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A simulation study was conducted to explore the performance of the independent means  $t$ -test, Satterthwaite's approximate  $t$ -test, and the conditional  $t$ -test under various conditions. Type I error control and statistical power of these testing approaches were examined and guidance provided on the proper selection among them.

*Keywords:* Type I error control, statistical power, parametric tests, independent means  $t$ -test, Satterthwaite's approximate  $t$ -test, conditional  $t$ -test

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## Introduction

In elementary statistics courses, the independent means  $t$ -test is followed by a discussion of statistical assumptions, robustness, Type I error control, and power. At the time of writing, the independent means  $t$ -test has been widely used in almost every discipline to this day. A search completed in June of 2014 with the key words "independent means  $t$ -test", with time period between 2013 and 2014, returned 1,740 articles from the Google Scholar database (excluding citation and patents) and 605 articles from the Web of Science database. Among the 605 articles in Web of Science, 170 out of the 202 most recent articles mentioned in the abstract that these studies utilized the independent means  $t$ -test.

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## PARAMETRIC TESTS FOR TWO POPULATION MEANS

The independent means  $t$ -test relies on the assumptions of population normality and equal variances (homoscedasticity). Alternative approaches such as the Satterthwaite's approximate  $t$ -test (Satterthwaite's test hereinafter) relax these assumptions, approximating the  $t$  distribution and the corresponding degrees of freedom. Although the independent means  $t$ -test is "the most powerful unbiased test" (Bridge & Sawilowsky, 1999, p. 229) for detecting true mean differences under the assumption of normality, statisticians to date are still evaluating the various conditions and factors for which this test is robust under the violation of the equality of variances assumption and departures from normality.

### Controversy about the Independent Means $t$ -Test

Many statistical textbooks (e.g., Cody & Smith, 1997; Schlotzhauer & Littell, 1997) continue recommending what Hayes and Cai (2007) call the "conditional decision rule" (p. 217), that researchers screen their samples for variance homogeneity by conducting preliminary tests (e.g., the Folded  $F$ -test). That is, the  $t$ -test assumes that the distributions of the two groups being compared are normal with equal variances. Although the authors of some statistics textbooks do not even mention the assumption of homogeneity of variance (e.g., Gravetter & Wallnau, 2011) as one required for the  $t$ -test, homoscedasticity is basic and necessary for hypothesis testing because the violations of this assumption "alter Type I error rates, especially when sample sizes are unequal" (Zimmerman, 2004, p. 173).

The preliminary test of the null hypothesis  $\sigma_1^2 = \sigma_2^2$  versus the alternative  $\sigma_1^2 \neq \sigma_2^2$  is conducted using the Folded  $F$ -test statistic  $F = S_1^2 / S_2^2$ . Common practice has been that if the Folded  $F$ -test is not statistically significant (e.g.,  $p \geq 0.05$ ), then the test of  $\mu_1 = \mu_2$  versus  $\mu_1 \neq \mu_2$  is calculated using the independent means  $t$ -test:

$$t = \frac{\bar{X}_1 - \bar{X}_2}{S_{\bar{X}_1\bar{X}_2} \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \quad (1)$$

On the other hand, if the preliminary test is statistically significant (e.g.,  $p < 0.05$ ) and in addition there are unequal sample sizes, the independent means  $t$ -test should be avoided and the Satterthwaite's test should be used instead (Moser, Stevens, & Watts, 1989):

$$t' = \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}} \quad (2)$$

with

$$df = \frac{\left(\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}\right)^2}{\frac{\left(\frac{S_1^2}{n_1}\right)^2}{n_1 - 1} + \frac{\left(\frac{S_2^2}{n_2}\right)^2}{n_2 - 1}} \quad (3)$$

Recently, researchers have questioned the robustness of the conditional  $t$ -test with respect to Type I error and statistical power when the assumptions of normality and homoscedasticity are not met. Besides the unlikelihood of encountering real data that are normally distributed (Micceri, 1989), it is reported that there are also serious disadvantages of performing preliminary tests of variances equality (e.g., Moser et al., 1989; Zimmerman, 2004). Specifically, Moser et al. (1989) evaluated the impact of a preliminary variance test on the power and Type I error rate of the regular  $t$ -test and the Satterthwaite's test. Based on calculations of power and Type I error, they suggested that for equal sample sizes ( $n_1 = n_2$ ), the  $t$ -test and the Satterthwaite's test had the same power and provided very stable Type I error rates close to the nominal alpha prescribed for the test of means. For unequal sample sizes ( $n_1 \neq n_2$ ), the Satterthwaite's test still provided reasonable and stable Type I error rates close to the nominal significance level. In conclusion, Moser et al. recommended applying directly the Satterthwaite's test for testing the equality of means from two independent and normally distributed populations where the ratio of the variance is unknown.

Rasch, Kubinger, and Moder (2011) conducted a simulation study to compare the performance of the regular (Student)  $t$ -test, Welch test, and Wilcoxon  $U$ -test to investigate if we should perform the  $t$ -test conditionally after testing the assumptions. These authors suggested not testing the underlying assumptions of the  $t$ -test because such testing was not effective. Zimmerman (2004) found similar optimal results for the Welch-Satterthwaite separate-variance  $t$ -test if applied unconditionally whenever sample sizes were unequal and noted that the power of this test deteriorated if it was conditioned by a preliminary test. Grissom (2000)

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argued that it is realistic to expect heteroscedasticity in data as well as outliers, and examined the effect of these factors on variance. He also addressed issues of robustness (i.e., control of Type I error rate) in the presence of heteroscedasticity and departures from normality, for which he suggested trimming as a way to stabilize variances.

