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Reliability Estimates of Generalized Poisson Distribution and Generalized Geometric Series Distribution

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Discrete distributions have played an important role in the reliability theory. In order to obtain Bayes estimators, researchers have adopted various conventional techniques. Generalizing the results of Maiti (1995), Chaturvadi and Tomer (2002) dealt with the problem of estimating $P\{X_1, X_2, \ldots, X_k \leq Y\}$, where random variables X and Y were assumed to follow a negative binomial distribution. Agit et al. obtained Bayesian estimates of the reliability functions and $P\{X_1, X_2, \ldots, X_k \leq Y\}$ considering X and Y following binomial and Poisson distributions. The reliability function of the generalized Poisson and generalized geometric distribution is investigated. The expression for $P\{X_1, X_2, \ldots, X_k \leq Y\}$ was obtained with X's and Y following a Poisson distribution and some particular cases are shown.

Keywords: Generalized Poisson distribution generalized geometric distribution, reliability function, Bayes estimators

Introduction

Much research exists in the literature for estimating various parametric functions of several discrete distributions through classical and Bayesian approaches. Cacoullos and Charalambildes (1975) obtained MVUE for truncated binomial and negative binomial distributions. Bayesian estimation of the parameter of binomial distribution has been considered by Chew (1971). Barton (1961) and Glasser (1962) obtained UMVUE of $P(X = x)$ for Poisson distribution. Blyth (1980) studied the absolute error of UMVUE of the probability of success of binomial distribution. For a random variable X following binomial distribution, Pulskanp (1990) has shown that the UMVUE of $P(X = x)$ is admissible under squared-error loss function

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when $X = 0$ or n. Irony (1992) developed Bayesian estimation procedures related to Poisson distribution. Guttman (1958) and Patil (1963) provided UMVUEs of parametric functions of negative binomial distribution. Patil & Wani (1966) obtained UMVEUs of distribution function of various distributions. Roy & Mitra (1957) considered the problem of minimum variance unbiased estimation of univariate power series distribution. Patil (1978) generalized their results to multivariate modified power series distribution. Patil & Bildikar (1966) derived MVUE for logarithmic series distribution.

Discrete distributions have played important role in reliability theory. Kumar & Bhattacharya (1989) considered negative binomial distribution as the life time modal and obtained UMVUEs of the mean life and reliability function. Another measure of reliability is under stress-strength setup in the probability $Pr\{X \leq Y\}$, under the assumption that X and Y followed geometric distribution and derived UMVUE & Bayes estimator. Chaturvedi & Tomer (2002) considered classical & Bayesian Estimation procedures for the reliability function of the negative binomial distribution from a different approach generalizing the results of Maiti (1995), they dealt with the problem of estimating $P\{X_1, X_2, \ldots, X_k \leq Y\}$, where random variables X & Y were assumed to follow negative binomial distributions. Chaturvadi, et al. (2007) considered Bionomail and Poisson distribution and obtained the Bayesian estimators of reliability function and dealt with the problem of estimating $P\{X_1, X_2, \ldots, X_k \leq Y\}$, where the random variables X & Y were assumed to follow binomial and Poisson distributions.

In order to obtain Bayes estimators of parameter and various parametric functions of different distributions, researchers have adopted a conventional technique, i.e., obtaining their posterior means. This article considers the Generalized Poisson and Generalized Geometric distributions and the problems of estimating reliability functions and $P = P\{X_1, X_2, \ldots, X_k \leq Y\}$ from a Bayesian viewpoint. Bayes estimators of these parametric functions are derived. It is worth mentioning that in contrary to conventional approach, only estimators of factorial moments are needed to estimate these parametric functions and no separate dealing is needed.

Generalized Poisson Distribution

The random variable follows Generalized Poisson distribution with parameter λ and β if its pmf is

$$P(X=x)=\frac{\lambda^{X}\left(1+x\beta\right)^{x-1}e^{-\lambda(\beta x+1)}}{x!},\ x=0,1,2,\ldots,\lambda>0,\ 0<\beta<\frac{1}{\lambda} \tag{1}$$

The reliability function at a specific mission time, for example, t_0 (≥ 0) is

$$R(t_0)=\sum_{x=t_0}^{\infty}\frac{\lambda^{x}\left(1+x\beta\right)^{x-1}e^{-\lambda(\beta x+1)}}{x!} \tag{2}$$

and the hazard rate function is

$$h(t_0)=\frac{P(t_0,\lambda,\beta)}{R(t_0)}=\left[\sum_{x=0}^{\infty}\frac{\lambda^{x}e^{-\lambda\beta x}\left(1+\left(x+t_0\right)\beta\right)^{x+t_0-1}}{\left(1+t_0\beta\right)^{t_0-1}\left(x+t_0\right)_{(x)}}\right]^{-1} \tag{3}$$

Let $\{X_i\}$, $i = 1, 2, 3, \ldots, k$ be k independent random variables following a generalized Poisson distribution (1) with parameters λ_i and β (known) and Y is a random variable, independent of X's following generalized Poisson distribution with parameter u. Denoting

$$X^{*}=\sum_{i=1}^{k}X_i \text{ and } \lambda^{*}=\sum_{i=1}^{k}\lambda_i \tag{4}$$

From additive property of generalized Poisson distribution

$$P=\sum_{x^{*}=0}^{\infty}\sum_{y=x^{*}}^{\infty}p\left(x^{*};\lambda^{*}\right).p\left(y;u\right) \tag{5}$$

Next, Bayes estimators of $R(t_0)$ and 'P' for generalized Poisson distribution are estimated.

