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Journal of Modern Applied Statistical Methods

Volume 13 | Issue 2 Article 16

11-2014

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Recommended Citation
Hassan, Anwar and Ahmad, Mehraj (2014) "Life Testing Analysis of Failure Censored Generalized Exponentiated Data," *Journal of Modern Applied Statistical Methods*: Vol. 13 : Iss. 2 , Article 16.
DOI: 10.22237/jmasm/1414815300
Available at: http://digitalcommons.wayne.edu/jmasm/vol13/iss2/16

Journal of Modern Applied Statistical Methods
November 2014, Vol. 13, No. 2, 307-323.

ISSN 1538 − 9472

# Life Testing Analysis of Failure Censored Generalized Exponentiated Data

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A generalized exponential distribution is considered for analyzing lifetime data; such statistical models are applicable when the observations are available in an ordered manner. This study examines failure censored data, which consist of testing n items and terminating the experiment when a pre-assigned number of items, for example $r$ ( < n), have failed. Due to scale and shape parameters, both have flexibility for analyzing different types of lifetime data. This distribution has increasing, decreasing and a constant hazard rate depending on the shape parameter. This study provides maximum likelihood estimation and uniformly minimum variance unbiased techniques for the estimation of reliability of a component. Numerical computation was conducted on a data set and a comparison of the performance of two different techniques is presented.

*Keywords:* Generalized exponential distribution, lifetime data, censored data, uniformly minimum variance unbiased estimation

## Introduction

Usually observations made on a random variable do not become available in an ordered manner. If $n$ items are taken from a machine and measured for some characteristics such as diameter, it would be an anomaly – as well as a cause for concern – if the first item taken had the smallest diameter; the second item, the second smallest diameter, etc. However, there exist numerous practical situations, for example, life testing fatigue and other kinds of destructive test situations, where the data become available in this way. If $n$ radio tubes are put through a life test, for example, then the weakest will fail first in time, the second weakest one fails next, etc. Based on this pattern, it seems clear that observations will naturally occur in an ordered manner in life test situations, regardless of whether the test is the life of electric bulbs, life of radio tubes, life of ball bearings, life of various kinds of

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physical equipment or length of life after some treatment performed on animals or human beings. There are other situations – for example, testing the current needed to blow out a fuse, the voltage needed to break down a condenser, the force needed to rupture some physical material, etc. – where observations become available in order if the test is arranged in such a way that every item in the sample is subjected to the same stimulus (current, voltage, stress, dosage, etc.), so that the first weakest item fails, then the second weakest item fails, and so on.

Put in general terms, if $n$ items drawn at random from some generalized exponential population are tested, and the data become available in such a way that the smallest observation comes first, the second smallest second, and so on until finally the largest observation is last, then it is possible to discontinue experimentation after observing the first $r$ failures in a life test. The two principal advantages associated with the possibility of stopping before all $n$ observations are made stem from the observations occurring in an ordered manner and the ability to reach a decision in a shorter time or with fewer observations than if utilizing a procedure that involves observing what happens to all items being tested. Thus, this study is devoted to failure censored data, which consists of putting $n$ items on test and terminating the experiment when a pre-assigned number of items, for example $r$ ($< n$), have failed. The data obtained from such experimentation is almost mandatory in dealing with high cost sophisticated items such as televisions.

The Generalized Exponential Distribution (GED), which more accurately represents time to failure, is used instead of the more commonly used exponential distribution. Although incorporation of the GED in life testing modeling adds to the complexity of modeling and estimation, it fits life data more accurately than the exponential distribution due to its flexibility.

The two parameter GED was proposed and studied extensively by Gupta and Kundu (1999, 2001a, 2001b, 2002), Raqab (2002), Raqab and Ahsanullah (2001) and Zheng (2002) and the two parameter GED distribution has: density function

$$f\left(x,\alpha,\lambda\right)=\alpha\lambda\left(1-e^{-\lambda x}\right)^{\alpha-1}e^{-\lambda x}, x>0, \alpha>0, \lambda>0, \tag{1}$$

cumulative distribution function (cdf)

$$F\left(x,\alpha,\lambda\right)=\left(1-e^{-\lambda x}\right)^{\alpha}, x>0, \alpha>0, \lambda>0, \tag{2}$$

survival function

$$S(x,\alpha,\lambda)=1-\left(1-e^{-\lambda x}\right)^{\alpha},\, x>0,\alpha>0,\lambda>0\,, \tag{3}$$

and hazard function

$$h(x,\alpha,\lambda)=\frac{\alpha\lambda\left(1-e^{-\lambda x}\right)^{\alpha-1}e^{-\lambda x}}{1-\left(1-e^{-\lambda x}\right)^{\alpha}},\, x>0,\alpha>0,\lambda>0\,. \tag{4}$$

Here $\alpha > 0$ and $\lambda > 0$ are the shape and scale parameters respectively. For different values of the shape parameter, the density function can take different shapes. Hereafter, the GED with shape parameter $\alpha$ and scale parameter $\lambda$ will be denoted by $GE(\alpha,\lambda)$. This article focuses on the maximum likelihood estimate and the minimum variance unbiased estimate of the shape parameter when the scale parameter is known.

