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Discrete Littlewood-Paley-Stein Theory And Wolff Potentials On Homogeneous Spaces And Multi-Parameter Hardy Spaces

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DISCRETE LITTLEWOOD-PALEY-STEIN THEORY AND WOLFF POTENTIALS ON HOMOGENEOUS SPACES AND MULTI-PARAMETER HARDY SPACES

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To my family

Xiang Xiao, Shouren Long, and Jie Long

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PREFACE

This dissertation consists of two parts:

- **Part I** We establish a new atomic decomposition of the multi-parameter Hardy spaces of homogeneous type and obtain the associated $H^p - L^p$ and $H^p - H^p$ boundedness criterions for singular integral operators. On the other hand, we compare the Wolff and Riesz potentials on spaces of homogenous type, followed by a Hardy-Littlewood-Sobolev type inequality. Then we drive integrability estimates of positive solutions to the Lane-Emden type integral systems on spaces of homogeneous type.

- **Part II** We establish a $(p, 2)$-atomic decomposition of the Hardy space associated with different homogeneities for $0 < p \leq 1$. In addition, We characterize the dual spaces of the weighted multi-parameter Hardy spaces associated with Zygmund dilations, i. e. $(H^p_{\mathcal{Z}}(w))^* = CMO^p_{\mathcal{Z}}(w)$ for $w \in A_\infty(\mathcal{Z})$ and $0 < p \leq 1$.

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Part II

Part I

1 Introduction

1.1 Background and main questions

For a set $\mathcal{X}$, we say that a function $\rho : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ is a quasi-metric on $\mathcal{X}$ if it satisfies that for any $x_1, x_2, x_3 \in \mathcal{X}$,

(i) $\rho(x_1, x_2) = \rho(x_2, x_1)$;

(ii) $\rho(x_1, x_2) = 0$ if and only if $x_1 = x_2$;

(iii) $\rho(x_1, x_2) \leq C[\rho(x_1, x_3) + \rho(x_2, x_3)]$, where $C \in [1, \infty)$ is a constant independent of x_1, x_2 and x_3.

Let $B(x, t)$ denote the ball $\{y \in \mathcal{X} : \ \rho(x, y) < t\}$ for all $x \in \mathcal{X}$ and $t > 0$, then such quasi-metric ρ defines a topology on $\mathcal{X}$, for which the balls $B(x, t)$ form a basis. Let μ be a nonnegative measure satisfying the doubling property, i.e.,

(iv) for all $t > 0$, there exists some constant C such that $\mu(B(x, 2t)) \leq C\mu(B(x, t))$;

then the set $\mathcal{X}$ together with a quasi-metric ρ and a nonnegative doubling measure μ on $\mathcal{X}$, $(\mathcal{X}, \rho, \mu)$, is called a space of homogeneous type, which was first introduced by R.Coifman and G.Weiss in [CW1] in order to extend the theory of Calderón-Zygmund singular integrals on $\mathbb{R}^n$ to a more general setting.

The project of developing theory of spaces of homogeneous type has received much attention due to its own difficulty, interest and applications. It has developed in significantly in the past four decades, there are many monographs and surveys available in the literature, among them we mention [ABI],[Cm],[CW2],[DH],[FS],[N].

On the other hand, The Hardy spaces H^p are important objects in classical harmonic analysis. For $p = 1$(or $p = \infty$), the Hardy space H^1(or its dual space) appears as a natural substitute of the classical Lebesgue space L^1(or L^∞). For $1 < p < \infty$, $\|f\|_{H^p} \sim \|f\|_p$ is well known as the Littlewood-Paley-Stein theory which implies that $H^p = L^p$. For $0 < p < 1$, while L^p have some undesirable properties, the H^p are much better behaved.

One of the principal interests of H^p theory is that it gives a natural extension of the results for singular integrals (originally developed for L^p, $p > 1$) to $0 < p \leq 1$. Broadly speaking, the $L^p(1 < p < \infty)$ boundedness theorems for singular integrals may be extended to H^p for all $0 < p \leq 1$. Therefore, one part of our research focus on using the discrete Littlewood-Paley theory to study the boundedness of singular integral operators on Hardy spaces of homogeneous type.

In Chapter 2, we first introduce the multi-parameter Hardy space of homogeneous type $H^p(\mathcal{X}_1 \times \mathcal{X}_2)$. By using Journe's covering lemma for spaces of homogeneous type, we derive a new atomic decomposition of $H^p(\mathcal{X}_1 \times \mathcal{X}_2)$ which converges in both the classical Lebesgue spaces L^q (for $1 < q < \infty$) and Hardy spaces H^p (for $0 < p \leq 1$). As an application, we prove boundedness criterions of operators from $H^p(\mathcal{X}_1 \times \mathcal{X}_2)$ to $L^p(\mathcal{X}_1 \times \mathcal{X}_2)$ and from $H^p(\mathcal{X}_1 \times \mathcal{X}_2)$ to iteself for $0 < p \leq 1$.

In Chapter 3, we get a Hardy-Littlewood-Sobolev type inequality on space of homogeneous type by comparing the associated Wolff and Riesz potentials. After that, by using the regularity lifting method, we derive integrablility estimates of positive solutions to Lane-Emden type integral system on spaces of homogeneous type.

At last, we would like to point it out that the spaces of homogeneous type include the classical Euclidean space $\mathbb{R}^n$, compact Lie groups, C^∞ manifolds with doubling volume measures for geodesic balls, Carnot-Caratheodory spaces, nilpotent Lie groups such as the Heisenberg group, and many other cases, so all the above results can be applied to these cases.

1.2 Some properties of spaces of homogeneous type

For any space of homogeneous type $(\mathcal{X}, \rho, \mu)$, R. A. Macias and C. Segovia [MS] have proved that the quasi-metric ρ can be replaced by another ρ^* such that $\rho^* \sim \rho$ and ρ^* yields the same topology on $\mathcal{X}$ as ρ. Moreover, let $B(x,t)$ denote the ball defined by ρ^*, $\{y \in \mathcal{X} : \rho^*(x,y) \leq t\}$, then for all $0 < t < \infty$, there exists some $d > 0$ such that

$$\mu(B(x,t)) \sim t^d,$$

and there exists a constant $A > 0$ such thatρ^* has the following regularity property

$$|\rho^*(x_1,x_2) - \rho^*(x_3,x_2)| \leq A\rho^*(x_1,x_3)^\theta[\rho^*(x_1,x_2) + \rho^*(x_3,x_2)]^{1-\theta}$$

for some regularity exponent $\theta \in (0,1)$ and all $x_1, x_2, x_3 \in \mathcal{X}$.

Therefore, a formal definition of homogeneous space in the sense of R. Coifman and G. Wiess can be given as follows.

Definition 1.1. *Let $d > 0$ and $\theta \in (0,1)$. A space of homogeneous type $(\mathcal{X}, \rho, \mu)_{d,\theta}$ is a set $\mathcal{X}$ equipped with a quasi-metric ρ and a nonnegative measure μ on $\mathcal{X}$, and there exists a* *constant* $A > 0$ *such that for all* $x_1,\ x_2,\ x_3 \in \mathcal{X}$ *and* $0 < t < \operatorname{diam} \mathcal{X}$,

$$\mu(B(x,t)) \sim t^d \tag{1.1}$$

and

$$|\rho(x_1,x_2) - \rho(x_3,x_2)| \leq A\rho(x_1,x_3)^\theta[\rho(x_1,x_2)+\rho(x_3,x_2)]^{1-\theta}. \tag{1.2}$$

1.3 Dyadic cubes on spaces of homogeneous type

For spaces of homogeneous type, an analogue of the grid of Euclidean dyadic cubes was given independently by M. Christ [Cm] and E. Sawyer and R. Wheeden [SW] as follows.

Lemma 1.2 (Dyadic cubes on homogeneous spaces [SW])**.** *Let* $(\mathcal{X},\rho,\mu)_{d,\theta}$ *be a space of homogeneous type. Then for every integer* $k \in \mathbb{Z}_+$, *there exists a collection of open subsets* $\{Q_\tau^k \subseteq \mathcal{X} : \tau \in I_k\}$, *where* I_k *denotes some index set depending on* k, *and positive constants* C_1, C_2 *such that*

(i) $\mu(\{\mathcal{X} \setminus \cup Q_\tau^k\}) = 0$;

(ii) If $l \geq k$, *then for all* $\tau' \in I_l$ *and* $\tau \in I_k$ *either* $Q_{\tau'}^l \subseteq Q_\tau^k$ *or* $Q_{\tau'}^l \cap Q_\tau^k = \emptyset$;

(iii) If $l < k$, *for each* $\tau \in I_k$, *there is a unique* $\tau' \in I_l$ *such that* $Q_\tau^k \subseteq Q_{\tau'}^l$, $\mathrm{diam}(\mathrm{Q}_\tau^\mathrm{k}) \leq \mathrm{C}_1 2^{-\mathrm{k}}$, *and each* Q_τ^k *contains some ball* $B(z_\tau^k, C_2 2^{-k})$.

With the settings defined above, in the following, we say that a cube $Q \subseteq \mathcal{X}$ is a dyadic cube if $Q = Q_\tau^k$ for some $k \in \mathbb{Z}_+$ and $\tau \in I_k$, denote it by $\mathrm{diam(Q)} \sim 2^{-\mathrm{k}}$.

1.4 Notational index

The following notations will be frequently used in the rest of Part I.

- ρ: a quasi-metric.
- μ: a nonnegative doubling measure.
- $(\mathcal{X}, \rho, \mu)$ or $(\mathcal{X}, \rho, \mu)_{d,\theta}$: a space of homogeneous type.
- $B(x,t)$: the ball centered at x and of radius t.
- Ω: an open and bounded domain in $\mathbb{R}^n$ or in $\mathcal{X}$.
- $\mathcal{M}(\Omega)$: the set of all maximal dyadic rectangles contained in Ω.
- $\mathcal{M}_i(\Omega)$: the set of all dyadic rectangles contained in Ω and maximal in the direction of x_i.
- $M_s f$: the strong maximal function of f.
- q': the conjugate index of the index $1 \le q \le \infty$, that is, $\frac{1}{q} + \frac{1}{q'} = 1$.
- Q_τ^k: a dyadic cube with $\mathrm{diam(Q)} \sim 2^{-\mathrm{k}}$ (See Lemma 1.2).
- $Q_\tau^{k,\,\nu}$: all dyadic cubes $Q_{\tau'}^{k+j} \subseteq Q_\tau^k$ for a fixed positive integer j, $\nu = 1, 2, ..., N(k,\tau)\}$.
- $y_\tau^{k,\nu}$: a point in $Q_\tau^{k,\nu}$.

2 Atomic decomposition of Multi-parameter Hardy spaces of Homogeneous Type

2.1 Introduction and Statements of Main results

It is well known that the elements in the classical Hardy space $H^p(\mathbb{R}^n)$ can be decomposed as the sum of an appropriate class of simple functions, that is, "atoms ". The "atoms" play an important role in proving the boundedness of operators on Hardy spaces by verifying their actions on such building blocks (see, for example, Coifman [CO] Coifman-Weiss [CW2], Grafakos [G], Latter [La], Lu [Lu], Meyer [M], Meyer-Coifman [MC], Stein [St], etc.).

In general, if a linear operator is bounded on the space of all atoms which is dense in Hardy space, it can be extended to a bounded operator on the whole Hardy space. However, this boundedness principle is not always true. M.Bownik [B] gave an example shows the boundedness principle is broken when considering $(1,\infty)$-atoms on $H^1(\mathbb{R}^n)$. Therefore, we need to proceed with caution when the above principle is used. And it is meaningful to ask under what circumstances the above fundamental principle can be applied. In [MSV], S. Meda, P. Sjogren and M. Vallarino have proved that this boundedness criterion holds on $H^1(\mathbb{R}^n)$ for $(1,q)$ atoms for $1 < q < \infty$ (see also [HZ] and [YZ] for related results). Furthermore, this criterion also holds on $H^p(\mathbb{R}^n)$ for $0 < p < 1$ when applying this principle for (p,∞)-atoms as shown by Ricci and Verdera [RV].

For the multi-parameter Hardy spaces, A. Chang and R. Fefferman ([CF1], [CF2], [CF3]) developed the product Hardy space $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ theory. In [CF2], they proved the following

Theorem. Let $0 < p < \infty$. $f \in H^p(\mathbb{R} \times \mathbb{R})$ if and only if $f(x,y) = \sum\limits_k \lambda_k a_k(x,y)$ where

$\sum_k |\lambda_k|^p < \infty$ and each $a_k(x,y)$ is a $(p,2)$-atom, that is, each $a_k(x,y)$ is supported in an open set Ω with finite measure and satisfies the following properties:

$$\|a_k\|_2 \le |\Omega|^{1/2-1/p};$$

each $a_k(x,y)$ can be further decomposed by

$$a_k(x,y) = \sum_{R\in\Omega} a_R(x,y)$$

where $R = I \times J \subset \Omega$ are dyadic rectangles in $\mathbb{R}^2$, and each $a_R(x,y)$ satisfies

$$\int_I a_R(x,y)x^\alpha dx = \int_J a_R(x,y)y^\beta dy = 0$$

for $0 \le |\alpha|, |\beta| \le N_p = [2/p - 4/3]$. Moreover, a_R is a C^η ($\eta \le N_p + 1$) function satisfying

$$|\frac{\partial^\eta}{\partial x^\eta} a_R(x,y)| \le d_R|I|^{-\eta}, \quad |\frac{\partial^\eta}{\partial y^\eta} a_R(x,y)| \le d_R|J|^{-\eta}$$

with

$$\sum_{R\in\mathcal{M}(\Omega)} |R|d_R^2 \le |\Omega|^{1-2/p},$$

where $\mathcal{M}(\Omega)$ is the set of all rectangles in Ω which are maximal in both directions of x and y.

The key tool that A. Chang and R. Fefferman used to prove the above $(p,2)$-atomic decomposition is the classical version of continuous Calderón's identity on the product space.

However, for $1 < q < \infty, q \neq 2$, the (p,q)-atomic decomposition on product Hardy spaces $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ cannot be established by using the classical Calderón's identity. Therefore, the (p,q) atomic decomposition for the product Hardy spaces becomes interesting for $q \neq 2$. This has been recently carried out by Han, Lu, and Zhao in [HLZk]. They established the (p,q)-product atoms on the multi-parameter Hardy spaces $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ for $0 < p \leq 1$ and $1 < q < \infty$ by using discrete Littlewood-Paley analysis and the discrete Calderón's identity. Since the spaces of homogeneous type is a generalized extension of the Euclidean spaces, it is natural to consider the atomic decomposition of Hardy spaces of homogeneous type.

We will derive a new (p,q)-atomic decomposition on the multi-parameter Hardy space $H^p(\mathcal{X}_1 \times \mathcal{X}_2)$ for $0 < p \leq 1$ and all $1 < q < \infty$, where $\mathcal{X}_1 \times \mathcal{X}_2$ is the product of two homogeneous type spaces in the sense of Coifman and Weiss ([CW1]). The series in $(p,2)$-atomic decomposition in [CF1] converges only in the sense of distributions. But the decomposition we get converges in both $L^q(\mathcal{X}_1 \times \mathcal{X}_2)$ $(1 < q < \infty)$ and $H^p(\mathcal{X}_1 \times \mathcal{X}_2)$ $(0 < p \leq 1)$. As an application, we prove that an operator T, which is bounded on $L^q(\mathcal{X}_1 \times \mathcal{X}_2)$ for some $1 < q < \infty$, is bounded from $H^p(\mathcal{X}_1 \times \mathcal{X}_2)$ to $L^p(\mathcal{X}_1 \times \mathcal{X}_2)$ if and only if T is bounded uniformly on all (p,q)-product atoms in $L^p(\mathcal{X}_1 \times \mathcal{X}_2)$. The similar boundedness criterion from $H^p(\mathcal{X}_1 \times \mathcal{X}_2)$ to $H^p(\mathcal{X}_1 \times \mathcal{X}_2)$ is also obtained. The main idea is establishing the Journé's covering lemma for spaces of homogeneous type, and using the Littlewood-Paley theory and a new discrete Calderón reproducing formulas on product spaces of homogeneous type (see [HL3]) to derive a (p,q)-atomic decompositon for $H^p(\mathcal{X}_1 \times \mathcal{X}_2)$. Then by the fact that $L^q(\mathcal{X}_1 \times \mathcal{X}_2) \cap H^p(\mathcal{X}_1 \times \mathcal{X}_2)$ is dense in $H^p(\mathcal{X}_1 \times \mathcal{X}_2)$ for $0 < p \leq 1 < q < \infty$ and $\|f\|_{L^q(\mathcal{X}_1 \times \mathcal{X}_2)} \leq C\|f\|_{H^p(\mathcal{X}_1 \times \mathcal{X}_2)}$(see [HLL-W]), we get the boundedness criterion of operators on $H^p(\mathcal{X}_1 \times \mathcal{X}_2)$. We would like to point out that this method is quite different from the classical product theory in Euclidean spaces (see [CF2], [Fr1], and [St]), which is not suitable for the Hardy space H^p $(0 < p \leq 1)$ on product spaces of homogeneous type. This method also works for the atomic decomposition of other Hardy spaces such as Hardy spaces associated with two different homogeneities, which we show in Chapter 4. For more general applications of the discrete Littlewood-Paley-Stein theory to the multi-parameter Hardy space theory on Carnot-Caratheodory spaces and product spaces of homogeneous type, please see [HLL2].

For $i = 1,\ 2$, let $(X_i, \rho_i, \mu_i)_{d_i,\, \theta_i}$ be a space of homogeneous type, and ρ_i satisfies (1.2) with A replaced by A_i . Then $R = Q_1 \times Q_2 \subset \mathcal{X}_1 \times \mathcal{X}_2$ is said to be a dyadic rectangle in product spaces of homogeneous type if Q_1 and Q_2 are dyadic cubes in $\mathcal{X}_1$ and $\mathcal{X}_2$ respectively, with $\mathrm{diam Q_1} \sim 2^{-\mathrm{k_1}}$ and $\mathrm{diam Q_2} \sim 2^{-\mathrm{k_2}}$ for some $k_1, k_2 \in \mathbb{Z}$.

Now we introduce the approximation to identity on the space of homogeneous type.

Definition 2.1 (approximation to the identity [HS]). *Let $(\mathcal{X}, \rho, \mu)_{d,\, \theta}$ be a space of homogeneous type. For $\epsilon \in (0, \theta]$, we call a sequence of linear operators $\{S_k\}_{k\in\mathbb{Z}}$ as an approximation to the identity of order ϵ on $\mathcal{X}$ if there exists $C_3 > 0$ such that for all $k \in \mathbb{Z}$, the kernel of S_k, $S_k(x_1, y_1)$, are functions from $\mathcal{X} \times \mathcal{X}$ into $\mathbb{C}$ satisfying that for all $x_1,\ x_2,\ y_1$ and $y_2 \in \mathcal{X}$,,*

(1) $|S_k(x_1, y_1)| \leq C_3 \dfrac{2^{-k\epsilon}}{(2^{-k} + \rho(x_1, y_1))^{d+\epsilon}}$;

(2) $|S_k(x_1, y_1) - S_k(x_2, y_1)| \leq C_3 \left(\dfrac{\rho(x_1, x_2)}{2^{-k} + \rho(x_1, y_1)}\right)^{\epsilon} \dfrac{2^{-k\epsilon}}{(2^{-k} + \rho(x_1, y_1))^{d+\epsilon}}$

$$for \quad \rho(x_1, x_2) \leq \frac{1}{2A}(2^{-k} + \rho(x_1, y_1));$$

(3) $|S_k(x_1, y_1) - S_k(x_1, y_2)| \le C_3 \left(\frac{\rho(y_1, y_2)}{2^{-k} + \rho(x_1, y_1)} \right)^\epsilon \frac{2^{-k\epsilon}}{(2^{-k} + \rho(x_1, y_1))^{d+\epsilon}}$

$$for \quad \rho(y_1, y_2) \le \frac{1}{2A}(2^{-k} + \rho(x_1, y_1));$$

(4) $|[S_k(x_1, y_1) - S_k(x_1, y_2)] - [S_k(x_2, y_1) - S_k(x_2, y_2)]| \le C_3 \left(\frac{\rho(x_1, x_2)}{2^{-k} + \rho(x_1, y_1)} \right)^\epsilon$

$$\times \left(\frac{\rho(y_1, y_2)}{2^{-k} + \rho(x_1, y_1)} \right)^\epsilon \frac{2^{-k\epsilon}}{(2^{-k} + \rho(x_1, y_1))^{d+\epsilon}}$$

for $\rho(x_1, x_2) \le \frac{1}{2A}(2^{-k} + \rho(x_1, y_1))$ *and* $\rho(y_1, y_2) \le \frac{1}{2A}(2^{-k} + \rho(x_1, y_1))$;

(5) $\int_{\mathcal{X}} S_k(x_1, y_1)\, d\mu(y_1) = 1$;

(6) $\int_{\mathcal{X}} S_k(x_1, y_1)\, d\mu(x_1) = 1$.

Moreover, we call a sequence of linear operators $\{S_k\}_{k\in\mathbb{Z}}$ *as an approximation to the identity of order* $\epsilon \in (0, \theta]$ *having compact support if there exist constants* $C_4,\ C_5 > 0$ *such that for all* $k \in \mathbb{Z}$, *the kernel of* S_k, $S_k(x_1, y_1)$, *are functions from* $\mathcal{X} \times \mathcal{X}$ *into* $\mathbb{C}$ *satisfying (1)-(6) and*

(7) $S_k(x_1, y_1) = 0$ *if* $\rho(x_1, y_1) \ge C_4 2^{-k}$ *and* $\|S_k\|_{L^\infty(\mathcal{X}\times\mathcal{X})} \le C_5 2^{kd}$;

(8) $|S_k(x_1, y_1) - S_k(x_2, y_1)| \le C_5 2^{k(d+\epsilon)} \rho(x_1, x_2)^\epsilon$;

(9) $|S_k(x_1, y_1) - S_k(x_1, y_2)| \le C_5 2^{k(d+\epsilon)} \rho(y_1, y_2)^\epsilon$;

(10) $|[S_k(x_1, y_1) - S_k(x_1, y_2)] - [S_k(x_2, y_1) - S_k(x_2, y_2)]| \le C_5 2^{k(d+2\epsilon)} \rho(x_1, x_2)^\epsilon \rho(y, y')^\epsilon$

for all $x_1,\ x_2,\ y_1$ *and* $y_2 \in \mathcal{X}$.

Remark. By Coifman's construction in [DJS], one can construct an approximation to the identity of order θ having compact support satisfying the above Definition 2.1.

To introduce the multi-parameter Hardy space of homogeneous type $H^p(\mathcal{X}_1 \times \mathcal{X}_2)$, we first need to introduce the space of test functions on the product space of homogeneous type $\mathcal{X}_1 \times \mathcal{X}_2$.

Definition 2.2. ([*HL3*]) *For $i = 1,\ 2$, fix $\gamma_i > 0$ and $\beta_i > 0$. A function f defined on $\mathcal{X}_1 \times \mathcal{X}_2$ is said to be a test function of type $(\beta_1, \beta_2, \gamma_1, \gamma_2)$ centered at $(x_0, y_0) \in \mathcal{X}_1 \times \mathcal{X}_2$ with width $r_1,\ r_2 > 0$ if for all $x,\ x' \in \mathcal{X}_1$, and $y,\ y' \in \mathcal{X}_2$, f satisfies the following conditions:*

(i) $|f(x,y)| \le C \frac{r_1^{\gamma_1}}{(r_1+\rho_1(x,x_0))^{d_1+\gamma_1}} \frac{r_2^{\gamma_2}}{(r_2+\rho_2(y,y_0))^{d_2+\gamma_2}}$;

(ii) $|f(x,y) - f(x',y)| \le C \left(\frac{\rho_1(x,x')}{r_1+\rho_1(x,x_0)}\right)^{\beta_1} \frac{r_1^{\gamma_1}}{(r_1+\rho_1(x,x_0))^{d_1+\gamma_1}} \frac{r_2^{\gamma_2}}{(r_2+\rho_2(y,y_0))^{d_2+\gamma_2}}$

$$for\ \ \rho_1(x,x') \le \frac{1}{2A_1}[r_1+\rho_1(x,x_0)];$$

(iii) $|f(x,y) - f(x,y')| \le C \frac{r_1^{\gamma_1}}{(r_1+\rho_1(x,x_0))^{d_1+\gamma_1}} \left(\frac{\rho_2(y,y')}{r_2+\rho_2(y,y_0)}\right)^{\beta_2} \frac{r_2^{\gamma_2}}{(r_2+\rho_2(y,y_0))^{d_2+\gamma_2}}$

$$for\ \ \rho_2(y,y') \le \frac{1}{2A_2}[r_2+\rho_2(y,y_0)];$$

(iv) $|[f(x,y) - f(x',y)] - [f(x,y') - f(x',y')]| \le C \left(\frac{\rho_1(x,x')}{r_1+\rho_1(x,x_0)}\right)^{\beta_1} \frac{r_1^{\gamma_1}}{(r_1+\rho_1(x,x_0))^{d_1+\gamma_1}}$

$$\times \left(\frac{\rho_2(y,y')}{r_2+\rho_2(y,y_0)}\right)^{\beta_2} \frac{r_2^{\gamma_2}}{(r_2+\rho_2(y,y_0))^{d_2+\gamma_2}}$$

$$for\ \ \rho_1(x,x') \le \frac{1}{2A_1}[r_1+\rho_1(x,x_0)]\ \ and\ \ \rho_2(y,y') \le \frac{1}{2A_2}[r_2+\rho_2(y,y_0)];$$

(v) $\int_{\mathcal{X}_1} f(x,y)\,d\mu_1(x) = 0$ *for all* $y \in \mathcal{X}_2$*;*

(vi) $\int_{\mathcal{X}_2} f(x,y)\,d\mu_2(y) = 0$ *for all* $x \in \mathcal{X}_1$*.*

If f *is a test function of type* $(\beta_1, \beta_2, \gamma_1, \gamma_2)$ *centered at* $(x_0, y_0) \in \mathcal{X}_1 \times \mathcal{X}_2$ *with width* $r_1,\ r_2 > 0$*, we write* $f \in \mathcal{G}(x_0, y_0; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2)$ *and we define the norm of* f *by*

$$\|f\|_{\mathcal{G}(x_0,y_0;r_1,r_2;\beta_1,\beta_2;\gamma_1,\gamma_2)} = \inf\{C :\ (i),\ (ii),\ (iii) \text{ and } (iv) \text{ hold}\}.$$

If $\beta_1 = \beta_2 = \beta$ *and* $\gamma_1 = \gamma_2 = \gamma$*, we will then simply write* $f \in \mathcal{G}(x_0, y_0; r_1, r_2; \beta; \gamma)$*. And we denote by* $\mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)$ *the class of* $\mathcal{G}(x_0, y_0; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2)$ *with* $r_1 = r_2 = 1$ *for fixed* $(x_0, y_0) \in \mathcal{X}_1 \times \mathcal{X}_2$*. Then if* $\beta_1 = \beta_2 = \beta$ *and* $\gamma_1 = \gamma_2 = \gamma$*, we will simply write* $f \in \mathcal{G}(\beta; \gamma)$*.*

Remark. It is easy to see that $\mathcal{G}(x_1, y_1; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2) = \mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)$ with an equivalent norm for all $(x_1, y_1) \in \mathcal{X}_1 \times \mathcal{X}_2$. We can easily check that the space $\mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)$ is a Banach space. Also, we denote by $(\mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2))'$ its dual space which is the set of all linear functionals $\mathcal{L}$ from $\mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)$ to $\mathbb{C}$ with the property that there exists $C \geq 0$ such that for all $f \in \mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)$,

$$|\mathcal{L}(f)| \leq C\|f\|_{\mathcal{G}(\beta_1,\beta_2;\gamma_1,\gamma_2)}.$$

Clearly, for all $h \in (\mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2))'$, $\langle h, f\rangle$ is well defined for all $f \in \mathcal{G}(x_0, y_0; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2)$ with $(x_0, y_0) \in \mathcal{X}_1 \times \mathcal{X}_2$, $r_1 > 0$ and $r_2 > 0$. By the same reason as the case of one-parameter spaces, we denote by $\mathring{\mathcal{G}}(\beta_1, \beta_2; \gamma_1, \gamma_2)$ the completion of the space $\mathcal{G}(\epsilon_1; \epsilon_2)$ in $\mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)$ when $0 < \beta_1,\ \gamma_1 < \epsilon_1$ and $0 < \beta_2,\ \gamma_2 < \epsilon_2$.

Next we recall Littlewood-Paley theorem on product spaces of homogeneous type, which can be stated as follows.

Lemma 2.3. ([*HL3*]) *For $i = 1,\ 2$, let $\epsilon_i \in (0, \theta_i]$, $\{S_{k_i}\}_{k_i \in \mathbb{Z}}$ be an approximation to the identity of order ϵ_i on $\mathcal{X}_i$, and $D_{k_i} = S_{k_i} - S_{k_i - 1}$ for all $k_i \in \mathbb{Z}$. If $1 < p < \infty$, then there is a constant $C_p > 0$ such that for all $f \in L^p(\mathcal{X}_1 \times \mathcal{X}_2)$,*

$$C_p^{-1} \|f\|_{L^p(\mathcal{X}_1 \times \mathcal{X}_2)} \le \|g_2(f)\|_{L^p(\mathcal{X}_1 \times \mathcal{X}_2)} \le C_p \|f\|_{L^p(\mathcal{X}_1 \times \mathcal{X}_2)},$$

where $g_q(f)$ for $q \in (0, \infty)$ is called the discrete Littlewood-Paley g-function on $\mathcal{X}_1 \times \mathcal{X}_2$ defined by

$$g_q(f)(x_1, x_2) = \left\{ \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} |D_{k_1} D_{k_2}(f)(x_1, x_2)|^q \right\}^{1/q}$$

for all $x_1 \in \mathcal{X}_1$ and $x_2 \in \mathcal{X}_2$.

Now we can introduce the multi-parameter Hardy spaces of homogeneous type $H^p(\mathcal{X}_1 \times \mathcal{X}_2)$ for some $p \le 1$ and establish their (p, q)-atomic decomposition characterization.

Definition 2.4. *For $i = 1,\ 2$, let $\epsilon_i \in (0, \theta_i]$, $\{D_{k_i}\}_{k_i \in \mathbb{Z}}$ be the same as in Lemma 2.3,*

$$\max \left\{ \frac{d_1}{d_1 + \epsilon_1}, \frac{d_2}{d_2 + \epsilon_2} \right\} < p < \infty$$

and

$$d_i(1/p - 1)_+ < \beta_i,\ \gamma_i < \epsilon_i. \tag{2.1}$$

The multi-parameter Hardy spaces of homogeneous type $H^p(\mathcal{X}_1 \times \mathcal{X}_2)$ *is the set defined by*

$$\left\{ f \in \left(\mathring{\mathcal{G}}(\theta_1, \theta_2; \gamma_1, \gamma_2) \right)' : \ \|g_2(f)\|_{L^p(\mathcal{X}_1 \times \mathcal{X}_2)} < \infty \right\},$$

and we define

$$\|f\|_{H^p(\mathcal{X}_1 \times \mathcal{X}_2)} = \|g_2(f)\|_{L^p(\mathcal{X}_1 \times \mathcal{X}_2)},$$

where $g_2(f)$ *is the discrete Littlewood-Paley square function defined as in Lemma 2.3.*

Remark. Here the definition of $H^p(\mathcal{X}_1 \times \mathcal{X}_2)$ is independent of the choice of the approximation to identity, see [HL3] for the proof.

We now can give the definition of (p, q)-atoms of $H^p(\mathcal{X}_1 \times \mathcal{X}_2)$ as follows. For the convenience, in the following, we use C to denote all constants only dependent on $\mathcal{X}_1$ and $\mathcal{X}_2$, which may vary from line to line.

Definition 2.5. *For* $0 < p \leq 1$ *and* $0 < q < \infty$, *a function* $a(x_1, x_2)$ *on* $\mathcal{X}_1 \times \mathcal{X}_2$ *is called a* (p, q)*-product atom of* $H^p(\mathcal{X}_1 \times \mathcal{X}_2)$, *if it satisfies the following conditions:*

(1) $\operatorname{supp} a \subset \Omega$, *where* Ω *is an open set in* $\mathcal{X}_1 \times \mathcal{X}_2$ *with finite measure;*

(2) $\|a\|_{L^q(\mathcal{X}_1 \times \mathcal{X}_2)} \leq \mu(\Omega)^{1/q - 1/p}$, *where* $\mu = \mu_1 \times \mu_2$.

