama discovers that conditional on the hypothesis that markets are rational and efficient most of the variation in forward exchange rates is related to variation in premiums, and more importantly, that the premium and expected future spot exchange rates are negatively correlated. It is from this last finding that the term "forward premium puzzle" was introduced and, according to Obstfeld and Rogoff (2000), has established itself as one of the great international macroeconomic anomalies.

A more formal interpretation of this anomaly begins with a simple empirical framework of the forward exchange rate and its relationship with the spot exchange rate or,

$$S_{t+1} = \beta_0 + \beta_1 F_t + \varepsilon_{t+1} \tag{1}$$

where in (1) S_{t+1} is the expected future spot exchange rate, $F_t = ln(f_t)$ is the log of the forward exchange rate, and ε_{t+1} is an i.i.d, normally distributed error term. The fundamental null hypothesis is to test $\beta_1 = 1$ where β_0 can be equated to any constant. Since it is assumed that $\beta_1 = 1$, the log of the current spot exchange rate, or $S_t = \ln(s_t)$, is subtracted from both sides of (1) yielding,

$$S_{t+1} - S_t = \beta_0 + \beta_1 (F_t - S_t) + \varepsilon_{t+1} \tag{2}$$

where (2) describes $F_t - S_t$ as the forward premium which explains variation in the gross rate of depreciation (*ex-post)* or $S_{t+1} - S_t$. Equation (2) is the form essentially estimated by Fama (1984) where not only is $\beta_1$different than unity, it is found to be *negative*, hence the advent of the forward premium puzzle as seen through lense of this short formal walk-through. The anomaly is connected to a failure in the uncovered interest rate parity (UIP) by first establishing covered interest rate parity or,

$$(F_t - S_t) = (R_t - R_t^*) \tag{3}$$

so that R_t is the return yielded to investors from domestic bonds and R_t^* is the return yielded to investors from foreign bonds so that excess returns from $(R_t - R_t^*)$ is in equilibrium with the forward exchange rate premium $(F_t - S_t)$, which is the definition of covered interest parity. Now, inserting (3) into (2) for $(F_t - S_t)$ results in,

$$S_{t+1} - S_t = \beta_0 + \beta_1 (R_t - R_t^*) + \varepsilon_{t+1} \tag{4}$$

hence (4) establishes the definition of UIP, which states excess returns are in equilibrium with the change in gross rate of depreciation, or $S_{t+1} - S_t = (R_t - R_t^*)$ which occurs if $\beta_0 = 0$ and $\beta_1 = 1$. Of course, (4) cannot be directly estimated and the closest proxy is (2), however using the discovery of $\beta_1 < 0$ from (2) and applying it to (4) results in $\beta_1 \neq 1$so that $S_{t+1} - S_t \neq (R_t - R_t^*)$ which is a violation in the definition of, and failure in, the UIP. In a survey by Froot and Thaler (1990), the average estimate of $\beta_1$over 75 publications is -0.88, and as stated by Engle (1996), "only a few of the estimates is greater than 0, and none is greater than 1 (p.125)." The breakdown in UIP, despite the validity of fluid markets under covered interest rate parity, could occur for a variety of reasons but the predominant scenarios as stated by Chinn (2007) are 1. Rational Expectations is an invalid assumption; 2. Risk Premium exists as an "unseen" component embedded within the forward premium estimator, β_1; and 3. Econometric Implementation.

From this point, the chapter is broken into sections that focus on major areas of research contributing to the solution of the forward premium puzzle namely Section 2 discusses works pertaining to the weakness of expectation formation within Rational Expectations, Section 3 encompasses articles focused on Risk Premium as a potential solution to the anomaly, Section 4 argues multiple econometric techniques that could be causing the forward premium puzzle, Section 5 introduces literature pertaining to Robust Control and its implementation within

Rational Expectations framework, resulting in "Animal Spirits" as being a candidate to solving the forward premium puzzle, and Section 6 concludes.

2 Rational Expectations

The hallmark of macroeconomic modeling, originally proposed by Muth (1961), Rational Expectations (RE) essentially has economic agents form expectations about future macroeconomic values by utilizing all information within a given time period, so that systematic bias is theoretically non-existent and all errors are strictly random. This feature of expectation formation has led to RE monumental success but also to its most significant weakness, which is especially prevalent within the context of forward premium puzzle literature. Engle (1996) and more recently Sarno (2005) both give an exhaustive survey of the state of forward premium literature within the context of RE (as well as within the realm of risk premium literature), however here important contributions are highlighted and focus on solving the anomaly by exploiting weaknesses within RE and develop alternative structures to households forming expectations.

Early notable works include Froot and Frankel (1989) which decompose a RE model into two sets of agents where the first set formulates expectations based on time-series data and the second set based on survey-data. The authors find that measures of $(S_{t+1} - S_t)$ differ greatly under each data set so that a survey-based data regime contributes to a greater forward premium bias than the time-series-based data regime suggesting the types of information agents utilize to formulate expectations causes the anomaly as opposed to the mechanic of expectation formulation itself. An alternative annex to the definition of RE described above is that expectations formulated by economic agents are correct over time on average, with no systematic bias so that agents are essentially learning from random perturbations, revising future forecasts of macroeconomic variables. In other words, as stated by Chinn (2007), it may not be that households are simply irrational, on the contrary economic agents are constantly learning so that forecasts may be embedded with long-run biases which contribute to the forward premium anomaly. Cheung (1993) incorporate a learning feature within economic agents through Kalman filtering to account for bias within forecasts. Cheung uses data in monthly frequency from July 1973 to December 1987 for the pound, the mark, and the yen. In his work, $(F_t - S_t)$ follows a low order ARMA process and ultimately exhibits a large amount of persistence, is negatively correlated with $(S_t - E_{t-1}(S_t))$, and very volatile emulating key features of the data.

A major weakness in RE is that agents are believed to have full information processing capabilities. In reality, this is hardly the case as there are a wide number of media outlets and information sources that the modern-day consumer cannot fully take into account when making economic decisions. To bring models closer to reality, Sims (2003) introduced Rational Inattention (RI) within macroeconomic modeling framework via Linear-Quadratic Control (LQC) techniques. Essentially, RI imposes a cost to the consumer for processing data so that a limited amount of information through a "channel-capacity" (measured in "bytes per time unit" or "bpt"). The wider the channel-capacity, the more households are able to process information and vice versa. Applied to the forward premium puzzle framework, although explained without formal models, Froot and Thaler (1990) and Lyons (2001) describe the ability of the anomaly to be solved by explaining economic agents' sluggishness in responding to new information. Bacchetta and Wincoop (2005) formally apply RI to a Dynamic Stochastic General Equilibrium (DSGE) model and discover that agents with low costs to processing information trade assets more actively, while other agents do no trade actively so only a small number of investors are attentive to new information. By imposing this restriction the authors, among many discoveries, find that inattention can account for most of the observed predictability of excess returns in the foreign exchange market, directly leading to a solution in the forward premium puzzle[1].

In macroeconomic models built on RE hypothesis, economic agents operate with no systematic bias, meaning forecasts about future macroeconomic variables are made with no systematic error. To state that households have a deep enough understanding about the evolutionary process of the economy is farfetched, prompting Gourchinas and Tornell (2004) to incorprate systematic distortions in investors' beliefs about interest rate processes, rendering economic agents irrational where agents believe shocks to the economic enviornment are excessively transitory contrary to the shocks actual duration. In their paper, using survey data from G-7 countries the author's account for a negative forward premium estimator and exchange rate overshooting, which may or may not occur given their model structure. Another interesting branch of literature related to implementing systematic error within macroeconomic models are works concerning sentiment-based explanations to the forward

[1]See Duffie and Sun (1990), Lynch (1996), Gabaix and Laibson (2003), and Peng and Xiong (2006) for more examples of models incorporating costs to information processing.

premium anomaly.

Yu (2013) describes agents perceiving higher domestic economic growth over foreign growth as having "high sentiment" which contributes to domestic interest rates exceeding foreign interest rates causing misperception about interest rate evolution and thus systematic bias in the decision of purchasing currency as an asset. Sentiment-based literature incorporates a behavioral component within economic modeling as economic agents having high sentiments is equivalent, according to Yu, as being optimistic. Using a 3-month forward exchange rate from 1973-2009 for G-10 countries and Baker and Wurgler's (2006) investor sentiment index from July 1965 to December 2007 as a proxy to sentiment elements mathematically embedded in the model, Yu accounts for the failure in UIP and low correlation between consumption growth differentials and exchange rate changes, a key feature of international market data[2].

As the field concerning manipulations to RE assumption continues to grow, other researchers prefer to maintain this hypothesis and instead focus on risk aversion manifesting as a "risk-premium," embedded within the forward premium estimator.

3 Risk Premium

A large body of literature focuses on embedding risk-averse agents with a risk-premium component as a way of solving the forward premium puzzle. The canonical interpretation of risk premium is that it drives a wedge between actual changes in the spot exchange rate,

[2]See Dumas, Kurshev, and Uppal (2009), and Xiong and Yan (2010) for further works relating to sentiment-based modeling.

$(S_{t+1} - S_t)$, and expected changes in the spot exchange rate, $(E_t(S_{t+1}) - S_t)$[3], where this discrepancy causes a negative relationship with $(F_t - S_t)$. As described in Engle (1996), to model risk premium proves to be complicating but the essential formal components follow from (2) reproduced here for convenience,

$$S_{t+1} - S_t = \beta_0 + \beta_1 (F_t - S_t) + \varepsilon_{t+1}$$

where, as stated above, the null hypothesis to be tested is $\beta_0 = 0$, $\beta_1 = 1$, and $\varepsilon_{t+1} \sim N(0, \sigma^2)$ where σ^2 is a constant variance term. Simplifying and using statistical operators, the estimator $\beta_1$becomes,

$$plim\left(\widehat{\beta_1}\right) = \frac{Cov\,(F_t - S_t, S_{t+1} - S_t)}{Var\,(F_t - S_t)} \tag{5}$$

where $\widehat{\beta}_1$is the estimate of $\beta_1$and is assumed to be consistent. Now, according to Engle, if expectations are rational then actual changes in the gross rate of depreciation are equivalent to expected changes in the gross rate of depreciation systematically speaking, with albeit random errors or more formally,

$$S_{t+1} - S_t = E_t(S_{t+1}) - S_t + \epsilon_{t+1} \tag{6}$$

where E_t is an economic agents' expectation given all information at time t and $\epsilon_{t+1} \sim N(0, \sigma^2)$. Using (6) the covariance term in (5) can be redefined as,

[3]The phrases "changes in spot exchange rate" and "gross rate of depreciation" are synonymous.

$$Cov\left(F_t - S_t, S_{t+1} - S_t\right) = Cov\left(F_t - S_t, E_t\left(S_{t+1}\right) - S_t\right) \tag{7}$$

now, to embed the risk-premium component within the right-hand side of (7) the expected change in spot exchange rates is altered by a simple manipulation,

$$\begin{aligned} E_t\left(S_{t+1}\right) - S_t &= E_t\left(S_{t+1}\right) - S_t + F_t - F_t \\ &= F_t - S_t - F_t + E_t\left(S_{t+1}\right) \\ &= F_t - S_t - rp_t^{re} \end{aligned} \tag{8}$$

where,

$$rp_t^{re} = F_t - E_t\left(S_{t+1}\right) \tag{9}$$

so that (9) is known as the risk-premium under the assumption of RE. The logic behind (9) is that if agents are risk neutral, the forward exchange rate F_t would be driven to equate with the expected future spot exchange rate $E_t\left(S_{t+1}\right)$, basically eliminating the risk-premium so that $rp_t^{re} = 0$. Alternatively, as explained by Engle, if agents are risk-averse and if $F_t > E_t\left(S_{t+1}\right)$ then the investor requires a premium for purchasing foreign currency forward at current time period t relative to the exchange-rates expected spot value at future time period $t+1$ so that $rp_t^{re} > 0$, basically to compensate the investor for purchasing a risky asset. Now, inserting (8) into the right-hand side of (7) results in,

$$
\begin{aligned}
Cov\left(F_t - S_t, E_t\left(S_{t+1}\right) - S_t\right) &= Var\left(F_t - S_t\right) - Cov\left(F_t - S_t, rp_t^{re}\right) \\
&= Var\left(F_t - S_t\right) - Cov\left(E_t\left(S_{t+1}\right) - S_t, rp_t^{re}\right) - Var\left(rp_t^{re}\right)
\end{aligned} \tag{10}
$$

finally, inserting (10) into the numerator of (5) and simplifying results in,

$$
plim\left(\widehat{\beta_1}\right) = 1 - \beta_{rp} \tag{11}
$$

where,

$$
\beta_{rp} = \frac{Cov\left(E_t\left(S_{t+1}\right) - S_t, rp_t^{re}\right) + Var\left(rp_t^{re}\right)}{Var\left(F_t - S_t\right)} \tag{12}
$$

where the bias term β_{rp} from (11) is an indirect function of rp_t^{re}. A common finding in earlier works using this methodology, such as Bilson (1981) and Fama (1984), is that $\widehat{\beta}_1 < 1$ and is not necessarily negative implying that (12) is a small positive when $Var\left(F_t - S_t\right)$ is large. Within the context of risk-premium literature, author's have introduced a variety of methods to ultimately enhance the general form of (11) and emulate the forward premium puzzle inherent within international data.

Campbell and Chochrane (1999) incorporate habit persistence within an investor's utility function and model consumption as an exogenous process that is used to explain a wealth of dynamic asset pricing related to international stock markets. Following Campbell and Chochrane, Verdelhan (2010) develops a model within the realm of RE but induces an external habit preference over consumption in economic agents. The intuition is that assuming Arrow-Debreau markets, the real exchange rate which is measured in terms of domestic goods relative to foreign goods is equivalent to the ratio of foreign to domestic pricing kernels. Thus, fluctuations in the exchange rate are dependent on the stochastic process underlying domestic and foreign consumption growth shocks. Furthermore, Verdelhan explains that if the conditional variance of the domestic stochastic discount factor is large relative to its foreign counterpart, then domestic growth shocks in consumption determine fluctuations in real exchange rates. Intuitively, if the economy experiences a negative consumption growth shock, this triggers an exchange rate depreciation which lowers the domestic investor's returns on purchased assets. Alternatively, if a positive consumption growth shock occurs, this causes appreciation in exchange rates translating as a higher return to domestic investor's. Conclusively, investor's carrying currencies are exposed to consumption growth risks so that the investor requires a risk-premium to compensate for such risk, adding a downward bias to the forward premium estimator[4]. Aside from models embedding habit persistence, another branch of literature popular within risk-premium modeling is incorporating nominal rigidities within RE framework.

The introduction of sticky prices within the context of DSGE models and RE induces risk premia to aid in explaining the forward premium anomaly. Lucas (1982), although no nominal rigidities are incorporated, pioneered risk premium inclusion within a DSGE , two-country, two-money model and showed structurally that risk premium drives a wedge between

[4]See Moore and Roche (2010) for further literature on habit preferences and its relation to forward premium bias.

the forward and spot exchange rates. Although a valiant attempt, the implementation of a risk premium within Lucas' model failed to account for the forward premium bias unless an extremely large parameter value of relative risk aversion is calibrated or the correlation between consumption and exchange rates must be high, as explained by Sarno (2005). In an open economy DSGE model analogous to Adolfson et al. (2007) that shares features of benchmark new Keynesian models set forth by Christiano, Eichenbaum and Evans (2005), Adolfson et al. (2008) incorporate nominal and real frictions that structurally enhances the risk premium used to explain failure in UIP. In their work, Adolfson et al. (2008) alter the structural representation of their open economy model used in Adolfson et al. (2007) and change the UIP condition to allow for the risk premium to be negatively correlated with the expected change in spot exchange rates. This type of modification to the UIP condition introduces a lagged dependence between exchange rates and interest rates in the model, a relationship otherwise absent under standard UIP framework. Using Swedish from 1980 to 2004, Adolfson et al. (2008) show that their log-linearlized system under the modified UIP framework matches data better than its unmodified counterpart, that is to say, given the overall structure of their model, the author's can account for the forward premium anomaly[5].

4 Econometric Implementation

Until now, the primary focus has been on structural literature pertaining to exploitation of weaknesses within and alteration of RE as well as preserving the RE hypothesis and

[5] See Alvarez et al. (2002) for further interesting forms of rigidities within DSGE framwork.

introducing risk premium to explain the forward premium puzzle. Here, attention is shifted to the noteworthy body of literature dealing with empirical issues behind the forward premium anomaly. Most of the before mentioned estimation has been done using simple ordinary least squares (OLS) analysis but such estimations suffer from omitted variable bias as observed by Fama (1984) and Liu and Maddala (1992), performing as a catalyst to the structural discussions above.

An early common belief [see Crowder (1994); Evans and Lewis (1993); and Mark et al. (1993)], with respect to time-series aspects, was that $(F_t - S_t)$ exhibited a non-stationary process leading to inconsistent estimates of β_1. According to Crowder (1994), a standard Augmented Dickey-Fuller (ADF) test is applied to the Canadian, German, and UK to US exchange rates and it is found that insufficient evidence exists to reject the hypothesis of a unit root within the forward premium (although this finding was mixed for the UK to US forward premium). Similarly, Crowder also implements the KPSS[6] test for stationarity and rejects the hypothesis of $I(0)$ (suggesting $I(1)$ behavior) for the list of previously mentioned forward premiums. The result of both ADF and KPSS tests imply that the forward premium indeed contains a unit root and is non-stationary.

Baillie and Bollerslev (1994) challenge this notion of non-stationarity and use autoregressive fractionally integrated moving average (ARFIMA) modeling on the forward premium, and is estimated using approximate maximum-likelihood (MLE) methodology. The reason for an exotic modeling technique is that Baillie and Bollerslev argue Crowder's choice in

[6] See Kwiatkowski, Phillips, Schmidt, and Shin (1992).

conducting the KPSS is rather weak, in that the econometrician only has a choice between selecting an $I(0)$ or $I(1)$ cointegrated process and there may be information missing between these degrees of cointegration. More general cointegration is allowed, namely the forward premium is allowed to follow a fractional cointegration process, or $I(d)$ where $0 < d < 1$, and with this identification modeled to an ARFIMA specification the forward premium becomes a mean-reverting process where random perturbations eventually die out. Baillie and Bollerslev also venture to argue that the KPSS test is rather powerful against fractional integration models so that to reject an $I(0)$ specification should not automatically lead an econometrician to believe a time series follows an $I(1)$ process, but should consider the implications of an $I(d)$ process as well. In more recent work, Maynard and Phillips (2001) utilize an ARFIMA specification with approximate MLE estimation for both short-horizon and long-horizon data and suggest that the forward premium anomaly maybe due to the discrepancy in persistence between both series. When a series is highly persistent, Maynard and Phillips argue that assuming $\beta_1 = 1$ and subtracting S_t from both sides of (1) to yield (2) causes severe distortions to the distribution of the forward premium estimator, contributing to its downward bias.

Another rich area of research is concerned with vector error correction modeling (VECM) where, depending on the cointegration of S_t and F_t, the relationship between each series can be split into short-run adjustment and long-run equilibrium components. A computational convenience to VECM is that the resulting empirical structure is independent of whether time-series data is stationary, which has led to its popularity in fitting forward premium data. The traditional VECM equation is represented as,

$$\triangle y_t = \delta + \Pi y_{t-1} + \sum_{i=1}^{p-1} \Phi_i \triangle y_{t-i} + \varepsilon_t \tag{13}$$

where $\triangle y_t = y_t - y_{t-1}$, δ is a constant, $\Pi = \alpha\beta'$ and is known as an "adjustment-matrix" whose value describes the amount of time (depending on a data sets frequency) a variable takes to reach its long-run equilibrium where α and β are $k \times r$ matrices, Φ_i is a $k \times k$ coefficient matrix denoting long-run equilibrium values of variables in vector y_{t-i} from lag $i = 1, \ldots, p-1$ and ε_t conforms to a vector of gaussian white-noise processes. Sarno (2005) explains that the spot and forward exchange rates are incorporated into (13) via the vector $y_t = \begin{bmatrix} S_t, & F_t^1, & F_t^2, & F_t^3, & \ldots, & F_t^l \end{bmatrix}'$, where S_t and F_t are defined as in Section 1and l is the number of periods a forward exchange rate comes to realization where the elements of y_t must be cointegrated with l unique cointegrating vectors where each of these unique vectors are given by the row of the matrix $\begin{bmatrix} \iota, & I_l \end{bmatrix}$ where ι is an l-dimensional column vector of ones and I_l is an $l \times l$ identity matrix. Clarida and Taylor (1997) utilize the form of [eq:13] and apply this modeling technique to weekly data on the dollar/sterling, dollar/mark, and dollar/yen data discovering that VECM fits the data features quite well. Specifically, dynamic out of sample forecasts up to one year ahead out-perform alternative forecasts utilizing random-walk and standard forward premium regression techniques. Clarida, Sarno, Taylor, and Valente (2003) introduce Markov-chain regime switching within a VECM of the form in (13) to create Markov-switching vector error correction models (MS-VECM) and fit a variety of exchange rates across a range of forward rate time-horizons and discover that, in terms of forecasting, the MS-VECM outperforms general VECM, random-walk models, and standard forward premium regression models[7].

Thus far, the discussion has been concerned with literature based on failure in UIP, or the forward premium puzzle, and its solution via weaknesses in RE, embedding risk-premia, or econometric implementation. Now, gears are shifted toward a phenomena from the realm of behavioral macroeconomics, better known as "Animal Spirits" and its potential in explaining the forward premium puzzle.

5 Animal Spirits

The provocative phrase originated with Keynes (1936) to describe the underlying set of deep human behavior that guides macroeconomic activity. The original excerpt from Keynes (1936) on Animal Spirits reads:

> "We should not conclude from this that everything depends on waves of irrational psychology. On the contrary, the state of long-term expectation is often steady, and, even when it is not, the other factors exert their compensating effects. We are merely reminding ourselves that human decisions affecting the future, whether personal or political or economic, cannot depend on strict mathematical expectation, since the basis for making such calculations does not exist; and that it is our innate urge to activity- or Animal Spirits, which makes the wheels go round, our rational selves choosing between the alternatives as best we

[7] See Sarno and Valente (2005) for more literature based in MS-VECM.

> are able, calculating where we can, but often falling back for our motive on whim or sentiment or chance." (Ch.12, p.162-163)

To paraphrase briefly (although it is with reluctance since the passage was stated beautifully), human beings do not behave in a strict mathematical (robotic) manner, but rather a mixture of rationality and evolutionary tendencies. The essence of Animal Spirits has fit within the realm of behavioral macroeconomics, pioneered by Akerlof (2002) who creates an agenda to be described by behavioral macroeconomics, 1. The existence of involuntary unemployment, 2. The impact of monetary policy on output and employment, 3. The failure of deflation to accelerate when unemployment is high, 4. The prevalence of undersaving for retirement, 5. The excessive volatility of stock prices relative to their fundamentals, and 6. The stubborn persistence of a self-destructive underclass. Akerlof's list of items have been failed by New Classical and Neo Classical models due to their lack of behavioral insights. To address his agenda, Akerlof describes behavioral macroeconomics as "the incorporation of realistic assumptions grounded in psychological and sociological observation, have produced models that comfortably account for each of these macroeconomic phenomena" (p.413). An issue behind behavioral macroeconomics, as De Grauwe (2012) denotes, is that the world of irrationality is dark and macroeconomists conform to the unanimity of RE theory since the question of "what is an irrational agent?" remains robust. However, as Akerlof (2002) states[8]:

[8] For further literature on Animal Spirits see Akerlof and Shiller (2009). Although it is aimed towards mass-consumption, their work gives numerous examples of how human behavior fits within the context of real-world macroeconomics.

> "Immediately after its publication, the economics profession tamed Keynesian economics. They domesticated it as they translated it into the "smooth" mathematics of classical econonomics. But economies, like lions, are wild and dangerous. Modern behavioral economics has rediscovered the wild side of macroeconomic behavior. Behavioral economists are becoming lion tamers. The task is as intellectually exciting as it is difficult." (p. 428)

Navigating the jungle, Thomas J. Sargent and Lars Peter Hansen have ventured to bridge a sub-theory of Optimal Control mathematics into behavioral macroeconomics in order to capture a specific type of Animal Spirit, namely *pessimism.*

5.1 Robust Control

Robust Control (RC) is a subset of mathematics derived from Optimal Control theory that deals with the problem of model misspecification. Control variables induced with a RC policy aims to achieve robust performance against bounded systematic modeling error terms[9]. Hansen and Sargent (2008) explain new control and estimation methods were sought after to improve upon adverse outcomes that come from applying ordinary control theory to a variety of engineering and physical problems. A theory is that model misspecification explains why actual outcomes were sometimes much worse than the results provided by control theory and thus decision rules and estimators acknowledging model misspecification were desired, hence the emergence of RC. A long standing issue in macroeconomics is that economic agents are as-

[9] See Whittle (1981), Whittle (1990), Basar and Bernhard (1995), and Whittle (1996) for works on Robust Control as a deviation from Rational Expectations.

sumed have no systematic biases in forecasting future economic variables (i.e. RE), meaning agents fully "trust" their models. As a way to relax this assumption, a "malevolent nature" is assumed to lurk subliminally within the economic environment, existing separately from households, and cause systematic perturbations to households' systematic understanding of the economy, whose objective is to minimize household utility. Economic agents, realizing what is systematically expected to happen deviates from what systematically actually happens[10], induces a fear that their understanding of the economy's evolution is quite correct or, in other words, they fear model misspecification so as to not fully trust their model. The discrepancy between what agents believe and what actually happens induces fear where this fear manifests itself as pessimism, captured via a parameter θ.

To formally show the difference between a RE and RC problem (as well as provide technical-intuition), we begin by introducing macroeconomics oriented in RE which can be cast in a Linear-Quadratic Control (LQC) framework[11], a technique that accommodates a wide ranging class of linear DSGE models. A typical linearized DSGE model cast within the framework of LQC takes the (simple) form,

$$\max_{\{u_t\}_{t=0}^{\infty}} E_t \sum_{t=0}^{\infty} \beta^t \left\{ \begin{bmatrix} x_t \\ u_t \end{bmatrix}' \begin{bmatrix} R & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} x_t \\ u_t \end{bmatrix} \right\} \tag{14}$$

[10] At this point, we deviate from traditional RE and behavioral macroeconomics is incorporated within expectation formation.

