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Construction of Efficiency-Balanced Design Using Factorial Design

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Two different methods are proposed for the construction of an efficiency balanced design. Method 1 discusses the construction of efficiency balanced design by deleting the control treatment and method 2 discusses the construction of efficiency balanced design by deleting the control treatment as well as all the main effect treatment combinations of a $2n$ symmetrical factorial experiment. Numerical examples are given.

Keywords: Block designs, C-matrix, M-matrix, efficiency factor, factorial design

Introduction

The concept of efficiency balance is due to Jones (1959) and the nomenclature "efficiency balance" is due to Puri and Nigam (1975) and Williams (1975). Block designs with v treatments and b blocks are considered. It is assumed that the i^{th} treatment is replicated r_i times, $i = 1, 2, 3, \ldots, v$ and the j^{th} block contains k_j (not necessarily distinct), treatments, $j = 1, 2, 3, \ldots, b$. Let $\mathbf{r} = [r_1, r_2, r_3, \ldots, r_v]'$, $\mathbf{k} = [k_1, k_2, k_3, \ldots, k_b]'$, $R = \text{diag}(r_1, r_2, r_3, \ldots, r_v)$, $K = \text{diag}(k_1, k_2, k_3, \ldots, k_b)$, and $\mathbf{N}$ be the $v \times b$ incidence matrix of the design. If $\mathbf{T}$ denotes the column vector of treatment totals then $\mathbf{s'T}$ is called a contrast of treatment totals if $\mathbf{s'r} = 0$, where $\underline{s}$ is a column vector. The intra-block component of $\mathbf{s'T}$ is defined by Jones (1959) as $\mathbf{s'Q}$ where $\mathbf{Q}$ is the vector of adjusted treatment totals, given by $\mathbf{Q} = \mathbf{T} - \mathbf{N}K^{-1}\mathbf{B}$, $\mathbf{B}$ being the vector of block totals.

Jones (1959) showed that if $\mathbf{s}$ is a right eigenvector of the matrix $\mathbf{M} = R^{-1}\mathbf{N}K^{-1}\mathbf{N}'$ corresponding to an eigenvalues $\mu(\neq 1)$, then the loss of information on the 'intra-block component' of $\mathbf{s'T}$ is μ so that the efficiency-factor of the 'intra-block component' is $1 - \mu$.

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Because **s'Q** (the intra-block component of **s'T**) is a function of observations and not of parameters (treatment effects) the concept of 'loss of information' or 'efficiency-factor' of **s'Q** is a little confusing when viewed from the classical definition of loss of information, referring to the loss incurred in estimating a certain contrast of treatment effects through a design, in relation to an orthogonal design.

A block design for which every contrast has the same loss of information (or, equivalently, same efficiency-factor) may be termed Efficiency Balanced. The concept of efficiency balance is different from the one used commonly, according to which design is balanced if every elementary contrast is estimated through the design with the same variance. To avoid confusion, the latter concept is called Variance-Balance (see e.g., Hedayat and Federer, 1974).

Calinski (1971) and Puri and Nigam (1975) established a sufficient condition for a design to be efficiency balanced is that its **M** matrix, given by

$$\mathbf{M} = \mu\mathbf{I} + (1-\mu)\mathbf{1r}' / n \tag{1}$$

where n is the total number of observations in the design. That (1) is necessary as well for a design to be efficiency balanced was shown by Williams (1975). He also showed that, for more than two varieties, an efficiency balanced design was also a variance balanced design if and only if the design is equi-replicated. Puri and Nigam (1975) gave a note on efficiency balanced design. Dey et al. (1981) proved that a necessary and sufficient condition for a design to be efficiency balanced is that (1) holds.

Mukerjee and Saha (1990) derived some optimality results on efficiency balanced designs. Gupta and Prasad (1991) gave a method for constructing general efficiency balanced designs with equal and unequal block sizes. Gupta (1992) gave a method for constructing efficiency balanced designs through BIB and GD designs.

