Journal of Modern Applied Statistical Methods

Volume 13 | Issue 2 Article 9

11-2014

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Journal of Modern Applied Statistical Methods
November 2014, Vol. 13, No. 2, 169-184.

ISSN 1538 − 9472

A Bivariate Distribution with Conditional Gamma and its Multivariate Form

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A bivariate distribution whose marginal are gamma and beta prime distribution is introduced. The distribution is derived and the generation of such bivariate sample is shown. Extension of the results are given in the multivariate case under a joint independent component analysis method. Simulated applications are given and they show consistency of our approach. Estimation procedures for the bivariate case are provided.

Keywords: Gamma distribution, Gamma function, Beta function, Beta distribution, generalized Beta prime distribution, incomplete gamma function

Introduction

The gamma and beta distributions are the two most commonly used distribution when it comes to analyzing skewed data. Since Kibble (1941), the bivariate gamma has gained considerable attention. The multivariate form of the gamma has been proposed in Johnson et al. (1997) and by many other authors, but there is no unifying formulation. Even in the multivariate exponential family of distributions, there is no known multivariate gamma (Joe, 1997). The simplest of the multivariate cases, the bivariate gamma distribution, is still raising debates, and has been proposed in Balakrishnan and Lai (2009). The marginal densities of the bavariate gamma can sometimes belong to other class of distributions. A modified version of Nadarajah (2009) bivariate distribution with Gamma and Beta marginals is considered, and a conditional component to the modeling is brought into account. Kotz et al (2004) proposed a bivariate gamma exponential distribution with gamma and Pareto distribution as marginals. In this article, a bivariate gamma distribution

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with gamma and beta prime as marginal distributions is defined. By including the dependence structure, more flexibility is added. Consider two random variables X, identified as the common measure, and Y related to X, and assuming that X is a gamma random variable with parameters α and β and the distribution of $Y \mid X$ is a gamma random variable with parameters a and X. The first section following this introduction shows the bivariate distribution with the conditional gamma. In the next section, 'Properties,' the main properties of the bivariate conditional gamma distribution are given. Extension to the multivariate setting is given in the next section, followed by a development of computational aspects in the inference. The calculations in this paper involve several special functions, including the incomplete gamma function defined by

$$\gamma(a,x) = \frac{1}{\Gamma(a)} \int_0^x t^{a-1} e^{-t} dt \,,$$

and the complementary gamma function defined as

$$\Gamma(a,x) = \int_x^\infty t^{a-1} e^{-t} dt \text{ , with } \Gamma(a) = \Gamma(a,0) \,.$$

Also, the beta function is defined as

$$B(a,b) = \int_0^1 t^{a-1} (1-t)^{b-1} dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \,,$$

for a, b, and x positive real values. For $x \in [0,1]$, $\alpha > 0$ and $\beta > 0$, the beta distribution can be defined as

$$f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}$$

Model Building and Density functions

Let X be a gamma rv's with shape and rate parameters denoted by α and β, respectively. The probability density function (pdf) of X is given by

$$f_x\left(x\right)=\frac{\beta^{\alpha}}{\Gamma\left(\alpha\right)}x^{\alpha-1}e^{-\beta x},\ x>0 \tag{1}$$

where $\alpha > 0$ and $\beta > 0$. Many authors have developed structural models with the underlying gamma distribution. Consider another random variable Y such that the distribution of the random variable Y given a realization of X at x is a gamma with the parameters a and x. That is the density of $Y \mid X$ is given by

$$f_{Y|X}\left(y \mid x\right)=\frac{x^{a}}{\Gamma\left(a\right)}y^{a-1}e^{-xy},\ y>0 \tag{2}$$