[11] A technique which is part of modern control theory.

$$s.t. \quad x_{t+1} = Ax_t + Bu_t + C\varepsilon_{t+1}$$

where R, Q, A, B, and C are coefficient matrices, x_t is a state variable vector, u_t is a control variable vector, and ε_{t+1} is a Gaussian white-noise vector. (14) is cast in a LQC setting where economic agents maximize their objective function via the control variable, u_t, and take the state-evolution constraint as an exogenous process. This optimization problem is within the framework of RE because as can be seen from the state variable's transition equation, only random errors perturb the evolutionary process of the economy. Once a DSGE model has been linearlized and put in the form of (14), the argument can be transformed into a linear-dynamic programming problem via a Bellman equation (assuming the Certainty Equivalence Principle[12]),

$$x_t' P x_t = \max_{\{u_t\}} \left\{ \begin{bmatrix} x_t \\ u_t \end{bmatrix}' \begin{bmatrix} R & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} x_t \\ u_t \end{bmatrix} + \beta x_{t+1}' P x_{t+1} \right\} \tag{15}$$

$$s.t. \quad x_{t+1} = Ax_t + Bu_t$$

where it is assumed that the value function takes on a quadratic form, or $V(x) = x_t' P x_t$ and P is a costate variable matrix which serves strictly as a mathematical construct used

[12]This rule is a mathematical convenience which states that a decision rule derived from a stochastic optimal control problem is equivalent to that derived from a static one. Technicaly, this is allows for setting $\varepsilon_{t+1} = 0$ and eliminating the expectations operator, essentially creating a static optimal control problem.

to find a stable steady-state solution which will be described in further detail below. Now, inserting the static constraint of (15) into its objective function and optimizing by choice of control variable u_t the following optimal decision rule is derived,

$$u_t = Fx_t \tag{16}$$

so that $F = -\beta \left(R + \beta B'PB\right)^{-1} B\prime PA$. (16) denotes the optimal decision made by households given the state of the economy x_t at time t. The Riccatti equation can be derived out of (15) where,

$$P = Q + \beta A \left[P - \beta PB \left(R + \beta B'PB\right)^{-1} B\prime P\right] A \tag{17}$$

where in (17) P is initialized by an identity matrix and iteration occurs until converging on a steady-state matrix, P^* which is inserted into F defining a stable optimal policy rule. Once a stable rule has been defined, economic agents' optimal decisions feed back into the evolution process of the economy so that,

$$x_{t+1} = A^0 x_t + C\varepsilon_{t+1} \tag{18}$$

and $A^0 = (A + BF)$. Thus, (18) is the equilibrium exogenous process generating dynamics within the economy that agents optimally respond to. The fundamental problem with this form of control theory is that actual results observed in nature are sometimes far worse than the simulations behind (16) using (18) causing an inquiry into other types of control analysis that deviate from RE specification in the state-evolution system of (14). In order for an optimal decision rule to accommodate model misspecification, RC is introduced within LQC context so that,

$$\max_{\{u_t\}_{t=0}^{\infty}} \min_{\{w_{t+1}\}_{t=0}^{\infty}} -E_t \sum_{t=0}^{\infty} \beta^t \left\{ \begin{bmatrix} x_t \\ u_t \end{bmatrix}' \begin{bmatrix} R & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} x_t \\ u_t \end{bmatrix} + \theta\beta w_{t+1}' w_{t+1} \right\} \tag{19}$$

$$s.t. \quad x_{t+1} = Ax_t + Bu_t + Cw_{t+1} + C\varepsilon_{t+1}$$

where now (19) represents a two-player Stacklberge game where the malevolent nature (minimizing player) that enters the optimization system quadratically and uses w_{t+1} as a way of minimizing household utility, indirectly manipulating the parameter θ, or household pessimism. The deviation from RE especially becomes apparent through the state-evolution system in (19) where x_{t+1} is not only randomly perturbed by ε_{t+1} but also systematically perturbed by w_{t+1} as well, implying that agents do not have a deep understanding of the economy's evolutionary tendencies and hence cannot make reliable systematic forecasts, casting doubt or distrust of their beliefs about actual economic values. Continuing, (19) can be converted into a Bellman equation and assuming Certainty Equivalence we have,

$$-x_t'Px_t = \max_{\{u_t\}} \min_{\{w_{t+1}\}} \left\{ - \begin{bmatrix} x_t \\ \\ u_t \end{bmatrix}' \begin{bmatrix} R & 0 \\ \\ 0 & Q \end{bmatrix} \begin{bmatrix} x_t \\ \\ u_t \end{bmatrix} + \theta\beta w_{t+1}' w_{t+1} - \beta x_{t+1}' P x_{t+1} \right\} \quad (20)$$

$$s.t. \quad x_{t+1} = Ax_t + Bu_t + Cw_{t+1}$$

unfortunately, the solution method is not quite as simple as (15). The problem presented in (20) is independent of sequencing[13] so that the minimizing player optimizes first and assumes that the maximizing player is carrying out their optimal decision rule (16) so that the malevolent nature's problem reduces to,

$$\min_{\{w_{t+1}\}} \left\{ \theta w_{t+1}' w_{t+1} - x_{t+1}' P x_{t+1} \right\} \quad (21)$$

$$s.t. \quad x_{t+1} = A^0 x_t + Cw_{t+1}$$

Inserting the constraint into its objective function results in an unconstrained minimization problem that yields the first order condition (FOC),

$$w_{t+1} = Kx_t \quad (22)$$

[13] Meaning the solution method is independent of whether minimization or maximization is chosen first.

where $K = \theta^{-1}\left(I - \theta^{-1}C\prime PC\right)^{-1}C\prime PA^{0}$. The intuition behind (22) is that based on economy's state x_t, a malevolent nature will choose a sequence of "worst-case" shocks, w_{t+1} that minimize the maximizing player's utility. Additionally, an artifact derived from (21) by taking the derivative with respect to x_t in the unconstrained optimization problem results in,

$$D(P) = P + PC\left(\theta I - C\prime PC\right)^{-1}C\prime P \tag{23}$$

where (23) represents a portion of the Ricatti equation derived from the minimization "part" of (20). Intuitively speaking, (23) is information the minimizing player uses to subliminally imbed their malevolent schematic within the maximizing player's optimization problem. Now that the minimizing player has optimized, the maximizing player chooses next and assumes the world is systematically unbias, implying $w_{t+1} = 0$ so that,

$$x_t' D(P) x_t = \max_{\{u_t\}} \left\{ \begin{bmatrix} x_t \\ u_t \end{bmatrix}' \begin{bmatrix} R & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} x_t \\ u_t \end{bmatrix} + \beta x_{t+1}' D(P) x_{t+1} \right\} \tag{24}$$

$$s.t. \quad x_{t+1} = Ax_t + Bu_t$$

where $V(x) = x_t' D(P) x_t$ is a minimized value function that households unknowingly maximize against (hence the malevolent nature "lurking in the background"). Taking the FOC of (24) results in,

$$u_t = Fx_t \tag{25}$$

and $F = -\beta \left(R + \beta B' D(P) B\right)^{-1} B' D(P) A$. (25) is similar to (16) but here (23) is embedded instead. The full Ricatti equation can be derived by taking the derivative of unconstrained optimization problem with respect to x_t or,

$$T(P) = Q + \beta A' \left[D(P) - \beta D(P) B \left(R + \beta B' D(P) B\right)^{-1} B' D(P) \right] A \tag{26}$$

now iterating on (23) until convergence and inserting into (22) and (25) for K and F respectively yield stable FOC's for the minimizing and maximizing players. Once stability is reached, insert both FOC's into the constraint from (19) resulting in an equilibrium transition equation,

$$x_{t+1} = \widetilde{A} x_t + C\varepsilon_{t+1} \tag{27}$$

where $\widetilde{A} = (A + BF + CK)$. Notice that the difference between equilibrium transition equations of (18) and (27) is that the latter accounts for systematic misspecification captured via the extra CK term embedded within the system. Now that the economy is in equilibrium households realize that there is discrepancy between what they systematically thought is happening (this belief is reflected by the constraint from (24) known as the "approximating system") and what is systematically actually happening (where this actuality is represented by the constraint in (20) known as the "perturbed system") which induces fear or pessimism through the parameter θ. Without too much technical detail, the further the economy deviates systematically from what is expected to occur, the higher a households' pessimism which is intuitive because, as human beings, if an event occurs worse than expected this causes pessimism which corrects our next forecast by discounting our expectations closer to what is observed in reality. To close this explanation, for an optimal value function iterate (26) until convergence and inserting into the value function results in,

$$V^*(x_t) = x_t^{*\prime} T \circ D(P) x_t^* \tag{28}$$

where $V^*(x_t)$ represents an optimized value function, x_t^* is an equilibrium state equation containing steady-state FOC's (22) and (25), and $T \circ D(P)$ is the Riccatti equation (26) evaluated at (23). Hopefully this segment has given the reader context and intuition behind RC problems and methodology.

Literature applying RC to the forward premium puzzle is extremely scarce. Li and Tornell (2008) apply RC to a simple investor model of exchange rates to account for the forward premium anomaly. According to the author's, in equilibrium optimizing investor's do not hold misperceptions about their model and distort their forecasts to in an attempt to attain robustness against potential misspecification. Additionally, this forecast distortion triggers delayed overreaction of exchange rates with respect to interest rate differential disturbances directly leading to a negative correlation between those exchange rates and interest rate differentials, hence an accounting of the forward premium anomaly.

6 Conclusion

This work is not exhaustive as the field pertaining to the famous forward premium puzzle and Animal Spirits is extremely large. The discussion presented above was an attempt to aim the reader towards a direction of foundational works and those literature that set the stage for present and future research. Macroeconomics has been a thriving and provocative enterprise contributing to great advancements of to our social complex. The advent of Behavioral Macroeconomics hopes to unite elements of economics, psychology, and sociology in order to gain a better understanding of our economic system. By introducing and explaining a well-known international puzzle, dissecting it into three branches of major research (those concerning RE, Risk Premia, and econometric implementation) and discussing RC, hopefully the reader has gained perspective behind Behavioral Macroeconomics and its application to difficult questions posed by the science.

CHAPTER 2 INCORPORATING ROBUST CONTROL IN AN INTERNATIONAL DSGE MODEL: CAN PESSIMISM EXPLAIN THE FORWARD PREMIUM PUZZLE?

7 Introduction

Keynes (1936) introduced the concept of "Animal Spirits" into economic literature, describing a set of human characteristics underlying macroeconomic activity. The advent and implementation of Robust Control (RC) by Hansen and Sargent (2008) in traditional Rational Expectations (RE) framework allows agents to realize initial systematic expectations differ from actual systematic occurrences, where this discrepancy induces fear of model misspecification or "pessimism", a specific type of Animal Spirit, in households. Building on research discussed in Shkrelja (Ch.1, 2014), this work uses a dynamic stochastic general equilibrium (DSGE) two-country two-money model from Lucas (1982), fitted with RC essentially equipping households with pessimism, the penultimate component in solving the well-documented forward premium puzzle[14] (FPP). Varying pessimism regimes both emulate data features and validate RE assumptions about movements in forward premium (discount) estimator, fully transmitting to movements in the gross rate of depreciation (appreciation).

This chapter is organized as follows: Section 8 defines the model, Section 9 calibrates pessimism, Section 10 derives FPP bias and estimation results, Section 11 concludes.

[14] See Fama (1984), Engle (1996), Sarno (2005), and Chinn (2007), among others.

8 Model

8.1 Inducing Malevolent Nature

Household agents ("maximizing player") believes the state of the world evolves according to AR(1) stochastic processes, known as the *approximating system* of stochastic equations, so

$$x_{t+1} = \overbrace{(1-\rho_1)+\rho_1 x_t}^{Systematic\ belief} + \varepsilon_{t+1}^{x} \tag{29}$$

$$y_{t+1} = (1-\rho_2)+\rho_2 y_t + \varepsilon_{t+1}^{y} \tag{30}$$

$$M_{t+1} = (1-\rho_3)+\rho_3 M_t + \varepsilon_{t+1}^{M} \tag{31}$$

$$N_{t+1} = (1-\rho_4)+\rho_4 N_t + \varepsilon_{t+1}^{N} \tag{32}$$

where x_{t+1} is domestic output, y_{t+1} is foreign output, M_{t+1} is gross domestic money supply, N_{t+1} is gross foreign money supply, and where the maximizing player believes the economy to evolve according to their systematic understanding of $(94)-(97)$. In addition the maximizing player believes error terms are distributed according to

$$\varepsilon_t^{j} \sim N(0,\sigma_j^2)\ for\ j = x, y, M, N \tag{33}$$

where σ_j^2 is constant variance term for $j = x, y, M, N$

Now, the malevolent nature ("minimizing player") perturbs approximating system through error terms

$$x_{t+1} = (1-\rho_1) + \rho_1 x_t + \widetilde{\varepsilon_{t+1}^{x}} \quad (34)$$

$$y_{t+1} = (1-\rho_2) + \rho_2 y_t + \widetilde{\varepsilon_{t+1}^{y}} \quad (35)$$

$$M_{t+1} = (1-\rho_3) + \rho_3 M_t + \widetilde{\varepsilon_{t+1}^{M}} \quad (36)$$

$$N_{t+1} = (1-\rho_4) + \rho_4 N_t + \widetilde{\varepsilon_{t+1}^{N}} \quad (37)$$

where $\widetilde{\varepsilon_t^j}$ for $j = x, y, M, N$ represents perturbed error terms distributed as

$$\widetilde{\varepsilon_t^j} \sim N(w_{t+1}, \sigma_j^2) \; for \; j = x, y, M, N \quad (38)$$

$$\widetilde{\varepsilon_t^j} - w_{t+1} \sim N(0, \sigma_j^2) \; for \; j = x, y, M, N \quad (39)$$

$$\varepsilon_t^j = \widetilde{\varepsilon_t^j} - w_{t+1} \quad (40)$$

$$\widetilde{\varepsilon_t^j} = \varepsilon_t^j + w_{t+1} \quad (41)$$

where w_{t+1} is the disruption used by the malevolent nature to systematically perturb (94) – (97). Inserting (154) for $j = x, y, M, N$ into (147) – (150) yields the *perturbed system* of stochastic equations

$$x_{t+1} = (1-\rho_1) + \rho_1 x_t + w_{t+1} + \varepsilon_{t+1}^{x} \quad (42)$$

$$y_{t+1} = (1-\rho_2) + \rho_2 y_t + w_{t+1} + \varepsilon_{t+1}^{y} \quad (43)$$

$$M_{t+1} = (1-\rho_3) + \rho_3 M_t + w_{t+1} + \varepsilon_{t+1}^{M} \quad (44)$$

$$N_{t+1} = (1-\rho_4) + \rho_4 N_t + w_{t+1} + \varepsilon_{t+1}^{N} \quad (45)$$

Ultimately, $(94)-(97)$ represents the state of nature that the maximizing player assumes to systematically exist and $(155)-(158)$ represents the state of nature that systematically actually exists. Next, the Lucas model is outfitted with RC.

8.2 Model with Robust Control

8.2.1 Optimization Problem

The international model fitted with robust control is represented as,

$$\max_{\substack{\{c_{xt}, c^*_{xt},\\ c_{yt}, c^*_{yt},\\ \omega_{xt}, \omega^*_{xt},\\ \omega_{yt}, \omega^*_{yt},\\ \psi_{Mt}, \psi^*_{Mt},\\ \psi_{Nt}, \psi^*_{Nt}\}_{t=0}^{\infty}}} \min_{\{w_{t+1}\}_{t=0}^{\infty}} E_t \sum_{t=0}^{\infty} \beta^t \left\{ \phi \left[-\frac{(c_{xt}^{\theta} c_{yt}^{1-\theta})^{1-\gamma}}{1-\gamma} + \beta\overline{\theta} w_{t+1}^2 \right] + (1-\phi) \left[-\frac{(c_{xt}^{*\theta} c_{yt}^{*1-\theta})^{1-\gamma}}{1-\gamma} + \beta\overline{\theta^*} w_{t+1}^2 \right] \right\}$$

$$s.t. \quad \phi \sum_{t=0}^{\infty} \beta^t \left\{ c_{xt} + \frac{S_t P_t^*}{P_t} c_{yt} + \omega_{xt} e_t + \omega_{yt} e_t^* + \psi_{Mt} r_t + \psi_{Nt} r_t^* - \frac{P_{t-1}}{P_t} \omega_{xt-1} x_{t-1} - \frac{S_t P_{t-1}^*}{P_t} \omega_{yt-1} y_{t-1} \right.$$
$$\left. - \frac{\psi_{Mt-1} \Delta M_t}{P_t} - \frac{\psi_{Nt-1} S_t \Delta N_t}{P_t} - \omega_{xt-1} e_t - \omega_{yt-1} e_t^* - \psi_{Mt-1} r_t - \psi_{Nt-1} r_t^* \right\} = 0$$

$$(1-\ \phi)\sum_{t=0}^{\infty}\beta^{t}\left\{c_{xt}^{*}+\frac{S_tP_t^{*}}{P_t}c_{yt}^{*}+\omega_{xt}^{*}e_t+\omega_{yt}^{*}e_t^{*}+\psi_{Mt}^{*}r_t+\psi_{Nt}^{*}r_t^{*}-\frac{P_{t-1}}{P_t}\omega_{xt-1}^{*}x_{t-1}-\frac{S_tP_{t-1}^{*}}{P_t}\omega_{yt-1}^{*}y_{t-1}\right.$$
$$\left.-\frac{\psi_{Mt-1}^{*}\Delta M_t}{P_t}-\frac{\psi_{Nt-1}^{*}S_t\Delta N_t}{P_t}-\omega_{xt-1}^{*}e_t-\omega_{yt-1}^{*}e_t^{*}-\psi_{Mt-1}^{*}r_t-\psi_{Nt-1}^{*}r_t^{*}\right\}=0$$

$$\overbrace{m_t=P_tc_{xt}}^{CIA}\qquad\overbrace{\omega_{xt}+\omega_{xt}^{*}=1}^{Equity\ Shares}\qquad\overbrace{c_{xt}+c_{xt}^{*}=x_t}^{\mathrm{Re}\,source\ Constraints}\qquad(46)$$

$$n_t=P_t^{*}c_{yt}\qquad\omega_{yt}+\omega_{yt}^{*}=1\qquad c_{yt}+c_{yt}^{*}=y_t\qquad(47)$$

$$m_t^{*}=P_tc_{xt}^{*}\qquad\psi_{Mt}+\psi_{Mt}^{*}=1\qquad m_t+m_t^{*}=M_t\qquad(48)$$

$$n_t^{*}\ =P_t^{*}c_{yt}^{*}\qquad\psi_{Nt}+\psi_{Nt}^{*}=1\qquad n_t+n_t^{*}=N_t\qquad(49)$$

Where c_{xt} and c_{yt} represent home good-x and good-y consumption respectively, m_t and n_t represent home holdings of home and foreign currency, ω_{xt} and ω_{yt} represent home output shares of good-x and good-y respectively valued at price e_t for ω_{xt} and e_t^{*} for ω_{yt} , ψ_{Mt} and ψ_{Nt} represent home holdings of home and foreign currency shares respectively valued at price r_t for ψ_{Mt} and r_t^{*} for ψ_{Nt}, P_t and S_t represent home prices and spot exchange rate respectively. Finally, all variables containing * are foreign variable counterparts to the preceding home variable list. Beginning with the objective function, domestic and foreign households optimize weighted average CRRA utility (consisting of domestic and foreign consumptions of goods x and y), so that β^t is discount factor, ϕ represents importance of country in decentralized world economy, $\overline{\theta}$and $\overline{\theta}^*$represent home and foreign pessimism respectively, w_{t+1} is malevolent nature's control variable, representing an intertemporal sequence of "worst-case" shocks that are used to perturb $(94) - (97)$. Constraints to the objective function include home and foreign budget constraints, the first column of $(159) - (160)$ represents CIA constraints, the second and third columns are adding up constraints to close the model where $(94) - (97)$ and $(155) - (158)$ are the stochastic systems which generate dynamics.

8.2.2 Maximizing Player Chooses First

Household agents choose first, maximizing their objective function, believing that the economy evolves according to the approximating system of stochastic equations. Setting up the Lagrangian, deriving FOC's, finding steady-states, and log-linearlizing around steady-states, and using matrix-algrebra to simplify results in

$$\widetilde{S}_t = \Pi X_t \tag{50}$$

$$\widetilde{F}_t = \Gamma X_t \tag{51}$$

Remark 1 *See Technical Appendix Section 20.2.1 for derivation of* (181) *and* (183)*.*

where $\Pi = \begin{bmatrix} 1 & -1 \end{bmatrix} U_S$, $\Gamma = \begin{bmatrix} \rho_3 & -\rho_4 \end{bmatrix} U_S$, and $U_S = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, representing a selection matrix that allows certain stochastic equations to effect (181) and (183), or spot and forward exchange rates respectively. Again, maximizing player believes (181) and (183) to evolve according to (94)-(97), represented in matrix algebra as

$$X_{t+1} = AX_t + C\varepsilon_{t+1} \tag{52}$$

where

$$X_t = \begin{bmatrix} \widetilde{x}_t \\ \widetilde{y}_t \\ \widetilde{M}_t \\ \widetilde{N}_t \end{bmatrix}, \quad \varepsilon_{t+1} = \begin{bmatrix} \varepsilon^x_{t+1} \\ \varepsilon^y_{t+1} \\ \varepsilon^M_{t+1} \\ \varepsilon^N_{t+1} \end{bmatrix}, \quad A = \begin{bmatrix} \rho_1 & 0 & 0 & 0 \\ 0 & \rho_2 & 0 & 0 \\ 0 & 0 & \rho_3 & 0 \\ 0 & 0 & 0 & \rho_4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

8.2.3 Minimizing Player Chooses Second

The malevolent nature assumes the maximizing player is both optimizing and in equilibrium[15]. Exploiting second-order taylor expansion of the objective function and Linear-Quadratic Gaussian Control (techniques the minimizing player's problem results in,

$$w_{t+1} = KX_t \tag{53}$$

$$P = R + \beta A'PA + A'PC\left(\overline{\overline{\theta}}I_4 - C'PC\right)^{-1}C'PA \tag{54}$$

Remark 2 *See Technical Appendix Section 20.2.2 for derivation of* (53) *and* (229).

where $K = \left(\overline{\overline{\theta}}I_4 - C'PC\right)^{-1}C'PA$. Equation (53) represents malevolent nature's intertemporal worst-case feed back rule, rather, based on the state of the economy X_t ,nature chooses a sequence of worst-case outcomes, w_{t+1}, that minimizes the maximizing player's utility. Equation (229) represents the Ricatti equation whose properties are well-known in control theory. Iterating on (229) until convergence to $\widehat{P}$ and inserting into K for (53) results in a stable worst-case feed back rule, where placing this stable w_{t+1} into a matrix-algebra representation of (155) – (158),

$$X_{t+1} = AX_t + Cw_{t+1} + C\varepsilon_{t+1} \tag{55}$$

[15] Symmetric pessimism is assumed across countries so that $\overline{\overline{\theta}} = \phi\overline{\theta} + (1-\phi)\overline{\theta}^*$ where $\overline{\theta} = \overline{\theta}^*$ and $\overline{\overline{\theta}}$ is essentially "world-pessimism."

results in,

$$X_{t+1} = A^0 X_t + C\varepsilon_{t+1} \tag{56}$$

where $A^0 = \left(A + CK(\bar{\bar{\theta}})\right)$.Ultimately, (234) is the dynamic-producing perturbed state-evolution system embedded with malevolent natures feedback rule $w_{t+1.}$Intuitively, nature selects its sequence of worst-case occurrences and places them in (55) so as the economy evolves agents now realize that (234) produces dynamics, so that what is systematically expected differs from what systematically happens where this discrepancy induces pessimism in households, where households are "doubtful" of their expectations of the economy, and is captured by parameter $\bar{\bar{\theta}}$.

9 Detection Error Probabilities

In order to calibrate or discipline the choice of $\bar{\bar{\theta}}$, detection error probabilities (DEP) must be used. Low values of DEP mean maximizing players are better at deciphering between evolution systems (52) and (234), implying households have low pessimism since they are better at determining what they think is going on and what is actually going on . Alternatively, high values of DEP mean households have difficulty in deciphering between evolution systems (52) and (234), invoking high amounts of pessimism in the household's psyche since they are worse at determining what they think is going on and what is actually going on.

Let model A denote the approximating stochastic system (52) and Let model B denote the perturbed stochastic system (234).Define,

$$p_A = \Pr\left(r|A < 0\right) \tag{57}$$

where $r|A$ is the log-likelihood ratio. When model A generates the data, p_A measures the probability that the maximizing player selects model B instead. Similarly define,

$$p_B = \Pr\left(r|B > 0\right) \tag{58}$$

where $r|B$ is the log-likelihood ratio. When model B generates the data, p_B measures the probability that the maximizing player selects model A instead. Following Hansen and Sargent (2008), the DEP, p, is defined as

$$p\left(\overline{\overline{\theta}}\right) = \frac{1}{2}\left(p_A + p_B\right) \tag{59}$$

where p is the probability of error in choosing the correct model which implies that $1-p$ is the probability of success in choosing the correct model and $\overline{\overline{\theta}}$ is the pessimism parameter used to generate model B. Since there lies a positive relationship between $\overline{\overline{\theta}}$ and p, as $\overline{\overline{\theta}}$ decreases its associated p will decrease as well so that the desired level of detection error will implicitly discipline the value of $\overline{\overline{\theta}}$.For this work, a $\overline{\overline{\theta}}$ is chosen that corresponds to $p = 0.100$, following Luo and Young (2010) and Hansen and Sargent (2008), meaning for the level of calibrated pessimism, the maximizing player will correctly decipher between model A and model B 90% of the time. Thus by choosing a specific $\overline{\overline{\theta}}$ that corresponds to a desired DEP, the choice of $\overline{\overline{\theta}}$ is disciplined.