Ceranka and Graczyk (2009) discussed some problems for a class of EB-BD based on balanced incomplete block designs with repeated blocks. Awad and Banerjee (2012) gave a method for constructing variance and efficiency balanced block designs with repeated blocks which are based on the incidence matrices of the known balanced incomplete block designs with repeated blocks. Sun and Tang (2010) gave the optimal efficiency balanced designs and their constructions.

Purpose of the Study

Efficiency balanced design using 2^n symmetrical factorial design by deleting control treatment

The construction of unequal block sizes and equi-replicated binary EB designs from symmetrical factorial designs are discussed. First, consider the following lemma without proof.

Lemma 1 In a 2^n symmetrical factorial experiment, delete the control treatment. For an example, let $n = 3$. The $2^3 = 8$ treatment combinations are

0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

Delete a treatment combination whose level of all factor is zero. That is, delete a control treatment. Keep the remaining treatment combinations as such. Finally, the treatment combinations are

0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

Let the matrix **N** be the combinations of $2^n - 1$ treatment combinations. Finally the matrix (transpose of **N**) becomes the incidence matrix of efficiency balanced design. Construction of an efficiency balanced design is shown in the theorem that follows.

Theorem If there exists a 2^n symmetrical factorial experiment then there always exists an unequal block sizes, equi-replicated, binary EB design, by deleting the control treatment with the following parameters

$$v=n,\ b=2^n-1,\ r=2^{n-1},\mathbf{k}=\left(\underbrace{1,...,1}_{{}^nC_1\text{ times}},\underbrace{2,...,2}_{{}^nC_2\text{ times}},\underbrace{3,...,3}_{{}^nC_3\text{ times}},...,\underbrace{v-1,...,v-1}_{{}^nC_{n-1}\text{ times}},\underbrace{v}_{{}^nC_n\text{ times}}\right)$$

Proof Consider a 2^n symmetrical factorial experiment. This has 2^n treatment combinations. Considering n factors as rows and 2^n treatment combinations as columns, and then using the Lemma 1, we have the following incidence matrix of a design d. The incidence matrix **N** is given as

$$\mathbf{N}=\begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots & 1 \\ 1 & 0 & 1 & \cdots & \cdots & 1 \\ 1 & 0 & 1 & \cdots & \cdots & 1 \\ 0 & 1 & 0 & \cdots & \cdots & 1 \\ \vdots & \vdots & \vdots & \cdots & \cdots & \vdots \\ 0 & 1 & 1 & \cdots & \cdots & 1 \\ 1 & 0 & 0 & \cdots & \cdots & 1 \end{bmatrix} \qquad (2)$$

Since we have n rows and considering these as treatments, obviously $v = n$.

For the incidence matrix **N**, among ${}^nC_1$ columns, in each column the element 1 will occur once and 0 will occur $(n-1)$ times. Similarly, for ${}^nC_2$ columns, the element 1 will occur two times and the element 0 will occur $(n-2)$ times in each column, and so on. Moreover, there will be one column whose elements are all unity. Hence the number of blocks is

$$\mathrm{b}={}^nC_1+{}^nC_2+{}^nC_3+\ldots+{}^nC_n$$

Because ${}^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots + {}^nC_n = 2^n$, $\mathrm{b} = 2^n - 1$.

Among $(2^n - 1)$ columns, ${}^nC_1$ columns have block size 1, ${}^nC_2$ columns have block size 2, and so on.

Hence

$$\mathbf{k} = \left(\underbrace{1,\ldots,1}_{{}^nC_1\text{ times}}, \underbrace{2,\ldots,2}_{{}^nC_2\text{ times}}, \underbrace{3,\ldots,3}_{{}^nC_3\text{ times}}, \ldots, \underbrace{n-1,\ldots,n-1}_{{}^nC_{n-1}\text{ times}}, \underbrace{n}_{{}^nC_n\text{ times}} \right)$$

Factors having level 0 occur $(2^{n-1} - 1)$ times, and factors having level 1 occur 2^{n-1} times. Since we have considered rows as treatments, there are $v = n$ treatments. Similarly, we have considered columns as blocks and hence we have $b = (2^n - 1)$ blocks. In each row, one occurs 2^{n-1} times, so the number of replications is $r = 2^{n-1}$.

