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Structural Properties of Transmuted Weibull Distribution

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The transmuted Weibull distribution, and a related special case, is introduced. Estimates of parameters are obtained by using a new method of moments.

Keywords: Transmuted Weibull distribution, moment generating function, sample coefficient of variation, Standard deviation, Skewness and kurtosis

Introduction

The Weibull distribution was introduced by the Swedish Physicist Waloddi Weibull in 1939. He applied this distribution to analyze the breaking strength of materials. This distribution has been extensively used in lifetime and reliability problem. The Weibull family is a generalization of the exponential family and can model data exhibiting monotone hazard rate behavior, i.e., it can accommodate three types of failure rates, namely increasing, decreasing and constant. Its application in connection with lifetimes of many types of manufactured items has been widely advocated (e.g., Weibull, 1951; Berrettoni, 1964), and it has been used as a model with diverse types of items such as vacuum tubes (Kao, 1959), ball bearings (Lieblein & Zelen, 1956), and electrical insulation. It is also widely used in biomedical applications.

A simple explanation of the Weibull distribution and its applications can be found in Franck (1988). A comprehensive review of this model is available in Johnson, Kotz, and Balakrishnan (1995). A generalization of the Weibull distribution with application to the analysis of survival data is given by Mudholkar, Srivastava, and Kollia (1996). Inferences from grouped data in the three-parameter Weibull models is introduced by Hirose and Lai (1997). Lawless

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(2002) provided statistical models and methods for lifetime data. Al-Athari (2011) and Hossain and Zimmer (2003) did some comparative studies on the estimation of Weibull parameters using complete and censored samples. Nadarajah and Kotz (2005) presented a procedure on some recent modifications of Weibull distribution.

For deriving new moment estimators of three parameters transmuted Weibull distribution, a similar approach to that of Huang and Hwang (2006) was used. Nadarajah and Kotz (2005) discussed products and ratios of Weibull random variables. Gokarna and Tsokos (2009) proposed a method on the transmuted extreme value distribution with application. Ahmad and Ahmad (2013) presented a procedure of Bayesian analysis of Weibull distribution.

A random variable x is said to have a Weibull distribution with parameters $\alpha > 0$ and $\beta > 0$ if its pdf is given by

$$g(x) = \frac{\beta}{\alpha} x^{\beta-1} \exp\left(-\frac{x^\beta}{\alpha}\right) \quad x \geq 0, \alpha > 0, \beta > 0$$

The cdf of Weibull distribution is given by

$$G(x) = \int_0^x g(x)\,dx$$

$$G(x) = \int_0^x \frac{\beta}{\alpha} x^{\beta-1} \exp\left(-\frac{x^\beta}{\alpha}\right) dx$$

$$\Rightarrow \qquad G(x) = 1 - \exp\left(-\frac{x^\beta}{\alpha}\right) \tag{1}$$

Transmuted Weibull distribution

In order to obtain the pdf of transmuted Weibull distribution, use the following cdf which is given by

$$F(x) = (1+\lambda)G(x) - \lambda G(x)^2 \tag{2}$$

where $G(x)$ is the cdf of base distribution. If $\lambda = 0$, we have the distribution of base random variable.

Now using equation (1) in equation (2),

$$F(x) = (1+\lambda)\left(1-\exp\left(-\frac{x^{\beta}}{\alpha}\right)\right) - \lambda\left(1-\exp\left(-\frac{x^{\beta}}{\alpha}\right)\right)^2$$

$$\Rightarrow \qquad F(x) = (1+\lambda)k - \lambda k^2$$

where

$$k = 1-\exp\left(-\frac{x^{\beta}}{\alpha}\right)$$

$$\Rightarrow \qquad F(x) = k(1+\lambda-\lambda k)$$

$$\Rightarrow \qquad F(x) = k\{1+\lambda(1-k)\}$$

$$\Rightarrow \qquad F(x) = \left\{1-\exp\left(-\frac{x^{\beta}}{\alpha}\right)\right\}\left\{1+\lambda\exp\left(-\frac{x^{\beta}}{\alpha}\right)\right\} \tag{3}$$

This is the required cdf of Transmuted Weibull distribution.