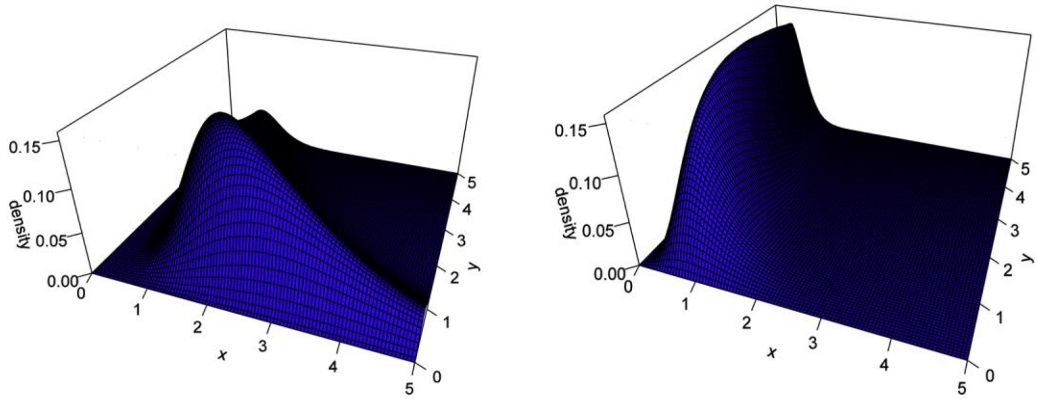
where $a > 0$ and $x > 0$ are the shape and rate parameters respectively. So the joint density of the random variables defined above is given by the expression below

$$\begin{aligned} f_{X,Y}\left(x,y\right) &= f_{Y|X=x}\left(y \mid X=x\right) * f_X\left(x\right) \\ &= \frac{\beta^{\alpha}x^{\alpha+a-1}y^{a-1}}{\Gamma(a)\Gamma(\alpha)}e^{-\beta x}e^{-xy},\ \ x>0 \text{ and } y>0, \end{aligned} \tag{3}$$

with parameters $\alpha > 0$, $\beta > 0$ and $a > 0$. Equation (3) integrates to 1, so this is a legitimate distribution. Figure 1 shows the plot of the joint distribution defined in Equation (3) for different values of α, β and a.

Thus the cumulative distribution of the random variable X and Y is

$$F_{X,Y}\left(x,y\right)=\frac{\gamma\left(\alpha,\beta x\right)\gamma\left(a,xy\right)}{\Gamma\left(a\right)\Gamma\left(\alpha\right)},\ x>0 \text{ and } y>0. \tag{4}$$



a) $\alpha = 2.5$, $\beta = 1.3$, $a = 3.2$ b) $\alpha = 1.5$, $\beta = 3.3$, $a = 2.2$

Figure 1. Joint Probability Density Function of (X,Y)

Properties

The main properties of the distribution as defined in (3), such as the marginal densities, their moments, their product products and covariance, are derived here.

Marginal Density and Moments:

The marginal density of X is given by (1). Marginal density of Y is given by the theorem below.

Theorem 1: If the joint density of (X,Y) is given in (3), then the marginal density of Y is given by

$$f_y(y) = \frac{1}{B(a,\alpha)} \frac{\beta^{\alpha} y^{a-1}}{(y+\beta)^{a+\alpha}}, \; y > 0, \, a > 0 \text{ and } \alpha > 0 \tag{5}$$

Proof. The marginal density of Y is given by

$$\begin{aligned} f_y(y) &= \frac{\beta^\alpha y^{a-1}}{\Gamma(\alpha)\Gamma(a)} \int_0^\infty x^{a+\alpha-1} e^{-(\beta x + xy)} dx \\ &= \frac{\beta^\alpha y^{a-1}}{\Gamma(\alpha)\Gamma(a)} \int_0^\infty \frac{z^{a+\alpha-1} e^{-z}}{(y+\beta)^{a+\alpha}} dz, \text{ with } z = x(y+\beta) \\ &= \frac{\Gamma(a+\alpha)}{\Gamma(a)\Gamma(\alpha)} \frac{\beta^\alpha y^{a-1}}{(y+\beta)^{a+\alpha}} \end{aligned}$$

Probability density function of Y is a special form of Generalized Beta prime density with shape parameter 1 and scale parameter β. Figure 2 describes its pdf for different values of α. Probability density of generalized beta prime distribution with scale p and shape q is given by

$$f(y;\alpha,\beta,p,q) = \frac{p\left(\frac{x}{q}\right)^{\alpha p-1}\left(1+\left(\frac{x}{q}\right)^{p}\right)^{-\alpha-\beta}}{qB(\alpha,\beta)} \tag{6}$$

Let T be a random variable such that $T \sim \text{Beta}(a, \alpha)$. Then $Y = \frac{\beta T}{1-T}$, has density given by (5).

Theorem 2: Let Y be a random variable with a pdf given in (5). The m^{th} moment of the random variable Y exists only if $\alpha > m$.