Using the incidence matrix **N** shown in (2), we have the following C-matrix.

$$\mathbf{C} = \begin{pmatrix} \alpha & \beta & \beta & \beta & \cdots & \beta \\ \beta & \alpha & \beta & \beta & \cdots & \beta \\ \beta & \beta & \alpha & \beta & \cdots & \beta \\ \beta & \beta & \beta & \alpha & \cdots & \beta \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \beta & \beta & \beta & \beta & \cdots & \alpha \end{pmatrix}_{(v \times v)} \tag{3}$$

where

$$\alpha = r_i - \sum_j \left[\frac{n_{ij}^2}{k_j} \right], \beta = -\sum_j \left[\frac{n_{ij} n_{i|j}}{k_j} \right]$$

The eigenvalue of the C-matrix is

$$\theta = \alpha \left[\frac{v}{v-1} \right]$$

with multiplicities $(v - 1)$, where v is the number of treatments. Also, $\mathbf{M} = \mathbf{I} - \mathbf{C}R^{-1}$.

After some simplifications, the matrix **M** is obtained as

$$\mathbf{M}=\begin{pmatrix} 1-\frac{\alpha}{r} & -\frac{\beta}{r} & -\frac{\beta}{r} & -\frac{\beta}{r} & \cdots & -\frac{\beta}{r} \\ -\frac{\beta}{r} & 1-\frac{\alpha}{r} & -\frac{\beta}{r} & -\frac{\beta}{r} & \cdots & -\frac{\beta}{r} \\ -\frac{\beta}{r} & -\frac{\beta}{r} & 1-\frac{\alpha}{r} & -\frac{\beta}{r} & \cdots & -\frac{\beta}{r} \\ -\frac{\beta}{r} & -\frac{\beta}{r} & -\frac{\beta}{r} & 1-\frac{\alpha}{r} & \cdots & -\frac{\beta}{r} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\frac{\beta}{r} & -\frac{\beta}{r} & -\frac{\beta}{r} & -\frac{\beta}{r} & \cdots & 1-\frac{\alpha}{r} \end{pmatrix}_{(v\times v)} \tag{4}$$

Thus **MJ** = **J**, where **J** is the unit vector of order ($v \times 1$), and the design having **N** as the incidence matrix is the efficiency balanced design.

The incidence matrix **N** in (2) gives unequal block sizes, equi-replicated and binary EB designs with parameters

$$v=n,\ b=2^n-1,\ r=2^{n-1}, \mathbf{k}=\left(\underbrace{1,\dots,1}_{{}^nC_1\text{ times}},\underbrace{2,\dots,2}_{{}^nC_2\text{ times}},\underbrace{3,\dots,3}_{{}^nC_3\text{ times}},\dots,\underbrace{n-1,\dots,n-1}_{{}^nC_{n-1}\text{ times}},\underbrace{n}_{{}^nC_n\text{ times}}\right)$$

Calculation of efficiency factor

The M-matrix of the efficiency balanced design is

$$\mathbf{M}=\mu\mathbf{I}+(1-\mu)\mathbf{Jr'}/n$$

where μ is the loss of information, **I** is the identity matrix of order ($v \times v$), **J** is the unit vector of order ($v \times 1$), **r'** is the row vector of order ($1 \times v$), and n is the number of observations.

