

11-1-2010

# On Bayesian Shrinkage Setup for Item Failure Data Under a Family of Life Testing Distribution

Gyan Prakash

S. N. Medical College, Agra, U. P., India, ggyanji@yahoo.com

Follow this and additional works at: <http://digitalcommons.wayne.edu/jmasm>

 Part of the [Applied Statistics Commons](#), [Social and Behavioral Sciences Commons](#), and the [Statistical Theory Commons](#)

## Recommended Citation

Prakash, Gyan (2010) "On Bayesian Shrinkage Setup for Item Failure Data Under a Family of Life Testing Distribution," *Journal of Modern Applied Statistical Methods*: Vol. 9 : Iss. 2 , Article 22.

DOI: 10.22237/jmasm/1288585260

Available at: <http://digitalcommons.wayne.edu/jmasm/vol9/iss2/22>

This Regular Article is brought to you for free and open access by the Open Access Journals at DigitalCommons@WayneState. It has been accepted for inclusion in Journal of Modern Applied Statistical Methods by an authorized editor of DigitalCommons@WayneState.

## On Bayesian Shrinkage Setup for Item Failure Data Under a Family of Life Testing Distribution

Gyan Prakash  
 S. N. Medical College, Agra, U. P., India

Properties of the Bayes shrinkage estimator for the parameter are studied of a family of probability density function when item failure data are available. The symmetric and asymmetric loss functions are considered for two different prior distributions. In addition, the Bayes estimates of reliability function and hazard rate are obtained and their properties are studied.

Key words: Bayes shrinkage estimator; squared error loss function (SELF); LINEX loss function (LLF); reliability function; hazard rate.

### Introduction

The probability density function (pdf) of a random variable  $x$  with parameter  $\theta$  and two known positive constants  $w$  and  $v$  for the proposed family of life testing distribution is given as

$$f(x; \theta, w, v) = \frac{v}{\Gamma_w} \left( \frac{x^{wv-1}}{\theta^w} \right) \exp\left(-\frac{x^v}{\theta}\right);$$

$$x > 0, \theta > 0, w, v > 0. \quad (1.1)$$

For the different values of  $w$  and  $v$ , the distributions are given as:

| $w$         | $v$ | Distribution          |
|-------------|-----|-----------------------|
| 1           | 1   | Exponential           |
|             | 1   | Two parameter Gamma   |
| +ve Integer | 1   | Erlang                |
| 1           |     | Two parameter Weibull |
| 1           | 2   | Rayleigh              |
| 3/2         | 2   | Maxwell               |

The use of SELF in the Bayes estimation may not be appropriate when positive and negative

Gyan Prakash is in the Department of Community Medicine. Email him at: ggyanji@yahoo.com.

errors have different consequences. To overcome this difficulty, an asymmetric loss function (LLF) was proposed by Varian (1975) and its invariant form for any parameter  $\theta$  is given by (see Singh, et al., 2007)

$$L(\Delta) = e^{a\Delta} - a\Delta - 1; a \neq 0$$

and

$$\Delta = (\hat{\theta} - \theta) / \theta. \quad (1.2)$$

where  $\hat{\theta}$  is any estimate of the parameter  $\theta$ . The sign and magnitude of 'a' represents the direction and degree of asymmetry respectively. The positive (negative) value of 'a' is used when overestimation is more (less) serious than underestimation. For small value of  $|a|$ , LLF is not far from SELF.

In many situations, the experimenter has some prior information about parameter in the form of a point guess value. Thompson (1968), Pandey and Singh (1977), Prakash and Singh (2006), Prakash and Singh (2008, 09) and others have suggested shrinkage estimators utilizing the point guess value of the parameter and have shown that they performed better when the guess value is in the vicinity of the true value. The shrinkage procedure has been applied in numerous problems, including mean survival time in epidemiological studies, forecasting of the money supply, estimating mortality rates and improving estimation in sample surveys.

## BAYESIAN SHRINKAGE SETUP FOR ITEM FAILURE DATA

The shrinkage estimator of the parameter  $\theta$  when a prior point guess value  $\theta_0$  is available, is given by

$$S = k \hat{\theta} + (1-k)\theta_0. \quad (1.3)$$

Here  $k \in [0,1]$  is the shrinkage factor and the experimenter according to his belief in the guess value specifies the values of the shrinkage factor.

In the item censored situations where  $n$  items are put to test without replacement and the test terminates as soon as the  $r^{\text{th}}$  item fails ( $r \leq n$ ).

Let  $x_1, x_2, \dots, x_r$  be the observed failure items for the first  $r$  components, then the likelihood function of  $r$  failure items  $\underline{x} = (x_1, x_2, \dots, x_r)$  is

$$L(\underline{x}; \theta) = h(\underline{x}) \frac{1}{\theta^{rw}} \exp\left(-\frac{T_r}{\theta}\right), \quad (1.4)$$

where  $T_r = \sum_{i=1}^r x_i^v + (n-r)x_{(r)}^v$  and the function  $h(\underline{x})$  is independent with the parameter  $\theta$ . The statistic  $T_r$  is sufficient for  $\theta$  and the UMVU estimator is  $U_r = T_r / rw$ . The risk of  $U_r$  under the SELF and LLF are obtained as

$$R_{(S)}(U_r) = \frac{\theta^2}{r}$$

and

$$R_{(L)}(U_r) = e^{-a} \left(1 - \frac{a}{r}\right)^{-r} - 1.$$

Here the suffixes S and L respectively show the risk taken under the SELF and LLF.