Proof: From the previous theorem it can be seen that if T: Beta(a, α) and $Y = \frac{\beta t}{1-t}$, then the density of Y will be same as defined in (5). And the m^{th} moment of Y is

$$\begin{aligned}
E\left[Y^m\right] &= E\left[\frac{\beta t}{1-t}\right]^m \\
&= \int_0^1 \frac{\beta^m t^m}{(1-t)^m} \frac{t^{a-1}(1-t)^{\alpha-1}}{Beta(a,\alpha)} \\
&= \frac{\beta^m}{Beta(a,\alpha)} \int_0^1 t^{a+m-1}(1-t)^{\alpha-m-1} \\
&= \frac{\beta^m}{Beta(a,\alpha)} Beta(a+m,\alpha-m), \text{ provided } \alpha > m \\
&\qquad \text{with } Beta(a,\alpha) = \frac{\Gamma(a)\Gamma(\alpha)}{\Gamma(a+\alpha)}
\end{aligned}$$

The choice of $\alpha > 2$ is made so that $E[Y]$ and $Var[Y]$ will both exist.

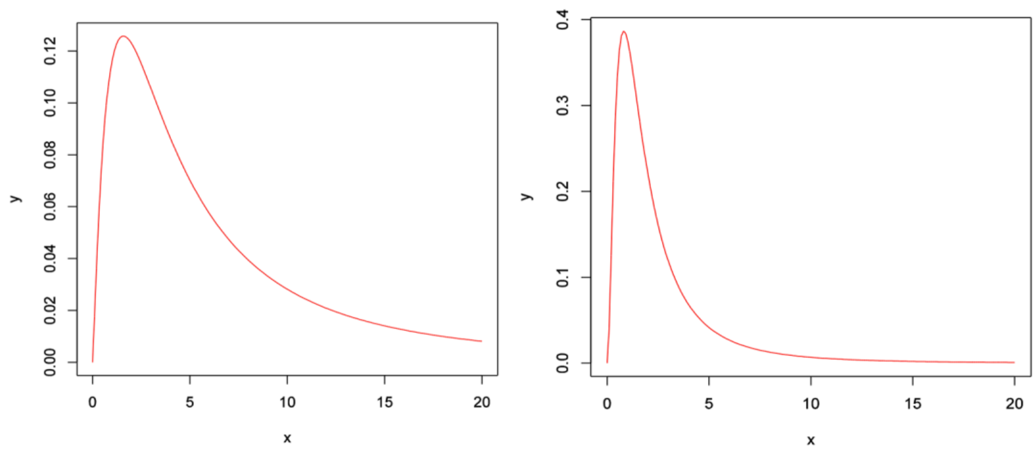


Figure 2. The Probability Density Function of Y as defined in (5)

Product Moments

Theorem 3: The product moment of the random variables (X,Y) associated with the pdf defined in (3) can be expressed as

$$E\left(X^mY^n\right)=\frac{\beta^{n-m}\Gamma(a+n)\Gamma(m-n+\alpha)}{\Gamma(a)}\text{ for } a>0, m>0, n>0 \text{ and } m+\alpha>n$$

Proof: For $m > 0$ and $n > 0$ one can write

$$\begin{aligned}
E\left(X^mY^n\right)&=\int_0^\infty\int_0^\infty x^m y^n \frac{\beta^\alpha x^{a+\alpha-1}y^{a-1}e^{-\beta x-xy}}{\Gamma(\alpha)\Gamma(a)}dxdy\\
&=\int_0^\infty \frac{\beta^\alpha e^{-\beta x}x^{a+m+\alpha-1}}{\Gamma(\alpha)\Gamma(a)}\int_0^\infty y^{a+n-1}e^{(-xy)}dydx\\
&=\int_0^\infty \frac{\beta^\alpha x^{a+m+\alpha-1}e^{-\beta x}}{\Gamma(\alpha)\Gamma(a)}\frac{\Gamma(a+n)}{x^{a+n}}dx\\
&=\frac{\beta^\alpha\Gamma(a+n)}{\Gamma(\alpha)\Gamma(a)}\int_0^\infty e^{-\beta x}x^{m-n+\alpha-1}\\
&=\frac{\beta^{n-m}\Gamma(a+n)\Gamma(m-n+a)}{\Gamma(\alpha)\Gamma(a)}
\end{aligned}$$

provided the integrals exist. Now for the m^{th} product moment by choosing $n = m$ in the above expression, one can write the product moment as

$$E\left(X^mY^n\right)=\frac{\Gamma(a+m)}{\Gamma(a)}$$

Note that the product moment depends only on a.