$$\mathbf{M}=\begin{bmatrix} \mu & 0 & 0 & \cdots & 0 \\ 0 & \mu & 0 & \cdots & 0 \\ 0 & 0 & \mu & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mu \end{bmatrix}+\left\{(1-\mu)\begin{bmatrix}1\\1\\1\\\vdots\\1\end{bmatrix}\begin{bmatrix} r & r & r & \cdots & r\end{bmatrix}\right\}\Big/\sum_{i=1}^{v} r_i$$

$$\mathbf{M} = \mu \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} + \begin{bmatrix} r/\sum r_i & r/\sum r_i & r/\sum r_i & \cdots & r/\sum r_i \\ r/\sum r_i & r/\sum r_i & r/\sum r_i & \cdots & r/\sum r_i \\ r/\sum r_i & r/\sum r_i & r/\sum r_i & \cdots & r/\sum r_i \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r/\sum r_i & r/\sum r_i & r/\sum r_i & \cdots & r/\sum r_i \end{bmatrix}$$

$$- \mu \begin{bmatrix} r/\sum r_i & r/\sum r_i & r/\sum r_i & \cdots & r/\sum r_i \\ r/\sum r_i & r/\sum r_i & r/\sum r_i & \cdots & r/\sum r_i \\ r/\sum r_i & r/\sum r_i & r/\sum r_i & \cdots & r/\sum r_i \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r/\sum r_i & r/\sum r_i & r/\sum r_i & \cdots & r/\sum r_i \end{bmatrix}$$

$$\begin{aligned} \mathbf{M} = \mu & \begin{bmatrix} 1-r/\sum r_i & 1-r/\sum r_i & \cdots & 1-r/\sum r_i \\ 1-r/\sum r_i & 1-r/\sum r_i & \cdots & 1-r/\sum r_i \\ \vdots & \vdots & \ddots & \vdots \\ 1-r/\sum r_i & 1-r/\sum r_i & \cdots & 1-r/\sum r_i \end{bmatrix} \\ & + \begin{bmatrix} r/\sum r_i & r/\sum r_i & \cdots & r/\sum r_i \\ r/\sum r_i & r/\sum r_i & \cdots & r/\sum r_i \\ \vdots & \vdots & \ddots & \vdots \\ r/\sum r_i & r/\sum r_i & \cdots & r/\sum r_i \end{bmatrix} \end{aligned} \tag{5}$$

Equating (4) and (5),

$$\mu = \left(\sum_{i=1}^{v} r_i - \frac{\alpha \sum_{i=1}^{v} r_i}{r} - r \right) \Bigg/ \sum_{i=1}^{v} r_i - r$$

The efficiency factor is $E = 1 - \mu$. By putting the value of μ, and after some simplifications, the efficiency factor E can be written as

$$E = \frac{\alpha \sum_{i=1}^{v} r_i \Big/ r}{\sum_{i=1}^{v} r_i - r}$$

Numerical Example

In a 2^4 factorial design, the incidence matrix after deleting the control treatment is given by

$$\mathbf{N} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Here $v = n = 4$. The above incidence matrix is of the order (4×15) which gives **R** and **K** matrices of order (4×4) and (15×15), respectively, where

$$\mathbf{R} = diag\begin{pmatrix} 8 & 8 & 8 & 8 \end{pmatrix}$$

$$\mathbf{K} = diag\left(\underbrace{1 \quad 1 \quad 1 \quad 1}_{{}^4C_1 \text{ times}} \underbrace{2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2}_{{}^4C_2 \text{ times}} \underbrace{3 \quad 3 \quad 3 \quad 3}_{{}^4C_3 \text{ times}} \underbrace{4}_{{}^4C_4 \text{ times}} \right)$$

The C-matrix is given by

$$\mathbf{C} = \begin{bmatrix} 17/4 & -17/12 & -17/12 & -17/12 \\ -17/12 & 17/4 & -17/12 & -17/12 \\ -17/12 & -17/12 & 17/4 & -17/12 \\ -17/12 & -17/12 & -17/12 & 17/4 \end{bmatrix}$$

The non-zero eigenvalue of the **C** matrix is $\theta = 17/3$.