The inverted Gamma distribution with parameters  $\alpha$  and  $\beta$  have been considered as the conjugate prior density for the parameter  $\theta$  with pdf is

$$g_1(\theta) = \frac{\beta^\alpha}{\Gamma\alpha} \left(\frac{1}{\theta}\right)^{\alpha+1} \exp\left(-\frac{\beta}{\theta}\right); \theta > 0, \alpha, \beta > 0 \quad (1.5)$$

having the prior mean is

$$\frac{\beta}{\alpha-1}; \alpha > 1$$

and the prior variance is

$$\frac{\beta^2}{(\alpha-1)^2(\alpha-2)}; \alpha > 2.$$

For the situation where life researchers have no prior information about the parameter  $\theta$ , the uniform, quasi or improper prior may be used. This study considered a class of quasi prior defined as

$$g_2(\theta) = \frac{1}{\theta^d} \exp\left(-\frac{pd}{\theta}\right); \theta > 0, p, d > 0. \quad (1.6)$$

If  $d=0$  then a diffuse prior results, and if  $d=1, p=0$  then a non-informative prior results. For a set of values of  $d$  and  $p$ , that satisfies the equality  $\Gamma(d-1) = (pd)^{d-1}$  makes  $g_2(\theta)$  a proper prior. The prior mean and prior variance are given as

$$\frac{\Gamma(d-2)}{(pd)^{d-2}}; d \geq 2$$

and

$$\frac{\Gamma(d-3)}{(pd)^{2d-4}} \left( (pd)^{d-1} - (d-3)\Gamma(d-2) \right); d > 3.$$

Some Bayes estimators and Bayes Shrinkage estimators are suggested for the parameter  $\theta$  when other parameters are known. The properties of these estimators are studied in terms of relative efficiencies empirically and by numerical example. The Bayes estimator of reliability function and hazard rate are obtained and their properties are studied.

Methodology

Bayes Shrinkage Estimators and their Properties

The posterior density  $Z_1(\theta)$  for the parameter  $\theta$  corresponding to prior  $g_1(\theta)$  is obtained as

$$Z_1(\theta) = \frac{(r U_r + \beta)^{r w + \alpha}}{\Gamma(r w + \alpha)} \frac{e^{-(r U_r + \beta)/\theta}}{\theta^{r w + \alpha + 1}}; \theta > 0. \quad (2.1)$$

which is an inverted Gamma distribution with parameters  $(r w + \alpha)$  and  $(r U_r + \beta)$ . The Bayes estimator of  $\theta$  under SELF is obtained as

$$\hat{\theta}_1 = E_p(\theta) = \varphi_1 (r U_r + \beta), \quad (2.2)$$

where  $\varphi_1 = (r w + \alpha - 1)^{-1}$  and suffix  $p$  indicates, the expectation is considered under the posterior density.

To utilize the prior information about the parameter  $\theta$  in the form of a point guess value  $\theta_0$ , the values of prior parameter  $\beta$  are chosen (Shirke & Nalawade, 2003) such as

$$E(\hat{\theta}_1) = \theta_0 \Rightarrow \varphi_1 \beta = (1 - \varphi_1 r) \theta_0. \quad (2.3)$$

Using (2.3) in (2.2), the shrinkage estimator takes the form (1.3) and is named the Bayes shrinkage estimator:

$$\bar{\theta}_1 = \lambda_1 U_r + (1 - \lambda_1) \theta_0; \lambda_1 = r \varphi_1. \quad (2.4)$$

The Bayes estimator  $\theta$  under the LLF is obtained by simplifying the given equality

$$E_p\left(\frac{1}{\theta} e^{a \hat{\theta}_2 / \theta}\right) = e^a E_p\left(\frac{1}{\theta}\right) \Rightarrow \hat{\theta}_2 = \varphi_2 (r U_r + \beta) \quad (2.5)$$

Similarly, the Bayes shrinkage estimator under the LLF is

$$\bar{\theta}_2 = \lambda_2 U_r + (1 - \lambda_2) \theta_0, \quad (2.6)$$

where

$$\lambda_2 = r \varphi_2, \varphi_2 = \frac{1}{a} \left(1 - \exp\left(\frac{-a}{r w + \alpha + 1}\right)\right).$$

