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Test for the Equality of the Number of Signals

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A likelihood ratio test for testing the equality of the ranks of two non-negative definite covariance matrices arising in the area of signal processing is derived. The asymptotic distribution of the test statistic follows a Chi-square distribution from the general theory of likelihood ratio test.

Key words: Likelihood ratio test; signal processing; white noise.

Introduction

In the area of signal processing, signals are observed at different sensors from different sources at different time points. Wax, Shan and Kailath (1984) and Whalen (1971) discussed models and varieties of problems in signal processing. In general, the signal processing model is as follows:

$$\mathbf{X}(t) = \mathbf{A}\mathbf{S}(t) + \mathbf{n}(t) \quad (1)$$

where, $\mathbf{X}(t) = (X_1(t), X_2(t), \dots, X_p(t))'$ is the $p \times 1$ observation vector at time t , $\mathbf{S}(t) = (S_1(t), S_2(t), \dots, S_q(t))'$ is the $q \times 1$ vector of unknown random signals at time t , $\mathbf{n}(t) = (n_1(t), n_2(t), \dots, n_p(t))'$ is the $p \times 1$ random noise vector at time t , $\mathbf{A} = (\mathbf{A}(\Phi_1), \mathbf{A}(\Phi_2), \dots, \mathbf{A}(\Phi_q))$ is the $p \times q$ matrix of unknown coefficients, and $\mathbf{A}(\Phi_r)$ is the $p \times 1$ vector of functions of the elements of unknown vector Φ_r associated with the r^{th} signal and $q < p$.

In model (1), $\mathbf{X}(t)$ is assumed to be distributed as p -variate normal with mean vector zero and dispersion matrix $\mathbf{A}\Psi\mathbf{A}' + \sigma^2\mathbf{I}_p = \Gamma + \sigma^2\mathbf{I}_p$, where $\Gamma = \mathbf{A}\Psi\mathbf{A}'$ is unknown n.n.d. matrix of rank $q (< p)$ and $\Psi =$

covariance matrix of $\mathbf{S}(t)$, $\sigma^2 (> 0)$ is unknown and $\sigma^2\mathbf{I}_p$ is the covariance matrix of the noise vector $\mathbf{n}(t)$. In this case, the model is called white noise model.

One important problem that arises in the area of signal processing is to estimate q , the number of signals transmitted. This problem is equivalent to estimating the multiplicity of the smallest eigenvalue of the covariance matrix of the observation vector. Anderson (1963), Krishnaiah (1976) and Rao (1983), among others, considered this problem. Wax and Kailath (1984) and Zhao, et al. (1986a, b) used information theoretic criteria proposed by Akaike (1972), Rissanen (1978) and Schwartz (1978) to estimate the number of signals.

More recently, Chen, et al. (2001), Chen (2002) and Kundu (2000) developed procedures for estimating the number of signals. This article considers the two sample problem of testing the equality of the number of signals between two sets of data from two populations. This problem is relevant in practice in the area of signal processing because it is important to know whether the total numbers of signals received are the same or not for two different days, which can be separated by a lengthy time. This problem is equivalent to testing the equality of multiplicity of the smallest eigenvalue of the covariance matrices of observation vectors of the two sets of data. Consider the following model:

$$X_i(t) = A_i S_i(t) + N_i(t); i = 1, 2$$

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where, $X_i(t)$ is a $p \times 1$ observation vector for the i^{th} population, $A_i = (A_i(\Phi_1^i), \dots, A_i(\Phi_{q_i}^i))$, $S_i(t) = (S_1^i(t), \dots, S_{q_i}^i(t))$, $i = 1, 2$ and $N_i(t)$ is a $p \times 1$ random noise vector for the i^{th} population. The following are assumed about $N_i(t)$ and $S_i(t)$:

$$N_i(t) \sim N_p(0, \Sigma_i),$$

$$S_i(t) \sim N_p(0, \Psi_i),$$

and $N_i(t)$ and $S_i(t)$ are independently distributed. The null hypothesis to test is $H_{0k} : q_1 = q_2 = k$, and the alternative hypothesis is $H_1 : q_1 \neq q_2, k = 0, 1, \dots, p-1$. At this point, the likelihood ratio test is derived next and asymptotic distribution of the test statistic is used to obtain the critical value.