Bayes Estimation Of $R(t_0)$ and 'P' For Generalized Poisson Distribution

The likelihood function given the random sample information $X = (X_1, X_2, \dots, X_n)$ is

$$L(\lambda, \beta / t) = \prod_{i=1}^{n} \left(\frac{(1 + x_i \beta)^{x_i - 1}}{x_i !} \right) \lambda^t e^{-\lambda(\beta t + n)} \tag{6}$$

where

$$t = \sum_{i=1}^{n} x_i$$

Because, $\lambda > 0$ consider the prior distribution for λ when β is known to be gamma with parameters (α, θ) and pdf

$$g(\lambda) = \frac{\theta^{\alpha} e^{-\theta \lambda} \lambda^{\alpha - 1}}{\Gamma(\alpha)}, \lambda, \alpha, \theta > 0 \tag{7}$$

From (6) and (7), the posterior density function of λ is given by

$$\prod(\lambda / t) = \frac{e^{-\lambda(\beta t + n + \theta)} \lambda^{t + \alpha - 1} (\beta t + n + \theta)^{t + \alpha}}{\Gamma(t + \alpha)} \tag{8}$$

The Bayesian estimator of λ^p, for $p > 0$, is given by

$$\begin{aligned} \hat{\lambda}^p &= \int_0^{\infty} \lambda^p \prod(\lambda / t)\, d\lambda \\ \hat{\lambda}^p &= \frac{\Gamma(t + \alpha + p)}{\Gamma(t + \alpha)} (\beta t + n + \theta)^{-p} \end{aligned} \tag{9}$$

Now, Equation (1) can be written as

$$P(x = x) = \frac{1}{x!} \sum_{r=0}^{\infty} (-1)^r \frac{(\beta x + 1)^{r + x - 1}}{r!} \lambda^{r + x}$$

On using (9) the Bayes estimator of $P(x;\lambda)$ at a specific point 'X' is

$$\begin{aligned}\hat{p}(x;\lambda) &= \frac{1}{x!}\sum_{r=0}^{\infty}(-1)^r \frac{(\beta x+1)^{r+x-1}}{r!}\frac{\Gamma(t+\alpha+r+x)}{\Gamma(t+\alpha)}(\beta t+n+\theta)^{-(r+x)} \\ &= \frac{(\beta x+1)^{x-1}}{(\beta t+n+\theta)^{x}}\cdot\frac{1}{x!}\sum_{r=0}^{\infty}(-1)^r\frac{(\beta x+1)^r}{r!}\frac{(t+\alpha+r+x-1)!}{(t+\alpha-1)!(\beta t+n+\theta)^r} \\ &= (\beta x+1)^{x-1}\binom{t+\alpha+x-1}{x}\frac{(\beta t+n+\theta)^{t+\alpha}}{(\beta(t+x)+n+\theta+1)^{t+\alpha+x}}\end{aligned} \tag{10}$$

Using (10) in (2), in order to obtain Bayesian estimator of $R(t_0)$, results in

$$\hat{R}(t_0) = \left(\frac{\beta t+n+\theta}{\beta(t+x)+n+\theta+1}\right)^{t+\alpha}\sum_{x=t_0}^{\infty}\binom{t+\alpha+x-1}{x}\frac{(\beta x+1)^{x-1}}{(\beta(t+x)+n+\theta+1)^{x}} \tag{11}$$

Also, for obtaining Bayesian estimator for 'P' we consider independent priors for λ^* and u to be gamma with parameters (α_1, θ_1) and (α_2, θ_2) respectively and using equations (4) and (10) is

$$\hat{P} = \frac{\sum_{x^*=0}^{\infty}\sum_{y=x^*}^{\infty}(\beta x^*+1)^{x^*-1}(\beta y+1)^{y-1}\binom{t_1+\alpha_1+x^*-1}{x^*}\binom{t_2+\alpha_2+y-1}{y}\cdot\left(\beta t_1+\sum_{i=1}^{k}n+\theta_1\right)^{t_1+\alpha_1}(\beta t_2+m+\theta_2)^{t_2+\alpha_2}}{\left(\beta(t_1+x^*)+\sum_{i=1}^{k}n+\theta_1+1\right)^{t_1+\alpha_1+x^*}(\beta(t_2+y)+m+\theta_2+1)^{t_2+\alpha_2+y}} \tag{12}$$