10 Forward Premium Puzzle

10.1 Downward Bias of Estimator

To generate the forward premium regression, use (181) as dependent variable and (183) independent variable so that a linear regression can be formed as,

$$\mathbf{S}_{t+1}-\mathbf{S}_t = \widehat{\beta_0} + \widehat{\beta_1}\left(\mathbf{F}_t-\mathbf{S}_t\right) + e_{t+1} \tag{60}$$

where is $\mathbf{S}_{t+1}-\mathbf{S}_t$ the gross rate of depreciation, $\mathbf{F}_t-\mathbf{S}_t$ is the forward premium,$\widehat{\beta_1}$ is the forward premium estimator , $\widehat{\beta_0}$is intercept estimator, and $e_{t+1} \sim N(0,\sigma^2)$ where σ^2 is constant variance term . To begin analyzing slope and intercept estimators we exploit relationships developed for gross rate of depreciation and forward premium by inserting (234) into (181) and (183) for X_t and this expansion into (60), utilize OLS optimization, and taking the probability limit of both β_0 and β_1 yields,

$$\text{plim}\left(\widehat{\beta_1}\right) = 1 - \frac{tr\left[\Psi' V' M_i \left(V - Q\right)\Psi\right]}{tr\left(\Psi' V' M_i V \Psi\right)} \tag{61}$$

Remark 3 *See Technical Appendix Section 22.1 for derivation of* (61) .

where,

$$\Psi = \begin{bmatrix} I & 0 & \cdots & 0 \\ -A^0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & -A^0 & I \end{bmatrix}^{-1} \begin{bmatrix} C & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & C \end{bmatrix}$$

$$V - Q = \begin{bmatrix} \Gamma - \Pi A^0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Gamma - \Pi A^0 \end{bmatrix}$$

$M_i = \left(I - \frac{ii'}{T}\right)$ and $A^0 = A + C\left(\overline{\overline{\theta}} I_4 - C'PC\right)^{-1} C'PA$ where we note that the intercept estimator $\widehat{\beta_0}$ disappears in the probability limit. In (61), the RHS second element is the bias term that forces $\widehat{\beta_1}$ downward from 1. Technically speaking, this bias term is an artefact of RC modeling with $\overline{\overline{\theta}}$ embedded in the term itself via Ψ and Q matrices, allowing for misspecification to an extent and ultimately resulting in a robust estimator. Intuitively speaking, now that economic agents face a malevolent nature that subliminally tries to reduce household utility, pessimism is induced in the agent to account for systematic evolutions in the economy that may not have been anticipated. Since the malevolent nature systematically perturbs the economy, this behavior induces households to be systematically pessimistic as well, given the state of the economy they face. Now, given that households are systematically pessimistic, this behavior ultimately maps into (61) as a downward bias. Agents, given their realization that what is systematically forecasted at an initial date may not be what systematically manifests at a later terminal date (as in RE), now understand that the economy systematically changes in the time between a forecast of gross rate of depreciation (forward premium) and the gross rate of depreciation *ex-post*. Taking state of economy as given and making a forecast based on that state, agents don't " systematically believe" or are doubtful of the initial forward premium (of course, because they're pessimistic) and will adjust domestic/foreign cash holdings *conservatively* meaning if, for example, the forward premium suggests appreciation of domestic currency against foreign then agents won't sell off the foreign currency (due, again, for fear that things may not go as expected) as much as they would have under RE (where there is complete confidence in things going as expected due obliviousness of any systematic perturbations) resulting in the *ex-post* gross rate of depreciation not appreciating as much as it would have under RE, hence the downward bias.

10.2 Consistency of Estimator

In this section, to show how biasness disappears so as to establish consistency, we begin with taking the limit of $\overline{\overline{\theta}} \to \infty$ to (61) or,

$$\lim_{\bar{\bar{\theta}}\to\infty}\left\{\text{plim}\left(\widehat{\beta_1}\right)\right\}=1 \tag{62}$$

Remark 4 *See Technical Appendix Section 22.2 for derivaiton of* (62) *.*

where (62) is a direct result of $(V-Q)=\mathbf{0}$ as $\bar{\bar{\theta}}\to\infty$. Intuitively, (62) implies that as pessimism becomes increasingly large, households become "systematically-oblivious" to any perturbations in their beliefs about the economy's evolution since they are abysmal at deciphering between what is believed and what is actually occurring. Continuing with this logic, if agents are unaware of perturbations then they do not behave in a systematically conservative manner, eliminating downward bias altogether.

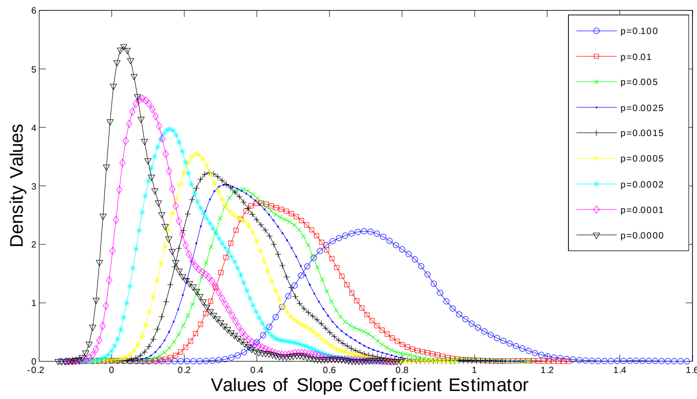
10.3 Estimation & Simulation

Using $/£ 1-month forward exchange rate data in quarterly frequency from 1986:III to 2013:III (Bank of England) and $/£ spot exchange rate data in quarterly frequency from 1986:III to 2013:III (St. Louis Federal Reserve) so as to create the gross rate of depreciation as well as the forward premium, and using OLS regression (60) yields,

$$\mathbf{S}_{t+1}-\mathbf{S}_t=\underset{(0.0042)}{-0.000897}-\underset{(0.0966)^{**}}{0.2037}\left(\mathbf{F}_t-\mathbf{S}_t\right)+e_{t+1} \tag{63}$$

where $\widehat{\beta_0}=-0.000897$ and $\widehat{\beta_1}=-0.2037$ with their respective standard deviations given in parenthesis where (**) implies significance at both 5% and 10% alpha levels. The slope coefficient of (63) means if the forward premium, $\mathbf{F}_t-\mathbf{S}_t$, increases by 1% then the change

Figure 1: Slope Coefficient Estimator PDF Simulation



in the spot exchange rate, $\mathbf{S}_{t+1} - \mathbf{S}_t$, will *decrease* (appreciate) by 0.2037% and the intercept term is statistically indifferent from 0. The puzzle is evident in (63) because the *a priori*

hypothesis implied above that $\beta_1 > 0$ is clearly violated as $\widehat{\beta_1} < 0$. Thus, the downward bias is prevalent in the data. Now, using pseudo-produced[16] observations from $(181) - (183)$ generated by (234) with pessimism corresponding to $p\left(\overline{\overline{\theta}}\right) = 10\%$, the regression becomes,

$$\mathbf{S}_{t+1} - \mathbf{S}_t = \underset{(0.1476)}{0.0668} + \underset{(0.1719)^{***}}{0.7712} \left(\mathbf{F}_t - \mathbf{S}_t\right) + e_{t+1} \tag{64}$$

where $\widehat{\beta_0} = 0.0668$ and $\widehat{\beta_1} = 0.7712$ with their respective standard deviations given in parenthesis where (***) implies significance at 1%, 5% and 10% alpha levels. The slope coefficient of (63) means if the forward premium, $\mathbf{F}_t - \mathbf{S}_t$, increases by 1% then the change in the gross rate of depreciation, $\mathbf{S}_{t+1} - \mathbf{S}_t$, *increases* (depreciates further *ex-post*) by 0.7712% and the intercept term is statistically indifferent from 0. Thus, $\overline{\overline{\theta}} \uparrow$ means downward bias diminishes resulting in $\widehat{\beta_1} \rightarrow 1$. Alternatively, using an identical experiment with $p\left(\overline{\overline{\theta}}\right) = 0\%$

[16] See Technical Appendix Section 6.4 for simulation procedure.

instead implies,

$$\mathbf{S}_{t+1}-\mathbf{S}_t = \underset{(0.2210)}{0.1432} - \underset{(0.0279)^{***}}{0.0640} (\mathbf{F}_t-\mathbf{S}_t) + e_{t+1} \tag{65}$$

where $\widehat{\beta_0} = 0.1432$ and $\widehat{\beta_1} = -0.0640$ .The slope coefficient of (63) means if the forward premium,$\mathbf{F}_t-\mathbf{S}_t$, increases by 1% then the change in the gross rate of depreciation,$\mathbf{S}_{t+1}-\mathbf{S}_t$, *decreases* (appreciates further *ex-post*) by 0.0640% and the intercept term is statistically indifferent from 0. Thus, $\overline{\overline{\theta}} \downarrow$ means downward bias exacerbates so much that it results in (for this case) $\widehat{\beta_1} < 0$. By implementing a lower amount of pessimism in the model, (65) captures a main feature of (63) in that $\widehat{\beta_1} < 0$ and verifies arguments developed above. For concreteness, Figure (1) shows Gaussian Monte-Carlo simulations conducted for normal PDF's of $\widehat{\beta_1}$ 5,000 times for various $\overline{\overline{\theta}}'s$ where the transition from positive to negative domain can be seen as $\overline{\overline{\theta}}$ decreases and hence $p\left(\overline{\overline{\theta}}\right)$ decreases, which again implies that as pessimism decreases in agents, forward premium bias is exacerbated.

11 Conclusion

Ultimately, Animal Spirits can play an important role in macroeconomic activity when expressed through appropriate models. In this paper, the FP bias was contained through exploiting relationships developed in an international DSGE model fitted with RC and it was shown that biasness disappears when households are "systematically-oblivious" or have high amounts of pessimism,$\overline{\overline{\theta}}$. By using varying regimes of pessimism, both features of the data (negative relationship between $(S_{t+1} - S_t)$ and $(F_t - S_t)$ exists) and unbiasness of uncovered interest parity (positive relationship between $(S_{t+1} - S_t)$ and $(F_t - S_t)$ exists) were produced in the forward premium estimator $\widehat{\beta_1}$.

Exploration within behavioral macroeconomic framework is beneficial in that this type of framework offers explanations into various economic inquiries. Implementing psychological behaviors of households within economic models can facilitate further understanding how complex systems, such as exchange rates, operate.

CHAPTER 3 AN INQUIRY INTO PESSIMISM: USING DETECTION ERROR PROBABILITIES TO CALIBRATE AN ANIMAL SPIRIT

12 Introduction

Since Keynes (1936) development of Animal Spirits within macroeconomic literature, economists such as Akerlof (2002), Hansen and Sargent (2008), and Akerlof and Shiller (2009) have strived to incorporate deviations from rational expectations (RE) into mainstream macroeconomics, hence the advent of behavioral macroeconomics. Hansen and Sargent venture from simple deviations of RE by incorporating robust control (RC) within a max-min optimization framework and press that one particular Animal Spirit, namely pessimism, is captured through a parameter,θ.Luo, Nie, and Young (2012) incorporate RC and use pessimism (or model uncertainty) in an intertemporal current account model which is grounded in linear-quadratic permanent income hypothesis[17] to account for international consumption and current account patterns. In growth theory literature, Bidder and Smith (2012) incorporate RC so that agents become pessimistic where under model uncertainty adverse shocks become more volatile, negatively impacting economic growth.

Outside of the general implementation of RC in dynamic-stochastic general equilibrium (DSGE) settings for studying specific variables , there lies scarce examination of using detection error probabilities (DEP)[18] as prescribed by Hansen and Sargent (2008) to, not only

[17] See Hall (1978).

[18] See Burnham and Anderson (1998) for an exhaustive study of detection error probabilities and the statistical theory of model selection in general.

calibrate pessimism, but to compare simulated pessimism to data-driven pessimism. This work sees to examine the proximity that a macroeconomic model is able to generate behavioral tendencies of real households, by way of DEP. In order to execute such a comparison, a Lucas (1982) two-country two money model is fitted with RC, so that international agents contain a fear of model misspecification, or pessimism. Ultimately, there lies two stochastic processes 1. The "approximating system" that preserves RE and 2. The "perturbed system" that incorporates a systematic sequence of worst-case shocks to the economy. These two models induce a selection problem to agents, and depending on the ability of the agent to select the correct data generating process (measured by DEP) will determine how pessimistic the agent is, where data-driven and simulation-driven pessimism are compared.

This chapter is organized as follows: Section 13 introduces the international DSGE model, Section 14 utilizes artefacts from the model into DEP methodology, Section 15 produces results from estimation and simulation, Section 16 concludes.

13 Model

In order to have a fruitful discussion about DEP, the Lucas (1982) international dynamic stochastic general equilibrium (DSGE) model embedded with Robust Control (RC) introduced in Shkrelja (Ch.2, 2014) is reproduced here for convenience,

$$\max_{\substack{\{c_{xt}, c^*_{xt},\\ c_{yt}, c^*_{yt},\\ \omega_{xt}, \omega^*_{xt},\\ \omega_{yt}, \omega^*_{yt},\\ \psi_{Mt}, \psi^*_{Mt},\\ \psi_{Nt}, \psi^*_{Nt}\}_{t=0}^{\infty}}} \min_{\{w_{t+1}\}_{t=0}^{\infty}} E_t \sum_{t=0}^{\infty} \beta^t \left\{ \phi \left[-\frac{(c_{xt}^{\theta} c_{yt}^{1-\theta})^{1-\gamma}}{1-\gamma} + \beta\overline{\theta} w_{t+1}^2 \right] + (1-\phi) \left[-\frac{(c_{xt}^{*\theta} c_{yt}^{*1-\theta})^{1-\gamma}}{1-\gamma} + \beta\overline{\theta^*} w_{t+1}^2 \right] \right\}$$

$$s.t. \quad \phi \sum_{t=0}^{\infty} \beta^t \left\{ c_{xt} + \frac{S_t P_t^*}{P_t} c_{yt} + \omega_{xt} e_t + \omega_{yt} e_t^* + \psi_{Mt} r_t + \psi_{Nt} r_t^* - \frac{P_{t-1}}{P_t} \omega_{xt-1} x_{t-1} - \frac{S_t P_{t-1}^*}{P_t} \omega_{yt-1} y_{t-1} \right.$$
$$\left. - \frac{\psi_{Mt-1} \Delta M_t}{P_t} - \frac{\psi_{Nt-1} S_t \Delta N_t}{P_t} - \omega_{xt-1} e_t - \omega_{yt-1} e_t^* - \psi_{Mt-1} r_t - \psi_{Nt-1} r_t^* \right\} = 0$$

$$(1-\ \phi)\sum_{t=0}^{\infty}\beta^{t}\left\{c_{xt}^{*}+\frac{S_{t}P_{t}^{*}}{P_{t}}c_{yt}^{*}+\omega_{xt}^{*}e_{t}+\omega_{yt}^{*}e_{t}^{*}+\psi_{Mt}^{*}r_{t}+\psi_{Nt}^{*}r_{t}^{*}-\frac{P_{t-1}}{P_{t}}\omega_{xt-1}^{*}x_{t-1}-\frac{S_{t}P_{t-1}^{*}}{P_{t}}\omega_{yt-1}^{*}y_{t-1}\right.$$
$$\left.-\frac{\psi_{Mt-1}^{*}\Delta M_{t}}{P_{t}}-\frac{\psi_{Nt-1}^{*}S_{t}\Delta N_{t}}{P_{t}}-\omega_{xt-1}^{*}e_{t}-\omega_{yt-1}^{*}e_{t}^{*}-\psi_{Mt-1}^{*}r_{t}-\psi_{Nt-1}^{*}r_{t}^{*}\right\}=0$$

$$\overbrace{m_{t}=P_{t}c_{xt}}^{CIA}\qquad\overbrace{\omega_{xt}+\omega_{xt}^{*}=1}^{Equity\ Shares}\qquad\overbrace{c_{xt}+c_{xt}^{*}=x_{t}}^{\text{Re}\,source\ Constraints}\tag{66}$$

$$n_{t}=P_{t}^{*}c_{yt}\qquad\omega_{yt}+\omega_{yt}^{*}=1\qquad c_{yt}+c_{yt}^{*}=y_{t}\tag{67}$$

$$m_{t}^{*}=P_{t}c_{xt}^{*}\qquad\psi_{Mt}+\psi_{Mt}^{*}=1\qquad m_{t}+m_{t}^{*}=M_{t}\tag{68}$$

$$n_{t}^{*}\ =P_{t}^{*}c_{yt}^{*}\qquad\psi_{Nt}+\psi_{Nt}^{*}=1\qquad n_{t}+n_{t}^{*}=N_{t}\tag{69}$$

Where c_{xt} and c_{yt} represent home good-x and good-y consumption respectively, m_t and n_t represent home holdings of home and foreign currency, ω_{xt} and ω_{yt} represent home output shares of good-x and good-y respectively valued at price e_t for ω_{xt} and e_t^* for ω_{yt} , ψ_{Mt} and ψ_{Nt} represent home holdings of home and foreign currency shares respectively valued at price r_t for ψ_{Mt} and r_t^* for ψ_{Nt}, P_t and S_t represent home prices and spot exchange rate respectively. Finally, all variables containing * are foreign variable counterparts to the preceding home variable list. Beginning with the objective function, domestic and foreign households optimize weighted average CRRA utility (consisting of domestic and foreign consumptions of goods x and y), so that β^t is discount factor, ϕ represents importance of country in decentralized world economy, $\overline{\theta}$and $\overline{\theta}^{*}$represent home and foreign pessimism respectively, w_{t+1} is malevolent nature's control variable, representing an intertemporal sequence of "worst-case" shocks that are used minimize household utility. Constraints to the objective function include home and foreign budget constraints, the first column of (66) − (69) represents CIA constraints, the second and third columns are adding up constraints. To close the model, the stochastic system pertaining to no systematic deviations, or the *approximating system*,is represented as,

$$x_{t+1} = (1-\rho_1)+\rho_1 x_t+\varepsilon^x_{t+1} \tag{70}$$

$$y_{t+1} = (1-\rho_2)+\rho_2 y_t+\varepsilon^y_{t+1} \tag{71}$$

$$M_{t+1} = (1-\rho_3)+\rho_3 M_t+\varepsilon^M_{t+1} \tag{72}$$

$$N_{t+1} = (1-\rho_4)+\rho_4 N_t+\varepsilon^N_{t+1} \tag{73}$$

where x_{t+1} is domestic output, y_{t+1} is foreign output, M_{t+1} is gross domestic money supply, N_{t+1} is gross foreign money supply. The stochastic system associated with systematic deviations, or the *perturbed system,* is represented as,

$$x_{t+1} = (1-\rho_1) + \rho_1 x_t + w_{t+1} + \varepsilon^x_{t+1} \quad (74)$$

$$y_{t+1} = (1-\rho_2) + \rho_2 y_t + w_{t+1} + \varepsilon^y_{t+1} \quad (75)$$

$$M_{t+1} = (1-\rho_3) + \rho_3 M_t + w_{t+1} + \varepsilon^M_{t+1} \quad (76)$$

$$N_{t+1} = (1-\rho_4) + \rho_4 N_t + w_{t+1} + \varepsilon^N_{t+1} \quad (77)$$

where w_{t+1} is represented as a systematic perturbation to an economic agents understanding of $(70)-(73)$. Now, as discussed in Shkrelja (Ch.2, 2014), out of the maximizing player's problem, log linearized first-order conditions (FOCs) in matrix form for both spot exchange rate, S_t and forward exchange rate, F_t can be derived. For purposes of this discussion, the approximating system $(70)-(73)$ can be represented log-linearized matrix-algebra form,

$$X_{t+1} = AX_t + C\varepsilon_{t+1} \quad (78)$$

where,

$$X_t = \begin{bmatrix} \widetilde{x}_t \\ \widetilde{y}_t \\ \widetilde{M}_t \\ \widetilde{N}_t \end{bmatrix}, \quad \varepsilon_{t+1} = \begin{bmatrix} \varepsilon_{t+1}^x \\ \varepsilon_{t+1}^y \\ \varepsilon_{t+1}^M \\ \varepsilon_{t+1}^N \end{bmatrix}, \quad A = \begin{bmatrix} \rho_1 & 0 & 0 & 0 \\ 0 & \rho_2 & 0 & 0 \\ 0 & 0 & \rho_3 & 0 \\ 0 & 0 & 0 & \rho_4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and $\widetilde{x}_t$, $\widetilde{y}_t$, $\widetilde{M}_t$, $\widetilde{N}_t$ are log-linearized counterparts to x_t, y_t, M_t, N_t respectively. Next, the optimal worst-case sequence (and its associated Riccatti equation for stabilization matrix P) chosen by the malevolent nature, w_{t+1} can be derived from the minimizing player's problem as in Shkrelja (2014b) which feeds back into the perturbed system (74) – (77) ultimately resulting in,

$$X_{t+1} = A^0 X_t + C\varepsilon_{t+1} \tag{79}$$

where $A^0 = \left(A + CK(\overline{\overline{\theta}})\right)$, which contains households' pessimism, $\overline{\overline{\theta}}$[19]. The purpose of the next section is to utilize (78) and (79) in calibrating pessimism via DEP methodology.

[19] Introduced from Shkrelja (2014b), "world-pessimism," $\overline{\overline{\theta}}$, is a weighted-average of home and foreign pessimism ($\overline{\theta}$ and $\overline{\theta}^*$ respectively) or, $\overline{\overline{\theta}} = \phi\overline{\theta} + (1-\phi)\,\overline{\theta}^*$. See the Technical Appendix Section 4.2.2 for more details.

14 Detection Error Probabilities

The intuition behind (78) and (79) play a role into the following DEP analysis. Both approximating and perturbed systems will essentially be "chosen" some fraction of the time, where depending on which model correctly generates the state of nature and that chosen by the economic agents, will determine to what percentage the agent is incorrect in deciphering the true state of nature. This will, indirectly, discipline the choice of pessimism used in the model from Section 2. To begin, the approximating stochastic system is specified as "Model A,"

$$X_{t+1} = AX_t + C\widehat{\varepsilon_{t+1}} \tag{80}$$

where (80) is essentially equivalent to (78) with the exception of $\widehat{\varepsilon_{t+1}}$, which are normally distributed error terms whose mean is implicitly perturbed by the malevolent nature's sequence of worst-case shocks, w_{t+1} or $\widehat{\varepsilon_{t+1}} \sim N\left(w_{t+1}, \sigma^2\right)$ where σ^2 is a constant variance[20]. Alternatively, the perturbed stochastic system will be defined as "Model B,"

$$X_{t+1} = A^0 X_t + C\varepsilon_{t+1} \tag{81}$$

where, again, (81) is equivalent to (79) where error terms are now normally distributed with a mean of zero or, $\varepsilon_{t+1} \sim N\left(0, \sigma^2\right)$ with constant variance σ^2. It is stressed that the discipline of pessimism is determined indirectly through (81) since A^0 is embedded with

[20] See Section 2.1 of Shkrelja (2014b) for futher details on perturbed error terms.

$\bar{\bar{\theta}}$.Both Model A and Model B are used in Log-Likelihood Ratio Tests (LRT), where through these tests pessimism can be indirectly calibrated. The intuition behind a LRT is that it is basically used to compare the fit of two models, where one of the models is a special case of the other model. The test, based primarily on the likelihood ratio, determines how likely a data set exists in one model over the other. In terms of this paper's analysis, (80) is a special case of (81) where $w_{t+1} = 0$ is assumed in the former. By adjusting $\bar{\bar{\theta}}$ the likelihood of which model exists given a set of data is manipulated to a degree, until a desired DEP is achieved.

14.1 Log-Likelihood Ratio Test with Model A

Initially, it is assumed that the malevolent natures sequence of worst-cast shocks are generated under Model A,

$$w_{t+1}^{A} = KX_{t+1}^{A} \tag{82}$$

where (82) is an FOC of the minimizing player's problem where $K = \left(\bar{\bar{\theta}} I_4 - C'PC\right)^{-1} C'PA$ and X_{t+1}^{A} is produced under Model A. Next, the innovations to (80) are defined as,

$$\begin{aligned} X_{t+1} &= AX_t + C\widehat{\varepsilon_{t+1}} \\ C\widehat{\varepsilon_{t+1}} &= X_{t+1} - AX_t \\ \widehat{\varepsilon_{t+1}} &= (C'C)^{-1} C' \left(X_{t+1} - AX_t\right) \end{aligned} \tag{83}$$

so that $\widehat{\varepsilon_{t+1}}$ is primarily a mapping of the discrepancy between a one-period in advance state of the economy, X_{t+1} and its dependency on the current state, AX_t. Similarly, the innovation to (81) are defined as,

$$\begin{aligned}
X_{t+1} &= A^0 X_t + C\varepsilon_{t+1} \\
C\varepsilon_{t+1} &= X_{t+1} - A^0 X_t \\
\varepsilon_{t+1} &= \left(C'C\right)^{-1} C' \left(X_{t+1} - A^0 X_t\right) \\
\varepsilon_{t+1} &= \left(C'C\right)^{-1} C' \left(X_{t+1} - AX_t\right) - \left(C'C\right)^{-1} C'CKX_t \qquad (84)
\end{aligned}$$

where $A^0 = (A + CK)$ was used. if the fact that $\widehat{\varepsilon_{t+1}} = \left(C'C\right)^{-1} C' \left(X_{t+1} - AX_t\right)$ and since $\left(C'C\right)^{-1} C'C = I_4$, the second element of the right-hand side of (84) can be equated to w_{t+1} so that (84) is ultimately consolidated to,

$$\varepsilon_{t+1} = \widehat{\varepsilon_{t+1}} - w_{t+1} \qquad (85)$$

so that ε_{t+1} in (85)accounts for w_{t+1}. Alternatively it can be easily seen, algebraically speaking, that the systematic error, w_{t+1}, in a linear combination with random error term, ε_{t+1}, forms the perturbed error term $\widehat{\varepsilon_{t+1}}$. Now that both (83) and (85) have been defined from (80) and (81) respectively, the log-likelihood tests can be constructed with respect to Model A and Model B as,

$$Log\ L_A = -\frac{1}{T}\sum_{t=0}^{T-1}\left\{n\log\sqrt{2\pi} + \frac{1}{2}\left(\widehat{\varepsilon_{t+1}}'\widehat{\varepsilon_{t+1}}\right)\right\} \tag{86}$$

$$Log\ L_B = -\frac{1}{T}\sum_{t=0}^{T-1}\left\{n\log\sqrt{2\pi} + \frac{1}{2}\left(\varepsilon_{t+1}'\varepsilon_{t+1}\right)\right\} \tag{87}$$

where $Log\ L_A$ is the log-likelihood associated with innovations constructed from Model A (as implied by the subscript A), $Log\ L_B$ is the log-likelihood associated with innovations constructed from Model B (as implied by the subscript B), and T is the total number of periods within a time-series data set ranging from time $t = 0, ..., T-1$. Using (86) and (87) the log-likelihood ratio test conditional on Model A can finally be constructed,

$$\begin{aligned}
r|A &= Log\ L_A - Log\ L_B \\
&= -\frac{1}{T}\sum_{t=0}^{T-1}\left\{n\log\sqrt{2\pi} + \frac{1}{2}\left(\widehat{\varepsilon_{t+1}}'\widehat{\varepsilon_{t+1}}\right)\right\} + \frac{1}{T}\sum_{t=0}^{T-1}\left\{n\log\sqrt{2\pi} + \frac{1}{2}\left(\varepsilon_{t+1}'\varepsilon_{t+1}\right)\right\} \\
&= \frac{1}{T}\sum_{t=0}^{T-1}\left\{\frac{1}{2}\left(\varepsilon_{t+1}'\varepsilon_{t+1}\right) - \frac{1}{2}\left(\widehat{\varepsilon_{t+1}}'\widehat{\varepsilon_{t+1}}\right)\right\} \\
&= \frac{1}{T}\sum_{t=0}^{T-1}\left\{\frac{1}{2}\left(\widehat{\varepsilon_{t+1}} - w_{t+1}^{A}\right)'\left(\widehat{\varepsilon_{t+1}} - w_{t+1}^{A}\right) - \frac{1}{2}\left(\widehat{\varepsilon_{t+1}}'\widehat{\varepsilon_{t+1}}\right)\right\} \\
&= \frac{1}{T}\sum_{t=0}^{T-1}\left\{\frac{1}{2}\left(\widehat{\varepsilon_{t+1}}'\widehat{\varepsilon_{t+1}} - \widehat{\varepsilon_{t+1}}'w_{t+1}^{A} - w_{t+1}^{A\prime}\widehat{\varepsilon_{t+1}} + w_{t+1}^{A\prime}w_{t+1}^{A}\right) - \frac{1}{2}\left(\widehat{\varepsilon_{t+1}}'\widehat{\varepsilon_{t+1}}\right)\right\}
\end{aligned}$$

hence,

$$r|A = \frac{1}{T}\sum_{t=0}^{T-1}\left\{\frac{1}{2}w_{t+1}^{A\prime}w_{t+1}^{A} - \frac{1}{2}w_{t+1}^{A\prime}\widehat{\varepsilon_{t+1}}\right\} \tag{88}$$

where $\varepsilon_{t+1} = \widehat{\varepsilon_{t+1}} - w_{t+1}$ was used and since Model A is assumed to be the true data generating process, malevolent nature's sequence of worst-case shocks can be reinterpreted as $w_{t+1} = w_{t+1}^{A}$. Basically, (88) is simulated through a large number of T observations assuming $w_{t+1}^{A} = KX_{t+1}^{A}$ and the number of times $r|A < 0$ is summed, resulting in a probability conditional on Model A or more formally[21],

$$p_A = \Pr\left(r|A < 0\right) \tag{89}$$

where p_A gives the probability associated with an economic agent choosing Model B, when in fact, the true data generating process is Model A. Thus, p_A denotes the fraction of time an economic agent incorrectly chooses Model B so that $1 - p_A$ specifies the fraction of time the agent correctly chooses Model A. The full DEP has not been fully crafted however, a similar exercise needs to be conducted for the log-likelihood with respect to Model B.