Moreover, a *can be decomposed into rectangle* (p, q)*-atoms* a_R *associated to the dyadic rectangle* $R = Q_1 \times Q_2$ *with* $\mathrm{diamQ_1} \sim 2^{-\mathrm{k_1}}$ *and* $\mathrm{diamQ_2} \sim 2^{-\mathrm{k_2}}$ *for some* $k_1,\ k_2 \in \mathbb{Z}_+$, *which is supported in* $B_1(z_1, C2^{-k_1}) \times B_2(z_2, C2^{-k_2})$, *where* z_i *is the center of* Q_i *for* $i = 1,\ 2$. *To be specify,* *(3a)* *For* $2 \leq q < \infty$, $a = \sum_{R \in \mathcal{M}(\Omega)} a_R$, *and*

$$\left\{ \sum_{R \in \mathcal{M}(\Omega)} \|a_R\|^q_{L^q(\mathcal{X}_1 \times \mathcal{X}_2)} \right\}^{1/q} \leq \mu(\Omega)^{1/q - 1/p}.$$

Here and in the sequel, $\mathcal{M}(\Omega)$ *is the set of all maximal dyadic rectangles contained in* Ω *in both directions of* x_1 *and* x_2*, that is,* $\mathcal{M}(\Omega) = \{R' \subset \Omega : R' = Q'_1 \times Q'_2, \mathrm{diam Q'_i} \sim 2^{-\mathrm{k'_i}}$ *for some* $k'_i \in \mathbb{Z}_+$ *and* Q'_i *is not contained in any other dyadic cube* $Q \in \Omega \cap \mathcal{X}_i$ *for* $i = 1, 2\}$*. And*

$$\widetilde{\Omega} = \{(x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2 : M_s(\chi_\Omega)(x_1, x_2) > C\},$$

Where

$$M_s f(x_1, x_2) = \sup_R \frac{1}{\mu(R))} \int_R |f(x_1, x_2)| d\mu(x_1, x_2)$$

is the strong maximal function, $\mu = \mu_1 \times \mu_2$*, and the above supremum is taken among all dyadic rectangles* R *in* $\mathcal{X}_1 \times \mathcal{X}_2$*,* C *is a small enough positive constant only depend on* $\mathcal{X}_1$ *and* $\mathcal{X}_2$*.*

(3b) *For* $1 < q < 2$, $a = \sum_{R \in \mathcal{M}_1(\Omega)} a_R + \sum_{R \in \mathcal{M}_2(\Omega)} a_R$*, and for any* $\delta > 0$*, there exists a constant* $C_{\delta,\, q} > 0$*, where* $C_{\delta,\, q}$ *only depends on* δ *and* q*, such that*

$$\left\{ \sum_{R \in \mathcal{M}_1(\Omega)} \gamma_2^{-\delta} \|a_R\|^q_{L^q(\mathcal{X}_1 \times \mathcal{X}_2)} + \sum_{R \in \mathcal{M}_2(\Omega)} \gamma_1^{-\delta} \|a_R\|^q_{L^q(\mathcal{X}_1 \times \mathcal{X}_2)} \right\}^{1/q} \leq C_{\delta,\, q}\, \mu(\Omega)^{1/q - 1/p}.$$

Here and in the sequel, $\mathcal{M}_1(\Omega)$ *is the set of all dyadic rectangles contained in* Ω *and maximal in the direction of* x_1 *and* $\mathcal{M}_2(\Omega)$ *is defined similarly.* γ_1 *is defined by* $\gamma_1(R) =$

$\frac{\mu_1(\widehat{Q}_1)}{\mu_1(Q_1)}$, *where* $R = Q_1 \times Q_2 \subset \mathcal{M}_2(\Omega)$ *and* $\widehat{Q}_1 = \widehat{Q}_1(Q_2)$ *be the "longest" dyadic cube containing* Q_1 *such that* $\mu(\widehat{Q}_1 \times Q_2 \cap \Omega) > \frac{1}{2}\mu(\widehat{Q}_1 \times Q_2)$. $\gamma_2(R)$ *is similarly defined.*

(4) For all $x_1 \in \mathcal{X}_1$,

$$\int_{\mathcal{X}_2} a_R(x_1, x_2)\, d\mu_2(x_2) = 0$$

and for all $x_2 \in \mathcal{X}_2$,

$$\int_{\mathcal{X}_1} a_R(x_1, x_2)\, d\mu_1(x_1) = 0.$$

Note that for $0 < p \le 1 < q < \infty$, $\|f\|_{L^p} \le C\|f\|_{H^p}$ for $f \in L^q \cap H^p$ and $L^q(\mathcal{X}_1 \times \mathcal{X}_2) \cap H^p(\mathcal{X}_1 \times \mathcal{X}_2)$ is dense in $H^p(\mathcal{X}_1 \times \mathcal{X}_2)$ (see [HLLW]). Therefore, it is sufficient to consider the atomic decomposition in the subspace $L^q(\mathcal{X}_1 \times \mathcal{X}_2) \cap H^p(\mathcal{X}_1 \times \mathcal{X}_2)$. Then One of our main results, atomic decomposition in terms of (p, q)-atoms for the multi-parameter product Hardy space of homogeneous type is as follows:

Theorem 2.6. *For* $i = 1,\ 2$, *let* $\epsilon_i \in (0, \theta_i]$, $f \in H^p(\mathcal{X}_1 \times \mathcal{X}_2) \cap L^q(\mathcal{X}_1 \times \mathcal{X}_2)$, *and*

$$0 < \max\left\{\frac{d_1}{d_1 + \epsilon_1}, \frac{d_2}{d_2 + \epsilon_2}\right\} < p \le 1 < q < \infty.$$

Then $f \in \left(\mathring{\mathcal{G}}(\beta_1, \beta_2; \gamma_1, \gamma_2)\right)'$ *for some* $\beta_i,\ \gamma_i$ *satisfying* (2.1) *for* $i = 1,\ 2$, *and there is a sequence of numbers,* $\{\lambda_k\}_{k\in\mathbb{Z}}$, *and a sequence of* (p, q)*-atoms of* $H^p(\mathcal{X}_1 \times \mathcal{X}_2)$, $\{a_k\}_{k\in\mathbb{Z}}$, *such that* $\sum_{k=-\infty}^{\infty} |\lambda_k|^p < \infty$ *and*

$$f = \sum_{k=-\infty}^{\infty} \lambda_k a_k,$$

where the series converges to f *in both* $H^p(\mathcal{X}_1 \times \mathcal{X}_2)$ *and* $L^q(\mathcal{X}_1 \times \mathcal{X}_2)$ *norms. Moreover, in* *this case,*

$$\|f\|_{H^p(\mathcal{X}_1\times\mathcal{X}_2)} \sim \inf\left\{\left[\sum_{k=-\infty}^{\infty}|\lambda_k|^p\right]^{1/p}\right\},$$

where the infimum is taken over all the decompositions as above.

2.2 Journé's Covering Lemma for spaces of homogeneous type

To prove Theorem 2.6, we need to establish Journé's covering lemma in the setting of spaces of homogeneous type.

For $i = 1,\ 2$, let $\{Q_{\tau_i}^{k_i} \subset \mathcal{X}_i : \tau_i \in I_{k_i}\}$ be the same as in Lemma 1.2, where $k_i \in \mathbb{Z}$; $\Omega \subset \mathcal{X}_1 \times \mathcal{X}_2$ be an open set with the finite measure and $\mathcal{M}_i(\Omega)$ be the same in Definition 2.5, that is the family of dyadic rectangles $R \subset \Omega$ which are maximal in the direction of x_i. In what follows, we denote by $R = Q_1 \times Q_2$ any dyadic rectangle of $\mathcal{X}_1 \times \mathcal{X}_2$. Given $R = Q_1 \times Q_2 \in \mathcal{M}_1(\Omega)$, let $\widehat{Q}_2 = \widehat{Q}_2(Q_1)$ be the "longest" dyadic cube containing Q_2 such that

$$\mu(Q_1 \times \widehat{Q}_2 \cap \Omega) > \frac{1}{2}\mu(Q_1 \times \widehat{Q}_2); \tag{2.2}$$

and given $R = Q_1 \times Q_2 \in \mathcal{M}_2(\Omega)$, let $\widehat{Q}_1 = \widehat{Q}_1(Q_2)$ be the "longest" dyadic cube containing Q_1 such that

$$\mu(\widehat{Q}_1 \times Q_2 \cap \Omega) > \frac{1}{2}\mu(\widehat{Q}_1 \times Q_2). \tag{2.3}$$

If $Q_i = Q_{\tau_i}^{k_i} \subset \mathcal{X}_i$ for some $k_i \in \mathbb{Z}$ and some $\tau_i \in I_{k_i}$, $(Q_i)_k$ for $k \in \mathbb{N}$ is used to denote any dyadic cube $Q_{\tau_i}^{k_i-k}$ containing $Q_{\tau_i}^{k_i}$ and $(Q_i)_0 = Q_i$, where $i = 1,\ 2$. Also, let $w(x)$ be any increasing function such that $\sum_{j=0}^{\infty} jw(C_6 2^{-j}) < \infty$, where $C_6 > 0$ is any given constant. In particular, we may take $w(x) = x^\theta$ for any $\theta > 0$.

The main idea of the following variant of Journé's covering lemma in the setting of spaces of homogeneous type comes from Pipher [P].

Lemma 2.7. *Assume that $\Omega \subset \mathcal{X}_1 \times \mathcal{X}_2$ is an open set with finite measure. Let all the notation be the same as above. Then*

$$\sum_{R=Q_1\times Q_2\in\mathcal{M}_1(\Omega)} \mu(R) w\left(\frac{\mu_2(Q_2)}{\mu_2(\widehat{Q}_2)}\right) \le C\mu(\Omega) \tag{2.4}$$

and

$$\sum_{R=Q_1\times Q_2\in\mathcal{M}_2(\Omega)} \mu(R) w\left(\frac{\mu_1(Q_1)}{\mu_1(\widehat{Q}_1)}\right) \le C\mu(\Omega). \tag{2.5}$$

Proof. We only verify (2.4) and the proof of (2.5) is similar. Let $R = Q_1 \times Q_2 \in \mathcal{M}_2(\Omega)$ and for $k \in \mathbb{N}$, let

$$A_{Q_1,k} = \bigcup \left\{ Q_2 :\ Q_1 \times Q_2 \in \mathcal{M}_2(\Omega) \quad \text{and} \quad \widehat{Q}_1 = (Q_1)_{k-1} \right\}. \tag{2.6}$$

Then

$$\begin{aligned}
&\sum_{R=Q_1\times Q_2\in\mathcal{M}_2(\Omega)} \mu(R) w\left(\frac{\mu_1(Q_1)}{\mu_1(\widehat{Q}_1)}\right) \qquad (2.7)\\
&= \sum_{R=Q_1\times Q_2\in\mathcal{M}_2(\Omega)} \mu_1(Q_1)\mu_2(Q_2) w\left(\frac{\mu_1(Q_1)}{\mu_1(\widehat{Q}_1)}\right)\\
&= \sum_{\{Q_1:\ Q_1\times Q_2\in\mathcal{M}_2(\Omega)\}} \mu_1(Q_1) \sum_{k=1}^{\infty} \sum_{\{Q_2:\ Q_2\in A_{Q_1,k}\}} \mu_2(Q_2) w\left(\frac{\mu_1(Q_1)}{\mu_1(\widehat{Q}_1)}\right)\\
&\le \sum_{\{Q_1:\ Q_1\times Q_2\in\mathcal{M}_2(\Omega)\}} \mu_1(Q_1) \sum_{k=1}^{\infty} w\left(C_6 2^{-k}\right) \sum_{\{Q_2:\ Q_2\in A_{Q_1,k}\}} \mu_2(Q_2)\\
&= \sum_{\{Q_1:\ Q_1\times Q_2\in\mathcal{M}_2(\Omega)\}} \mu_1(Q_1) \sum_{k=1}^{\infty} w\left(C_6 2^{-k}\right) \mu_2\left(A_{Q_1,k}\right),
\end{aligned}$$

since $\{Q_2:\ Q_2 \in A_{Q_1,k}\}$ are disjoint by their "maximality", where $C_6 > 0$ depends only on the doubling measure μ_1 and the constants C_1 and C_2 in Lemma 1.2 for $\mathcal{X}_1$.

Set

$$E_{Q_1}(\Omega) = \bigcup \{Q_2:\ Q_1 \times Q_2 \subset \Omega\}.$$

If $x_2 \in A_{Q_1,k}$, then there is some dyadic cube $Q_1 \times Q_2 \in \mathcal{M}_2(\Omega)$ and some $k \in \mathbb{N}$ such that $x_2 \in Q_2$ and $\widehat{Q}_1 = (Q_1)_{k-1}$ by (2.6). By (2.3) and the maximality of $\widehat{Q}_1$, we have

$$\mu\left((Q_1)_{k-1} \times Q_2 \cap \Omega\right) > \frac{1}{2}\mu\left((Q_1)_{k-1} \times Q_2\right)$$

and

$$\mu\left((Q_1)_k \times Q_2 \cap \Omega\right) \le \frac{1}{2}\mu\left((Q_1)_k \times Q_2\right),$$

which implies that

$$\mu\left((Q_1)_k \times Q_2 \cap \left((Q_1)_k \times E_{(Q_1)_k}\right)\right) \le \frac{1}{2}\mu\left((Q_1)_k \times Q_2\right)$$

and further

$$\mu\left((Q_1)_k \times \left(Q_2 \cap E_{(Q_1)_k}\right)\right) \le \frac{1}{2}\mu\left((Q_1)_k \times Q_2\right).$$

Therefore,

$$\mu_2\left(Q_2 \cap E_{(Q_1)_k}\right) \le \frac{1}{2}\mu_2(Q_2),$$

which in turn tells us that

$$\mu_2\left(Q_2 \cap \left(E_{(Q_1)_k}\right)^c\right) > \frac{1}{2}\mu_2(Q_2), \tag{2.8}$$

where $\left(E_{(Q_1)_k}\right)^c = \mathcal{X}_2 \setminus E_{(Q_1)_k}$. From (2.8), it follows that

$$M_2\left(\chi_{E_{Q_1}\setminus E_{(Q_1)_k}}\right)(x_2) > \frac{1}{2}$$

and therefore

$$A_{Q_1,k} \subset \left\{x_2 \in \mathcal{X}_2 :\ M_2\left(\chi_{E_{Q_1}\setminus E_{(Q_1)_k}}\right)(x_2) > \frac{1}{2}\right\},$$

which implies that

$$\mu_2\left(A_{Q_1,k}\right) \le \mu_2\left(\left\{x_2 \in \mathcal{X}_2 :\ M_2\left(\chi_{E_{Q_1}\setminus E_{(Q_1)_k}}\right)(x_2) > \frac{1}{2}\right\}\right) \tag{2.9}$$

$$\le C\mu_2\left(E_{Q_1} \setminus E_{(Q_1)_k}\right).$$

Combining (2.7) with (2.9) yields that

$$\begin{aligned}
&\sum_{R=Q_1\times Q_2\in\mathcal{M}_2(\Omega)} \mu(R) w\left(\frac{\mu_1(Q_1)}{\mu_1(\widehat{Q}_1)}\right) \\
&\le C \sum_{\{Q_1:\ Q_1\times Q_2\in\mathcal{M}_2(\Omega)\}} \mu_1(Q_1) \sum_{k=1}^{\infty} w\left(C_6 2^{-k}\right) \mu_2\left(E_{Q_1}\setminus E_{(Q_1)_k}\right) \\
&\le C \sum_{\{Q_1:\ Q_1\times Q_2\in\mathcal{M}_2(\Omega)\}} \mu_1(Q_1) \sum_{k=1}^{\infty} w\left(C_6 2^{-k}\right) \\
&\quad \times \left\{\mu_2\left(E_{Q_1}\setminus E_{(Q_1)_1}\right) + \cdots + \mu_2\left(E_{(Q_1)_{k-1}}\setminus E_{(Q_1)_k}\right)\right\} \\
&\le C \sum_{\{Q_1:\ Q_1\times Q_2\in\mathcal{M}_2(\Omega)\}} \mu_1(Q_1) \sum_{k=1}^{\infty} w\left(C_6 2^{-k}\right) \\
&\quad \times \sum_{\substack{\{Q_0 \text{ dyadic cube: } Q_1\subseteq Q_0\subsetneq (Q_1)_k \\ Q_0\times(E_{Q_0}\setminus E_{(Q_0)_1})\subset\Omega\}}} \mu_2\left(E_{Q_0}\setminus E_{(Q_0)_1}\right)
\end{aligned}$$

$$\begin{aligned}
&\le C\sum_{k=1}^{\infty} w\left(C_6 2^{-k}\right) \sum_{\substack{Q_0 \ dyadic\ cube \\ Q_0\times(E_{Q_0}\setminus E_{(Q_0)_1})\subset\Omega}} \mu_1(Q_0)\mu_2\left(E_{Q_0}\setminus E_{(Q_0)_1}\right) \\
&\quad\times \sum_{\{Q_1\ dyadic\ cube:\ Q_1\subset Q_0\subset (Q_1)_k\}} \frac{\mu_1(Q_1)}{\mu_1(Q_0)} \\
&\le C\sum_{k=1}^{\infty} w\left(C_6 2^{-k}\right) \sum_{\substack{Q_0 \ dyadic\ cube \\ Q_0\times(E_{Q_0}\setminus E_{(Q_0)_1})\subset\Omega}} \mu_1(Q_0)\mu_2\left(E_{Q_0}\setminus E_{(Q_0)_1}\right) \\
&\quad\times \sum_{j=1}^{k} \sum_{\{Q_1\ dyadic\ cube:\ \mu_1(Q_1)\sim 2^{-jd}\mu_1(Q_0)\}} \frac{\mu_1(Q_1)}{\mu_1(Q_0)} \\
&\le C\sum_{k=1}^{\infty} kw\left(C_6 2^{-k}\right) \sum_{\substack{Q_0 \ dyadic\ cube \\ Q_0\times(E_{Q_0}\setminus E_{(Q_0)_1})\subset\Omega}} \mu_1(Q_0)\mu_2\left(E_{Q_0}\setminus E_{(Q_0)_1}\right) \\
&\le C\sum_{k=1}^{\infty} kw\left(C_6 2^{-k}\right)\mu(\Omega),
\end{aligned}$$

since

$$\sum_{\substack{Q_0 \ dyadic\ cube \\ Q_0\times(E_{Q_0}\setminus E_{(Q_0)_1})\subset\Omega}} \mu_1(Q_0)\mu_2\left(E_{Q_0}\setminus E_{(Q_0)_1}\right) \le C\mu(\Omega)$$

by noting that the sets $\left\{Q_0\times\left(E_{Q_0}\setminus E_{(Q_0)_1}\right)\subset\Omega:\ Q_0 \text{ is any dyadic cube}\right\}$ are disjoint, which finishes the proof of Lemma 2.7. □

2.3 Proof of the atomic decompostion of $H^p(\mathcal{X}_1\times\mathcal{X}_2)$

We begin with recalling the following discrete Calderón reproducing formula.

Lemma 2.8. *(See[HLY, HL3]) For $i=1,2$, let $\epsilon_i\in(0,\theta_i]$, $\{S_{k_i}\}_{k_i\in\mathbb{Z}}$ be an approximation to the identity of order ϵ_i, $D_{k_i}=S_{k_i}-S_{k_i-1}$ for $k_i\in\mathbb{Z}$, $\{Q_{\tau_1}^{k_1,\nu_1}:\ k_1\in\mathbb{Z},\ \tau_1\in I_{k_1},\ \nu_1=1,\cdots,N(k_1,\tau_1)\}$ and $\{Q_{\tau_2}^{k_2,\nu_2}:\ k_2\in\mathbb{Z},\ \tau_2\in I_{k_2},\ \nu_2=1,\cdots,N(k_2,\tau_2)\}$ respectively be the dyadic cubes of $\mathcal{X}_1$ and $\mathcal{X}_2$ defined in Lemma 1.2 with $j_1,\ j_2\in\mathbb{N}$ large enough. Then there are families of linear operators $\{\overline{D}_{k_i}\}_{k_i\in\mathbb{Z}}$ on $\mathcal{X}_i$ such that for all $f\in\mathcal{G}(\beta_1,\beta_2;\gamma_1,\gamma_2)$ with $\beta_i,\ \gamma_i\in(0,\epsilon_i)$ for $i=1,\ 2$, and any point $y_{\tau_1}^{k_1,\nu_1}\in Q_{\tau_1}^{k_1,\nu_1}$ and $y_{\tau_2}^{k_2,\nu_2}\in Q_{\tau_2}^{k_2,\nu_2}$,*

$$
\begin{aligned}
f(x_1,x_2) \ &= \sum_{k_1=-\infty}^{\infty}\sum_{\tau_1\in I_{k_1}}\sum_{\nu_1=1}^{N(k_1,\tau_1)}\sum_{k_2=-\infty}^{\infty}\sum_{\tau_2\in I_{k_2}}\sum_{\nu_2=1}^{N(k_2,\tau_2)}\mu_1(Q_{\tau_1}^{k_1,\,\nu_1})\mu_2(Q_{\tau_2}^{k_2,\,\nu_2})\\
&\quad\times D_{k_1}(x_1,y_{\tau_1}^{k_1,\,\nu_1})D_{k_2}(x_2,y_{\tau_2}^{k_2,\,\nu_2})\overline{D}_{k_1}\overline{D}_{k_2}(f)(y_{\tau_1}^{k_1,\,\nu_1},y_{\tau_2}^{k_2,\,\nu_2}),
\end{aligned}
$$

where the series converge in the norm of both the space $\mathcal{G}(\beta_1',\beta_2';\gamma_1',\gamma_2')$ with $\beta_i'\in(0,\beta_i)$ and $\gamma_i'\in(0,\gamma_i)$ for $i=1,\ 2,$ and $L^p(\mathcal{X}_1\times\mathcal{X}_2)$ with $p\in(1,\infty)$.

Now we can first establish the atomic decomposition into (p,q)-atoms for $0<p\le 1<q<\infty$, namely Theorem 2.6.

Let $f\in H^p(\mathcal{X}_1\times\mathcal{X}_2)$, then by Definition 2.4, $f\in\Big(\mathring{\mathcal{G}}(\beta_1,\beta_2;\gamma_1,\gamma_2)\Big)'$ for some $\beta_i,\ \gamma_i$ satisfying (2.1) for $i=1,\ 2$. We will use Lemma 2.8 to get the atomic decomposition of f. For any $k\in\mathbb{Z}$, let

$$
\Omega_k=\big\{(x_1,x_2)\in\mathcal{X}_1\times\mathcal{X}_2:\ g_2(f)(x_1,x_2)>2^k\big\}
$$

and

$$
\widetilde{\Omega}_k=\{(x_1,x_2)\in\mathcal{X}_1\times\mathcal{X}_2: M_s(\chi_{\Omega_k})(x_1,\mathcal{X}_2)>C\}
$$

with a small enough constant C only depending on $\mathcal{X}_i$, $i=1,2$, here and in the sequel, M_s is the strong Hardy-Littlewood maximal function on $\mathcal{X}_1\times\mathcal{X}_2$ defined as in Definition 2.5. Then, the $L^q(\mathcal{X}_1\times\mathcal{X}_2)$-boundedness of M_s (see [DH]) implies that $\mu(\widetilde{\Omega}_k)\le C\mu(\Omega_k)$.

Let $\mathcal{R}$ be the set of all dyadic rectangles of $\mathcal{X}_1\times\mathcal{X}_2$, that is

$$
\mathcal{R}=\{R=Q_1\times Q_2:\ Q_1 \text{ and } Q_2 \text{ are dyadic cubes, respectively, of } \mathcal{X}_1 \text{ and } \mathcal{X}_2\},
$$

and for $k \in \mathbb{Z}$,

$$\mathcal{R}_k = \left\{ R \in \mathcal{R} : \ \mu(R \cap \Omega_k) > \frac{1}{2}\mu(R) \text{ and } \mu(R \cap \Omega_{k+1}) \le \frac{1}{2}\mu(R) \right\}.$$

Obviously, for any $R \in \mathcal{R}$, there is a unique $k \in \mathbb{Z}$ such that $R \in \mathcal{R}_k$. Thus, we can reclassify the set of all dyadic rectangles in $\mathcal{X}_1 \times \mathcal{X}_2$ by

$$\bigcup_{R \in \mathcal{R}} R = \bigcup_{k \in \mathbb{Z}} \bigcup_{R \in \mathcal{R}_k} R. \tag{2.10}$$

In what follows, for $i = 1,\ 2$, if Q_{k_i} is a dyadic cube and $\mathrm{diam Q_{k_i}} \sim 2^{-\mathrm{k_i}}$, we rewrite D_{k_i} and $\overline{D}_{k_i}$, respectively, by $D_{Q_{k_i}}$ and $\overline{D}_{Q_{k_i}}$. And denote by y_{k_i} a point in Q_{k_i}. Then, by lemma 2.8, we have

$$f(x_1, x_2) = \sum_{k_1=-\infty}^{\infty} \sum_{\tau_1 \in I_{k_1}} \sum_{\nu_1=1}^{N(k_1,\tau_1)} \sum_{k_2=-\infty}^{\infty} \sum_{\tau_2 \in I_{k_2}} \sum_{\nu_2=1}^{N(k_2,\tau_2)} \mu_1(Q_{\tau_i}^{k_i,\,\nu_i})\mu_2(Q_{\tau_2}^{k_2,\,\nu_2}) \tag{2.11}$$

$$\begin{aligned}
&\times D_{k_1}(x_1, y_{\tau_1}^{k_1,\,\nu_1}) D_{k_2}(x_2, y_{\tau_2}^{k_2,\,\nu_2}) \overline{D}_{k_1} \overline{D}_{k_2}(f)(y_{\tau_1}^{k_1,\,\nu_1}, y_{\tau_2}^{k_2,\,\nu_2}) \\
&= \sum_{k_1=-\infty}^{\infty} \sum_{\mathrm{diam Q_{k_1}} \sim 2^{-\mathrm{k_1}}} \sum_{k_2=-\infty}^{\infty} \sum_{\mathrm{diam Q_{k_2}} \sim 2^{-\mathrm{k_2}}} \mu_1(Q_{k_1})\mu_2(Q_{k_2}) D_{k_1}(x_1, y_{k_1}) D_{k_2}(x_2, y_{k_2}) \\
&\overline{D}_{k_1} \overline{D}_{k_2}(f)(y_{k_1}, y_{k_2}) \\
&= \sum_{k=-\infty}^{\infty} \sum_{R = Q_{k_1} \times Q_{k_2} \in \mathcal{R}_k} \mu_1(Q_{k_1})\mu_2(Q_{k_2}) D_{Q_{k_1}}(x_1, y_{k_1}) D_{Q_{k_2}}(x_2, y_{k_2}) \overline{D}_{Q_{k_1}} \overline{D}_{Q_{k_2}}(f)(y_{k_1}, y_{k_2}) \\
&= \sum_{k=-\infty}^{\infty} \sum_{R = Q_{k_1} \times Q_{k_2} \in \mathcal{R}_k} \mu(R) D_{Q_{k_1}}(x_1, y_{k_1}) D_{Q_{k_2}}(x_2, y_{k_2}) \overline{D}_{Q_{k_1}} \overline{D}_{Q_{k_2}}(f)(y_{k_1}, y_{k_2}) \\
&= \sum_{k=-\infty}^{\infty} \lambda_k a_k(x_1, x_2),
\end{aligned}$$

where

$$a_k(x_1, x_2) = \frac{1}{\lambda_k} \sum_{R=Q_{k_1}\times Q_{k_2}\in\mathcal{R}_k} \mu(R) D_{Q_{k_1}}(x_1, y_{k_1}) D_{Q_{k_2}}(x_2, y_{k_2}) \overline{D}_{Q_{k_1}} \overline{D}_{Q_{k_2}}(f)(y_{k_1}, y_{k_2}),$$

and when $2 \le q < \infty$ we let

$$\lambda_k = C \left\| \left\{ \sum_{R=Q_{k_1}\times Q_{k_2}\in\mathcal{R}_k} |\overline{D}_{Q_{k_1}} \overline{D}_{Q_{k_2}}(f)(y_{k_1}, y_{k_2})|^2 \chi_R(\cdot\ ,\ \cdot) \right\}^{1/2} \right\|_q \mu(\widetilde{\Omega}_k)^{\frac{1}{p}-\frac{1}{q}},$$

while $1 < q < 2$ we let,

$$\lambda_k = C \left\| \left\{ \sum_{R=Q_{k_1}\times Q_{k_2}\in\mathcal{R}_k} |\overline{D}_{Q_{k_1}} \overline{D}_{Q_{k_2}}(f)(y_{k_1}, y_{k_2})|^2 \chi_R(\cdot\ ,\ \cdot) \right\}^{1/2} \right\|_q \mu(\widetilde{\Omega}_k)^{\frac{1}{p}-\frac{1}{2}}.$$

We now verify that $\{\lambda_k\}_{k\in\mathbb{Z}}$ and $\{a_k\}_{k\in\mathbb{Z}}$ satisfy the requirement of the Theorem 2.6. First, note that in the above expressions we have set

$$\widetilde{\Omega}_k = \{(x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2 : M_s(\chi_{\Omega_k})(x_1, x_2) > C\},$$

where C is only dependent on $\mathcal{X}_1$ and $\mathcal{X}_2$ and is chosen to be small enough. It is easy to check that $\operatorname{supp} a_k \subset \widetilde{\Omega}_k$, since $R \in \mathcal{R}_k$ implies $R \in \widetilde{\Omega}$. Thus a_k is supported in an open set, and hence satisfies (1) of Definition 2.5.

To see that a_k satisfies (2) of Definition 2.5, let $h \in L^{q'}(\mathcal{X}_1 \times \mathcal{X}_2) \cap L^2(\mathcal{X}_1 \times \mathcal{X}_2)$, where

$\frac{1}{q}+\frac{1}{q\,'}=1$. By Hölder inequality and Lemma 2.7, we have

$$\begin{aligned}
&\left\|\sum_{R=Q_{k_1}\times Q_{k_2}\in\mathcal{R}_k}\mu(R)D_{Q_{k_1}}(x_1,y_{k_1})D_{Q_{k_2}}(x_2,y_{k_2})\overline{D}_{Q_{k_1}}\overline{D}_{Q_{k_2}}(f)(y_{k_1},y_{k_2})\right\|_q\\
&=\sup_{\|h\|_{q'}\le 1}\left|\left\langle\sum_{R=Q_{k_1}\times Q_{k_2}\in\mathcal{R}_k}\mu(R)D_{Q_{k_1}}(x_1,y_{k_1})D_{Q_{k_2}}(x_2,y_{k_2})\overline{D}_{Q_{k_1}}\overline{D}_{Q_{k_2}}(f)(y_{k_1},y_{k_2}),h\right\rangle\right|\\
&=\sup_{\|h\|_{q'}\le 1}\left|\sum_{R=Q_{k_1}\times Q_{k_2}\in\mathcal{R}_k}\int_{X_1\times X_2}\int_R D_{Q_{k_1}}(x_1,y_{k_1})D_{Q_{k_2}}(x_2,y_{k_2})\overline{D}_{Q_{k_1}}\overline{D}_{Q_{k_2}}(f)(y_{k_1},y_{k_2})\right.\\
&\quad\left. h(x_1,x_2)\,d\mu_1(y_1)\,d\mu_2(y_2)d\mu_1(x_1)\,d\mu_2(x_2)\right|\\
&\le\sup_{\|h\|_{q'}\le 1}\int_{X_1\times X_2}\sum_{R=Q_{k_1}\times Q_{k_2}\in\mathcal{R}_k}D_{Q_{k_1}}D_{Q_{k_2}}(h)(y_{k_1},y_{k_2})\overline{D}_{Q_{k_1}}\overline{D}_{Q_{k_2}}(f)(y_{k_1},y_{k_2})\\
&\quad\chi_R(y_1,y_2)d\mu_1(y_1)\,d\mu_2(y_2)\\
&\le\sup_{\|h\|_{q'}le1}\left\|\left\{\sum_{R=Q_{k_1}\times Q_{k_2}\in\mathcal{R}_k}\left|\overline{D}_{Q_{k_1}}\overline{D}_{Q_{k_2}}(f)(y_{k_1},y_{k_2})\right|^2\chi_R(\cdot\,,\,\cdot)\right\}^{\frac{1}{2}}\right\|_q\\
&\quad\left\|\left\{\sum_{R=Q_{k_1}\times Q_{k_2}\in\mathcal{R}_k}\left|D_{Q_{k_1}}D_{Q_{k_2}}(h)(y_{k_1},y_{k_2})\right|^2\chi_R(\cdot\,,\,\cdot)\right\}^{\frac{1}{2}}\right\|_{q'}\\
&\le\sup_{\|h\|_{q'}\le 1}\left\|\left\{\sum_{R=Q_{k_1}\times Q_{k_2}\in\mathcal{R}_k}\left|\overline{D}_{Q_{k_1}}\overline{D}_{Q_{k_2}}(f)(y_{k_1},y_{k_2})\right|^2\chi_R(\cdot\,,\,\cdot)\right\}^{\frac{1}{2}}\right\|_q\\
&\quad\left\|\left\{\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\left|D_{Q_{k_1}}D_{Q_{k_2}}(h)(y_{k_1},y_{k_2})\right|^2\chi_R(\cdot\,,\,\cdot)\right\}^{\frac{1}{2}}\right\|_{q'}\\
&\le\sup_{\|h\|_{q'}\le 1}\|g_2(h)\|_{q'}\left\|\left\{\sum_{R=Q_{k_1}\times Q_{k_2}\in\mathcal{R}_k}\left|\overline{D}_{Q_{k_1}}\overline{D}_{Q_{k_2}}(f)(y_{k_1},y_{k_2})\right|^2\chi_R(\cdot\,,\,\cdot)\right\}^{\frac{1}{2}}\right\|_q\\
&\le C\sup_{\|h\|_{q'}\le 1}\|(h)\|_{q'}\left\|\left\{\sum_{R=Q_{k_1}\times Q_{k_2}\in\mathcal{R}_k}\left|\overline{D}_{Q_{k_1}}\overline{D}_{Q_{k_2}}(f)(y_{k_1},y_{k_2})\right|^2\chi_R(\cdot\,,\,\cdot)\right\}^{\frac{1}{2}}\right\|_q\\
&=C\left\|\left\{\sum_{R=Q_{k_1}\times Q_{k_2}\in\mathcal{R}_k}\left|\overline{D}_{Q_{k_1}}\overline{D}_{Q_{k_2}}(f)(y_{k_1},y_{k_2})\right|^2\chi_R(\cdot\,,\,\cdot)\right\}^{\frac{1}{2}}\right\|_q.
\end{aligned}$$

Then the above estimate yields that when $2 \le q < \infty$,

$$
\begin{aligned}
\|a_k\|_q &= \left(C \left\| \left\{ \sum_{R=Q_{k_1}\times Q_{k_2}\in \mathcal{R}_k} |\overline{D}_{Q_{k_1}}\overline{D}_{Q_{k_2}}(f)(y_{k_1},y_{k_2})|^2 \chi_R(\cdot\ ,\ \cdot) \right\}^{1/2} \right\|_q \mu(\widetilde{\Omega}_k)^{\frac{1}{p}-\frac{1}{q}} \right)^{-1} \\
&\quad \times \left\| \sum_{R=Q_{k_1}\times Q_{k_2}\in \mathcal{R}_k} \mu(R) D_{Q_{k_1}}(x_1,y_{k_1}) D_{Q_{k_2}}(x_2,y_{k_2}) \overline{D}_{Q_{k_1}}\overline{D}_{Q_{k_2}}(f)(y_{k_1},y_{k_2}) \right\|_q \\
&\le \mu(\widetilde{\Omega}_k)^{\frac{1}{q}-\frac{1}{p}}
\end{aligned}
$$

Note that a_k is supported in $\widetilde{\Omega}_k$. Thus if $1 < q < 2$, the similar estimate and the definition of λ_k yield

$$
\begin{aligned}
\|a_k\|_q &= \left(C \left\| \left\{ \sum_{R=Q_{k_1}\times Q_{k_2}\in \mathcal{R}_k} |\overline{D}_{Q_{k_1}}\overline{D}_{Q_{k_2}}(f)(y_{k_1},y_{k_2})|^2 \chi_R(\cdot\ ,\ \cdot) \right\}^{1/2} \right\|_2 \mu(\widetilde{\Omega}_k)^{\frac{1}{p}-\frac{1}{2}} \right)^{-1} \\
&\quad \times \left\| \sum_{R=Q_{k_1}\times Q_{k_2}\in \mathcal{R}_k} \mu(R) D_{Q_{k_1}}(x_1,y_{k_1}) D_{Q_{k_2}}(x_2,y_{k_2}) \overline{D}_{Q_{k_1}}\overline{D}_{Q_{k_2}}(f)(y_{k_1},y_{k_2}) \right\|_q \\
&\le \left(C \left\| \left\{ \sum_{R=Q_{k_1}\times Q_{k_2}\in \mathcal{R}_k} |\overline{D}_{Q_{k_1}}\overline{D}_{Q_{k_2}}(f)(y_{k_1},y_{k_2})|^2 \chi_R(\cdot\ ,\ \cdot) \right\}^{1/2} \right\|_2 \mu(\widetilde{\Omega}_k)^{\frac{1}{p}-\frac{1}{2}} \right)^{-1} \\
&\quad \times \mu(\widetilde{\Omega}_k)^{\frac{1}{q}-\frac{1}{2}} \left\| \sum_{R=Q_{k_1}\times Q_{k_2}\in \mathcal{R}_k} \mu(R) D_{Q_{k_1}}(x_1,y_{k_1}) D_{Q_{k_2}}(x_2,y_{k_2}) \overline{D}_{Q_{k_1}}\overline{D}_{Q_{k_2}}(f)(y_{k_1},y_{k_2}) \right\|_q \\
&\le \mu(\widetilde{\Omega}_k)^{\frac{1}{q}-\frac{1}{p}},
\end{aligned}
$$

which implies that a_k satisfies the size condition (2) of (p,q)-atoms.