Covariance Matrix

With the density of X and Y as given in Equations (1) and (5), respectively, the variance-covariance matrix of X and Y is given by the following theorem

Theorem 4: Denote the Variance-Covariance matrix of X and Y by $Cov(X,Y)$, then

$$Cov(X,Y)=\begin{pmatrix} \dfrac{\alpha}{\beta^2} & \dfrac{-a}{\alpha-1} \\ \dfrac{-a}{\alpha-1} & \dfrac{a\beta^2(a+\alpha-1)}{(\alpha-1)^2(\alpha-2)} \end{pmatrix} \tag{7}$$

Proof: Using Theorem 2, the variance and expectation of the random variable Y can be computed

$$\begin{aligned} Var(Y) &= E(Y^2)-(E(Y))^2 \\ &= \frac{a\beta^2(a+1)}{(\alpha-1)(\alpha-2)}-\left[\frac{a\beta}{(\alpha-1)}\right]^2 \\ &= \frac{a\beta^2(a+\alpha-1)}{(\alpha-1)^2(\alpha-2)} \end{aligned} \tag{8}$$

Equation (2) implies that the distribution of X is a Gamma distribution with shape α and rate β. So variance of X is given by

$$Var(X)=\frac{\alpha}{\beta^2} \tag{9}$$

Now the covariance between X and Y can be written as

$$\begin{aligned}
Cov(X,Y) &= E(XY) - E(X)E(Y) \\
&= E\left(XE(Y \mid X)\right) - E(X)E(Y) \\
&= E\left(X\frac{a}{X}\right) - E(X)E(Y) \text{ as } Y \mid X : Gamma(a,x) \\
&= a - \frac{\alpha}{\beta}\frac{a\beta}{(\alpha-1)} \\
&= -\frac{a}{(\alpha-1)}
\end{aligned} \tag{10}$$

Using the Equations (8), (9) and (10) the result follows. Note that the covariance between X and Y exists only when $\alpha \neq 1$, and is positive when $\alpha < 1$. Variance of Y only exists when $\alpha > 2$.

Multivariate Extension Case

Consider the multivariate case of the model: take $n + 1$ random variables as follows:

$$\begin{gathered}
X_0 : Gamma(\alpha, \beta) \\
X_1 \mid X_0 : Gamma(a_1, b_1 x_0) \\
X_2 \mid X_0 : Gamma(a_2, b_2 x_0) \\
\vdots \\
X_n \mid X_0 : Gamma(a_n, b_n x_0)
\end{gathered}$$

where $X_i \mid X_0$ and $X_j \mid X_0$ are independent components for $i \neq j$ and $(i, j) \in \{1, 2, \ldots, n\}$. Then using the same argument as in 'Properties,' the joint independent component model is built and the marginal density function for each random variable X_i is derived. In general, the density function of X_i is given by

$$f(x_i) = \frac{\Gamma(\alpha + a_i)}{\Gamma(a_i)\Gamma(\alpha)} \frac{b_i^{a_i} \beta^{\alpha} x_i^{a_i - 1}}{(\beta + b_i x_i)^{\alpha + a_i}}, \text{for } i = 1, 2, \ldots, n \tag{11}$$

Using the independence assumption of the above model, the joint density of $X_0, X_1, \ldots, X_n$ is then derived. The derived joint density will be of the form

$$f\left(x_0, x_1, ..., x_n\right) = f\left(x_0\right) \prod_{i=1}^{i=n} f\left(x_i \mid x_0\right)$$

The density of the joint distribution (X_1, X_2, …, X_n) and its variance covariance expression are derived next.

For the density of (X_1, X_2, …, X_n), the integration of the joint density with respect to the variable X_0 is needed.