### **Purpose of the Present Study**

The purpose of this study was to explore the performance of the independent means *t*-test and two alternatives, Satterthwaite's test and the conditional *t*-test, by conducting a series of simulations under various manipulated conditions. The current study extended previous studies on the independent means *t*-test and its alternatives by taking into account the non-normality of population distribution and various levels of heteroscedasticity. Accordingly, extensive simulation conditions were included in this study: a wide range of total sample sizes (from 10 to 400 in contrast with 10 to 100 in Rasch et al., 2011 and 30 and 60 in Zimmerman, 2004); various variance ratios between populations up to 1:20 (beyond the realistic maximum sample variance ratio of 1:12 suggested in Grissom, 2000 and the great variance ratio of 1:16 mentioned in Wilcox, 1987); wide range of alpha set for testing treatment effects and testing homogeneity assumption for the conditional *t*-test; large range of non-normality (skewness from 0 to 6 and kurtosis from 0 to 25). In the study of Rasch et al. (2011), skewness and kurtosis were examined from 0 to 3 and 0 to 15, respectively. In addition, this paper provides some guidelines for researchers on the selection of an appropriate test for the equality of two population means.

### **Methodology**

A simulation approach was used to explore and compare the behaviors of the independent means *t*-test, Satterthwaite's test, and the conditional *t*-test for two means because simulation allows for the controlling of designed factors.

### **Manipulated Factors**

A crossed factorial mixed design included seven factors: (a) total sample size (10, 20, 50, 100, 200, 300, and 400), (b) sample size ratio between groups (1:1, 2:3, and 1:4), (c) variance ratio between populations (1:1, 1:2, 1:4, 1:8, 1:12, 1:16, and 1:20), (d) effect size for mean difference between populations ( $\Delta = 0, .2, .5, \text{ and } .8$ ), (e) alpha set for testing treatment effects ( $\alpha = 0.01, 0.05, 0.10, 0.15, 0.20, \text{ and } 0.25$ ),

(f) alpha set for testing homogeneity assumption for the conditional  $t$ -test ( $\alpha = 0.01, 0.05, 0.10, 0.15, 0.20, 0.25, 0.30, 0.40, 0.45,$  and  $0.50$ ), and (g) population distributions with varying kurtosis and skewness values (i.e.,  $\gamma_1 = 1.0$  and  $\gamma_2 = 3.0, \gamma_1 = 1.5$  and  $\gamma_2 = 5.0, \gamma_1 = 2.0$  and  $\gamma_2 = 6.0, \gamma_1 = 0.0$  and  $\gamma_2 = 25.0,$  as well as  $\gamma_1 = 0.0$  and  $\gamma_2 = 0.0$  for the normal distribution, where  $\gamma_1$  and  $\gamma_2$  represent skewness and kurtosis, respectively). This crossed factorial design provided a total of 176,400 conditions for the conditional  $t$ -test and 17,640 conditions for the Satterthwaite's test and the regular  $t$ -test.

### Data Generation

A random number generator, RANNOR in SAS/IML statistical software, was employed with a different seed value for each execution of the simulation program to generate data for this study. For each condition in the simulation, 100,000 samples were generated. The use of 100,000 replications provides a maximum standard error of an observed proportion (i.e., Type I error rate estimate) of .00158, and a 95% confidence interval no wider than  $\pm .003$  (Robey & Barcikowski, 1992).

### Statistically Analytical Procedures

For each sample generated, both the independent means  $t$ -test and Satterthwaite's test, each at a range of nominal alpha levels (i.e., 0.01 through 0.25), were conducted. The independent means  $t$ -test and Satterthwaite's test were investigated under a total of 17,640 conditions. In addition, the conditional  $t$ -test was conducted. The testing procedures for the conditional  $t$ -test were as follows. Firstly, the Folded  $F$ -test was implemented to examine the variance homogeneity assumption using a range of nominal alpha levels (i.e., 0.01 through 0.50). Based upon the results of the Folded  $F$ -test, either the independent means  $t$ -test or Satterthwaite's test was applied. Thus, for the conditional  $t$ -test, a total of 176,400 conditions were examined.

The simulation focused on Type I error rates and power. Type I error was examined when the population effect size (or two-group mean difference) was simulated zero; otherwise power was computed. Type I error rates were evaluated on the basis of the liberal criterion for robustness suggested by Bradley (1978). Given a nominal alpha level, Bradley's liberal criterion provides a plausible range of Type I error rates in which a test can be considered robust. The liberal criterion for the robustness is set at  $0.5\alpha$  around the nominal alpha. For example, when  $\alpha = 0.05$ , a test is considered robust when the Type I error rate falls between 0.025, which is given by  $0.5 \cdot 0.05$ , and .075, which is given by  $1.5 \cdot 0.05$ . When there was

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considerable variability in the estimated Type I error and power across simulation conditions, eta-square analyses were conducted to examine the design factors related to the variability.

### Results

The results of simulation are reported in the following order: (a) power of the Folded  $F$ -test for the test of equal variances, (b) Type I error control for the test of means, and (c) power for the test of means. Under the Type I error control for the test of means, an overview of Type I error rates, an analysis of design factors associated with Type I error control, and an analysis based on Bradley's liberal criterion for robustness are presented.