## Estimation Based on MLE

### Maximum Likelihood Estimation

Suppose $n$ items are subjected to test without replacement and the test is terminated after $r$ items have failed. If the failure censored data consist of the lifetimes of the $r$ items that failed ($X_{(1)} < X_{(2)} < \ldots < X_{(r)}$) and the fact that ($n - r$) items have survived beyond $X_{(r)}$. The likelihood of the ordered sample failure times is given below if the failure times are generalized exponentially distributed with pdf (1).

For given ordered failures times when it is desired to estimate $\alpha$ when $\lambda$ is known:

$$L(\alpha,\lambda\,|\,x)=\binom{n}{r}r!\prod_{i=1}^{r}\alpha\lambda\left(1-e^{-\lambda x_i}\right)^{\alpha-1}e^{-\lambda x_i}\prod_{i=r+1}^{n}\left(1-e^{-\lambda x_r}\right)^{\alpha}$$

$$L(\alpha,\lambda\,|\,x)=\binom{n}{r}r!\alpha^{r}\lambda^{r}\prod_{i=1}^{r}\left(1-e^{-\lambda x_i}\right)^{\alpha-1}e^{-\lambda\sum_{i=1}^{r}x_i}\left(1-e^{-\lambda x_r}\right)^{\alpha(n-r)}.$$

The log likelihood function of the observed sample is

$$L(\alpha) = \ln(C) + r\ln(\alpha) + r\ln(\lambda) \\ -\lambda\sum_{i=1}^{r} x_i + (\alpha - 1)\sum_{i=1}^{r}\ln\left(1 - e^{-\lambda x_i}\right)\alpha(n-r)\ln\left(1 - e^{-\lambda x_r}\right) \qquad (5)$$

where $C = \binom{n}{r} r!$

The MLE of $\alpha$, for example, $\hat{\alpha}$ for known $\lambda$ is

$$\hat{\alpha} = -\frac{r}{\sum_{i=1}^{r}\ln\left(1 - e^{-\lambda x_i}\right) + (n-r)\ln\left(1 - e^{-\lambda x_r}\right)}$$

$$= \frac{r}{\sum_{i=1}^{r} T_i + (n-r)T_r} = \frac{r}{T' + (n-r)T''}$$

where $T_i = \ln\left(1 - e^{-\lambda x_i}\right)^{-1}$ and $T_r = \ln\left(1 - e^{-\lambda x_r}\right)^{-1}$

and $T' = \sum_{i=1}^{r}\ln\left(1 - e^{-\lambda x_i}\right)^{-1}, T'' = \left(1 - e^{-\lambda x_r}\right)^{-1}$.

## Unbiasedness of $\hat{\alpha}$

If the $n$ items are tested and observation continues until $r$ units have failed then ($T_{(1)}$, $T_{(2)}$, …, $T_{(r)}$) are the transferred failure time from exponential population with mean life $\alpha$. Because $X_1$, $X_2$, …, $X_r$ are independently and identically distributed (*iid*) $GED(\alpha,\lambda)$, then $T_{(i)}$, the transformed ordered failures, are *iid* as Expo($\alpha$). In this plan the number of items exposed at any time is $n$, the joint distribution of $T_{(1)}$, $T_{(2)}$, …, $T_{(r)}$, that is, the number of failed items out of $n$ items tested is given by

$$g\left(t_{(1)}, t_{(2)}, \ldots, t_{(r)} \mid \alpha\right) = \binom{n}{r} r!\,\alpha^r e^{-\sum_{i=1}^{r} t_{(i)}}; 0 < T_{(1)} < T_{(2)} < \cdots < T_{(r)} < \infty$$

Using transformation $Z_i = (n - i + 1)(T_{(i)} - T_{(i-1)})$, $i = 1,2,3, \ldots, r$, with $T_{(o)} = 0$, then

$$\sum_{i=1}^{r} Z_i = \sum_{i=1}^{r} T_{(i)} + (n-r) T_{(r)} \tag{6}$$

$$\frac{\partial(Z_1, Z_2, \dots, Z_r)}{\partial(T_{(1)}, T_{(2)}, \dots, T_{(r)})} = \binom{n}{r} r! \text{ and } |J| = \frac{1}{\binom{n}{r} r!}$$