To verify that a_k satisfies the condition (3) and (4) of Definition 2.5, note that if $R \in \mathcal{R}_k$, then $R \subset \widetilde{\Omega}_k$. From this, it is easy to see that we can further decompose $a_k(x_1, x_2)$ into

$$\begin{aligned}
a_k(x_1,x_2) &= \frac{1}{\lambda_k} \sum_{R=Q_{k_1}\times Q_{k_2}\in\mathcal{R}_k} \mu(R) D_{Q_{k_1}}(x_1,y_{k_1}) D_{Q_{k_2}}(x_2,y_{k_2}) \overline{D}_{Q_{k_1}} \overline{D}_{Q_{k_2}}(f)(y_{k_1},y_{k_2}) \\
&= \frac{1}{\lambda_k} \sum_{\widetilde{R}\in\mathcal{M}(\widetilde{\Omega}_k)} \sum_{\substack{R=Q_{k_1}\times Q_{k_2}\in\mathcal{R}_k \\ R\subset\widetilde{R}}} \mu(R) D_{Q_{k_1}}(x_1,y_{k_1}) D_{Q_{k_2}}(x_2,y_{k_2}) \overline{D}_{Q_{k_1}} \overline{D}_{Q_{k_2}}(f)(y_{k_1},y_{k_2}) \\
&= \sum_{\widetilde{R}\in\mathcal{M}(\widetilde{\Omega}_k)} \frac{1}{\lambda_k} \sum_{\substack{R=Q_{k_1}\times Q_{k_2}\in\mathcal{R}_k \\ R\subset\widetilde{R}}} \mu(R) D_{Q_{k_1}}(x_1,y_{k_1}) D_{Q_{k_2}}(x_2,y_{k_2}) \overline{D}_{Q_{k_1}} \overline{D}_{Q_{k_2}}(f)(y_{k_1},y_{k_2}) \\
&= \sum_{\widetilde{R}\in\mathcal{M}(\widetilde{\Omega}_k)} \alpha_{\widetilde{R}}(x_1,x_2).
\end{aligned}$$

Let $\widetilde{R} = Q_1 \times Q_2$ with $\mathrm{diam Q_1} \sim 2^{-\mathrm{k}_1'}$ and $\mathrm{diam Q_2} \sim 2^{-\mathrm{k}_2'}$ and z_i be the center of Q_i with $i = 1,\ 2$. Then $k_i' \leq k_i$ for $i = 1,\ 2$. From this, it is easy to verify that

$$\mathrm{supp\ a_{\widetilde{R}} \subset B_1(z_1, C2^{-k_1'}) \times B_2(z_2, C2^{-k_2'})}.$$

Obviously, we have that for all $x_2 \in \mathcal{X}_2$,

$$\int_{\mathcal{X}_1} \alpha_{\widetilde{R}}(x_1,x_2)\, d\mu_1(x_1) = 0,$$

and for all $x_1 \in \mathcal{X}_1$,

$$\int_{\mathcal{X}_2} \alpha_{\widetilde{R}}(x_1,x_2)\, d\mu_2(x_2) = 0.$$

Then it remains to show (3). To see that when $2 \leq q < \infty$, $\alpha_{\widetilde{R}}$ satisfies the estimate of

(3a), by the same proof for the estimate of $\|a_k\|_q$, we have

$$\|\alpha_{\widetilde{R}}\|_q \le \frac{C}{\lambda_k} \left\| \left\{ \sum_{\substack{R=Q_{k_1}\times Q_{k_2}\in\mathcal{R}_k \\ R\subset\widetilde{R}}} |\overline{D}_{Q_{k_1}}\overline{D}_{Q_{k_2}}(f)(y_{k_1},y_{k_2})|^2 \chi_R(\cdot\,,\,\cdot) \right\}^{1/2} \right\|_q,$$

hence, the fact that $2\le q<\infty$ and the definition of λ_k yield

$$\textstyle\sum_{\widetilde{R}\in\mathcal{M}(\widetilde{\Omega}_k)} \|\alpha_{\widetilde{R}}\|_q \le \|a_k\|_q^q,$$

which, by the estimate (2) for a_k, implies that a_k satisfies (3a) of Definition 2.5. When $1<q<2$, we have

$$\begin{aligned}
&\textstyle\sum_{R\in\mathcal{M}_1(\widetilde{\Omega}_k)} \gamma_2^{-\delta}\|a_R\|^q_{L^q(\mathcal{X}_1\times\mathcal{X}_2)}\\
&\le \frac{C}{\lambda_k^q} \textstyle\sum_{R\in\mathcal{M}_1(\widetilde{\Omega}_k)} \gamma_2^{-\delta}(R) \left\| \left\{ \displaystyle\sum_{\substack{R=Q_{k_1}\times Q_{k_2}\in\mathcal{R}_k \\ R\subset\widetilde{R}}} |\overline{D}_{Q_{k_1}}\overline{D}_{Q_{k_2}}(f)(y_{k_1},y_{k_2})|^2 \chi_R(\cdot\,,\,\cdot) \right\}^{1/2} \right\|_q^q\\
&\le \frac{C}{\lambda_k^q} \textstyle\sum_{R\in\mathcal{M}_1(\widetilde{\Omega}_k)} \gamma_2^{-\delta}(R)\mu(R)^{1-\frac{2}{q}} \left\{ \displaystyle\int_{\mathcal{X}_1\times\mathcal{X}_2} \sum_{\substack{R=Q_{k_1}\times Q_{k_2}\in\mathcal{R}_k \\ R\subset\widetilde{R}}} |\overline{D}_{Q_{k_1}}\overline{D}_{Q_{k_2}}(f)(y_{k_1},y_{k_2})|^2 \right.\\
&\quad \left. \times\chi_R(x_1,\ x_2)\,d\mu_1(x_1)\,d\mu_2(x_2) \right\}^{q/2}\\
&\le \frac{C}{\lambda_k^q} \{\textstyle\sum_{R\in\mathcal{M}_1(\widetilde{\Omega}_k)} \gamma_2^{-\delta'}(R)\mu(R)\}^{1-\frac{2}{q}} \left\{ \displaystyle\int_{\mathcal{X}_1\times\mathcal{X}_2} \sum_{R=Q_{k_1}\times Q_{k_2}\in\mathcal{R}_k} |\overline{D}_{Q_{k_1}}\overline{D}_{Q_{k_2}}(f)(y_{k_1},y_{k_2})|^2 \right.\\
&\quad \left. \times\chi_R(x_1,\ x_2)\,d\mu_1(x_1)\,d\mu_2(x_2) \right\}^{q/2}\\
&\le C_{q,\delta'}\mu(\widetilde{\Omega}_k)^{1-\frac{2}{q}}\mu(\widetilde{\Omega}_k)^{\frac{q}{2}-\frac{q}{p}} \le C_{\delta'}\mu(\widetilde{\Omega}_k)^{1-\frac{q}{p}}.
\end{aligned}$$

where the last inequality follows from Lemma 2.7. The other summation in (3b) can be proved by the same manner. This shows that a_k satisfies (3b) of Definition 2.5.

Note that by the maximal theorem $\mu(\widetilde{\Omega}_k) \le C\mu(\Omega_k)$. Since if $(x_1,\ x_2) \in R \in \mathcal{R}_k$ then $M_s\left(\chi_{R\cap\widetilde{\Omega}_k\setminus\Omega_{k+1}}\right)(x_1, x_2) > \frac{1}{2}$, we have $\chi_R(x_1, x_2) \le 2M_s\left(\chi_{R\cap\widetilde{\Omega}_k\setminus\Omega_{k+1}}\right)(x_1, x_2)$. Thus, by the Fefferman-Stein vector valued inequality , for all $1 < q < \infty$,

$$\begin{aligned}
&\sum_{k=-\infty}^{\infty} \left\| \left\{ \sum_{R=Q_{k_1}\times Q_{k_2}\in\mathcal{R}_k} |\overline{D}_{Q_{k_1}}\overline{D}_{Q_{k_2}}(f)(y_{k_1}, y_{k_2})|^2 \chi_R(\cdot\ ,\ \cdot) \right\}^{1/2} \right\|_q^q \\
&= \int_{\mathcal{X}_1\times\mathcal{X}_2} \left\{ \sum_{R=Q_{k_1}\times Q_{k_2}\in\mathcal{R}_k} |\overline{D}_{Q_{k_1}}\overline{D}_{Q_{k_2}}(f)(y_{k_1}, y_{k_2})|^2 \chi_R(x_1,\ x_2) \right\}^{q/2} d\mu_1(x_1)\, d\mu_2(x_2) \\
&\le C \int_{\mathcal{X}_1\times\mathcal{X}_2} \left\{ \sum_{R=Q_{k_1}\times Q_{k_2}\in\mathcal{R}_k} |\overline{D}_{Q_{k_1}}\overline{D}_{Q_{k_2}}(f)(y_{k_1}, y_{k_2})| M_s\left(\chi_{R\cap\widetilde{\Omega}_k\setminus\Omega_{k+1}}\right)(x_1, x_2)|^2 \right\}^{q/2} \\
&\quad d\mu_1(x_1)\, d\mu_2(x_2) \\
&\le C \int_{\widetilde{\Omega}_k\setminus\Omega_{k+1}} \left\{ \sum_{R=Q_{k_1}\times Q_{k_2}\in\mathcal{R}_k} |\overline{D}_{Q_{k_1}}\overline{D}_{Q_{k_2}}(f)(y_{k_1}, y_{k_2})|^2 \chi_R(x_1,\ x_2) \right\}^{q/2} d\mu_1(x_1)\, d\mu_2(x_2) \\
&\le C 2^{kq} \mu(\widetilde{\Omega}_k).
\end{aligned}$$

Therefore, when $2 \le q < \infty$, we have

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} |\lambda_k|^p &= \sum_{k=-\infty}^{\infty} \left\| \left\{ \sum_{R=Q_{k_1}\times Q_{k_2}\in\mathcal{R}_k} |\overline{D}_{Q_{k_1}}\overline{D}_{Q_{k_2}}(f)(y_{k_1}, y_{k_2})|^2 \chi_R(\cdot\ ,\ \cdot) \right\}^{1/2} \right\|_q^p \|\widetilde{\Omega}_k\|^{1-\frac{p}{q}} \\
&\le C \sum_{k=-\infty}^{\infty} 2^{kp} \mu(\widetilde{\Omega}_k)^{\frac{p}{q}} \mu(\widetilde{\Omega}_k)^{1-\frac{p}{q}} = C \sum_{k=-\infty}^{\infty} 2^{kp} \mu(\widetilde{\Omega}_k) \\
&\le C \sum_{k=-\infty}^{\infty} 2^{kp} \mu(\Omega_k) \le C \sum_{k=-\infty}^{\infty} 2^{kp} \mu(\Omega_k \setminus \Omega_{k+1}) \\
&\le C \|g_2(f)\|^p_{L^p(\mathcal{X}_1\times\mathcal{X}_2)},
\end{aligned}$$

and when $1 < q < 2$,

$$
\begin{aligned}
\sum_{k=-\infty}^{\infty} |\lambda_k|^p &= \sum_{k=-\infty}^{\infty} \left\| \left\{ \sum_{R=Q_{k_1}\times Q_{k_2}\in\mathcal{R}_k} |\overline{D}_{Q_{k_1}} \overline{D}_{Q_{k_2}}(f)(y_{k_1}, y_{k_2})|^2 \chi_R(\cdot\ ,\ \cdot) \right\}^{1/2} \right\|_q^2 \|\widetilde{\Omega}_k\|^{1-\frac{p}{2}} \\
&\leq C \sum_{k=-\infty}^{\infty} 2^{kp} \mu(\widetilde{\Omega}_k)^{\frac{p}{2}} \mu(\widetilde{\Omega}_k)^{1-\frac{p}{2}} = C \sum_{k=-\infty}^{\infty} 2^{kp} \mu(\widetilde{\Omega}_k) \\
&\leq C \, \|g_2(f)\|^p_{L^p(\mathcal{X}_1\times\mathcal{X}_2)},
\end{aligned}
$$

which is a desired estimate. Finally, note the fact that the atomic decomposition converges in $L^q(\mathcal{X}_1 \times \mathcal{X}_2)$ follows from the same proof of the convergence of Lemma 2.1 in [HL3]. This ends the proof of Theorem 2.6.

2.4 Boundedness criterions of operators

For an operator on multi-parameter Hardy spaces of homogeneous type, by considering its action on (p, q)-atoms, we are able to prove a uniform boundedness criterion as follows.

Theorem 2.9. *Suppose that T is a bounded linear operator on $L^q(\mathcal{X}_1 \times \mathcal{X}_2)$ for some $1 < q < \infty$. Let $\epsilon_i \in (0, \theta_i]$ and*

$$
\max\left\{\frac{d_1}{d_1+\epsilon_1}, \frac{d_2}{d_2+\epsilon_2}\right\} < p \leq 1.
$$

Then

(1) T is bounded from $H^p(\mathcal{X}_1 \times \mathcal{X}_2)$ to $L^p(\mathcal{X}_1 \times \mathcal{X}_2)$ if and only if $\|Ta\|_{L^p(\mathcal{X}_1,\mathcal{X}_2)} \leq C$ for all (p, q)-atoms of $H^p(\mathcal{X}_1 \times \mathcal{X}_2)$;

(2) T is bounded on $H^p(\mathcal{X}_1 \times \mathcal{X}_2)$ if and only if $\|Ta\|_{H^p(\mathcal{X}_1\times\mathcal{X}_2)} \le C$ for all (p,q)-atoms of $H^p(\mathcal{X}_1 \times \mathcal{X}_2)$, where the constant C is independent of a.

To prove Theorem 2.9, we first claim that for any (p,q)-atom ($0 < p \le 1 < q < \infty$) of $H^p(\mathcal{X}_1 \times \mathcal{X}_2)$, a, there is a constant $C > 0$ such that

$$\|g_2(a)\|_{L^p(\mathcal{X}_1\times\mathcal{X}_2)} \le C,$$

where g_2 is the discrete Littlewood-Paley square function on $\mathcal{X}_1 \times \mathcal{X}_2$ defined in Lemma 2.3.

We give the outline of the proof of the claim that if a is an (p,q)-product atom for $1 < q < 2$, then $\|a\|_{H^p(\mathcal{X}_1\times\mathcal{X}_2)} \le C$, where C is a constant independent of a. The proof for this fact when $q > 2$ is easier and we omit it here.

In fact, to show $\|a\|_{H^p(\mathcal{X}_1\times\mathcal{X}_2)} \le C$, it suffices to show $\|g_2(a)\|_{L^p(\mathcal{X}_1\times\mathcal{X}_2)} \le C$.

Recall that a is an atom supported in Ω satisfying conditions (1), (2) in Definition 2.5 and the following (i.e. (3b) in Definition 2.5):

$$a = \sum_{R\in\mathcal{M}_1(\Omega)} a_R + \sum_{R\in\mathcal{M}_2(\Omega)} a_R,$$

and for any $\delta > 0$, there exists a constant $C_{q,\,\delta}$ which only depends on q and δ, and a_R satisfying (4), such that

$$\left\{ \sum_{R\in\mathcal{M}_1(\Omega)} \gamma_2^{-\delta}\|a_R\|^q_{L^q(X_1\times X_2)} + \sum_{R\in\mathcal{M}_2(\Omega)} \gamma_1^{-\delta}\|a_R\|^q_{L^q(\mathcal{X}_1\times\mathcal{X}_2)} \right\}^{1/q} \le C_{q,\delta}\mu(\Omega)^{1/q-1/p}.$$

We will follow the similar outline as given on page 120 of [Fr1]. Let

$$\widetilde{\Omega} = \{(x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2 : M_s(\chi_\Omega)(x_1, x_2) > C\},$$

and $\widetilde{\widetilde{\Omega}} = \widetilde{(\widetilde{\Omega})}$. Then by Holder inequality and the boundedness of g_2 on $L^q(\mathcal{X}_1 \times \mathcal{X}_2)$ in Lemma 2.3, we have

$$\begin{aligned}\int_{\widetilde{\widetilde{\Omega}}} |g_2(a)|^p d\mu_1(x_1) d\mu_2(x_2) &\leq \{\int_{\widetilde{\widetilde{\Omega}}} \|g_2(a)\|^q d\mu_1(x_1)\mu_2(x_2)\}^{\frac{p}{q}} \mu(\widetilde{\widetilde{\Omega}})^{1-\frac{p}{q}} \ . \\ &\leq C\|a\|_q^p \mu(\Omega)^{1-\frac{p}{q}} \leq C,\end{aligned}$$

To estimate $\int_{\widetilde{\widetilde{\Omega}}^c} |g_2(a)|^p d\mu_1(x_1) d\mu_2(x_2)$, let $R = Q_1 \times Q_2 \subset \mathcal{M}_2(\Omega)$, Q be the "longest" dyadic cube containing Q_1 such that $\mu(Q \times Q_2 \cap \Omega) > \frac{1}{2}\mu(Q \times Q_2)$, $\widetilde{Q}$ be the double of Q and $(\widetilde{Q})^c$ be the complement of $\widetilde{Q}$, we have

$$\int_{(\widetilde{Q})^c \times \mathcal{X}_2} |g_2(a_R)|^p d\mu_1(x_1)\mu_2(x_2) \leq C(\gamma_1(R))^{-\delta} \|a_R\|_q^p \mu(R)^{1-\frac{p}{q}}.$$

Summing over R gives

$$\begin{aligned}&\sum_{R \in \mathcal{M}_2(\Omega)} (\gamma_1(R))^{-\delta} \|a_R\|_q^p \mu(R)^{1-\frac{p}{q}} \\ &\leq \{\sum_{R \in \mathcal{M}_2(\Omega)} (\gamma_1(R))^{-\delta} \|a_R\|_q^q\}^{\frac{p}{q}} \{\sum_{R \in \mathcal{M}_2(\Omega)} (\gamma_1(R))^{-\delta''} \mu(R)\}^{1-\frac{p}{q}} \leq C,\end{aligned}$$

where δ', δ'' are constants only dependent on δ, q, p and $\mathcal{X}_1$. Here the last inequality above from the condition (3b) of (p,q)-atom a in Definition 2.5 and Lemma 2.7.

Hence for any (p, q)-atom a, there exists a constant C, such that $\|a\|_{H^p(\mathcal{X}_1 \times \mathcal{X}_2)} \leq C$. Then we only need to prove the "if" part of Theorem 2.9. If $\|Ta\|_{L^p(\mathcal{X}_1\times\mathcal{X}_2)} \le C$ uniformly on all (p,q)-atoms of $H^p(\mathcal{X}_1\times\mathcal{X}_2)$ in $L^p(\mathcal{X}_1\times\mathcal{X}_2)$, then by Theorem 2.6, for $f\in H^p(\mathcal{X}_1\times\mathcal{X}_2)\cap L^q(\mathcal{X}_1,\mathcal{X}_2)$,

$$Tf = \sum_k \lambda_k T a_k.$$

Since T is bounded on $L^q(\mathcal{X}_1\times\mathcal{X}_2)$, and $f=\sum_k \lambda_k a_k$ on $L^q(\mathcal{X}_1\times\mathcal{X}_2)$. Thus

$$\|Tf\|_p^p \le \sum_k |\lambda_k|^p \|Ta_k\|_p^p \le C^p \sum_k |\lambda_k|^p \le C\|f\|_{H^p}^p.$$

(2) If $\|Ta\|_{H^p(\mathcal{X}_1\times\mathcal{X}_2)} \le C$ uniformly on all (p,q)-atoms of $H^p(\mathcal{X}_1\times\mathcal{X}_2)$ in $H^p(\mathcal{X}_1\times\mathcal{X}_2)$, then by Theorem 1.1, for $f\in H^p(\mathcal{X}_1\times\mathcal{X}_2)\cap L^q(\mathcal{X}_1,\mathcal{X}_2)$,

$$\|Tf\|_{H^p(\mathcal{X}_1\times\mathcal{X}_2)}^p \le \sum_k |\lambda_k|^p \|Ta_k\|_{H^p(\mathcal{X}_1\times\mathcal{X}_2)}^p \le C^p \sum_k |\lambda_k|^p \le C\|f\|_{H^p}^p.$$

Since $H^p(\mathcal{X}_1\times\mathcal{X}_2)\cap Ł^q(\mathcal{X}_1\times\mathcal{X}_2)$ is dense in $H^p(\mathcal{X}_1\times\mathcal{X}_2)$, the proof of Theorem 2.9 is complete.

3 Wolff potentials and regularity of solutions to integral systems on spaces of homogenous type

3.1 Introduction and statements of main results

Wolff potentials on $\mathbb{R}^n$ were originally studied by Hedberg and Wolff [HW]: Given $\omega\in\mathbb{M}^+(\mathbb{R}^n)$, the class of all positive locally finite Borel measure on $\mathbb{R}^n$, the (continuous) Wolff potential $W_{\alpha,p}\omega(x)$ for $\alpha > 0$ and $p > 1$ is defined as

$$W_{\alpha,p}\omega(x) = \int_0^\infty \left[\frac{\omega(B_t(x))}{t^{n-\alpha p}}\right]^{p'-1} \frac{dt}{t}$$

for $x \in \mathbb{R}^n$, where $\omega(B_t(x)) = \int_{B_t(x)} d\omega$ and p' is the conjugate index of p. They also introduced the discrete version of Wolff potentials as

$$W^D_{\alpha,p}\omega(x) = \sum_{Q\in\mathbb{D}} \left[\frac{\omega(Q)}{|Q|^{1-\frac{\alpha p}{n}}}\right]^{\frac{1}{p-1}} \chi_Q(x),$$

where $\mathbb{D}$ is the set of all the dyadic cubes $Q \subseteq \mathbb{R}^n$ and $|Q|$ denotes its volume. We define the (continuous) Riesz potential of ω for $0 < \lambda < n$ as

$$I_\lambda\omega(x) = c\int_{\mathbb{R}^n} |x-y|^{\lambda-n} d\omega = \int_0^\infty \frac{\omega(B_t(x))}{t^{n-\lambda}}\frac{dt}{t}.$$

It is evident that $I_\lambda\omega = W_{\frac{\lambda}{2},2}\omega$, and the discrete version of Riesz potentials can be similarly established. Wolff's theorem ([HW], see also §4.5 in [AH]) states

Theorem 3.1 (Wolff's theorem)**.** *Let $\alpha > 0$, $1 < p < \infty$, $0 < \alpha p < n$ and $\omega \in \mathbb{M}^+(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n} W^D_{\alpha,p}\omega(x)d\omega \simeq \int_{\mathbb{R}^n} (I_\alpha\omega(x))^{p'}dx.$$

The brilliant work [HW] of Hedberg and Wolff was originally carried out to fill the gap in the study of Sobolev spaces, however it also has important applications in other areas. Here we mention some interesting examples among them. Note that if $u \geq 0$ is measurable on $\mathbb{R}^n$, then $d\omega = udx \in \mathbb{M}^+(\mathbb{R}^n)$.

Example 1.

$$u(x) = W_{\frac{\lambda}{2},2}(u^{\frac{n+\lambda}{n-\lambda}})(x) = I_\lambda(u^{\frac{n+\lambda}{n-\lambda}})(x),$$

and its corresponding semilinear partial differential equation

$$(-\Delta)^\lambda u = u^{\frac{n+\lambda}{n-\lambda}}.$$

This family of equations are closely related to optimizers of sharp Hardy-Littlewood-Sobolev[i] inequality. See [FL, Lu] for the study of this inequality, and [HLZj] provides some recent results about sharp HLS inequalities on homogeneous spaces of Heisenberg type.

Example 2 (p-Laplacian equations).

$$u(x) = W_{1,p}(u^q)(x),$$

and its corresponding p-Laplacian equation

$$-\Delta_p u = -div(\nabla u|\nabla u|^{p-2}) = u^q.$$

Example 3 (Hessian equations).

$$u(x) = W_{\frac{2k}{k+1},k+1}(u^q)(x).$$

[i]We use HLS to denote Hardy-Littlewood-Sobolev in the following content.

and its corresponding k-Hessian equation

$$F_k[-u] = u^q.$$

Phuc and Verbitsky [PV1] studied Examples 2 and 3, based on systematic use of Wolff potentials. They gave the existence and pointwise estimate of the positive solutions, in terms of the corresponding Wolff potentials. Recently, Ma, Chen and Li [MCL] proved regularity for positive solutions of an integral system associated with Wolff potentials. In this chapter, we shall concentrate on some analogous results on homogeneous spaces, and first record truncated version of Wolff potentials defined above for $0 < r \leq \infty$ as

$$W^r_{\alpha,p}\omega(x) = \int_0^r \left[\frac{\omega(B_t(x))}{t^{n-\alpha p}}\right]^{\frac{1}{p-1}} \frac{dt}{t},$$

thus $W^r_{\alpha,p}\omega$ and $W_{\alpha,p}\omega$ coincide when $r = \infty$.

Proposition 3.2 ([PV1]). *Let $\alpha > 0$, $1 < p < \infty$, $q > p - 1$, $\omega \in \mathbb{M}^+(\mathbb{R}^n)$ and $0 < r \leq \infty$, then the following quantities are equivalent.*

$$\left\|W^r_{\alpha p,\frac{q}{q-p+1}}\omega\right\|_{L^1(d\omega)} = \int_{\mathbb{R}^n}\int_0^r \left[\frac{\omega(B_t(x))}{t^{n-\frac{\alpha p q}{q-p+1}}}\right]^{\frac{q-p+1}{p-1}} \frac{dt}{t} d\omega, \tag{3.1}$$

$$\|W^r_{\alpha,p}\omega\|^q_{L^q(dx)} = \int_{\mathbb{R}^n}\left\{\int_0^r \left[\frac{\omega(B_t(x))}{t^{n-\alpha p}}\right]^{\frac{1}{p-1}} \frac{dt}{t}\right\}^q dx, \tag{3.2}$$

$$\|I^r_{\alpha p}\omega\|^{\frac{q}{p-1}}_{L^{\frac{q}{p-1}}(dx)} = \int_{\mathbb{R}^n}\left[\int_0^r \frac{\omega(B_t(x))}{t^{n-\alpha p}}\frac{dt}{t}\right]^{\frac{q}{p-1}} dx. \tag{3.3}$$

Remark. In Proposition 3.2, (3.1) $\simeq$ (3.3) is the truncated version of Wolff's theorem, while we call (3.2) $\simeq$ (3.3) a HLS type inequality.

In the following sections of this chapter, we will switch our attention to spaces of homogeneous type. We extend (3.2) $\simeq$ (3.3) in Proposition 3.2 to spaces of homogeneous type, followed by an associated HLS inequality for Wolff potentials on spaces of homogenous type.

We define the continuous truncated version of Wolff potentials on spaces of homogeneous type for $\omega \in \mathbb{M}^+(\mathcal{X})$ as

$$W^r_{\alpha,p}\omega(x) = \int_0^r \left[\frac{\omega(B_t(x))}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}}\right]^{\frac{1}{p-1}} \frac{dt}{t}.$$

One can similarly define the continuous version $W_{\alpha,p}\omega = W^\infty_{\alpha,p}\omega$ and the discrete version $W^D_{\alpha,p}\omega$, using the dyadic construction on spaces of homogeneous type by Christ [Cm] and Sawyer and Wheeden [SW] (see in Lemma 1.2).

One of our main results about the Wolff potentials on spaces of homogeneous type is as follows. Similar result on Garnot groups of arbitrary steps has also been independently obtained by N. Phuc and I. Verbitsky in [PV2].

Theorem 3.3. *Let $\alpha > 0$, $1 < p < \infty$, $q > p-1$, $\omega \in \mathbb{M}^+(\mathcal{X})$ and $0 < r \le \infty$, then*

$$\|W^r_{\alpha,p}\omega\|^q_{L^q(d\mu)} = \int_{\mathcal{X}} \left\{\int_0^r \left[\frac{\omega(B_t(x))}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}}\right]^{\frac{1}{p-1}} \frac{dt}{t}\right\}^q d\mu \tag{3.4}$$

$$\simeq \quad \|I^r_{\alpha p}\omega\|^{\frac{q}{p-1}}_{L^{\frac{q}{p-1}}(d\mu)} = \int_{\mathcal{X}} \left[\int_0^r \frac{\omega(B_t(x))}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \frac{dt}{t}\right]^{\frac{q}{p-1}} d\mu. \tag{3.5}$$

We point out that Wolff's theorem on spaces of homogeneous type, i.e., the parallel result of (3.1) $\simeq$ (3.3) in Proposition 3.2 on homogeneous spaces was proved by Cascante and Ortega (Theorems 2.7 and 3.1 in [CO]). By a HLS inequality proved by Sawyer and

Wheeden [SW] (see also Sawyer, Wheeden and Zhao [SWZ]) for Riesz potentials on spaces of homogeneous type(i.e., fractional integrals, and they proved weighted version therein), it is not difficult (We also provide the proof in the next section.) to derive the following HLS type inequality for Wolff potentials.

Theorem 3.4 (HLS type inequality for Wolff potentials)**.** *Let $\alpha > 0$, $1 < p < \infty$, $q > p-1$ and $\alpha p < N$. If $f \in L^s(d\mu)$ for $s > 1$, then*

$$\|W_{\alpha,p}(f)\|_{L^q(d\mu)} \leq C\|f\|_{L^s(d\mu)}^{\frac{1}{p-1}},$$

where $\frac{p-1}{q} = \frac{1}{s} - \frac{\alpha p}{N}$.

We apply this inequality to study a Lane-Emden type integral system, that is,

$$\begin{cases} u = W_{\alpha,p}(v^{q_2}), \\ v = W_{\alpha,p}(u^{q_1}), \end{cases} \tag{3.6}$$

under the (critical) condition

$$\frac{p-1}{q_1+p-1} + \frac{p-1}{q_2+p-1} = \frac{N-\alpha p}{N}, \tag{3.7}$$

and when $u = v$ and $q_1 = q_2 = q$, (3.6) is reduced to

$$u = W_{\alpha,p}(u^q),$$

which is the Lane-Emden type integral equation, and deduces Examples 1, 2 and 3 above on homogeneous spaces, given special pairs of α and p. Our main regularity theorems state

Theorem 3.5 (Integrability estimates)**.** *Let $\alpha > 0$, $1 < p \le 2$, $\alpha p < N$ and $q_1, q_2 > 1$, assume that (u, v) is a pair of positive solutions of (3.6) and (3.7) satisfying $(u, v) \in L^{q_1+p-1}(d\mu) \times L^{q_2+p-1}(d\mu)$, then $(u, v) \in L^{s_1}(d\mu) \times L^{s_2}(d\mu)$ for all s_1 and s_2 such that*

$$\frac{1}{s_1} \in \left(0, \frac{p}{q_1+p-1}\right) \cap \left(-\frac{1}{q_2+p-1} + \frac{1}{q_1+p-1}, \frac{p-1}{q_2+p-1} + \frac{1}{q_1+p-1}\right)$$

and

$$\frac{1}{s_2} \in \left(0, \frac{p}{q_2+p-1}\right) \cap \left(-\frac{1}{q_1+p-1} + \frac{1}{q_2+p-1}, \frac{p-1}{q_1+p-1} + \frac{1}{q_2+p-1}\right).$$

Theorem 3.6 (L^∞ estimates)**.** *Under the same conditions in Theorem 3.5, u and v are both uniformly bounded on $\mathcal{X}$.*

3.2 Comparison of Wolff and Reize potentials

For $\alpha > 0$, $1 < p < \infty$ and $\omega \in \mathbb{M}^+(\mathcal{X})$, we define the discrete Wolff potentials on homogeneous space $\mathcal{X}$ by

$$W^D_{\alpha,p}\omega(x) = \sum_k \sum_{\mathrm{diam(Q)\sim 2^{-k}}} \left[\frac{\omega(Q)}{\mu(Q)^{1-\frac{\alpha p}{N}}}\right]^{\frac{1}{p-1}} \chi_Q(x).$$

and when $\alpha = \lambda/2$ and $p = 2$, the discrete Riesz follows as

$$I^D_\lambda\omega(x) = \sum_k \sum_{\mathrm{diam(Q)\sim 2^{-k}}} \frac{\omega(Q)}{\mu(Q)^{1-\frac{\lambda}{N}}} \chi_Q(x),$$

Next we will prove the discrete version of Theorem 3.3, that is

Theorem 3.7 (Discrete version of Theorem 3.3)**.** *Let $\alpha > 0$, $1 < p < \infty$, $q > p-1$ and $\omega \in \mathbb{M}^+(\mathcal{X})$, then*

$$\left\|W^D_{\alpha,p}\omega\right\|^q_{L^q(d\mu)} = \int_{\mathcal{X}} \left\{ \sum_k \sum_{\mathrm{diam(Q)} \sim 2^{-\mathrm{k}}} \left[\frac{\omega(Q)}{\mu(Q)^{1-\frac{\alpha p}{N}}} \right]^{\frac{1}{p-1}} \chi_Q(x) \right\}^q d\mu \tag{3.8}$$

$$\simeq \quad \left\|I^D_{\alpha p}\omega\right\|^{\frac{q}{p-1}}_{L^{\frac{q}{p-1}}(d\mu)} = \int_{\mathcal{X}} \left[\sum_k \sum_{\mathrm{diam(Q)} \sim 2^{-\mathrm{k}}} \frac{\omega(Q)}{\mu(Q)^{1-\frac{\alpha p}{N}}} \chi_Q(x) \right]^{\frac{q}{p-1}} d\mu. \tag{3.9}$$

In order to prove Theorem 3.7, discrete version of Theorem 3.3, we need to introduce an equivalent recording of discrete Riesz potentials.