Likelihood Ratio Test: Case 1

Consider $\Sigma_i = \sigma^2 I_p, i = 1, 2$. The test hypotheses are:

$$H_{0k} : q_1 = q_2 = k$$

and

$$H_1 : q_1 \neq q_2, k = 0, 1, \dots, p-1.$$

The observations from the two populations are as follows: $X_1(t_1), \dots, X_1(t_{n_1})$ are i.i.d. $\sim N_p(0, A_1 \Psi_1 A_1' + \sigma^2 I_p)$ and $X_2(t_{n_1+1}), \dots, X_2(t_{n_1+n_2})$ are i.i.d. $\sim N_p(0, A_2 \Psi_2 A_2' + \sigma^2 I_p)$.

Let $R_i = A_i \Psi_i A_i' + \sigma^2 I_p, i = 1, 2$. It may be stated that testing H_{0k} is equivalent to testing the rank of $A_i \Psi_i A_i' = k, i = 1, 2$.

Let $R_i = R_i^{(k)}$ under $H_{0k}, i = 1, 2$. Using spectral decomposition of $R_1^{(k)}$ and $R_2^{(k)}$, it can be written that

$$R_1^{(k)} = (\lambda_1 - \sigma^2)U_1 U_1' + \dots + (\lambda_k - \sigma^2)U_k U_k' + \sigma^2 I_p$$

and

$$R_2^{(k)} = (\mu_1 - \sigma^2)V_1 V_1' + \dots + (\mu_k - \sigma^2)V_k V_k' + \sigma^2 I_p$$

where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k > \sigma^2$ are the eigenvalues of $R_1^{(k)}$ and U_1, \dots, U_k are the corresponding orthonormal eigenvectors of $A_1 \Psi_1 A_1'$ and similarly, $\mu_1 \geq \mu_2 \geq \dots \geq \mu_k > \sigma^2$ are the eigenvalues of $R_2^{(k)}$ and V_1, \dots, V_k are the corresponding orthonormal eigen vectors of $A_2 \Psi_2 A_2'$.

Thus, under H_{0k} the log-likelihood (apart from a constant term) is

$$\begin{aligned} \log L &= -\frac{n_1}{2} \log |R_1^{(k)}| - \frac{n_1}{2} \text{tr} . (\hat{R}_1 (R_1^{(k)})^{-1}) \\ &\quad - \frac{n_2}{2} \log |R_2^{(k)}| - \frac{n_2}{2} \text{tr} . (\hat{R}_2 (R_2^{(k)})^{-1}) \\ &= -\frac{n_1}{2} \text{tr} . (\hat{R}_1 (R_1^{(k)})^{-1}) - \frac{n_2}{2} \text{tr} . (\hat{R}_2 (R_2^{(k)})^{-1}) \\ &\quad - \frac{n_1}{2} \sum_{i=1}^k \log \lambda_i - \frac{n_2}{2} \sum_{i=1}^k \log \mu_i \\ &\quad - \frac{n}{2} (p-k) \log \sigma^2 \end{aligned} \tag{2}$$

where

$$\hat{R}_1 = \frac{1}{n_1} \sum_{i=1}^{n_1} X_1(t_i) X_1'(t_i),$$

$$\hat{R}_2 = \frac{1}{n_2} \sum_{i=n_1+1}^{n_1+n_2} X_2(t_i) X_2'(t_i)$$

and

$$n = n_1 + n_2.$$

Rather than maximizing (2), equivalently minimize

$$\begin{aligned} \log L^* &= n_1 \text{tr} . (\hat{R}_1 (R_1^{(k)})^{-1}) + n_2 \text{tr} . (\hat{R}_2 (R_2^{(k)})^{-1}) \\ &\quad + n_1 \sum_{i=1}^k \log \lambda_i + n_2 \sum_{i=1}^k \log \mu_i \\ &\quad + n(p-k) \log \sigma^2 \end{aligned} \tag{3}$$