Density Function

To derive the density function, the integral below is computed

$$f\left(x_1, ..., x_n\right) = \int_0^{\infty} f\left(x_0\right) \prod_{i=1}^{i=n} f\left(x_i \mid x_0\right) dx_0 \tag{12}$$

And solving the integral in (12), the joint density is as follows

$$f\left(x_1, ..., x_n\right) = \frac{\Gamma\left(\alpha + \sum_{i=1}^{n} a_i\right)}{\Gamma\left(\alpha\right) \sum_{i=1}^{n} \Gamma\left(a_i\right)} \frac{\beta^{\alpha} \prod_{i=1}^{n} \left(b_i^{a_i} x_i^{a_i - 1}\right)}{\left\{\beta + \sum_{i=1}^{n} \left(b_i x_i\right)\right\}^{\sum_{i=1}^{n} a_i + \alpha}} \tag{13}$$

where $x_i > 0$, $a_i > 0$ for all $i = 1, 2, \ldots, n$, and $\alpha > 0$, $\beta > 0$. In the distribution obtained from (13), if the choices of $\beta = 1$ and $b_i = 1$ for all $i = 1, 2, \ldots, n$ are made, then the inverted Dirichlet distribution is obtained. The application of this distribution can be found in many places in the literature. Taio and Cuttman (1965) introduced this type of distribution and discussed about their applications.

Covariance

The covariance between X_i and X_j for $i \neq j$ is derived in Theorem 5.

Theorem 5: If the random variables $X_1, X_2, \ldots, X_n$ have the density function defined in (13), then the covariance between X_i and X_j for $i \neq j$ is given by the expression below

$$Cov\left(X_i, X_j\right) = \frac{a_i a_j \beta^2}{b_i b_j \left(\alpha-1\right)^2 \left(\alpha-2\right)} \text{ for } i \neq j, b_i \text{ can be equal to } b_j$$

Proof: Using the same arguments in Theorem 2, the m^{th} moments of X_i are derived. Based on the density of X_i defined by (11)

$$E\left(X_i^m\right) = \frac{\Gamma\left(a_i+m\right)\Gamma\left(\alpha-m\right)}{\Gamma\left(a_i\right)\Gamma\left(\alpha\right)} \frac{\beta^m}{b_i^m} \text{ with } \alpha > m \tag{14}$$

From (13) this useful identity is obtained

$$\int_0^\infty \cdots \int_0^\infty \frac{\prod_{i=1}^n x_i^{a_i-1}}{\left(\beta+\sum_{i=1}^n b_i x_i\right)^{\alpha+\sum_{i=1}^n a_i}} = \frac{\Gamma\left(\alpha\right)\prod_{i=1}^n \Gamma\left(a_i\right)}{\beta^\alpha \sum_{i=1}^n b_i^{a_i} \Gamma\left(\alpha+\sum_{i=1}^n a_i\right)} \tag{15}$$

Using the identity in (15), the $(m_1, m_2, \ldots, m_n)^{\text{th}}$ mixed moment is given as

$$E\left(X_i^{m_1} \ldots X_i^{m_n}\right) = \frac{\Gamma\left(\alpha-\sum_{i=1}^n m_i\right)}{\Gamma\left(\alpha\right)} \prod_{i=1}^n \frac{\Gamma\left(a_i+m_i\right)}{b_i^{m_i}\Gamma\left(a_i\right)} \tag{16}$$

provided $\alpha > \sum_{i=1}^n m_i$. In particular, the covariances between X_i and X_j, for $i = 1, 2, \ldots, n$, is as

$$Cov\left(X_i, X_j\right) = \frac{a_i a_j \beta^2}{b_i b_j \left(\alpha-1\right)^2 \left(\alpha-2\right)}$$

Note that the covariance between X_0 and X_i for $i = 1, 2, \ldots, n$ is also derived as

$$\begin{aligned} Cov\left(X_0, X_i\right) &= E\left(X_0 X_i\right) - E\left(X_0\right)E\left(X_i\right) \\ &= E\left(X_0 E\left(X_i \mid X_0\right)\right) - E\left(X_0\right)E\left(X_i\right) \\ &= E\left(X_0 \frac{a_i}{X_0 b_i}\right) - E\left(X_0\right)E\left(X_i\right) \text{ as } X_i \mid X_0 : Gamma\left(a_i, b_i x\right) \quad (17) \\ &= \frac{a_i}{X_0 b_i} - \frac{\alpha}{\beta}\frac{\beta\Gamma\left(a_i+1\right)\Gamma\left(\alpha-1\right)}{b_i\Gamma\left(a_i\right)\Gamma\left(\alpha\right)} \\ &= -\frac{a_i}{b_i\left(\alpha-1\right)} \end{aligned}$$

Bivariate cases will reduce to Equation (10).

Likelihood and Estimation for Bivariate Case

In this section, the maximum likelihood estimation process and Fisher information matrix for the bivariate model are introduced. Statistical analysis software (SAS) is used to generate data and R is used to get the maximum likelihood estimates (MLEs).