Lemma 3.8. *Assume the same conditions in Theorem 3.7, define*

$$\Lambda(\omega, \mu) := \int_{\mathcal{X}} \left[\sup_{k \in \mathbb{Z}_+,\, \mathrm{diam(Q)} \sim 2^{-\mathrm{k}},\, \mathrm{x} \in \mathrm{Q}} \frac{1}{\mu(Q')^{1-\frac{\alpha p}{N}}} \sum_{Q' \subseteq Q} \omega(Q') \right]^{\frac{q}{p-1}} d\mu.$$

Then we have

$$\Lambda(\omega, \mu) \simeq \left\|I^D_{\alpha p}\omega\right\|^{\frac{q}{p-1}}_{L^{\frac{q}{p-1}}(d\mu)}. \tag{3.10}$$

Proof of Lemma 3.8.

- $\Lambda(\omega, \mu) \lesssim \left\|I^D_{\alpha p}\omega\right\|^{\frac{q}{p-1}}_{L^{\frac{q}{p-1}}(d\mu)}$

We need dyadic Hardy-Littlewood maximal function M^d on $\mathcal{X}$, which is defined for all

$\nu \in \mathbb{M}^+(\mathcal{X})$ by

$$M^d(\nu)(x) = \sup_{x\in Q} \frac{\nu(Q)}{\mu(Q)}.$$

Since for $d\nu = |f|d\mu$, the operator M^d is bounded on $L^{\frac{q}{p-1}}(\mu)$ for $q > p-1$, (See, e.g. Theorem 3.1(c) in [ABI].) we have

$$\begin{aligned}
\Lambda(\omega,\mu) &= \int_{\mathcal{X}} \left[\sup_{k\in\mathbb{Z}_+,\ \mathrm{diam(Q)}\sim 2^{-k},\ \mathrm{x}\in\mathrm{Q}} \sum_{Q'\subseteq Q} \frac{\omega(Q')}{\mu(Q)^{1-\frac{\alpha p}{N}}} \right]^{\frac{q}{p-1}} d\mu \\
&\leq \int_{\mathcal{X}} M^d \left[\sum_k \sum_{\mathrm{diam(Q)}\sim 2^{-k}} \frac{\omega(Q)}{\mu(Q)^{1-\frac{\alpha p}{N}}} \right]^{\frac{q}{p-1}} d\mu \\
&\leq C \left\| I^D_{\alpha p}\omega \right\|^{\frac{q}{p-1}}_{L^{\frac{q}{p-1}}(d\mu)},
\end{aligned}$$

which finishes the proof $\Lambda(\omega,\mu) \lesssim \left\| I^D_{\alpha p}\omega \right\|^{\frac{q}{p-1}}_{L^{\frac{q}{p-1}}(d\mu)}$.

- $\Lambda(\omega,\mu) \gtrsim \left\| I^D_{\alpha p}\omega \right\|^{\frac{q}{p-1}}_{L^{\frac{q}{p-1}}(d\mu)}$

First we show that for all $x \in \mathcal{X}$,

$$\begin{aligned}
&\left[\sum_k \sum_{\mathrm{diam(Q)}\sim 2^{-k}} \omega(Q)\chi_Q(x) \right]^{\frac{q}{p-1}} \qquad (3.11)\\
\leq\ & \frac{q}{p-1} \sum_k \sum_{\mathrm{diam(Q)}\sim 2^{-k}} \omega(Q)\chi_Q(x) \left[\sum_{Q'\subseteq Q} \omega(Q')\chi_{Q'}(x) \right]^{\frac{q}{p-1}-1}.
\end{aligned}$$

in three cases.

Case I: If

$$\sum_k \sum_{\mathrm{diam(Q)}\sim 2^{-k}} \omega(Q)\chi_Q(x) \leq \infty.$$

Note that for a fixed $x \in \mathcal{X}$, the dyadic cubes containing x form a nested family of cubes. Hence using the elementary $b^t - a^t \le t(b-a)b^{t-1}$ for $0 \le a \le b$ and $1 \le t < \infty$, we have

$$\left[\sum_{Q' \subseteq Q} \omega(Q')\chi_{Q'}(x)\right]^{\frac{q}{p-1}} - \left[\sum_{Q' \subsetneq Q} \omega(Q')\chi_{Q'}(x)\right]^{\frac{q}{p-1}}$$
$$\le \frac{q}{p-1} \sum_k \sum_{\mathrm{diam(Q)} \sim 2^{-\mathrm{k}}} \omega(Q)\chi_Q(x) \left[\sum_{Q' \subseteq Q} \omega(Q')\chi_{Q'}(x)\right]^{\frac{q}{p-1}-1}.$$

From this (3.11) follows by a telescoping sum argument, taking the sums of both sides over all dyadic cubes Q that contain x.

Case II: If

$$\sum_k \sum_{\mathrm{diam(Q)} \sim 2^{-\mathrm{k}}} \omega(Q)\chi_Q(x) = \infty,$$

but

$$\sum_{Q \subseteq Q_0} \omega(Q)\chi_Q(x) \le \infty$$

for some (and hence every) dyadic cube Q_0 which contains x, then (3.11) follows by the same argument as in Case I taking the sums over all $Q \subseteq Q_0$ and then letting $\mu(Q_0) \to \infty$.

Case III: If

$$\sum_k \sum_{\mathrm{diam(Q)} \sim 2^{-\mathrm{k}}} \omega(Q)\chi_Q(x) = \infty,$$

but

$$\sum_{Q \subseteq Q_0} \omega(Q)\chi_Q(x) = \infty$$

for some Q_0, then both side of (3.10) are obviously infinite. This completes the proof of (3.11).

Next we use induction on $\frac{q}{p-1} > 1$ to prove

$$\begin{aligned} & \left\| I_{\alpha p}^D \omega \right\|_{L^{\frac{q}{p-1}}(d\mu)}^{\frac{q}{p-1}} \\ \leq \;& C \sum_k \sum_{\mathrm{diam(Q)\sim 2^{-k}}} \frac{\omega(Q)}{\mu(Q)^{-\frac{\alpha p}{N}}} \left[\frac{1}{\mu(Q)^{1-\frac{\alpha p}{N}}} \sum_{Q' \subseteq Q} \omega(Q') \right]^{\frac{q}{p-1}-1}, \end{aligned} \tag{3.12}$$

where C only depends on $\mathcal{X}$, p and q.

Step 1: To verify (3.12) is true if $1 < \frac{q}{p-1} \leq 2$. By (3.11),

$$\begin{aligned} & \left\| I_{\alpha p}^D \omega \right\|_{L^{\frac{q}{p-1}}(d\mu)}^{\frac{q}{p-1}} \\ = \;& \int_{\mathcal{X}} \left[\sum_k \sum_{\mathrm{diam(Q)\sim 2^{-k}}} \frac{\omega(Q)}{\mu(Q)^{1-\frac{\alpha p}{N}}} \chi_Q(x) \right]^{\frac{q}{p-1}} d\mu \\ \leq \;& \frac{q}{p-1} \sum_k \sum_{\mathrm{diam(Q)\sim 2^{-k}}} \frac{\omega(Q)}{\mu(Q)^{1-\frac{\alpha p}{N}}} \int_Q \left[\sum_{Q' \subseteq Q} \frac{\omega(Q')}{\mu(Q')^{1-\frac{\alpha p}{N}}} \chi_{Q'}(x) \right]^{\frac{q}{p-1}-1} d\mu, \end{aligned}$$

then by Hölder's inequality with exponents $\frac{p-1}{q-p+1}$ and $\frac{p-1}{2p-2-q}$, we have

$$\begin{aligned} & \int_Q \left[\sum_{Q' \subseteq Q} \frac{\omega(Q')}{\mu(Q')^{1-\frac{\alpha p}{N}}} \chi_{Q'}(x) \right]^{\frac{q}{p-1}-1} d\mu \\ \leq \;& \left[\int_Q \sum_{Q' \subseteq Q} \frac{\omega(Q')}{\mu(Q')^{1-\frac{\alpha p}{N}}} \chi_{Q'}(x) d\mu \right]^{\frac{q-p+1}{p-1}} \mu(Q)^{\frac{2p-2-q}{p-1}} \\ \leq \;& \left[\int_Q \sum_{Q' \subseteq Q} \frac{\omega(Q')}{\mu(Q')^{1-\frac{\alpha p}{N}}} \chi_{Q'}(x) d\mu \right]^{\frac{q}{p-1}-1} \left[\frac{1}{\mu(Q)} \right]^{\frac{q}{p-1}-1} \mu(Q) \end{aligned}$$

$$
\begin{aligned}
&= \mu(Q)\left[\frac{1}{\mu(Q)}\int_Q \sum_{Q'\subseteq Q}\frac{\omega(Q')}{\mu(Q')^{1-\frac{\alpha p}{N}}}\chi_{Q'}(x)d\mu\right]^{\frac{q}{p-1}-1}\\
&\leq \mu(Q)\left[\frac{1}{\mu(Q)}\sum_{Q'\subseteq Q}\frac{\omega(Q')}{\mu(Q')^{-\frac{\alpha p}{N}}}\right]^{\frac{q}{p-1}-1}\\
&\leq \mu(Q)\left[\frac{1}{\mu(Q)^{1-\frac{\alpha p}{N}}}\sum_{Q'\subseteq Q}\omega(Q')\right]^{\frac{q}{p-1}-1}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\left\|I^D_{\alpha p}\omega\right\|^{\frac{q}{p-1}}_{L^{\frac{q}{p-1}}(d\mu)}\\
\leq\ & \frac{q}{p-1}\sum_k\sum_{\mathrm{diam(Q)}\sim 2^{-k}}\frac{\omega(Q)}{\mu(Q)^{1-\frac{\alpha p}{N}}}\int_Q\left[\sum_{Q'\subseteq Q}\frac{\omega(Q')}{\mu(Q')^{1-\frac{\alpha p}{N}}}\chi_{Q'}(x)\right]^{\frac{q}{p-1}-1}d\mu\\
\leq\ & \frac{q}{p-1}\sum_k\sum_{\mathrm{diam(Q)}\sim 2^{-k}}\frac{\omega(Q)}{\mu(Q)^{-\frac{\alpha p}{N}}}\left[\frac{1}{\mu(Q)^{1-\frac{\alpha p}{N}}}\sum_{Q'\subseteq Q}\omega(Q')\right]^{\frac{q}{p-1}-1},
\end{aligned}
$$

which means (3.12) holds for $1<\frac{q}{p-1}\leq 2$.

Step 2: Given an integer $m\geq 2$, we assume that (3.12) holds for any $\frac{q}{p-1}\leq m$, then and we show that it also holds for $\frac{q}{p-1}\leq m+1$.

By (3.11) and the induction hypothesis, we have

$$
\begin{aligned}
&\left\|I^D_{\alpha p}\omega\right\|^{\frac{q}{p-1}}_{L^{\frac{q}{p-1}}(d\mu)}\\
\leq\ & \frac{q}{p-1}\sum_k\sum_{\mathrm{diam(Q)}\sim 2^{-\mathrm{k}}}\frac{\omega(Q)}{\mu(Q)^{1-\frac{\alpha p}{N}}}\int_Q\left[\sum_{Q'\subseteq Q}\frac{\omega(Q')}{\mu(Q')^{1-\frac{\alpha p}{N}}}\chi_{Q'}(x)\right]^{\frac{q}{p-1}-1}d\mu\\
\leq\ & C\frac{q}{p-1}\left(\frac{q}{p-1}-1\right)\sum_k\sum_{\mathrm{diam(Q)}\sim 2^{-k}}\frac{\omega(Q)}{\mu(Q)^{1-\frac{\alpha p}{N}}}\sum_{Q'\subseteq Q}\frac{\omega(Q')}{\mu(Q')^{-\frac{\alpha p}{N}}}\left[\frac{1}{\mu(Q')^{1-\frac{\alpha p}{N}}}\sum_{Q''\subseteq Q'}\omega(Q'')\right]^{\frac{q}{p-1}-2}
\end{aligned}
$$

$$
\begin{aligned}
&= C\frac{q(q-p+1)}{(p-1)^2}\sum_{k'}\sum_{\mathrm{diam(Q')\sim 2^{-k'}}}\frac{\omega(Q')}{\mu(Q')^{-\frac{\alpha p}{N}}}\left[\frac{1}{\mu(Q')^{1-\frac{\alpha p}{N}}}\sum_{Q''\subseteq Q'}\omega(Q'')\right]^{\frac{q}{p-1}-2}\sum_{Q'\subseteq Q}\frac{\omega(Q)}{\mu(Q)^{1-\frac{\alpha p}{N}}} \\
&\le C\frac{q(q-p+1)}{(p-1)^2}\int_{\mathcal{X}}\sum_{k'}\sum_{\mathrm{diam(Q')\sim 2^{-k'}}}\frac{\omega(Q')}{\mu(Q')^{1-\frac{\alpha p}{N}}}\chi_{Q'}(x)\left[\frac{1}{\mu(Q')^{1-\frac{\alpha p}{N}}}\sum_{Q''\subseteq Q'}\omega(Q'')\right]^{\frac{q}{p-1}-2} \\
&\quad \times\left[\sum_{k}\sum_{\mathrm{diam(Q)\sim 2^{-k}}}\frac{\omega(Q)}{\mu(Q)^{1-\frac{\alpha p}{N}}}\chi_{Q}(x)\right]d\mu.
\end{aligned}
$$

Note that $\frac{q}{p-1}-1>m-1\ge 2$, by Hölder's inequality with exponents $\frac{q}{p-1}-1=\frac{q-p+1}{p-1}$ and $\frac{q-p+1}{q-2p+2}$, we have

$$
\begin{aligned}
&\sum_{k'}\sum_{\mathrm{diam(Q')\sim 2^{-k'}}}\frac{\omega(Q')}{\mu(Q')^{1-\frac{\alpha p}{N}}}\chi_{Q'}(x)\left[\frac{1}{\mu(Q')^{1-\frac{\alpha p}{N}}}\sum_{Q''\subseteq Q'}\omega(Q'')\right]^{\frac{q}{p-1}-2} \\
&= \sum_{k'}\sum_{\mathrm{diam(Q')\sim 2^{-k'}}}\left[\frac{\omega(Q')}{\mu(Q')^{1-\frac{\alpha p}{N}}}\chi_{Q'}(x)\right]^{\frac{p-1}{q-p+1}}\left[\frac{\omega(Q')}{\mu(Q')^{1-\frac{\alpha p}{N}}}\chi_{Q'}(x)\right]^{\frac{q-2p+2}{q-p+1}} \\
&\quad \times\left[\frac{1}{\mu(Q')^{1-\frac{\alpha p}{N}}}\sum_{Q''\subseteq Q'}\omega(Q'')\right]^{\frac{q}{p-1}-2} \\
&\le \left[\sum_{k'}\sum_{\mathrm{diam(Q')\sim 2^{-k'}}}\frac{\omega(Q')}{\mu(Q')^{1-\frac{\alpha p}{N}}}\chi_{Q'}(x)\left(\frac{1}{\mu(Q')^{1-\frac{\alpha p}{N}}}\sum_{Q''\subseteq Q'}\omega(Q'')\right)^{\frac{q}{p-1}-1}\right]^{\frac{q-2p+2}{q-p+1}} \\
&\quad \times\left[\sum_{k'}\sum_{\mathrm{diam(Q')\sim 2^{-k'}}}\frac{\omega(Q')}{\mu(Q')^{1-\frac{\alpha p}{N}}}\chi_{Q'}(x)\right]^{\frac{p-1}{q-p+1}}.
\end{aligned}
$$

Therefore,

$$
\left\|I^{D}_{\alpha p}\omega\right\|^{\frac{q}{p-1}}_{L^{\frac{q}{p-1}}(d\mu)}
$$

$$\leq C\frac{q(q-p+1)}{(p-1)^2}\int_{\mathcal{X}}\left[\sum_k\sum_{\mathrm{diam(Q)\sim 2^{-k}}}\frac{\omega(Q)}{\mu(Q)^{1-\frac{\alpha p}{N}}}\chi_Q(x)\right]\left[\sum_{k'}\sum_{\mathrm{diam(Q')\sim 2^{-k'}}}\frac{\omega(Q')}{\mu(Q')^{1-\frac{\alpha p}{N}}}\chi_{Q'}(x)\right]^{\frac{p-1}{q-p+1}}$$

$$\times\left[\sum_{k'}\sum_{\mathrm{diam(Q')\sim 2^{-k'}}}\frac{\omega(Q')}{\mu(Q')^{1-\frac{\alpha p}{N}}}\chi_{Q'}(x)\left(\frac{1}{\mu(Q')^{1-\frac{\alpha p}{N}}}\sum_{Q''\subseteq Q'}\omega(Q'')\right)^{\frac{q}{p-1}-1}\right]^{\frac{q-2p+2}{q-p+1}}d\mu$$

$$\leq C\frac{q(q-p+1)}{(p-1)^2}\int_{\mathcal{X}}\left[\sum_{k'}\sum_{\mathrm{diam(Q')\sim 2^{-k'}}}\frac{\omega(Q')}{\mu(Q')^{1-\frac{\alpha p}{N}}}\chi_{Q'}(x)\right]^{\frac{q}{q-p+1}}$$

$$\times\left[\sum_{k'}\sum_{\mathrm{diam(Q')\sim 2^{-k'}}}\frac{\omega(Q')}{\mu(Q')^{1-\frac{\alpha p}{N}}}\chi_{Q'}(x)\left(\frac{1}{\mu(Q')^{1-\frac{\alpha p}{N}}}\sum_{Q''\subseteq Q'}\omega(Q'')\right)^{\frac{q}{p-1}-1}\right]^{\frac{q-2p+2}{q-p+1}}d\mu.$$

By using Hölder's inequality with exponents $\frac{q-p+1}{p-1}$ and $\frac{q-p+1}{q-2p+2}$ again, we have

$$\begin{aligned}
&\left\|I^D_{\alpha p}\omega\right\|^{\frac{q}{p-1}}_{L^{\frac{q}{p-1}}(d\mu)}\\
\leq\ & C\frac{q(q-p+1)}{(p-1)^2}\left[\int_{\mathcal{X}}\left(\sum_{k'}\sum_{\mathrm{diam(Q')\sim 2^{-k'}}}\frac{\omega(Q')}{\mu(Q')^{1-\frac{\alpha p}{N}}}\chi_{Q'}(x)\right)^{\frac{q}{p-1}}d\mu\right]^{\frac{p-1}{q-p+1}}\\
&\times\left[\int_{\mathcal{X}}\sum_{k'}\sum_{\mathrm{diam(Q')\sim 2^{-k'}}}\frac{\omega(Q')}{\mu(Q')^{1-\frac{\alpha p}{N}}}\chi_{Q'}(x)\left(\frac{1}{\mu(Q')^{1-\frac{\alpha p}{N}}}\sum_{Q''\subseteq Q'}\omega(Q'')\right)^{\frac{q}{p-1}-1}d\mu\right]^{\frac{q-2p+2}{q-p+1}}\\
=\ & C\frac{q(q-p+1)}{(p-1)^2}\left[\left\|I^d_{\alpha p}\omega\right\|^{\frac{q}{p-1}}_{L^{\frac{q}{p-1}}(d\mu)}\right]^{\frac{p-1}{q-p+1}}\\
&\times\left[\int_{\mathcal{X}}\sum_{k'}\sum_{\mathrm{diam(Q')\sim 2^{-k'}}}\frac{\omega(Q')}{\mu(Q')^{1-\frac{\alpha p}{N}}}\chi_{Q'}(x)\left(\frac{1}{\mu(Q')^{1-\frac{\alpha p}{N}}}\sum_{Q''\subseteq Q'}\omega(Q'')\right)^{\frac{q}{p-1}-1}d\mu\right]^{\frac{q-2p+2}{q-p+1}}\\
\leq\ & C\frac{q(q-p+1)}{(p-1)^2}\left[\left\|I^d_{\alpha p}\omega\right\|^{\frac{q}{p-1}}_{L^{\frac{q}{p-1}}(d\mu)}\right]^{\frac{p-1}{q-p+1}}\\
&\times\left[\sum_{k'}\sum_{\mathrm{diam(Q')\sim 2^{-k'}}}\frac{\omega(Q')}{\mu(Q')^{-\frac{\alpha p}{N}}}\left(\frac{1}{\mu(Q')^{1-\frac{\alpha p}{N}}}\sum_{Q''\subseteq Q'}\omega(Q'')\right)^{\frac{q}{p-1}-1}\right]^{\frac{q-2p+2}{q-p+1}}.
\end{aligned}$$

From the above inequality it follows that (3.12) holds for $m < \frac{q}{p-1} \leq m+1$, where C only depends on $\mathcal{X}$, p and q, and then (3.12) is verified for every $1 < \frac{q}{p-1} < \infty$.

With the help of (3.12), we compute

$$
\begin{aligned}
&\left\|I_{\alpha p}^{D}\omega\right\|_{L^{\frac{q}{p-1}}(d\mu)}^{\frac{q}{p-1}} \\
\leq\ & C\int_{\mathcal{X}}\sum_{k}\sum_{\mathrm{diam(Q)}\sim 2^{-\mathrm{k}}}\frac{\omega(Q)}{\mu(Q)^{1-\frac{\alpha p}{N}}}\chi_Q(x)\left[\frac{1}{\mu(Q)^{1-\frac{\alpha p}{N}}}\sum_{Q'\subseteq Q}\omega(Q')\right]^{\frac{q}{p-1}-1} d\mu \\
\leq\ & C\int_{\mathcal{X}}\left[\sum_{k}\sum_{\mathrm{diam(Q)}\sim 2^{-\mathrm{k}}}\frac{\omega(Q)}{\mu(Q)^{1-\frac{\alpha p}{N}}}\chi_Q(x)\right]\left[\sup_{k,\ \mathrm{diam(Q)}\sim 2^{-\mathrm{k}},\ \mathrm{x}\in \mathrm{Q}}\frac{1}{\mu(Q)^{1-\frac{\alpha p}{N}}}\sum_{Q'\subseteq Q}\omega(Q')\right]^{\frac{q}{p-1}-1} d\mu \\
\leq\ & C\left(\left\|I_{\alpha p}^{d}\omega\right\|_{L^{\frac{q}{p-1}}(d\mu)}^{\frac{q}{p-1}}\right)^{\frac{p-1}{q}}\left[\Lambda(\omega,\mu)\right]^{1-\frac{p-1}{q}},
\end{aligned}
$$

where the last estimate we have used Hölder's inequality with exponents $\frac{q}{p-1}$ and $\frac{q}{q-p+1}$. Thus

$$
\left\|I_{\alpha p}^{D}\omega\right\|_{L^{\frac{q}{p-1}}(d\mu)}^{\frac{q}{p-1}} \leq C^{\frac{q}{q-p+1}}\Lambda(\omega,\mu) \lesssim \Lambda(\omega,\mu),
$$

and the proof of Lemma 3.8 is completed. $\square$

Next we will use Lemma 3.8 to prove Theorem 3.7.

Proof of Theorem 3.7.

- $\left\|W_{\alpha,p}^{D}\omega\right\|_{L^q(d\mu)}^{q} \gtrsim \left\|I_{\alpha p}^{D}\omega\right\|_{L^{\frac{q}{p-1}}(d\mu)}^{\frac{q}{p-1}}$

It becomes obvious once one notices that

$$
\begin{aligned}
& \left\|I_{\alpha p}^{D}\omega\right\|_{L^{\frac{q}{p-1}}(d\mu)}^{\frac{q}{p-1}} \simeq \Lambda(\omega,\mu) \\
= \; & \int_{\mathcal{X}} \left[\sup_{k\in\mathbb{Z}_+,\, \mathrm{diam(Q)}\sim 2^{-k},\, \mathrm{x}\in\mathrm{Q}} \frac{1}{\mu(Q')^{1-\frac{\alpha p}{N}}} \sum_{Q'\subseteq Q} \omega(Q') \right]^{\frac{q}{p-1}} d\mu \\
\leq \; & \int_{\mathcal{X}} \left\{ \sum_{k} \sum_{\mathrm{diam(Q)}\sim 2^{-k}} \left[\frac{\omega(Q)}{\mu(Q)^{1-\frac{\alpha p}{N}}} \right]^{\frac{1}{p-1}} \chi_Q(x) \right\}^{q} d\mu \\
= \; & \left\|W_{\alpha,p}^{D}\omega\right\|_{L^q(d\mu)}^{q}.
\end{aligned}
$$

- $\left\|W_{\alpha,p}^{D}\omega\right\|_{L^q(d\mu)}^{q} \lesssim \left\|I_{\alpha p}^{D}\omega\right\|_{L^{\frac{q}{p-1}}(d\mu)}^{\frac{q}{p-1}}$

The proof of this direction follows the same line to that given in [PV1]. We only show $p > 2$, since $\left\|W_{\alpha,p}^{D}\omega\right\|_{L^q(d\mu)}^{q} \lesssim \left\|I_{\alpha p}^{D}\omega\right\|_{L^{\frac{q}{p-1}}(d\mu)}^{\frac{q}{p-1}}$ is trivial when $p \leq 2$ by using Minkowski's inequality. Write $t = \frac{p-1}{p-2}$ and $0 < \varepsilon < \frac{\alpha p}{(p-1)n}$, then $t' = p-1 > 1$ and

$$
-t\left(1-\frac{\alpha p}{N}\right)\frac{1}{p-1} + t - t\varepsilon > 1.
$$

By Hölder's inequality, we have

$$
\begin{aligned}
& \sum_{Q'\subseteq Q} \frac{\omega(Q')^{\frac{1}{p-1}}}{\mu(Q')^{(1-\frac{\alpha p}{N})\frac{1}{p-1}-1}} \\
= \; & \sum_{Q'\subseteq Q} \left[\omega(Q')^{\frac{1}{p-1}}\mu(Q')^{\varepsilon}\right] \mu(Q')^{-(1-\frac{\alpha p}{N})\frac{1}{p-1}+1-\varepsilon} \\
\leq \; & \left[\sum_{Q'\subseteq Q} \omega(Q')^{\frac{1}{p-1}}\mu(Q')^{\varepsilon t'}\right]^{\frac{1}{t'}} \left[\sum_{Q'\subseteq Q} \mu(Q')^{-t(1-\frac{\alpha p}{N})\frac{1}{p-1}+t-t\varepsilon}\right]^{\frac{1}{t}}
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\omega(Q)^{\frac{1}{p-1}}\mu(Q)^{\varepsilon}\mu(Q)^{-(1-\frac{\alpha p}{n})\frac{1}{p-1}+1-\varepsilon} \\
&= C\frac{\omega(Q)^{\frac{1}{p-1}}}{\mu(Q)^{(1-\frac{\alpha p}{N})\frac{1}{p-1}-1}}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\left\|W^D_{\alpha,p}\omega\right\|^q_{L^q(d\mu)} \\
\leq\ & C\sum_k \sum_{\mathrm{diam(Q)}\sim 2^{-\mathrm{k}}} \frac{\omega(Q)^{\frac{1}{p-1}}}{\mu(Q)^{(1-\frac{\alpha p}{N})\frac{1}{p-1}+q-2}}\left[\frac{\omega(Q)^{\frac{1}{p-1}}}{\mu(Q)^{(1-\frac{\alpha p}{N})\frac{1}{p-1}-1}}\right]^{q-1} \\
=\ & C\sum_k \sum_{\mathrm{diam(Q)}\sim 2^{-\mathrm{k}}} \frac{\omega(Q)^{\frac{q}{p-1}}}{\mu(Q)^{(1-\frac{\alpha p}{N})\frac{q}{p-1}-1}} \\
=\ & C\int_{\mathcal{X}}\left[\sum_k \sum_{\mathrm{diam(Q)}\sim 2^{-\mathrm{k}}} \frac{\omega(Q)}{\mu(Q)^{1-\frac{\alpha p}{N}}}\chi_Q(x)\right]^{\frac{q}{p-1}} d\mu \\
\lesssim\ & \left\|I^D_{\alpha p}\omega\right\|^{\frac{q}{p-1}}_{L^{\frac{q}{p-1}}(d\mu)},
\end{aligned}
$$

which completes the proof of the Theorem 3.4.

$\square$

3.3 Proof of HLS inequality

Theorem 3.3 follows evidently from its discrete counterpart, and we give a short proof of the HLS type inequality for Wolff potentials in Theorem 3.4.

Proof of Theorem 3.4. From [SW], one have for Riesz potentials

$$
\|I_\lambda(f)\|_{L^q(d\mu)} \leq C\|f\|_{L^s(d\mu)},
$$

where $1 < s \leq q < \infty$, $0 < \lambda < N$, $\frac{1}{q} = \frac{1}{s} - \frac{\lambda}{N}$ and $f \in L^s(d\mu)$. Thus by taking $\alpha > 0$, $1 < p < \infty$ and $\lambda = \alpha p$, we have

$$\|I_{\alpha p}(f)\|_{L^{\frac{q}{p-1}}(d\mu)} \leq C\|f\|_{L^s(d\mu)},$$

where

$$\frac{p-1}{q} = \frac{1}{s} - \frac{\alpha p}{N}.$$

Then by comparison of Wolff and Riesz potentials in Theorem 3.3, we arrive at

$$\|W_{\alpha,p}(f)\|_{L^q(d\mu)} \leq C\|I_{\alpha p}(f)\|_{L^{\frac{q}{p-1}}(d\mu)}^{\frac{1}{p-1}} \leq C\|f\|_{L^s(d\mu)}^{\frac{1}{p-1}},$$

and Theorem 3.4 is verified. □

3.4 Proof of the integrability and L^∞ estimates

In this section, we prove regularity estimates in Theorems 3.5 and 3.6. The tool is regularity lifting, and let us begin with setting the frame, that is, suppose V is a topological vector space with two extended norms,

$$\|\cdot\|_X, \|\cdot\|_Y : V \to [0, \infty],$$

let $X := \{v \in V : \|v\|_X < \infty\}$ and $Y := \{v \in V : \|v\|_Y < \infty\}$. The operator $T : X \to Y$ is said to be contracting if

$$\|Tf - Th\|_Y \leq \eta\|f - h\|_X,$$

$\forall f, h \in X$ and some $0 < \eta < 1$. And T is said to be shrinking if

$$\|Tf\|_Y \leq \theta \|f\|_X,$$

$\forall f \in X$ and some $0 < \theta < 1$.

Remark. It is obvious that for a linear operator T, these two conditions above are equivalent. Thus the following theorem is also true for linear shrinking operators.

Theorem 3.9 (Regularity lifting by contracting operators ([HaL, MCL]))**.** *Let T be a contracting operator from X to itself and from Y to itself, and assume that X, Y are both complete. If $f \in X$, and there exists $g \in Z := X \cap Y$ such that $f = Tf + g$ in X, then $f \in Z$.*

Now we can prove Theorem 3.5 by using the above lifting Theorem. Without causing any confusion, we simply denote $\|\cdot\|_{L^q(d\mu)}$ by $\|\cdot\|_q$, and $L^q(d\mu)$ by L^q in the following proof.

Proof of Theorem 3.5. For a fixed real number $a > 0$, define

$$v_a(u) = \begin{cases} v(x) & \text{if } |v(x)| > a, \text{ or } |x| > a, \\ 0 & \text{otherwise.} \end{cases}$$

Let $v_b(u) = v(u) - v_a(u)$, and similarly we define u_a and u_b, then v_b and u_b are uniformly bounded by a in $B_a(0)$ obviously. It is evident that $v_a \cdot v_b = 0$ and $v^r = (v_a + v_b)^r = v_a^r + v_b^r$ for all $r > 0$. Define the linear operator T_1,

$$T_1 h(x) = \int_0^\infty \left[\frac{\int_{B_t(x)} v^{q_2} d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right]^{\frac{2-p}{p-1}} \left[\frac{\int_{B_t(x)} v_a^{q_2 - 1} h d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right] \frac{dt}{t}.$$

Since u satisfies (3.6), $u = W_{\alpha,p}(v^{q_2})$, we have

$$\begin{aligned}
u(x) &= W_{\alpha,p}(v^{q_2})(x) \\
&= \int_0^\infty \left[\frac{\int_{B_t(x)} v^{q_2} d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}}\right]^{\frac{2-p}{p-1}} \left[\frac{\int_{B_t(x)} (v_a^{q_2} + v_b^{q_2}) d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}}\right] \frac{dt}{t} \\
&= T_1 v(x) + \int_0^\infty \left[\frac{\int_{B_t(x)} v^{q_2} d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}}\right]^{\frac{2-p}{p-1}} \left[\frac{\int_{B_t(x)} v_b^{q_2} d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}}\right] \frac{dt}{t} \\
&:= T_1 v(x) + F(x),
\end{aligned}$$

and thus $u = T_1 v + F$, in which

$$F(x) = \int_0^\infty \left[\frac{\int_{B_t(x)} v^{q_2} d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}}\right]^{\frac{2-p}{p-1}} \left[\frac{\int_{B_t(x)} v_b^{q_2} d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}}\right] \frac{dt}{t}.$$

Similarly, we define

$$T_2 h(x) = \int_0^\infty \left[\frac{\int_{B_t(x)} u^{q_1} d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}}\right]^{\frac{2-p}{p-1}} \left[\frac{\int_{B_t(x)} u_a^{q_1 - 1} h d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}}\right] \frac{dt}{t}$$

and

$$G(x) = \int_0^\infty \left[\frac{\int_{B_t(x)} u^{q_1} d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}}\right]^{\frac{2-p}{p-1}} \left[\frac{\int_{B_t(x)} u_b^{q_1} d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}}\right] \frac{dt}{t}.$$

Then we have $v = T_2 u + G$. Define the operator $T(f,g) = (T_1 g, T_2 f)$, equip the product space $L^{q_1+p-1} \times L^{q_2+p-1}$ with norm $\|(f,g)\|_{q_1+p-1,q_2+p-1} = \|f\|_{q_1+p-1} + \|g\|_{q_2+p-1}$, and $L^{s_1} \times L^{s_2}$ with norm $\|(f,g)\|_{s_1,s_2} = \|f\|_{s_1} + \|g\|_{s_2}$. It is easy to see they are both complete under these norms respectively.

Thus we immediately observe that (u,v) solves the equation $(f,g) = T(f,g) + (F,G)$. In order to apply regularity lifting by contracting operators (Theorem 3.9), we fix the indices s_1 and s_2 satisfying

$$\frac{1}{s_1} - \frac{1}{s_2} = \frac{1}{q_1 + p - 1} - \frac{1}{q_2 + p - 1}. \tag{3.13}$$

Note that the interval conditions in Theorem 3.5 guarantee the existence of such pairs (s_1, s_2). Then to arrive at the conclusion that $(f, g) \in L^{s_1} \times L^{s_2}$, we need to verify the following conditions, for sufficiently large a. (Here T is linear, by the remark above we only need to verify that it is shrinking.)