This results in the joint distribution of $Z_1, Z_2, \dots, Z_r$ as $g(Z_1, Z_2, \dots, Z_r \mid \alpha) = \alpha^r e^{-\sum_{i=1}^{r} z_i}$, thus $Z_1, Z_2, \dots, Z_r$ are *iid* as $g(z \mid \alpha) = \alpha e^{-\alpha z}$; $z, \alpha \geq 0$

$$\hat{\alpha} = \frac{r}{\sum_{i=1}^{r} Z_i}$$

$$\frac{\hat{\alpha}}{r\alpha} = \frac{1}{\alpha \sum_{i=1}^{r} Z_i} \text{ or } \frac{r\alpha}{\hat{\alpha}} = \alpha \sum_{i=1}^{r} Z_i \text{ follows Gamma}(r)$$

and $Y = \frac{\hat{\alpha}}{r\alpha} = \frac{1}{\alpha \sum_{i=1}^{r} Z_i}$ follows inverted gamma density of Raiffa and Schlaifer (1961) as

$$f(y) = \frac{1}{\Gamma r} e^{-\frac{1}{y}} \left(\frac{1}{y}\right)^{r+1}; y \geq 0, r \geq 0, \tag{7}$$

and the pdf of $\hat{\alpha}$ is $f(\hat{\alpha} \mid \alpha) = \frac{1}{\alpha \Gamma(r+1)} e^{\frac{-r\alpha}{\hat{\alpha}}} \left(\frac{r\alpha}{\hat{\alpha}}\right)^{r+1}; \hat{\alpha} \geq 0, r \geq 0.$ (8)

## Moments of $\hat{\alpha}$

It is necessary to extract the first two moments of $\hat{\alpha}$, to find in general the $k^{\text{th}}$ moment of $\hat{\alpha}$ as

$$\mu_k' = E(Y^k) = \int_0^\infty y^k \frac{1}{\Gamma r} e^{-\frac{1}{y}} \left(\frac{1}{y}\right)^{r+1} dy = \frac{\Gamma(r-k)}{\Gamma r}$$

$$\mu_1' = \frac{1}{(r-1)}, \mu_2' = \frac{1}{(r-1)(r-2)}, \text{ and } \mu_2 = \frac{1}{(r-1)^2(r-2)}$$

$$\mu_1' = E(Y) = E\left(\frac{\hat{\alpha}}{r\alpha}\right) = \frac{1}{(r-1)} \text{ results in } E(\hat{\alpha}) = \frac{r\alpha}{r-1}$$

$$\mu_2 = V(Y) = V\left(\frac{\hat{\alpha}}{r\alpha}\right) = \frac{1}{(r-1)^2(r-2)} \text{ results in } V(\hat{\alpha}) = \frac{r^2\alpha^2}{(r-1)^2(r-2)}$$

Thus $E(\hat{\alpha}) \neq \alpha$, which clearly shows that the MLE of $\alpha$ is not an unbiased estimate of $\alpha$, but instead it is asymptotically unbiased estimate of $\alpha$.

### Sufficiency of $\hat{\alpha}$

$$L(x_1, x_2, \ldots, x_r \mid \alpha, \lambda) = \frac{n!}{(n-r)!}\alpha^r \lambda^r \prod_{i=1}^{r}\left(1-e^{-\lambda x_i}\right)^{\alpha-1} \prod_{i=1}^{r} e^{-\lambda x_i}\left(1-e^{-\lambda x_r}\right)^{\alpha(n-r)}$$

Using the transformation as in Lemma 1 (see Appendix A) results in,

$$L(x_1, x_2, \ldots, x_r \mid \alpha, \lambda) = \frac{n!}{(n-r)!}\alpha^r \lambda^r e^{-\alpha\left[\sum_{i=1}^{r} t_i + (n-r)T_r\right]} e^{\sum_{i=1}^{r} t_i} \prod_{i=1}^{r}\left(1-e^{-t_i}\right)$$

Using the transformation as in (6) results in,

$$L(x_1, x_2, \ldots, x_r \mid \alpha, \lambda) = \alpha^r \lambda^r e^{-\alpha \sum_{i=1}^{r} z_i} \frac{n!}{(n-r)!} e^{\sum_{i=1}^{r} t_i} \prod_{i=1}^{r}\left(1-e^{-t_i}\right)$$

Using $\hat{\alpha} = \frac{r}{\sum_{i=1}^{r} T_i} = \frac{r}{\sum_{i=1}^{r} Z_i}$ results in,

$$\frac{L(x_1, x_2, \ldots, x_r \mid \alpha, \lambda)}{f(\hat{\alpha} \mid \alpha)} = \frac{n!}{(n-r)!}\frac{\Gamma r}{r^r}\lambda^r \prod_{i=1}^{r}\left(1-e^{-t_i}\right) e^{\sum_{i=1}^{r} t_i} (\hat{\alpha})^{(r+1)}$$

which is independent of the unknown parameter $\alpha$, thus $\hat{\alpha}$ is a sufficient estimator for $\alpha$.

### MLE of Reliability

Because the MLE of $\alpha$ i,e $\hat{\alpha} = \dfrac{r}{\sum_{i=1}^{r} T_i + (n-r)T_r} = \dfrac{r}{\sum_{i=1}^{r} Z_i}$ has been calculated using a property of MLE, that function of an MLE is also an MLE, thus the MLE of reliability of GED is denoted by $\hat{R}(t)$ and is given as $\hat{R}(t) = 1 - \left(1 - e^{-\lambda t}\right)^{\hat{\alpha}}$.