The risks under the SELF and LLF of these estimators are obtained as

$$R_{(S)}(\hat{\theta}_i) = \theta^2 (r(r+1)\varphi_i^2 - 2r\varphi_i + 1) + 2\theta\beta\varphi_i(r\varphi_i - 1) + \beta^2\varphi_i^2,$$

$$R_{(L)}(\hat{\theta}_i) = \frac{e^{C_i}}{(1 - a\varphi_i)^r} - (a\varphi_i r + C_i + 1),$$

$$C_i = a \left( \frac{\varphi_i \beta}{\theta} - 1 \right),$$

$$R_{(S)}(\bar{\theta}_i) = \theta^2 \left\{ \lambda_i^2 \left( \frac{r+1}{r} + \delta(\delta-2) \right) + (1-\delta)^2 (1-2\lambda_i) \right\}$$

and

$$R_{(L)}(\bar{\theta}_i) = \frac{r^r e^{a(\delta(1-\lambda_i)-1)}}{(r - a\lambda_i)^r} - 1 + a(1-\delta)(1-\lambda_i);$$

$$\delta = \frac{\theta_0}{\theta}, i=1, 2.$$

Similarly, the Bayes risks of these estimators are

$$R_{(BS)}(\hat{\theta}_i) = \beta^2 \left( \frac{r(r+1)\varphi_i^2 - 2r\varphi_i + 1}{(\alpha-1)(\alpha-2)} + \frac{2\varphi_i(r\varphi_i - 1)}{(\alpha-1)} + \varphi_i^2 \right),$$

$$R_{(BL)}(\hat{\theta}_i) = \frac{e^{-a}}{(1 - a\varphi_i)^{\alpha+r}} - a\varphi_i(\alpha+r) + a - 1,$$

$$R_{(BS)}(\bar{\theta}_i) = (1 - \lambda_i)^2 \left( \theta_0^2 - \frac{2\beta\theta_0}{\alpha-1} + \frac{\beta^2}{(\alpha-1)(\alpha-2)} \right) + \frac{\beta^2 \lambda_i^2}{r(\alpha-1)(\alpha-2)}$$

and

$$R_{(BL)}(\bar{\theta}_i) = \frac{r^r e^{-a}}{(r - a \lambda_i)^r} \left( 1 - \frac{a(1 - \lambda_i)\theta_0}{\beta} \right)^{-a} \\ - 1 + \frac{a(\beta - \alpha\theta_0)}{\beta(1 - \lambda_i)^{-1}} \\ - 1 + \frac{a(\beta - \alpha\theta_0)}{\beta(1 - \lambda_i)^{-1}}.$$

The relative efficiency of  $\bar{\theta}_i$  with respect to  $U_r$  under the SELF and LLF loss criterions are defined as

$$RE_{(S)}(\bar{\theta}_i, U_r) = \frac{R_{(S)}(U_r)}{R_{(S)}(\bar{\theta}_i)}$$

and

$$RE_{(L)}(\bar{\theta}_i, U_r) = \frac{R_{(L)}(U_r)}{R_{(L)}(\bar{\theta}_i)}; i = 1, 2.$$

The expressions of relative efficiencies are the functions of  $r, a, v, \delta$  and  $\alpha$ . For the selected values of  $r = 04(02)10$ ;  $a = 0.25, 0.50, 1.00, 1.50$ ;  $v = 1.00, 1.50$ ;  $\delta = 0.40, 0.50(0.25) 1.50, 1.60$  and  $\alpha = 1.50, 03, 05, 07, 10, 15$ . The relative efficiencies have been calculated and are presented in Tables 1–4 for selected parametric values.

The positive values of 'a' are considered because overestimation in mean life is more serious than underestimation. To guard against the large risk, the large values of 'a' may be ignored and the smaller values of  $a (\leq 2)$  are considered (see Singh, et al., 2007).

Results

Tables 1 and 2 show that the estimator  $\bar{\theta}_1$  is more efficient than the UMVU estimator  $U_r$  under the SELF and LLF for all selected parametric set of values when  $r \leq 06$ . For large  $r \geq 08$ , the effective interval decreases with large  $\alpha \geq 10$ . The efficiency attains maximum at the point  $\delta = 1.00$  (except  $\alpha = 1.50$  when the loss criterion is LLF) and the efficiency

decreases as  $r$  increases for all considered values of  $\delta$ .

In addition, under LLF loss criterion the efficiency increases with 'a' increases for all considered values of  $\delta$  for small  $r \leq 06$ , and for large  $r$  in the interval  $\delta \leq 1.00$ . The estimator  $\bar{\theta}_2$  performs uniformly well with respect to  $U_r$  for all considered values of the parametric space when  $r$  is small and in the interval  $0.50 \leq \delta \leq 1.25$  when  $r \geq 06$  (under the SELF-criterion).

In addition, the effective interval decreases as  $r$  or  $\alpha$  increases (Table 3). The increasing trend in efficiency is also observed when 'a' increases in the interval  $\delta \in [0.75, 1.25]$  for other fixed parametric values.

The estimator  $\bar{\theta}_2$  performs uniformly well with respect to  $U_r$  under LLF loss criterion for all considered values of the parameter space (Table 4). The increasing trend in efficiency is observed when 'a' increases for all  $\delta$  when  $r$  is small and in the range  $\delta \leq 1.25$  when  $r \geq 08$ .