Log likelihood

Let (x_i, y_i), for $i = 1, 2, \ldots, n$, be a sample of size n from the bivariate gamma distribution as defined in Equation (3). Then, the log likelihood function is

$$\begin{aligned} L\left(x, y; \alpha, \beta, a\right) = n\alpha\log\left(\beta\right) + \left(\alpha-1\right)\sum_{i=1}^{n}\log\left(x_i\right) - n\beta\overline{x} - n\log\left[\Gamma\left(\alpha\right)\right] \\ + a\sum_{i=1}^{n} x_i + \left(\alpha-1\right)\sum_{i=1}^{n}\log\left(y_i\right) - \sum_{i=1}^{n} x_i y_i - n\log\Gamma\left(a\right) \end{aligned} \quad (18)$$

The first order derivatives of the log likelihood with respect to the three parameters are

$$\frac{\partial Log(\alpha, \beta, a)}{\partial\alpha} = nlog(\beta) + \sum_{i=0}^{n} log(x_i) - n\psi(\alpha) \quad (19)$$

$$\frac{\partial Log(\alpha,\beta,a)}{\partial\beta}=\frac{n\alpha}{\beta}-n\overline{x} \tag{20}$$

$$\frac{\partial Log(\alpha,\beta,a)}{\partial a}=n\overline{x}+\sum_{i=0}^{n}log(x_i)-n\psi(a) \tag{21}$$

where $\psi(x)=\frac{d}{dx}\ln(\Gamma(x))$ is the *Digamma* function.

Solving above Equations (19-21) simultaneously, the MLEs of the parameters can be formulated. As the MLEs are not in a closed form, an R code is developed to get the estimates.

Fisher Information Matrix

The Fisher information matrix g is given by the expectation of the covariance of partial derivatives of the log likelihood function. Let (θ_1, θ_2, θ_3) = (α, β, a); then the components of the Fisher information matrix are given by

$$g_{ij}=\int_0^\infty\int_0^\infty\frac{\partial^2\log f(x,y,\alpha,\beta,a)}{\partial\theta_i\partial\theta_j}$$

$$\text{Hence, } g=\begin{pmatrix}\frac{\partial^2 log\Gamma(\alpha)}{\partial\alpha} & \frac{-1}{\beta} & 0\\ \frac{-1}{\beta} & \frac{\alpha}{\beta^2} & 0\\ 0 & 0 & \frac{\partial^2 log\Gamma(\alpha)}{\partial\alpha}\end{pmatrix} \tag{22}$$

Inverting the fisher information matrix, the asymptotic standard errors of the maximum-likelihood estimates can be obtained.

Example Using Simulated Data

A number of simulations are performed to evaluate the statistical properties and the estimation are computed using *maximum likelihood method*. Because of the complexity of the target density and of the likelihood, there is no closed form of the estimators. Effective sample sizes will be directly impacting the estimates. R program is used to do the optimization, but *SAS 9.3* version is used to simulate data with samples of sizes n = 200 and 25.

Accordingly, for each set of parameters and sample size, X_0 is simulated from a gamma distribution with parameters α and β. Then, for each X_0, generate X_1 based on X_0 according to Equation (3) with the same value of a.

The simulation results presented under the table give the estimates of the parameters. Figure 3 gives the plot of log likelihood and shows the uniqueness of the solution estimate for each parameter at sample size 200.

The results show that the larger the sample size, the more accurate the estimates are. A plot of the estimates versus sample size is given in Figure 4.

Simulation results

Table 1. Estimation of parameters for different sample sizes

	Estimates (SE)	
Actual Values	n = 200	n = 25
α = 2.5	$\hat{\alpha}$ = 2.326 (0.819)	$\hat{\alpha}$ = 2.001 (1.175)
β = 1.3	$\hat{\beta}$ = 1.206 (0.474)	$\hat{\beta}$ = 1.061 (0.708)
a = 3.2	$\hat{a}$ = 3.229 (0.442)	$\hat{a}$ = 3.075 (0.772)
α = 6.3	$\hat{\alpha}$ = 6.816 (2.503)	$\hat{\alpha}$ = 5.287 (3.243)
β = 2.1	$\hat{\beta}$ = 2.225 (0.848)	$\hat{\beta}$ = 1.757 (1.131)
a = 1.2	$\hat{a}$ = 1.169 (0.232)	$\hat{a}$ = 1.280 (0.415)