### Expectation of Reliability and its Standard Error

To evaluate the expectation of reliability and its standard error, results from Watson (1952) viz $\int_0^{\infty} x^{-r} e^{-\left(ax + \frac{b}{x}\right)} dx = 2\left(\dfrac{a}{b}\right)^{\frac{r-1}{2}} K_{r-1}\left(2\sqrt{ab}\right)$ are used where $K_r$ (.) is the modified Bessels function of the second kind of order $r$.

$$E\left(\hat{R}(t)\right) = 1 - \frac{2}{\Gamma r}\left(\ln A_o^{-r\alpha}\right)^{\frac{r}{2}} K_{-r}\left(2\sqrt{\ln A_o^{-r\alpha}}\right)$$

and

$$E\left(\hat{R}(t)\right)^2 = 2E\left(\hat{R}(t)\right) + \frac{2}{\Gamma r}\left(\ln A_o^{-2r\alpha}\right)^{\frac{r}{2}} K_{-r}\left(2\sqrt{\ln A_o^{-2r\alpha}}\right) - 1.$$

## Estimation Based on Minimum Variance Unbiased Estimate

### Minimum Variance Unbiased Estimate

The Minimum Variance Unbiased Estimate (MVUE) approach is now considered. Note that $\hat{\alpha}$ is biased, but the bias can be easily corrected as

$$\tilde{\alpha} = \frac{r-1}{r}\hat{\alpha} = \frac{r-1}{r}\frac{r}{\sum_{i=1}^{r} Z_i} = \frac{r-1}{\sum_{i=1}^{r} Z_i}$$

this implies, $$\frac{\tilde{\alpha}}{(r-1)\alpha} = \frac{1}{\alpha\sum_{i=1}^{r} Z_i}$$

Recall the proven result that $\alpha\sum_{i=1}^{r} Z_i$ follows Gamma($r$) and $Y = \dfrac{\tilde{\alpha}}{(r-1)\alpha} = \dfrac{1}{\alpha\sum_{i=1}^{r} Z_i}$ follows the inverted gamma density of Raiffa and Schlaifer (1961) as

$$f(y) = \frac{1}{\Gamma r} e^{-\frac{1}{y}} \left(\frac{1}{y}\right)^{r+1}; y \geq 0, r \geq 0;$$

the pdf of $\tilde{\alpha}$ is then:

$$f(\tilde{\alpha} \mid \alpha) = \frac{1}{\alpha(r-1)\Gamma r} e^{\frac{-(r-1)\alpha}{\tilde{\alpha}}} \left(\frac{(r-1)\alpha}{\tilde{\alpha}}\right)^{r+1}; \tilde{\alpha} \geq 0, r \geq 0$$

$$\mu_k' = E(Y^k) = \frac{\Gamma(r-k)}{\Gamma r}$$

$$\mu_1' = \frac{1}{(r-1)},\ \mu_2' = \frac{1}{(r-1)(r-2)} \text{ and } \mu_2 = \frac{1}{(r-1)^2(r-2)}$$

$$\mu_1' = E(Y) = E\left(\frac{\tilde{\alpha}}{(r-1)\alpha}\right) = \frac{1}{(r-1)} \text{ results in } E(\tilde{\alpha}) = \alpha$$

$$\mu_2 = V(Y) = V\left(\frac{\tilde{\alpha}}{(r-1)\alpha}\right) = \frac{1}{(r-1)^2(r-2)} \text{ results in } V(\tilde{\alpha}) = \frac{\alpha^2}{(r-2)}$$

Clearly $V(\hat{\alpha}) \geq V(\tilde{\alpha})$. However, equality holds for $r$ = ½ which is not an integer, thus it implies that this inequality never holds for integral value of $n$.

## Sufficiency of $\tilde{\alpha}$

$$L\left(x_1, x_2, \ldots, x_r \mid \alpha, \lambda\right) = \frac{n!}{(n-r)!} \alpha^r \lambda^r \prod_{i=1}^{r} \left(1 - e^{-\lambda x_i}\right)^{\alpha - 1} \prod_{i=1}^{r} e^{-\lambda x_i} \left(1 - e^{-\lambda x_r}\right)^{\alpha(n-r)}$$

Using the transformation in Lemma 1 results in

$$L\left(x_1, x_2, \ldots, x_r \mid \alpha, \lambda\right) = \frac{n!}{(n-r)!} \alpha^r \lambda^r e^{-\alpha\left[\sum_{i=1}^{r} t_i + (n-r) T_r\right]} e^{\sum_{i=1}^{r} t_i} \prod_{i=1}^{r} \left(1 - e^{-t_i}\right)$$