Using Tables 3 and 4, it may be concluded that the efficiency reaches its maximum at the point  $\delta = 1.00$ . The efficiency decreases as  $r$  increases for all considered values of parametric space.

Further, as  $v$  increases, the gain in efficiency is recorded only in close vicinity of the guess value and true value of the parameter but the effective interval becomes smaller for both the Bayes shrinkage estimators.

Remark

Note that the posterior density with respect to the quasi prior  $g_2(\theta)$  is

$$Z_2(\theta) = \frac{(r U_r + p d)^{r w + d - 1} e^{-(r U_r + p d)/\theta}}{\Gamma(r w + d - 1) \theta^{(r w + d)}}.$$

The only changes in the posterior (2.1) are replacement  $\alpha$  and  $\beta$  by  $d - 1$  and  $p d$  respectively. Hence, all the results are valid by substitution of these two.

PRAKASH

Table 1: Relative Efficiency for the Bayes Shrinkage Estimator  $\bar{\theta}_1$  with Respect to  $U_r$  Under the SELF

| r  | $\delta$ | $\alpha$ |        |        |        |        |        |
|----|----------|----------|--------|--------|--------|--------|--------|
|    |          | 1.50     | 3.00   | 5.00   | 7.00   | 10.00  | 15.00  |
| 04 | 0.40     | 1.2378   | 1.6544 | 1.6393 | 1.4741 | 1.2741 | 1.0864 |
|    | 0.50     | 1.2462   | 1.8000 | 2.0000 | 1.9231 | 1.7423 | 1.5283 |
|    | 0.75     | 1.2607   | 2.1176 | 3.2000 | 4.0000 | 4.6621 | 4.9846 |
|    | 1.00     | 1.2656   | 2.2500 | 4.0000 | 6.2500 | 10.562 | 20.250 |
|    | 1.25     | 1.2607   | 2.1176 | 3.2000 | 4.0000 | 4.6621 | 4.9846 |
|    | 1.50     | 1.2462   | 1.8000 | 2.0000 | 1.9231 | 1.7423 | 1.5283 |
|    | 1.60     | 1.2378   | 1.6544 | 1.6393 | 1.4741 | 1.2741 | 1.0864 |

|    |      |        |        |        |        |        |        |
|----|------|--------|--------|--------|--------|--------|--------|
| 06 | 0.40 | 1.1563 | 1.4337 | 1.4172 | 1.2658 | 1.0866 | 1.0708 |
|    | 0.50 | 1.1615 | 1.5238 | 1.6667 | 1.6000 | 1.4286 | 1.2121 |
|    | 0.75 | 1.1706 | 1.7067 | 2.3810 | 2.9091 | 3.3898 | 3.6530 |
|    | 1.00 | 1.1736 | 1.7778 | 2.7778 | 4.0000 | 6.2500 | 11.111 |
|    | 1.25 | 1.1706 | 1.7067 | 2.3810 | 2.9091 | 3.3898 | 3.6530 |
|    | 1.50 | 1.1615 | 1.5238 | 1.6667 | 1.6000 | 1.4286 | 1.2121 |
|    | 1.60 | 1.1563 | 1.4337 | 1.4172 | 1.2658 | 1.0866 | 1.0708 |

|    |      |        |        |        |        |        |        |
|----|------|--------|--------|--------|--------|--------|--------|
| 10 | 0.40 | 1.1163 | 1.3242 | 1.3081 | 1.1689 | 1.0721 | 0.7701 |
|    | 0.50 | 1.1202 | 1.3889 | 1.5000 | 1.4412 | 1.2788 | 1.0614 |
|    | 0.75 | 1.1267 | 1.5152 | 2.0000 | 2.3902 | 2.7656 | 2.9877 |
|    | 1.00 | 1.1289 | 1.5625 | 2.2500 | 3.0625 | 4.5156 | 7.5625 |
|    | 1.25 | 1.1267 | 1.5152 | 2.0000 | 2.3902 | 2.7656 | 2.9877 |
|    | 1.50 | 1.1202 | 1.3889 | 1.5000 | 1.4412 | 1.2788 | 1.0614 |
|    | 1.60 | 1.1163 | 1.3242 | 1.3081 | 1.1689 | 1.0721 | 0.7701 |

|    |      |        |        |        |        |        |        |
|----|------|--------|--------|--------|--------|--------|--------|
| 15 | 0.40 | 1.0927 | 1.2587 | 1.2437 | 1.1150 | 1.0219 | 0.7150 |
|    | 0.50 | 1.0957 | 1.3091 | 1.4000 | 1.3474 | 1.1934 | 0.9763 |
|    | 0.75 | 1.1008 | 1.4049 | 1.7818 | 2.0898 | 2.3967 | 2.5888 |
|    | 1.00 | 1.1025 | 1.4400 | 1.9600 | 2.5600 | 3.6100 | 5.7600 |
|    | 1.25 | 1.1008 | 1.4049 | 1.7818 | 2.0898 | 2.3967 | 2.5888 |
|    | 1.50 | 1.0957 | 1.3091 | 1.4000 | 1.3474 | 1.1934 | 0.9763 |
|    | 1.60 | 1.0927 | 1.2587 | 1.2437 | 1.1150 | 1.0219 | 0.7150 |