#### Power for the Folded $F$ -Test

The distributions of statistical power estimates for the Folded  $F$ -test were examined across all conditions simulated in which population variances were not equal. As expected, when the alpha level used for the Folded  $F$ -test was small (e.g., 0.01 or 0.05), the average power was low. However, the power of the Folded  $F$ -test increased when the applied alpha level increased.

Nominal alpha levels of 0.05 and 0.25 for the Folded  $F$ -test were selected for further analysis of power. The average power of the Folded  $F$ -test based on simulation design factors is presented in Table 1. As seen in the table, the power remained stable regardless of distribution shapes; yet using the alpha level of 0.25 consistently yielded more power. The average powers for 0.05 and 0.25 alpha levels were around .81 and .90, respectively, across normal and non-normal distributions. Further, as the value of variance ratio increased, the power of the Folded  $F$ -test increased as well. Using the alpha level of 0.25 provided substantially more power when the variance ratios were small (i.e., variance ratio = 1:2 and 1:4). As the variance ratios increased, the power differences between the two nominal alpha levels decreased.

It is well-known that the power increases when the sample size increases. Using an alpha level of 0.05 for the Folded  $F$ -test yielded average power of .80 with sample size of 50 and of 0.90 with 100. In contrast, the average power reached .80 with as few as 20 observations and 0.90 with 50 observations using an alpha level of 0.25. The use of extremely unbalanced samples (sample size ratios of 1:4 or 4:1) reduced the power of the Folded  $F$ -test, but power advantages of the more liberal alpha level remained evident.

**Table 1.** The power of the Folded *F*-Test using  $\alpha = .05$  and  $\alpha = .25$

Condition	$\alpha = .05$	$\alpha = .25$	Condition	$\alpha = .05$	$\alpha = .25$
Total <i>N</i>			Variance ratio		
10	0.36	0.62	1:2	0.55	0.73
20	0.64	0.82	1:4	0.76	0.87
50	0.85	0.92	1:8	0.85	0.93
100	0.92	0.96	1:12	0.89	0.93
200	0.96	0.98	1:16	0.91	0.96
300	0.98	0.99	1:20	0.92	0.97
400	0.99	0.99			
<i>N</i> ratio			Shape		
1:04	0.74	0.86	0.0, 0	0.82	0.91
2:03	0.83	0.92	1.0, 3	0.81	0.90
1:01	0.85	0.93	1.5, 5	0.81	0.90
3:02	0.85	0.92	2.0, 6	0.81	0.89
4:01	0.80	0.82	0.0, 25	0.81	0.91

For Shape, the two values indicate skewness and kurtosis, respectively

### Type I Error Control for the Test of Means

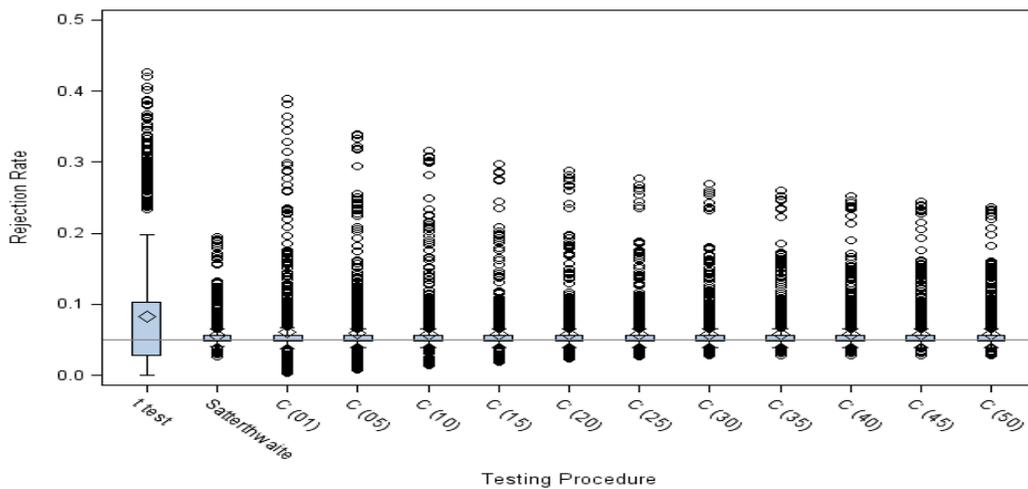
An overall view of the Type I error control of the tests is provided in Figures 1 and 2. These boxplots describe the distributions of the Type I error rate estimates under a nominal alpha level of 0.05 across all conditions in which the population means were identical. The first two plots are for the independent means *t*-test and Satterthwaite’s test, respectively. The remaining plots delineate the Type I error rate estimates for the conditional *t*-test across the different conditioning rules (i.e., the alpha levels for the Folded *F*-test) that were investigated. For instance, the plot for C(01) provides the distribution of the Type I error rates for the conditional *t*-test when an alpha level of 0.01 was used with the Folded *F*-test as the rule to choose between the independent means *t*-test and Satterthwaite’s test.