In order to find the pdf of Transmuted Weibull distribution, first differentiate equation (3) w.r.t. x which is given by

$$f(x) = \frac{d}{dx}\{F(x)\}$$

$$\Rightarrow \qquad f(x) = \frac{d}{dx}\left[\left\{1-\exp\left(-\frac{x^{\beta}}{\alpha}\right)\right\}\left\{1+\lambda\exp\left(-\frac{x^{\beta}}{\alpha}\right)\right\}\right]$$

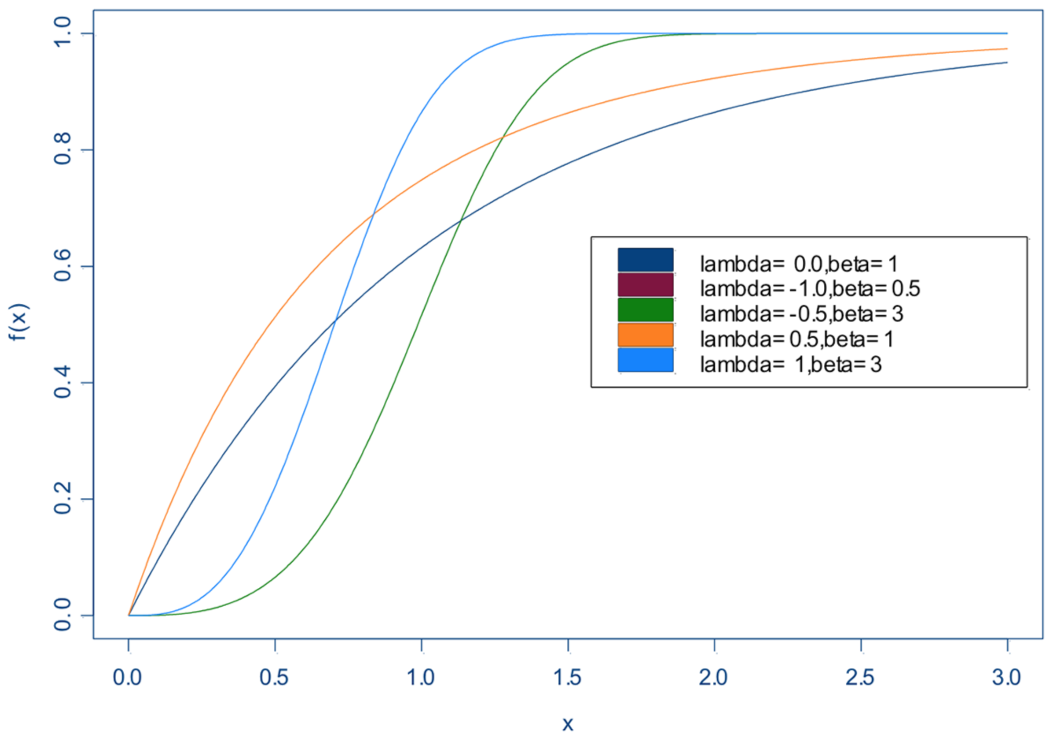

Figure 1. The cdfs of various transmuted Weibull distributions.

After differentiating the above equation w.r.t. x,

$$f(x)=\frac{\beta}{\alpha}x^{\beta-1}\exp\left(-\frac{x^{\beta}}{\alpha}\right)\left\{1-\lambda+2\lambda\exp\left(-\frac{x^{\beta}}{\alpha}\right)\right\} \tag{4}$$

which is the required pdf of Transmuted Weibull distribution with parameters α, β and λ.