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Orthogonal matrices P_1, P_2, U_1, U_2 exist such that

$$\hat{R}_1 = P_1 D_1 P_1', \quad R_1^{(k)} = U_1 \Lambda_1 U_1'$$

and

$$\hat{R}_2 = P_2 D_2 P_2', \quad R_2^{(k)} = U_2 \Lambda_2 U_2'$$

where, $D_1 = \text{Diag.}(l_1, \dots, l_p)$, $l_1 \geq l_2 \geq \dots \geq l_p$ are the eigenvalues of \hat{R}_1 and similarly, $D_2 = \text{Diag.}(\xi_1, \dots, \xi_p)$, $\xi_1 \geq \xi_2 \geq \dots \geq \xi_p$ are the eigenvalues of \hat{R}_2
 $\Lambda_1 = \text{Diag.}(\lambda_1, \dots, \lambda_k, \sigma^2, \dots, \sigma^2)$, and
 $\Lambda_2 = \text{Diag.}(\mu_1, \dots, \mu_k, \sigma^2, \dots, \sigma^2)$. Thus, (3) can be rewritten as follows:

$$\begin{aligned} \log L^* &= n_1 \text{tr.}(P_1 D_1 P_1' U_1 \Lambda_1^{-1} U_1') \\ &\quad + n_2 \text{tr.}(P_2 D_2 P_2' U_2 \Lambda_2^{-1} U_2') \\ &\quad + n_1 \sum_{i=1}^k \log \lambda_i + n_2 \sum_{i=1}^k \log \mu_i \\ &\quad + n(p-k) \log \sigma^2 \\ &= n_1 \text{tr.}(D_1 V_1 \Lambda_1^{-1} V_1') + n_2 \text{tr.}(D_2 V_2 \Lambda_2^{-1} V_2') \\ &\quad + \text{term independent of } V_1 \text{ and } V_2 \end{aligned} \quad (4)$$

where, $V_1 = P_1' U_1$ and $V_2 = P_2' U_2$ and hence V_1 and V_2 are orthogonal.

Differentiating (4) with respect to V_1 subject to $V_1 V_1' = I_p$ and equating it to 0, results in

$$\begin{aligned} D_1 [\Lambda_1^{-1} V_1' - V_1 \Lambda_1^{-1} V_1^{-2}] &= 0 \\ \text{i.e., } V_1 &= I_p \end{aligned}$$

Similarly, $V_2 = I_p$ is obtained. Hence, given λ_i 's, μ_i 's and σ^2 ,

$$\text{Inf}_{H_{0k}} \log L^* =$$

$$\begin{aligned} &n_1 \left(\sum_{i=1}^k \frac{l_i}{\lambda_i} + \frac{\sum_{i=k+1}^p l_i}{\sigma^2} \right) + n_2 \left(\sum_{i=1}^k \frac{\xi_i}{\mu_i} + \frac{\sum_{i=k+1}^p \xi_i}{\sigma^2} \right) \\ &+ n_1 \sum_{i=1}^k \log \lambda_i + n_2 \sum_{i=1}^k \log \mu_i + n(p-k) \log \sigma^2 \end{aligned} \quad (5)$$

Differentiating (5) with respect to λ_i 's and equating it to 0, results in

$$\begin{aligned} -n_1 \frac{l_i}{\lambda_i^2} + \frac{n_1}{\lambda_i} &= 0 \\ \text{i.e., } \hat{\lambda}_i &= l_i \end{aligned}$$

Similarly, $\hat{\mu}_i = \xi_i; i = 1, \dots, k$.

Differentiating (5) with respect to σ^2 and equating it to 0, results in

$$\hat{\sigma}^2 = \frac{n_1 \sum_{i=k+1}^p l_i + n_2 \sum_{i=k+1}^p \xi_i}{n(p-k)},$$

hence,

$$\begin{aligned} \text{Sup}_{H_{0k}} \log L &= -nk - n_1 \frac{\sum_{i=k+1}^p l_i}{\sigma^2} - n_2 \frac{\sum_{i=k+1}^p \xi_i}{\sigma^2} \\ &\quad - n_1 \sum_{i=1}^k \log l_i - n_2 \sum_{i=1}^k \log \xi_i - n(p-k) \log \hat{\sigma}^2 \\ &= -np - n_1 \sum_{i=1}^k \log l_i - n_2 \sum_{i=1}^k \log \xi_i \\ &\quad - n(p-k) \log \left(\frac{n_1 \sum_{i=k+1}^p l_i + n_2 \sum_{i=k+1}^p \xi_i}{n(p-k)} \right) \\ &= L_1 \text{ (say)} \end{aligned}$$