14.2 Log-Likelihood Ratio Test with Model B

A strategy similar to the exercise for constructing $r|A$ can be also be done for log-likelihood conditional on Model B instead, where now the malevolent nature's worst-case sequence is assumed to be generated from Model B,

[21] See Technical Appendix Section 5.4 for further details on simulation strategy for p_A.

$$w_{t+1}^{B} = KX_{t+1}^{B} \tag{90}$$

where X_{t+1}^{B} is produced from (81). Additionally, (86) and (87) can be used to construct the log-likelihood ratio test under Model B (following a similar procedure to constructing $r|A$),

$$r|B = -\frac{1}{T}\sum_{t=0}^{T-1}\left\{\frac{1}{2}w_{t+1}^{B\prime}w_{t+1}^{B} + \frac{1}{2}w_{t+1}^{B\prime}\widehat{\varepsilon_{t+1}}\right\} \tag{91}$$

from (91) a large sequence of T observations can be simulated from (90) and implemented into (91) and the occurrences of $r|B > 0$ are summed which results in a probability conditional on Model B being the true data generating process[22],

$$p_B = \Pr\left(r|B > 0\right) \tag{92}$$

where p_B gives the probability that an economic agent incorrectly selects Model A when the true data generating process is Model B. Alternatively, 1-p_B is the probability that the economic agent correctly selects Model B.

14.3 The Detection Error Probability

Now that p_A and p_B have been constructed, both are linearly averaged as in Hansen et. al (2002), Hansen and Sargent (2008), and Luo, Nie, and Young (2012) so that,

[22] See Technical Appendix Section 5.4 for further details on simulation strategy for p_B.

$$p\left(\overline{\overline{\theta}}\right) = \frac{1}{2}\left(p_A + p_B\right) \tag{93}$$

where $p\left(\overline{\overline{\theta}}\right)$ is the objective DEP that describes the probability international households will incorrectly choose the true data generating process with respect to the model introduced in Section 2. Alternatively, $1- p\left(\overline{\overline{\theta}}\right)$ implies the fraction of time that international households correctly choose the true data generating process. Values of $\overline{\overline{\theta}}$ are initially very large which in turn determine $p\left(\overline{\overline{\theta}}\right)$ and pessimism is then decreased until the desired DEP is reached. Theoretically speaking, there lies a positive relationship between $\overline{\overline{\theta}}$ and $p\left(\overline{\overline{\theta}}\right)$ where $p\left(\overline{\overline{\theta}}\right) \rightarrow 0.5$ as $\overline{\overline{\theta}} \rightarrow \infty$ meaning as pessimism becomes increasingly high, international households become increasingly worse at deciphering which stochastic process is the true data generating process, where agents incorrectly select the true state of nature 50% of the time. Using the theoretical excursion given through this section, both simulated and actual data values are applied to ultimately produce DEP from (93) and its implied pessimism value, $\overline{\overline{\theta}}$.

15 Calibrated Pessimism

15.1 Data-Based Pessimism

To generate a comparison between data-driven and simulated pessimism parameters, a brief overview of the data is necessary. The state vector X_t contains x_t designated as aggregate U.S. real GDP observations in millions of chained 2009 dollars (Federal Reserve Bank of St. Louis) in quarterly frequency from 1986:III to 2013:III; y_t is defined as U.K. real GDP in millions of pounds deflated by UK CPI (with base year 2010) in quarterly frequency from 1986:III to 2013:III; M_t is home currency designated as a time-series data vector containing aggregate U.S. Household Financial Assets and Currency observations (Federal Reserve Bank of St. Louis) in quarterly frequency from 1986:III to 2013:III; and N_t is foreign currency defined as U.K. Household Outstanding Holdings of Notes/Coin (Bank of England) observations in quarterly frequency from 1986:III to 2013:III. The data vector X_t is then applied to (88) and (91) in order to generate (89) and (92) respectively. Finally, both (89) and (92) are linearly averaged to generate (93) where a data-driven world pessimism value is generated with its associated DEP. Table 1 below gives calibrated $\overline{\overline{\theta}}$ values and their associated DEP values.

As $\overline{\overline{\theta}}$ increases, it can be seen that its associated DEP, $p\left(\overline{\overline{\theta}}\right)$ increases as well, where for large values of $\overline{\overline{\theta}}$ the DEP converges towards 0.5000. Within the context of the data and starting with low values of DEP, $p\left(500.0000\right) = 0.0000$ implies that agents are incorrectly choosing the true data generating process 0% of the time which alternatively implies that $1-\ p\left(500.0000\right)$ means households correctly choose the state of nature 100%. Intuitively, households are fully aware of the systematic bias within the economic environment and factor this perturbation within their expectations when making forecasts about future economic

Table 1: Data-based calibration of pessimism

$p(\bar{\bar{\theta}})$	$\bar{\bar{\theta}}$	$p(\bar{\bar{\theta}})$	$\bar{\bar{\theta}}$
0.0000	500.0000	0.4450	1,569.2000
0.0413	699.0000	0.4679	2,000.0000
0.1743	700.0000	0.4771	3,000.0000
0.2982	750.0000	0.4817	4,000.0000
0.3853	798.8450	0.4908	5,000.0000
0.3945	798.8800	0.4908	6,000.0000
0.3991	798.9200	0.4954	7,000.0000
0.4128	798.9500	0.4954	8,000.0000
0.4220	798.9900	0.5000	9,000.0000
0.4358	799.9800	0.5000	10,000.0000
0.4404	838.9940	0.5000	11,000.0000

*Note: Pessimism and DEP are in increasing order from the left to right.

values. On the other hand, for large values of pessimism where $\bar{\bar{\theta}} = 11,000.0000$, its associated DEP yields $p\,(11,000.0000) = 0.5000$ meaning households incorrectly choose the state of nature 50% of the time. It can easily be seen that this value also implies $1-p\,(11,000.0000)$, or households correctly choose the true data generating process 50% of the time. As the malevolent nature's systematic perturbations become more miniscule, it becomes increasingly difficult for the agent to decipher between an approximating stochastic system and its perturbed system counterpart. This difficulty translates as increased pessimism in households where there lies an equal chance in correctly and incorrectly realizing what is actually occurring within the economic environment.

15.2 Simulation-Based Pessimism

In this section, pessimism and associated DEP values are calibrated using simulations from systems (80) and (81) where the coefficient matrix A of both systems are estimated from the same data described in Section 4.1[23]. The resulting state vectors of both systems are then

[23] See Technical Appendix Section 7.2 for details on estimation procedure.

used to generate (93) where Table 2 lists calibrated values of both pessimism and related DEP values[24].

Table 2: Simulation-based calibration of pessimism

$p(\bar{\bar{\theta}})$	$\bar{\bar{\theta}}$	$p(\bar{\bar{\theta}})$	$\bar{\bar{\theta}}$
0.0000	798.8450	0.3539	4,000.0000
0.0001	798.8560	0.3850	5,000.0000
0.0002	798.8800	0.4038	6,000.0000
0.0005	798.9200	0.4200	7,000.0000
0.0015	798.9500	0.4130	8,000.0000
0.0025	798.9900	0.4379	9,000.0000
0.0050	799.9800	0.4462	10,000.0000
0.0100	838.9943	0.4519	11,000.0000
0.1000	1,569.2000	0.4562	12,000.0000
0.1838	2,000.0000	0.4603	13,000.0000
0.2974	3,000.0000	0.4639	14,000.0000

*Note: Pessimism and DEP are in increasing order from the left to right.

Same principles apply to Table 2 as in Table 1 where as increasingly large values of pessimism lead associated DEP values converging towards 0.5000. Here, simulated pessimism of $\bar{\bar{\theta}} = 798.8450$ results in $p\,(798.8450) = 0.0000$ implying as before that agents incorrectly choose the true data generating process 0% of the time implying agents correctly choose the true data generating process 100% of the time. Alternatively, simulated pessimism of $\bar{\bar{\theta}} = 14,000.0000$ implies $p\,(14,000.0000) = 0.4693$ where households incorrectly choose the state of nature 46.93% of the time and $1 - p\,(14,000.0000)$ implies agents correctly choose 53.07% of the time.

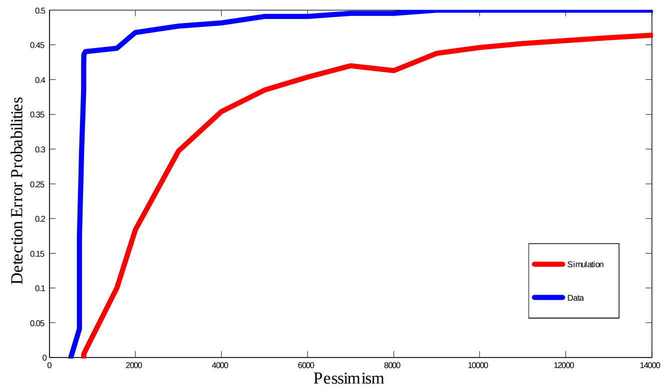
To clearly see the difference between data-driven and simulated pessimism values, Figure 1 plots values of Table 1 and Table 2 within the same diagram. The blue line indicates values generated from the data and the red-line from simulation and it can be seen that as pessimism continues into larger values, both lines converge towards 0.5000. Clearly, for

[24]See Technical Appendix Section 5.4 for simulation strategy.

each $\overline{\overline{\theta}}$ value, its associated $p\left(\overline{\overline{\theta}}\right)$ is understated in the simulation when compared to their data-driven counterparts, essentially inflating the selection ability of agents. This implies that agents within the simulation can better decipher between states of nature than what the data suggests, for all values of pessimism. For low values of pessimism where $p\left(\overline{\overline{\theta}}\right) < 0.1000$, the simulated DEP's perform rather well against the data as there is not a large discrepancy present, however shortly beyond the $p\left(\overline{\overline{\theta}}\right) = 0.1000$ threshold, there lies a large distance between simulated and data-based DEP's, until high values of pessimism are reached where both lines eventually converge.

This "inflated-selction ability" of the simulated model over its data counterpart suggests that the behavioral nature of international economic agents is not well described by the model in Section 2 for a broad range of pessimism, especially those for $p\left(\overline{\overline{\theta}}\right) > 0.1000$. The data suggests that for values of pessimism, people are actually not very good at selecting between what they think and what actually is going on (which, through natural observation, is not far from truth), when comparing to simulated pessimism. Through a simple thought experiment, if the behavioral nature of international agents can be accounted for, estimators produced from FOC artefacts of Section 2's model (when estimated under some statistical procedure) could produce results closer to reality. It must be noted that both $\overline{\overline{\theta}}$ and $p\left(\overline{\overline{\theta}}\right)$ are context-specific as far as what data is used and how the coefficient matrix A is estimated. Changing the methodology alters the outcome of both pessimism and DEP, suggesting a different stochastic representation of (70) $-$ (73) or estimation procedure to A could dampen the behavioral discrepancy between data and simulation. Additionally, it is assumed that "world-

Figure 2: Simulated vs. Data-Based Pessimism



pessimism" or $\overline{\overline{\theta}}$ has been assumed to be a weighted average of domestic (US) pessimism, $\overline{\theta}$ and foreign (UK) pessimism, $\overline{\theta}^*$ so that both are essentially homogenous ($\overline{\theta} = \overline{\theta}^*$), when calibrating $\overline{\overline{\theta}}$. Perhaps separating each term into non-homogenous components so that $\overline{\theta} \neq \overline{\theta}^*$ would be more appropriate in trying to match simulated pessimism with what data suggests.

16 Conclusion

This work has strived to capture the behavioral nature of international economic agents with respect to pessimism. Through a two-country, two-money international DSGE model incorporated with RC, two stochastic systems (approximating and perturbed systems) were assumed exist causing a selection problem for households. Through detection error probability estimation methodology, the ability of agents to realize which state of nature is the true data generating process is reflected in how pessimistic an agent is so that the lower an agents' ability to correctly select a true data generating process, the higher the agents' pessimism. To account for this behavioral phenomenon empirically, US and UK data was taken and applied to DEP methodology to generate data and simulation-based pessimism values and their associated DEP's. The simulated model inflated households' ability to select appropriate models when compared to the data, so the stylized international economy perhaps is not the most effective in trying to capture context-specific macroeconomic behaviors.

Other issues associated with the discrepancy between simulated and data-based pessimism is the stochastic structure of approximating and perturbed systems, estimation methodology of the coefficient matrix A, and the homogeneity assumption of $\overline{\overline{\theta}}$.These avenues for future research could potentially enhance proximity of simulated pessimism to its data counterpart, and bring behavioral macroeconomics closer to explaining Animal Spirits.

APPENDIX

17 Elements

- Home Country Utility Function

$$u(c_{xt}, c_{yt}) = \frac{(c_{xt}^{\theta} c_{yt}^{1-\theta})^{1-\gamma}}{1-\gamma} \tag{94}$$

- Foreign Country Utility Function

$$u(c_{xt}^{*}, c_{yt}^{*}) = \frac{(c_{xt}^{*\theta} c_{yt}^{*1-\theta})^{1-\gamma}}{1-\gamma} \tag{95}$$

- Home Country Budget Constraint

$$\begin{aligned} c_{xt} + \frac{S_t P_t^{*}}{P_t} c_{yt} + \omega_{xt} e_t + \omega_{yt} e_t^{*} + \psi_{Mt} r_t + \psi_{Nt} r_t^{*} = \frac{P_{t-1}}{P_t} \omega_{xt-1} x_{t-1} \\ + \frac{S_t P_{t-1}^{*}}{P_t} \omega_{yt-1} y_{t-1} + \frac{\psi_{Mt-1} \Delta M_t}{P_t} + \frac{\psi_{Nt-1} S_t \Delta N_t}{P_t} + \omega_{xt-1} e_t + \omega_{yt-1} e_t^{*} \\ + \psi_{Mt-1} r_t + \psi_{Nt-1} r_t^{*} \end{aligned} \tag{96}$$

- Foreign Country Budget Constraint

$$\begin{aligned} c_{xt}^{*} + \frac{S_t P_t^{*}}{P_t} c_{yt}^{*} + \omega_{xt}^{*} e_t + \omega_{yt}^{*} e_t^{*} + \psi_{Mt}^{*} r_t + \psi_{Nt}^{*} r_t^{*} = \frac{P_{t-1}}{P_t} \omega_{xt-1}^{*} x_{t-1} \\ + \frac{S_t P_{t-1}^{*}}{P_t} \omega_{yt-1}^{*} y_{t-1} + \frac{\psi_{Mt-1}^{*} \Delta M_t}{P_t} + \frac{\psi_{Nt-1}^{*} S_t \Delta N_t}{P_t} + \omega_{xt-1}^{*} e_t + \omega_{yt-1}^{*} e_t^{*} \\ + \psi_{Mt-1}^{*} r_t + \psi_{Nt-1}^{*} r_t^{*} \end{aligned} \tag{97}$$

- Cash-in-Advance Constraints

$$m_t \geq P_t c_{xt}$$

$$n_t \geq P_t^* c_{yt} \quad (98)$$

$$m_t^* \geq P_t c_{xt}^* \quad (99)$$

$$n_t^* \geq P_t^* c_{yt}^* \quad (100)$$

- Adding-Up Constraints

$$\omega_{xt} + \omega_{xt}^* = 1 \quad (101)$$

$$\omega_{yt} + \omega_{yt}^* = 1 \quad (102)$$

$$\psi_{Mt} + \psi_{Mt}^* = 1 \quad (103)$$

$$\psi_{Nt} + \psi_{Nt}^* = 1 \quad (104)$$

$$c_{xt} + c_{xt}^* = x_t \quad (105)$$

$$c_{yt} + c_{yt}^* = y_t \quad (106)$$

$$m_t + m_t^* = M_t \quad (107)$$

$$n_t + n_t^* = N_t \quad (108)$$

- Stochastic Processes

$$x_{t+1} = (1 - \rho_1) + \rho_1 x_t + \varepsilon_{t+1}^x \quad (109)$$

$$y_{t+1} = (1 - \rho_2) + \rho_2 y_t + \varepsilon_{t+1}^y \quad (110)$$

$$M_{t+1} = (1 - \rho_3) + \rho_3 M_t + \varepsilon_{t+1}^M \quad (111)$$

$$N_{t+1} = (1 - \rho_4) + \rho_4 N_t + \varepsilon_{t+1}^N \quad (112)$$

18 Equilibrium

18.1 Centralized Economy

For simplicity, the log utility functions of (94) and (95) respectively can be defined as

$$\log u(c_{xt}, c_{yt}) = \theta \log(c_{xt}) + (1-\theta)\log(c_{yt}) \tag{113}$$

$$\log u(c^*_{xt}, c^*_{yt}) = \theta \log(c^*_{xt}) + (1-\theta)\log(c^*_{yt}) \tag{114}$$

The central planner maximizes a weighted average of (113) and (114) subject to (105) and (106).

$$\max_{\{c_{xt}, c_{yt}, c^*_{xt}, c^*_{yt}\}_{t=0}^{\infty}} E_t \sum_{t=0}^{\infty} \beta^t \left\{\phi\left[\theta \log(c_{xt}) + (1-\theta)\log(c_{yt})\right] + (1-\phi)\left[\theta \log(c^*_{xt}) + (1-\theta)\log(c^*_{yt})\right]\right\}$$

$$s.t.\ c_{xt} + c^*_{xt} = x_t$$
$$c_{yt} + c^*_{yt} = y_t$$

The central planner's problem can be reduced to a static problem (where $t = 0$)

$$\max_{c_x, c_y, c^*_x, c^*_y} \phi\left[\theta \log(c_x) + (1-\theta)\log(c_y)\right] + (1-\phi)\left[\theta \log(c^*_x) + (1-\theta)\log(c^*_y)\right]$$

$$s.t.\ c_x + c^*_x = x$$
$$c_y + c^*_y = y$$

Inserting (106) and (107) into the object function yields an unconstrained maximization problem

$$\max_{c_x, c_y} \phi \left[\theta \log(c_x) + (1-\theta)\log(c_y)\right] + (1-\phi)\left[\theta \log(c_x - x) + (1-\theta)\log(c_y - y)\right]$$

Optimizing the unconstrained objective function w.r.t control variables yields F.O.C's

$$\frac{\partial}{\partial c_x} = 0 : \frac{\phi\theta}{c_x} - \frac{(1-\phi)\,\theta}{x - c_x} = 0 \tag{115}$$

$$\frac{\partial}{\partial c_y} = 0 : \frac{\phi(1-\theta)}{c_y} - \frac{(1-\phi)\,(1-\theta)}{y - c_y} = 0 \tag{116}$$

Using $(115) - (116)$,$(106) - (107)$, and allowing time to evolve the pareto efficient allocations become

$$c_{xt} = (1-\phi)\, x_t \tag{117}$$

$$c_{yt} = (1-\phi)\, y_t \tag{118}$$

$$c^*_{xt} = \phi x_t \tag{119}$$

$$c^*_{yt} = \phi y_t \tag{120}$$

If both countries are weighted equally by the central planner so that $\phi = \frac{1}{2}$ then this results in a perfect sharing pareto efficient allocation

$$
\begin{aligned}
c_{xt} &= \frac{x_t}{2} && (121)\\
c_{yt} &= \frac{y_t}{2} && (122)\\
c^*_{xt} &= \frac{x_t}{2} && (123)\\
c^*_{yt} &= \frac{y_t}{2} && (124)
\end{aligned}
$$

19 Decentralized Economy

19.1 Home Country Decentralized Economy

Home country maximizes (113) subject to (96)

$$\max_{\{c_{xt},c_{yt},\omega_{xt},\omega_{yt},\psi_{Mt},\psi_{Nt}\}_{t=0}^{\infty}} E_t \sum_{t=0}^{\infty} \beta^t \left\{\theta \log(c_{xt}) + (1-\theta)\log(c_{yt})\right\}$$

$$
\begin{aligned}
s.t.\quad & c_{xt} + \frac{S_t P_t^*}{P_t} c_{yt} + \omega_{xt} e_t + \omega_{yt} e_t^* + \psi_{Mt} r_t + \psi_{Nt} r_t^* = \frac{P_{t-1}}{P_t}\omega_{xt-1} x_{t-1} \\
& + \frac{S_t P_{t-1}^*}{P_t}\omega_{yt-1} y_{t-1} + \frac{\psi_{Mt-1}\Delta M_t}{P_t} + \frac{\psi_{Nt-1} S_t \Delta N_t}{P_t} + \omega_{xt-1} e_t + \omega_{yt-1} e_t^* \\
& + \psi_{Mt-1} r_t + \psi_{Nt-1} r_t^*
\end{aligned}
$$

The decentralized problem can be reduced to static form (where $t = 0$)

$$\max_{c_x,c_y,\omega_x,\omega_y,\psi_M,\psi_N} [\theta \log(c_x) + (1-\theta)\log(c_y)]$$

$$s.t.\qquad c_x + \frac{SP^*}{P}c_y + \omega_x e + \omega_y e^* + \psi_M r + \psi_N r^* = \omega_x x + \frac{SP^*}{P}\omega_y y + \omega_x e + \omega_y e^* + \psi_M r + \psi_N r^*$$

To "peg" a decentralized allocation that matches the centralized allocation, set $\omega_x = \omega_y =$

$\psi_M = \psi_N = \frac{1}{2}$ where inserting these values into (96) for $t = 0$, the problem further reduces to

$$\max_{c_x, c_y}[\theta \log(c_x) + (1-\theta)\log(c_y)] \tag{125}$$

$$s.t. \qquad c_x + \frac{SP^*}{P}c_y = \frac{1}{2}\left(x + \frac{SP^*}{P}y\right) \tag{126}$$

Inserting (126) into (125) the unconstrained maximization problem becomes

$$\max_{c_y}\left[\theta \log\left(\frac{1}{2}\left(x + \frac{SP^*}{P}y\right) - \frac{SP^*}{P}c_y\right) + (1-\theta)\log(c_y)\right]$$

$$\frac{\partial}{\partial c_y} = 0 : \frac{-(\frac{SP^*}{P})\theta}{c_x} + \frac{(1-\theta)}{c_y} = 0 \tag{127}$$

Rearranging (127), applying to (126),and redefining $\frac{SP^*}{P} = q$ (real exchange rate) the resulting Home demand functions are

$$c_x = \frac{\theta\,(x + qy)}{2} \tag{128}$$

$$c_y = \frac{\theta\,(x + qy)\,(1-\theta)}{2q} \tag{129}$$

19.2 Foreign Country Decentralized Economy

Foreign country maximizes (114) subject to (97)

$$\max_{\left\{c^*_{xt}, c^*_{yt}, \omega^*_{xt}, \omega^*_{yt}, \psi^*_{Mt}, \psi^*_{Nt}\right\}_{t=0}^{\infty}} E_t \sum_{t=0}^{\infty} \beta^t \left\{\theta \log(c^*_{xt}) + (1-\theta)\log(c^*_{yt})\right\}$$

$$s.t. \quad c^*_{xt} + \frac{S_t P^*_t}{P_t} c^*_{yt} + \omega^*_{xt} e_t + \omega^*_{yt} e^*_t + \psi^*_{Mt} r_t + \psi^*_{Nt} r^*_t = \frac{P_{t-1}}{P_t} \omega^*_{xt-1} x_{t-1}$$
$$+ \frac{S_t P^*_{t-1}}{P_t} \omega^*_{yt-1} y_{t-1} + \frac{\psi^*_{Mt-1} \Delta M_t}{P_t} + \frac{\psi^*_{Nt-1} S_t \Delta N_t}{P_t} + \omega^*_{xt-1} e_t + \omega^*_{yt-1} e^*_t$$
$$+ \psi^*_{Mt-1} r_t + \psi^*_{Nt-1} r^*_t$$

The decentralized problem can be reduced to static form (where $t = 0$)

$$\max_{c^*_x, c^*_y, \omega^*_x, \omega^*_y, \psi^*_M, \psi^*_N} [\theta \log(c^*_x) + (1-\theta) \log(c^*_y)]$$

$$s.t. \qquad c^*_x + \frac{SP^*}{P} c^*_y + \omega^*_x e + \omega^*_y e^* + \psi^*_M r + \psi^*_N r^* = \omega^*_x x + \frac{SP^*}{P} \omega^*_y y + \omega^*_x e + \omega^*_y e^* + \psi^*_M r + \psi^*_N r^*$$

To "peg" a decentralized allocation that matches the centralized allocation, set $\omega^*_x = \omega^*_y = \frac{1}{2} = \psi^*_M = \frac{1}{2} = \psi^*_N = \frac{1}{2}$ where inserting these values into (97) for $t = 0$, the problem further reduces to

$$\max_{c^*_x, c^*_y} [\theta \log(c^*_x) + (1-\theta) \log(c^*_y)] \tag{130}$$

$$s.t. \qquad c^*_x + \frac{SP^*}{P} c^*_y = \frac{1}{2} \left(x + \frac{SP^*}{P} y \right) \tag{131}$$

Inserting (131) into (130) the unconstrained maximization problem becomes

$$\max_{c^*_y} \left[\theta \log \left(\frac{1}{2} \left(x + \frac{SP^*}{P} y \right) - \frac{SP^*}{P} c^*_y \right) + (1-\theta) \log(c^*_y) \right]$$

$$\frac{\partial}{\partial c^*_y} = 0 : \frac{-(\frac{SP^*}{P})\theta}{c^*_x} + \frac{(1-\theta)}{c^*_y} = 0 \tag{132}$$