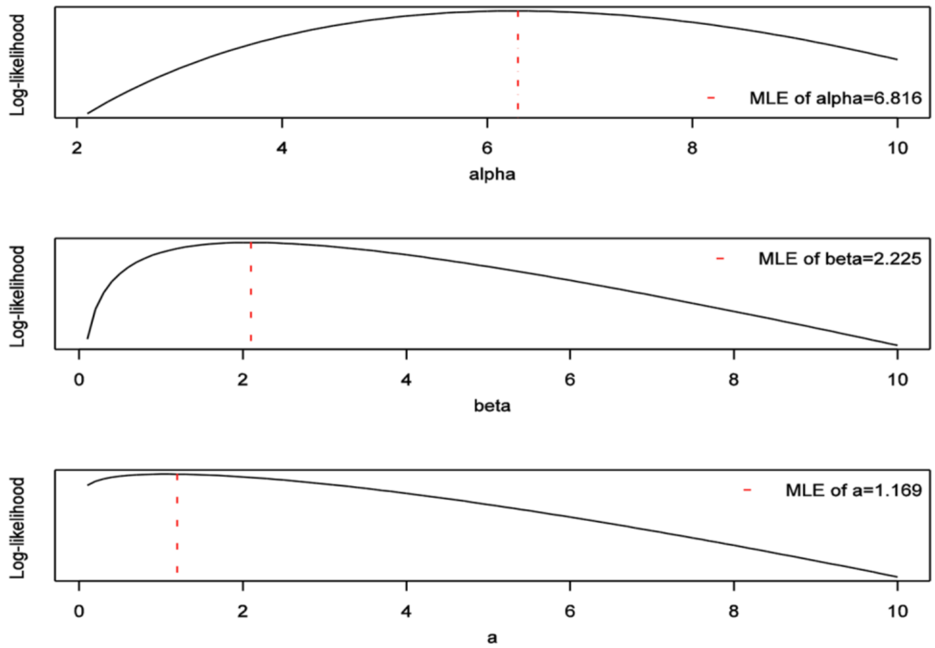


Figure 3. MLE estimates of parameters for a sample of 200

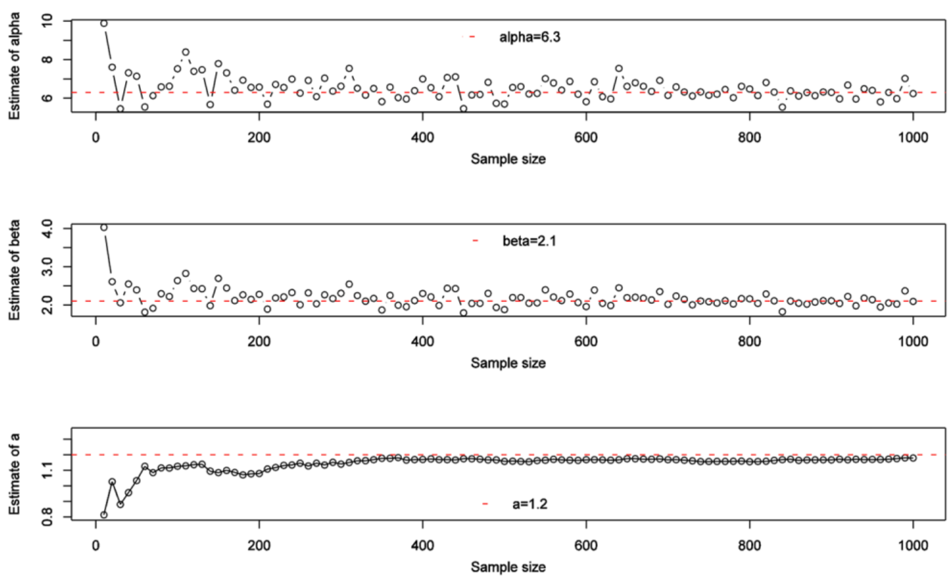


Figure 4. Parameter estimation for increasing sample size

Conclusion

In this paper, a bivariate conditional gamma and its multivariate form are proposed. Their associated properties are presented and the simulation studies have shown significant improvement in the parameter estimations, taking into account the intra-correlation and dependence among the observed mixing random variables. While our proposed model process is guided by a formal fit criteria, Bayesian approach is another option to determine the parameters. However, the proposed approach has the advantage of giving a simple implementation for mixed outcome data.

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