The M-matrix of the efficiency balanced design can be obtained by substituting all the values of **N**, R, and K in $\mathbf{M} = R^{-1}\mathbf{N}K^{-1}\mathbf{N}'$ and, after some simplifications, the matrix **M** of the required design is

$$\mathbf{M} = \begin{bmatrix} 15/32 & 17/96 & 17/96 & 17/96 \\ 17/96 & 15/32 & 17/96 & 17/96 \\ 17/96 & 17/96 & 15/32 & 17/96 \\ 17/96 & 17/96 & 17/96 & 15/32 \end{bmatrix} \tag{6}$$

Obviously this matrix satisfies the conditions of efficiency balanced design i.e. **MJ** = **J**, where **J** is the (v × 1) unit vector.

The efficiency factor is calculated by using the formula

$$\mathbf{M} = \mu\mathbf{I} + (1-\mu)\,\mathbf{Jr'}/n$$

$$\mathbf{M} = \begin{bmatrix} \mu & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & \mu \end{bmatrix} + \left\{ (1-\mu) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 8 & 8 & 8 & 8 \end{bmatrix} \right\} \Big/ 32$$

$$\begin{aligned} \mathbf{M} = {} & \begin{bmatrix} \mu & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & \mu \end{bmatrix} + \begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix} \\ & -\mu \begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix} \end{aligned} \tag{7}$$

Equating (6) and (7), we get $\mu = 7/24$.

The design with the above incidence matrix gives efficiency balanced design with parameters

$$v = 4,\ b = 15,\ \mathbf{r} = (8 \quad 8 \quad 8 \quad 8),$$

$$\mathbf{k} = \left(\underbrace{1 \quad 1 \quad 1 \quad 1}_{^4C_1 \text{ times}} \underbrace{2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2}_{^4C_2 \text{ times}} \underbrace{3 \quad 3 \quad 3 \quad 3}_{^4C_3 \text{ times}} \underbrace{4}_{^4C_4 \text{ times}} \right)$$

having efficiency factor $E = (1 - \mu) = {}^{17}\!/_{24}$.

Efficiency balanced design using 2^n symmetrical factorial design by deleting control and all main effect treatments

Lemma 2 Consider a 2^n symmetrical factorial experiment. From the 2^n treatment combinations, delete the control treatments as well as all main effects. Hence we have $2^n - n - 1$ treatment combinations as the blocks in the required design.

As an example, let $n = 3$. The $2^3 = 8$ treatment combinations are

0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

Delete any treatment combinations where the levels of all factors is zero. That is, delete any control treatments. Next, delete all treatment combinations where the level of one factor is one while the levels of all other factors are zero. That is, delete the main effects. Keep the remaining treatment combinations as such. Finally, the treatment combinations are

0	1	1
1	0	1
1	1	0
1	1	1

Let the matrix **N** be the combinations of $2^n - n - 1$ treatment combinations. Finally, the matrix (transpose of **N**) becomes the incidence matrix of an efficiency balanced design. Construction of an efficiency balanced design is given in the theorem that follows.

Theorem If there exists 2^n symmetrical factorial experiments then there always exists an unequal block sizes, equi-replicated, binary EB design by deleting the control treatment and main effects with the following parameters:

$$v = n,\ b = 2^n - n - 1,\ r = 2^{n-1} - 1, \text{ and}$$
$$\mathbf{k} = \left(\underbrace{2,\ldots,2}_{^nC_2 \text{ times}}, \underbrace{3,\ldots,3}_{^nC_3 \text{ times}}, \ldots, \underbrace{n-1,\ldots,n-1}_{^nC_{n-1} \text{ times}}, \underbrace{n}_{^nC_n \text{ times}} \right)$$

Proof Consider a 2^n symmetrical factorial experiment. This has 2^n treatment combinations. Considering n factors as rows and 2^n treatment combinations as columns and then using the Lemma 2 we have the following incidence matrix of a design d. The incidence matrix **N** is given as

$$\mathbf{N} = \begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots & 1 \\ 1 & 0 & 1 & \cdots & \cdots & 1 \\ 1 & 0 & 1 & \cdots & \cdots & 1 \\ 0 & 1 & 0 & \cdots & \cdots & 1 \\ \vdots & \vdots & \vdots & \cdots & \cdots & \vdots \\ 0 & 1 & 1 & \cdots & \cdots & 1 \\ 1 & 0 & 0 & \cdots & \cdots & 1 \end{bmatrix} \tag{8}$$

Since we have n rows and, considering these as treatments, $v = n$.