1. T is shrinking from $L^{q_1+p-1} \times L^{q_2+p-1}$ to itself.

2. T is shrinking from $L^{s_1} \times L^{s_2}$ to itself.

3. $(F, G) \in L^{q_1+p-1} \times L^{q_2+p-1} \cap L^{s_1} \times L^{s_2}$, i.e., $F \in L^{q_1+p-1} \cap L^{s_1}$ and $G \in L^{q_2+p-1} \cap L^{s_2}$.

(1). T is shrinking from $L^{q_1+p-1} \times L^{q_2+p-1}$ to itself.

First, we show that $\|T_1 h\|_{q_1+p-1} \leq \frac{1}{2}\|h\|_{q_2+p-1}$ for all $h \in L^{q_2+p-1}$. By choosing $\frac{1}{2-p}$ and $\frac{1}{p-1}$ as two conjugate indices in Hölder's inequality, we have

$$\begin{aligned}
|T_1 h(x)| &= \left| \int_0^\infty \left[\frac{\int_{B_t(x)} v^{q_2} d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right]^{\frac{2-p}{p-1}} \left[\frac{\int_{B_t(x)} v_a^{q_2-1} h d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right] \frac{dt}{t} \right| \\
&\leq \left\{ \int_0^\infty \left[\frac{\int_{B_t(x)} v^{q_2} d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right]^{\frac{1}{p-1}} \frac{dt}{t} \right\}^{2-p} \left\{ \int_0^\infty \left[\frac{\int_{B_t(x)} v_a^{q_2-1} |h| d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}} \right]^{\frac{1}{p-1}} \frac{dt}{t} \right\}^{p-1} \\
&= \Big[W_{\alpha,p}(v^{q_2})(x) \Big]^{2-p} \Big[W_{\alpha,p}(u_a^{q_2-1}|h|)(x) \Big]^{p-1} \\
&= u^{2-p}(x) \Big[W_{\alpha,p}(v_a^{q_2-1}|h|)(x) \Big]^{p-1}.
\end{aligned}$$

Thus, applying Hölder's inequality again,

$$
\begin{aligned}
&\|T_1 h\|_{q_1+p-1} \\
\leq &\|u^{2-p}\|_{\frac{q_1+p-1}{2-p}} \left\| \left[W_{\alpha,p}(v_a^{q_2-1}|h|)\right]^{p-1} \right\|_s \\
= &\|u\|_{q_1+p-1}^{2-p} \|W_{\alpha,p}(v_a^{q_2-1}|h|)\|_{s(p-1)}^{p-1} \\
\leq &C\|u\|_{q_1+p-1}^{2-p} \|v_a^{q_2-1}|h|\|_{\frac{q_2+p-1}{q_2}} \\
\leq &C\|u\|_{q_1+p-1}^{2-p} \|v_a^{q_2-1}\|_{\frac{q_2+p-1}{q_2-1}} \|h\|_{q_2+p-1} \\
= &C\|u\|_{q_1+p-1}^{2-p} \|v_a\|_{q_2+p-1}^{q_2-1} \|h\|_{q_2+p-1},
\end{aligned}
$$

in which we used HLS type inequality for Wolff potentials in Theorem 3.4 and have

$$
\frac{1}{q_1+p-1} = \frac{2-p}{q_1+p-1} + \frac{1}{s}
$$

and

$$
\frac{1}{s} = \frac{q_2}{q_2+p-1} - \frac{\alpha p}{N},
$$

which is ensured by the condition (3.7). Thus we choose a sufficiently large that

$$
C\|u\|_{q_1+p-1}^{2-p} \|v_a\|_{q_2+p-1}^{q_2-1} \leq \frac{1}{2},
$$

since $u \in L^{q_1+p-1}$ and $v \in L^{q_2+p-1}$. Then $\|T_1 h\|_{q_1+p-1} \leq \frac{1}{2}\|h\|_{q_2+p-1}$ is verified. Similarly we can prove that $\|T_2 h\|_{q_2+p-1} \leq \frac{1}{2}\|h\|_{q_1+p-1}$ for all $h \in L^{q_1+p-1}$ by choosing a large enough.

Combining them together, we have no difficulty to get

$$\begin{aligned}
& \|T(f,g)\|_{q_1+p-1,q_2+p-1} \\
= \;& \|T_1 g\|_{q_1+p-1} + \|T_2 f\|_{q_2+p-1} \\
\leq \;& \frac{1}{2}(\|g\|_{q_2+p-1} + \|f\|_{q_1+p-1}) \\
= \;& \frac{1}{2}\|(f,g)\|_{q_1+p-1,q_2+p-1},
\end{aligned}$$

and this shows that T is shrinking from $L^{q_1+p-1} \times L^{q_2+p-1}$ to itself.

(2). T is shrinking from $L^{s_1} \times L^{s_2}$ to itself.

We use the same tool as we did in (1), that is, HLS type inequality for Wolff potentials in Theorem 3.4 with assistance of Hölder's inequality, by properly choosing the indices. Here, we prove that $\|T_2 h\|_{s_2} \leq \frac{1}{2}\|h\|_{s_1}$ first,

$$\begin{aligned}
& \|T_2 h\|_{s_2} \\
\leq & \|v^{2-p}\|_{\frac{q_2+p-1}{2-p}} \|\Big[W_{\alpha,p}(u_a^{q_1-1}|h|)\Big]^{p-1}\|_{t_1} \\
= & \|v\|_{q_2+p-1}^{2-p} \|W_{\alpha,p}(u_a^{q_1-1}|h|)\|_{t_1(p-1)}^{p-1} \\
\leq & C\|v\|_{q_2+p-1}^{2-p} \|u_a^{q_1-1}|h|\|_{t_2} \\
\leq & C\|v\|_{q_2+p-1}^{2-p} \|u_a^{q_1-1}\|_{\frac{q_1+p-1}{q_1-1}} \|h\|_{s_1} \\
= & C\|v\|_{q_2+p-1}^{2-p} \|u_a\|_{q_1+p-1}^{q_1-1} \|h\|_{s_1},
\end{aligned}$$

in which we choose a sufficiently large such that

$$C\|v\|_{q_2+p-1}^{2-p}\|u_a\|_{q_1+p-1}^{q_1-1} \leq \frac{1}{2},$$

since $v \in L^{q_2+p-1}$ and $u \in L^{q_1+p-1}$. Thus, $\|T_2 h\|_{s_2} \leq \frac{1}{2}\|h\|_{s_1}$ for all $h \in L^{s_1}$. The indices s_1, s_2, t_1 and t_2 above satisfy

$$\frac{1}{s_2} = \frac{2-p}{q_2+p-1} + \frac{1}{t_1},$$

$$\frac{1}{t_2} = \frac{q_1-1}{q_1+p-1} + \frac{1}{s_1}$$

and by (3.13) and (3.7),

$$\begin{aligned}
\frac{1}{t_1} &= \frac{1}{s_2} - \frac{2-p}{q_2+p-1} \\
&= \frac{1}{s_1} - \frac{1}{q_1+p-1} + \frac{1}{q_2+p-1} - \frac{2-p}{q_2+p-1} \\
&= \frac{1}{s_1} - \frac{1}{q_1+p-1} + \frac{p-1}{q_2+p-1} \\
&= \frac{1}{s_1} - \frac{1}{q_1+p-1} + \frac{N-\alpha p}{N} - \frac{p-1}{q_1+p-1} \\
&= \frac{1}{s_1} + \frac{q_1-1}{q_1+p-1} - \frac{\alpha p}{N} \\
&= \frac{1}{t_2} - \frac{\alpha p}{N},
\end{aligned}$$

which ensures us to use HLS type inequality for Wolff potentials in Theorem 3.4, and we need

$$\frac{1}{t_2} = \frac{q_1-1}{q_1+p-1} + \frac{1}{s_1} < 1,$$

that is

$$\frac{1}{s_1} < \frac{p}{q_1 + p - 1}.$$

Similarly we estimate T_1 for $h \in L^{s_2}$ if

$$\frac{1}{s_2} < \frac{p}{q_2 + p - 1},$$

and easily pass the results to $L^{s_1} \times L^{s_2}$, i.e.,

$$\|T(f,g)\|_{s_1,s_2} \leq \frac{1}{2}\|(f,g)\|_{s_1,s_2},$$

which shows that T is shrinking from $L^{s_1} \times L^{s_2}$ to itself.

(3). $F \in L^{q_1+p-1} \cap L^{s_1}$ and $G \in L^{q_2+p-1} \cap L^{s_2}$.

We only estimate F, one notices that v_b is uniformly bounded by a in $B_a(0)$, thus $v_b \in L^{q_2+p-1} \cap L^{s_2}$. Because T_1 is bounded from L^{q_2+p-1} to L^{q_1+p-1} by (1), then $F = T_1 v_b \in L^{q_1+p-1}$. Because T_1 is bounded from L^{s_2} to L^{s_1} by (2), then $F = T_1 v_b \in L^{s_1}$, and we conclude $F \in L^{q_1+p-1} \cap L^{s_1}$.

Applying regularity lifting we finish the proof of Theorem 3.5.

$\square$

Now we are able to prove L^∞ estimate.

Proof of Theorem 3.6. It is sufficient to show for u, then the estimate of v can be proved similarly. For any $x \in \mathcal{X}$, we divide

$$
\begin{aligned}
u(x) &= W_{\alpha,p}(v^{q_2})(x) \\
&= \int_0^1 \left[\frac{\int_{B_t(x)} v^{q_2} d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}}\right]^{\frac{1}{p-1}} \frac{dt}{t} + \int_1^\infty \left[\frac{\int_{B_t(x)} v^{q_2} d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}}\right]^{\frac{1}{p-1}} \frac{dt}{t} \\
&:= I_1(x) + I_2(x),
\end{aligned}
$$

in which the first integral

$$
\begin{aligned}
I_1(x) &= \int_0^1 \left[\frac{\int_{B_t(x)} v^{q_2} d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}}\right]^{\frac{1}{p-1}} \frac{dt}{t} \\
&\le \int_0^1 \left[\frac{\left(\int_{B_t(x)} 1^{s'} d\mu\right)^{\frac{1}{s'}} \left(\int_{B_t(x)} v^{q_2 s} d\mu\right)^{\frac{1}{s}}}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}}\right]^{\frac{1}{p-1}} \frac{dt}{t} \\
&\le \|v\|_{q_2 s}^{\frac{q_2}{p-1}} \int_0^1 [\mu(B_t(x))]^{\frac{1}{p-1}(\frac{1}{s'}-1+\frac{\alpha p}{N})} \frac{dt}{t} \\
&\lesssim \|v\|_{q_2 s}^{\frac{q_2}{p-1}} \int_0^1 t^{\frac{N}{p-1}(\frac{1}{s'}-\frac{N-\alpha p}{N})-1} dt \\
&\le C_1,
\end{aligned}
$$

as we choose s such that $\|v\|_{q_2 s} < \infty$ and $\frac{1}{s'} - \frac{N-\alpha p}{N} > 0$, that is, $\frac{1}{q_2 s} < \frac{\alpha p}{q_2 N}$. By integrability estimate of v in Theorem 3.5, we only need to check

$$
\frac{\alpha p}{q_2 N} > -\frac{1}{q_1 + p - 1} + \frac{1}{q_2 + p - 1},
$$

this is plain by a simple computation.

We notice that C_1 is independent of x. To estimate the second integral I_2, given $\delta > 0$, for all $y \in \mathcal{X}$ such that $d(x,y) \leq \delta$, thus we have $d(z,y) \leq k_1(d(z,x)+d(x,y)) \leq k_1(t+\delta)$ for all $z \in B_t(x)$. (Recall the definition of quasi-metric on homogeneous spaces.) We compute

$$\begin{aligned}
I_2(x) &= \int_1^\infty \left[\frac{\int_{B_t(x)} v^{q_2}\,d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}}\right]^{\frac{1}{p-1}} \frac{dt}{t} \\
&\leq \int_1^\infty \left[\frac{\int_{B_{k_1(t+\delta)}(y)} v^{q_2}\,d\mu}{\mu(B_t(x))^{1-\frac{\alpha p}{N}}}\right]^{\frac{1}{p-1}} \frac{dt}{t} \\
&\leq \int_1^\infty \left[\frac{\int_{B_{k_1(t+\delta)}(y)} v^{q_2}\,d\mu}{\mu(B_{k_1(t+\delta)}(y))^{1-\frac{\alpha p}{N}}}\right]^{\frac{1}{p-1}} \left[\frac{\mu(B_{k_1(t+\delta)}(y))}{\mu(B_t(x))}\right]^{\frac{1}{p-1}(1-\frac{\alpha p}{N})} \frac{dt}{t} \\
&\leq \int_1^\infty \left[\frac{\int_{B_{k_1(t+\delta)}(y)} v^{q_2}\,d\mu}{\mu(B_{k_1(t+\delta)}(y))^{1-\frac{\alpha p}{N}}}\right]^{\frac{1}{p-1}} \left[\frac{k_1(t+\delta)}{t}\right]^{\frac{N-\alpha p}{p-1}+1} \frac{dt}{k_1(t+\delta)} \\
&\leq k_1^{\frac{N-\alpha p}{p-1}+1}(1+\delta)^{\frac{N-\alpha p}{p-1}+1} \int_{k_1(1+\delta)}^\infty \left[\frac{\int_{B_{k_1(t+\delta)}(y)} v^{q_2}\,d\mu}{\mu(B_{k_1(t+\delta)}(y))^{1-\frac{\alpha p}{N}}}\right]^{\frac{1}{p-1}} \frac{dt}{k_1(t+\delta)} \\
&\leq k_1^{\frac{N-\alpha p}{p-1}}(1+\delta)^{\frac{N-\alpha p}{p-1}+1} \int_{k_1(1+\delta)}^\infty \left[\frac{\int_{B_t(y)} v^{q_2}\,d\mu}{\mu(B_t(y))^{1-\frac{\alpha p}{N}}}\right]^{\frac{1}{p-1}} \frac{dt}{t} \\
&\leq C_2 W_{\alpha,p}(v^{q_2})(y) \\
&= C_2 u(y),
\end{aligned}$$

in which C_2 is independent of x and y. Thus, combining I_1 and I_2, we have

$$u(x) \leq C_1 + C_2 u(y),$$

for any x and y such that $d(x,y) \leq \delta$. s-th powering and integrating both sides,

$$\int_{B_\delta(x)} u^s(x) d\mu \leq \int_{B_\delta(x)} (C_1 + C_2 u(y))^s d\mu \lesssim C\|u\|_s^s$$

by choosing $s > 1$ in the integrability interval such that $\|u\|_s < \infty$. Then we finish L^∞ estimate by noticing that C is independent of x.

$\square$

Part II

4 The atomic decomposition of Hardy spaces associated with different homogeneities

4.1 Introduction and statements of main results

For all functions and operators defined on $\mathbb{R}^m = \mathbb{R}^{m-1} \times \mathbb{R}$ with $x = (x', x_m)$ where $x' \in \mathbb{R}^{m-1}$ and $x_m \in \mathbb{R}$, we denote $|x|_e = (|x'|^2 + |x_m|^2)^{\frac{1}{2}}$ and $|x|_h = (|x'|^2 + |x_m|)^{\frac{1}{2}}$.

Let $K_1 \in L^1_{loc}(\mathbb{R}^m \backslash \{0\})$ and satisfying

$$|\frac{\partial^\alpha}{\partial x^\alpha} K_1(x)| \leq A|x|_e^{-m-|\alpha|} \quad for\ all\ |\alpha| \geq 0$$

and

$$\int_{r<|x|_e<R} K_1(x)dx = 0$$

for all $0 < r < R < \infty$. We say that the operator T_1 defined by

$$T_1(f)(x) = p.v.(K_1 * f)(x)$$

is a Calderón-Zygmund singular integral operator associated with isotropic homogeneity.

Let $K_2 \in L^1_{loc}(\mathbb{R}^m \backslash \{0\})$ and satisfying

$$|\frac{\partial^\alpha}{\partial (x')^\alpha} \frac{\partial^\beta}{\partial (x^m)^\beta} K_2(x)| \leq B|x|_h^{-m-1-|\alpha|-2\beta} \quad for\ all\ |\alpha| \geq 0,\ \beta \geq 0$$

and

$$\int_{r<|x|_h<R} K_2(x)dx = 0$$

for all $0 < r < R < \infty$. We say that the operator T_2 defined by $T_2(f)(x) = p.v.(K_2 * f)(x)$ is a Calderón-Zygmund singular integral operator associated with non-isotropic homogeneity.

It is well known that both T_1 and T_2 are bounded on $L^p(\mathbb{R}^m)$ for $1 < p < \infty$ and of weak type $(1,1)$. In addition, T_1 is bounded on the classical isotropic Hardy space, i.e., the classical Hardy space $H^p(\mathbb{R}^m)$ introduced in [FcS], and T_2 is bounded on the non-isotropic Hardy spaces $H^p_{non}(\mathbb{R}^m)$. Consider the composition of these two Calderón-Zygmund operators which arise from the $\bar{\partial}$- Neumann problem, D. H. Phong, E. M. Stein [PS] show that $T_1 \circ T_2$ is of weak-type $(1,1)$, which answered the question asked by Rivieré in [WW]. However, $T_1 \circ T_2$ is bounded neither on the classical Hardy space $H^p(\mathbb{R}^m)$ nor the non-isotropic Hardy space $H^p_{non}(\mathbb{R}^m)$. Therefore, Y. Han, C. Lin, G. Lu, Z. Ruan and E. Sawyer in [HLLRS] develop a new Hardy space theory and prove that the composition $T_1 \circ T_2$ is bounded on these new Hardy spaces. In this chapter we will establish the atomic decomposition of these new Hardy spaces associated with different homogeneities which are defined as follows.

Let $\psi^{(1)} \in \mathcal{S}(\mathbb{R}^m)$ with $supp\ \widehat{\psi^{(1)}} \subseteq \{(\xi', \xi_m) \in \mathbb{R}^{m-1} \times \mathbb{R} : \frac{1}{2} \leq |\xi|_e \leq 2\}$ and

$$|\sum_{j\in\mathbb{Z}} |\widehat{\psi^{(1)}}(2^{-j}\xi', 2^{-j}\xi_m)|^2 = 1 \quad for\ all\ (\xi', \xi_m) \in \mathbb{R}^{m-1} \times \mathbb{R}\backslash\{(0,0)\},$$

$\psi^{(2)} \in \mathcal{S}(\mathbb{R}^m)$ with $supp\ \widehat{\psi^{(2)}} \subseteq \{(\xi', \xi_m) \in \mathbb{R}^{m-1} \times \mathbb{R} : \frac{1}{2} \leq |\xi|_h \leq 2\}$ and

$$|\sum_{k\in\mathbb{Z}} |\widehat{\psi^{(2)}}(2^{-k}\xi', 2^{-2k}\xi_m)|^2 = 1 \quad for\ all\ (\xi', \xi_m) \in \mathbb{R}^{m-1} \times \mathbb{R}\backslash\{(0,0)\}|.$$

For $j,k\in\mathbb{Z}$, let $\psi_j^{(1)}(x)=\psi_j^{(1)}(x',x_m)=2^{jm}\psi^{(1)}(2^jx',2^jx_m)$, $\psi_k^{(2)}(x)=\psi_k^{(2)}(x',x_m)=2^{j(m+1)}\psi^{(2)}(2^kx',2^{2k}x_m)$, and $\psi_{j,k}(x)=\psi_j^{(1)}*\psi_k^{(2)}(x)$. Then a discrete Littlewood-Paley-Stein square function $g^d_{\psi,com}$ is defined by

$$g^d_{\psi,\ com}(f)(x',x_m)=\{\sum_{j,k\in\mathbb{Z}}\sum_{(l',l_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}}|(\psi_{j,k}*f(2^{-(j\wedge k)}l',2^{-(j\wedge 2k)}l_m)|^2\chi_I(x')\chi_J(x_m)\}^{\frac{1}{2}},$$

where I are dyadic cubes in $\mathbb{R}^{m-1}$ and J are dyadic intervals in $\mathbb{R}$ with the side length $\ell(I)=2^{-(j\wedge k)}$ and $\ell(J)=2^{-(j\wedge 2k)}$, and the left lower corners of I and the left end points of J are $2^{-(j\wedge k)}l'$ and $2^{-(j\wedge 2k)}l_m$, respectively.

Let $\mathcal{S}_0(\mathbb{R}^m)=\{f\in\mathcal{S}(\mathbb{R}^m):\ \int_{\mathbb{R}^m}f(x)x^\alpha dx=0\ for\ any\ |\alpha|\geq 0\}$. Now we can define the Hardy spaces associated with two different homogeneities by the following

Definition 4.1. *Let $0<p\leq 1$. $H^p_{com}(\mathbb{R}^m)=\{f\in\mathcal{S}_0'(\mathbb{R}^m):\ \ g^d_{\psi,\ com}(f)\in\ \ L^p(\mathbb{R}^m)\}$. If $f\in H^p_{com}(\mathbb{R}^m)$, the norm of f is defined by $\|f\|_{H^p_{com}(\mathbb{R}^m)}=\|g^d_{\psi,\ com}(f)\|_{L^p(\mathbb{R}^m)}$.*

$H^p_{com}(\mathbb{R}^m)$ is independent of the choice of the function $\psi^{(1)}$ and $\psi^{(2)}$ and thus it is well-defined. Moreover, for all $0<p<\infty$, we have $\|g^d_{\psi,com}(f)\|_{L^p}\sim\|f\|_{L^p}$. In fact, it can also be shown that $\|g^d_{\psi,com}(f)\|_{L^p}\sim\|g_{com}(f)\|_{L^p}$ holds for all $0<p<\infty$ by a similar argument in [FJ], where $g_{com}(f)(x)=\{\sum_{j,k}|\psi_{j,k}*f(x)|^2\}^{\frac{1}{2}}$.

Now we can introduce the $(p,2)$-atom of $H^p_{com}(\mathbb{R}^m)$ for $0<p\leq 1$.

Definition 4.2. *A function $a(x',x_m)$ on $\mathbb{R}^{m-1}\times\mathbb{R}$ is called a $(p,2)-$ atom of $H^p_{com}(\mathbb{R}^m)$ for $0<p\leq 1$, if it satisfies*

(1) supp $a\subset\Omega$, where Ω is an open set of $\mathbb{R}^m$ with finite measure;

(2) $\|a\|_{L^2(\mathbb{R}^m)} \leq |\Omega|^{\frac{1}{2}-\frac{1}{p}}$. *Moreover, a can be further decomposed into rectangle atom a_R associated with the rectangle $R = I \times J \subset \mathbb{R}^{m-1} \times \mathbb{R}$. To be precise,*

(3) $a = \sum_{R=I\times J\in\mathcal{M}(\Omega)} a_R$ *and*

$$\left\{ \sum_{R\in\mathcal{M}(\Omega)} \|a_R\|^2_{L^2(\mathbb{R}^m)} \right\}^{\frac{1}{2}} \leq |\Omega|^{\frac{1}{2}-\frac{1}{p}}.$$

(4) For all $x' \in \mathbb{R}^{m-1}$,

$$\int_{\mathbb{R}} a_R(x', x_m)dx_m = 0$$

and for all $x_m \in \mathbb{R}$,

$$\int_{\mathbb{R}^{m-1}} a_R(x', x_m)dx' = 0.$$

Theorem 4.3. *For* $0 < p \leq 1$ *and* $f \in L^2(\mathbb{R}^m) \cap H^p_{com}(\mathbb{R}^m)$, *there is a sequence of numbers,* $\{\lambda_k\}_{k\in\mathbb{Z}}$, *and a sequence of* $(p, 2)$*-atoms of* $H^p_{com}(\mathbb{R}^m)$, $\{a_k\}_{k\in\mathbb{Z}}$, *such that*

$$(\sum_{k=-\infty}^{\infty} |\lambda_k|^p)^{\frac{1}{p}} \leq C\|f\|_{H^p_{com}(\mathbb{R}^m)}$$

with the constant C independent of f and

$$f = \sum_{k=-\infty}^{\infty} \lambda_k a_k,$$

where the series converges to f in both the $L^2(\mathbb{R}^m)$ and $H^p_{com}(\mathbb{R}^m)$ norms.

4.2 Proof of the atomic decomposition of $H^p_{com}(\mathbb{R}^m)$

For $i=1,2$, let $\phi^{(i)} \in \mathcal{S}(\mathbb{R}^m)$ with *supp* $\phi^{(i)} \subseteq B(0,1)$,

$$\sum_{j\in\mathbb{Z}} |\widehat{\phi^{(1)}}(2^{-j}\xi)|^2 = 1 \quad \text{for all } \xi \in \mathbb{R}^m \backslash \{0\},$$

and

$$\sum_{k\in\mathbb{Z}} |\widehat{\phi^{(2)}}(2^{-k}\xi', 2^{-2k}\xi_m)|^2 = 1 \quad \text{for all } (\xi', \xi_m) \in \mathbb{R}^{m-1} \times \mathbb{R}\backslash\{(0,0)\}.$$

Moreover,

$$\int_{\mathbb{R}^m} \phi^{(1)}(x) x^\alpha dx = 0 \quad \text{for all } |\alpha| \leq 10M$$

and

$$\int_{\mathbb{R}^m} \phi^{(2)}(x) x^\beta dx = 0 \quad \text{for all } |\beta| \leq 10M,$$

where M is a fixed large positive integer depending on p. Set $\phi_{j,k} = \phi_j^{(1)} * \phi_k^{(2)}$, where $\phi_j^{(1)}(x) = 2^{jm}\phi^{(1)}(2^j, x)$ and $\phi_k^{(2)}(x', x_m) = 2^{k(m+1)}\phi^{(2)}(2^k x', 2^{2k} x_m)$. To show Theorem 4.3, we need the following two lemmas.

Lemma 4.4 ([HLLRS]). *For any $f \in L^2(\mathbb{R}^m) \cap H^p_{com}(\mathbb{R}^m)$, there exists $\tilde{f} \in L^2(\mathbb{R}^m) \cap H^p_{com}(\mathbb{R}^m)$ such that for a sufficiently large $N \in \mathbb{N}$,*

$$\begin{aligned} f(x', x_m) &= \sum_{j,k\in\mathbb{Z}} \sum_{l=(l',l_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}} |I||J| \phi_{j,k}(x' - 2^{-j\wedge k-N} l', x_m - 2^{-j\wedge 2k-N} l_m) \\ &\quad \times (\phi_{j,k} * \tilde{f})(2^{-j\wedge k-N} l', 2^{-j\wedge 2k-N} l_m), \end{aligned}$$

where the series converges in L^2, I are dyadic cubes in $\mathbb{R}^{m-1}$ and J are dyadic intervals in

$\mathbb{R}$ *with the side length* $\ell(I) = 2^{-(j\wedge k)-N}$ *and* $\ell(J) = 2^{-(j\wedge 2k)-N}$, *and the left lower corners of* I *and the left end points of* J *are* $2^{-(j\wedge k)-N}l'$ *and* $2^{-(j\wedge 2k)-N}l_m$, *respectively. Moreover,*

$$\|f\|_{L^2(\mathbb{R}^m)} \approx \|\tilde{f}\|_{L^2(\mathbb{R}^m)},$$

and

$$\|f\|_{H^p_{com}(\mathbb{R}^m)} \approx \|\tilde{f}\|_{H^p_{com}(\mathbb{R}^m)}.$$

Lemma 4.5 ([HLLRS]). *Let* $0 < p \leq 1$ *and all the notation be the same as in Lemma 4.4. Then for* $f \in L^2(\mathbb{R}^m) \cap H^p_{com}(\mathbb{R}^m)$,

$$\|f\|_{H^p_{com}} \approx \|(\sum_{j,k\in\mathbb{Z}} \sum_{(l',l_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}} |(\phi_{j,k} * f)(2^{-j\wedge k-N}l', 2^{-j\wedge 2k-N}l_m)|^2 \chi_I \chi_J)^{\frac{1}{2}}\|_{L^p}.$$

Now we can prove Theorem 4.3.

Proof. For any $f \in L^2(\mathbb{R}^m) \cap H^p_{com}(\mathbb{R}^m)$, let $\phi_{j,\ k}$, $\tilde{f}$, I and J are the same as in Lemma 4.4. For any $i \in \mathbb{Z}$, set

$$\Omega_i = \{(x', x_m) \in \mathbb{R}^{m-1} \times \mathbb{R} : \ \widetilde{g}^d_\phi(f)(x', x_m) > 2^i\},$$

and

$$\mathcal{B}_i = \{(j,k,l) : |(I \times J) \cap \Omega_i| > \frac{1}{2}|I||J|,\ |(I \times J) \cap \Omega_{i+1}| \leq \frac{1}{2}|I||J|\},$$

where

$$\widetilde{g}_{\phi}^{d}(f)(x',x_m)=\{\sum_{j,k\in\mathbb{Z}}\sum_{(l',l_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}}|(\phi_{j,k}*\tilde{f}(2^{-(j\wedge k)-N}l',2^{-(j\wedge 2k)-N}l_m)|^2\chi_I(x')\chi_J(x_m)\}^{\frac{1}{2}},$$

and for fixed $N>0$, $I\subset\mathbb{R}^{m-1}$, $J\subset\mathbb{R}$ are dyadic cubes (intervals) determined by $j,k\in\mathbb{Z}$ and $l=(l',l_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}$ as in Lemma 4.4, that is, $\ell(I)=2^{-(j\wedge k)-N}$ and $\ell(J)=2^{-(j\wedge 2k)-N}$, $2^{-(j\wedge k)-N}l'$ and $2^{-(j\wedge 2k)-N}l_m$ are the left lower corners of I and J, respectively.

By Lemma 4.4, we can write

$$\begin{aligned}f(x',x_m)&=\sum_i\sum_{(j,k,l)\in\mathcal{B}_i}|I||J|\phi_{j,k}(x'-2^{-j\wedge k-N}l',x_m-2^{-j\wedge 2k-N}l_m)\\&\qquad\times(\phi_{j,k}*\tilde{f})(2^{-j\wedge k-N}l',2^{-j\wedge 2k-N}l_m)\\&=\sum_i\lambda_i a_i(x',x_m),\end{aligned}$$

where

$$\begin{aligned}a_i(x',x_m)&=\frac{1}{\lambda_i}\sum_{(j,k,l)\in\mathcal{B}_i}|I||J|\phi_{j,k}(x'-2^{-j\wedge k-N}l',x_m-2^{-j\wedge 2k-N}l_m)\\&\qquad\times(\phi_{j,k}*\tilde{f})(2^{-j\wedge k-N}l',2^{-j\wedge 2k-N}l_m)\end{aligned}$$

and

$$\lambda_i=\left\|\left\{\sum_{(j,k,l)\in\mathcal{B}_i}|(\phi_{j,k}*\tilde{f})(2^{-j\wedge k-N}l',2^{-j\wedge 2k-N}l_m)|^2\chi_{I\times J}\right\}^{1/2}\right\|_2|\widetilde{\Omega}_i|^{\frac{1}{p}-\frac{1}{2}};$$

where

$$\widetilde{\Omega}_i=\{x=(x',x_m)\in\mathbb{R}^{m-1}\times\mathbb{R}:M_s(\chi_{\Omega_i})(x)>\frac{1}{2^{N+1}}\}.$$

Note that *supp* $\phi^{(1)}\subseteq B(0,1)$ and *supp* $\phi^{(2)}\subseteq B(0,1)$, $\phi_{j,k}(x)=\phi_j^{(1)}*\phi_k^{(2)}(x)$, $\phi_j^{(1)}(x)=\phi_j^{(1)}(x',x_m)=2^{jm}\phi^{(1)}(2^jx',2^jx_m)$, and $\phi_k^{(2)}(x)=\phi_k^{(2)}(x',x_m)=2^{k(m+1)}\phi^{(2)}(2^kx',2^{2k}x_m)$, then for any $j, k \in \mathbb{Z}$, $\phi_{j,k}$ is supported in $B(0, 2^{-(j\wedge k)}) \times B(0, 2^{-(j\wedge 2k)}) \subset \mathbb{R}^{m-1} \times \mathbb{R}$.

Since for $(j, k, l) \in B_i$, we have $|(I \times J) \cap \Omega_i| > \frac{1}{2}|I||J| \Rightarrow 2^N(I \times J) \subset \widetilde{\Omega}_i$, this implies a_i is supported in

$$\bigcup_{(j,k,l)\in B_i} (2^{-(j\wedge k)-N}l' + B(0, 2^{-(j\wedge k)}), 2^{-(j\wedge 2k)-N}l_m + B(0, 2^{-(j\wedge 2k)})) \subseteq \bigcup_{(j,k,l)\in B_i} 2^N(I \times J) \subseteq \widetilde{\Omega}_i,$$

and hence a_i satisfies (1) in Definition 4.2.