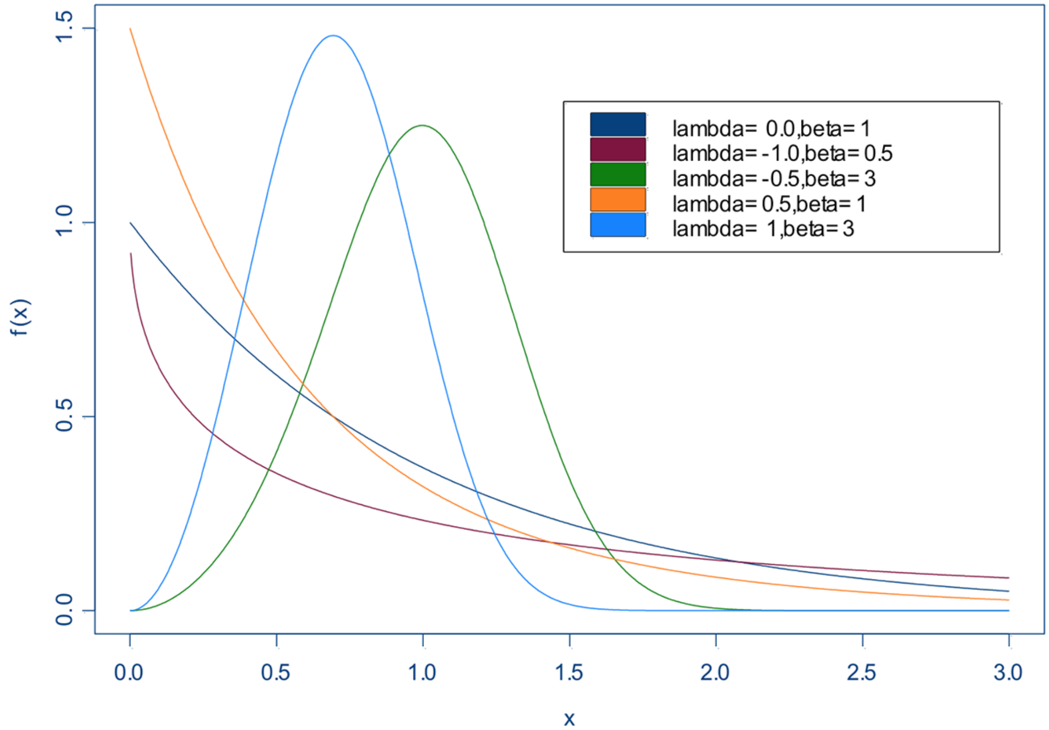

Figure 2. The pdfs of various Transmuted Weibull distributions.

Special cases

1) If $\lambda = 0$, then Transmuted Weibull distribution reduced to two parameter Weibull distribution with parameters α and β.

$$f\left(x;\alpha,\beta\right)=\frac{\beta}{\alpha}x^{\beta-1}\exp\left(-\frac{x^{\beta}}{\alpha}\right)\ x\geq 0\ \ \alpha,\beta>0$$

2) If $\lambda = 0$ and $\beta = 1$, then Transmuted Weibull distribution reduced to exponential distribution with parameter $\left(\frac{1}{\alpha}\right)$, i.e.

$$f\left(x\right)=\frac{1}{\alpha}\exp\left(-\frac{x}{\alpha}\right)\ x>0,\alpha>0$$

3) If $\lambda = 0$ and $\alpha = \beta = 1$, then Transmuted Weibull distribution reduced to standard exponential distribution, i.e.

$$f(x) = \exp(-x) \quad x > 0$$

Moments of Transmuted Weibull distribution

Moments are the expected values of certain functions of a random variable. They serve to numerically describe the variable with respect to given characteristics for location, variation, skewness and kurtosis, to name a few. The expected value of x^r is termed as r^{th} moment about origin of the random variable x which is given by

$$\mu_r' = E(x)^r$$

Thus the r^{th} moment of Transmuted Weibull distribution is given by

$$\mu_r' = \int_0^\infty x^r f(x;\alpha,\beta,\lambda)\,dx$$

$$\mu_r' = \int_0^\infty x^r \frac{\beta}{\alpha} x^{\beta-1} \exp\left(-\frac{x^\beta}{\alpha}\right)\left\{1-\lambda+2\lambda\exp\left(-\frac{x^\beta}{\alpha}\right)\right\}dx$$

After solving the above equation,

$$\mu_r' = \alpha^{\frac{r}{\beta}}\Gamma\left(\frac{r}{\beta}+1\right)\left(1-\lambda+\lambda 2^{-\frac{r}{\beta}}\right) \tag{5}$$

Mean of the Transmuted Weibull distribution

Setting $r = 1$ in equation (5) leads to the mean of the Transmuted Weibull distribution, which is given by

$$\mu_1' = \alpha^{\frac{1}{\beta}}\Gamma\left(\frac{1}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-1}{\beta}}\right) \tag{6}$$

Second moment of the Transmuted Weibull distribution

Setting $r = 2$ in equation (5),

$$\mu_2' = \alpha^{\frac{2}{\beta}}\Gamma\left(\frac{2}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-2}{\beta}}\right) \tag{7}$$