Using the transformation in (6) results in

$$L\left(x_1, x_2, \ldots, x_r \mid \alpha, \lambda\right) = \alpha^r \lambda^r e^{-\alpha \sum_{i=1}^{r} Z_i} \frac{n!}{(n-r)!} e^{\sum_{i=1}^{r} t_i} \prod_{i=1}^{r} \left(1 - e^{-t_i}\right)$$

Using $\frac{\tilde{\alpha}}{(r-1)} = \frac{1}{\sum_{i=1}^{r} Z_i}$ results in

$$\frac{L\left(x_1, x_2, \ldots, x_r \mid \alpha, \lambda\right)}{f\left(\tilde{\alpha} \mid \alpha\right)} = \frac{n!}{(n-r)!} \frac{\Gamma r}{(r-1)^r} \lambda^r \prod_{i=1}^{r} \left(1 - e^{-t_i}\right) e^{\sum_{i=1}^{r} t_i} \left(\tilde{\alpha}\right)^{(r+1)}$$

which is independent of the unknown parameter $\alpha$ and, thus, $\tilde{\alpha}$ is a sufficient estimator for $\alpha$.

## Completeness

A family of density functions $f\left(\boldsymbol{X}, \boldsymbol{\alpha}\right), \boldsymbol{\alpha} \in \boldsymbol{H}$ (Parametric Space) is called complete if $E\left(u(\boldsymbol{x})\right) = 0$ for all $\boldsymbol{\alpha} \in \boldsymbol{H}$ implies $u(\boldsymbol{x}) = 0$ with probability 1, for all $\boldsymbol{\alpha} \in \boldsymbol{H}$.

That is, there are no two different functions of $\boldsymbol{X}$ which have the same expected value for all $\boldsymbol{\alpha} \in \boldsymbol{H}$. Thus, for example, if a sufficient statistic is complete, there will be only one unbiased estimator of $\boldsymbol{\alpha}$ which is a function of the sufficient statistic.

$$E_\alpha\left(\psi\left(\tilde{\alpha}\right)\right)=\int_0^\infty \psi\left(\tilde{\alpha}\right)\frac{1}{\alpha\left(r-1\right)\Gamma r}e^{\frac{-(r-1)\alpha}{\tilde{\alpha}}}\left(\frac{\left(r-1\right)\alpha}{\tilde{\alpha}}\right)^{r+1}d\tilde{\alpha}$$
$$=\int_0^\infty \psi^*\left(\tilde{\alpha}\right)e^{\frac{-(r-1)\alpha}{\tilde{\alpha}}}d\tilde{\alpha}$$

where $\psi^*\left(\tilde{\alpha}\right)$ includes all other terms.

Now using Laplas transformation that $\int_0^\infty h\left(x\right)e^{-\theta x}dx=0\Rightarrow h\left(x\right)=0$

$$E_\alpha\left(\psi\left(\tilde{\alpha}\right)\right)=0\Rightarrow\psi^*\left(\tilde{\alpha}\right)=0\Rightarrow\psi\left(\tilde{\alpha}\right)=0;\alpha\geq 0,r>0$$

Thus $\tilde{\alpha}$ is also a complete estimate of $\alpha$ because $E(\tilde{\alpha}) = \alpha$, it follows that $\tilde{\alpha}$ is a uniformly minimum variance unbiased estimate (UMVUE) of $\alpha$.

## UMVUE of Reliability

Previously $\tilde{\alpha}=\frac{r-1}{\sum_{i=1}^{r}z_i}$, which is an UMVUE was estimated, the UMVUE of reliability $\tilde{R}\left(t\right)$ is derived next. The general method of finding the UMVUE is to search for any unbiased statistics $T\left(x_1,x_2,\ldots,x_n\right)$ and a complete and sufficient statistic if one exists. Consider a function $T\left(x_1,x_2,\ldots,x_n\right)$ such that $T\left(x_1\right)=1$ if $x_1\geq t$ and = 0 otherwise. Thus, $T$ is a function of $x_1$ alone, denoted by $T\left(x_1\right)$.

$$E\left\{T\left(x_1\right)\right\}=1.P\left(X_1\geq t\right)+0.P\left(X_1<t\right)=P\left(X_1\geq t\right)=\tilde{R}\left(t\mid\alpha\right)$$

For its derivation it is necessary to derive the conditional distribution of $T_{(1)}$ given $\tilde{\alpha}$ and split the random sample $\left(T_{(1)},T_{(2)},\ldots,T_{(r)}\right)$ into two independent components $T=T_{(1)}$ of sample size one and $\left(T_{(2)},T_{(3)},\ldots,T_{(r)}\right)$ of sample size $\left(r-1\right)$. Because $T_{(i)}$ are *iid* as exponential with parameter $\alpha$ as proved in Lemma 1, then

$$f\left(t_{(i)}\mid\alpha\right)=\alpha e^{-\alpha t_{(i)}},\alpha\geq 0;t_{(i)}\geq 0$$