To see that a_i satisfies (2) in Definition 4.2, let $h \in L^2(\mathbb{R}^m)$, by Hölder's inequality, Lemma 4.4, and Lemma 4.5, we have

$$\begin{aligned}
&\left\| \sum_{(j,k,l)\in\mathcal{B}_i} |I||J|\phi_{j,k}(x' - 2^{-j\wedge k-N}l', x_m - 2^{-j\wedge 2k-N}l_m)(\phi_{j,k} * \tilde{f})(2^{-j\wedge k-N}l', 2^{-j\wedge 2k-N}l_m) \right\|_2 \\
&= \sup_{\|h\|_2\le 1} \left| < \sum_{(j,k,l)\in\mathcal{B}_i} |I||J|\phi_{j,k}(x' - 2^{-j\wedge k-N}l', x_m - 2^{-j\wedge 2k-N}l_m)(\phi_{j,k} * \tilde{f})(2^{-j\wedge k-N}l', 2^{-j\wedge 2k-N}l_m), h > \right| \\
&= \sup_{\|h\|_2\le 1} \int_{\mathbb{R}^{m-1}\times\mathbb{R}} \int_{I\times J} \sum_{(j,k,l)\in\mathcal{B}_i} \phi_{j,k}(x' - 2^{-j\wedge k-N}l', x_m - 2^{-j\wedge 2k-N}l_m)(\phi_{j,k} * \tilde{f})(2^{-j\wedge k-N}l', 2^{-j\wedge 2k-N}l_m) \\
&\qquad h(x', x_m)dy'dy_m dx'dx_m \\
&= \sup_{\|h\|_2\le 1} \int_{I\times J} \sum_{(j,k,l)\in\mathcal{B}_i} \phi_{j,k} * h(2^{-j\wedge k-N}l', 2^{-j\wedge 2k-N}l_m)(\phi_{j,k} * \tilde{f})(2^{-j\wedge k-N}l', 2^{-j\wedge 2k-N}l_m)dy'dy_m \\
&\le \sup_{\|h\|_2\le 1} \|\{ \sum_{(j,k,l)\in\mathcal{B}_i} |\phi_{j,k} * h(2^{-j\wedge k-N}l', 2^{-j\wedge 2k-N}l_m)|^2 \chi_{I\times J}\}^{\frac{1}{2}}\|_2 \\
&\qquad \times \|\{ \sum_{(j,k,l)\in\mathcal{B}_i} |\phi_{j,k} * \tilde{f}(2^{-j\wedge k-N}l', 2^{-j\wedge 2k-N}l_m)|^2 \chi_{I\times J}\}^{\frac{1}{2}}\|_2 \\
&\le C \sup_{\|h\|_2\le 1} \|\widetilde{g}^d_\phi(h)\|_2 \|\{ \sum_{(j,k,l)\in\mathcal{B}_i} |\phi_{j,k} * \tilde{f}(2^{-j\wedge k-N}l', 2^{-j\wedge 2k-N}l_m)|^2 \chi_{I\times J}\}^{\frac{1}{2}}\|_2 \\
&\le C \|\{ \sum_{(j,k,l)\in\mathcal{B}_i} |\phi_{j,k} * \tilde{f}(2^{-j\wedge k-N}l', 2^{-j\wedge 2k-N}l_m)|^2 \chi_{I\times J}\}^{\frac{1}{2}}\|_2.
\end{aligned}$$

The above estimate implies the size condition (2) of a_i, since

$$\begin{aligned} \|a_i\|_2 &= \left(\left\| \left\{ \sum_{(j,k,l)\in\mathcal{B}_i} |(\phi_{j,k} * \tilde{f})(2^{-j\wedge k-N}l', 2^{-j\wedge 2k-N}l_m)|^2 \chi_{I\times J} \right\}^{1/2} \right\|_2 |\widetilde{\Omega}_i|^{\frac{1}{p}-\frac{1}{2}} \right)^{-1} \\ &\quad \times \|\{ \sum_{(j,k,l)\in\mathcal{B}_i} |\phi_{j,k} * \tilde{f}(2^{-j\wedge k-N}l', 2^{-j\wedge 2k-N}l_m)|^2 \chi_{I\times J}\}^{\frac{1}{2}}\|_2 \\ &\leq |\widetilde{\Omega}_i|^{\frac{1}{2}-\frac{1}{p}}. \end{aligned}$$

To verify a_i satisfies conditions (3) and (4), note that if $(j,k,l) \in B_i$, then $R = I \times J \in \widetilde{\Omega}_i$ and there exists a $\tilde{R} \in \mathcal{M}(\widetilde{\Omega}_i)$ such that $R \subseteq \tilde{R}$. Therefore, we can further decompose a_i into

$$a_i(x', x_m) = \sum_{\widetilde{R}\in\mathcal{M}(\widetilde{\Omega}_i)} a_{\widetilde{R}}(x', x_m),$$

where

$$\begin{aligned} a_{\widetilde{R}}(x', x_m) = \frac{1}{\lambda_i} \sum_{\substack{(j,k,l)\in\mathcal{B}_i,\\ R=I\times J\subset\widetilde{R}\in\mathcal{M}(\widetilde{\Omega}_i)}} & |I||J|\phi_{j,k}(x' - 2^{-j\wedge k-N}l', x_m - 2^{-j\wedge 2k-N}l_m) \\ & \times(\phi_{j,k} * \tilde{f})(2^{-j\wedge k-N}l', 2^{-j\wedge 2k-N}l_m). \end{aligned}$$

We can see that *supp* $a_{\widetilde{R}} \subset \sum_{(j,k,l)\in\mathcal{B}_i} 2^N(I \times J) \subset 2^N\widetilde{R}$ and the cancellation conditions (4) follow directly from the conditions on $\phi^{(1)}$ and $\phi^{(2)}$. On the other hand, by the same proof for the estimate of $\|a_i\|_2$, we have

$$\|a_{\widetilde{R}}\|_2 \leq \frac{C}{\lambda_i} \|\{ \sum_{\substack{(j,k,l)\in\mathcal{B}_i\\ R=I\times J\subset\widetilde{R}\in\mathcal{M}(\widetilde{\Omega}_i)}} |\phi_{j,k} * \tilde{f}(2^{-j\wedge k-N}l', 2^{-j\wedge 2k-N}l_m)|^2 \chi_{I\times J}\}^{\frac{1}{2}}\|_2.$$

Therefore, by the definition of λ_i we have $\sum_{\widetilde{R}\in\mathcal{M}(\tilde{\Omega}_i)} \|a_{\widetilde{R}}\|_2 \le \|a_i\|_2 \le |\widetilde{\Omega}_i|^{\frac{1}{2}-\frac{1}{p}}$.

Note that by the maximal theorem $|\tilde{\Omega}_i| \le C|\Omega|$. Since if $(j,k,l)\in\mathcal{B}_i$ and $x=(x',x_m)$ belongs to the corresponding $R=I\times J$, then $M_s(\chi_{R\cap\tilde{\Omega}_i\setminus\Omega_{i+1}}(x',x_m) > \frac{1}{2}$, we have $\chi_R(x',x_m)\le 2M_s(\chi_{R\cap\tilde{\Omega}_i\setminus\Omega_{i+1})}(x',x_m)$. Thus,

$$
\begin{aligned}
&\left\|\left\{\sum_{(j,k,l)\in\mathcal{B}_i} |(\phi_{j,k}*\tilde{f})(2^{-j\wedge k-N}l', 2^{-j\wedge 2k-N}l_m)|^2\chi_{I\times J}\right\}^{1/2}\right\|_2^2 \\
&= \int_{\mathbb{R}^{m-1}\times\mathbb{R}} \sum_{(j,k,l)\in\mathcal{B}_i} |(\phi_{j,k}*\tilde{f})(2^{-j\wedge k-N}l', 2^{-j\wedge 2k-N}l_m)|^2\chi_{I\times J}(x',x_m)dx'dx_m
\end{aligned}
$$

$$
\begin{aligned}
&\le C\int_{\mathbb{R}^{m-1}\times\mathbb{R}} \sum_{(j,k,l)\in\mathcal{B}_i} |(\phi_{j,k}*\tilde{f})(2^{-j\wedge k-N}l', 2^{-j\wedge 2k-N}l_m)|^2 M_s(\chi_{R\cap\tilde{\Omega}_i\setminus\Omega_{i+1}})(x',x_m)dx'dx_m \\
&\le C\int_{\tilde{\Omega}_i\setminus\Omega_{i+1}} \sum_{(j,k,l)\in\mathcal{B}_i} |(\phi_{j,k}*\tilde{f})(2^{-j\wedge k-N}l', 2^{-j\wedge 2k-N}l_m)|^2\chi_R(x',x_m)dx'dx_m \\
&\le C2^i|\widetilde{\Omega}_i|.
\end{aligned}
$$

Therefore, by the definition of Ω_i, we have

$$
\begin{aligned}
\sum_{i=-\infty}^{\infty}|\lambda_i|^p \le\ & C\sum_{i=-\infty}^{\infty} 2^i|\widetilde{\Omega}_i|^{\frac{p}{2}}|\widetilde{\Omega}_i|^{1-\frac{p}{2}} \\
&= C\sum_{i=-\infty}^{\infty} 2^i|\widetilde{\Omega}_i| \le C\sum_{i=-\infty}^{\infty} 2^i|\Omega_i| \\
&\le C\sum_{i=-\infty}^{\infty} 2^i|\Omega_i\setminus\Omega_{i+1}| \\
&\le C\|\widetilde{g}_\phi^d(f)\|_p^p \le C\|f\|_{H^p_{com}(\mathbb{R}^m)}.
\end{aligned}
$$

□

For a L^2-bounded linear operator on $H^p_{com}(\mathbb{R}^m)$, consider its action on $(p,2)$-atoms, we have the following boundedness criterion.

Theorem 4.6. *Let T is bounded linear operator on $L^2(\mathbb{R}^m)$, then T is bounded from $H^p_{com}(\mathbb{R}^m)$ to $L^p(\mathbb{R}^m)$ if and only if $\|Ta\|_{L^p(\mathbb{R}^m)} \leq C$ for all $(p,2)-$ atoms of $H^p_{com}(\mathbb{R}^m)$.*

Here we omit the proof of Theorem 4.6 because it is same with the proof of Theorem 2.9.

5 The duality theorem of weighted multi-parameter Hardy spaces associated with Zygmund dilation

5.1 Introduction and statements of main results

The celebrated $H^1(\mathbb{R}^n) - BMO(\mathbb{R}^n)$ duality theorem was proved by C. Fefferman and Stein [Fc, FcS] in one-parameter case. In multi-parameter setting, S-Y. A. Chang and R. Fefferman [CF1, CF3] proved that the dual space of the product $H^1(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$ is the product $BMO(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$ using the bi-Hilbert transform.

Among the multi-parameter analysis, the Zygmund dilations are the simplest after pure product space dilations. (See R. Fefferman's survey [Fr2].) Recently, Y. Han and G. Lu [HL2, HL3] developed a unified approach of multi-parameter Hardy space theory using the discrete multi-parameter Littlewood-Paley-Stein analysis, and the $H^p_{\mathcal{Z}} - CMO^p_{\mathcal{Z}}$ duality theorem (Theorem 1.6 in [HL2]) is one of their subsequent work, where $H^p_{\mathcal{Z}}$ is the multi-parameter Hardy space associated with Zygmund dilations and $CMO^p_{\mathcal{Z}}$ is the Carleson measure spaces associated with Zygmund dilations.

We will characterize the dual spaces of the weighted multi-parameter Hardy spaces associated with Zygmund dilations, that is, $(H^p_{\mathcal{Z}}(\omega))^* = CMO^p_{\mathcal{Z}}(\omega)$ for all $0 < p \leq 1$ and $\omega \in A_\infty(\mathcal{Z})$. Such Carleson measure spaces $CMO^p_{\mathcal{Z}}(\omega)$ play the same role as the John-

Nirenberg BMO spaces in the duality $H^1(\mathbb{R}^n) - BMO(\mathbb{R}^n)$ in the one-parameter setting.

Let us first establish the preliminaries for Zygmund dilations and recall the related background briefly. In $\mathbb{R}^3$, the Zygmund dilation is given by $\rho_{s,t}(x,y,z) = (sx, ty, stz)$ for $s, t > 0$, and the maximal operator associated to Zygmund dilations is defined by

$$\mathcal{M}_{\mathcal{Z}} f(x,y,z) = \sup_{\substack{(x,y,z)\in Q \\ Q\in\mathcal{R}_{\mathcal{Z}}}} \frac{1}{|Q|} \int_Q |f|, \tag{5.1}$$

where $\mathcal{R}_{\mathcal{Z}}$ is the class of rectangles whose sides are parallel to the axes and have side lengths of the form s, t, and st. As a special case of Córdoba's solution [Ca] of Zygmund's conjecture, the operator $\mathcal{M}_{\mathcal{Z}}$ is bounded from the Orlicz space $L\log^+ L(Q_1)$ to weak $L^1(Q_1)$. (Q_1 is the unit cube in $\mathbb{R}^3$.) The weighted L^p boundedness of $\mathcal{M}_{\mathcal{Z}}$ for $1 < p < \infty$ was proved by R. Fefferman [Fr1], see also [FP] and various generalizations in [JT].

Write $\mathcal{S}(\mathbb{R}^n)$ as the space of Schwartz functions in $\mathbb{R}^n$. The test function defined on $\mathbb{R}^3$ is given by

$$\psi(x,y,z) = \psi^{(1)}(x)\psi^{(2)}(y,z),$$

where $\psi^{(1)} \in \mathcal{S}(\mathbb{R})$ and $\psi^{(2)} \in \mathcal{S}(\mathbb{R}^2)$ satisfy

$$\sum_{j\in\mathbb{Z}} |\widehat{\psi^{(1)}}(2^{-j}\xi_1)|^2 = 1 \text{ for all } \xi_1 \in \mathbb{R}\backslash\{0\},$$

$$\sum_{j\in\mathbb{Z}} |\widehat{\psi^{(2)}}(2^{-k}\xi_2, 2^{-k}\xi_3)|^2 = 1 \text{ for all } (\xi_2,\xi_3) \in \mathbb{R}^2\backslash\{(0,0)\},$$

and the moment conditions

$$\int_{\mathbb{R}} x^\alpha \psi^{(1)}(x)dx = \int_{\mathbb{R}^2} y^\beta z^\gamma \psi^{(2)}(y,z)dydz = 0$$

for all integers $\alpha, \beta, \gamma \geq 0$. By taking Fourier transform, it is easy to see the continuous version of Calderón's identity

$$f(x,y,z) = \sum_{j,k} \psi_{j,k} * \psi_{j,k} * f(x,y,z), \tag{5.2}$$

where

$$\psi_{j,k}(x,y,z) = 2^{2(j+k)}\psi^{(1)}(2^j x)\psi^{(2)}(2^k y, 2^{j+k} z), \tag{5.3}$$

and the series converges in L^2. Ricci and Stein [RS] introduced the what is now called Ricci-Stein singular integral operator $T_{\mathcal{Z}}$ as $T_{\mathcal{Z}} = K * f$, and

$$K(x,y,z) = \sum_{j,k} 2^{-2(j+k)} \psi_{j,k}\left(\frac{x}{2^j}, \frac{y}{2^k} \frac{z}{2^{j+k}}\right),$$

where the functions $\psi_{j,k}$ are test functions in $\mathcal{S}(\mathbb{R}^3)$. They also gave the L^p $(1 < p < \infty)$ boundedness of the operator $T_{\mathcal{Z}}$. The weighted L^p boundedness theorem was proved by R. Fefferman and Pipher (Theorem 2.4 in [FP]) when $w \in A_p(\mathcal{Z})$. The authors in [HL2] proved that both the convolution and non-convolution type Ricci-Stein operators are bounded on $H^p_{\mathcal{Z}}$ and $BMO_{\mathcal{Z}}$[i]. While the other paper [HLX2] will show the boundedness result on weighted $H^p_{\mathcal{Z}}$ spaces when $w \in A_\infty(\mathcal{Z})$, it is interesting to note that we only require $w \in A_\infty(\mathcal{Z})$ which

[i]The multi-parameter Hardy space associated with Zygmund dilations $H^p_{\mathcal{Z}}$ is defined in the following content, see [HL2] for more information, where one can also find a nice historical note in the introductory section.

is much weaker than the usual requirement $w \in A_1$ for boundedness of singular integral operators on weighted Hardy spaces. Using the A_∞ weight to consider the boundedness of singular integrals on weighted multiparameter Hardy spaces seems to be first used in [DHLW]. (See also [R] for the case of more parameters.)

Now we define the Littlewood-Paley-Stein square function of f associated with the Zygmund dilation,

$$g_{\mathcal{Z}}(f)(x,y,z) = \left\{ \sum_{j,k} |\psi_{j,k} * f(x,y,z)|^2 \right\}^{1/2}. \tag{5.4}$$

From Ricci and Stein's L^p boundedness of the operator $T_{\mathcal{Z}}$, together with the L^2 convergence of Calderón's identity, one can obtain the L^p estimate of $g_{\mathcal{Z}}$ as $\|g_{\mathcal{Z}}(f)\|_{L^p} \approx \|f\|_{L^p}$ for $1 < p < \infty$. Precisely, there exist two constants $C_1, C_2 > 0$ independent of f such that

$$C_1\|f\|_{L^p} \le \|g_{\mathcal{Z}}(f)\|_{L^p} \le C_2\|f\|_{L^p}. \tag{5.5}$$

To pass these Littlewood-Paley-Stein analysis to Hardy spaces and weighted Hardy spaces, we need to introduce a proper distribution space.

Definition 5.1. *A Schwartz test function $f(x,y,z)$ defined on $\mathbb{R}^3$ is said to be a product test function on $\mathbb{R} \times \mathbb{R}^2$ if $f \in \mathcal{S}(\mathbb{R}^3)$ and*

$$\int_{\mathbb{R}} x^\alpha f(x,y,z)dx = \int_{\mathbb{R}^2} y^\beta z^\gamma f(x,y,z)dydz = 0$$

for all indices α, β, and γ of nonnegative integers.

If f is a product test function on $\mathbb{R} \times \mathbb{R}^2$, we denote $f \in \mathcal{S}_{\mathcal{Z}}(\mathbb{R}^3)$ and the norm of f is defined by the norm of Schwartz test function. We denote the dual of $\mathcal{S}_{\mathcal{Z}}(\mathbb{R}^3)$ by $(\mathcal{S}_{\mathcal{Z}}(\mathbb{R}^3))'$.

Since the functions $\psi_{j,k}$ constructed above belong to $\mathcal{S}_{\mathcal{Z}}(\mathbb{R}^3)$, so the Littlewood-Paley-Stein square function $g_{\mathcal{Z}}$ can be defined for all distributions in $(\mathcal{S}_{\mathcal{Z}}(\mathbb{R}^3))'$. Thus for $0 < p < \infty$, the multi-parameter Hardy space associated with Zygmund dilations is defined as

$$H^p_{\mathcal{Z}}(\mathbb{R}^3) = \{f \in (\mathcal{S}_{\mathcal{Z}}(\mathbb{R}^3))' : g_{\mathcal{Z}}(f) \in L^p(\mathbb{R}^3)\},$$

and $H^p_{\mathcal{Z}}(\mathbb{R}^3) = L^p(\mathbb{R}^3)$ for $1 < p < \infty$ follows immediately from (1.5) above. See the work [HL2] for the thorough study of such $H^p_{\mathcal{Z}}$ spaces including the duality theory and boundedness of convolution and non-convolution operators.

Given $1 < p < \infty$, a nonnegative function ω on $\mathbb{R}^3$ is said to belong to $A_p(\mathcal{Z})$ if

$$\sup_{Q \in R_{\mathcal{Z}}} \left(\frac{1}{|Q|}\int_Q \omega\right)\left(\frac{1}{|Q|}\int_Q \omega^{-\frac{1}{p-1}}\right)^{p-1} = \|\omega\|_{A_p(\mathcal{Z})} < \infty.$$

When $p = 1$, $\omega \in A_1(\mathcal{Z})$ if there exists $C > 0$ such that $\mathcal{M}_{\mathcal{Z}}(\omega)(x) \le C\omega(x)$ for almost every $x \in \mathbb{R}^3$. Finally, we define

$$A_\infty(\mathcal{Z}) = \bigcup_{1 \le p < \infty} A_p(\mathcal{Z}).$$

Notice that if $\omega \in A_\infty(\mathcal{Z})$, then $\omega \in A_{q_\omega}(\mathcal{Z})$, where $q_\omega = \inf\{q : \omega \in A_q(\mathcal{Z})\}$. Now let us introduce the two spaces that we study.

Definition 5.2 ($H^p_{\mathcal{Z}}(\omega)$)**.** *Let $0 < p < \infty$ and $\omega \in A_\infty(\mathcal{Z})$, the multi-parameter Hardy space associated with the Zygmund dilation is defined as $H^p_{\mathcal{Z}}(\omega) = \{f \in (\mathcal{S}_{\mathcal{Z}}(\mathbb{R}^3))' : g_{\mathcal{Z}}(f) \in L^p_\omega)\}$. If $f \in H^p_{\mathcal{Z}}(\omega)$, the norm of f is defined by $\|f\|_{H^p_{\mathcal{Z}}(\omega)} = \|g_{\mathcal{Z}}(f)\|_{L^p_\omega}$.*

Definition 5.3 ($CMO^p_{\mathcal{Z}}(\omega)$)**.** *Let $0 < p \le 1$, $\omega \in A_\infty(\mathcal{Z})$ and $\psi_{j,k}$ be the same as in (1.3), we say that $f \in CMO^p_{\mathcal{Z}}(\omega)$ if $f \in (\mathcal{S}_{\mathcal{Z}}(\mathbb{R}^3))'$ with the finite norm $\|f\|_{CMO^p_{\mathcal{Z}}(\omega)}$ defined by*

$$\sup_{\Omega}\left\{\frac{1}{\omega(\Omega)^{\frac{2}{p}-1}}\sum_{j,k}\sum_{I\times J\times R\subseteq\Omega}|\psi_{j,k}*f(x_I,y_J,z_R)|^2\frac{|I\times J\times R|^2}{\omega(I\times J\times R)}\right\}^{\frac{1}{2}}$$

for all open sets Ω in $\mathbb{R}^3$ with finite weighted measures and any fixed points x_I, y_J, and z_R in $I \subseteq \mathbb{R}$, $J \subseteq \mathbb{R}$, and $R \subseteq \mathbb{R}$, where I, J, and R are dyadic intervals with interval-length $\ell(I) = 2^{-j-N}, \ell(J) = 2^{-k-N}$, and $\ell(R) = 2^{-j-k-2N}$ for a fixed large positive integer N.

Remark. In Definitions 5.2 and 5.3 above, the definitions of $H^p_{\mathcal{Z}}(\omega)$ and $CMO^p_{\mathcal{Z}}(\omega)$ involve $\psi_{j,k}$, to show these definitions are well defined, we need to prove that they are independent of the choice of functions $\psi_{j,k}$. Precisely, we use sup-inf comparison principle of first kind as Theorem 5.7 to show that $H^p_{\mathcal{Z}}(\omega)$ is well-defined. While we state sup-inf comparison principle of second kind as one of our major theorems below, to prove that $CMO^p_{\mathcal{Z}}(\omega)$ is well-defined.

Theorem 5.4 (Sup-inf comparison principle of second kind)**.** *Let $0 < p \le 1$ and $\omega \in A_\infty(\mathcal{Z})$, suppose $\psi^{(1)}, \phi^{(1)} \in \mathcal{S}(\mathbb{R})$, $\psi^{(2)}, \phi^{(2)} \in \mathcal{S}(\mathbb{R}^2)$, and $\psi_{j,k}, \phi_{j,k}$ satisfy the condition in (1.3). Then for $f \in (\mathcal{S}_{\mathcal{Z}}(\mathbb{R}^3))'$,*

$$\begin{aligned}&\sup_{\Omega}\left\{\frac{1}{\omega(\Omega)^{\frac{2}{p}-1}}\sum_{j,k}\sum_{I\times J\times R\subseteq\Omega}\sup_{u\in I,v\in J,w\in R}|\psi_{j,k}*f(u,v,w)|^2\frac{|I\times J\times R|^2}{\omega(I\times J\times R)}\right\}^{\frac{1}{2}}\\ \approx\ &\sup_{\Omega}\left\{\frac{1}{\omega(\Omega)^{\frac{2}{p}-1}}\sum_{j,k}\sum_{I\times J\times R\subseteq\Omega}\inf_{u\in I,v\in J,w\in R}|\phi_{j,k}*f(u,v,w)|^2\frac{|I\times J\times R|^2}{\omega(I\times J\times R)}\right\}^{\frac{1}{2}}.\end{aligned}$$

where $I \subseteq \mathbb{R}$, $J \subseteq \mathbb{R}$, and $R \subseteq \mathbb{R}$ are dyadic intervals with interval-length $\ell(I) = 2^{-j-N}$, $\ell(J) = 2^{-k-N}$, and $\ell(R) = 2^{-j-k-2N}$ for a fixed large positive integer N, and Ω are all open *sets in* $\mathbb{R}^3$ *with finite weighted measures.*

Then we state that the space $CMO^p_{\mathcal{Z}}$ is exactly the dual space of $H^p_{\mathcal{Z}}(\omega)$ for $0 < p \le 1$.

Theorem 5.5 ($H^p_{\mathcal{Z}}(\omega) - CMO^p_{\mathcal{Z}}(\omega)$)**.** *Let* $0 < p \le 1$ *and* $\omega \in A_\infty(\mathcal{Z})$*. Then* $(H^p_{\mathcal{Z}}(\omega))^* = CMO^p_{\mathcal{Z}}(\omega)$*, namely the dual space of* $H^p_{\mathcal{Z}}(\omega)$ *is* $CMO^p_{\mathcal{Z}}(\omega)$*. More precisely, if* $g \in CMO^p_{\mathcal{Z}}(\omega)$*, the map* ℓ_g *given by* $\ell_g(f) =< f, g >$*, defined initially for* $f \in \mathcal{S}_{\mathcal{Z}}(\mathbb{R}^3)$*, extends to a continuous linear functional on* $H^p_{\mathcal{Z}}(\omega)$ *with* $\|\ell_g\| \approx \|g\|_{CMO^p_{\mathcal{Z}}(\omega)}$*. Conversely, for every* $\ell \in (H^p_{\mathcal{Z}}(\omega))^*$ *there exists some* $g \in CMO^p_{\mathcal{Z}}(\omega)$ *so that* $\ell = \ell_g$*. In particular,* $(H^1_{\mathcal{Z}}(\omega))^* = BMO_{\mathcal{Z}}(\omega)$*.*

In Section 5.2, we collect several known results on the discrete Calderón's identity and sup-inf comparison principle of first kind which are used to prove that $H^p_{\mathcal{Z}}(\omega)$ is well-defined. In Chapter 6, we show the well-definition of $CMO^p_{\mathcal{Z}}(\omega)$ using sup-inf comparison of second kind and almost orthogonality estimate. While Chapter 7 is devoted to prove the duality theory Theorem 5.5.

We shall point out in the end of the introduction that the main tool in this part, the discrete multi-parameter Littlewood-Paley-Stein analysis, is a relatively unified theory with a whole scheme, some theorems and lemmas we use here originate from the work [HL2]. An interested reader should consult the papers [HL1, HL2, HL3, HLL1] and related works mentioned therein. (See also [DHLW] and [R] where some nice application of the discrete Littlewood-Paley-Stein analysis was given in weighted setting.)

5.2 Discrete Calderón identity

To show the definition of $H^p_{\mathcal{Z}}(\omega)$ is independent of the choice of functions $\psi_{j,k}$ and thus well defined in Definition 5.2, we need to recall some results associated with the Zygmund dilation. First, we require the discrete version of Calderón's identity.

Theorem 5.6 (Discrete Calderón's identity). *Suppose that $\psi_{j,k}$ are the same as in (5.3). Then*

$$f(x,y,z) = \sum_{j,k} \sum_{I,J,R} |I||J||R| \widetilde{\psi}_{j,k}(x,y,z,x_I,y_J,z_R)(\psi_{j,k} * f)(x_I,y_J,z_R), \tag{5.6}$$

converges in the norm of $\mathcal{S}_{\mathcal{Z}}(\mathbb{R}^3)$ and in the dual space $(\mathcal{S}_{\mathcal{Z}}(\mathbb{R}^3))'$, where $\widetilde{\psi}_{j,k}(x,y,z,x_I,y_J,z_R) \in \mathcal{S}_{\mathcal{Z}}(\mathbb{R}^3)$, $I \subseteq \mathbb{R}$, $J \subseteq \mathbb{R}$, and $R \subseteq \mathbb{R}$ are dyadic intervals with interval-length $\ell(I) = 2^{-j-N}$, $\ell(J) = 2^{-k-N}$, and $\ell(R) = 2^{-j-k-2N}$ for a fixed large integer N, and x_I, y_J, z_R are any fixed points in I, J, R, respectively.

The complete proof is contained in §2.2 of [HL2], for the reader's convenience, we provide a sketch of the proof here. An observation shows that the continuous version of Calderón's identity (5.2) converges in the norm of $\mathcal{S}_{\mathcal{Z}}(\mathbb{R}^3)$ and in the dual space $(\mathcal{S}_{\mathcal{Z}}(\mathbb{R}^3))'$. Then it can be decomposed in dyadic form and we only need to estimate the remainder term as the difference. The explicit expression of $\widetilde{\psi}_{j,k}(x,y,z,x_I,y_J,z_R)$ can also be found in [HL2].

The discrete Calderón's identity enables us to derive the following weighted version sup-inf comparison principle of first kind, whose proof is included in [HLX2] (Theorem 1.1). It is an extension of the non-weighted one first derived in [HL2].

Theorem 5.7 (Sup-inf comparison principle of first kind). *Let $0 < p < \infty$ and $\omega \in A_\infty(\mathcal{Z})$, suppose $\psi^{(1)}, \phi^{(1)} \in \mathcal{S}(\mathbb{R})$, $\psi^{(2)}, \phi^{(2)} \in \mathcal{S}(\mathbb{R}^2)$, and $\psi_{j,k}, \phi_{j,k}$ satisfy the condition in (1.3).*

Then for $f \in (\mathcal{S}_{\mathcal{Z}}(\mathbb{R}^3))'$,

$$\begin{aligned} &\| \left\{ \sum_{j,k} \sum_{I,J,R} \sup_{u\in I, v\in J, w\in R} |\psi_{j,k} * f(u,v,w)|^2 \chi_I(\cdot)\chi_J(\cdot)\chi_R(\cdot) \right\}^{\frac{1}{2}} \|_{L^p_\omega} \\ \approx\ &\| \left\{ \sum_{j,k} \sum_{I,J,R} \inf_{u\in I, v\in J, w\in R} |\phi_{j,k} * f(u,v,w)|^2 \chi_I(\cdot)\chi_J(\cdot)\chi_R(\cdot) \right\}^{\frac{1}{2}} \|_{L^p_\omega}, \end{aligned} \tag{5.7}$$

where $I \subseteq \mathbb{R}$, $J \subseteq \mathbb{R}$, *and* $R \subseteq \mathbb{R}$ *are dyadic intervals with interval-length* $\ell(I) = 2^{-j-N}$, $\ell(J) = 2^{-k-N}$, *and* $\ell(R) = 2^{-j-k-2N}$ *for a fixed large positive integer* N, χ_I, χ_J, *and* χ_R *are indicator functions of* I, J, *and* R, *respectively.*

From this sup-inf comparison principle, we introduce the discrete Littlewood-Paley-Stein square function

$$g^d_{\mathcal{Z}}(f)(x,y,z) = \left\{ \sum_{j,k} \sum_{I,J,R} |(\psi_{j,k} * f)(x_I, y_J, z_R)|^2 \chi_I(x)\chi_J(y)\chi_R(z) \right\}^{\frac{1}{2}}, \tag{5.8}$$

where we admit all the settings in Theorem 5.7, and the $H^p_{\mathcal{Z}}(\omega)$ norm of f can be characterized using a discrete form

$$\|f\|_{H^p_{\mathcal{Z}}(\omega)} \approx \|g^d_{\mathcal{Z}}(f)\|_{L^p_\omega}.$$

Thus, we conclude that $H^p_{\mathcal{Z}}(\omega)$ is well-defined by Theorem 5.7.

5.3 Sup-inf comparison principle of second kind

The purpose of this section is to get the sup-inf comparison principle of second kind, i.e., Theorem 5.4, to ensure that the space $CMO^p_{\mathcal{Z}}(\omega)$ in Definition 5.3 is well-defined. First, we recall an "almost orthogonality lemma", and refer the reader to Corollary 2.6 in [HL2] for its detailed proof.

Lemma 5.8 (Almost orthogonality estimate)**.** *If $\psi, \phi \in \mathcal{S}_{\mathcal{Z}}(\mathbb{R}^3)$, define*

$$\psi_{t,s}(x,y,z) = t^{-2}s^{-2}\psi(\frac{x}{t}, \frac{y}{s}, \frac{z}{ts})$$

and $\phi_{t',s'}$ is defined similarly. Then, for any positive integers L and M, there exists a constant $C = C(L,M) > 0$ such that

$$\begin{aligned} & |\psi_{t,s} * \phi_{t',s'}(x,y,z)| \\ \leq \; & C(\frac{t}{t'} \wedge \frac{t'}{t})^L (\frac{s}{s'} \wedge \frac{s'}{s})^L \frac{(t \vee t')^M}{(t \vee t' + |x|)^{M+1}} \frac{(s \vee s')^M}{t^*(s \vee s' + |y| + \frac{|z|}{t^*})^{M+2}}, \end{aligned} \tag{5.9}$$

where $t^* = t$ if $s > s'$, $t^* = t'$ if $s \leq s'$, $t \wedge s = \min(t,s)$, and $t \vee s = \max(t,s)$.

Together with the discrete Calderón identity and some geometric properties of multi-parameter rectangles, Theorem 5.4 can be proved by a delicate study of the Zygmund rectangles.

Proof of Theorem 5.4. For simplicity, we denote $f_{j,k} = f_Q$, where $Q = I \times J \times R \subseteq \mathbb{R}^3$, $I \subseteq \mathbb{R}$, $J \subseteq \mathbb{R}$, and $R \subseteq \mathbb{R}$ are dyadic intervals with interval-length $\ell(I) = 2^{-j-N}$, $\ell(J) = 2^{-k-N}$, and $\ell(R) = 2^{-j-k-2N}$ for a fixed large positive integer N. While x_I, y_J, and z_R are any fixed points in I, J, and R, respectively.

Then, we can rewrite the discrete Calderón identity on $(\mathcal{S}_{\mathcal{Z}}(\mathbb{R}^3))'$ as

$$f(x,y,z) = \sum_{i,j} \sum_{Q=I'\times J'\times R'} |I'||J'||R'|\widetilde{\phi}_{Q'}(x,y,z,x_{I'},y_{J'},z_{R'})(\phi_{Q'} * f)(x_{I'},y_{J'},z_{R'}).$$

Thus, for all $(x, y, z) \in Q$,

$$\psi_Q * f(x,y,z) = \sum_{i,j} \sum_{Q'=I'\times J'\times R'} |Q'|\psi_Q * \widetilde{\phi}_{Q'}(x,y,z,x_{I'},y_{J'},z_{R'})(\phi_{Q'} * f)(x_{I'},y_{J'},z_{R'}),$$

where $I' \subseteq \mathbb{R}$, $J' \subseteq \mathbb{R}$, and $R' \subseteq \mathbb{R}$ are dyadic intervals with interval-length $\ell(I') = 2^{-j'-N}$, $\ell(J') = 2^{-k'-N}$, and $\ell(R') = 2^{-j'-k'-2N}$ for a fixed large positive integer N. While $x_{I'}$, $y_{J'}$, and $z_{R'}$ are any fixed points in I', J', and R', respectively.