Variance of Transmuted Weibull distribution

The variance of Transmuted Weibull distribution is given by

$$\mu_2 = \alpha^{\frac{2}{\beta}}\left\{\Gamma\left(\frac{2}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-2}{\beta}}\right)-\Gamma^2\left(\frac{1}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-1}{\beta}}\right)^2\right\} \tag{8}$$

Third and fourth moments of Transmuted Weibull distribution

Setting $r = 3$ in equation (5),

$$\mu_3' = \alpha^{\frac{3}{\beta}}\Gamma\left(\frac{3}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-3}{\beta}}\right)$$

and

$$\mu_3 = \alpha^{\frac{3}{\beta}}\left[\begin{array}{l}\Gamma\left(\frac{3}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-3}{\beta}}\right)\\ -\Gamma\left(\frac{1}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-1}{\beta}}\right)\left\{\begin{array}{l}3\Gamma\left(\frac{2}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-2}{\beta}}\right)\\ -2\Gamma^2\left(\frac{1}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-1}{\beta}}\right)^2\end{array}\right\}\end{array}\right] \tag{9}$$

If $r = 4$ in equation (5),

$$\mu_4' = \alpha^{\frac{4}{\beta}}\Gamma\left(\frac{4}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-4}{\beta}}\right)$$

thus

$$\mu_4 = \alpha^{\frac{4}{\beta}} \left[\begin{array}{l} \Gamma\left(\frac{4}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-4}{\beta}}\right) \\ -\Gamma\left(\frac{1}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-1}{\beta}}\right) \left\{ \begin{array}{l} 4\Gamma\left(\frac{3}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-3}{\beta}}\right) \\ -6\Gamma^2\left(\frac{1}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-1}{\beta}}\right) \\ \Gamma\left(\frac{2}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-2}{\beta}}\right) \\ +3\Gamma^3\left(\frac{1}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-1}{\beta}}\right)^3 \end{array} \right\} \end{array} \right] \tag{10}$$

MGF of Transmuted Weibull distribution

The mgf of Transmuted Weibull distribution is given by

$$M_x(t) = \int_0^\infty e^{tx} f(x)\,dx$$

$$M_x(t) = \int_0^\infty \left\{ 1 + tx + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \cdots + \frac{(tx)^n}{n!} + \cdots \right\} f(x)\,dx$$

$$M_x(t) = \int_0^\infty \sum_{r=0}^\infty \frac{t^r x^r}{r!} f(x)\,dx$$

$$M_x(t) = \sum_{r=0}^\infty \frac{t^r}{r!} \int_0^\infty x^r f(x)\,dx$$

$$M_x(t) = \sum_{r=0}^\infty \frac{t^r}{r!} \mu_r'$$

Now by using the equation (5) in the above equation, we have

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \alpha^{\frac{r}{\beta}} \Gamma\left(\frac{r}{\beta}+1\right)\left(1-\lambda+\lambda 2^{-\frac{r}{\beta}}\right) \tag{11}$$

This is the required mgf of Transmuted Weibull distribution.

Standard deviation of Transmuted Weibull distribution

The positive square root of the variance is called standard deviation. Symbolically, $\sigma = \sqrt{\sigma^2}$. From equation (8), the variance of Transmuted Weibull distribution is given as

$$\sigma^2 = \alpha^{\frac{2}{\beta}} \left\{ \Gamma\left(\frac{2}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-2}{\beta}}\right) - \Gamma^2\left(\frac{1}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-1}{\beta}}\right)^2 \right\}$$

$$\Rightarrow \sigma = \alpha^{\frac{1}{\beta}} \sqrt{\left\{ \Gamma\left(\frac{2}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-2}{\beta}}\right) - \Gamma^2\left(\frac{1}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-1}{\beta}}\right)^2 \right\}}$$

$$\Rightarrow \qquad \sigma = \alpha^{\frac{1}{\beta}} \sqrt{\sigma_2 - \sigma_1^2}$$

where

$$\sigma_k = \Gamma\left(\frac{k}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-k}{\beta}}\right) \tag{12}$$