BAYESIAN SHRINKAGE SETUP FOR ITEM FAILURE DATA

Table 2: Relative Efficiency for the Bayes Shrinkage Estimator  $\bar{\theta}_1$  with respect to  $U_r$  under the LLF

| r = 04 |          | $\alpha$ |        |        |        |        |        |
|--------|----------|----------|--------|--------|--------|--------|--------|
| a      | $\delta$ | 1.50     | 3.00   | 5.00   | 7.00   | 10.00  | 15.00  |
| 0.25   | 0.40     | 1.2666   | 1.7672 | 1.7814 | 1.6073 | 1.3911 | 1.1874 |
|        | 0.50     | 1.2715   | 1.9093 | 2.1609 | 2.0867 | 1.8925 | 1.6607 |
|        | 0.75     | 1.2770   | 2.2010 | 3.3852 | 4.2651 | 4.9906 | 5.3386 |
|        | 1.00     | 1.2729   | 2.2882 | 4.1003 | 6.4364 | 10.923 | 21.021 |
|        | 1.25     | 1.2594   | 2.1145 | 3.2063 | 4.0280 | 4.7259 | 5.0860 |
|        | 1.50     | 1.2307   | 1.7608 | 1.9872 | 1.9241 | 1.7524 | 1.5412 |
|        | 1.60     | 1.2257   | 1.6068 | 1.6256 | 1.4709 | 1.2763 | 1.0894 |
| 0.50   | 0.40     | 1.2997   | 1.9024 | 1.9552 | 1.7709 | 1.5347 | 1.3115 |
|        | 0.50     | 1.3007   | 2.0404 | 2.3580 | 2.2880 | 2.0777 | 1.8240 |
|        | 0.75     | 1.2968   | 2.3032 | 3.6144 | 4.5947 | 5.4006 | 5.7815 |
|        | 1.00     | 1.2835   | 2.3423 | 4.2400 | 6.6931 | 11.414 | 22.060 |
|        | 1.25     | 1.2613   | 2.1256 | 3.2413 | 4.0963 | 4.8409 | 5.2460 |
|        | 1.50     | 1.2311   | 1.7640 | 1.9921 | 1.9437 | 1.7805 | 1.5706 |
|        | 1.60     | 1.2310   | 1.6108 | 1.6262 | 1.4818 | 1.2912 | 1.1036 |
| 1.00   | 0.40     | 1.3830   | 2.2709 | 2.4450 | 2.2347 | 1.9422 | 1.6636 |
|        | 0.50     | 1.3757   | 2.3977 | 2.9135 | 2.8603 | 2.6054 | 2.2894 |
|        | 0.75     | 1.3510   | 2.5174 | 4.2662 | 5.5404 | 6.5833 | 7.0633 |
|        | 1.00     | 1.3182   | 2.5891 | 4.6862 | 7.5059 | 12.955 | 25.292 |
|        | 1.25     | 1.2782   | 2.2041 | 3.4259 | 4.3950 | 5.2801 | 5.8092 |
|        | 1.50     | 1.2321   | 1.7870 | 2.0697 | 2.0563 | 1.9090 | 1.6956 |
|        | 1.60     | 1.2323   | 1.6214 | 1.6820 | 1.5583 | 1.3718 | 1.1766 |
| 1.50   | 0.40     | 1.5036   | 2.8418 | 3.2760 | 3.0281 | 2.6406 | 2.2668 |
|        | 0.50     | 1.4861   | 2.8655 | 3.8543 | 3.8407 | 3.5120 | 3.0895 |
|        | 0.75     | 1.4372   | 2.9731 | 5.3708 | 7.1652 | 8.6319 | 9.2915 |
|        | 1.00     | 1.3822   | 3.0604 | 5.5108 | 9.0029 | 15.782 | 31.192 |
|        | 1.25     | 1.3225   | 2.4017 | 3.8550 | 5.0453 | 6.1813 | 6.9176 |
|        | 1.50     | 1.2594   | 1.9032 | 2.2881 | 2.3249 | 2.1938 | 1.9654 |
|        | 1.60     | 1.2335   | 1.7157 | 1.8501 | 1.7503 | 1.5608 | 1.3455 |

