Journal of Modern Applied Statistical Methods

Volume 8 | Issue 2 Article 25

11-1-2009

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Journal of Modern Applied Statistical Methods
November 2009, Vol. 8, No. 2, 594-596

1538 – 9472/09/$95.00

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Norton (1984) presented a calculation of the MLE for the parameter of the double exponential distribution based on the calculus. An inductive approach is presented here.

Key words: MLE, median, double exponential.

Introduction

Norton (1984) derived the MLE using a calculus argument. This article shows how to obtain it using a simple induction argument that depends only on knowing the shape of a function of sums of absolute values. Some introductory mathematical statistics textbooks, such as Hogg and Craig (1970) give the answer to be the median – although correct, this does not tell the whole story as Norton points out; this is emphasized here.

Methodology

It is useful to review the behavior of linear absolute value functions and sums of linear absolute value functions. For example, consider the function

$$g(x) = |1.8 - x|.$$

Its graph is shown in Figure 1. Note that it has a V-shape with a minimum at x = 1.8. Now consider a sum of two linear absolute value terms:

$$h(x) = |1.8 - x| + |3.2 - x|.$$

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Plots of this function and its components, |1.8 - x| and |3.2 - x|, are shown in Figure 2. Note that $h(x)$ takes a minimum at all points in the interval $1.8 \le x \le 3.2$.

The MLE

The double exponential distribution is given by

$$f(x) = \frac{1}{2} e^{-|x-\theta|}, \quad -\infty < x < \infty.$$

For the sample $\{x_1, x_2, ..., x_n\}$, the log-likelihood function is

$$\ell(\theta) = n \ln(1/2) - \sum_i |x_i - \theta|.$$

Maximizing this function with respect to θ is equivalent to minimizing

$$g_n(\theta) = \sum_i |x_i - \theta|.$$

To obtain the MLE for general n, begin with the case n = 1 where $g_1(\theta) = |x_1 - \theta|$. This function has a minimum at $\theta = x_1$, hence, for n = 1, the MLE is

$$\theta^{MLE} = x_1.$$

Figure 1: Plot of g(x) = | x - 1.8|

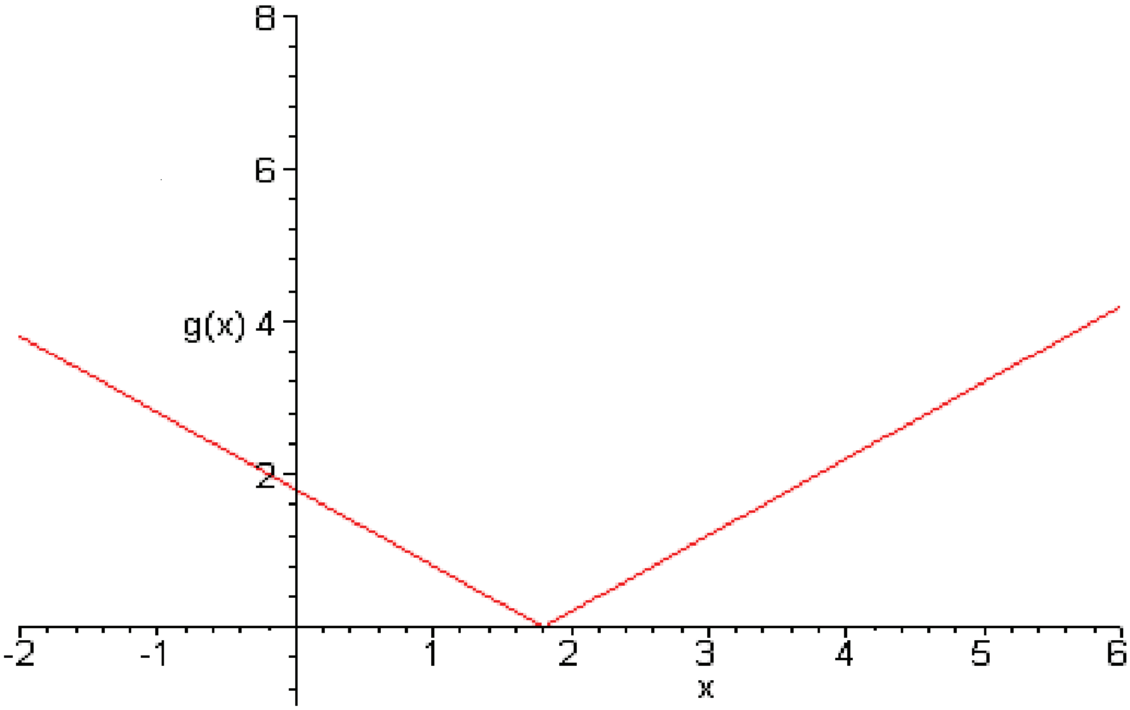
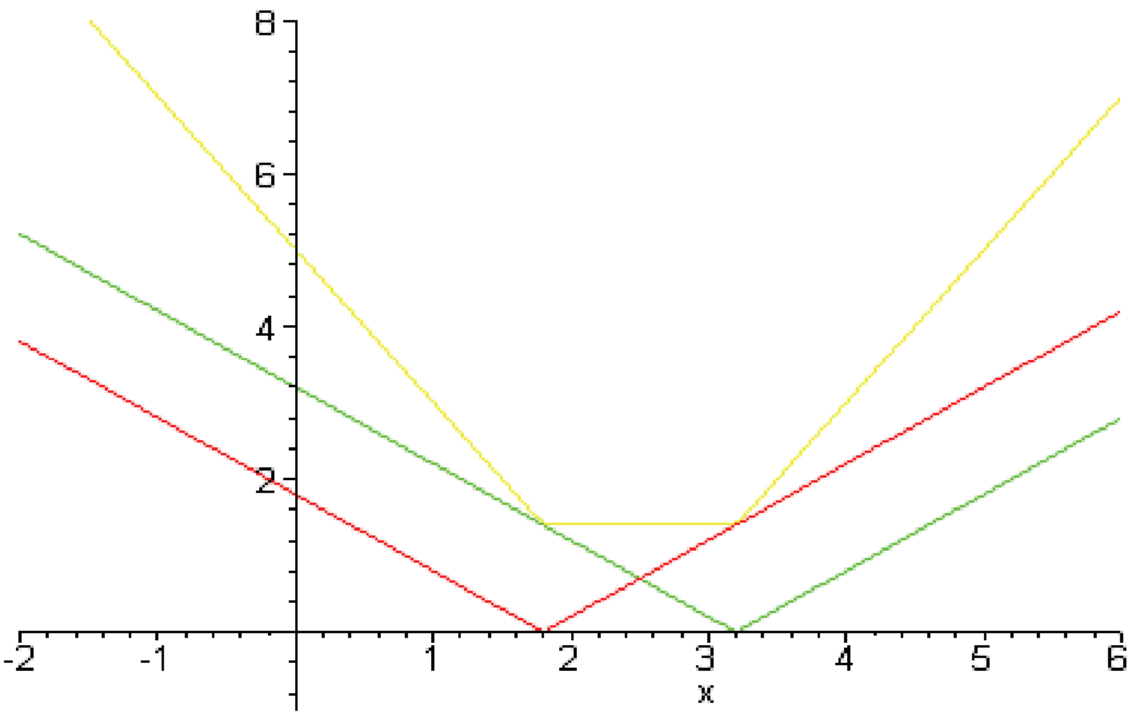