Coefficient of variation of Transmuted Weibull distribution

This is the ratio of standard deviation and mean. Usually, it is denoted by C.V. and is given by

$$C.V. = \frac{\sigma}{\mu}$$

$$\Rightarrow \; C.V. = \frac{\alpha^{\frac{1}{\beta}} \sqrt{\left\{ \Gamma\left(\frac{2}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-2}{\beta}}\right) - \Gamma^2\left(\frac{1}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-1}{\beta}}\right)^2 \right\}}}{\alpha^{\frac{1}{\beta}} \Gamma\left(\frac{1}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-1}{\beta}}\right)}$$

$$\Rightarrow \; C.V. = \frac{\sqrt{\sigma_2 - \sigma_1^2}}{\sigma_1} \tag{13}$$

where $\sigma_k = \Gamma\left(\frac{k}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-1}{\beta}}\right)$

Skewness and kurtosis of Transmuted Weibull distribution

The most popular way to measure the skewness and kurtosis of a distribution function rests upon ratios of moments. Lack of symmetry of tails (about mean) of frequency distribution curve is known as skewness. The formula for measure of skewness given by Karl Pearson in terms of moments of frequency distribution is given by

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3}$$

After using equation (8) and equation (9) in the above equation, we have

$$\beta_1 = \frac{\left[\begin{array}{l} \Gamma\left(\frac{3}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-3}{\beta}}\right) \\ -\Gamma\left(\frac{1}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-1}{\beta}}\right) \left\{ \begin{array}{l} 3\Gamma\left(\frac{2}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-2}{\beta}}\right) \\ -2\Gamma^2\left(\frac{1}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-1}{\beta}}\right)^2 \end{array} \right\} \end{array} \right]^2}{\left[\Gamma\left(\frac{2}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-2}{\beta}}\right) - \Gamma^2\left(\frac{1}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-1}{\beta}}\right)^2 \right]^3}$$

$$\Rightarrow \ \beta_1 = \frac{\left\{\sigma_3 - \sigma_1\left(3\sigma_2 - \sigma_1^2\right)\right\}^2}{\left(\sigma_2 - \sigma_1^2\right)^3}$$

where

$$\sigma_k = \Gamma\left(\frac{k}{\beta} + 1\right)\left(1 - \lambda + \lambda 2^{\frac{-1}{\beta}}\right)$$

Therefore

$$\gamma_1 = \sqrt{\beta_1}$$

$$\Rightarrow \ \gamma_1 = \frac{\left\{\sigma_3 - \sigma_1\left(3\sigma_2 - \sigma_1^2\right)\right\}}{\left(\sigma_2 - \sigma_1^2\right)^{\frac{3}{2}}}$$

If $\gamma_1 < 0$, then the frequency curve is negatively skewed. If $\gamma_1 > 0$, then the frequency curve is positively skewed.

Kurtosis

The formula for measure of kurtosis is given by

$$\beta_2 = \frac{\mu_4}{\mu_2^2}$$

After using equation (8) and equation (10) in the above equation,

$$\beta_2 = \frac{\left[\begin{array}{l}\Gamma\left(\frac{4}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-4}{\beta}}\right) \\ -\Gamma\left(\frac{1}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-1}{\beta}}\right)\left\{4\Gamma\left(\frac{3}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-3}{\beta}}\right)\right. \\ \quad -6\Gamma\left(\frac{1}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-1}{\beta}}\right)\Gamma\left(\frac{2}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-2}{\beta}}\right) \\ \quad \left. +3\Gamma^3\left(\frac{1}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-1}{\beta}}\right)^3 \right\}\end{array}\right.}{\left[\Gamma\left(\frac{2}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-2}{\beta}}\right)-\Gamma^2\left(\frac{1}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-1}{\beta}}\right)^2\right]^2}$$

$$\Rightarrow \beta_2 = \frac{\left\{\sigma_4-\sigma_1\left(4\sigma_3-6\sigma_1\sigma_2+3\sigma_1\right)\right\}^3}{\left(\sigma_2-\sigma_1\right)^2}$$

where

$$\sigma_k = \Gamma\left(\frac{k}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-1}{\beta}}\right)$$

and

$$\gamma_2 = \beta_2 - 3$$

$$\Rightarrow \quad \gamma_2 = \frac{\left\{\sigma_4-\sigma_1\left(4\sigma_3-6\sigma_1\sigma_2+3\sigma_1\right)\right\}^3}{\left(\sigma_2-\sigma_1\right)^2} - 3$$

If $\gamma_2 > 0$, then the frequency curve is leptokurtic. If $\gamma_2 < 0$, then the frequency curve is platykurtic. If $\gamma_2 = 0$, then the frequency curve is mesokurtic, or we can say that there is no kurtosis.