PRAKASH

Table 3: Relative Efficiency for the Bayes Shrinkage Estimator  $\bar{\theta}_2$  with Respect to  $U_r$  Under the SELF

| r = 04 |          | $\alpha$ |        |        |        |        |        |
|--------|----------|----------|--------|--------|--------|--------|--------|
| a      | $\delta$ | 1.50     | 3.00   | 5.00   | 7.00   | 10.00  | 15.00  |
| 0.25   | 0.40     | 1.6932   | 1.6299 | 1.4640 | 1.3237 | 1.1778 | 1.0374 |
|        | 0.50     | 1.9175   | 1.9995 | 1.9156 | 1.7925 | 1.6368 | 1.4674 |
|        | 0.75     | 2.4770   | 3.2597 | 4.0396 | 4.5231 | 4.8735 | 5.0000 |
|        | 1.00     | 2.7438   | 4.1266 | 6.4079 | 9.1891 | 14.298 | 25.314 |
|        | 1.25     | 2.4770   | 3.2597 | 4.0396 | 4.5231 | 4.8735 | 5.0000 |
|        | 1.50     | 1.9175   | 1.9995 | 1.9156 | 1.7925 | 1.6368 | 1.4674 |
|        | 1.60     | 1.6932   | 1.6299 | 1.4640 | 1.3237 | 1.1778 | 1.0374 |

|      |      |        |        |        |        |        |        |
|------|------|--------|--------|--------|--------|--------|--------|
| 0.50 | 0.40 | 1.6944 | 1.6199 | 1.4540 | 1.3161 | 1.1730 | 1.0348 |
|      | 0.50 | 1.9341 | 1.9981 | 1.9079 | 1.7850 | 1.6312 | 1.4642 |
|      | 0.75 | 2.5485 | 3.3188 | 4.0782 | 4.5455 | 4.8817 | 4.9998 |
|      | 1.00 | 2.8504 | 4.2566 | 6.5691 | 9.3816 | 14.537 | 25.631 |
|      | 1.25 | 2.5485 | 3.3188 | 4.0782 | 4.5455 | 4.8817 | 4.9998 |
|      | 1.50 | 1.9341 | 1.9981 | 1.9079 | 1.7850 | 1.6312 | 1.4642 |
|      | 1.60 | 1.6944 | 1.6199 | 1.4540 | 1.3161 | 1.1730 | 1.0348 |

|      |      |        |        |        |        |        |        |
|------|------|--------|--------|--------|--------|--------|--------|
| 1.00 | 0.40 | 1.6917 | 1.5990 | 1.4342 | 1.3013 | 1.1636 | 1.0298 |
|      | 0.50 | 1.9611 | 1.9928 | 1.8922 | 1.7701 | 1.6203 | 1.4578 |
|      | 0.75 | 2.6919 | 3.4349 | 4.1529 | 4.5883 | 4.8971 | 4.9992 |
|      | 1.00 | 3.0737 | 4.5267 | 6.9016 | 9.7765 | 15.026 | 26.276 |
|      | 1.25 | 2.6919 | 3.4349 | 4.1529 | 4.5883 | 4.8971 | 4.9992 |
|      | 1.50 | 1.9611 | 1.9928 | 1.8922 | 1.7701 | 1.6203 | 1.4578 |
|      | 1.60 | 1.6917 | 1.5990 | 1.4342 | 1.3013 | 1.1636 | 1.0298 |

|      |      |        |        |        |        |        |        |
|------|------|--------|--------|--------|--------|--------|--------|
| 1.50 | 0.40 | 1.6830 | 1.5770 | 1.4147 | 1.2868 | 1.1544 | 1.0249 |
|      | 0.50 | 1.9806 | 1.9846 | 1.8760 | 1.7554 | 1.6096 | 1.4516 |
|      | 0.75 | 2.8351 | 3.5477 | 4.2240 | 4.6284 | 4.9113 | 4.9983 |
|      | 1.00 | 3.3113 | 4.8108 | 7.2479 | 10.185 | 15.528 | 26.934 |
|      | 1.25 | 2.8351 | 3.5477 | 4.2240 | 4.6284 | 4.9113 | 4.9983 |
|      | 1.50 | 1.9806 | 1.9846 | 1.8760 | 1.7554 | 1.6096 | 1.4516 |
|      | 1.60 | 1.6830 | 1.5770 | 1.4147 | 1.2868 | 1.1544 | 1.0249 |



BAYESIAN SHRINKAGE SETUP FOR ITEM FAILURE DATA

Table 4: Relative Efficiency for the Bayes Shrinkage Estimator  $\bar{\theta}_2$  with respect to  $U_r$  under the LLF