and define $\bar{Y}=\frac{1}{r-1}\sum_{i=2}^{r}Z_{(i)}$

$$\left(r-1\right)\bar{Y}+T_{(1)}=\sum_{i=1}^{r}T_{(i)}=r\bar{T}$$

This yields $\left(r-1\right)\bar{y}$ follows $\text{Gamma}\left(\alpha,r-1\right)$ because $\sum_{i=1}^{r}T_{(i)}=\sum_{i=1}^{r}Z_{i}\sim\text{Gamma}\left(\alpha,r\right)$ and assumes $\left(r-1\right)\bar{y}=S\sim\text{Gamma}\left(\alpha,r-1\right)$

$$f\left(\bar{y}\right)=\frac{\alpha^{r-1}}{\Gamma\left(r-1\right)}e^{-\alpha(r-1)\bar{y}}\left(\left(r-1\right)\bar{y}\right)^{r-2}\left(r-1\right)$$

$$f\left(\bar{y}\right)=\frac{\alpha^{r-1}\left(r-1\right)^{(r-1)}}{\Gamma\left(r-1\right)}e^{-\alpha(r-1)\bar{y}}\left(\bar{y}\right)^{r-2}$$

and then the joint distribution of $T_{(1)}$ and $\bar{y}$ is given by

$$f\left(t_{(1)},\bar{y}\,|\,\alpha\right)=\alpha e^{-\alpha t_{(1)}}\,\frac{\alpha^{r-1}\left(r-1\right)^{(r-1)}}{\Gamma\left(r-1\right)}e^{-\alpha(r-1)\bar{y}}\left(\bar{y}\right)^{r-2}$$
$$=\frac{\alpha^{n}\left(r-1\right)^{(r-1)}}{\Gamma\left(r-1\right)}e^{-\alpha\left\{(r-1)\bar{y}+t_{(1)}\right\}}\left(\bar{y}\right)^{r-2}$$

Because $t_{(1)}$ and $\bar{y}$ are independently distributed, the joint distribution of $\bar{T}$ and $T_{(1)}$ can be obtained by using the transformation

$$\left(r-1\right)\bar{Y}+T_{(1)}=\sum_{i=1}^{r}T_{(i)}=r\bar{T},\text{ resulting in }\left|J\right|=\frac{d\bar{y}}{d\bar{t}}=\frac{r}{r-1}$$

Therefore, $f\left(t_{(1)},\bar{t}\,|\,\alpha\right)=\frac{\alpha^{r}r\left(r-1\right)^{(r-2)}}{\Gamma\left(r-1\right)}e^{-\alpha r\bar{t}}\left[\frac{r\bar{t}}{r-1}-\frac{t_{(1)}}{r-1}\right]^{r-2};0<t_{(i)}<r\bar{t}$

$$\text{and } f\left(\bar{t}\,|\,\alpha\right)=\frac{\left(r\alpha\right)^{r}}{\Gamma r}\left(\bar{t}\right)^{r-1}e^{-r\alpha\bar{t}}$$

The conditional distribution of $t_{(1)}$ is then obtained as $f\left(t_{(1)} \mid \bar{t}, \alpha\right) = \dfrac{f\left(t_{(1)}, \bar{t} \mid \alpha\right)}{f\left(\bar{t} \mid \alpha\right)}$

$$f\left(t_{(1)} \mid \bar{t}, \alpha\right) = \frac{r-1}{r\bar{t}}\left[1 - \frac{t_{(1)}}{r\bar{t}}\right]^{r-2} ;0 < t_{(1)} < r\bar{t}$$

Using $\tilde{\alpha} = \dfrac{r-1}{\sum_{i=1}^{r} T_i}$ results in $\bar{t} = \dfrac{r-1}{nr}$ and using this value in the above pdf of $t_{(1)}$

$$f\left(t_{(1)} \mid \tilde{\alpha}\right) = \tilde{\alpha}\left[1 - \frac{t_{(1)}\tilde{\alpha}}{r-1}\right]^{r-2} ;0 < t_{(1)} < \frac{r-1}{\tilde{\alpha}}$$

Thus UMVUE of reliability $\tilde{R}(t)$ is obtained as

$$\begin{aligned}\tilde{R}(t) &= P\left(X_1 \geq t_o \mid \tilde{\alpha}\right) = P\left\{-\ln\left(1 - e^{-\lambda x_1}\right) < -\ln\left(1 - e^{-\lambda t_o}\right) \mid \tilde{\alpha}\right\} \\ &= P\left\{T_1 < -\ln\left(1 - e^{-\lambda t_o}\right) \mid \tilde{\alpha}\right\} = \int_0^{-\ln\left(1-e^{-\lambda t_o}\right)} \tilde{\alpha}\left[1 - \frac{t_{(1)}\tilde{\alpha}}{r-1}\right]^{r-2} dt_{(1)}\end{aligned}$$

$$\tilde{R}(t) = 1 - \left\{1 + \frac{\tilde{\alpha}\ln\left(1 - e^{-\lambda t_o}\right)}{r-1}\right\}^{r-1}$$