Note that in Figure 1 the independent means *t*-test has great dispersion of Type I error rates. In some conditions, this testing approach provides appropriate control of the Type I error probability while in others the Type I error rate is very different from the nominal alpha level. In contrast, Satterthwaite’s approximate *t*-test provides adequate Type I error control in nearly all of the conditions simulated. The series of plots for the conditional *t*-test illustrate that the conditional test provides a notable improvement in Type I error control relative to the independent means *t*-test and the improvement increases as the alpha level for the Folded *F*-test increases. This improvement occurs because the statistical power of the Folded *F*-test increases as the alpha level increases. That is, the ability of the Folded *F*-test

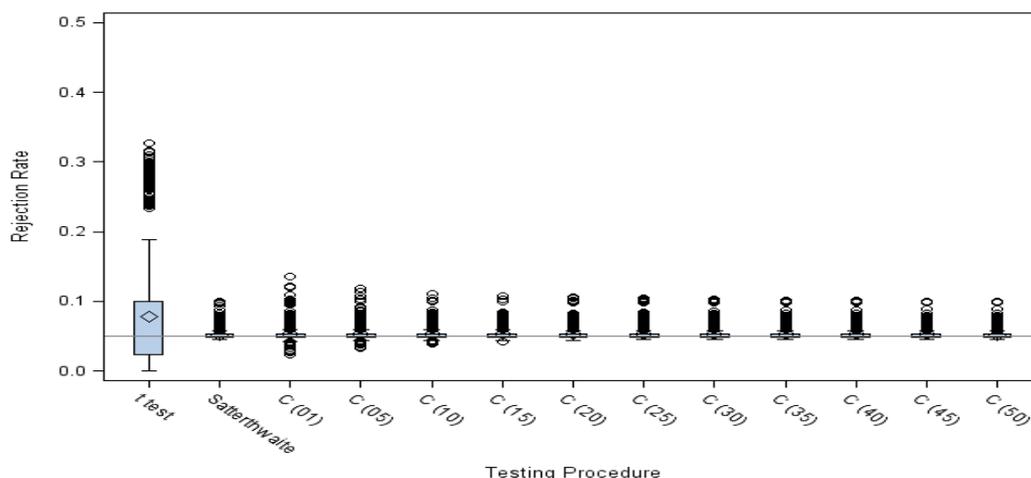
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to detect variance heterogeneity (and to subsequently steer us away from the independent means  $t$ -test and steer us to Satterthwaite's test) increases with the alpha level for this test, which supports the argument of insufficient power when using a more conservative alpha level for a preliminary analysis.

Considering that the power of Folded  $F$ -test was substantially lower when the total sample size was 10 or 20 (see Table 1) and the behavior of conditional  $t$ -test heavily depended on the power of the Folded  $F$ -test, we inspected the Type I error rates of the conditional  $t$ -test only for total sample size greater than 20. As speculated, the Type I error rates of the conditional  $t$ -test across different alpha levels are almost identical to that of Satterthwaite's test if the decision of conditional  $t$ -test is made at  $\alpha = 0.10$  or greater (see Figure 2).



**Figure 1.** Distributions of estimated Type I error rates for independent means  $t$ -test ( $t$ -test), Satterthwaite's test, and conditional  $t$ -test ( $\alpha = 0.05$ ) for all sample size conditions. C(01) denotes the conditional  $t$ -test at  $\alpha = 0.01$  of the Folded  $F$ -test



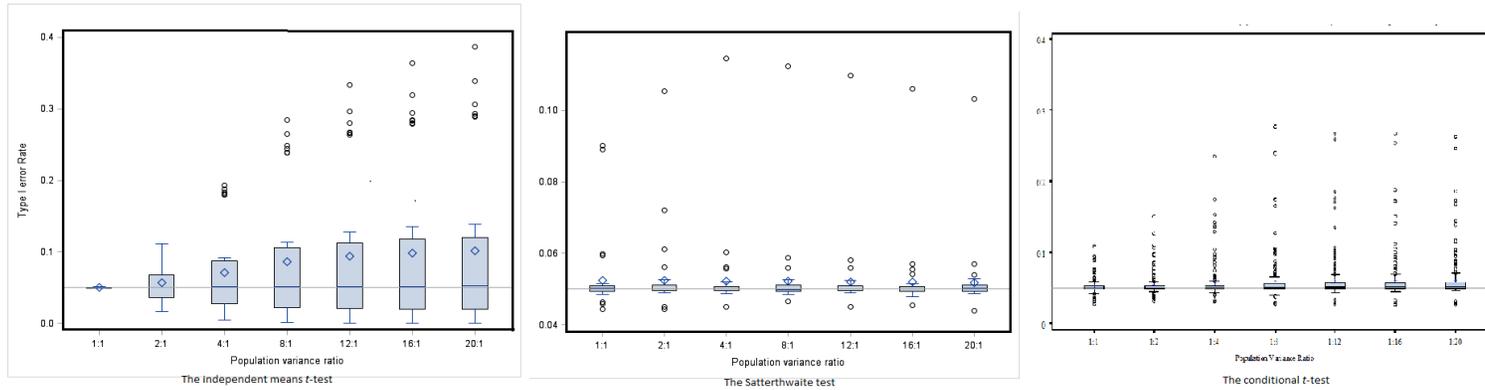
**Figure 2.** Distributions of estimated Type I error rates for independent means  $t$ -test ( $t$ -test), Satterthwaite’s test, and conditional  $t$ -test ( $\alpha = 0.05$ ) for  $N > 20$ . C(01) denotes the conditional  $t$ -test at  $\alpha = 0.01$  of the Folded  $F$ -test