For the incidence matrix **N**, among $^nC_2$ columns, the element 1 will occur two times and the element 0 will occur $(n - 2)$ times. Similarly for $^nC_3$ columns, the element 1 will occur three times and the element 0 will occur $(n - 3)$ times in each column, and so on. Moreover there will be one column whose all elements are unity. Hence the number of blocks is

$$b = {}^nC_2 + {}^nC_3 + \ldots + {}^nC_n$$

Because $^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots + {}^nC_n = 2^n$, $b = 2^n - 1$.

Among $2^n - n - 1$ columns, $^nC_2$ columns have block size 2 while $^nC_3$ columns have block size 3, and so on.

Hence

$$\mathbf{k} = \left(\underbrace{2,\ldots,2}_{{}^nC_2 \text{ times}}, \underbrace{3,\ldots,3}_{{}^nC_3 \text{ times}}, \ldots, \underbrace{n-1,\ldots,n-1}_{{}^nC_{n-1} \text{ times}}, \underbrace{n}_{{}^nC_n \text{ times}} \right)$$

Factors having level 0 occur n times, while factors having level 1 occur $2^{n-1} - 1$ times. Because rows are considered as treatments, there are $v = n$ treatments. Similarly, columns are considered as blocks, so there are $b = 2^n - n - 1$ blocks. In each row, 1 occurs $2^{n-1} - 1$ times, so the number of replication is $r = 2^{n-1} - 1$.

Using the incidence matrix **N** shown in (8), we have the following C-matrix.

$$\mathbf{C} = \begin{pmatrix} \alpha & \beta & \beta & \beta & \cdots & \beta \\ \beta & \alpha & \beta & \beta & \cdots & \beta \\ \beta & \beta & \alpha & \beta & \cdots & \beta \\ \beta & \beta & \beta & \alpha & \cdots & \beta \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \beta & \beta & \beta & \beta & \cdots & \alpha \end{pmatrix}_{(v\times v)} \tag{9}$$

where

$$\alpha = r_i - \sum_j \left[\frac{n_{ij}^2}{k_j} \right], \beta = -\sum_j \left[\frac{n_{ij} n_{i|j}}{k_j} \right]$$

The eigenvalue of the **C** matrix in (9) is

$$\theta = \alpha \left[\frac{v}{v-1} \right]$$

with multiplicities $(v - 1)$, where v is the number of treatments. Also, $\mathbf{M} = \mathbf{I} - \mathbf{C}R^{-1}$.

After some simplifications, the matrix **M** is obtained as

$$\mathbf{M}=\begin{pmatrix} 1-\frac{\alpha}{r} & -\frac{\beta}{r} & -\frac{\beta}{r} & -\frac{\beta}{r} & \cdots & -\frac{\beta}{r} \\ -\frac{\beta}{r} & 1-\frac{\alpha}{r} & -\frac{\beta}{r} & -\frac{\beta}{r} & \cdots & -\frac{\beta}{r} \\ -\frac{\beta}{r} & -\frac{\beta}{r} & 1-\frac{\alpha}{r} & -\frac{\beta}{r} & \cdots & -\frac{\beta}{r} \\ -\frac{\beta}{r} & -\frac{\beta}{r} & -\frac{\beta}{r} & 1-\frac{\alpha}{r} & \cdots & -\frac{\beta}{r} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\frac{\beta}{r} & -\frac{\beta}{r} & -\frac{\beta}{r} & -\frac{\beta}{r} & \cdots & 1-\frac{\alpha}{r} \end{pmatrix}_{(v\times v)} \tag{10}$$

It can be verified that **MJ** = **J** where **J** is the unit vector of order ($v \times 1$) and hence the design having **N** as the incidence matrix is the efficiency balanced design.