## Expectation of MVUE Reliability and Its Standard Error

$$E\left(\tilde{R}(t)\right) = 1 - \sum_{j=0}^{r-1} \frac{(\alpha A)^j}{j!}$$

$$E\left(\tilde{R}(t)\right)^2 = 2E\left(\tilde{R}(t)\right) + \frac{1}{\Gamma r}\sum_{j=0}^{2r-2}\binom{2r-2}{j}\Gamma(r-j)(\alpha A)^j - 1$$

## Data Analysis

Sixty items were tested and the test was terminated after the first 10 items failed. The failure times (in months) were recorded as 0.12, 0.21, 0.39, 0.52, 0.68, 0.72, 0.87, 0.99, 1.14, 1.27. Assume that failure times are distributed as generalized exponentially distributed.

The mean value of failure times is 0.69 months. The parameter $\alpha$ and reliability was estimated using the MLE and MVUE for various known values of $\lambda$ and the behavior of two different estimations on the estimation of reliability and parameter estimation was studied; results are shown in Tables 1 and 2 (see Appendix A).

## Acknowledgements

The project was supported by the Research Center, College of Science, King Saud University, Saudi Arabia.

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## Appendix A

### Lemma 1

***Part 1*** If $X_i$ are random variables independently and identically generalized exponentially distributed $GED(\alpha,\lambda)$, with $\lambda$ known, then $T_i = -\ln\left(1-e^{-\lambda x_i}\right) = \ln\left(1-e^{-\lambda x_i}\right)^{-1}$ follows Expo($\alpha$).

***Part 2*** Prove that $\frac{1}{T'}$ follows the inverted gamma distribution, that is $\frac{1}{T'} = y$ follows inverted gamma density (Raiffa & Schlaifer, 1961) as

$$f\left(y\right) = \frac{1}{\alpha \Gamma r} e^{\frac{\alpha}{y}} \left(\frac{\alpha}{y}\right)^{r+1} ;\ y \geq 0, \alpha, r \geq 0 \tag{9}$$

$$E\left(\frac{1}{T'}\right) = \frac{\alpha}{r-1} \text{ and } E\left(\frac{1}{T'}\right)^2 = \frac{\alpha^2}{\left(r-1\right)\left(r-2\right)} \text{ then } V\left(\frac{1}{T'}\right) = \frac{\alpha^2}{\left(r-1\right)^2\left(r-2\right)}$$

where $T' = -\sum_{i-1}^{n} \ln\left(1-e^{-\lambda x_i}\right) = \sum_{i=1}^{n} \ln\left(1-e^{-\lambda x_i}\right)^{-1}$.

## Tables 1 and 2

**Table 1.** Estimate of parameter *α* and reliability using the MLE for various known values of *λ*

| $\lambda$ | $\hat{\alpha}$ | $S.E(\hat{\alpha})$ | $\hat{R}(t)$ | $E(\hat{R}(t))$ | $S.E(\hat{R}(t))$ |
|---|---|---|---|---|---|
| 0.01 | 0.03697 | 0.01452 | 0.68107 | 0.82608 | 0.08094 |
| 0.02 | 0.04362 | 0.01714 | 0.70618 | 0.82435 | 0.08484 |
| 0.03 | 0.04872 | 0.01914 | 0.72496 | 0.82303 | 0.08778 |
| 0.04 | 0.05309 | 0.02086 | 0.74074 | 0.82192 | 0.09027 |
| 0.05 | 0.05704 | 0.02241 | 0.75471 | 0.82091 | 0.09248 |
| 0.06 | 0.06071 | 0.02385 | 0.76745 | 0.81999 | 0.07015 |
| 0.07 | 0.06418 | 0.02521 | 0.77928 | 0.81912 | 0.07197 |
| 0.08 | 0.06750 | 0.02652 | 0.79041 | 0.81830 | 0.07369 |
| 0.09 | 0.07070 | 0.02778 | 0.80099 | 0.81751 | 0.07533 |
| 0.10 | 0.07382 | 0.02900 | 0.81111 | 0.81675 | 0.07690 |
| 0.12 | 0.07987 | 0.03137 | 0.83027 | 0.81530 | 0.07989 |
| 0.14 | 0.08573 | 0.03368 | 0.84830 | 0.81391 | 0.08273 |
| 0.15 | 0.08862 | 0.03481 | 0.85697 | 0.81324 | 0.08409 |
| 0.18 | 0.09716 | 0.03817 | 0.88194 | 0.81127 | 0.08805 |
| 0.20 | 0.10279 | 0.04038 | 0.89786 | 0.80999 | 0.09059 |
| 0.30 | 0.13107 | 0.05149 | 0.90123 | 0.80388 | 0.06835 |
| 0.40 | 0.16051 | 0.06305 | 0.90110 | 0.79796 | 0.07581 |
| 0.50 | 0.19190 | 0.07539 | 0.85735 | 0.79209 | 0.08300 |
| 0.60 | 0.22575 | 0.08868 | 0.85331 | 0.78618 | 0.09004 |
| 0.70 | 0.26245 | 0.10310 | 0.84784 | 0.78022 | 0.09695 |
| 0.80 | 0.30232 | 0.11876 | 0.84001 | 0.77419 | 0.07782 |
| 0.90 | 0.34569 | 0.13580 | 0.82882 | 0.76812 | 0.08284 |
| 1.00 | 0.39284 | 0.15432 | 0.80001 | 0.76201 | 0.08778 |
| 2.00 | 1.12735 | 0.44286 | 0.71003 | 0.70934 | 0.01114 |
| 3.00 | 2.42090 | 0.95102 | 0.63922 | 0.70982 | 0.00853 |
| 4.00 | 4.15967 | 1.63407 | 0.60843 | 0.59425 | 0.10358 |
| 5.00 | 6.12946 | 2.40788 | 0.45982 | 0.63671 | 0.23600 |
| 10.00 | 19.33348 | 7.59491 | 0.24577 | 0.39212 | 0.20864 |
| 20.00 | 90.37409 | 35.50230 | 0.24575 | 0.35531 | 0.28917 |
| 30.00 | 338.40470 | 132.93700 | 0.24565 | 0.29058 | 0.29601 |