### Impact of Simulation Design Factors on Type I Error Controls

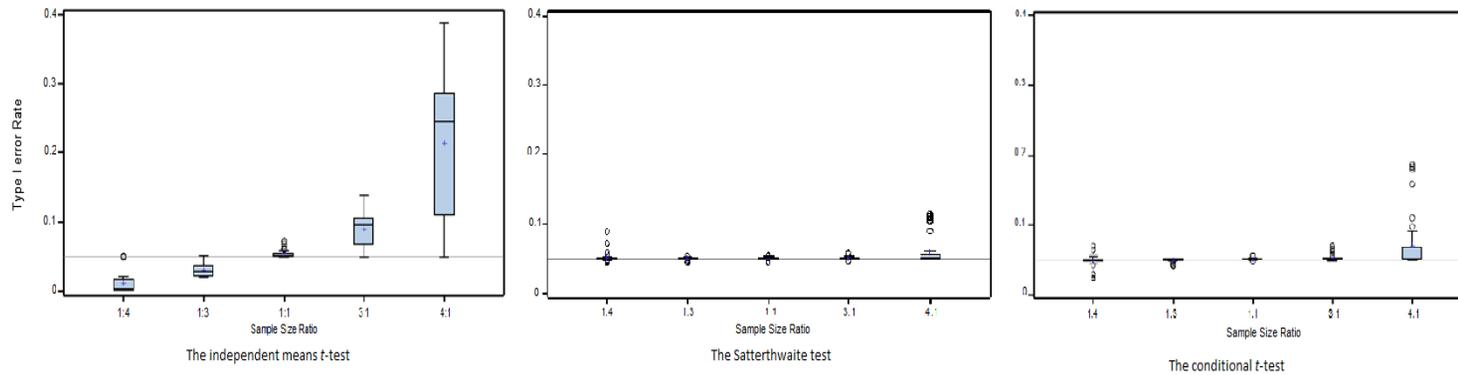
**Variance heterogeneity.** The large dispersion of Type I error rates for the independent means  $t$ -test resulted in large part from the variance heterogeneity that was included in the simulation conditions. Figure 3 presents the distributions of Type I error rates for the independent means  $t$ -test with the results disaggregated by population variance ratio. Note that as the population variance ratio increases, both the average Type I error rate and the dispersion of Type I error rates increase. On the other hand, both Satterthwaite’s test and the conditional  $t$ -test provide good control of Type I error rate even if the population variances in the two groups are heterogeneous (Figure 3).

Of course, the independent means  $t$ -test is known to be relatively robust to violations of the assumption of variance homogeneity if the sample sizes in the two groups are equal. This phenomenon is illustrated in Figure 4. Note that the Type I error rate for the independent means  $t$ -test is maintained near the nominal 0.05 level if sample sizes are equal. With disparate sample sizes in the two groups, the independent means  $t$ -test either becomes conservative (Type I error rates lower than the nominal alpha level) or liberal (Type I error rates higher than the nominal level) depending upon the relationship between sample size and population variance. In contrast, both Satterthwaite’s test and the conditional  $t$ -test evidence much

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**Figure 3.** Distributions of estimated Type I error rates by variance ratio (1:1, 2:1, 4:1, 8:1, 12:1, 16:1, 20:1) at  $\alpha = 0.05$ .



**Figure 4.** Distributions of estimated Type I error rates by sample size ratio (1:4, 1:3, 1:1, 3:1, 4:1) at  $\alpha = 0.05$ .

improved Type I error control under variance heterogeneity when samples sizes are unequal.

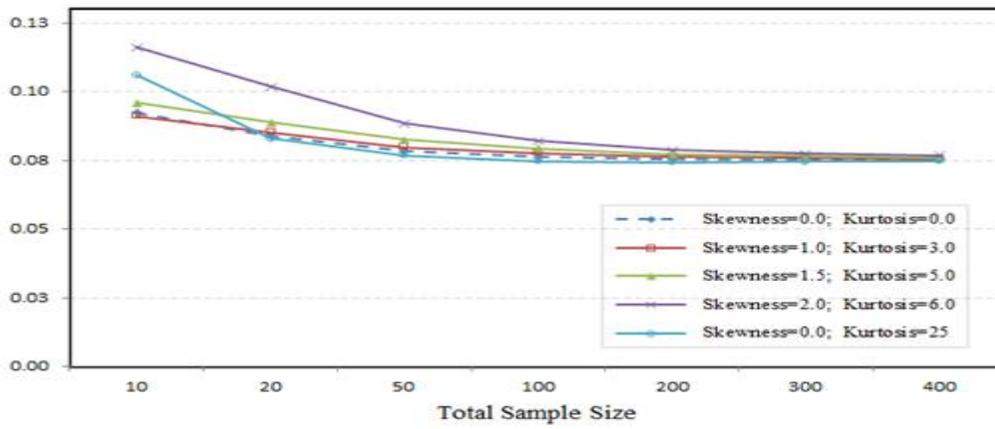
***Other design factors related to Type I error control.*** The variability in the estimated Type I error rates for the three tests of means was analyzed by computing the eta-squared value associated with each simulation design factor and the first-order interactions. For the independent means *t*-test, the factors associated with variability in estimated Type I error rates were sample size ratio ( $\eta^2 = 0.69$ ) and the interaction between sample size ratio and variance ratio ( $\eta^2 = 0.22$ ). For Satterthwaite's test and the conditional *t*-test respectively, the major factors were sample size ratio ( $\eta^2 = 0.15$ ;  $\eta^2 = 0.18$ ), total sample size ( $\eta^2 = 0.18$ ;  $\eta^2 = 0.14$ ), and the interaction between sample size ratio and total sample size ( $\eta^2 = 0.26$ ;  $\eta^2 = 0.36$ ). An analysis of the sole impact of distribution shape on Type I error rates of the three tests showed that Type I error rate of Satterthwaite's test was most affected ( $\eta^2 = 0.07$ ). While Type I error rate of the independent means *t*-test was least impacted by the distribution shape ( $\eta^2 = 0.001$ ), Type I error rate of the conditional *t*-test was also impacted ( $\eta^2 = 0.04$ ) but to a much lesser degree in comparison with that of Satterthwaite's test.