| r = 04 |          | $\alpha$ |        |        |        |        |        |
|--------|----------|----------|--------|--------|--------|--------|--------|
| a      | $\delta$ | 1.50     | 3.00   | 5.00   | 7.00   | 10.00  | 15.00  |
| 0.25   | 0.40     | 1.8240   | 1.7718 | 1.5965 | 1.4448 | 1.2866 | 1.1344 |
|        | 0.50     | 2.0515   | 2.1615 | 2.0788 | 1.9469 | 1.7782 | 1.5948 |
|        | 0.75     | 2.5926   | 3.4508 | 4.3086 | 4.8389 | 5.2200 | 5.3538 |
|        | 1.00     | 2.7988   | 4.2316 | 6.6005 | 9.4934 | 14.815 | 26.304 |
|        | 1.25     | 2.4751   | 3.2671 | 4.0691 | 4.5772 | 4.9570 | 5.1097 |
|        | 1.50     | 1.8947   | 1.9878 | 1.9172 | 1.8010 | 1.6490 | 1.4802 |
|        | 1.60     | 1.6636   | 1.6171 | 1.4613 | 1.3250 | 1.1808 | 1.0401 |
| 0.50   | 0.40     | 1.9892   | 1.9353 | 1.7475 | 1.5846 | 1.4144 | 1.2504 |
|        | 0.50     | 2.2395   | 2.3606 | 2.2710 | 2.1283 | 1.9461 | 1.7481 |
|        | 0.75     | 2.8215   | 3.7598 | 4.6905 | 5.2598 | 5.6626 | 5.7964 |
|        | 1.00     | 2.9917   | 4.5192 | 7.0418 | 10.119 | 15.778 | 27.986 |
|        | 1.25     | 2.5656   | 3.3663 | 4.1820 | 4.7057 | 5.1063 | 5.2799 |
|        | 1.50     | 1.9049   | 1.9946 | 1.9310 | 1.8208 | 1.6731 | 1.5056 |
|        | 1.60     | 1.6644   | 1.6175 | 1.4633 | 1.3322 | 1.1910 | 1.0510 |
| 1.00   | 0.40     | 2.4607   | 2.4012 | 2.1776 | 1.9825 | 1.7780 | 1.5802 |
|        | 0.50     | 2.7745   | 2.9280 | 2.8197 | 2.6462 | 2.4254 | 2.1857 |
|        | 0.75     | 3.4681   | 4.6392 | 5.7837 | 6.4691 | 6.9391 | 7.0775 |
|        | 1.00     | 3.5344   | 5.3440 | 8.3263 | 11.959 | 18.629 | 32.999 |
|        | 1.25     | 2.8427   | 3.7029 | 4.5902 | 5.1748 | 5.6420 | 5.8720 |
|        | 1.50     | 1.9870   | 2.0828 | 2.0351 | 1.9345 | 1.7908 | 1.6201 |
|        | 1.60     | 1.6975   | 1.6564 | 1.5234 | 1.3986 | 1.2587 | 1.1151 |
| 1.50   | 0.40     | 3.2712   | 3.2019 | 2.9164 | 2.6657 | 2.4019 | 2.1455 |
|        | 0.50     | 3.6926   | 3.9030 | 3.7633 | 3.5375 | 3.2503 | 2.9387 |
|        | 0.75     | 4.4534   | 6.1474 | 7.6678 | 8.5593 | 9.1522 | 9.3049 |
|        | 1.00     | 4.5695   | 6.7637 | 10.564 | 15.186 | 23.661 | 41.896 |
|        | 1.25     | 3.3496   | 4.3562 | 5.4076 | 6.1173 | 6.7074 | 7.0303 |
|        | 1.50     | 2.2055   | 2.3275 | 2.3004 | 2.2058 | 2.0581 | 1.8726 |
|        | 1.60     | 1.8476   | 1.8227 | 1.6995 | 1.5742 | 1.4268 | 1.2692 |

Example: Exponential Failure Model

Two hundred electronic tubes were tested under the exponential failure model with the parameter  $\theta=04$  and the test was terminated after the first six items failed. The failure times (in hours) were recorded as follows:

83.5, 221, 356, 478, 535, 632

The relative efficiencies and Bayes risks of the proposed estimators were obtained for  $\alpha = 5.00, 8.50, 10$ ;  $\beta = 20, 32, 50$ ;  $a = 0.50, 1.00, 1.50$  and are presented in Table 5 for  $\alpha = 8.50, \beta = 32$  and  $a = 0.50$ .

It may be concluded that the relative efficiency attains maximum at point  $\theta = \theta_0$  for all considered values. Under the LLF criterion the gain in efficiency is larger than the SELF-criterion when  $\delta \leq 1.25$ . Further, the Bayes risks are nominal when the loss criterion is LLF. The risks have the tendency to be smaller when  $\theta > \theta_0$  and attains minimum when  $\theta = \theta_0$  and then increases. Further, both the risk and Bayes risk decreases (increases) when 'a'( $\alpha$ ) increases under both loss criterions when other parametric values are fixed.

The Bayes Estimator of Reliability Function and Hazard Rate

The Reliability function  $\Psi(t)$  at time  $t(>0)$  is defined as

$$\Psi(t) = \frac{1}{\Gamma(w)} \int_{t^{v/\theta}}^{\infty} e^{-S} S^{w-1} dS. \quad (3.1)$$

Similarly, the Hazard rate at time  $t(>0)$  is given by

$$\rho(t) = \frac{v t^{wv-1}}{\theta^w e^{t^{v/\theta}}} \left\{ \int_{t^{v/\theta}}^{\infty} e^{-S} S^{w-1} dS \right\}^{-1}. \quad (3.2)$$

In particular, for the exponential distribution ( $w = v = 1$ ) the Reliability function and Hazard rate are given as

$$\Psi(t) = \exp(-t/\theta) \text{ and } \rho(t) = 1/\theta. \quad (3.3)$$

The Bayes estimator of the reliability function and hazard rates under the SELF, corresponding to the posterior density  $Z_1(\theta)$  are obtained as

$$\Psi_1 = G(0, \infty, J_1) \text{ and } \rho_1 = G(0, \infty, J_2); \quad (3.4)$$

where

$$G(u_1, u_2, \xi) = \frac{(r U_r + \beta)^{r_1} u_2}{\Gamma(r_1)} \int_{u_1}^{\infty} e^{-(r U_r + \beta)z} z^{r_1-1}(\xi) dz,$$