Rearranging (132), applying to (131), and redefining $\frac{SP^*}{P} = q$ (real exchange rate) the resulting Foreign demand functions are

$$c_x^* = \frac{\theta(x+qy)}{2} \tag{133}$$

$$c_y^* = \frac{\theta(x+qy)(1-\theta)}{2q} \tag{134}$$

19.3 World Decentralized Equilibrium

The static version of (101) – (106) necessary for world economy equilibrium are

$$\omega_x + \omega_x^* = 1 \tag{135}$$

$$\omega_y + \omega_y^* = 1 \tag{136}$$

$$\psi_M + \psi_M^* = 1 \tag{137}$$

$$\psi_N + \psi_N^* = 1 \tag{138}$$

$$c_x + c_x^* = x \tag{139}$$

$$c_y + c_y^* = y \tag{140}$$

using (128) – (129) with (139), or (133) – (134) with (140), and allowing time to evolve yields the equilibrium real exchange rate associated with decentralized economy

$$\frac{S_t P_t^*}{P_t} = q_t = \frac{x_t(1-\theta)}{y_t\theta} \tag{141}$$

Finally, Constraints (135) – (140) are satisfied due to $\omega_x = \omega_x^* = \omega_y = \omega_y^* = \psi_M = \psi_M^* = \psi_N = \psi_N^* = \frac{1}{2}$ (perfect risk pooling equilibrium). Using (141), (128) – (129) (133) – (134), (139) – (140), and allowing time to evolve the resulting decentralized world equilibrium allocation is

$$c_{xt} = \frac{x_t}{2} \tag{142}$$

$$c_{yt} = \frac{y_t}{2} \tag{143}$$

$$c^*_{xt} = \frac{x_t}{2} \tag{144}$$

$$c^*_{yt} = \frac{y_t}{2} \tag{145}$$

20 Model

20.1 Inducing Malevolent Nature

Maximizing player believes state of the world to evolve according to the approximating system of stochastic equations, restated for convenience

$$\begin{aligned} x_{t+1} &= (1-\rho_1) + \rho_1 x_t + \varepsilon^x_{t+1} \\ y_{t+1} &= (1-\rho_2) + \rho_2 y_t + \varepsilon^y_{t+1} \\ M_{t+1} &= (1-\rho_3) + \rho_3 M_t + \varepsilon^M_{t+1} \\ N_{t+1} &= (1-\rho_4) + \rho_4 N_t + \varepsilon^N_{t+1} \end{aligned}$$

where the maximizing player believes the error terms are distributed according to

$$\varepsilon^j_t \sim N(0, \sigma^2_j) \ for \ j = x, y, M, N \tag{146}$$

where σ^2_j is constant variance term for $j = x, y, M, N$
The minimizing player perturbs approximating system through error terms

$$x_{t+1} = (1-\rho_1) + \rho_1 x_t + \widetilde{\varepsilon^x_{t+1}} \tag{147}$$
$$y_{t+1} = (1-\rho_2) + \rho_2 y_t + \widetilde{\varepsilon^y_{t+1}} \tag{148}$$
$$M_{t+1} = (1-\rho_3) + \rho_3 M_t + \widetilde{\varepsilon^M_{t+1}} \tag{149}$$
$$N_{t+1} = (1-\rho_4) + \rho_4 N_t + \widetilde{\varepsilon^N_{t+1}} \tag{150}$$

where $\widetilde{\varepsilon^j_t}$ for $j = x, y, M, N$ represents perturbed error terms distributed as

$$\widetilde{\varepsilon^j_t} \sim N(w_{t+1}, \sigma_j^2) \ for \ j = x, y, M, N \tag{151}$$
$$\widetilde{\varepsilon^j_t} - w_{t+1} \sim N(0, \sigma_j^2) \ for \ j = x, y, M, N \tag{152}$$
$$\varepsilon^j_t = \widetilde{\varepsilon^j_t} - w_{t+1} \tag{153}$$
$$\widetilde{\varepsilon^j_t} = \varepsilon^j_t + w_{t+1} \tag{154}$$

inserting (154) for $j = x, y, M, N$ into (147)−(150) yields the perturbed stochastic system

$$x_{t+1} = (1-\rho_1) + \rho_1 x_t + w_{t+1} + \varepsilon^x_{t+1} \tag{155}$$
$$y_{t+1} = (1-\rho_2) + \rho_2 y_t + w_{t+1} + \varepsilon^y_{t+1} \tag{156}$$
$$M_{t+1} = (1-\rho_3) + \rho_3 M_t + w_{t+1} + \varepsilon^M_{t+1} \tag{157}$$
$$N_{t+1} = (1-\rho_4) + \rho_4 N_t + w_{t+1} + \varepsilon^N_{t+1} \tag{158}$$

20.2 Max-Min Optimization Problem

- Objective Function and Budget Constraints

$$\max_{\substack{\{c_{xt}, c^*_{xt},\\ c_{yt}, c^*_{yt},\\ \omega_{xt}, \omega^*_{xt},\\ \omega_{yt}, \omega^*_{yt},\\ \psi_{Mt}, \psi^*_{Mt},\\ \psi_{Nt}, \psi_{Nt}\}_{t=0}^{\infty}}} \min_{\{w_{t+1}\}_{t=0}^{\infty}} E_t \sum_{t=0}^{\infty} \beta^t \left\{ \phi \left[-\frac{(c_{xt}^{\theta} c_{yt}^{1-\theta})^{1-\gamma}}{1-\gamma} + \beta\overline{\theta} w_{t+1}^2 \right] + (1-\phi) \left[-\frac{(c_{xt}^{*\theta} c_{yt}^{*1-\theta})^{1-\gamma}}{1-\gamma} + \beta\overline{\theta^*} w_{t+1}^2 \right] \right\}$$

$$\begin{aligned} s.t. \quad \phi \sum_{t=0}^{\infty} \beta^t \ \Bigg\{ & c_{xt} + \frac{S_t P_t^*}{P_t} c_{yt} + \omega_{xt} e_t + \omega_{yt} e_t^* + \psi_{Mt} r_t + \psi_{Nt} r_t^* - \frac{P_{t-1}}{P_t} \omega_{xt-1} x_{t-1} - \frac{S_t P_{t-1}^*}{P_t} \omega_{yt-1} y_{t-1} \\ & - \frac{\psi_{Mt-1} \Delta M_t}{P_t} - \frac{\psi_{Nt-1} S_t \Delta N_t}{P_t} - \omega_{xt-1} e_t - \omega_{yt-1} e_t^* - \psi_{Mt-1} r_t - \psi_{Nt-1} r_t^* \Bigg\} = 0 \end{aligned}$$

$$(1-\ \phi)\sum_{t=0}^{\infty}\beta^t\left\{c^*_{xt}+\frac{S_tP^*_t}{P_t}c^*_{yt}+\omega^*_{xt}e_t+\omega^*_{yt}e^*_t+\psi^*_{Mt}r_t+\psi^*_{Nt}r^*_t-\frac{P_{t-1}}{P_t}\omega^*_{xt-1}x_{t-1}-\frac{S_tP^*_{t-1}}{P_t}\omega^*_{yt-1}y_{t-1}\right.$$
$$\left.-\frac{\psi^*_{Mt-1}\Delta M_t}{P_t}-\frac{\psi^*_{Nt-1}S_t\Delta N_t}{P_t}-\omega^*_{xt-1}e_t-\omega^*_{yt-1}e^*_t-\psi^*_{Mt-1}r_t-\psi^*_{Nt-1}r^*_t\right\}=0$$

where $\overline{\theta}$and $\overline{\theta^*}$represent Home and Foreign pessimism respectively, w_{t+1} is malevolent nature's control variable used to perturb (109) – (112), and ϕ represents importance of country in decentralized world economy. Remaining constraints are reproduced below for convenience

- Cash-in-Advance Constraints

$$m_t = P_tc_{xt} \tag{159}$$
$$n_t = P^*_tc_{yt} \tag{160}$$
$$m^*_t = P_tc^*_{xt} \tag{161}$$
$$n^*_t = P^*_tc^*_{yt} \tag{162}$$

note that CIA constraints bind in equilibrium.

- Approximating Stochastic Processes

$$x_{t+1} = (1-\rho_1)+\rho_1x_t+\varepsilon^x_{t+1}$$
$$y_{t+1} = (1-\rho_2)+\rho_2y_t+\varepsilon^y_{t+1}$$
$$M_{t+1} = (1-\rho_3)+\rho_3M_t+\varepsilon^M_{t+1}$$
$$N_{t+1} = (1-\rho_4)+\rho_4N_t+\varepsilon^N_{t+1}$$

- Perturbed Stochastic Processes

$$
\begin{aligned}
x_{t+1} &= (1-\rho_1) + \rho_1 x_t + w_{t+1} + \varepsilon^x_{t+1} \\
y_{t+1} &= (1-\rho_2) + \rho_2 y_t + w_{t+1} + \varepsilon^y_{t+1} \\
M_{t+1} &= (1-\rho_3) + \rho_3 M_t + w_{t+1} + \varepsilon^M_{t+1} \\
N_{t+1} &= (1-\rho_4) + \rho_4 N_t + w_{t+1} + \varepsilon^N_{t+1}
\end{aligned}
$$

- Adding-Up Constraints

$$
\begin{aligned}
\omega_{xt} + \omega^*_{xt} &= 1 \\
\omega_{yt} + \omega^*_{yt} &= 1 \\
\psi_{Mt} + \psi^*_{Mt} &= 1 \\
\psi_{Nt} + \psi^*_{Nt} &= 1 \\
c_{xt} + c^*_{xt} &= x_t \\
c_{yt} + c^*_{yt} &= y_t \\
m_t + m^*_t &= M_t \\
n_t + n^*_t &= N_t
\end{aligned}
$$

20.2.1 Maximizing Player Chooses First

The optimization problem is broken into two components with maximizing player choosing first

$$\max_{\substack{\{c_{xt}, c^*_{xt}, \\ c_{yt}, c^*_{yt}, \\ \omega_{xt}, \omega^*_{xt}, \\ \omega_{yt}, \omega^*_{yt}, \\ \psi_{Mt}, \psi^*_{Mt}, \\ \psi_{Nt}, \psi_{Nt}\}_{t=0}^{\infty}}} \quad - E_t \sum_{t=0}^{\infty} \beta^t \left\{ \phi \left[\frac{(c_{xt}^{\theta} c_{yt}^{1-\theta})^{1-\gamma}}{1-\gamma} \right] + (1-\phi) \left[\frac{(c_{xt}^{*\theta} c_{yt}^{*1-\theta})^{1-\gamma}}{1-\gamma} \right] \right\}$$

$$s.t. \quad \phi \sum_{t=0}^{\infty} \beta^t \left\{ c_{xt} + \frac{S_t P_t^*}{P_t} c_{yt} + \omega_{xt} e_t + \omega_{yt} e_t^* + \psi_{Mt} r_t + \psi_{Nt} r_t^* - \frac{P_{t-1}}{P_t} \omega_{xt-1} x_{t-1} - \frac{S_t P_{t-1}^*}{P_t} \omega_{yt-1} y_{t-1} \right.$$
$$\left. - \frac{\psi_{Mt-1} \Delta M_t}{P_t} - \frac{\psi_{Nt-1} S_t \Delta N_t}{P_t} - \omega_{xt-1} e_t - \omega_{yt-1} e_t^* - \psi_{Mt-1} r_t - \psi_{Nt-1} r_t^* \right\} = 0$$

$$(1-\ \phi) \sum_{t=0}^{\infty} \beta^t \left\{ c_{xt}^* + \frac{S_t P_t^*}{P_t} c_{yt}^* + \omega_{xt}^* e_t + \omega_{yt}^* e_t^* + \psi_{Mt}^* r_t + \psi_{Nt}^* r_t^* - \frac{P_{t-1}}{P_t} \omega_{xt-1}^* x_{t-1} - \frac{S_t P_{t-1}^*}{P_t} \omega_{yt-1}^* y_{t-1} \right.$$
$$\left. - \frac{\psi_{Mt-1}^* \Delta M_t}{P_t} - \frac{\psi_{Nt-1}^* S_t \Delta N_t}{P_t} - \omega_{xt-1}^* e_t - \omega_{yt-1}^* e_t^* - \psi_{Mt-1}^* r_t - \psi_{Nt-1}^* r_t^* \right\} = 0$$

where (159)−(162), (101)−(108), and (109)−(112) completes maximizing player problem specification. Forming the lagrangian,

$$
\begin{aligned}
\pounds = E_t \sum_{t=0}^{\infty} \beta^t \Bigg\{ \phi \Bigg[\frac{(c_{xt}^{\theta} c_{yt}^{1-\theta})^{1-\gamma}}{1-\gamma} - \lambda_t^1 \Bigg(c_{xt} + \frac{S_t P_t^*}{P_t} c_{yt} + \omega_{xt} e_t + \omega_{yt} e_t^* + \psi_{Mt} r_t + \psi_{Nt} r_t^* - \frac{P_{t-1}}{P_t} \omega_{xt-1} x_{t-1} \\
- \frac{S_t P_{t-1}^*}{P_t} \omega_{yt-1} y_{t-1} - \frac{\psi_{Mt-1} \Delta M_t}{P_t} - \frac{\psi_{Nt-1} S_t \Delta N_t}{P_t} - \omega_{xt-1} e_t \\
- \omega_{yt-1} e_t^* - \psi_{Mt-1} r_t - \psi_{Nt-1} r_t^* \Bigg) \Bigg] \\
+ (1-\phi) \Bigg[\frac{(c_{xt}^{*\theta} c_{yt}^{*1-\theta})^{1-\gamma}}{1-\gamma} - \lambda_t^2 \Bigg(c_{xt}^* + \frac{S_t P_t^*}{P_t} c_{yt}^* + \omega_{xt}^* e_t + \omega_{yt}^* e_t^* + \psi_{Mt}^* r_t + \psi_{Nt}^* r_t^* - \frac{P_{t-1}}{P_t} \omega_{xt-1}^* x_{t-1} \\
- \frac{S_t P_{t-1}^*}{P_t} \omega_{yt-1}^* y_{t-1} - \frac{\psi_{Mt-1}^* \Delta M_t}{P_t} - \frac{\psi_{Nt-1}^* S_t \Delta N_t}{P_t} - \omega_{xt-1}^* e_t \\
- \omega_{yt-1}^* e_t^* - \psi_{Mt-1}^* r_t - \psi_{Nt-1}^* r_t^* \Bigg) \Bigg] \Bigg\}
\end{aligned}
$$

$$\frac{\partial \pounds}{\partial c_{xt}} = 0 : \left(c_{xt}^{\theta} c_{yt}^{1-\theta}\right)^{-\gamma} \theta c_{xt}^{\theta-1} c_{yt}^{1-\theta} - \lambda_t^1 = 0 \tag{163}$$

$$\frac{\partial \pounds}{\partial c_{yt}} = 0 : \left(c_{xt}^{\theta} c_{yt}^{1-\theta}\right)^{-\gamma} (1-\theta)\, c_{xt}^{\theta} c_{yt}^{-\theta} - \lambda_t^1 \frac{S_t P_t^*}{P_t} = 0 \tag{164}$$

$$\frac{\partial \pounds}{\partial \omega_{xt}} = 0 : -\lambda_t^1 e_t + \beta E_t \left(\lambda_{t+1}^1 e_{t+1} + \lambda_{t+1}^1 \frac{P_t}{P_{t+1}} x_t \right) = 0 \tag{165}$$

$$\frac{\partial \pounds}{\partial \omega_{yt}} = 0 : -\lambda_t^1 e_t^* + \beta E_t \left(\lambda_{t+1}^1 e_{t+1}^* + \lambda_{t+1}^1 \frac{S_{t+1} P_t^*}{P_{t+1}} x_t \right) = 0 \tag{166}$$

$$\frac{\partial \pounds}{\partial \psi_{Mt}} = 0 : -\lambda_t^1 r_t + \beta E_t \left(\lambda_{t+1}^1 r_{t+1} + \lambda_{t+1}^1 \frac{\Delta M_{t+1}}{P_{t+1}} x_t \right) = 0 \tag{167}$$

$$\frac{\partial \pounds}{\partial \psi_{Nt}} = 0 : -\lambda_t^1 r_t^* + \beta E_t \left(\lambda_{t+1}^1 r_{t+1}^* + \lambda_{t+1}^1 \frac{S_{t+1} \Delta N_{t+1}}{P_{t+1}} x_t \right) = 0 \tag{168}$$

Foreign has an analogous set of F.O.C's. Using (159) – (162) and (107) – (108) derive aggregate relative prices

$$P_t = \frac{M_t}{x_t} \tag{169}$$

$$P_t^* = \frac{N_t}{y_t} \tag{170}$$

Solving (163) for λ_t^1, insert the result along with (169) – (170) into (164) – (168) and rearranging yields

$$S_t = \frac{(1-\theta)\,M_t}{\theta N_t} \tag{171}$$

$$\frac{e_t}{x_t} = \beta E_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{1-\gamma}\left(\frac{e_{t+1}}{e_t} + \frac{M_t}{M_{t+1}}\right)\right] \tag{172}$$

$$\frac{e_t^*}{q_t y_t} = \beta E_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{1-\gamma}\left(\frac{e_{t+1}^*}{q_{t+1}y_{t+1}} + \frac{N_t}{N_{t+1}}\right)\right] \tag{173}$$

$$\frac{r_t}{x_t} = \beta E_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{1-\gamma}\left(\frac{r_{t+1}}{x_{t+1}} + \frac{\Delta M_{t+1}}{M_{t+1}}\right)\right] \tag{174}$$

$$\frac{r_t^*}{x_t} = \beta E_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{1-\gamma}\left(\frac{r_{t+1}}{x_{t+1}} + \frac{(1-\theta)}{\theta}\frac{\Delta N_{t+1}}{N_{t+1}}\right)\right] \tag{175}$$

where $C_t = c_{xt}^{\theta} c_{yt}^{1-\theta}$. (171) – (175) represent maximizing player's Euler equations in terms of exogenous state variables which evolve according to the approximating system (109) – (112).Although no explicit foreign exchange market exists in the model , the forward exchange rate can be written as

$$F_t = S_t \frac{\beta E_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{1-\gamma} \frac{N_t}{N_{t+1}} \right]}{\beta E_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{1-\gamma} \frac{M_t}{M_{t+1}} \right]} \tag{176}$$

The focus will be on (171) and (176).

Steady-State Beginning with (171) and eliminating time subscripts,

$$\overline{S} = \frac{(1-\theta)\overline{M}}{\theta \overline{N}} \tag{177}$$

where the bar above each variable represents its steady-state counterpart and $\overline{M}$ aswell as $\overline{N}$ are steady-state Home and Foreign money respectively whose values are developed in minimizing player's section.

Next given (176) and removing time subscripts in addition to expectations operators results in

$$\overline{F} = \overline{S} \frac{\beta \left[\left(\frac{\overline{C}}{\overline{C}} \right)^{1-\gamma} \frac{\overline{N}}{\overline{N}} \right]}{\beta \left[\left(\frac{\overline{C}}{\overline{C}} \right)^{1-\gamma} \frac{\overline{M}}{\overline{M}} \right]}$$

$$\overline{F} = \overline{S} \tag{178}$$

where $\overline{C} = \overline{c}_x^{\theta} \overline{c}_y^{1-\theta}$ and both $\overline{c}_x$ aswell as $\overline{c}_y$ values are developed in minimizing player's section.

Log-Linearlization

Spot Exchange Rate Beginning with and applying Uhlig's Method to log-linearlize (171),

$$\begin{aligned} S_t &= \frac{(1-\theta)\,M_t}{\theta N_t} \\ \overline{S}e^{\widetilde{S}_t} &= \frac{(1-\theta)\,\overline{M}e^{\widetilde{M}_t}}{\theta\overline{N}e^{\widetilde{N}_t}} \\ \frac{\theta\overline{SN}}{(1-\theta)\,\overline{M}} &= e^{\widetilde{M}_t-\widetilde{N}_t-\widetilde{S}_t} \\ 1 &= e^{\widetilde{M}_t-\widetilde{N}_t-\widetilde{S}_t} \end{aligned} \tag{179}$$

where (177) was used.Taking the first-order taylor expansion to natural exponential in (179)

$$e^{\widetilde{M}_t-\widetilde{N}_t-\widetilde{S}_t} \simeq e^{\widetilde{M}-\widetilde{N}-\widetilde{S}} + e^{\widetilde{M}-\widetilde{N}-\widetilde{S}}\left(\widetilde{M}_t-\widetilde{M}\right) - e^{\widetilde{M}-\widetilde{N}-\widetilde{S}}\left(\widetilde{N}_t-\widetilde{N}\right) - e^{\widetilde{M}-\widetilde{N}-\widetilde{S}}\left(\widetilde{S}_t-\widetilde{S}\right)$$

$$e^{\widetilde{M}_t-\widetilde{N}_t-\widetilde{S}_t} \simeq 1+\widetilde{M}_t-\widetilde{N}_t-\widetilde{S}_t \tag{180}$$

where $\widetilde{M}=\widetilde{N}=\widetilde{S}=0$ was used. Inserting (180) into (179) and simplifying results in

$$\widetilde{S}_t = \widetilde{M}_t - \widetilde{N}_t \tag{181}$$

Forward Exchange Rate Next, simplifying (176)

$$
\begin{aligned}
F_t &= S_t \frac{\beta E_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{1-\gamma} \frac{N_t}{N_{t+1}} \right]}{\beta E_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{1-\gamma} \frac{M_t}{M_{t+1}} \right]} \\
F_t &= S_t \frac{N_t}{M_t} E_t \left[\frac{M_{t+1}}{N_{t+1}} \right] \\
F_t &= \frac{(1-\theta)}{\theta} E_t \left[\frac{M_{t+1}}{N_{t+1}} \right] \qquad (182)
\end{aligned}
$$

where (171) was used. Applying Uhlig's method to (182) and first order taylor expansion results in

$$
\begin{aligned}
F_t &= \frac{(1-\theta)}{\theta} E_t \left[\frac{M_{t+1}}{N_{t+1}} \right] \\
\overline{F} e^{\widetilde{F}_t} &= \frac{(1-\theta)}{\theta} E_t \left[\frac{\overline{M} e^{\widetilde{M}_{t+1}}}{\overline{N} e^{\widetilde{N}_{t+1}}} \right] \\
1 &= e^{E_t \left[\widetilde{M}_{t+1} \right] - E_t \left[\widetilde{N}_{t+1} \right] - \widetilde{F}_t}
\end{aligned}
$$

$$
\widetilde{F}_t = E_t \left[\widetilde{M}_{t+1} \right] - E_t \left[\widetilde{N}_{t+1} \right] \qquad (183)
$$

where (177) and (178) were used. Thus (181) and (183) are expressed in terms of log-linearlized exogenous stochastic state variables.

Approximating Stochastic Processes Finally, the log-linearlized counterparts to (109) – (112) are

$$\begin{aligned}
\widetilde{x_{t+1}} &= \rho_1 \widetilde{x}_t + \varepsilon^{x}_{t+1} && (184)\\
\widetilde{y_{t+1}} &= \rho_2 \widetilde{y}_t + \varepsilon^{y}_{t+1} && (185)\\
\widetilde{M_{t+1}} &= \rho_3 \widetilde{M}_t + \varepsilon^{M}_{t+1} && (186)\\
\widetilde{N_{t+1}} &= \rho_4 \widetilde{N}_t + \varepsilon^{N}_{t+1} && (187)
\end{aligned}$$

System (184) – (187) can be converted into linear algebra

$$X_{t+1} = AX_t + C\varepsilon_{t+1} \tag{188}$$

where

$$X_t = \begin{bmatrix} \widetilde{x}_t \\ \widetilde{y}_t \\ \widetilde{M}_t \\ \widetilde{N}_t \end{bmatrix}, \qquad \varepsilon_{t+1} = \begin{bmatrix} \varepsilon^{x}_{t+1} \\ \varepsilon^{y}_{t+1} \\ \varepsilon^{M}_{t+1} \\ \varepsilon^{N}_{t+1} \end{bmatrix}$$

$$A = \begin{bmatrix} \rho_1 & 0 & 0 & 0 \\ 0 & \rho_2 & 0 & 0 \\ 0 & 0 & \rho_3 & 0 \\ 0 & 0 & 0 & \rho_4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

20.2.2 Minimizing Player Chooses Second

The malevolent nature assumes the maximizing player is both optimizing and in equilibrium so that perfect risk sharking equilibrium, (142) – (145), (171) – (175) all hold, and (96) – (97) bind so that the minimizing player's problem is presented as

$$\min_{\{w_{t+1}\}_{t=0}^{\infty}} E_t \sum_{t=0}^{\infty} \beta^t \left\{ \phi \left[-\frac{(c_{xt}^{\theta} c_{yt}^{1-\theta})^{1-\gamma}}{1-\gamma} + \beta\overline{\theta} w_{t+1}^2 \right] + (1-\phi) \left[-\frac{(c_{xt}^{*\theta} c_{yt}^{*1-\theta})^{1-\gamma}}{1-\gamma} + \beta\overline{\theta^*} w_{t+1}^2 \right] \right\}$$

$$\begin{aligned} s.t. \quad x_{t+1} &= (1-\rho_1) + \rho_1 x_t + w_{t+1} + \varepsilon_{t+1}^x \\ y_{t+1} &= (1-\rho_2) + \rho_2 y_t + w_{t+1} + \varepsilon_{t+1}^y \\ M_{t+1} &= (1-\rho_3) + \rho_3 M_t + w_{t+1} + \varepsilon_{t+1}^M \\ N_{t+1} &= (1-\rho_4) + \rho_4 N_t + w_{t+1} + \varepsilon_{t+1}^N \end{aligned}$$

(105) – (108), (159) – (162), aswell as aforementioned conditions complete specification of minimizing player's problem.