The mean Type I error rates by total sample size and distribution shape for the independent means *t*-test and Satterthwaite's test under the nominal alpha level of .05 are presented in Figures 5 and 6. The graph for the conditional *t*-test is similar to that for Satterthwaite's test. The mean Type I error rates of the independent means *t*-test are much above the nominal alpha level in all conditions of distribution shapes and total sample sizes (see Figure 5). Although the mean estimated Type I error rates decrease with larger samples, they remain substantially greater than 0.05. In contrast, both Satterthwaite's test and the conditional *t*-test provided much better Type I error control except for extremely small sample sizes (i.e., total sample size of 10 or 20) or the extremely skewed distribution (e.g., skewness of 2) (see Figure 6 for Satterthwaite's test).

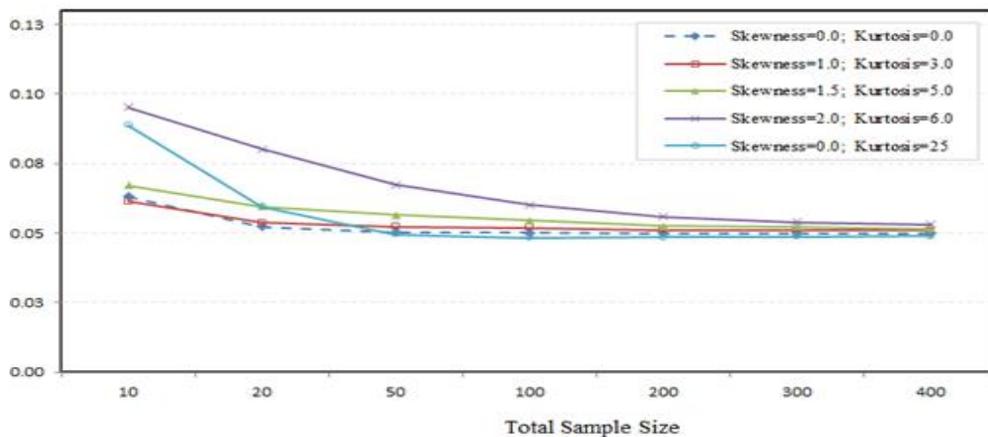
The factors related to the Bradley proportions vary across tests (Table 2). For the independent means *t*-test, sample size ratio and variance ratio between the two populations emerged as primary factors making an impact on the Type I error control. Although the overall proportions of cases meeting the Bradley's criterion for the independent means *t*-test were very low (below 50%), the Type I error rates were perfectly controlled when the homogeneity of variance assumption was met (i.e., variance ratio between groups = 1:1). As the disproportion of two group variances became larger to 1:20, the Type I error control of independent means *t*-test diminished considerably. When the two groups have equal sample size, the

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independent means  $t$ -test adequately controlled the Type I error rates within the Bradley's criterion for 91% of the conditions. The imbalance of sample size between groups worsened the Type I error control noticeably. On the other hand, the adequacy of Type I error control of the independent means  $t$ -test appears independent of total sample size and the shape of distribution. That is, the proportions meeting the Bradley's criterion were consistently low irrespective of total sample size and distribution shape.



**Figure 5.** Mean Type I error rate by total sample size for the independent means  $t$ -test



**Figure 6.** Mean Type I error rate by total sample size for the Satterthwaite's test

**Table 2.** Proportions of cases meeting the Bradley’s Liberal Criterion at  $\alpha = 0.05$

Condition	t-test	Conditional	Satterthwaite	Condition	t-test	Conditional	Satterthwaite
Total <i>N</i>				Variance ratio			
10	0.45	0.68	0.65	1:1	1.00	0.94	0.92
20	0.49	0.76	0.82	1:2	0.62	0.93	0.95
50	0.45	0.93	0.95	1:4	0.40	0.91	0.93
100	0.43	0.97	0.97	1:8	0.29	0.90	0.91
200	0.42	1.00	1.00	1:12	0.27	0.89	0.89
300	0.41	1.00	1.00	1:16	0.25	0.89	0.89
400	0.41	1.00	1.00	1:20	0.23	0.89	0.90
<i>N</i> ratio				Shape			
1:4	0.18	0.98	0.97	0.0, 0	0.43	0.96	0.97
2:3	0.67	0.98	0.98	1.0, 3	0.44	0.96	0.97
1:1	0.91	0.97	0.97	1.5, 5	0.44	0.93	0.94
3:2	0.28	0.91	0.91	2.0, 6	0.46	0.77	0.78
4:1	0.14	0.70	0.74	0.0, 25	0.41	0.91	0.91

Conditional indicates conditional t-test at  $\alpha = .25$  of Folded *F*-test. Shape values indicate skewness and kurtosis, respectively.