Special Cases

For $\beta = 0$, equation (1) reduces to Poisson distribution, therefore for $\beta = 0$, equations (11) and (12) give the Bayesian estimators for $R(t_0)$ and P (see Chaturvedi, et al., 2007) and are

$$\hat{R}_1(t_0) = \left(\frac{n+\theta}{n+\theta+1}\right)^{t+\alpha} \sum_{x=t_0}^{\infty} \binom{t+\alpha+x-1}{x} \frac{1}{(n+\theta+1)^x}$$

$$\hat{P} = \frac{\sum_{x^*=0}^{\infty} \sum_{y=x^*}^{\infty} \binom{t_1+\alpha_1+x^*-1}{x^*} \binom{t_2+\alpha_2+y-1}{y} \left(\sum_{i=1}^{k} n+\theta_1\right)^{t_1+\alpha_1} (m+\theta_2)^{t_2+\alpha_2}}{\left(\sum_{i=1}^{k} n+\theta_1+1\right)^{t_1+\alpha_1+x^*} (m+\theta_2+1)^{t_2+\alpha_2+y}}$$

Generalized Geometric Distribution

The random variable 'X' follows Generalized Geometric distribution with parameters α and β if its pmf is

$$P(x=x) = \frac{1}{\beta x+1} \binom{\beta x+1}{x} \alpha^x (1-\alpha)^{1+(\beta-1)x} \qquad (13)$$

$$0 < \alpha < 1, |\alpha\beta| < 1, x = 0, 1, 2...$$

The reliability function at a specific mission time, for example, t_0 (≥ 0) is

$$R(t_0) = \sum_{x=t_0}^{\infty} \frac{1}{\beta x+1} \binom{\beta x+1}{x} \alpha^x (1-\alpha)^{1+(\beta-1)x} \qquad (14)$$

and the hazard rate function is

$$h(t_0) = \frac{P(t_0, \alpha, \beta)}{R(t_0)}$$

Bayes Estimation Of $R(t_0)$ For Generalized Geometric Distribution

The likelihood function given the random sample information $X = (X_1, X_2, \ldots, X_n)$ is

$$L(\alpha, \beta / t) = \prod_{i=1}^{n} \left(\frac{1}{\beta x_i + 1} \binom{\beta x_i + 1}{x_i} \right) \alpha^t (1-\alpha)^{n+(\beta-1)t} \tag{15}$$

where

$$t = \sum_{i=1}^{n} x_i$$

Because, $0 < \alpha < 1$, it is assumed that the prior information about α when β is known from Beta distribution with pdf

$$g(\alpha) = \frac{\alpha^{a-1} (1-\alpha)^{b-1}}{B(a,b)}, 0 < \alpha < 1, a > 0, b > 0 \tag{16}$$

The posterior distribution from (15) and (16) can be written as

$$\prod(\alpha / t) = \frac{\alpha^{t+a-1} (1-\alpha)^{n+(\beta-1)t+b-1}}{B(t+a, n+(\beta-1)t+b)} \tag{17}$$

The Bayesian estimator of α^p, for $p > 0$, is given by

$$\begin{aligned}\hat{\alpha}^p &= \int_0^\infty \alpha^p \prod(\alpha/t)\,d\alpha \\ \hat{\alpha}^p &= \frac{B\left(t+a+p, n+(\beta-1)t+b\right)}{B\left(t+a, n+(\beta-1)t+b\right)}\end{aligned} \tag{18}$$

Now, equation (13) can be written as

$$P(x=x) = \frac{1}{\beta x+1}\binom{\beta x+1}{x}\sum_{r=0}^{\infty}(-1)^r\binom{\beta x+1-x}{r}\alpha^{x+r}$$

On using (18) the Bayes estimator of $P(x;\lambda)$ at a specific point 'X' is

$$\begin{aligned}\hat{P}(x;\alpha) &= \frac{1}{\beta x+1}\binom{\beta x+1}{x}\sum_{r=0}^{\infty}(-1)^r\binom{\beta x+1-x}{r} \\ &\quad\times\frac{B\left(t+a+x+r, n+(\beta-1)t+b\right)}{B\left(t+a, n+(\beta-1)t+b\right)} \\ &= \frac{1}{\beta x+1}\binom{\beta x+1}{x}\frac{B\left(t+a+x, n+1+(\beta-1)t+b\right)}{B\left(t+a, n+1+(\beta-1)t+b\right)}\end{aligned} \tag{19}$$

Using (19) in (14), in order to obtain Bayesian estimator of $R(t_0)$, results in

$$\hat{R}(t_0) = \sum_{x=t_0}^{\infty}\frac{1}{\beta x+1}\binom{\beta x+1}{x}\frac{B\left(t+a+x, n+1+(\beta-1)t+b\right)}{B\left(t+a, n+1+(\beta-1)t+b\right)}.$$

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