Harmonic mean of Transmuted Weibull distribution

$$\frac{1}{H}=\int_0^\infty f\left(x;\alpha,\beta,\lambda\right)dx$$

$$\frac{1}{H}=\int_0^\infty \frac{1}{x}\frac{\beta}{\alpha}x^{\beta-1}\exp\left(-\frac{x^\beta}{\alpha}\right)\left\{1-\lambda+2\lambda\exp\left(-\frac{x^\beta}{\alpha}\right)\right\}dx$$

$$\frac{1}{H}=\left(1-\lambda\right)\int_0^\infty \frac{1}{x}\frac{\beta}{\alpha}x^{\beta-1}\exp\left(-\frac{x^\beta}{\alpha}\right)dx+2\lambda\int_0^\infty \frac{1}{x}\frac{\beta}{\alpha}x^{\beta-1}\exp\left(-\frac{2x^\beta}{\alpha}\right)dx$$

After substitution,

$$\frac{1}{H}=\frac{\left(1-\lambda\right)}{\alpha}\int_0^\infty \frac{1}{z^{\frac{1}{\beta}}}\exp\left(-\frac{z}{\alpha}\right)dz+2\frac{\lambda}{\alpha}\int_0^\infty \frac{1}{z^{\frac{1}{\beta}}}\exp\left(-\frac{2z}{\alpha}\right)dz$$

After solving the above equation

$$\frac{1}{H}=\alpha^{\frac{1}{\beta}}\Gamma\left(1-\frac{1}{\beta}\right)\left(1-\lambda+\lambda 2^{\frac{-1}{\beta}}\right)$$

$$\Rightarrow\ H=\frac{1}{\alpha^{\frac{1}{\beta}}\Gamma\left(1-\frac{1}{\beta}\right)\left(1-\lambda+\lambda 2^{\frac{-1}{\beta}}\right)} \tag{14}$$

New moment estimator of the Transmuted Weibull distribution

For deriving new moment estimators of three parameters transmuted Weibull distribution, we need the following theorem obtained by using the similar approach of Huang and Hwang (2006).

Theorem 1. Let $n\geq 3$ and let $X_1, X_2, X_3, \ldots, X_n$ be n positive identical independently random variables having probability density function $f(x)$. Then the independence of the sample mean $\bar{X}_n$ and the sample coefficient of variance

$V_n = \dfrac{S_n}{\bar{X}_n}$ is equivalent to that $f(x)$ is a Transmuted Weibull density where S_n is the sample standard deviation.

The next theorem requires the derivation of the expectation and the variance of $V_n^{\,2} = \left(\dfrac{S_n}{\bar{X}_n}\right)^2$, where $\bar{X}_n$ and S_n are respectively the sample mean and the sample standard deviation.

Theorem 2. Let $X_1, X_2, X_3, \ldots, X_n$ be n positive identical independently distributed random samples drawn from a population having Transmuted Weibull density

$$f(x;\alpha,\beta,\lambda) = \frac{\beta}{\alpha} x^{\beta-1} \exp\left(-\frac{x^\beta}{\alpha}\right)\left\{1-\lambda+2\lambda\exp\left(-\frac{x^\beta}{\alpha}\right)\right\}$$

then

$$E\left(S_n^2\right) = \alpha^{\frac{2}{\beta}}\left\{\Gamma\left(\frac{2}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-2}{\beta}}\right) - \Gamma^2\left(\frac{1}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-1}{\beta}}\right)^2\right\}$$

Proof: Because the r^{th} moment of a random variable x about origin is given by

$$\mu_r' = \int_0^\infty x^r f(x;\alpha,\beta,\lambda)\,dx$$

$$\mu_r' = \int_0^\infty x^r \frac{\beta}{\alpha} x^{\beta-1} \exp\left(-\frac{x^\beta}{\alpha}\right)\left\{1-\lambda+2\lambda\exp\left(-\frac{x^\beta}{\alpha}\right)\right\}dx$$

After solving the above equation,

$$\mu_r' = \alpha^{\frac{r}{\beta}}\Gamma\left(\frac{r}{\beta}+1\right)\left(1-\lambda+\lambda 2^{-\frac{r}{\beta}}\right)$$