The impact of variance ratio and sample size ratio on the Type I error control appears minimal for the Satterthwaite’s test and the conditional *t*-test. Both tests showed adequate levels of Type I error control in the majority of conditions regardless of variance ratio and sample size ratio. Instead, total sample size and the skewness of the distribution were associated with the Bradley proportions of both Satterthwaite’s test and the conditional *t*-test. When the total sample size was 10, the proportions meeting the criterion dropped to about 65%. In this total sample size condition ( $N = 10$ ), the conditional *t*-test showed slightly better control of Type I error (68%) than the Satterthwaite’s test. Interestingly, for both tests the proportions meeting the Bradley’s criterion were affected by skewness but not by kurtosis (see Table 2).

### Statistical Power Analysis

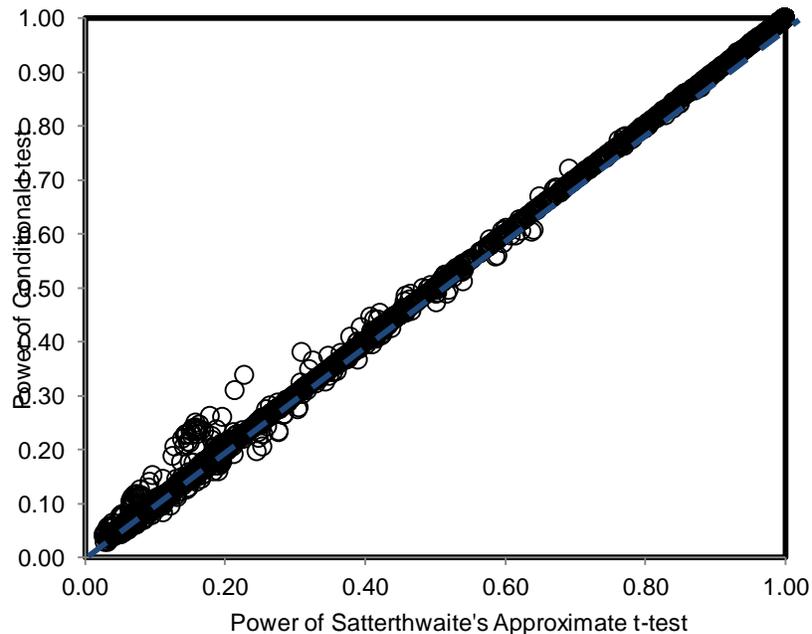
Although Satterthwaite’s test generally provides superior Type I error control, it is not always the best test to select because of the potential for power differences. When the assumptions are met, the independent means *t*-test is the most powerful test for mean differences. For this simulation study, power comparisons were made only for conditions in which both Satterthwaite’s test and the conditional *t*-test procedures evidenced adequate Type I error control by Bradley’s (1978) benchmark. The distributions of power estimates (at a nominal alpha level of 0.05

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for the tests of means) for Satterthwaite's test and the conditional  $t$ -test (using an alpha level of 0.25 for the Folded  $F$ -test) showed that the power differences between the tests were small.

Figure 7 presents a scatter plot of the power estimates for Satterthwaite's test and the conditional  $t$ -test (using an alpha level of 0.25 for the Folded  $F$ -test). Data points above the line represent conditions in which the conditional  $t$ -test was more powerful than Satterthwaite's test, while those below the line are conditions in which Satterthwaite's test is more powerful. Overall, the conditional  $t$ -test, using an alpha level of 0.25 for the Folded  $F$ -test of variances, was more powerful in 29% of the conditions, while Satterthwaite's test was more powerful in only 23% of the conditions (identical power estimates were obtained in the other conditions).

To identify research design factors associated with power differences between these two tests, the percentages of conditions in which each test evidenced power advantages were disaggregated by the simulation design factors (Table 3). For conditions with homogeneous variances, the conditional  $t$ -test evidenced more power than the Satterthwaite's test in 61.64% of the conditions, while the Satterthwaite's test was more powerful in 20.55% of the conditions (in the



**Figure 7.** Scatterplot of power estimates for the conditional  $t$ -test and Satterthwaite's approximate  $t$ -test

remaining conditions, the two tests evidenced equal power). As the variance ratios increased, the power advantages of the conditional *t*-test diminished, such that the Satterthwaite’s test was more often the more powerful test when the population variance ratio was 1:8 or larger.

With balanced samples the conditional *t*-test was more powerful in 35.38% of the conditions and Satterthwaite’s test was never more powerful. With unbalanced samples in which the larger sample is drawn from the population with the larger variance (in heterogeneous populations), the Satterthwaite’s test presents notable power advantages (44.77% and 51.03% of the cases with sample size ratios of 2:3 and 1:4, respectively). In contrast, when the larger sample is drawn from the population with the smaller variance, the conditional *t*-test evidences more power than the Satterthwaite’s test (47.68% and 48.52% of the conditions with sample size ratios of 3:2 and 4:1, respectively). The results by total sample size show that the conditional *t*-test is more powerful in more conditions, except for the smallest sample sizes examined ( $N = 10$ ). Finally, the conditional *t*-test is more powerful in more conditions for all distribution shapes except for the most skewed distribution examined (i.e. skewness of 2, kurtosis of 6).