Lemma 1 *Home and Foreign have homogenous pessimism so that* $\overline{\theta} = \overline{\theta^*}$

The resulting minimization problem with homogenous pessimism is

$$\min_{\{w_{t+1}\}_{t=0}^{\infty}} E_t \sum_{t=0}^{\infty} \beta^t \left\{ -\left[\phi \frac{(c_{xt}^{\theta} c_{yt}^{1-\theta})^{1-\gamma}}{1-\gamma} + (1-\phi) \frac{(c_{xt}^{*\theta} c_{yt}^{*1-\theta})^{1-\gamma}}{1-\gamma} \right] + \beta\overline{\overline{\theta}} w_{t+1}^2 \right\}$$

$$\begin{aligned} s.t. \quad x_{t+1} &= (1-\rho_1) + \rho_1 x_t + w_{t+1} + \varepsilon_{t+1}^x \\ y_{t+1} &= (1-\rho_2) + \rho_2 y_t + w_{t+1} + \varepsilon_{t+1}^y \\ M_{t+1} &= (1-\rho_3) + \rho_3 M_t + w_{t+1} + \varepsilon_{t+1}^M \\ N_{t+1} &= (1-\rho_4) + \rho_4 N_t + w_{t+1} + \varepsilon_{t+1}^N \end{aligned}$$

where $\overline{\overline{\theta}} = \phi\overline{\theta}+(1-\phi)\overline{\theta^*}$, in other words world pessimism $\overline{\overline{\theta}}$ is an implicit weighted average of $\overline{\theta}$ and $\overline{\theta^*}$ so since $\overline{\theta} = \overline{\theta^*}$ and $\phi = \frac{1}{2}, \overline{\overline{\theta}}$ becomes a parameter used by both Home and Foreign as a concern for model misspecification.To solve the minization problem, Linear-Quadratic Approximation Control is used

Steady-State The steady-states needed for log-linearlization of minimizing player problem involve $(142) - (145), (155) - (158)$, and $(169) - (170)$. Initially, eliminating time subscripts and turning off both perturbation and error terms of $(155) - (158)$ yields

$$\begin{aligned} x &= (1-\rho_1) + \rho_1 x \\ y &= (1-\rho_2) + \rho_2 y \\ M &= (1-\rho_3) + \rho_3 M \\ N &= (1-\rho_4) + \rho_4 N \end{aligned}$$

simplifying results in

$$\overline{x} = 1 \tag{189}$$
$$\overline{y} = 1 \tag{190}$$
$$\overline{M} = 1 \tag{191}$$
$$\overline{N} = 1 \tag{192}$$

where $(189)-(192)$ are steady-state versions of their time-varying stochastic counterparts. Next, eliminating time subscripts from $(142) - (145)$ and inserting $(189) - (190)$ where appropriate results in

$$\overline{c_x} = \frac{1}{2} \tag{193}$$

$$\overline{c_y} = \frac{1}{2} \tag{194}$$

$$\overline{c_x^*} = \frac{1}{2} \tag{195}$$

$$\overline{c_y^*} = \frac{1}{2} \tag{196}$$

where (193)−(196) are steady-state versions of their time-varying stochastic counterparts. Finally, following similar steps for (169) – (170) and using (189) – (192) results in

$$\overline{P} = 1 \tag{197}$$

$$\overline{P^*} = 1 \tag{198}$$

where (197) – (198) represent steady-state home price and foreign price respectively.

Log-Linearlization To properly use LQA-Control techniques, the minimizing agents problem must be transfored using log-linearlization techniques .

Minimizing Agent's Weighted Average Component Beginning with (94) and using Uhlig's Method

$$\frac{(c_{xt}^{\theta}c_{yt}^{1-\theta})^{1-\gamma}}{1-\gamma} = \frac{\left[\left(\overline{c_x^{\theta}}e^{\theta\widetilde{c_{xt}}}\right)\left(\overline{c_y^{(1-\theta)}}e^{(1-\theta)\widetilde{c_{yt}}}\right)\right]^{1-\gamma}}{1-\gamma}$$
$$= \frac{\left(\overline{c_x^{\theta}}\overline{c_y^{(1-\theta)}}\right)^{(1-\gamma)} e^{(1-\gamma)(\theta\widetilde{c_{xt}}+(1-\theta)\widetilde{c_{yt}})}}{1-\gamma} \quad (199)$$

Performing a second order taylor expansion (Kim and Kim 2007) on the natural exponential component of (199)

$$e^{(1-\gamma)(\theta\widetilde{c_{xt}}+(1-\theta)\widetilde{c_{yt}})} \simeq e^{(1-\gamma)(\theta\widetilde{c_x}+(1-\theta)\widetilde{c_y})} + e^{(1-\gamma)(\theta\widetilde{c_x}+(1-\theta)\widetilde{c_y})}(1-\gamma)\theta\left(\widetilde{c_{xt}}-\widetilde{c_x}\right)$$
$$+ e^{(1-\gamma)(\theta\widetilde{c_x}+(1-\theta)\widetilde{c_y})}(1-\gamma)(1-\theta)\left(\widetilde{c_{yt}}-\widetilde{c_y}\right)$$
$$+ \frac{1}{2}\left[e^{(1-\gamma)(\theta\widetilde{c_x}+(1-\theta)\widetilde{c_y})}(1-\gamma)^2\theta^2\left(\widetilde{c_{xt}}-\widetilde{c_x}\right)^2\right.$$
$$+2e^{(1-\gamma)(\theta\widetilde{c_x}+(1-\theta)\widetilde{c_y})}(1-\gamma)^2(1-\theta)\theta\left(\widetilde{c_{xt}}-\widetilde{c_x}\right)\left(\widetilde{c_{yt}}-\widetilde{c_y}\right)$$
$$\left.+e^{(1-\gamma)(\theta\widetilde{c_x}+(1-\theta)\widetilde{c_y})}(1-\gamma)^2(1-\theta)^2\left(\widetilde{c_{yt}}-\widetilde{c_y}\right)^2\right] \quad (200)$$

Inserting $\widetilde{c_x} = \widetilde{c_y} = 0$ into (200) and simplifying results in

$$e^{(1-\gamma)(\theta\widetilde{c_{xt}}+(1-\theta)\widetilde{c_{yt}})} \simeq 1 + (1-\gamma)\theta\widetilde{c_{xt}} + (1-\gamma)(1-\theta)\widetilde{c_{yt}} + \frac{1}{2}\left[(1-\gamma)^2\theta^2\widetilde{c_{xt}}^2\right.$$
$$\left.+2(1-\gamma)^2(1-\theta)\theta\widetilde{c_{xt}}\widetilde{c_{yt}} + (1-\gamma)^2(1-\theta)^2\widetilde{c_{yt}}^2\right] \quad (201)$$

Inserting $\widetilde{c_{xt}}\widetilde{c_{yt}} = 0$ into (201) for the cross-product term

$$e^{(1-\gamma)(\theta\widetilde{c_{xt}}+(1-\theta)\widetilde{c_{yt}})} \simeq 1 + (1-\gamma)\,\theta\widetilde{c_{xt}} + (1-\gamma)\,(1-\theta)\,\widetilde{c_{yt}} + \frac{1}{2}\left[(1-\gamma)^2\,\theta^2\widetilde{c_{xt}}^2 + (1-\gamma)^2\,(1-\theta)^2\,\widetilde{c_{yt}}^2\right] \quad (202)$$

$$\therefore \frac{(c_{xt}^{\theta}c_{yt}^{1-\theta})^{1-\gamma}}{1-\gamma} \simeq \frac{\left(\overline{c_x^{\theta}c_y^{(1-\theta)}}\right)^{(1-\gamma)}}{1-\gamma} + \left(\overline{c_x^{\theta}c_y^{(1-\theta)}}\right)^{(1-\gamma)}\left[\theta\widetilde{c_{xt}} + (1-\theta)\,\widetilde{c_{yt}} + \frac{1}{2}\left((1-\gamma)\,\theta^2\widetilde{c_{xt}}^2 + (1-\gamma)\,(1-\theta)^2\,\widetilde{c_{yt}}^2\right)\right] \quad (203)$$

where (202) is inserted into (199) for the natural exponential term, and simplified resulting in (203). The constant term implies that (203) is expressed in terms of levels but the expression must be in terms of log-deviations so subtract the constant term from both sides to yield

$$\widetilde{U}_H \simeq \left(\overline{c_x^{\theta}c_y^{(1-\theta)}}\right)^{(1-\gamma)}\left[\theta\widetilde{c_{xt}} + (1-\theta)\,\widetilde{c_{yt}} + \frac{1}{2}\left((1-\gamma)\,\theta^2\widetilde{c_{xt}}^2 + (1-\gamma)\,(1-\theta)^2\,\widetilde{c_{yt}}^2\right)\right] \quad (204)$$

where $\widetilde{U}_H = \frac{(c_{xt}^{\theta}c_{yt}^{1-\theta})^{1-\gamma}}{1-\gamma} - \frac{\left(\overline{c_x^{\theta}c_y^{(1-\theta)}}\right)^{(1-\gamma)}}{1-\gamma}$. Analogously for Foreign,

$$\widetilde{U_F} \simeq \left(\overline{c_x^{*\theta}}\overline{c_y^{*(1-\theta)}}\right)^{(1-\gamma)} \left[\theta\widetilde{c_{xt}^*} + (1-\theta)\widetilde{c_{yt}^*} + \frac{1}{2}\left((1-\gamma)\theta^2\widetilde{c_{xt}^*}^2 + (1-\gamma)(1-\theta)^2\widetilde{c_{yt}^*}^2\right)\right] \tag{205}$$

where $\widetilde{U_F} = \frac{(c_{xt}^{*\theta}c_{yt}^{*1-\theta})^{1-\gamma}}{1-\gamma} - \frac{\left(\overline{c_x^{*\theta}}\overline{c_y^{*(1-\theta)}}\right)^{(1-\gamma)}}{1-\gamma}$.

The minimizing player's objective function contains a weighted average component of (94) and (95) or

$$\phi\frac{(c_{xt}^{\theta}c_{yt}^{1-\theta})^{1-\gamma}}{1-\gamma} + (1-\phi)\frac{(c_{xt}^{*\theta}c_{yt}^{*1-\theta})^{1-\gamma}}{1-\gamma} \tag{206}$$

Inserting (204) for (94) and (205) for (95) in (206)

$$\phi\widetilde{U_H} + (1-\phi)\widetilde{U_F} \tag{207}$$

Setting $\phi = \frac{1}{2}$, expanding, and simplifying (207)

$$\alpha\left[\theta\left(\widetilde{c_{xt}} + \widetilde{c_{xt}^*}\right) + (1-\theta)\left(\widetilde{c_{yt}} + \widetilde{c_{yt}^*}\right) + \frac{1}{2}(1-\gamma)\theta^2\left(\widetilde{c_{xt}^2} + \widetilde{c_{xt}^{2*}}\right) + \frac{1}{2}(1-\gamma)(1-\theta)^2\left(\widetilde{c_{yt}^2} + \widetilde{c_{yt}^{2*}}\right)\right] \tag{208}$$

where $\alpha = \frac{\left(\overline{c_x^{\theta}}\overline{c_y^{(1-\theta)}}\right)^{(1-\gamma)}}{2}$ and (208) ultimately is the log-linearlized (in terms of deviations froms steady-state) form of (206). Now in order to express (208) in terms of exogenous variables x_t, y_t, M_t, and N_t the remainder of minimizing player's specification is used namely $(105)-(108)$, and $(159)-(162)$. Log-linearlize (105) using Uhlig's method and second-order taylor approximation,

$$\begin{aligned}
c_{xt} + c^*_{xt} &= x_t \\
\overline{c_x} e^{\widetilde{c_{xt}}} + \overline{c_x} e^{\widetilde{c^*_{xt}}} &= \overline{x} e^{\widetilde{x_t}} \\
\overline{c_x}\left(1 + \widetilde{c_{xt}} + \frac{1}{2}\widetilde{c_{xt}}^2\right) + \overline{c^*_x}\left(1 + \widetilde{c^*_{xt}} + \frac{1}{2}\widetilde{c^*_{xt}}^2\right) &= \overline{x}\left(1 + \widetilde{x}_t + \frac{1}{2}\widetilde{x}_t^2\right) \\
\overline{c_x} + \overline{c_x}\widetilde{c_{xt}} + \frac{\overline{c_x}}{2}\widetilde{c_{xt}}^2 + \overline{c^*_x} + \overline{c^*_x}\widetilde{c_{xt}}^* + \frac{\overline{c^*_x}}{2}\widetilde{c^*_{xt}}^2 &= \overline{x} + \overline{x}\widetilde{x}_t + \frac{\overline{x}}{2}\widetilde{x}_t^2 \\
\overline{c_x}\widetilde{c_{xt}} + +\overline{c^*_x}\widetilde{c_{xt}}^* &= \overline{x}\widetilde{x}_t + \frac{\overline{x}}{2}\widetilde{x}_t^2 - \frac{\overline{c_x}}{2}\widetilde{c_{xt}}^2 - \frac{\overline{c^*_x}}{2}\widetilde{c^*_{xt}}^2 \\
\widetilde{c_{xt}} + \widetilde{c_{xt}}^* &= \frac{\overline{x}}{\overline{c_x}}\widetilde{x}_t + \frac{\overline{x}}{2\overline{c_x}}\widetilde{x}_t^2 - \frac{1}{2}\widetilde{c_{xt}}^2 - \frac{1}{2}\widetilde{c^*_{xt}}^2
\end{aligned}$$

$$\widetilde{c_{xt}} + \widetilde{c_{xt}}^* = \frac{\overline{x}}{\overline{c_x}}\widetilde{x}_t + \frac{\overline{x}}{2\overline{c_x}}\widetilde{x}_t^2 - \frac{1}{2}\left(\widetilde{c_{xt}}^2 + \widetilde{c^*_{xt}}^2\right) \tag{209}$$

where $\overline{c_x} = \overline{c^*_x}$ is used. Following a similar procedure for (106) results in

$$\widetilde{c_{yt}} + \widetilde{c_{yt}}^* = \frac{\overline{y}}{\overline{c_y}}\widetilde{y}_t + \frac{\overline{y}}{2\overline{c_y}}\widetilde{y}_t^2 - \frac{1}{2}\left(\widetilde{c_{yt}}^2 + \widetilde{c^*_{yt}}^2\right) \tag{210}$$

where $\overline{c_y} = \overline{c^*_y}$ is used. Plug (209) and (210) into (208) for linear consumptions

$$\begin{aligned}
\alpha\Bigg[\theta\left(\frac{\overline{x}}{\overline{c_x}}\widetilde{x}_t + \frac{\overline{x}}{2\overline{c_x}}\widetilde{x}_t^2\right) - \frac{\theta}{2}\left(\widetilde{c_{xt}}^2 + \widetilde{c^*_{xt}}^2\right) + (1-\theta)\left(\frac{\overline{y}}{\overline{c_y}}\widetilde{y}_t + \frac{\overline{y}}{2\overline{c_y}}\widetilde{y}_t^2\right) - \frac{(1-\theta)}{2}\left(\widetilde{c_{yt}}^2 + \widetilde{c^*_{yt}}^2\right) \\
+\frac{1}{2}(1-\gamma)\,\theta^2\left(\widetilde{c^2_{xt}} + \widetilde{c^{2*}_{xt}}\right) + \frac{1}{2}(1-\gamma)(1-\theta)^2\left(\widetilde{c^2_{yt}} + \widetilde{c^{2*}_{yt}}\right)\Bigg]
\end{aligned}$$

further simplification yields

$$\alpha\left[\theta\left(\frac{\overline{x}}{\overline{c_x}}\widetilde{x}_t+\frac{\overline{x}}{2\overline{c_x}}\widetilde{x}_t^{\,2}\right)+(1-\theta)\left(\frac{\overline{y}}{\overline{c_y}}\widetilde{y}_t+\frac{\overline{y}}{2\overline{c_y}}\widetilde{y}_t^{\,2}\right)+\left(\frac{1}{2}(1-\gamma)\theta^2-\frac{\theta}{2}\right)\left(\widetilde{c_{xt}^2}+\widetilde{c_{xt}^{2*}}\right)\right.$$
$$\left.+\left(\frac{1}{2}(1-\gamma)(1-\theta)^2-\frac{(1-\theta)}{2}\right)\left(\widetilde{c_{yt}^2}+\widetilde{c_{yt}^{2*}}\right)\right] \quad (211)$$

Next, combining (107) – (108) with (159) – (162), log-linearlizing using Uhlig's method with second-order taylor expansion so that a similar process to (209) and (210) is followed results in

$$\widetilde{c_{xt}^2}+\widetilde{c_{xt}^{*2}}=\frac{2\overline{M}}{\overline{P}\overline{c_x}}\widetilde{M_t}+\frac{\overline{M}}{\overline{P}\overline{c_x}}\widetilde{M_t^2}-2\left(\widetilde{c_{xt}}+\widetilde{c_{xt}^*}\right) \quad (212)$$

and analogously,

$$\widetilde{c_{yt}^2}+\widetilde{c_{yt}^{*2}}=\frac{2\overline{N}}{\overline{P^*}\overline{c_y}}\widetilde{N_t}+\frac{\overline{N}}{\overline{P^*}\overline{c_y}}\widetilde{N_t^2}-2\left(\widetilde{c_{yt}}+\widetilde{c_{yt}^*}\right) \quad (213)$$

Inserting (212) and (213) into (211) for quadratic consumption terms yields

$$\alpha\left[\theta\left(\frac{\overline{x}}{\overline{c_x}}\widetilde{x}_t+\frac{\overline{x}}{2\overline{c_x}}\widetilde{x}_t^{\,2}\right)+(1-\theta)\left(\frac{\overline{y}}{\overline{c_y}}\widetilde{y}_t+\frac{\overline{y}}{2\overline{c_y}}\widetilde{y}_t^{\,2}\right)\right.$$
$$+\left(\frac{1}{2}(1-\gamma)\theta^2-\frac{\theta}{2}\right)\left(\frac{2\overline{M}}{\overline{P}\overline{c_x}}\widetilde{M_t}+\frac{\overline{M}}{\overline{P}\overline{c_x}}\widetilde{M_t^2}-2\left(\widetilde{c_{xt}}+\widetilde{c_{xt}^*}\right)\right)$$
$$\left.+\left(\frac{1}{2}(1-\gamma)(1-\theta)^2-\frac{(1-\theta)}{2}\right)\left(\frac{2\overline{N}}{\overline{P^*}\overline{c_y}}\widetilde{N_t}+\frac{\overline{N}}{\overline{P^*}\overline{c_y}}\widetilde{N_t^2}-2\left(\widetilde{c_{yt}}+\widetilde{c_{yt}^*}\right)\right)\right] \quad (214)$$

Since the minimizing player assumes the maximizing player is in equilibrium, use log-linear versions of (142) – (145) or

$$
\begin{aligned}
\widetilde{c_{xt}} &= \frac{\widetilde{x_t}}{2} \\
\widetilde{c_{yt}} &= \frac{\widetilde{y_t}}{2} \qquad (215) \\
\widetilde{c^*_{xt}} &= \frac{\widetilde{x_t}}{2} \qquad (216) \\
\widetilde{c^*_{yt}} &= \frac{\widetilde{y_t}}{2} \qquad (217)
\end{aligned}
$$

Insert (215) – (217) into (214) for remaining linear consumption terms and expanding results in

$$
\begin{aligned}
&\alpha\left[\frac{\theta\overline{x}}{\overline{c_x}}\widetilde{x_t} + \frac{\theta\overline{x}}{2\overline{c_x}}\widetilde{x_t}^2 + \frac{(1-\theta)\,\overline{y}}{\overline{c_y}}\widetilde{y_t} + \frac{(1-\theta)\,\overline{y}}{2\overline{c_y}}\widetilde{y_t}^2\right.\\
&+\frac{2\overline{M}}{\overline{P}\overline{c_x}}\left(\frac{1}{2}(1-\gamma)\,\theta^2 - \frac{\theta}{2}\right)\widetilde{M_t} + \frac{\overline{M}}{\overline{P}\overline{c_x}}\left(\frac{1}{2}(1-\gamma)\,\theta^2 - \frac{\theta}{2}\right)\widetilde{M_t^2} - 2\left(\frac{1}{2}(1-\gamma)\,\theta^2 - \frac{\theta}{2}\right)\widetilde{x_t}\\
&+\frac{2\overline{N}}{\overline{P^*}\overline{c_y}}\left(\frac{1}{2}(1-\gamma)(1-\theta)^2 - \frac{(1-\theta)}{2}\right)\widetilde{N_t} + \frac{\overline{N}}{\overline{P^*}\overline{c_y}}\left(\frac{1}{2}(1-\gamma)(1-\theta)^2 - \frac{(1-\theta)}{2}\right)\widetilde{N_t^2}\\
&\left.-2\left(\frac{1}{2}(1-\gamma)(1-\theta)^2 - \frac{(1-\theta)}{2}\right)\widetilde{y_t}\right] \qquad (218)
\end{aligned}
$$

consolidating linear and quadratic terms in (218)

$$\alpha\left\{\left[\frac{\theta\overline{x}}{\overline{c_x}}-2\left(\frac{1}{2}\left(1-\gamma\right)\theta^2-\frac{\theta}{2}\right)\right]\widetilde{x}_t+\left[\frac{\left(1-\theta\right)\overline{y}}{\overline{c_y}}-2\left(\frac{1}{2}\left(1-\gamma\right)\left(1-\theta\right)^2-\frac{\left(1-\theta\right)}{2}\right)\right]\widetilde{y}_t\right.$$
$$+\frac{2\overline{M}}{\overline{P}\overline{c_x}}\left(\frac{1}{2}\left(1-\gamma\right)\theta^2-\frac{\theta}{2}\right)\widetilde{M}_t+\frac{2\overline{N}}{\overline{P^*}\overline{c_y}}\left(\frac{1}{2}\left(1-\gamma\right)\left(1-\theta\right)^2-\frac{\left(1-\theta\right)}{2}\right)\widetilde{N}_t$$
$$\left.+\frac{\theta\overline{x}}{2\overline{c_x}}\widetilde{x}_t^{\,2}+\frac{\left(1-\theta\right)\overline{y}}{2\overline{c_y}}\widetilde{y}_t^{\,2}+\frac{\overline{M}}{\overline{P}\overline{c_x}}\left(\frac{1}{2}\left(1-\gamma\right)\theta^2-\frac{\theta}{2}\right)\widetilde{M_t^2}+\frac{\overline{N}}{\overline{P^*}\overline{c_y}}\left(\frac{1}{2}\left(1-\gamma\right)\left(1-\theta\right)^2-\frac{\left(1-\theta\right)}{2}\right)\widetilde{N_t^2}\right\} \tag{219}$$

For small deviations in each state variable, the linear terms in (219) can be ignored (Levine and Pearlman 2006) yielding

$$\alpha\left[\frac{\theta\overline{x}}{2\overline{c_x}}\widetilde{x}_t^{\,2}+\frac{\left(1-\theta\right)\overline{y}}{2\overline{c_y}}\widetilde{y}_t^{\,2}+\frac{\overline{M}}{\overline{P}\overline{c_x}}\left(\frac{1}{2}\left(1-\gamma\right)\theta^2-\frac{\theta}{2}\right)\widetilde{M_t^2}+\frac{\overline{N}}{\overline{P^*}\overline{c_y}}\left(\frac{1}{2}\left(1-\gamma\right)\left(1-\theta\right)^2-\frac{\left(1-\theta\right)}{2}\right)\widetilde{N_t^2}\right] \tag{220}$$

Distributing α and redefining each coefficient where

$$\begin{aligned}
r_1 &= \alpha\frac{\theta\overline{x}}{2\overline{c_x}}\\
r_2 &= \alpha\frac{\left(1-\theta\right)\overline{y}}{2\overline{c_y}}\\
r_3 &= \alpha\frac{\overline{M}}{\overline{P}\overline{c_x}}\left(\frac{1}{2}\left(1-\gamma\right)\theta^2-\frac{\theta}{2}\right)\\
r_4 &= \alpha\frac{\overline{N}}{\overline{P^*}\overline{c_y}}\left(\frac{1}{2}\left(1-\gamma\right)\left(1-\theta\right)^2-\frac{\left(1-\theta\right)}{2}\right)
\end{aligned}$$

(220) is reduced to

$$r_1\widetilde{x_t}^2 + r_2\widetilde{y_t}^2 + r_3\widetilde{M_t^2} + r_4\widetilde{N_t^2} \tag{221}$$

(221) expresses the weighted average component of minizing agent's objective fuction as a quadratic in terms exogenous state variables.