From the almost orthogonality estimates (5.9) in Lemma 5.8, by choosing $t = 2^{-j}$, $t' = 2^{-j'}$, $s = 2^{-k}$, and $s' = 2^{-k'}$,

$$\begin{aligned} & |\psi_Q * f(x,y,z)|^2 \\ \leq\ & C \sum_{Q'=I'\times J'\times R'} |Q'| \left(\frac{|I|}{|I'|} \wedge \frac{|I'|}{|I|}\right)^L \left(\frac{|J|}{|J'|} \wedge \frac{|J'|}{|J|}\right)^L \frac{(|I| \vee |I'|)^M}{(|I| \vee |I'| + d(I,I'))^{M+1}} \\ & \times \frac{(|J| \vee |J'|)^M}{t^*(|J| \vee |J'| + d(J,J') + \frac{d(R,R')}{t^*})^{M+2}} |\phi_{Q'} * f(x_{I'},y_{J'},z_{R'})|^2, \end{aligned} \tag{5.10}$$

where $|Q'| = |I'||J'||R'|$, $t^* = |I|$ when $|J| \geq |J'|$, and $t^* = |I'|$ when $|J| < |J'|$, the constant C depends only on M, L, and functions ψ and ϕ. Write

$$P_Q = \sup_{x\in I, y\in J, z\in R} |\psi_Q * f(x,y,z)|^2,$$

and

$$F_Q = \inf_{x\in I, y\in J, z\in R} |\phi_Q * f(x,y,z)|^2.$$

Since $x_{I'}$, $y_{J'}$, and $z_{R'}$ in (3.2) are arbitrary in I', J' and R', we have

$$\sum_{Q\subseteq\Omega}\frac{|I\times J\times R|^2}{\omega(I\times J\times R)}P_Q\leq C\sum_{Q\subseteq Q'\subseteq\Omega}\widetilde{r}(Q,Q')P(Q,Q')\frac{|I'\times J'\times R'|^2}{\omega(I'\times J'\times R')}F_{Q'},\tag{5.11}$$

where

$$\widetilde{r}(Q,Q')=\left(\frac{|I|}{|I'|}\wedge\frac{|I'|}{|I|}\right)^{L-2}\left(\frac{|J|}{|J'|}\wedge\frac{|J'|}{|J|}\right)^{L-2}\left(\frac{|R|}{|R'|}\wedge\frac{|R'|}{|R|}\right)^{-2}\frac{\omega(I'\times J'\times R')}{\omega(I\times J\times R)},$$

and

$$\begin{aligned}P(Q,Q') &= \frac{(|I|\vee|I'|)^{M+1}}{(|I|\vee|I'|+d(I,I'))^{M+1}}\frac{(|J|\vee|J'|)^{M+1}}{(|J|\vee|J'|+d(J,J')+\frac{d(R,R')}{t^*})^{M+1}}\\ &\quad\times\frac{|R|\vee|R'|}{t^*(|J|\vee|J'|)+t^*d(J,J')+d(R,R')}.\end{aligned}$$

Since $\omega\in A_\infty(\mathcal{Z})$ and $Q\subseteq Q'$, there exists q_ω and $1\leq q_\omega<\infty$ such that

$$\frac{\omega(I'\times J'\times R')}{\omega(I\times J\times R)}\leq C\left(\frac{|I'\times J'\times R'|}{|I\times J\times R|}\right)^{q_\omega}.$$

Thus,

$$\widetilde{r}(Q,Q')\leq r(Q,Q'),$$

where

$$r(Q,Q')=\left(\frac{|I|}{|I'|}\wedge\frac{|I'|}{|I|}\right)^{L-q_\omega-2}\left(\frac{|J|}{|J'|}\wedge\frac{|J'|}{|J|}\right)^{L-q_\omega-2}\left(\frac{|R|}{|R'|}\wedge\frac{|R'|}{|R|}\right)^{-q_\omega-2}.$$

We have

$$\frac{1}{\omega(\Omega)^{\frac{2}{p}-1}}\sum_{Q\subseteq\Omega}P_Q\frac{|Q|^2}{\omega(Q)}\le C\frac{1}{\omega(\Omega)^{\frac{2}{p}-1}}\sum_{Q\subseteq Q'\subseteq\Omega}r(Q,Q')P(Q,Q')\cdot F_{Q'}\frac{|Q'|^2}{\omega(Q')}. \tag{5.12}$$

To show Theorem 5.4, we only need to estimate the right hand side of (5.12). That is, to prove that it can be controlled by

$$\sup_{\Omega^*}\frac{1}{\omega(\Omega^*)^{\frac{2}{p}-1}}\sum_{Q'\subseteq\Omega^*}F_{Q'}\frac{|Q'|^2}{\omega(Q')}.$$

For $i,l\ge 0$, set

$$\Omega^{i,l}=\bigcup_{Q=I\times J\times R\subseteq\Omega}3(2^iI\times 2^lJ\times 2^{i+l}R).$$

Then, write

$$B_{0,0}=\{Q'=I'\times J'\times R':3Q'\cap\Omega^{0,0}\ne\emptyset\},$$

and for $i,l\ge 1$,

$$B_{i,0}=\{Q'=I'\times J'\times R':3(2^iI'\times J'\times 2^iR')\cap\Omega^{i,0}\ne\emptyset,3(2^{i-1}I'\times J'\times 2^{i-1}R')\cap\Omega^{i,0}=\emptyset\},$$

$$B_{0,l}=\{Q'=I'\times J'\times R':3(I'\times 2^lJ'\times 2^lR')\cap\Omega^{0,l}\ne\emptyset,3(I'\times 2^{l-1}J'\times 2^{l-1}R')\cap\Omega^{0,l}=\emptyset\},$$

$$B_{i,l}=\{Q'=I'\times J'\times R':3(2^iI'\times 2^lJ'\times 2^{i+l}R')\cap\Omega^{i,l}\ne\emptyset,\ 3(2^{i-1}I'\times 2^{l-1}J'\times 2^{i+l-2}R')\cap\Omega^{i,l}=\emptyset\}.$$

Note that since $\bigcup_{Q'} = \bigcup_{i,l\geq 0} \bigcup_{Q'\in B_{i,l}}$, the right hand of (5.12) can be bounded by

$$\begin{aligned}
&\frac{1}{\omega(\Omega)^{\frac{2}{p}-1}} \sum_{Q\subseteq\Omega} \left[\sum_{Q'\in B_{0,0}} + \sum_{i\geq 1} \sum_{Q'\in B_{i,0}} + \sum_{l\geq 1} \sum_{Q'\in B_{0,l}} + \sum_{i,l\geq 1} \sum_{Q'\in B_{i,l}} \right] \\
&\times r(Q,Q')P(Q,Q') \cdot F_{Q'} \frac{|Q'|^2}{\omega(Q')} \\
\triangleq\ & I + II + III + IV.
\end{aligned} \tag{5.13}$$

Here we only show the estimate of I, then the estimates for the other three can follow similarly. Notice that if $Q' \in B_{0,0}$, then $3Q' \cap \Omega^{0,0} \neq \emptyset$. Let

$$\mathcal{F}_h^{0,0} = \{Q' \in B_{0,0} : |3Q' \cap \Omega^{0,0}| \geq \frac{1}{2^h} |3Q'|\},$$

$$\mathcal{D}_h^{0,0} = \mathcal{F}_h^{0,0} \backslash \mathcal{F}_{h-1}^{0,0},$$

and

$$\Omega_h^{0,0} = \bigcup_{Q'\in D_h^{0,0}} Q',$$

where $h \geq 0$ and $\mathcal{F}_{-1}^{0,0} = \emptyset$. Without loss of generality we may assume that for any open set $\Omega \subset \mathbb{R}^3$,

$$\sum_{Q=I\times J\times R\subseteq\Omega} \frac{|I\times J\times R|^2}{\omega(I\times J\times R)} F_Q \leq C\omega(\Omega)^{\frac{2}{p}-1}. \tag{5.14}$$

Since $\bigcup_{h\geq 0} \mathcal{D}_h^{0,0} = B_{0,0}$, we have

$$I \leq \frac{1}{\omega(\Omega)^{\frac{2}{p}-1}} \sum_h \sum_{Q'\in\mathcal{D}_h^{0,0}} \sum_{Q\subset\Omega} r(Q,Q')P(Q,Q') \cdot F_{Q'} \frac{|Q'|^2}{\omega(Q')}. \tag{5.15}$$

To estimate (5.15), for each $Q' \in B_{0,0}$ and $i', l', v' \geq 1$, we decompose $\{Q \subset \Omega\}$ into 8 pieces as follows,

$$A_{0,0,0}(Q') = \{Q \subset \Omega : d(I,I') \leq |I| \vee |I'|,\ d(J,J') \leq |J| \vee |J'|, d(R,R') \leq |R| \vee |R'|\},$$

$$A_{i',0,0}(Q') = \{Q \subset \Omega : 2^{i'-1}(|I| \vee |I'|) < d(I,I') \leq 2^{i'}(|I| \vee |I'|), d(J,J') \leq |J| \vee |J'|,$$
$$d(R,R') \leq |R| \vee |R'|\},$$

$$A_{0,l',0}(Q') = \{Q \subset \Omega : d(I,I') \leq |I| \vee |I'|,\ 2^{l'-1}(|J| \vee |J'|) < d(J,J') \leq 2^{l'}(|J| \vee |J'|),$$
$$d(R,R') \leq |R| \vee |R'|\},$$

$$A_{0,0,v'}(Q') = \{Q \subset \Omega : d(I,I') \leq |I| \vee |I'|,\ d(J,J') \leq |J| \vee |J'|,$$
$$2^{v'-1}(|R| \vee |R'|) < d(R,R') \leq 2^{v'}(|R| \vee |R'|)\},$$

$$A_{i',l',0}(Q') = \{Q \subset \Omega : 2^{i'-1}(|I| \vee |I'|) < d(I,I') \leq 2^{i'}(|I| \vee |I'|),$$
$$2^{l'-1}(|J| \vee |J'|) < d(J,J') \leq 2^{l'}(|J| \vee |J'|),\ d(R,R') \leq |R| \vee |R'|\},$$

$$A_{i',0,v'}(Q') = \{Q \subset \Omega : 2^{i'-1}(|I| \vee |I'|) < d(I,I') \leq 2^{i'}(|I| \vee |I'|), d(J,J') \leq |J| \vee |J'|,$$
$$2^{v'-1}(|R| \vee |R'|) < d(R,R') \leq 2^{v'}(|R| \vee |R'|)\},$$

$$A_{0,l',v'}(Q') = \{Q \subset \Omega : d(I,I') \leq |I| \vee |I'|,\ 2^{l'-1}(|J| \vee |J'|) < d(J,J') \leq 2^{l'}(|J| \vee |J'|),$$
$$2^{v'-1}(|R| \vee |R'|) < d(R,R') \leq 2^{v'}(|R| \vee |R'|)\},$$

$$A_{i',l',v'}(Q') = \{Q \subset \Omega : 2^{i'-1}(|I| \vee |I'|) < d(I,I') \leq 2^{i'}(|I| \vee |I'|),$$
$$2^{l'-1}(|J| \vee |J'|) < d(J,J') \leq 2^{l'}(|J| \vee |J'|),$$
$$2^{v'-1}(|R| \vee |R'|) < d(R,R') \leq 2^{v'}(|R| \vee |R'|)\}.$$

Then, (5.15) becomes

$$
\begin{aligned}
I \quad &\le \quad \frac{1}{\omega(\Omega)^{\frac{2}{p}-1}} \sum_h \sum_{Q' \in \mathcal{D}_h^{0,0}} \sum_{Q \subset \Omega} r(Q,Q')P(Q,Q') \cdot m_{Q'} \frac{|Q'|^2}{\omega(Q')} \\
&= \quad \frac{1}{\omega(\Omega)^{\frac{2}{p}-1}} \sum_h \sum_{Q' \in \mathcal{D}_h^{0,0}} \Big(\sum_{Q \in A_{0,0,0}(Q')} + \sum_{i' \ge 1} \sum_{Q \in A_{i',0,0}(Q')} \\
&\quad + \sum_{l' \ge 1} \sum_{Q \in A_{0,l',0}(Q')} + \sum_{v' \ge 1} \sum_{Q \in A_{0,0,v'}(Q')} + \sum_{i',l' \ge 1} \sum_{Q \in A_{i',l',0}(Q')} \\
&\quad + \sum_{i',v' \ge 1} \sum_{Q \in A_{i',0,v'}(Q')} + \sum_{l',v' \ge 1} \sum_{Q \in A_{0,l',v'}(Q')} + \sum_{i',l',v' \ge 1} \sum_{Q \in A_{i',l',v'}(Q')} \Big) \\
&\quad \cdot r(Q,Q')P(Q,Q') \cdot F_{Q'} \frac{|Q'|^2}{\omega(Q')} \\
&\triangleq \quad I_1 + \cdots + I_8.
\end{aligned}
$$

In the following proof, we will give the estimates for I_1 and I_4 separately and the estimates for I_2, I_3, I_5, I_6, I_7, and I_8 can be showed similarly. (**i**). To estimate

$$
I_1 = \frac{1}{\omega(\Omega)^{\frac{2}{p}-1}} \sum_h \sum_{Q' \in \mathcal{D}_h^{0,0}} \sum_{Q \in A_{0,0,0}(Q')} r(Q,Q')P(Q,Q') \cdot F_{Q'} \frac{|Q'|^2}{\omega(Q')}, \tag{5.16}
$$

we divide $\{Q \in A_{0,0,0}(Q')\}$ into 6 cases, and note that $3Q \cap 3Q' \neq \emptyset$ for $Q \in A_{0,0,0}(Q')$.

Case 1. $|I'| \ge |I|$, $|J'| \le |J|$, and $|R'| \le |R|$.

We will use a similar idea of analyzing geometric properties of the intervals (such analysis is similar to what was used in [CF2] in a less complicated situation). Since

$$
\frac{|I|}{|3I'|}|3Q'| = |I| \times |3J'| \times |3R'| \le 9|3Q \cap 3Q'| \le |3Q' \cap \Omega^{0,0}| < \frac{9}{2^{h-1}}|3Q'|,
$$

then $|I| \le 2^{-h+5}|I'|$ and thus $|I'| \sim 2^{h-5+n}|I|$ for some $n \ge 0$. Moreover, for each given such

n, the number of such I's is no more than $5 \cdot 2^{h-5+n}$. As for J, we have $|J| \sim 2^m|J'|$ for some $m \geq 0$ and for each m, the number of such J's is no more than 5. Since $|R| = |I| \times |J|$ and $|R'| = |I'| \times |J'|$, we have $|R| \sim 2^{-(h-5+n)}2^m|R'|$. Note that $|R| \geq |R'|$, thus $m > h-5+n$. Furthermore, for each fixed n and m, the number of such R's is no more than 5. Thus,

$$\begin{aligned}
&\sum_{\text{Case 1}} r(Q,Q')P(Q,Q') \\
\leq\ & C\sum_{\text{Case 1}} \left(\frac{|I|}{|I'|}\right)^{L-q_\omega-2} \left(\frac{|J'|}{|J|}\right)^{L-q_\omega-2} \left(\frac{|R'|}{|R|}\right)^{-q_\omega-2} \frac{|R|}{|I||J|} \\
\leq\ & C\sum_{n\geq 0}\sum_{m\geq 0} 2^{(5-h-n)(L-q_\omega-2)} \cdot 2^{-m(L-q_\omega-2)} \cdot 2^{(h-5+n-m)(-q_\omega-2)} \cdot 2^n \\
\leq\ & C\sum_{n\geq 0}\sum_{m\geq 0} 2^{-hL} \cdot 2^{5L} \cdot 2^{-m(L-2q_\omega-4)} \cdot 2^{-4n} \\
\leq\ & C2^{-hL}.
\end{aligned}$$

Case 2: $|I'| \geq |I|$, $|J'| \leq |J|$, and $|R'| \geq |R|$.

Since

$$\frac{|I||R|}{|3I'||3R'|}|3Q'| = |I| \times |3J'| \times |R| \leq 3|3Q' \cap 3Q| \leq 3|3Q' \cap \Omega^{0,0}| < \frac{3}{2^{h-1}}|3Q'|,$$

then $|I'||R'| \sim 2^{h-5+n}|I||R|$. As for J, $|J| \sim 2^m|J'|$. So for each m, the number of such J's is no more than 5. Noting that $|R| = |I| \times |J|$ and $|R'| = |I'| \times |J'|$, we have $|I'||I'||J'| \sim 2^{h-5+n}|I||I||J|$, which yields that $|I'|^2 \sim 2^{h-5+n+m}|I|^2$, that is, $|I'| \sim 2^{(h-5+n+m)/2}|I|$. Hence for each n and m, the number of such I's is less than $5 \cdot 2^{(h+m+n)/2}$. Since we can obtain that $|R'| \sim 2^{(h-5+n-m)/2}|R|$, and $|R'| \geq |R|$, we have $m \leq h-5+n$. For each fixed n and m, the number of such R's is less than $5 \cdot 2^{(h-5+n-m)/2}$. Thus,

$$\begin{aligned}
&\sum_{\text{Case } 2} r(Q,Q')P(Q,Q') \\
\leq\ & C \sum_{\text{Case } 2} \left(\frac{|I|}{|I'|}\right)^{L-q_\omega-2} \left(\frac{|J'|}{|J|}\right)^{L-q_\omega-2} \left(\frac{|R|}{|R'|}\right)^{-q_\omega-2} \frac{|R'|}{|I||J|} \\
\leq\ & C \sum_{n\geq 0}\sum_{m\geq 0} 2^{\frac{1}{2}(5-h-n-m)(L-q_\omega-2)} \cdot 2^{-m(L-q_\omega-2)} \cdot 2^{\frac{1}{2}(5-h-n+m)(-q_\omega-2)} 2^{h+n} \\
=\ & \sum_{n\geq 0}\sum_{m\geq 0} 2^{-h(\frac{L}{2}-q_\omega-3)} \cdot 2^{\frac{5}{2}L-5q_\omega-10} \cdot 2^{-n(\frac{L}{2}-q_\omega-3)} \cdot 2^{-m(\frac{3}{2}L-2q_\omega-4)} \\
\leq\ & C2^{-h(\frac{L}{2}-q_\omega-3)}.
\end{aligned}$$

Case 3: $|I'| \leq |I|$, $|J'| \geq |J|$, and $|R'| \leq |R|$.

This can be handled in a way similar to that of Case 1, and we have

$$\sum_{\text{Case } 3} r(Q,Q')P(Q,Q') \leq C2^{-hL}.$$

Case 4: $|I'| \leq |I|$, $|J'| \geq |J|$, and $|R'| \geq |R|$.

This can be handled in a similar way to that of Case 2, and we have

$$\sum_{\text{Case } 4} r(Q,Q')P(Q,Q') \leq C2^{-h(\frac{L}{2}-q_\omega-2)}.$$

Case 5: $|I'| \geq |I|$, $|J'| \geq |J|$, and thus $|R'| \geq |R|$.

Since

$$|I| \times |J| \times |R| \leq |3Q' \cap 3Q| \leq |3Q' \cap \Omega^{0,0}| < \frac{1}{2^{h-1}}|3Q'|,$$

then $|Q'| \sim 2^{h-1+n}|Q|$ for some $n \geq 0$. Note that for each n, the number of such Q's is less than $(2^n)^3 = 2^{3n}$. More precisely, we have $(|I'||J'|)^2 \sim 2^{h-1+n}(|I||J|)^2$. Thus,

$$\begin{aligned}
& \sum_{\text{Case 5}} r(Q,Q')P(Q,Q') \\
\leq\ & C \sum_{\text{Case 5}} \left(\frac{|I||J|}{|I'||J'|}\right)^{L-q_\omega-2} \left(\frac{|R|}{|R'|}\right)^{-q_\omega-2} \frac{|R'|}{|I'||J'|} \\
\leq\ & C \sum_{n\geq 0} 2^{-\frac{h-1+n}{2}(L-q_\omega-2)} \cdot 2^{-\frac{h-1+n}{2}(-q_\omega-2)} \cdot 2^{3n} \\
=\ & C \sum_{n\geq 0} 2^{-h(\frac{L}{2}-q_\omega-2)} \cdot 2^{\frac{L}{2}-q_\omega-2} \cdot 2^{-n(\frac{L}{2}-q_\omega-5)} \\
\leq\ & C 2^{-h(\frac{L}{2}-q_\omega-2)}.
\end{aligned}$$

Case 6: $|I'| \leq |I|$, $|J'| \leq |J|$, and thus $|R'| \leq |R|$.

Since

$$|I'| \times |J'| \times |R'| \leq |3Q' \cap \Omega^{0,0}| < \frac{1}{2^{h-1}}|3Q'|,$$

then we can see that in this case, h must be less than 3. From $|I'| \leq |I|$, we have $|I| \sim 2^n|I'|$ for some $n \geq 0$ and for each given such n, the number of such I's is less than 5. Similarly, from $|J'| \leq |J|$, we have $|J| \sim 2^m|J'|$ and for each m, the number of such J's is less than 5. Hence we have $|R| \sim 2^{n+m}|R'|$, and for each n and m, the number of such R's is less than 5. Thus,

$$\begin{aligned}
& \sum_{\text{Case 6}} r(Q,Q')P(Q,Q') \\
\leq\ & C \sum_{\text{Case 6}} \left(\frac{|I'||J'|}{|I||J|}\right)^{L-q_\omega-2} \left(\frac{|R'|}{|R|}\right)^{-q_\omega-2} \frac{|R|}{|I||J|} \\
\leq\ & C \sum_{n\geq 0}\sum_{n\geq 0} (2^{-n-m})^{L-q_\omega-2} \cdot (2^{-n-m})^{-q_\omega-2} \cdot 2^{-mL} \\
=\ & C \sum_{n\geq 0}\sum_{n\geq 0} 2^{-n(L-2q_\omega-4)} \cdot 2^{-m(2L-2q_\omega-4)} \leq C.
\end{aligned}$$

Before we combine these 6 cases above, observe that since $|\Omega_h^{0,0}| \le Ch2^h|\Omega^{0,0}|$, $|\Omega^{0,0}| \le C|\Omega|$, and $\omega \in A_\infty(\mathcal{Z})$, which is a doubling measure, together with (3.6), we have

$$\begin{aligned}
&\sum_h 2^{-h(\frac{L}{2}-q_\omega-3)}\omega(\Omega_h^{0,0})^{\frac{2}{p}-1}\\
\le\ &\sum_h 2^{-h(\frac{L}{2}-q_\omega-3)}\omega(Ch^2 2^h\Omega^{0,0})^{\frac{2}{p}-1}\\
\le\ &C\omega(\Omega^{0,0})^{\frac{2}{p}-1} \le C\omega(\Omega)^{\frac{2}{p}-1}.
\end{aligned}$$

Thus, combining the above 6 cases, I_1 in (5.16) can be estimated as

$$\begin{aligned}
I_1 \le\ & \frac{1}{\omega(\Omega)^{\frac{2}{p}-1}}\sum_h \sum_{Q'\in\mathcal{D}_h^{0,0}}\left(\sum_{\text{Case 1}}+\cdots+\sum_{\text{Case 5}}\right) r(Q,Q')P(Q,Q')\cdot F_{Q'}\frac{|Q'|^2}{\omega(Q')}\\
&+\frac{1}{\omega(\Omega)^{\frac{2}{p}-1}}\sum_h \sum_{Q'\in\mathcal{D}_h^{0,0}}\sum_{\text{Case 6}} r(Q,Q')P(Q,Q')\cdot F_{Q'}\frac{|Q'|^2}{\omega(Q')}\\
\le\ & C\frac{1}{\omega(\Omega)^{\frac{2}{p}-1}}\sum_h \sum_{Q'\in\mathcal{D}_h^{0,0}} 2^{-h(\frac{L}{2}-q_\omega-3)}\cdot F_{Q'}\frac{|Q'|^2}{\omega(Q')}\\
&+\frac{1}{\omega(\Omega)^{\frac{2}{p}-1}}\sum_{h=0}^{3} \sum_{Q'\in\mathcal{D}_h^{0,0}} F_{Q'}\frac{|Q'|^2}{\omega(Q')}\\
\le\ & C\frac{1}{\omega(\Omega)^{\frac{2}{p}-1}}\sum_h 2^{-h(\frac{L}{2}-q_\omega-3)}\omega(\Omega_h^{0,0})^{\frac{2}{p}-1}\frac{1}{\omega(\Omega_h^{0,0})^{\frac{2}{p}-1}}\sum_{Q'\subseteq\Omega_h^{0,0}} F_{Q'}\frac{|Q'|^2}{\omega(Q')}\\
&+\frac{1}{\omega(\Omega)^{\frac{2}{p}-1}}\sum_{h=0}^{3}\omega(\Omega_h^{0,0})^{\frac{2}{p}-1}\frac{1}{\omega(\Omega_h^{0,0})^{\frac{2}{p}-1}}\sum_{Q'\subseteq\Omega_h^{0,0}} F_{Q'}\frac{|Q'|^2}{\omega(Q')}\\
\le\ & C\frac{1}{\omega(\Omega)^{\frac{2}{p}-1}}\sum_h 2^{-h(\frac{L}{2}-q_\omega-3)}(h^2 2^h)^{\frac{2}{p}-1}\omega(\Omega)^{\frac{2}{p}-1}\sup_{\Omega^*}\frac{1}{\omega(\Omega^*)^{\frac{2}{p}-1}}\sum_{Q'\subseteq\Omega^*} F_{Q'}\frac{|Q'|^2}{\omega(Q')}\\
&+\frac{1}{\omega(\Omega)^{\frac{2}{p}-1}}\sum_{h=0}^{3}(h^2 2^h)^{\frac{2}{p}-1}\omega(\Omega)^{\frac{2}{p}-1}\sup_{\Omega^*}\frac{1}{\omega(\Omega^*)^{\frac{2}{p}-1}}\sum_{Q'\subseteq\Omega^*} F_{Q'}\frac{|Q'|^2}{\omega(Q')}\\
\le\ & C\sup_{\Omega^*}\frac{1}{\omega(\Omega^*)^{\frac{2}{p}-1}}\sum_{Q'\subseteq\Omega^*} F_{Q'}\frac{|Q'|^2}{\omega(Q')},
\end{aligned}$$

where we choose L large enough, and the estimate of I_1 is finished. Next we move our attention to the estimate of I_4.

(**iv**) To estimate

$$I_4 = \frac{1}{\omega(\Omega)^{\frac{2}{p}-1}} \sum_h \sum_{Q' \in \mathcal{D}_h^{0,0}} \sum_{v' \geq 1} \sum_{Q \in A_{0,0,v'}(Q')} r(Q,Q')P(Q,Q') \cdot F_{Q'} \frac{|Q'|^2}{\omega(Q')}, \tag{5.17}$$

similar to what we did in (**i**), we divide $\{Q \in A_{0,0,v'}(Q')\}$ into 6 cases for each $v' \geq 1$.

Case 1: $|I'| \geq |I|$, $|J'| \leq |J|$, and $|R'| \leq |R|$.

Note that in this case, $3(I' \times J' \times R') \cap 3(I \times J \times 2^{v'}R) \neq \emptyset$. Since

$$\frac{|I|}{|3I'|}|3Q'| = |I| \times |3J'| \times |3R'| \leq |3Q' \cap \Omega^{0,0}| < \frac{1}{2^{h-1}}|3Q'|,$$

then $|I'| \sim 2^{h-1+n}|I|$ for some $n \geq 0$, and for each n, the number of such I''s is no more than $5 \cdot 2^n$. As for J, $|J| \sim 2^m|J'|$. And for each m, the number of such J's is no more than 5. Note that $2^{v'-1}|R| < d(R,R') \leq 2^{v'}|R|$, which yields that $3R' \cap 3 \cdot 2^{v'}R \neq \emptyset$. Moreover, from $|R| \sim 2^{-(h-1+n)}2^m|R'|$ and $|R| \geq |R'|$, we have $m > h-1+n$ and for each fixed v', n and m, the number of such R's is less than $5 \cdot 2^{v'}$. Thus,

$$\begin{aligned}
&\sum_{\text{Case 1}} r(Q,Q')P(Q,Q') \\
\leq\ & C \sum_{\text{Case 1}} C \sum_{\text{Case 1}} \left(\frac{|I|}{|I'|}\right)^{L-q_\omega-2} \left(\frac{|J'|}{|J|}\right)^{L-q_\omega-2} \left(\frac{|R'|}{|R|}\right)^{-q_\omega-2} \\
& \frac{(|J|)^{M+1}}{\left(|J| + \frac{2^{v'-1}|R|}{|I|}\right)^{M+1}} \frac{|R|}{|I||J| + 2^{v'-1}|R|}
\end{aligned}$$

$$
\begin{aligned}
&\leq C\sum_{n\geq 0}\sum_{m\geq 0} 2^{-(h-1+n)(L-q_\omega-2)}\cdot 2^{(h-1+n-m)(-q_\omega-2)}\cdot 2^{-m(L-q_\omega-2)}\cdot 2^{n}\cdot 2^{-v'(M+2)}\\
&\leq C\sum_{n\geq 0}\sum_{m\geq 0} 2^{-hL}\cdot 2^{-m(L-2q_\omega-4)}\cdot 2^{-n(L-1)}\cdot 2^{L}\cdot 2^{-v'(M+2)}\\
&\leq C2^{-hL}2^{-v'(M+2)}.
\end{aligned}
$$

Case 2: $|I'|\geq|I|$, $|J'|\leq|J|$, and $|R'|\geq|R|$.

Note that in this case, $3(I'\times J'\times 2^{v'}R')\cap 3(I\times J\times R)\neq\emptyset$. Since

$$
\frac{|I||R|}{|3I'||3R'|}|3Q'| = |I|\times|3J'|\times|R| \leq |3Q'\cap\Omega^{0,0}| < \frac{1}{2^{h-1}}|3Q'|,
$$

then $|I'||R'|\sim 2^{h-1+n}|I||R|$. As for J, $|J|\sim 2^m|J'|$. So for each m, the number of such J's is no more than 5. Noting that $|R|=|I|\times|J|$ and $|R'|=|I'|\times|J'|$, we have $|I'||I'||J'|\sim 2^{h-1+n}|I||I||J|$, which yields that $|I'|^2\sim 2^{h-1+n+m}|I|^2$, that is, $|I'|\sim 2^{(h-1+n+m)/2}|I|$. Hence for each n and m, the number of such I's is less than $5\cdot 2^{(h+m+n)/2}$. Also we can obtain that $|R'|\sim 2^{(h-1+n-m)/2}|R|$. Since $|R'|\geq|R|$, we have $m\leq h-1+n$. Moreover, note that $2^{v'-1}|R'|<d(R,R')\leq 2^{v'}|R'|$, which yields that $3\cdot 2^{v'}R'\cap 3R\neq\emptyset$. For each fixed v', n and m, the number of such R's is less than $5\cdot 2^{(h+n-m)/2}2^{v'}$. Thus,

$$
\begin{aligned}
&\sum_{\text{Case 2}} r(Q,Q')P(Q,Q')\\
\leq\ & C\sum_{\text{Case 1}}\left(\frac{|I|}{|I'|}\right)^{L-q_\omega-2}\left(\frac{|J'|}{|J|}\right)^{L-q_\omega-2}\left(\frac{|R'|}{|R|}\right)^{-q_\omega-2}\\
&\cdot\frac{(|J|)^{M+1}}{\left(|J|+\frac{2^{v'-1}|R'|}{|I|}\right)^{M+1}}\frac{|R'|}{|I||J|+2^{v'-1}|R'|}
\end{aligned}
$$

$$\begin{aligned}
&\leq C\sum_{n\geq 0}\sum_{m\geq 0} 2^{-\frac{h-1+n}{2}(L-q_\omega-2)}\cdot 2^{-m(L-q_\omega-2)}\cdot 2^{-\frac{h-1+n-m}{2}(-q_\omega-2)} \\
&\quad \cdot 2^{h+n}2^{v'}2^{-v'(M+2)} \\
&= C\sum_{n\geq 0}\sum_{m\geq 0} 2^{-h(\frac{L}{2}-q_\omega-1)}\cdot 2^{-m(\frac{3L}{2}-q_\omega-4)}\cdot 2^{-n(\frac{L}{2}-q_\omega-1)}\cdot 2^{\frac{L}{2}-q} \\
&\leq C2^{-h(\frac{L}{2}-q_\omega-1)}2^{-v'(M+1)}.
\end{aligned}$$

Case 3: $|I'|\leq |I|$, $|J'|\geq |J|$, and $|R'|\leq |R|$.

This can be handled similarly as Case 1, and we have

$$\sum_{\text{Case 3}} r(Q,Q')P(Q,Q')\leq C2^{-hL}2^{-v'(M+2)}.$$

Case 4: $|I'|\leq |I|$, $|J'|\geq |J|$, and $|R'|\geq |R|$.

This can be handled similarly as Case 2, and we have

$$\sum_{\text{Case 4}} r(Q,Q')P(Q,Q')\leq C2^{-h(\frac{L}{2}-q_\omega-1)}2^{-v'(M+1)}.$$

Case 5: $|I'|\geq |I|$, $|J'|\geq |J|$, and thus $|R'|\geq |R|$.

Since

$$|I|\times|J|\times|R|\leq |3Q'\cap\Omega^{0,0}|<\frac{1}{2^{h-1}}|3Q'|,$$

then $|Q'|\sim 2^{h-1+n}|Q|$. And for each v' and n, the number of such Q's is less than $2^{v'}(2^n)^3$.

More precisely, we have $(|I'||J'|)^2 \sim 2^{h-1+n}(|I||J|)^2$. Thus,

$$
\begin{aligned}
& \sum_{\text{Case 5}} r(Q,Q')P(Q,Q') \\
\leq\ & C \sum_{\text{Case 5}} \left(\frac{|I||J|}{|I'||J'|}\right)^{L-q_\omega-2} \left(\frac{|R|}{|R'|}\right)^{-q_\omega-2} \\
& \cdot \frac{(|J'|)^{M+1}}{(|J'|+\frac{2^{v'-1}|R'|}{|I'|})^{M+1}} \frac{|R'|}{|I'||J'|+2^{v'-1}|R'|} \\
\leq\ & C \sum_{n\geq 0} 2^{\frac{-(h-1+n)}{2(L-q_\omega-2)}} \cdot 2^{\frac{-(h-1+n)}{2(-q_\omega-2)}} \cdot 2^{v'} 2^{3n} 2^{-v'(M+2)} \\
\leq\ & C 2^{-h(\frac{L}{2}-q_\omega-2)} \cdot 2^{-n(\frac{L}{2}-q_\omega-5)} \cdot 2^{\frac{L}{2}-q_\omega-2} \cdot 2^{-v'(M+1)} \\
\leq\ & C 2^{-h\frac{L}{2-q_\omega-2}} 2^{-v'(M+1)}.
\end{aligned}
$$

Case 6: $|I'| \leq |I|$, $|J'| \leq |J|$, and thus $|R'| \leq |R|$.