**Table 3.** Percentage of simulation conditions by simulation design factors in which the conditional *t*-test and Satterthwaite’s test were more powerful

Condition	Conditional	Satterthwaite	Condition	Conditional	Satterthwaite
Total <i>N</i>			Variance ratio		
10	48.97	51.03	1:1	61.64	20.55
20	54.89	44.11	1:2	43.15	34.97
50	44.79	28.83	1:4	31.03	29.35
100	29.22	20.98	1:8	21.02	22.08
200	19.43	12.38	1:12	16.56	18.06
300	11.62	9.52	1:16	14.62	17.63
400	8.19	7.81	1:20	13.25	15.81
<i>N</i> ratio			Shape		
1:4	10.59	53.95	0.0, 0	29.20	15.67
2:3	9.21	44.77	1.0, 3	29.60	21.98
1:1	35.58	0.00	1.5, 5	28.17	24.63
3:2	47.68	4.04	2.0, 6	21.69	24.51
4:1	48.52	4.54	0.0, 25	34.98	27.50

Conditional indicates conditional *t*-test at  $\alpha = 0.25$  of Folded *F*-test. Shape values indicate skewness and kurtosis, respectively

## Conclusion

The testing of differences between two population means is a fundamental statistical application, but controversy about the appropriate test to use has been evident for many years. When conducting the independent means  $t$ -test, major statistical software programs (e.g., SAS and SPSS) automatically produce the results of the independent means  $t$ -test and the alternative Satterthwaite's test. Depending on the statistical significance of homogeneous variance testing (Folded  $F$ -test in SAS and Levene's  $F$  test in SPSS), researchers are recommended to follow one of the options; this has been a common practice in studies comparing two population means. However, recent studies on the conditional  $t$ -tests in comparison to Satterthwaite's test have strongly supported the Satterthwaite's test over the conditional  $t$ -test and suggested even abandoning the conventional practice of selecting one of the options based on the results of the homogeneity of variance test. Considering the ongoing controversy surrounding these tests and the frequency with which two means are compared in applied research, in this simulation study we investigated the performance of the independent means  $t$ -test, Satterthwaite's approximate  $t$ -test, and the conditional  $t$ -test under the manipulated conditions of population distribution shape, total sample size, sample size ratio between groups, variance ratio between populations, difference in means between populations, alpha level for testing the treatment effect, and alpha level for testing the homogeneity assumption for the conditional  $t$ -test. Type I error control and power analysis were used to examine the performance of these testing procedures.

As expected, the independent means  $t$ -test performed very well on Type I error control when the homogeneity assumption was met regardless of the tenability of the normality assumption. This reminds us of the long-known property that the independent means  $t$ -test requires the homogeneity assumption to be met and this test is robust to violations of the normality assumption when two population variances are equal. Furthermore, the independent means  $t$ -test showed adequate Type I error control when sample sizes in the two groups were equal under the normal distribution. This re-emphasizes another well-known property that the independent means  $t$ -test is robust to violations of the homogeneity assumption when the sample sizes are equal under the normal distribution. Under these conditions, the independent means  $t$ -test is the best method to test the difference between two independent means. This testing procedure also provides more statistical power. On the other hand, the  $t$ -test evidenced poor Type I error control under heterogeneous variances with non-normal distributions. Thus, two

alternatives, Satterthwaite's test and the conditional  $t$ -test, were considered in this study.

It was also found that the Type I error rate of the conditional  $t$ -test was affected by the alpha level for the Folded  $F$ -test that was used to test the homogeneity assumption of population variances. The more conservative alpha levels for the Folded  $F$ -test resulted in larger Type I error rates for the conditional test because of lower statistical power, such that the Folded  $F$ -test is less likely to detect the true difference between population variances. This leads us to re-consider the conventional procedures for examining the difference between two population means. Thus, the conditional  $t$ -test using a relatively large alpha level for the Folded  $F$ -test may be an appropriate alternative.

Overall, Satterthwaite's test performed best in control of Type I error rate but the conditional  $t$ -test also yielded comparable results using a large alpha level of .25 for the Folded  $F$ -test. Both alternatives made a tremendous improvement in Type I error control, compared to the independent means  $t$ -test, when group variances were unequal. Extreme skewness (e.g. skewness of 2) contaminated the Type I error control for both alternative testing procedures. Kurtosis seemed not to have this kind of impact. Increasing total sample size was found in this study to improve Type I error control for both testing procedures, but not for the independent  $t$ -test. When total sample size was 200 or more, Bradley's rates were 100% for both alternative testing procedures. Although Satterthwaite's test provides slightly better Type I error control, the use of the conditional  $t$ -test may have a slight power advantage.

**Recommendations.** With equal sample size the independent means  $t$ -test is the appropriate testing procedure to examine the difference of two independent group means because it provides adequate Type I error control and more statistical power. With unequal sample size the Folded  $F$ -test can provide reasonable guidance in the choice between the independent  $t$ -test and Satterthwaite's test. A large alpha level of .25 is recommended to evaluate the results of the Folded  $F$ -test. If the  $F$  value is not statistically significant at this large alpha level, then the independent means  $t$ -test should be used. In contrast, if the  $F$  value is statistically significant at this large alpha level, then Satterthwaite's test should be chosen. Finally, the confidence in this conditional testing procedure increases as the sample sizes become larger. To adequately control for Type I error rate in the conditional testing procedure, a total sample size of at least 200 is recommended with extremely skewed populations (e.g. skewness of 2). For less skewed populations, a total sample size of at least 100 is recommended. With a total sample size smaller than these

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recommended in the corresponding conditions, the Type I error control resulting from any of these testing procedures may be questionable.

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