Perturbed Stochastic Processes The log-linearlization of (155) − (158) yields

$$\begin{aligned} \widetilde{x_{t+1}} &= \rho_1\widetilde{x_t} + w_{t+1} + \varepsilon_{t+1}^x && (222)\\ \widetilde{y_{t+1}} &= \rho_2\widetilde{y_t} + w_{t+1} + \varepsilon_{t+1}^y && (223)\\ \widetilde{M_{t+1}} &= \rho_3\widetilde{M_t} + w_{t+1} + \varepsilon_{t+1}^M && (224)\\ \widetilde{N_{t+1}} &= \rho_4\widetilde{N_t} + w_{t+1} + \varepsilon_{t+1}^N && (225) \end{aligned}$$

Replacing the weighted average component of minimizing player's problem with (221) and using (222) − (225) as constraints instead, the malevolent nature's problem becomes

$$\min_{\{w_{t+1}\}_{t=0}^{\infty}} E_t \sum_{t=0}^{\infty} \beta^t \left\{ -\left(r_1\widetilde{x_t}^2 + r_2\widetilde{y_t}^2 + r_3\widetilde{M_t^2} + r_4\widetilde{N_t^2} \right) + \beta\bar{\bar{\theta}} w_{t+1}^2 \right\}$$

$$\begin{aligned} s.t. \quad \widetilde{x_{t+1}} &= \rho_1\widetilde{x_t} + w_{t+1} + \varepsilon_{t+1}^x\\ \widetilde{y_{t+1}} &= \rho_2\widetilde{y_t} + w_{t+1} + \varepsilon_{t+1}^y\\ \widetilde{M_{t+1}} &= \rho_3\widetilde{M_t} + w_{t+1} + \varepsilon_{t+1}^M\\ \widetilde{N_{t+1}} &= \rho_4\widetilde{N_t} + w_{t+1} + \varepsilon_{t+1}^N \end{aligned}$$

Linear Quadratic Approximation Control To cast in LQA framework, transform the above minimization problem in terms of matrix algebra

$$\min_{\{w_{t+1}\}_{t=0}^{\infty}} E_t \sum_{t=0}^{\infty} \beta^t \left\{ - \begin{bmatrix} \widetilde{x}_t \\ \widetilde{y}_t \\ \widetilde{M}_t \\ \widetilde{N}_t \end{bmatrix}' \begin{bmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 \\ 0 & 0 & r_3 & 0 \\ 0 & 0 & 0 & r_4 \end{bmatrix} \begin{bmatrix} \widetilde{x}_t \\ \widetilde{y}_t \\ \widetilde{M}_t \\ \widetilde{N}_t \end{bmatrix} + \beta\overline{\overline{\theta}} I_4 w_{t+1}' w_{t+1} \right\}$$

$$s.t. \quad \begin{bmatrix} \widetilde{x_{t+1}} \\ \widetilde{y_{t+1}} \\ \widetilde{M_{t+1}} \\ \widetilde{N_{t+1}} \end{bmatrix} = \begin{bmatrix} \rho_1 & 0 & 0 & 0 \\ 0 & \rho_2 & 0 & 0 \\ 0 & 0 & \rho_3 & 0 \\ 0 & 0 & 0 & \rho_4 \end{bmatrix} \begin{bmatrix} \widetilde{x}_t \\ \widetilde{y}_t \\ \widetilde{M}_t \\ \widetilde{N}_t \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} w_{t+1} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{t+1}^x \\ \varepsilon_{t+1}^y \\ \varepsilon_{t+1}^M \\ \varepsilon_{t+1}^N \end{bmatrix}$$

or

$$\min_{\{w_{t+1}\}_{t=0}^{\infty}} E_t \sum_{t=0}^{\infty} \beta^t \left\{ -X_t' R X_t + \beta\overline{\overline{\theta}} I_4 w_{t+1}' w_{t+1} \right\}$$

$$s.t. \quad X_{t+1} = A X_t + C w_{t+1} + C \varepsilon_{t+1}$$

where

$$X_t = \begin{bmatrix} \widetilde{x}_t \\ \widetilde{y}_t \\ \widetilde{M}_t \\ \widetilde{N}_t \end{bmatrix}, \qquad \varepsilon_{t+1} = \begin{bmatrix} \varepsilon^x_{t+1} \\ \varepsilon^y_{t+1} \\ \varepsilon^M_{t+1} \\ \varepsilon^N_{t+1} \end{bmatrix}, \quad R = \begin{bmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 \\ 0 & 0 & r_3 & 0 \\ 0 & 0 & 0 & r_4 \end{bmatrix},$$

$$A = \begin{bmatrix} \rho_1 & 0 & 0 & 0 \\ 0 & \rho_2 & 0 & 0 \\ 0 & 0 & \rho_3 & 0 \\ 0 & 0 & 0 & \rho_4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, and \qquad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where the minimizing player's LQA-C problem is expressed in terms of exogenous state variables (i.e. excludes maximizing player's control variables), which is natural to purely exogenous endowment economies, and malevolent nature's control variable w_{t+1}.The problem can be translated into a Bellman equation,

$$-X_t' P X_t - p = \min_{w_{t+1}} \left\{ -X_t' R X_t + \beta \overline{\overline{\theta}} I_4 w_{t+1}' w_{t+1} + \beta E_t \left(-X_{t+1}' P X_{t+1} - p \right) \right\}$$

$$s.t. \quad X_{t+1} = A X_t + C w_{t+1} + C \varepsilon_{t+1}$$

where $V(X_t) = -X_t' P X_t - p$ is a specific quadratic structure of the value function and the symmetric matrix P is a fixed point used to find a stable solution that minimizes malevolent nature's problem . Using certainty equivalence, all uncertainty in the problem is eliminated so that $\varepsilon_{t+1} = 0$ which in turn eliminates p,

$$-X_t' P X_t = \min_{w_{t+1}} \left\{ -X_t' R X_t + \beta \bar{\bar{\theta}} I_4 w_{t+1}' w_{t+1} + \beta \left(-X_{t+1}' P X_{t+1} \right) \right\}$$

$$s.t. \quad X_{t+1} = A X_t + C w_{t+1}$$

Inserting state-evolution constraint into the objective function yields the unconstrained minimization problem,

$$-X_t' P X_t = \min_{w_{t+1}} \left\{ -X_t' R X_t + \beta \bar{\bar{\theta}} I_4 w_{t+1}' w_{t+1} + \beta \left[- \left(A X_t + C w_{t+1} \right)' P \left(A X_t + C w_{t+1} \right) \right] \right\}$$

$$\frac{\partial V}{\partial w_{t+1}} = 0 : 2\beta \bar{\bar{\theta}} I_4 w_{t+1} - 2\beta C' P A X_t - 2\beta C' P C w_{t+1} = 0 \tag{226}$$

$$\frac{\partial V}{\partial X_t} = 0 : -2 R X_t - 2\beta A' P A X_t - 2 A' P C w_{t+1} = 0 \tag{227}$$

Beginning with (226),

$$\begin{aligned} 2\beta \bar{\bar{\theta}} I_4 w_{t+1} - 2\beta C' P A X_t - 2\beta C' P C w_{t+1} &= 0 \\ \bar{\bar{\theta}} I_4 w_{t+1} - C' P A X_t - C' P C w_{t+1} &= 0 \\ w_{t+1} &= \left(\bar{\bar{\theta}} I_4 - C' P C \right)^{-1} C' P A X_t \end{aligned}$$

$$w_{t+1} = KX_t \tag{228}$$

where $K = \left(\overline{\overline{\theta}} I_4 - C'PC\right)^{-1} C'PA$. (228) represents movement of malevolent nature's perturbation w_{t+1} as a function of the state vector X_t. Next, utilizing (227) and inserting (228) for expanded K,

$$\begin{aligned} -2RX_t - 2\beta A'PAX_t - 2A'PC\left(\overline{\overline{\theta}} I_4 - C'PC\right)^{-1} C'PAX_t &= 0 \\ R + \beta A'PA + A'PC\left(\overline{\overline{\theta}} I_4 - C'PC\right)^{-1} C'PA &= 0 \end{aligned}$$

$$P = R + \beta A'PA + A'PC\left(\overline{\overline{\theta}} I_4 - C'PC\right)^{-1} C'PA \tag{229}$$

(229) represents the Riccati equation whose properites are well-known in linear-quadratic control theory. The fixed-point matrix P is used to find a stable solution so that (229) becomes

$$P_{k+1} = R + \beta A'P_kA + A'P_kC\left(\overline{\overline{\theta}} I_4 - C'P_kC\right)^{-1} C'P_kA \tag{230}$$

Setting $P_0 = 0$ as an initial condition and iterating (230) until convergence yields

$$\widehat{P} = R + \beta A'\widehat{P}A + A'\widehat{P}C\left(\overline{\overline{\theta}} I_4 - C'\widehat{P}C\right)^{-1} C'\widehat{P}A \tag{231}$$

where $\widehat{P}$ is a steady-state fixed point that when inserted into (228) for expanded K,

$$w_{t+1} = \left(\overline{\overline{\theta}} I_4 - C'\widehat{P}C\right)^{-1} C'\widehat{P}AX_t \quad (232)$$

$$w_{t+1} = K(\widehat{P})X_t \quad (233)$$

where $K(\widehat{P}) = \left(\overline{\overline{\theta}} I_4 - C'\widehat{P}C\right)^{-1} C'\widehat{P}A$ and (233) is malevolent nature's perturbation encoded with fixed-point matrix $\widehat{P}$.

(233) feeds back into the perturbed state-evolution system

$$X_{t+1} = AX_t + Cw_{t+1} + C\varepsilon_{t+1}$$

$$X_{t+1} = AX_t + CKX_t + C\varepsilon_{t+1}$$

$$X_{t+1} = A^0 X_t + C\varepsilon_{t+1} \quad (234)$$

where $A^0 = (A + CK)$ .Ultimately, (234) is the perturbed state-evolution system embedded with malevolent natures feedback rule w_{t+1}.

20.3 Spot & Forward Exchange Rates

Generating inference involves (181), (183), and (234) but recall that both (181) and (183) are in terms of two exogenous state variables and (234) is in terms of four exogenous state variables. To remedy the issue, convert (181) into linear-algebra form

$$\begin{aligned}
\widetilde{S}_t &= \widetilde{M}_t - \widetilde{N}_t \\
\widetilde{S}_t &= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} \widetilde{M}_t \\ \widetilde{N}_t \end{bmatrix} \\
\widetilde{S}_t &= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \widetilde{x}_t \\ \widetilde{y}_t \\ \widetilde{M}_t \\ \widetilde{N}_t \end{bmatrix} \\
\widetilde{S}_t &= \begin{bmatrix} 1 & -1 \end{bmatrix} U_s X_t
\end{aligned}$$

$$\widetilde{S}_t = \Pi X_t \tag{235}$$

where $\Pi = \begin{bmatrix} 1 & -1 \end{bmatrix} U_s$ and $U_s = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is known as a selection matrix that creates dependence between $\widetilde{S}_t$ and $\widetilde{M}_t, \widetilde{N}_t$.

Next, insert (186) and (187) into (183) ,distribute expectation operator, and convert into linear algebra

$$
\begin{aligned}
\widetilde{F}_t &= E_t\left[\widetilde{M}_{t+1}\right] - E_t\left[\widetilde{N}_{t+1}\right] \\
\widetilde{F}_t &= \rho_3 \widetilde{M}_t - \rho_4 \widetilde{N}_t \\
\widetilde{F}_t &= \begin{bmatrix} \rho_3 & -\rho_4 \end{bmatrix} \begin{bmatrix} \widetilde{M}_t \\ \widetilde{N}_t \end{bmatrix} \\
\widetilde{F}_t &= \begin{bmatrix} \rho_3 & -\rho_4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \widetilde{x}_t \\ \widetilde{y}_t \\ \widetilde{M}_t \\ \widetilde{N}_t \end{bmatrix} \\
\widetilde{F}_t &= \begin{bmatrix} \rho_3 & -\rho_4 \end{bmatrix} U_s X_t
\end{aligned}
$$

$$
\widetilde{F}_t = \Gamma X_t \qquad (236)
$$

where $\Gamma = \begin{bmatrix} \rho_3 & -\rho_4 \end{bmatrix} U_s$ and U_s is defined as above.

20.4 Gross Rate of Depreciation & Forward Premium

The *Gross Rate of Depreciation* can be constructed using (235) and a lead of (235) or

$$\widetilde{S}_{t+1} - \widetilde{S}_t \tag{237}$$

where $\widetilde{S}_{t+1}$ is *ex-post.*

and the *Forward Premium* can be constructed using both (235) and (236)

$$\widetilde{F}_t - \widetilde{S}_t \tag{238}$$

Again, the motivator for dynamics of (235) − (238) is perturbed stochastic system

$$X_{t+1} = A^0 X_t + C\varepsilon_{t+1} \tag{239}$$

21 Detection Error Probabilities

To discipline the choice of $\overline{\overline{\theta}}$, detection error probability method is enforced so state the approximating state-evolution system (188)

$$X_{t+1} = AX_t + C\widehat{\varepsilon_{t+1}} \tag{240}$$

which will be known as "Model A." State the perturbed state-evolution system (234)

$$X_{t+1} = A^0 X_t + C\varepsilon_{t+1} \tag{241}$$

which will be known as "Model B."

21.1 Log-Likelihood Ratio Test with Model A

Assume the worst-case shock is generated under Model A,

$$w_{t+1}^{A} = KX_{t+1}^{A} \tag{242}$$

Define Model A innovations,

$$\begin{aligned} X_{t+1} &= AX_t + C\widehat{\varepsilon_{t+1}} && (243)\\ C\widehat{\varepsilon_{t+1}} &= X_{t+1} - AX_t && (244)\\ \widehat{\varepsilon_{t+1}} &= \left(C'C\right)^{-1} C' \left(X_{t+1} - AX_t\right) && (245) \end{aligned}$$

Define Model B innovations,

$$\begin{aligned} X_{t+1} &= A^0 X_t + C\varepsilon_{t+1} && (246)\\ C\varepsilon_{t+1} &= X_{t+1} - A^0 X_t && (247)\\ \varepsilon_{t+1} &= \left(C'C\right)^{-1} C' \left(X_{t+1} - A^0 X_t\right) && (248)\\ \varepsilon_{t+1} &= \left(C'C\right)^{-1} C' \left(X_{t+1} - AX_t\right) - \left(C'C\right)^{-1} C'CKX_t && (249)\\ \varepsilon_{t+1} &= \widehat{\varepsilon_{t+1}} - w_{t+1} && (250) \end{aligned}$$

where $A^0 = (A + CK)$ was used. Next, define both

$$Log\ L_A = -\frac{1}{T}\sum_{t=0}^{T-1}\left\{n\log\sqrt{2\pi}+\frac{1}{2}\left(\widehat{\varepsilon_{t+1}}'\widehat{\varepsilon_{t+1}}\right)\right\} \tag{251}$$

$$Log\ L_B = -\frac{1}{T}\sum_{t=0}^{T-1}\left\{n\log\sqrt{2\pi}+\frac{1}{2}\left(\varepsilon_{t+1}'\varepsilon_{t+1}\right)\right\} \tag{252}$$

where (251) is generated under innovations from (245) and (252) is generated under innovations from (250). To construct log-likelihood ratio test w.r.t Model A define,

$$\begin{aligned}
r|A &= Log\ L_A - Log\ L_B \\
&= -\frac{1}{T}\sum_{t=0}^{T-1}\left\{n\log\sqrt{2\pi}+\frac{1}{2}\left(\widehat{\varepsilon_{t+1}}'\widehat{\varepsilon_{t+1}}\right)\right\}+\frac{1}{T}\sum_{t=0}^{T-1}\left\{n\log\sqrt{2\pi}+\frac{1}{2}\left(\varepsilon_{t+1}'\varepsilon_{t+1}\right)\right\} \\
&= \frac{1}{T}\sum_{t=0}^{T-1}\left\{\frac{1}{2}\left(\varepsilon_{t+1}'\varepsilon_{t+1}\right)-\frac{1}{2}\left(\widehat{\varepsilon_{t+1}}'\widehat{\varepsilon_{t+1}}\right)\right\} \\
&= \frac{1}{T}\sum_{t=0}^{T-1}\left\{\frac{1}{2}\left(\widehat{\varepsilon_{t+1}}-w_{t+1}^A\right)'\left(\widehat{\varepsilon_{t+1}}-w_{t+1}^A\right)-\frac{1}{2}\left(\widehat{\varepsilon_{t+1}}'\widehat{\varepsilon_{t+1}}\right)\right\} \\
&= \frac{1}{T}\sum_{t=0}^{T-1}\left\{\frac{1}{2}\left(\widehat{\varepsilon_{t+1}}'\widehat{\varepsilon_{t+1}}-\widehat{\varepsilon_{t+1}}'w_{t+1}^A-w_{t+1}^{A\prime}\widehat{\varepsilon_{t+1}}+w_{t+1}^{A\prime}w_{t+1}^A\right)-\frac{1}{2}\left(\widehat{\varepsilon_{t+1}}'\widehat{\varepsilon_{t+1}}\right)\right\}
\end{aligned}$$

$$r|A = \frac{1}{T}\sum_{t=0}^{T-1}\left\{\frac{1}{2}w_{t+1}^{A\prime}w_{t+1}^A-\frac{1}{2}w_{t+1}^{A\prime}\widehat{\varepsilon_{t+1}}\right\} \tag{253}$$

The objective of this section is to give the probability associated with incorrectly choosing Model B when the true data driving process is Model A or

$$p_A = \Pr\left(r|A<0\right) \tag{254}$$

21.2 Log-Likelihood Ratio Test with Model B

Assume the worst-case shock is generated under Model B,

$$w_{t+1}^{B} = KX_{t+1}^{B} \tag{255}$$

(251) and (252) can be used to construct the log-likelihood ratio test under Model B,

$$r|B = -\frac{1}{T}\sum_{t=0}^{T-1}\left\{\frac{1}{2}w_{t+1}^{B\prime}w_{t+1}^{B} + \frac{1}{2}w_{t+1}^{B\prime}\widehat{\varepsilon_{t+1}}\right\} \tag{256}$$

where (256) is produced under a similar process to (253) except that the worst-case shocks are generated by (255). The objective of this section is to give the probability associated with incorrectly choosing Model A when the true data driving process is Model B or

$$p_B = \Pr\left(r|B > 0\right) \tag{257}$$

21.3 Probability of Detection Error

Using (254) and (257) yields the formula

$$p = \frac{1}{2}\left(p_A + p_B\right) \tag{258}$$

where p is the probability of error in choosing the correct model which implies that $1-p$ is the probability of success in choosing the correct model.

Remark 5 *There lies a positive relationship between p and $\overline{\overline{\theta}}$*

Thus, as $\overline{\overline{\theta}}$ decreases its associated p will decrease as well so that the desired level of detection error will implicitly discipline the value of $\overline{\overline{\theta}}$.

21.4 Probability of Detection Error Estimation

The following steps denote detection error probability estimation:

For p_A

Step 1) Set prior about $\overline{\overline{\theta}}$ and generate $\{\widehat{\varepsilon_{t+1}}\}_{t=0}^{T}$ from $\widehat{\varepsilon}_t \tilde{} N(0_4, I_4)$ for data observations of length T where 0_4 is a $4x4$ matrix of zero's and I_4 is $4x4$ identity matrix.

Step 2) Use pseudo $\{\widehat{\varepsilon_{t+1}}\}_{t=0}^{T}$ to iterate on (240) resulting in $\left\{X_t^A\right\}_{t=0}^{T}$

Step 3) Use $\left\{X_t^A\right\}_{t=0}^{T}$ to iterate on (242) resulting in $\left\{w_{t+1}^A\right\}_{t=0}^{T}$

Step 4) Insert pseudo-generated $\{\widehat{\varepsilon_{t+1}}\}_{t=0}^{T}$ and $\left\{w_{t+1}^A\right\}_{t=0}^{T}$ into (253) and sum

Step 5) Simulate (Repeat steps 1-4) for large sample size

Step 6) Count number of times $r|A < 0$ from simulation and average by number of times simulated: The result is p_A

For p_B

Step 1) Using same prior $\overline{\overline{\theta}}$ generate $\{\varepsilon_{t+1}\}_{t=0}^{T}$ from $\varepsilon_t \tilde{} N(0_4, I_4)$ for data observations of length T where 0_4 is a $4x4$ matrix of zero's and I_4 is $4x4$ identity matrix.

Step 2) Use pseudo $\{\varepsilon_{t+1}\}_{t=0}^{T}$ to iterate on (241) resulting in $\left\{X_t^B\right\}_{t=0}^{T}$

Step 3) Use $\left\{X_t^B\right\}_{t=0}^{T}$ to iterate on (255) resulting in $\left\{w_{t+1}^B\right\}_{t=0}^{T}$

Step 4) Insert pseudo-generated $\{\varepsilon_{t+1}\}_{t=0}^{T}$ and $\left\{w_{t+1}^B\right\}_{t=0}^{T}$ into (256) and sum

Step 5) Simulate (Repeat steps 1-4) for large sample size

Step 6) Count number of times $r|B > 0$ from simulation and average by number of times simulated: The result is p_B

Once p_A and p_B are obtained insert into (258) to obtain p. When estimating p using the above process, begin with prior about $\overline{\overline{\theta}}$ and drive down $\overline{\overline{\theta}}$ until desired level of p is achieved. Reaching this desired level of p in turn disciplines the choice of $\overline{\overline{\theta}}$.

22 Forward Premium Puzzle

22.1 Downward Bias of Estimator

To derive the downward bias of the forward premium estimator, we begin by using an OLS regression of (237) on to (238) so that,

$$\widetilde{S}_{t+1} - \widetilde{S}_t = \widehat{\beta_0} + \widehat{\beta_1}\left(\widetilde{F}_t - \widetilde{S}_t\right) + \epsilon_{t+1} \tag{259}$$

where $e_{t+1} \sim N(0, \sigma^2)$. Inserting (235) and (236) into (259) results in,

$$\begin{aligned} \Pi X_{t+1} - \Pi X_t &= \widehat{\beta_0} + \widehat{\beta_1}\left(\widetilde{F}_t - \widetilde{S}_t\right) + \epsilon_{t+1} \\ \Pi A^0 X_t + \Pi C \epsilon_{t+1} - \Pi X_t &= \widehat{\beta_0} + \widehat{\beta_1}\left(\widetilde{F}_t - \widetilde{S}_t\right) + \epsilon_{t+1} \\ \Pi (A - I) X_t + \Pi C \epsilon_{t+1} &= \widehat{\beta_0} + \widehat{\beta_1} (\Gamma - \Pi) X_t + \epsilon_{t+1} \end{aligned}$$

Next, expand each matrix and vector by the number of observations from $t = 1, ..., T$ so that,

$$\underbrace{\begin{bmatrix} \Pi\left(A-I\right) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Pi\left(A-I\right) \end{bmatrix}}_{Q} \begin{bmatrix} X_1 \\ \vdots \\ X_T \end{bmatrix} + \underbrace{\begin{bmatrix} \Pi C & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Pi C \end{bmatrix}}_{R} \begin{bmatrix} \epsilon_2 \\ \vdots \\ \epsilon_{T+1} \end{bmatrix} =$$

$$\widehat{\beta_0} + \widehat{\beta_1} \overbrace{\underbrace{\begin{bmatrix} (\Gamma-\Pi) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & (\Gamma-\Pi) \end{bmatrix}}_{V} \begin{bmatrix} X_1 \\ \vdots \\ X_T \end{bmatrix}}^{z} + \begin{bmatrix} \epsilon_2 \\ \vdots \\ \epsilon_{T+1} \end{bmatrix}$$

$$\underbrace{QX + R\epsilon}_{Y} = \underbrace{\begin{bmatrix} i & z \end{bmatrix}}_{W} \underbrace{\begin{bmatrix} \widehat{\beta_0} \\ \widehat{\beta_1} \end{bmatrix}}_{\widehat{\beta}} + \epsilon$$

Where subscripts have been eliminated from X and ϵ which imply current time-period. Utilizing ordinary least squares optimal slope estimator formula,

$$\begin{aligned}
\widehat{\beta} &= (W'W)^{-1} W'Y \\
\widehat{\beta} &= \left\{ \begin{bmatrix} i' \\ \\ z' \end{bmatrix} \begin{bmatrix} i & z \end{bmatrix} \right\}^{-1} \begin{bmatrix} i' \\ \\ z' \end{bmatrix} (QX + R\epsilon) \\
\widehat{\beta} &= \begin{bmatrix} i'i & i'z \\ \\ z'i & z'z \end{bmatrix}^{-1} \begin{bmatrix} i' \\ \\ z' \end{bmatrix} (QX + R\epsilon)
\end{aligned}$$

$$\widehat{\beta} = \begin{bmatrix} i'i & i'z \\ \\ z'i & z'z \end{bmatrix}^{-1} \begin{bmatrix} i'QX + i'R\epsilon \\ \\ z'QX + z'R\epsilon \end{bmatrix} \tag{260}$$

To rid the X term on the RHS of (260), exploit equation (241) where if $t = 0$,

$$\begin{aligned}
X_1 &= A^0 X_0 + C\epsilon_1 \qquad (261) \\
X_1 - A^0 X_0 &= C\epsilon_1
\end{aligned}$$

$$X_1 = C\epsilon_1 \tag{262}$$

where it's assumed $X_0 = 0$ in (262). "Pushing" (261) forward in time results in,

$$\begin{aligned} X_2 &= A^0 X_1 + C\epsilon_2 \\ X_2 - A^0 X_1 &= C\epsilon_2 \end{aligned}$$

continuing in this manner results from $t = 1, ..., T$ results in a system of equations,

$$\begin{aligned} X_1 &= C\epsilon_1 \\ X_2 - A^0 X_1 &= C\epsilon_2 \\ X_3 - A^0 X_2 &= C\epsilon_3 \\ &\vdots \\ X_T - A^0 X_{T-1} &= C\epsilon_T \end{aligned}$$

converting to matrix form,

$$\begin{bmatrix} I & 0 & 0 & \cdots & 0 \\ -A^0 & I & 0 & \ddots & \vdots \\ 0 & -A^0 & I & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -A^0 & I \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_T \end{bmatrix} = \begin{bmatrix} C & 0 & 0 & \cdots & 0 \\ 0 & C & 0 & \ddots & \vdots \\ 0 & 0 & C & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & C \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \vdots \\ \epsilon_T \end{bmatrix}$$

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_T \end{bmatrix} = \underbrace{\begin{bmatrix} I & 0 & 0 & \cdots & 0 \\ -A^0 & I & 0 & \ddots & \vdots \\ 0 & -A^0 & I & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -A^0 & I \end{bmatrix}^{-1} \begin{bmatrix} C & 0 & 0 & \cdots & 0 \\ 0 & C & 0 & \ddots & \vdots \\ 0 & 0 & C & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & C \end{bmatrix}}_{\Psi} \underbrace{\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \vdots \\ \epsilon_T \end{bmatrix}}_{\epsilon_{-1}}$$

or,

$$X = \Psi \epsilon_{-1} \tag{263}$$

Inserting (263) into (260) results in,

$$\widehat{\beta} = \begin{bmatrix} i'i & i'z \\ z'i & z'z \end{bmatrix}^{-1} \begin{bmatrix} i'Q\Psi\epsilon_{-1} + i'R\epsilon \\ z'Q\Psi\epsilon_{-1} + z'R\epsilon \end{bmatrix}$$

$$\widehat{\beta} = \begin{bmatrix} i'i & i'z \\ z'i & z'z \end{bmatrix}^{-1} \begin{bmatrix} i'Q\Psi\epsilon_{-1} + i'R\epsilon \\ (VX)'\,Q\Psi\epsilon_{-1} + (VX)'\,R\epsilon \end{bmatrix}$$

$$\widehat{\beta} = \begin{bmatrix} i'i & i'z \\ z'i & z'z \end{bmatrix}^{-1} \begin{bmatrix} i'Q\Psi\epsilon_{-1} + i'R\epsilon \\ (V\Psi\epsilon_{-1})'\,Q\Psi\epsilon_{-1} + (V\Psi\epsilon_{-1})'\,R\epsilon \end{bmatrix}$$

$$\widehat{\beta} = \begin{bmatrix} i'i & i'z \\ z'i & z'z \end{bmatrix}^{-1} \begin{bmatrix} i'Q\Psi\epsilon_{-1} + i'R\epsilon \\ \epsilon'_{-1}\Psi'V'Q\Psi\epsilon_{-1} + \epsilon'_{-1}\Psi'V'R\epsilon \end{bmatrix}$$

$$\widehat{\beta} = \begin{bmatrix} i'i & i'z \\ z'i & z'z \end{bmatrix}^{-1} \begin{bmatrix} i'Q\Psi\epsilon_{-1} + i'R\epsilon \\ \epsilon'_{-1}\Psi'V'Q\Psi\epsilon_{-1} + \epsilon'_{-1}\Psi'V'R\epsilon \end{bmatrix}$$

$$\widehat{\beta} = \frac{1}{(i'i)\,(z'z) - (i'z)\,(z'i)} \begin{bmatrix} z'z & -i'z \\ -z'i & i'i \end{bmatrix} \begin{bmatrix} i'Q\Psi\epsilon_{-1} + i'R\epsilon \\ \epsilon'_{-1}\Psi'V'Q\Psi\epsilon_{-1} + \epsilon'_{-1}\Psi'V'R\epsilon \end{bmatrix}$$

$$\widehat{\beta} = \frac{1}{T\left[(VX)'VX\right]-(i'VX)\left[(VX)'i\right]}\begin{bmatrix} (VX)'VX & -i'VX \\ -(VX)'i & i'i \end{bmatrix}\begin{bmatrix} i'Q\Psi\epsilon_{-1}+i'R\epsilon \\ \epsilon_{-1}'\Psi'V'Q\Psi\epsilon_{-1}+\epsilon_{-1}'\Psi'V'R\epsilon \end{bmatrix}$$

$$\widehat{\beta} = \frac{1}{TX'V'VX-i'VXX'V'i}\begin{bmatrix} X'V'VX & -i'VX \\ -X'V'i & i'i \end{bmatrix}\begin{bmatrix} i'Q\Psi\epsilon_{-1}+i'R\epsilon \\ \epsilon_{-1}'\Psi'V'Q\Psi\epsilon_{-1}+\epsilon_{-1}'\Psi'V'R\epsilon \end{bmatrix}$$