This indicates the incidence matrix **N** in (8) gives unequal block sizes, equi-replicated and binary EB designs with parameters

$$v=n,\ b=2^n-n-1,\ r=2^{n-1}-1, \mathbf{k}=\left(\underset{{}^nC_2\text{ times}}{2,\ldots,2},\underset{{}^nC_3\text{ times}}{3,\ldots,3},\ldots,\underbrace{v-1,\ldots,v-1}_{{}^nC_{n-1}\text{ times}},\underset{{}^nC_n\text{ times}}{v}\right)$$

Numerical Example

In a 2^4 factorial design, the incidence matrix after deleting the control treatment and main effects is given as

$$\mathbf{N}=\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Here $v = n = 4$. The above incidence matrix is of the order (4×11) which gives R and K matrices of order (4×4) and (11×11), respectively, where

$$\mathbf{R} = diag\begin{pmatrix}7 & 7 & 7 & 7\end{pmatrix}$$

$$\mathbf{K} = diag\left(\underbrace{2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2}_{{}^4C_2 \text{ times}}\underbrace{3 \quad 3 \quad 3 \quad 3}_{{}^4C_3 \text{ times}}\underbrace{4}_{{}^4C_4 \text{ times}}\right)$$

The C-matrix is given by

$$\mathbf{C} = \begin{bmatrix} 17/4 & -17/12 & -17/12 & -17/12 \\ -17/12 & 17/4 & -17/12 & -17/12 \\ -17/12 & -17/12 & 17/4 & -17/12 \\ -17/12 & -17/12 & -17/12 & 17/4 \end{bmatrix}$$

The non-zero eigenvalue of the C-matrix is $\theta = {}^{17}\!/_{3}$.

The matrix **M** of the efficiency balanced design can be obtained by substituting all the values of **N**, R, and K in $\mathbf{M} = R^{-1}\mathbf{N}K^{-1}\mathbf{N}'$ and, after some simplifications, the M-matrix of the required design is

$$\mathbf{M} = \begin{bmatrix} 11/28 & 17/84 & 17/84 & 17/84 \\ 17/84 & 11/28 & 17/84 & 17/84 \\ 17/84 & 17/84 & 11/28 & 17/84 \\ 17/84 & 17/84 & 17/84 & 11/28 \end{bmatrix} \tag{11}$$

This matrix satisfies the conditions of efficiency balanced design, i.e. $\mathbf{MJ} = \mathbf{J}$, where **J** is the $(v \times 1)$ unit vector.

The efficiency factor is calculated by using the formula

$$\mathbf{M} = \mu\mathbf{I} + (1-\mu)\,\mathbf{Jr}'/n$$

$$\mathbf{M} = \begin{bmatrix} \mu & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & \mu \end{bmatrix} + \left\{(1-\mu)\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\begin{bmatrix}7 & 7 & 7 & 7\end{bmatrix}\right\}\Big/ 28$$

$$\mathbf{M} = \begin{bmatrix} \mu & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & \mu \end{bmatrix} + \begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix} - \mu \begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix} \tag{12}$$

Equating (11) and (12) we get $\mu = {}^{4}\!/_{21}$.

The design with the above incidence matrix gives an efficiency balanced design with parameters

$$v = 4,\ b = 11,\ \mathbf{r} = \begin{pmatrix} 7 & 7 & 7 & 7 \end{pmatrix} \text{ and}$$

$$\mathbf{k} = \left(\underbrace{2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2}_{{}^{4}C_2 \text{ times}} \underbrace{3 \quad 3 \quad 3 \quad 3}_{{}^{4}C_3 \text{ times}} \underbrace{4}_{{}^{4}C_4 \text{ times}} \right)$$

having efficiency factor $E = (1 - \mu) = {}^{17}\!/_{21}$.

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