If $r = 1$ in the above equation,

$$E\left(\bar{X}_n\right)=\alpha^{\frac{1}{\beta}}\Gamma\left(\frac{1}{\beta}+1\right)\left(1-\lambda+\lambda 2^{-\frac{1}{\beta}}\right)$$

Also if $r = 2$ in the above equation,

$$E\left(\bar{X}_n^2\right)=\frac{\alpha^{\frac{2}{\beta}}\left[\begin{array}{l}\Gamma\left(\frac{2}{\beta}+1\right)\left(1-\lambda+\lambda 2^{-\frac{2}{\beta}}\right)\\ +(n-1)\Gamma^2\left(\frac{1}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-1}{\beta}}\right)^2\end{array}\right]}{n} \qquad (15)$$

and

$$V\left(\bar{X}_n\right)=\frac{\alpha^{\frac{2}{\beta}}\left\{\begin{array}{l}\Gamma\left(\frac{2}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-2}{\beta}}\right)\\ -\Gamma^2\left(\frac{1}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-1}{\beta}}\right)^2\end{array}\right\}}{n}.$$

Thus

$$E\left(S_n^2\right)=nV\left(\bar{X}_n\right)$$

$$E\left(S_n^2\right)=\alpha^{\frac{2}{\beta}}\left\{\begin{array}{l}\Gamma\left(\frac{2}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-2}{\beta}}\right)\\ -\Gamma^2\left(\frac{1}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-1}{\beta}}\right)^2\end{array}\right\} \qquad (16)$$

where $\bar{X}_n$ and S_n^2 are respectively the sample mean and the sample variance.

Theorem 3. Let $X_1, X_2, X_3, \ldots, X_n$ be n positive identical independently distributed random samples drawn from a population having Transmuted Weibull density

$$f(x;\alpha,\beta,\lambda)=\frac{\beta}{\alpha}x^{\beta-1}\exp\left(-\frac{x^{\beta}}{\alpha}\right)\left\{1-\lambda+2\lambda\exp\left(-\frac{x^{\beta}}{\alpha}\right)\right\}$$

then

$$E\left(\frac{S_n^2}{\bar{X}_n^2}\right)=\frac{n\left\{\begin{array}{l}\Gamma\left(\frac{2}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-2}{\beta}}\right)\\ -\Gamma^2\left(\frac{1}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-1}{\beta}}\right)^2\end{array}\right\}}{\left[\begin{array}{l}\Gamma\left(\frac{2}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-2}{\beta}}\right)\\ +(n-1)\Gamma^2\left(\frac{1}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-1}{\beta}}\right)^2\end{array}\right]}$$

where $\bar{X}_n$ and S_n^2 are respectively the sample mean and the sample variance.

Proof: By using the theorem (1), we have

$$E\left(S_n^2\right)=E\left(\frac{S_n^2}{\bar{X}_n^2}\bar{X}_n^2\right)=E\left(\frac{S_n^2}{\bar{X}_n^2}\right)E\left(\bar{X}_n^2\right)$$

$$\Rightarrow\qquad E\left(\frac{S_n^2}{\bar{X}_n^2}\right)=\frac{E\left(S_n^2\right)}{E\left(\bar{X}_n^2\right)}\qquad(17)$$

Now using equations (15) and (16) in equation (17), we have

$$E\left(\frac{S_n^2}{\bar{X}_n^2}\right)=\frac{n\left\{\begin{array}{l}\Gamma\left(\frac{2}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-2}{\beta}}\right)\\-\Gamma^2\left(\frac{1}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-1}{\beta}}\right)^2\end{array}\right\}}{\left[\begin{array}{l}\Gamma\left(\frac{2}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-2}{\beta}}\right)\\+(n-1)\Gamma^2\left(\frac{1}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-1}{\beta}}\right)^2\end{array}\right]}$$

$$E\left(\frac{S_n^2}{\bar{X}_n^2}\right)\rightarrow\frac{\left\{\Gamma\left(\frac{2}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-2}{\beta}}\right)\right\}}{\left[\Gamma^2\left(\frac{1}{\beta}+1\right)\left(1-\lambda+\lambda 2^{\frac{-1}{\beta}}\right)^2\right]}-1$$

as $n\rightarrow\infty$ and that this limit is the square of the population coefficient of variation. Thus, $\frac{S_n^2}{\bar{X}_n^2}$ is an asymptotically unbiased estimator of the square of the population coefficient of variation.

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