**Table 2.** Estimate of parameter $\alpha$ and reliability using the UMVUE for various known values of $\lambda$

| $\lambda$ | $\tilde{\alpha}$ | $S.E(\tilde{\alpha})$ | $\tilde{R}(t)$ | $E(\tilde{R}(t))$ | $S.E(\tilde{R}(t))$ |
|---|---|---|---|---|---|
| 0.01 | 0.03327 | 0.01176 | 0.53961 | 0.52654 | 0.08460 |
| 0.02 | 0.03926 | 0.01388 | 0.56303 | 0.54956 | 0.08372 |
| 0.03 | 0.04385 | 0.01550 | 0.58055 | 0.56679 | 0.08306 |
| 0.04 | 0.04778 | 0.01689 | 0.59529 | 0.58126 | 0.08251 |
| 0.05 | 0.05134 | 0.01815 | 0.60834 | 0.59408 | 0.08202 |
| 0.06 | 0.05464 | 0.01932 | 0.62023 | 0.60577 | 0.08157 |
| 0.07 | 0.05776 | 0.02042 | 0.63129 | 0.61662 | 0.08115 |
| 0.08 | 0.06075 | 0.02148 | 0.64169 | 0.62684 | 0.08076 |
| 0.09 | 0.06363 | 0.02250 | 0.65158 | 0.63655 | 0.08039 |
| 0.10 | 0.06644 | 0.02349 | 0.66104 | 0.64584 | 0.08004 |
| 0.12 | 0.07188 | 0.02541 | 0.67896 | 0.66344 | 0.07936 |
| 0.14 | 0.07716 | 0.02728 | 0.69583 | 0.68000 | 0.07873 |
| 0.15 | 0.07975 | 0.02820 | 0.70396 | 0.68797 | 0.07842 |
| 0.18 | 0.08744 | 0.03091 | 0.72733 | 0.71091 | 0.07755 |
| 0.20 | 0.09251 | 0.03271 | 0.74224 | 0.72553 | 0.07699 |
| 0.30 | 0.11796 | 0.04171 | 0.81143 | 0.79338 | 0.07439 |
| 0.40 | 0.14446 | 0.05107 | 0.87503 | 0.85569 | 0.07200 |
| 0.50 | 0.17271 | 0.06106 | 0.93523 | 0.91465 | 0.06974 |
| 0.60 | 0.20318 | 0.07183 | 0.94094 | 0.97121 | 0.06757 |
| 0.70 | 0.23620 | 0.08351 | 0.94992 | 0.90141 | 0.06548 |
| 0.80 | 0.27209 | 0.09620 | 0.95738 | 0.89999 | 0.06345 |
| 0.90 | 0.31112 | 0.11000 | 0.96834 | 0.89320 | 0.02318 |
| 1.00 | 0.35356 | 0.12500 | 0.95637 | 0.81830 | 0.02318 |
| 2.00 | 1.01461 | 0.35872 | 0.82882 | 0.79209 | 0.02318 |
| 3.00 | 2.17881 | 0.77033 | 0.80001 | 0.78618 | 0.02318 |
| 4.00 | 3.74370 | 1.32360 | 0.61843 | 0.78022 | 0.02318 |
| 5.00 | 5.51652 | 1.95038 | 0.49867 | 0.43011 | 0.02318 |
| 10.00 | 17.40010 | 6.15188 | 0.36759 | 0.42673 | 0.02318 |
| 20.00 | 81.33660 | 28.75680 | 0.32793 | 0.33867 | 0.02318 |
| 30.00 | 304.56400 | 107.67900 | 0.27546 | 0.31526 | 0.02318 |