Since

$$
|I'| \times |J'| \times |R'| \leq |3Q' \cap \Omega^{0,0,}| < \frac{1}{2^{h-1}}|3Q'|,
$$

then we can see that in this case, h must be less than 3. And from $|I'| \leq |I|$, we have $|I| \sim 2^n|I'|$ and for each n, the number of such I is less than 5. Similarly, from $|J'| \leq |J|$, we have $|J| \sim 2^m|J'|$ and for each m, the number of such J is less than 5. Hence we have

$|R| \sim 2^{n+m}|R'|$, and for each v', n, m, the number of such R is less than $5 \cdot 2^{v'}$. Thus,

$$
\begin{aligned}
& \sum_{\text{Case } 6} r(Q,Q')P(Q,Q') \\
\leq \; & C \sum_{\text{Case } 6} \left(\frac{|I'||J'|}{|I||J|}\right)^{L-q_\omega-2} \left(\frac{|R'|}{|R|}\right)^{-q_\omega-2} \\
& \cdot \frac{(|J|)^{M+1}}{(|J| + \frac{2^{v'-1}|R|}{|I|})^{M+1}} \frac{|R|}{|I||J| + 2^{v'-1}|R|} \\
\leq \; & C \sum_{n\geq 0} \sum_{m \geq 0} 2^{-n(L-q_\omega-2)} \cdot 2^{-m(L-q_\omega-2)} \cdot 2^{(-n-m)(-q_\omega-2)} \cdot 2^{v'} 2^{-v'(M+2)} \\
\leq \; & C 2^{-v'(M+1)}.
\end{aligned}
$$

Thus, combining the above 6 cases, by choosing L and M large enough, I_4 in (5.17) becomes

$$I_4 \leq C \sup_{\Omega^*} \frac{1}{\omega(\Omega^*)^{\frac{2}{p}-1}} \sum_{Q' \subseteq \Omega^*} F_{Q'} \frac{|Q'|^2}{\omega(Q')}.$$

Using the same techniques, we are able to control the 8 integrates for I, therefore give the estimate for I, that is,

$$I \leq C \sup_{\Omega^*} \frac{1}{\omega(\Omega^*)^{\frac{2}{p}-1}} \sum_{Q' \subseteq \Omega^*} F_{Q'} \frac{|Q'|^2}{\omega(Q')}.$$

Without any difficulty, II, III, IV in (5.13) can be calculated similarly and bounded by the right hand side of the above inequality. Hence the proof of the theorem 5.4 is complete.

$\square$

Finally we show that $CMO^p_{\mathcal{Z}}(\omega)$ is well defined as a corollary of sup-inf comparison principle of second kind.

Corollary 5.9. *The definition of $CMO^p_{\mathcal{Z}}(\omega)$ in Definition 5.3 is independent of the choice of $\psi_{j,k}$, therefore it is well-defined.*

5.4 Proof of the duality theorem

In this section, we prove the $(H^p_{\mathcal{Z}}(\omega))^* - CMO^p_{\mathcal{Z}}(\omega)$ duality theorem, i.e., Theorem 5.5, and we need to introduce two sequence spaces.

Definition 5.10 ($s^p_{\mathcal{Z}}(\omega)$ and $c^p_{\mathcal{Z}}(\omega)$). *Let $\omega \in A_\infty(\mathcal{Z})$, $j,k \in \mathbb{Z}$, and $I \subseteq \mathbb{R}$, $J \subseteq \mathbb{R}$, and $R \subseteq \mathbb{R}$ are dyadic intervals with interval-length $\ell(I) = 2^{-j-N}, \ell(J) = 2^{-k-N}$, and $\ell(R) = 2^{-j-k-2N}$ for a fixed large positive integer N. The sequence $s = \{s_{I\times J\times R}\}$ is said to be in the sequence space $s^p_{\mathcal{Z}}(\omega)$ if*

$$\|s\|_{s^p_{\mathcal{Z}}(\omega)} = \| \left\{ \sum_{j,k} \sum_{I\times J\times R} |s_{I\times J\times R}|^2 |I|^{-1}|J|^{-1}|R|^{-1} \chi_I(x)\chi_J(y)\chi_R(z) \right\}^{\frac{1}{2}} \|_{L^p_\omega} < \infty, \tag{5.18}$$

and the sequence $t = \{t_{I\times J\times R}\}$ is said to be in the sequence space $c^p_{\mathcal{Z}}(\omega)$ if

$$\|t\|_{c^p_{\mathcal{Z}}(\omega)} = \sup_\Omega \left\{ \frac{1}{\omega(\Omega)^{\frac{2}{p}-1}} \sum_{j,k} \sum_{I\times J\times R\subseteq\Omega} |t_{I\times J\times R}|^2 \frac{|I\times J\times R|}{\omega(I\times J\times R)} \right\}^{\frac{1}{2}} < \infty, \tag{5.19}$$

for all open sets Ω in $\mathbb{R}^3$ with finite weighted measures, and $I\times J\times R$ run over all dyadic cubes with side-lengths defined above.

We now derive the following duality theorem for these sequence spaces.

Theorem 5.11 ($s^p_{\mathcal{Z}}(\omega) - c^p_{\mathcal{Z}}(\omega)$). *$(s^p_{\mathcal{Z}}(\omega))^* = c^p_{\mathcal{Z}}(\omega)$, precisely, let $\omega \in A_\infty(\mathcal{Z})$ and $0 < p \le 1$, the map which maps $s = \{s_{I\times J\times R}\}$ to $< s,t >= \sum_{I\times J\times R} s_{I\times J\times R}\bar{t}_{I\times J\times R}$ defines a continuous* *linear functional on* $s^p_{\mathcal{Z}}(\omega)$ *with operator norm* $\|t\|_{(s^p_{\mathcal{Z}}(\omega))^*} \approx \|t\|_{c^p_{\mathcal{Z}}(\omega)}$, *and every* $\ell \in (s^p_{\mathcal{Z}}(\omega))^*$ *is of this form for some* $t \in c^p_{\mathcal{Z}}(\omega)$.

Proof of Theorem 5.11. First we show that $c^p_{\mathcal{Z}}(\omega) \subseteq (s^p_{\mathcal{Z}}(\omega))^*$. Suppose $t = \{t_{I\times J\times R}\} \in c^p_{\mathcal{Z}}(\omega)$ and $s = \{s_{I\times J\times R}\} \in s^p_{\mathcal{Z}}(\omega)$, set

$$h(x,y,z) = \left\{ \sum_{j,k} \sum_{I\times J\times R} |s_{I\times J\times R}|^2 |I|^{-1}|J|^{-1}|R|^{-1}\chi_I(x)\chi_J(y)\chi_R(z) \right\}^{\frac{1}{2}},$$

which means $\|s\|_{s^p_{\mathcal{Z}}(\omega)} = \|h\|_{L^p_\omega}$. Then we write $\Omega_i = \{(x,y,z) : h(x,y,z) > 2^i\}$, and

$$B_i = \{I\times J\times R : \omega(I\times J\times R \cap \Omega_i) > \frac{1}{2}\omega(I\times J\times R), \omega(I\times J\times R\cap \Omega_{i+1}) \le \frac{1}{2}\omega(I\times J\times R)\}.$$

Thus,

$$\sum_{j,k}\sum_{I\times J\times R} s_{I\times J\times R}\bar{t}_{I\times J\times R} = \sum_i \sum_{I\times J\times R\in B_i} s_{I\times J\times R}\bar{t}_{I\times J\times R}.$$

Note that $0 < p \le 1$, by Cauchy-Schwartz's inequality,

$$\begin{aligned} & |\sum_i \sum_{I\times J\times R\in B_i} s_{I\times J\times R}\bar{t}_{I\times J\times R}| \\ \le\ & \sum_i \left(\sum_{I\times J\times R\in B_i} |s_{I\times J\times R}|^2 \frac{\omega(I\times J\times R)}{|I\times J\times R|} \right)^{\frac{1}{2}} \left(\sum_{I\times J\times R\in B_i} |\bar{t}_{I\times J\times R}|^2 \frac{|I\times J\times R|}{\omega(I\times J\times R)} \right)^{\frac{1}{2}} \end{aligned}$$

$$\leq \left\{ \sum_i \left(\sum_{I\times J\times R\in B_i} |s_{I\times J\times R}|^2 \frac{\omega(I\times J\times R)}{|I\times J\times R|} \right)^{\frac{p}{2}} \left(\sum_{I\times J\times R\in B_i} |t_{I\times J\times R}|^2 \frac{|I\times J\times R|}{\omega(I\times J\times R)} \right)^{\frac{p}{2}} \right\}^{\frac{1}{p}}$$
$$\leq C\|t\|_{c^p_{\mathcal{Z}}(\omega)} \left\{ \sum_i \omega(\Omega_i)^{1-\frac{p}{2}} \left(\sum_{I\times J\times R\in B_i} |s_{I\times J\times R}|^2 \frac{\omega(I\times J\times R)}{|I\times J\times R|} \right)^{\frac{p}{2}} \right\}^{\frac{1}{p}}.$$

Where the last inequality follows from the fact that if $I\times J\times R\in B_i$, then there exists $0<\theta<1$ such that

$$I\times J\times R\subseteq \{(x,y,z): \mathcal{M}_{\mathcal{Z}}(\chi_{\Omega_i})(x,y,z)>\theta\} \triangleq \widetilde{\Omega}_i,$$

together with $\omega(\widetilde{\Omega}_i)\leq C\omega(\Omega_i)$, imply

$$\left(\sum_{I\times J\times R\in B_i} |t_{I\times J\times R}|^2 \frac{|I\times J\times R|}{\omega(I\times J\times R)} \right)^{\frac{1}{2}} \leq C\|t\|_{c^p_{\mathcal{Z}}(\omega)} \omega(\Omega_i)^{\frac{1}{p}-\frac{1}{2}}.$$

We claim for now

$$\sum_{I\times J\times R\in B_i} |s_{I\times J\times R}|^2 \frac{\omega(I\times J\times R)}{|I\times J\times R|} \leq C2^{2i}\omega(\Omega_i). \tag{5.20}$$

Assume this claim for the moment, then

$$\begin{aligned} & |\sum_i \sum_{I\times J\times R\in B_i} s_{I\times J\times R}\bar{t}_{I\times J\times R}| \\ \leq\ & C\|t\|_{c^p_{\mathcal{Z}}(\omega)} (\sum_i 2^{ip}\omega(\Omega_i))^{\frac{1}{p}} \\ \leq\ & C\|t\|_{c^p_{\mathcal{Z}}(\omega)} \|h\|_{L^p_\omega} \\ \leq\ & C\|t\|_{c^p_{\mathcal{Z}}(\omega)} \|s\|_{s^p_{\mathcal{Z}}(\omega)}, \end{aligned}$$

therefore $c^p_{\mathcal{Z}}(\omega) \subseteq (s^p_{\mathcal{Z}}(\omega))^*$. To show the claim (5.20), it is sufficient to prove

$$\sum_{I\times J\times R\in B_i} |s_{I\times J\times R}|^2 \frac{\omega(I\times J\times R)}{|I\times J\times R|} \leq C\int_{\widetilde{\Omega}_i\backslash\Omega_{i+1}} h^2(x,y,z)\omega dxdydz$$

because

$$\int_{\widetilde{\Omega}_i\backslash\Omega_{i+1}} h^2(x,y,z)\omega dxdydz \leq 2^{2(i+1)}\omega(\widetilde{\Omega}_i) \leq C2^{2i}\omega(\Omega_i).$$

However,

$$\begin{aligned}
& \int_{\widetilde{\Omega}_i\backslash\Omega_{i+1}} h^2(x,y,z)\omega dxdydz \\
= \quad & \int_{\widetilde{\Omega}_i\backslash\Omega_{i+1}} \sum_{I\times J\times R} |s_{I\times J\times R}|^2 |I|^{-1}|J|^{-1}|R|^{-1}\chi_I(x)\chi_J(y)\chi_R(z)\omega dxdydz \\
\geq \quad & \sum_{I\times J\times R\in B_i} |s_{I\times J\times R}|^2 \frac{\omega((\widetilde{\Omega}_i\backslash\Omega_{i+1})\cap(I\times J\times R))}{\omega(I\times J\times R)} \frac{\omega(I\times J\times R)}{|I\times J\times R|} \\
\geq \quad & \frac{1}{2}\sum_{I\times J\times R\in B_i} |s_{I\times J\times R}|^2 \frac{\omega(I\times J\times R)}{|I\times J\times R|},
\end{aligned}$$

since for $I\times J\times R\in B_i$,

$$\omega((\widetilde{\Omega}_i \cap I\times J\times R)) > \frac{1}{2}\omega(I\times J\times R),$$

and

$$\omega((\Omega_{i+1} \cap I\times J\times R)) \leq \frac{1}{2}\omega(I\times J\times R).$$

Then $I\times J\times R\in\widetilde{\Omega}_i$, hence

$$\omega((\widetilde{\Omega}_i\backslash\Omega_{i+1})\cap(I\times J\times R)) > \frac{1}{2}\omega(I\times J\times R).$$

The claim is verified.

Next we shall prove that $(s^p_{\mathcal{Z}}(\omega))^* \subseteq c^p_{\mathcal{Z}}(\omega)$. Let $\ell \in (s^p_{\mathcal{Z}}(\omega))^*$, then there exists some $t = \{t_{I\times J\times R}\}$ such that $\forall s = \{s_{I\times J\times R}\} \in s^p_{\mathcal{Z}}(\omega)$,

$$\ell(s) = \sum_{I\times J\times R} s_{I\times J\times R}\bar{t}_{I\times J\times R}.$$

For an open set Ω in $\mathbb{R}^3$ with $\omega(\Omega) < \infty$, define

$$\|s\|_{s^p_{\mathcal{Z},\Omega}(\omega)} = \left\{\int_\Omega \left(\sum_{I\times J\times R\subseteq\Omega} |s_{I\times J\times R}|^2|I|^{-1}|J|^{-1}|R|^{-1}\chi_I(x)\chi_J(y)\chi_R(z)\right)^{\frac{p}{2}} \omega dxdydz\right\}^{\frac{1}{p}},$$

and

$$\|s\|_{\ell^2_{\mathcal{Z},\Omega}(\omega)} = \left(\sum_{I\times J\times R\subseteq\Omega} |s_{I\times J\times R}|^2\frac{\omega(I\times J\times R)}{|I\times J\times R|}\right)^{\frac{1}{2}}.$$

Then, by Hölder's inequality,

$$\begin{aligned}
&\|s\|_{s^p_{\mathcal{Z},\Omega}(\omega)}\\
= \;& \left\{\int_\Omega \left(\sum_{I\times J\times R\subseteq\Omega} |s_{I\times J\times R}|^2|I|^{-1}|J|^{-1}|R|^{-1}\chi_I(x)\chi_J(y)\chi_R(z)\right)^{\frac{p}{2}} \omega dxdydz\right\}^{\frac{1}{p}}\\
\leq \;& \omega(\Omega)^{\frac{1}{p}-\frac{1}{2}}\left\{\int_\Omega \sum_{I\times J\times R\subseteq\Omega} |s_{I\times J\times R}|^2|I|^{-1}|J|^{-1}|R|^{-1}\chi_I(x)\chi_J(y)\chi_R(z)\omega dxdydz\right\}^{\frac{1}{2}}\\
= \;& \omega(\Omega)^{\frac{1}{p}-\frac{1}{2}}\left(\sum_{I\times J\times R\subseteq\Omega} |s_{I\times J\times R}|^2\frac{\omega(I\times J\times R)}{|I\times J\times R|}\right)^{\frac{1}{2}}\\
= \;& \omega(\Omega)^{\frac{1}{p}-\frac{1}{2}}\|s\|_{\ell^2_{\mathcal{Z},\Omega}(\omega)}.
\end{aligned}$$

Thus, we compute

$$
\begin{aligned}
& \left\{ \frac{1}{\omega(\Omega)^{\frac{2}{p}-1}} \sum_{j,k} \sum_{I\times J\times R\subseteq\Omega} |t_{I\times J\times R}|^2 \frac{|I\times J\times R|}{\omega(I\times J\times R)} \right\}^{\frac{1}{2}} \\
= \;& \frac{1}{\omega(\Omega)^{\frac{1}{p}-\frac{1}{2}}} \sup_{\|s\|_{\ell^2_{\mathcal{Z},\Omega}(\omega)}\leq 1} \Big| \sum_{I\times J\times R\subseteq\Omega} s_{I\times J\times R}\bar{t}_{I\times J\times R} \Big| \\
\leq \;& \frac{1}{\omega(\Omega)^{\frac{1}{p}-\frac{1}{2}}} \sup_{\|s\|_{\ell^2_{\mathcal{Z},\Omega}(\omega)}\leq 1} \|t\|_{(s^p_{\mathcal{Z}}(\omega))^*} \|s_{I\times J\times R}\|_{s^p_{\mathcal{Z},\Omega}(\omega)} \\
= \;& \|t\|_{(s^p_{\mathcal{Z}}(\omega))^*} \sup_{\|s\|_{\ell^2_{\mathcal{Z},\Omega}(\omega)}\leq 1} \frac{1}{\omega(\Omega)^{\frac{1}{p}-\frac{1}{2}}} \|s_{I\times J\times R}\|_{s^p_{\mathcal{Z},\Omega}(\omega)} \\
= \;& \|t\|_{(s^p_{\mathcal{Z}}(\omega))^*} \sup_{\|s\|_{\ell^2_{\mathcal{Z},\Omega}(\omega)}\leq 1} \|s_{I\times J\times R}\|_{\ell^2_{\mathcal{Z},\Omega}(\omega)} \\
\leq \;& \|t\|_{(s^p_{\mathcal{Z}}(\omega))^*}
\end{aligned}
$$

for all Ω. Therefore, by taking the superium, $t \in c^p_{\mathcal{Z}}(\omega)$ and $\|t\|_{c^p_{\mathcal{Z}}(\omega)} \leq \|t\|_{(s^p_{\mathcal{Z}}(\omega))^*}$, which implies $(s^p_{\mathcal{Z}}(\omega))^* \subseteq c^p_{\mathcal{Z}}(\omega)$, and thus the proof of Theorem 5.11 is complete.

□

In order to pass the duality theory from sequence spaces to $H^p_{\mathcal{Z}}(\omega)$ and $CMO^p_{\mathcal{Z}}(\omega)$, we need the following lemmas.

Lemma 5.12. *Given large positive integer N and integers $j, k, j', k' \in \mathbb{Z}$. Let $I, J, R, I', J', R' \subseteq \mathbb{R}$ are dyadic intervals with interval-length $\ell(I) = 2^{-j-N}$, $\ell(J) = 2^{-k-N}$, $\ell(R) = 2^{-j-k-2N}$, $\ell(I') = 2^{-j'-N}$, $\ell(J') = 2^{-k'-N}$, and $\ell(R') = 2^{-j'-k'-2N}$. Let $\{a_{I',J',R'}\}$ be any given sequence, $x_{I'} \in I'$, $y_{J'} \in J'$, and $z_{J'} \in R'$ be any points. Then for any $u, u^* \in I$, $v, v^* \in J$, $w, w^* \in R$*

we have

$$\sum_{I',J',R'} \frac{2^{-(j\wedge j')M_1}2^{-(k\wedge k')M_2}|I'||J'||R'|}{(2^{-(j\wedge j')}+|u-x_{I'}|)^{1+M_1}2^{-j^*}(2^{-(k\wedge k')}+|v-y_{J'}|+\frac{|w-z_{R'}|}{2^{-j^*}})^{2+M_2}}|a_{I',J',R'}|$$
$$\leq \quad C2^{4N(\frac{1}{r}-1)}2^{\tau}\left\{\mathcal{M}_{\mathcal{Z}}(\sum_{I',J',R'}|a_{I',J',R'}|^r\chi_{I'}\chi_{J'}\chi_{R'})(u^*,v^*,w^*)\right\}^{1/r},$$

where $j^*=j$ *if* $k<k'$, *and* $j^*=j'$ *if* $k\geq k'$. $\mathcal{M}_{\mathcal{Z}}$ *is the maximal operator associated with Zygmund dilations defined in (5.1), and* $\max\{\frac{2}{1+M_1},\frac{2}{2+M_2}\}<r\leq 1$. *The summation is taken for all* I',J',R' *with the fixed side-length.* τ *is defined as follows,*

$$\tau=\begin{cases}(\frac{2}{r}-2)(j'+k'-j-k) & \text{if } j<j' \text{ and } k<k',\\ (\frac{2}{r}-1)(j'-j) & \text{if } j<j' \text{ and } k\geq k',\\ j-j'+(\frac{2}{r}-2)(k'-k) & \text{if } j\geq j' \text{ and } k<k',\\ 0 & \text{if } j\geq j' \text{ and } k\geq k'.\end{cases}$$

The detailed proof of Lemma 5.12 can be found in [HL2].

Lemma 5.13. *Let* $\omega\in A_\infty(\mathcal{Z})$, $j,k\in\mathbb{Z}$, $\psi_{j,k}$ *be same as in (5.3) and* $I\subseteq\mathbb{R}$, $J\subseteq\mathbb{R}$, *and* $R\subseteq\mathbb{R}$ *are dyadic intervals with interval-length* $\ell(I)=2^{-j-N},\ell(J)=2^{-k-N}$, *and* $\ell(R)=2^{-j-k-2N}$ *for a fixed large positive integer* N. *Define a map* S *on* $(\mathcal{S}_{\mathcal{Z}}(\mathbb{R}^3))'$ *by*

$$h(f)=\left\{|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}|R|^{\frac{1}{2}}\psi_{j,k}*f(x_I,y_J,z_R)\right\}.$$

For any sequence $s = \{s_{I\times J\times R}\}$*, we define the map* T *by*

$$T(s) = \sum_{j,k}\sum_{I\times J\times R} s_{I\times J\times R}|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}|R|^{\frac{1}{2}}\widetilde{\psi}_{j,k}(x,y,z,x_I,y_J,z_R),$$

where $\widetilde{\psi}_{j,k}$ *are same as in the discrete Calderón's identity in Theorem 5.6.*

Then, S *is bounded from* $H^p_{\mathcal{Z}}(\omega)$ *to* $s^p_{\mathcal{Z}}(\omega)$*, and from* $CMO^p_{\mathcal{Z}}(\omega)$ *to* $c^p_{\mathcal{Z}}(\omega)$*. While* T *is bounded from* $s^p_{\mathcal{Z}}(\omega)$ *to* $H^p_{\mathcal{Z}}(\omega)$*, and from* $c^p_{\mathcal{Z}}(\omega)$ *to* $CMO^p_{\mathcal{Z}}(\omega)$*. Moreover,* $T\circ S$ *is the identity map on both* $H^p_{\mathcal{Z}}(\omega)$ *and* $CMO^p_{\mathcal{Z}}(\omega)$*.*

Proof of Lemma 5.13. If $f \in H^p_{\mathcal{Z}}(\omega)$, then by the definition of $H^p_{\mathcal{Z}}(\omega)$ in Definition 5.2 together with discrete Littlewood-Paley-Stein square function (5.8),

$$\begin{aligned}
&\|h(f)\|_{s^p_{\mathcal{Z}}(\omega)}\\
=\ &\|\left\{\sum_{j,k}\sum_{I\times J\times R}|h(f)_{I\times J\times R}|^2|I|^{-1}|J|^{-1}|R|^{-1}\chi_I(x)\chi_J(y)\chi_R(z)\right\}^{\frac{1}{2}}\|_{L^p_\omega}\\
=\ &\|\left\{\sum_{j,k}\sum_{I,J,R}|(\psi_{j,k}*f)(x_I,y_J,z_R)|^2\chi_I(x)\chi_J(y)\chi_R(z)\right\}^{\frac{1}{2}}\|_{L^p_\omega}\\
\le\ &C\|f\|_{H^p_{\mathcal{Z}}(\omega)}.
\end{aligned}$$

Similarly, by the aid of sup-inf comparison principle of second kind in Theorem 5.4, we can show S is bounded from $CMO^p_{\mathcal{Z}}(\omega)$ to $c^p_{\mathcal{Z}}(\omega)$.

To show T is bounded from $s^p_{\mathcal{Z}}(\omega)$ to $H^p_{\mathcal{Z}}(\omega)$, by using almost orthogonality estimate in

Lemma 5.8, we get

$$\begin{aligned}
&\sum_{j',k'}\sum_{I',J',R'}|\psi_{j',k'}*T(s)(x_{I'},y_{J'},z_{R'})|^2\chi_{I'}(x)\chi_{J'}(y)\chi_{R'}(z)\\
=\;&\sum_{j',k'}\sum_{I',J',R'}|\psi_{j',k'}*(\sum_{j,k}\sum_{I\times J\times R}s_{I\times J\times R}|I|^{\frac12}|J|^{\frac12}|R|^{\frac12}\widetilde{\psi}_{j,k}(x,y,z,x_I,y_J,z_R))\\
&(x_{I'},y_{J'},z_{R'})|^2\chi_{I'}(x)\chi_{J'}(y)\chi_{R'}(z)\\
\le\;&\sum_{j',k'}\sum_{I',J',R'}\sum_{j,k}\sum_{I,J,R}2^{-|j-j'|L}2^{-|k-k'|L}|I|^{-\frac12}|J|^{-\frac12}|R|^{-\frac12}\frac{2^{-(j\wedge j')M}}{(2^{-j\wedge j'}+|x_{I'}-x_I|)^{M+1}}\\
&\times\frac{2^{-(k\wedge k')M}}{2^{-j^*}(2^{-k\wedge k'}+|y_{J'}-y_J|+2^{j^*}|z_{R'}-z_R|)^{M+2}}|s_{I\times J\times R}|\chi_{I'}(x)\chi_{J'}(y)\chi_{R'}(z)\\
\le\;&C\sum_{j,k}2^{-|j-j'|L}2^{-|k-k'|L}2^{\tau}\left\{\mathcal{M}_{\mathcal{Z}}(\sum_{I,J,R}|I|^{-\frac12}|J|^{-\frac12}|R|^{-\frac12}|s_{I\times J\times R}|\chi_I\chi_J\chi_R)^r\right\}^{\frac1r},
\end{aligned}$$

in which we applied Lemma 5.12 to get the last inequality, and use the weighted inequalities for vector-valued maximal operator associated with Zygmund dilations, we will be able to derive that

$$\begin{aligned}
&\|T(s)\|_{H^p_{\mathcal{Z}}(\omega)}\\
=\;&\|\left\{\sum_{j',k'}\sum_{I',J',R'}|(\psi_{j',k'}*T(s))(x_I',y_J',z_R')|^2\chi_I'(x)\chi_J'(y)\chi_R'(z)\right\}^{\frac12}\|_{L^p_\omega}\\
\le\;&C\|\sum_{j,k}2^{-|j-j'|L}2^{-|k-k'|L}2^{\tau}\left\{\mathcal{M}_{\mathcal{Z}}(\sum_{I,J,R}|I|^{-\frac12}|J|^{-\frac12}|R|^{-\frac12}|s_{I\times J\times R}|\chi_I\chi_J\chi_R)^r\right\}^{\frac1r}\|_{L^p_\omega}\\
\le\;&C\|\left\{\sum_{j,k}\sum_{I\times J\times R}|s_{I\times J\times R}|^2|I|^{-1}|J|^{-1}|R|^{-1}\chi_I(x)\chi_J(y)\chi_R(z)\right\}^{\frac12}\|_{L^p_\omega}\\
=\;&C\|s\|_{s^p_{\mathcal{Z}}(\omega)}.
\end{aligned}$$

Similarly, we can prove T is bounded from $c^p_{\mathcal{Z}}(\omega)$ to $CMO^p_{\mathcal{Z}}(\omega)$, and it is evident that $T\circ S$ is the identity map on $H^p_{\mathcal{Z}}(\omega)$ and $CMO^p_{\mathcal{Z}}(\omega)$. $\square$

Combining Theorem 5.11 and Lemma 5.13, we are able to prove Theorem 5.5.

Proof of Theorem 5.5. First, if $g \in CMO^p_{\mathcal{Z}}(\omega)$, the map ℓ_g is given by $\ell_g(f) =< f, g >$ for $f \in \mathcal{S}_{\mathcal{Z}}(\mathbb{R}^3)$

$$
\begin{aligned}
& \ell_g(f) = | < f, g > | \\
= \; & | < \sum_{j,k} \sum_{I,J,R} |I||J||R| \widetilde{\psi}_{j,k}(x, y, z, x_I, y_J, z_R)(\psi_{j,k} * f)(x_I, y_J, z_R), g > | \\
= \; & | < S(f), S(g) > | \\
\leq \; & \|S(f)\|_{s^p_{\mathcal{Z}}(\omega)} \|S(g)\|_{c^p_{\mathcal{Z}}(\omega)} \\
\leq \; & C\|f\|_{H^p_{\mathcal{Z}}(\omega)} \|g\|_{CMO^p_{\mathcal{Z}}(\omega)}.
\end{aligned}
$$

Since $S_{\mathcal{Z}}(\mathbb{R}^3)$ is dense in $H^p_{\mathcal{Z}}(\omega)$ (see [HLX2]), Hahn-Banach Theorem implies that the map $\ell_g =< f, g >$ can be extended to a continuous linear functional on $H^p_{\mathcal{Z}}(\omega)$, and $\|\ell_g\| \leq C\|g\|_{CMO^p_{\mathcal{Z}}(\omega)}$.

Conversely, for every $\ell \in (H^p_{\mathcal{Z}}(\omega))^*$, consider $\ell_T = \ell \circ T$ defined on $s^p_{\mathcal{Z}}(\omega)$, and for every $s \in s^p_{\mathcal{Z}}(\omega)$,

$$|\ell_T(s)| = |\ell(T(s))| \leq \|\ell\| \|T(s)\|_{H^p_{\mathcal{Z}}(\omega)} \leq C\|\ell\| \|s\|_{s^p_{\mathcal{Z}}(\omega)},$$

which implies $\ell_T \in (s^p_{\mathcal{Z}}(\omega))^*$, then by Theorem 5.11, there exists $t = \{t_{I\times J\times R}\} \in c^p_{\mathcal{Z}}(\omega)$ such that

$$\ell_T(s) =< s, t >= \sum_{I\times J\times R} s_{I\times J\times R} \bar{t}_{I\times J\times R},$$

for all $s \in s^p_{\mathcal{Z}}(\omega)$, and

$$\|t\|_{c^p_{\mathcal{Z}}(\omega)} \approx \|\ell_T\| \leq C\|\ell\|.$$

From Lemma 5.13, $T \circ S$ is the identity map on $H^p_{\mathcal{Z}}(\omega)$, thus $\ell = \ell \circ T \circ S = \ell_T \circ S$, and

$$\ell(f) = \ell_T(S(f)) =< S(f), t >=< f, g >,$$

in which $g = T(t)$. This shows $\ell = \ell_g$ for $g \in CMO^p_{\mathcal{Z}}(\omega)$, and $\|g\|_{CMO^p_{\mathcal{Z}}(\omega)} \leq C\|t\|_{c^p_{\mathcal{Z}}(\omega)} \leq C\|\ell\|$, which completes the proof of Theorem 5.5.

$\square$

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ABSTRACT

DISCRETE LITTLEWOOD-PALEY-STEIN THEORY AND WOLFF POTENTIALS ON HOMOGENEOUS SPACES AND MULTI-PARAMETER HARDY SPACES

by

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AUGUST 2013

Advisor: Dr. Guozhen Lu

Major: Mathematics

Degree: Doctor of Philosophy

Part I Let $\mathcal{X}, \mathcal{X}_1$ and $\mathcal{X}_2$ be the spaces of homogeneous type, by using the discrete harmonic analysis, we

- derive a new (p, q) - atomic decomposition on the multi-parameter Hardy space $H^p(\mathcal{X}_1 \times \mathcal{X}_2)$ for $0 < p \leq 1$ and all $1 < q < \infty$, where this decomposition converges in both $L^q(\mathcal{X}_1 \times \mathcal{X}_2)$ (for $1 < q < \infty$) and Hardy space $H^p(\mathcal{X}_1 \times \mathcal{X}_2)$ (for $0 < p \leq 1$).

- prove that an operator T, which is bounded on $L^q(\mathcal{X}_1 \times \mathcal{X}_2)$ for some $1 < q < \infty$, is bounded from $H^p(\mathcal{X}_1 \times \mathcal{X}_2)$ to $L^p(\mathcal{X}_1 \times \mathcal{X}_2)$ if and only if T is bounded uniformly on all (p, q)-product atoms in $L^p(\mathcal{X}_1 \times \mathcal{X}_2)$. The similar boundedness criterion from $H^p(\mathcal{X}_1 \times \mathcal{X}_2)$ to $H^p(\mathcal{X}_1 \times \mathcal{X}_2)$ is also obtained.

- compare the Wolff and Riesz potentials on $\mathcal{X}$ and get an associated Hardy-Littlewood-Sobolev type inequality. Applying this inequality, we derive integrability estimates of positive solutions to the Lane-Emden type integral system on $\mathcal{X}$. system,

Part II By applying the discrete Littlewood-Paley-Stein analysis, we establish a $(p, 2)$-atomic decomposition of Hardy spaces associated with different homogeneities. In addition, we prove the duality theorem of weighted multi-parameter Hardy spaces associated with Zygmund dilations, i.e., $(H^p_{\mathcal{Z}}(\omega))^* = CMO^p_{\mathcal{Z}}(\omega)$ for $0 < p \leq 1$. Our theorems extend the weighted Hardy spaces the $H^p_{\mathcal{Z}} - CMO^p_{\mathcal{Z}}$ duality established in [HL2] for non-weighted multi-parameter Hardy spaces associated with the Zygmund dilation.

AUTOBIOGRAPHICAL STATEMENT

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Education

- 2013, Ph.D. in Mathematics (*expected*)
  Wayne State University (WSU), Detroit, Michigan, USA.
- 2012, M.A. in Mathematical Statistics, WSU, Detroit, Michigan, USA
- 2007, M.S. in Mathematics
  Beijing Normal University (BSU), Beijing, China.
- 2004, B.S. in Mathematics and Applied Mathematics, BSU, Beijing, China.

Awards and honors

- 2013, The Wladmimir Seidel Award, WSU
- 2012, Travel Award for 2012 JMM, American Mathematical Society
- 2012-13, Thomas C. Rumble University Graduate Fellowship, WSU
- 2012, The Maurice J. Zelonka Endowed Mathematics Scholarship, WSU
- 2011-13, Graduate Student Professional Travel Award, WSU

Papers and preprints

1. With G. Lu and X. Han, Dual Spaces of Weighted Multi-parameter Hardy Spaces Associated with the Zygmund Dilation, *Advanced Nonlinear Studies*, 12 (2012), 533-553.
2. With G. Lu, Atomic Decomposition and Boundedness Criterion of Operators on Multi-parameter Hardy Space of Homogeneous Type, *Acta Mathematica Sinica*, Vol. 28, No. 7(2012), 1329-1346.
3. With Q. Wang, H. Chen, and Q. Zou, Scanning and Analysis of Mismatch Distribution on Human Genome, *International Journal of Biomathematics*, Vol.5, No.3 (2012),101-108.
4. With Y. Ding, The Carleson Measure with Rough Kernels, *Chinese Annals of Mathematics*, Vol. 29, No. 6(2008), 801-809.
5. With G. Lu and X. Han, Wolff Potential and Integral Systems on Homogeneous Spaces, submitted.
6. With G. Lu and X. Han, Weighted multi-parameter Hardy spaces associated with the Zygmund dilation and boundedness of singular integrals, submitted.
7. Boundedness of Para-product on Spaces of Homogeneous Type, prepint.