In the above expression of L_1 , the unknown k can be estimated by using Zhao, Krishnaiah and Bai's (1986a,b) information criterion as follows: Estimate k by \hat{k} such that

$$I(\hat{k}, c_n) = \min_{0 \leq k \leq p-1} I(k, c_n),$$

where

$$I(k, c_n) = -L_1 + c_n \left\{ (2k+1) + 2 \left(pk - k - \frac{k(k-1)}{2} \right) \right\}$$

and c_n is such that

$$(i) \lim_{n \rightarrow \infty} \frac{c_n}{n} = 0$$

$$(ii) \lim_{n \rightarrow \infty} \frac{c_n}{\log \log n} = \infty$$

For practical purposes, choose $c_n = \log n$ which satisfies conditions (i) and (ii). Hence,

$$L_1^* = -np - n_1 \sum_{i=1}^{\hat{k}} \log l_i - n_2 \sum_{i=1}^{\hat{k}} \log \xi_i - n(p - \hat{k}) \log \left(\frac{n_1 \sum_{i=\hat{k}+1}^p l_i + n_2 \sum_{i=\hat{k}+1}^p \xi_i}{n(p - \hat{k})} \right) \tag{6}$$

Similarly,

$$L_2^* = \underset{H_1}{\text{Sup}} \log L = -np - n_1 \sum_{i=1}^{\hat{q}_1} \log l_i - n_2 \sum_{i=1}^{\hat{q}_2} \log \xi_i - [n_1(p - \hat{q}_1) + n_2(p - \hat{q}_2)] \log \left(\frac{n_1 \sum_{i=\hat{q}_1+1}^p l_i + n_2 \sum_{i=\hat{q}_2+1}^p \xi_i}{n_1(p - \hat{q}_1) + n_2(p - \hat{q}_2)} \right) \tag{7}$$

where, \hat{q}_1 and \hat{q}_2 are obtained such that

$$I(\hat{q}_1, \hat{q}_2, c_n) = \min_{\substack{0 \leq q_1 \leq p-1 \\ 0 \leq q_2 \leq p-1}} I(q_1, q_2, c_n)$$

and

$$I(q_1, q_2, c_n) = np + n_1 \sum_{i=1}^{q_1} \log l_i + n_2 \sum_{i=1}^{q_2} \log \xi_i + \left[\frac{n_1(p - q_1)}{+n_2(p - q_2)} \right] \log \left(\frac{n_1 \sum_{i=q_1+1}^p l_i + n_2 \sum_{i=q_2+1}^p \xi_i}{n_1(p - q_1) + n_2(p - q_2)} \right) + c_n \left\{ (q_1 + q_2) + 1 + (p - 1)(q_1 + q_2) - \frac{q_1(q_1 - 1)}{2} - \frac{q_2(q_2 - 1)}{2} \right\}$$

and c_n is defined the same as previously.

Hence log of likelihood ratio statistic is $L_1^* - L_2^*$, where L_1^* and L_2^* are given by (6) and (7) respectively. The critical value for this test can be approximated from the fact that asymptotically, $-2(L_1^* - L_2^*) \sim \chi^2_{\gamma(\hat{q}_1, \hat{q}_2, \hat{k})}$ under H_0 where,

$$\gamma(\hat{q}_1, \hat{q}_2, \hat{k}) = (\hat{q}_1 + \hat{q}_2 - 2\hat{k}) + (p - 1)(\hat{q}_1 + \hat{q}_2 - 2\hat{k}) + \hat{k}(\hat{k} - 1) - \frac{\hat{q}_1(\hat{q}_1 - 1)}{2} - \frac{\hat{q}_2(\hat{q}_2 - 1)}{2}$$

Likelihood Ratio Test: Case 2

Consider, $\Sigma_i = \sigma_i^2 I_p, i = 1, 2$. For case 2, the problem can be solved similarly and the problem is easier than that in case 1.

Likelihood Ratio Test: Case 3

Consider, σ^2 is known in case 1. Without loss of generality, $\sigma^2 = 1$ can be assumed and in that case, the log likelihood must be maximized with respect to the eigenvalues subject to the condition that the eigenvalues are greater than 1, in which case the technique presented by Zhao, Krishnaiah and Bai (1986a, b) can be used.

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