Thus,

$$\widehat{\beta} = \frac{1}{T\epsilon_{-1}'\Psi'V'V\Psi\epsilon_{-1}-i'V\Psi\epsilon_{-1}\epsilon_{-1}'\Psi'V'i}\begin{bmatrix} \epsilon_{-1}'\Psi'V'V\Psi\epsilon_{-1} & -i'V\Psi\epsilon_{-1} \\ -\epsilon_{-1}'\Psi'V'i & T \end{bmatrix}\begin{bmatrix} i'Q\Psi\epsilon_{-1}+i'R\epsilon \\ \epsilon_{-1}'\Psi'V'Q\Psi\epsilon_{-1}+\epsilon_{-1}'\Psi'V'R\epsilon \end{bmatrix} \tag{264}$$

where extensive use of $z = VX$ and $X = \Psi\epsilon_{-1}$ was made. Taking the probability limit of (264) results in,

$$p\lim\left(\widehat{\beta}\right) = E\{\cdot\}$$

where the $\cdot$ inside the brackets represents RHS terms of (264) .Distributing the expectations term,

$$p\lim\left(\widehat{\beta}\right)=\Omega E\left\{\begin{bmatrix}\epsilon_{-1}'\Psi'V'V\Psi\epsilon_{-1} & -i'V\Psi\epsilon_{-1}\\ \\ -\epsilon_{-1}'\Psi'V'i & T\end{bmatrix}\begin{bmatrix}i'Q\Psi\epsilon_{-1}+i'R\epsilon\\ \\ \epsilon_{-1}'\Psi'V'Q\Psi\epsilon_{-1}+\epsilon_{-1}'\Psi'V'R\epsilon\end{bmatrix}\right\}$$

where $\Omega=1/\left[Ttr(\Psi'V'V\Psi)\sigma^2+\epsilon_{-1}^{m\prime}\Psi'V'V\Psi\epsilon_{-1}^{m}-tr\left(i'V\Psi\Psi'V'i\right)\sigma^2+i'V\Psi\epsilon_{-1}^{m}\epsilon_{-1}^{m\prime}\Psi'V'i\right]$. Further consolidation yields,

$$p\lim\left(\widehat{\beta}\right)=\Omega^{*}E\left\{\begin{bmatrix}\epsilon_{-1}'\Psi'V'V\Psi\epsilon_{-1}\left(i'Q\Psi\epsilon_{-1}+i'R\epsilon\right)-i'V\Psi\epsilon_{-1}\left(\epsilon_{-1}'\Psi'V'Q\Psi\epsilon_{-1}+\epsilon_{-1}'\Psi'V'R\epsilon\right)\\ \\ -\epsilon_{-1}'\Psi'V'i\left(i'Q\Psi\epsilon_{-1}+i'R\epsilon\right)+T\left(\epsilon_{-1}'\Psi'V'Q\Psi\epsilon_{-1}+\epsilon_{-1}'\Psi'V'R\epsilon\right)\end{bmatrix}\right\}$$

where $\Omega^{*}=1/\sigma^2\left[Ttr(\Psi'V'V\Psi)-tr\left(i'V\Psi\Psi'V'i\right)\right]$.Distributing E operator through the matrix results in,

$$p\lim\left(\widehat{\beta}\right)=\Omega^{*}\begin{bmatrix}0\\ \\ E\left\{-\epsilon_{-1}'\Psi'V'i\left(i'Q\Psi\epsilon_{-1}+i'R\epsilon\right)+T\left(\epsilon_{-1}'\Psi'V'Q\Psi\epsilon_{-1}+\epsilon_{-1}'\Psi'V'R\epsilon\right)\right\}\end{bmatrix}$$

where further simplification finally results in,

$$p\lim\left(\beta_1\right)=\frac{E\left\{-\epsilon_{-1}'\Psi'V'i\left(i'Q\Psi\epsilon_{-1}+i'R\epsilon\right)+T\left(\epsilon_{-1}'\Psi'V'Q\Psi\epsilon_{-1}+\epsilon_{-1}'\Psi'V'R\epsilon\right)\right\}}{Ttr(\Psi'V'V\Psi)\sigma^2-tr\left(i'V\Psi\Psi'V'i\right)\sigma^2}\qquad(265)$$

where the superscript m in $\epsilon^{m'}_{-1}$ denotes the mean of ϵ'_{-1} so that $\epsilon^{m'}_{-1} = 0$. Additionally, manipulations using $E(\epsilon_{-1}) = E(\epsilon) = E(\epsilon_{-1}\epsilon') = 0$ aided in eliminating $\widehat{\beta_0}$ from the $\widehat{\beta}$ vector. Further simplification of (265) results in,

$$
\begin{aligned}
p\lim\left(\widehat{\beta_1}\right) &= \frac{E\left\{-\epsilon'_{-1}\Psi'V'ii'Q\Psi\epsilon_{-1} - \epsilon'_{-1}\Psi'V'ii'R\epsilon + T\left(\epsilon'_{-1}\Psi'V'Q\Psi\epsilon_{-1} + \epsilon'_{-1}\Psi'V'R\epsilon\right)\right\}}{\sigma^2\left[Ttr(\Psi'V'V\Psi) - tr\left(i'V\Psi\Psi'V'i\right)\right]} \\
p\lim\left(\widehat{\beta_1}\right) &= \frac{\sigma^2\left[-tr\left(\Psi'V'ii'Q\Psi\right) + Ttr\left(\Psi'V'Q\Psi\right)\right]}{\sigma^2\left[Ttr(\Psi'V'V\Psi) - tr\left(i'V\Psi\Psi'V'i\right)\right]} \\
p\lim\left(\widehat{\beta_1}\right) &= \frac{tr\left[T\Psi'V'Q\Psi - \Psi'V'ii'Q\Psi\right]}{tr\left[T\Psi'V'V\Psi - i'V\Psi\Psi'V'i\right]} \\
p\lim\left(\widehat{\beta_1}\right) &= \frac{tr\left[T\Psi'V'Q\Psi - \Psi'V'ii'Q\Psi\right]}{tr\left[T\Psi'V'V\Psi - \Psi'V'ii'V\Psi\right]} \\
p\lim\left(\widehat{\beta_1}\right) &= \frac{tr\left[\Psi'V'\left(TI - ii'\right)Q\Psi\right]}{tr\left[\Psi'V'\left(TI - ii'\right)V\Psi\right]} \\
p\lim\left(\beta_1\right) &= \frac{Ttr\left[\Psi'V'\left(I - \frac{ii'}{T}\right)Q\Psi\right]}{Ttr\left[\Psi'V'\left(I - \frac{ii'}{T}\right)V\Psi\right]} \\
p\lim\left(\widehat{\beta_1}\right) &= \frac{tr\left(\Psi'V'M_iQ\Psi\right)}{tr\left(\Psi'V'M_iV\Psi\right)} \\
p\lim\left(\widehat{\beta_1}\right) &= 1 - 1 + \frac{tr\left(\Psi'V'M_iQ\Psi\right)}{tr\left(\Psi'V'M_iV\Psi\right)} \\
p\lim\left(\widehat{\beta_1}\right) &= 1 - 1 + \frac{tr\left(\Psi'V'M_iQ\Psi\right)}{tr\left(\Psi'V'M_iV\Psi\right)} \\
p\lim\left(\widehat{\beta_1}\right) &= 1 - \left[\frac{tr\left(\Psi'V'M_iV\Psi\right) - tr\left(\Psi'V'M_iQ\Psi\right)}{tr\left(\Psi'V'M_iV\Psi\right)}\right]
\end{aligned}
$$

$$
\therefore \quad p\lim\left(\widehat{\beta_1}\right) = 1 - \frac{tr\left[\Psi'V'M_i\left(V - Q\right)\Psi\right]}{tr\left(\Psi'V'M_iV\Psi\right)} \tag{266}
$$

22.2 Consistency of Estimator

To show consistency of the slope estimator, take the limit of $\overline{\overline{\theta}} \to \infty$ to $p\lim\left(\widehat{\beta_1}\right)$ or,

$$
\lim_{\overline{\overline{\theta}}\to\infty}\left\{p\lim\left(\widehat{\beta_1}\right)\right\} = \lim_{\overline{\overline{\theta}}\to\infty}\left\{1 - \frac{tr\left[\Psi'V'M_i\left(V - Q\right)\Psi\right]}{tr\left(\Psi'V'M_iV\Psi\right)}\right\} \tag{267}
$$

Focusing on the $(V-Q)$ matrix,

$$(V-Q) = \begin{bmatrix} (\Gamma-\Pi) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & (\Gamma-\Pi) \end{bmatrix} - \begin{bmatrix} \Pi\left(A^0-I\right) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Pi\left(A^0-I\right) \end{bmatrix}$$

$$(V-Q) = \begin{bmatrix} \Gamma-\Pi A^0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Gamma-\Pi A^0 \end{bmatrix}$$

$$\lim_{\bar{\bar{\theta}}\to\infty}(V-Q) = \begin{bmatrix} \lim_{\bar{\bar{\theta}}\to\infty}\left\{\Gamma-\Pi A^0\right\} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lim_{\bar{\bar{\theta}}\to\infty}\left\{\Gamma-\Pi A^0\right\} \end{bmatrix}$$

$$\lim_{\bar{\bar{\theta}}\to\infty}(V-Q) = \begin{bmatrix} \Gamma-\Pi A & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Gamma-\Pi A \end{bmatrix} \tag{268}$$

where we recall that $A^0 = A + C\left[\bar{\bar{\theta}} I_{4x4} - C'PC\right]^{-1} C'PA$ and $C\left[\bar{\bar{\theta}} I_{4x4} - C'PC\right]^{-1} C'PA \rightarrow 0$ as $\bar{\bar{\theta}} \rightarrow \infty$. Now, to show that (268) is essentially a null matrix, we show that $\Gamma = \Pi A$ where recall,

$$\Gamma = \begin{bmatrix} \rho_3 & -\rho_4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Gamma = \begin{bmatrix} 0 & 0 & \rho_3 & -\rho_4 \end{bmatrix}$$

and

$$\Pi A = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho_1 & 0 & 0 & 0 \\ 0 & \rho_2 & 0 & 0 \\ 0 & 0 & \rho_3 & 0 \\ 0 & 0 & 0 & \rho_4 \end{bmatrix}$$

$$\Pi A = \begin{bmatrix} 0 & 0 & \rho_3 & -\rho_4 \end{bmatrix}$$

Implementing this null matrix back in (267) eliminates the bias term reducing (267) to

$$\lim_{\overline{\overline{\theta}}\to\infty}\left\{p\lim\left(\beta_1\right)\right\}=1 \tag{269}$$

Hence an unbiased slope estimator.

22.3 Estimation

Step 1) Generate sequence of pseudo errors $\{\varepsilon_t\}_{t=1}^{T}$ where $\varepsilon_t \sim N\left(0_4, I_4\right)$ and T is equivalent to length of observational data

Step 2) Insert generated $\{\varepsilon_t\}_{t=1}^{T}$ into (239) and iterate to generate $\{X_t\}_{t=1}^{T}$

Step 3) Insert $\{X_t\}_{t=1}^{T}$ into (235) and (236) to generate $\left\{\widetilde{S}_t\right\}_{t=1}^{T}$ and $\left\{\widetilde{F}_t\right\}_{t=1}^{T}$ respectively

Step 4) Construct $\left\{\widetilde{S}_{t+1}-\widetilde{S}_t\right\}_{t=1}^{T-1}=\left\{\widetilde{S}_{t+1}\right\}_{t=1}^{T-1}-\left\{\widetilde{S}_t\right\}_{t=1}^{T-1}$ and $\left\{\widetilde{F}_t-\widetilde{S}_t\right\}_{t=1}^{T-1}=\left\{\widetilde{F}_t\right\}_{t=1}^{T-1}-\left\{\widetilde{S}_t\right\}_{t=1}^{T-1}$

Step 5) Convert into vector so that

$$\mathbf{S}_{t+1}-\mathbf{S}_t=\begin{bmatrix}\widetilde{S}_2-\widetilde{S}_1\\ \vdots\\ \widetilde{S}_T-\widetilde{S}_{T-1}\end{bmatrix}$$

$$\mathbf{F}_t-\mathbf{S}_t=\begin{bmatrix}\widetilde{F}_1-\widetilde{S}_1\\ \vdots\\ \widetilde{F}_{T-1}-\widetilde{S}_{T-1}\end{bmatrix}$$

From Step 5, utilize vectors and perform the following regression

$$\mathbf{S}_{t+1} - \mathbf{S}_t = \widehat{\beta_0} + \widehat{\beta_1} \left(\mathbf{F}_t - \mathbf{S}_t\right) + e_{t+1} \tag{270}$$

where (273) uses standard ordinary least squares. If $\mathbf{F}_t - \mathbf{S}_t$ and $\mathbf{S}_{t+1} - \mathbf{S}_t$ are temporarily defined as $\mathbf{X}$ and $\mathbf{Y}$ respectively then the estimators $\widehat{\beta_0}$ and $\widehat{\beta_1}$ are

$$\widehat{\beta_1} = \left(\mathbf{X}'\mathbf{X}\right)^{-1} \left(\mathbf{X}'\mathbf{Y}\right) \tag{271}$$

$$\widehat{\beta_0} = \overline{\mathbf{Y}} - \widehat{\beta_1} \overline{\mathbf{X}} \tag{272}$$

where $\overline{\mathbf{X}}$ and $\overline{\mathbf{Y}}$ represent the mean of $\mathbf{X}$ and $\mathbf{Y}$ respectively. $\overline{\overline{\theta}}$ which correspond to detection error probabilities of $p(\overline{\overline{\theta}}) = 0.100$ and $p(\overline{\overline{\theta}}) = 0.000$ can be inserted within Step 2) in order to yield $\widehat{\beta_1} > 0$ and $\widehat{\beta_1} < 0$ respectively.

22.4 Simulation

Parameter Values Model parameters used for simulation are,

$$\begin{aligned} \beta &= 0.99 \\ \gamma &= 10 \\ \theta &= 0.5 \\ \phi &= 0.5 \end{aligned}$$

Step 1) Generate sequence of pseudo errors $\{\varepsilon_t\}_{t=1}^{T}$ where $\varepsilon_t \sim N\left(0_4, I_4\right)$ and T is equivalent to length of observational data

Step 2) Insert generated $\{\varepsilon_t\}_{t=1}^{T}$ into (239) and iterate to generate $\{X_t\}_{t=1}^{T}$

Step 3) Insert $\{X_t\}_{t=1}^{T}$ into (235) and (236) to generate $\left\{\widetilde{S}_t\right\}_{t=1}^{T}$ and $\left\{\widetilde{F}_t\right\}_{t=1}^{T}$ respectively

Step 4) Construct $\left\{\widetilde{S}_{t+1} - \widetilde{S}_t\right\}_{t=1}^{T-1} = \left\{\widetilde{S}_{t+1}\right\}_{t=1}^{T-1} - \left\{\widetilde{S}_t\right\}_{t=1}^{T-1}$ and $\left\{\widetilde{F}_t - \widetilde{S}_t\right\}_{t=1}^{T-1} = \left\{\widetilde{F}_t\right\}_{t=1}^{T-1} - \left\{\widetilde{S}_t\right\}_{t=1}^{T-1}$

Step 5) Convert into vector so that

$$\mathbf{S}_{t+1} - \mathbf{S}_t = \begin{bmatrix} \widetilde{S}_2 - \widetilde{S}_1 \\ \vdots \\ \widetilde{S}_T - \widetilde{S}_{T-1} \end{bmatrix}$$

$$\mathbf{F}_t - \mathbf{S}_t = \begin{bmatrix} \widetilde{F}_1 - \widetilde{S}_1 \\ \vdots \\ \widetilde{F}_{T-1} - \widetilde{S}_{T-1} \end{bmatrix}$$

From Step 5, utilize vectors and perform the following regression

$$\mathbf{S}_{t+1} - \mathbf{S}_t = \widehat{\beta_0} + \widehat{\beta_1}\left(\mathbf{F}_t - \mathbf{S}_t\right) + e_{t+1} \tag{273}$$

where (273) uses standard ordinary least squares. If $\mathbf{F}_t - \mathbf{S}_t$ and $\mathbf{S}_{t+1} - \mathbf{S}_t$ are temporarily defined as $\mathbf{X}$ and $\mathbf{Y}$ respectively then the estimators $\widehat{\beta_0}$ and $\widehat{\beta_1}$ are

$$
\begin{aligned}
\widehat{\beta_1} &= (\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{Y}) && (274)\\
\widehat{\beta_0} &= \overline{\mathbf{Y}} - \widehat{\beta_1}\overline{\mathbf{X}} && (275)
\end{aligned}
$$

where $\overline{\mathbf{X}}$ and $\overline{\mathbf{Y}}$ represent the mean of $\mathbf{X}$ and $\mathbf{Y}$ respectively. Repeat Step 1)-Step 5) for large simulation size N where the simulation size is conducted under a specific $\overline{\overline{\theta}}$ corresponding to detection error probability $p(\overline{\overline{\theta}})$. This will generate N number of $\widehat{\beta_1}$ under a specific $\overline{\overline{\theta}}$ regime, ultimately yielding desired PDF.

23 Data

23.1 Description

1. Home Output: Defined as

$$
\mathbf{x}_t^D = \begin{bmatrix} x_{1986:III} \\ \vdots \\ x_{2013:III} \end{bmatrix} \tag{276}
$$

where $\mathbf{x}_t^D$ represents a time-series data vector containing aggregate U.S. real G.D.P observations in millions of chained 2009 dollars (from Federal Reserve Bank of St.Louis) and quarterly frequency from 3rd quarter of 1986 to 3rd quarter of 2013.

2. Foreign Output: This variable is constructed using

$$y_t^D = \frac{y_t^n}{p_t} \tag{277}$$

where y_t represents U.K. real G.D.P constructed from y_t^n which is aggregate U.K. nominal G.D.P observations in millions of pounds (from Federal Reserve Bank of St. Louis) and p_t which is U.K. C.P.I (with 2010=100 and from Federal Reserve Bank of St. Louis). Using (277) the following can be formed

$$\mathbf{y}_t^D = \begin{bmatrix} y_{1986:III} \\ \vdots \\ y_{2013:III} \end{bmatrix} \tag{278}$$

where $\mathbf{y}_t^D$ represents a time-series data vector containing constructed aggregate U.K. real G.D.P observations in quarterly frequency from 3rd quarter of 1986 to 3rd quarter of 2013.

3. Home Money: Defined as

$$\mathbf{M}_t^D = \begin{bmatrix} M_{1986:III} \\ \vdots \\ M_{2013:III} \end{bmatrix} \tag{279}$$

where $\mathbf{M}_t^D$ is a time-series data vector containing aggregate U.S. Household Financial Assets and Currency observations (from Federal Reserve Bank of St.Louis) in quarterly frequency from 3rd quarter of 1986 to 3rd quarter of 2013.

4. Foreign Money: Defined as

$$\mathbf{N}_t^D = \begin{bmatrix} N_{1986:III} \\ \vdots \\ N_{2013:III} \end{bmatrix} \tag{280}$$

where $\mathbf{N}_t$ is a time-series data vector containing aggregate U.K. Household Outstanding Holdings of Notes/Coin observations in millions of pounds (from Bank of England) and quarterly frequency from 3rd quarter of 1986 to 3rd quarter of 2013.

5. Spot Exchange Rate: Defined as

$$\mathbf{S}_t^D = \begin{bmatrix} S_{1986:III} \\ \vdots \\ S_{2013:III} \end{bmatrix} \tag{281}$$

where $\mathbf{S}_t^D$ is a time-series data vector containing \$/£ spot exchange rate observations (from Federal Reserve Bank of St.Louis) in quarterly frequency from 3rd quarter of 1986 to 3rd quarter of 2013.

6. Forward Exchange Rate: Defined as

$$\mathbf{F}_t^D = \begin{bmatrix} F_{1986:III} \\ \vdots \\ F_{2013:III} \end{bmatrix} \tag{282}$$

where $\mathbf{F}_t^D$ is a time-series data vector containing \$/£ 1-month forward exchange rate observations (from Bank of England) in quarterly frequency from 3rd quarter of 1986 to 3rd quarter of 2013.

23.2 Stochastic Processes Estimation with Data

In this paper's model, the stochastic process are assumed to evolve according to $(109)-(112)$ restated below for convenience

$$\begin{aligned} x_{t+1} &= (1-\rho_1) + \rho_1 x_t + \varepsilon_{t+1}^x \\ y_{t+1} &= (1-\rho_2) + \rho_2 y_t + \varepsilon_{t+1}^y \\ M_{t+1} &= (1-\rho_3) + \rho_3 M_t + \varepsilon_{t+1}^M \\ N_{t+1} &= (1-\rho_4) + \rho_4 N_t + \varepsilon_{t+1}^N \end{aligned}$$

The following steps indicate how to process data:

Step 1) Take log of each element in (276), (278), (279), and (280) producing $\log(\mathbf{x}_t^D), \log(\mathbf{y}_t^D), \log(\mathbf{M}_t^D)$, and $\log(\mathbf{N}_t^D)$ respectively.

Step 2) HP-Filter each logged data vectors which separates trend and cyclical components producing $\log(\mathbf{x}_t^D)^C, \log(\mathbf{y}_t^D)^C, \log(\mathbf{M}_t^D)^C$, and $\log(\mathbf{N}_t^D)^C$ where each superscript C denotes the logged cyclical component of the superscripts respective data vector.

Step 3) Insert $\log(\mathbf{x}_t^D)^C, \log(\mathbf{y}_t^D)^C, \log(\mathbf{M}_t^D)^C$, and $\log(\mathbf{N}_t^D)^C$ in place of x_t, y_t, M_t, and N_t respectively yielding

$$\log(\mathbf{x}_{t+1}^D)^C = (1-\rho_1) + \rho_1 \log(\mathbf{x}_t^D)^C + \varepsilon_{t+1}^x \quad (283)$$
$$\log(\mathbf{y}_{t+1}^D)^C = (1-\rho_2) + \rho_2 \log(\mathbf{y}_t^D)^C + \varepsilon_{t+1}^y \quad (284)$$
$$\log(\mathbf{M}_{t+1}^D)^C = (1-\rho_3) + \rho_3 \log(\mathbf{M}_t^D)^C + \varepsilon_{t+1}^M \quad (285)$$
$$\log(\mathbf{N}_{t+1}^D)^C = (1-\rho_4) + \rho_4 \log(\mathbf{N}_t^D)^C + \varepsilon_{t+1}^N \quad (286)$$

Step 4) Convert system (283) – (286) into VAR form

$$\mathbb{X}_{t+1} = \rho_C + \rho_S \mathbb{X}_t + \varepsilon_{t+1} \quad (287)$$

where

$$\mathbb{X}_t = \begin{bmatrix} \log(\mathbf{x}_t^D)^C \\ \log(\mathbf{y}_t^D)^C \\ \log(\mathbf{M}_t^D)^C \\ \log(\mathbf{N}_t^D)^C \end{bmatrix}, \quad \varepsilon_{t+1} = \begin{bmatrix} \varepsilon_{t+1}^x \\ \varepsilon_{t+1}^y \\ \varepsilon_{t+1}^M \\ \varepsilon_{t+1}^N \end{bmatrix}, \quad \rho_C = \begin{bmatrix} (1-\rho_1) \\ (1-\rho_2) \\ (1-\rho_3) \\ (1-\rho_4) \end{bmatrix}, \quad \rho_S = \begin{bmatrix} \rho_1 & 0 & 0 & 0 \\ 0 & \rho_2 & 0 & 0 \\ 0 & 0 & \rho_3 & 0 \\ 0 & 0 & 0 & \rho_4 \end{bmatrix}$$

Step 5) Estimate (287) via ordinary least squares resulting in

$$\widehat{\rho_S} = \begin{bmatrix} \widehat{\rho}_1 & \widehat{\rho}_2 & \widehat{\rho}_3 & \widehat{\rho}_4 \end{bmatrix} \tag{288}$$

$$\widehat{\rho_C} = (1 - \widehat{\rho_S})\,\overline{\mathbb{X}} \tag{289}$$

where the elements of (288) and (289) are used in place of their true counterparts in system (109) – (112)

23.3 Forward Premium Regression with Data

Above, pseudo observations were generated for the regression of forward premium on gross rate of depreciation so for comparison to data-driven regression, the following steps are performed:

Step 1) Take the log of each element in (281)and (282) producing $\log(\mathbf{S}_t)$ and $\log(\mathbf{F}_t)$ respectively

Step 2) Construct $\log(\mathbf{F}_t^D) - \log(\mathbf{S}_t^D)$ which is forward premium

Step 3) Construct $\log(\mathbf{S}_{t+1}^D) - \log(\mathbf{S}_t^D)$ which is gross rate of depreciation

Step 4) Define $\mathbb{F}_t - \mathbb{S}_t = \log(\mathbf{F}_t^D) - \log(\mathbf{S}_t^D)$, $\mathbb{S}_{t+1} - \mathbb{S}_t = \log(\mathbf{S}_{t+1}^D) - \log(\mathbf{S}_t^D)$, and perform the following regression via ordinary least squares

$$\mathbb{S}_{t+1} - \mathbb{S}_t = \alpha_0 + \alpha_1\,(\mathbb{F}_t - \mathbb{S}_t) + e_{t+1} \tag{290}$$

where $\widehat{\alpha_0}$ and $\widehat{\alpha_1}$ are estimators to α_0 and α_1 respectively.

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ABSTRACT

CAN ANIMAL SPIRITS SOLVE THE FORWARD PREMIUM PUZZLE?

by

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This dissertation decomposes, discusses, and ventures to solve an international macroeconomic anomaly known as the "Forward Premium Puzzle" into three main chapters. Chapter 1 explores the state of research pertaining to the Forward Premium Puzzle, which derives from a failure in the Uncovered Interest Parity (UIP). The body of literature is split into three branches i. Those works advocating the presence of an anomaly due to assumption of Rational Expectations, ii. Works preserving the assumption of Rational Expectations and instead discuss a bias due to Risk Premia, and iii. Research focused on econometric implementation of the forward premium estimator. Furthermore, a tour of Animal Spirits is given and how applications out of control theory, or Robust Control, serves as a vehicle of implementing Animal Spirits within modern macroeconomics as a potential resolution to the anomaly.

In Chapter 2, a dynamic stochastic general equilibrium (DSGE) two-country two-money model is fitted with Robust Control, inducing fear of model misspecification, or pessimism, in international households. The forward premium bias, produced from a collapse in uncovered interest parity (UIP), is a direct function of pessimism. Under various regimes of pessimism, the forward premium estimator emulates both features of international data and unbias UIP.

In Chapter 3 pessimism, via a parameter θ, is implemented in a two-country, two-money dynamic stochastic general equilibrium (DSGE) model fitted with robust control (RC). Using detection error probability methodology, pessimism is calibrated using international data and simulations from the RC-DSGE model. Data-driven pessimism and its simulation-based counterpart are compared to determine how the animal spirit, produced from the model, performs against pessimism implied by the data.

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