Figure 2: Plot of h(x) = |1.8 - x| + |3.2 - x|



Now, consider the case n = 2. For the purposes herein it is useful to order the observations, thus, suppose that the sample is $\{x_{(1)}, x_{(2)}\}$ where $x_{(1)} < x_{(2)}$. The value of θ which minimizes must now be found using

$$g_2(\theta) = |x_{(1)} - \theta| + |x_{(2)} - \theta|.$$

Based on the above, this function takes the form

$$g_2(\theta) = \begin{cases} -2\theta + x_{(1)} + x_{(2)} & \theta \le x_{(1)} \\ x_{(2)} - x_{(1)} & x_{(1)} \le \theta \le x_{(2)} \\ 2\theta - x_{(1)} - x_{(2)} & \theta \ge x_{(2)} \end{cases}$$

and has a minimum at any point θ in the interval $x_{(1)} \le \theta \le x_{(2)}$. Hence the MLE for n = 2 is

$$\theta^{MLE} = \lambda x_{(1)} + (1-\lambda) x_{(2)}, \quad 0 \le \lambda \le 1.$$

For this case, the median is defined $(x_{(1)} + x_{(2)})/2$ and is a solution, but it is not the only solution.

Next, consider the case n = 3 with an ordered sample $x_{(1)} \le x_{(2)} \le x_{(3)}$. Using the same graphical analysis, it can be shown that

$$g_3(\theta) = |x_{(1)} - \theta| + |x_{(2)} - \theta| + |x_{(3)} - \theta|$$

has a unique minimum at $\theta = x_{(2)}$, the median. In the case n = 4, the solution is

$$\theta^{MLE} = \lambda x_{(2)} + (1-\lambda) x_{(3)}, \quad 0 \le \lambda \le 1.$$

Thus, the median is a solution, but not the only solution.

Conclusion

Extending the argument for general n is straightforward. It is the median, $x_{((x+1)/2)}$, if n is odd and the generalized median, $\lambda x_{(n/2)} + (1-\lambda) x_{(n/2+1)}$, when n is even.

References

Hogg, R. V., & Craig, A. T. (1970). *Introduction to Mathematical Statistics*, (*3rd Ed.*). New York, NY: MacMillan Publishing Company.

Norton, R. M. (1984). The double exponential distribution: Using calculus to find a maximum likelihood estimator. *The American Statistician*, *38*(*2*), 135-136.