$\xi$  be the function of  $z$ ,  $r_1 = r w + \alpha$ ,

$$J_1 = \int_{t^{vz}}^{\infty} \frac{e^{-S} S^{w-1}}{\Gamma(w)} dS \text{ and } J_2 = \frac{v z^w t^{wv-1}}{e^{t^{vz}}} \left\{ \int_{t^{vz}}^{\infty} e^{-S} S^{w-1} dS \right\}^{-1}.$$

Similarly, the Bayes estimate of the reliability function  $\Psi_2$ , and the Hazard rate  $\rho_2$  under the LLF for the given posterior  $Z_1(\theta)$  are obtained by solving the given equality

$$G(0, \infty, (J_1^{-1} \exp(a J_1^{-1} \Psi_{(L)}(t)))) = e^a G(0, \infty, J_1^{-1})$$

and

$$G(0, \infty, (J_2^{-1} \exp(a J_2^{-1} \rho_{(L)}(t)))) = e^a G(0, \infty, J_2^{-1}) \quad (3.5)$$

The close form of the Bayes estimators  $\Psi(t)$  and  $\rho(t)$  under the LLF are nonexistent, therefore, the risk and Bayes risks do not exist in the closed form. For convenience, consider Varian's (1975) asymmetric loss function defined for any parameter  $\theta$  as

$$L(\Delta') = e^{a \Delta'} - a \Delta' - 1; \Delta' = \hat{\theta} - \theta.$$

Hence, the Bayes estimators  $\Psi(t)$  and  $\rho(t)$  under the LLF are given by

$$\Psi_3 = -\frac{1}{a} \ln \left\{ \frac{(r U_r + \beta)^{r_1}}{\Gamma(r_1)} G(0, \infty, e^{-aJ_1}) \right\}$$

and

$$\rho_3 = -\frac{1}{a} \ln \left\{ \frac{(r U_r + \beta)^{r_1}}{\Gamma(r_1)} G(0, \infty, e^{-aJ_2}) \right\}. \tag{3.6}$$

The risk and Bayes risk of these Bayes estimators under the SELF and LLF do not exist in a closed form. However, the numerical findings of the risk and Bayes risk for these Bayes estimators under SELF and LLF  $R_{(S)}(\Psi_i)$ ,  $R_{(L)}(\Psi_i)$ ,  $R_{(BS)}(\Psi_i)$ ,  $R_{(BL)}(\Psi_i)$ ,  $R_{(S)}(\rho_i)$ ,  $R_{(L)}(\rho_i)$ ,  $R_{(BS)}(\rho_i)$  and  $R_{(BL)}(\rho_i)$ ;  $i=1,3$  are obtained for a particular case when  $v = w = 1$ .

Example: Risks and Bayes Risks

Consider the above example with  $t = 250h$  the Bayes estimates for the reliability function and hazard rate, risks and Bayes risks as obtained are presented in Table 6.

The risk of the estimator  $\Psi_1$  increases as  $\beta$  increases when  $\alpha \geq 8.50$  under the LLF. A similar trend is observed for the risk and Bayes risk of  $\Psi_3$  as  $\beta$  increases when  $\alpha \geq 8.50$  under the LLF.

Further, the Bayes risk of  $\Psi_1$  and  $\Psi_3$  increases when 'a' increases under both loss criterions. The risk and Bayes risk decreases when  $\alpha$  increases (except  $\beta \geq 50.00$ ) when other parametric values are fixed for  $\Psi_1$  (LLF-criterion) and  $\Psi_3$  (SELF and LLF criterions).

The risk and Bayes risk for the estimators  $\rho_1$  and  $\rho_3$  increases as  $\beta$  increases for all the considered values of  $\alpha$  under the SELF and LLF (except  $\alpha = 5.00$ ) when other parametric values are fixed. The risk and Bayes risk of  $\rho_1$  and  $\rho_3$  also increases

(decreases) under both loss criterions when 'a' ( $\alpha$ ) increases.

References

Pandey, B. N., & Singh, J. (1977). Estimation of variance of Normal population using apriori information. *Journal of Indian Statistical Association*, 15, 141-150.

Prakash, G., & Singh, D. C. (2006). Shrinkage testimators for the inverse dispersion of the inverse Gaussian distribution under the LINEX loss function. *Austrian Journal of Statistics*, 35(4), 463-470.

Prakash, G., and Singh, D.C. (2008). Shrinkage Estimation in Exponential Type-II Censored Data under LINEX Loss. *Journal of the Korean Statistical Society*, 37 (1), 53-61.

Prakash, G., and Singh, D. C. (2009). A Bayesian Shrinkage Approach in Weibull Type-II Censored Data Using Prior Point Information. *REVSTAT – Statistical Journal*, 7 (2), 171-187.

Shirke, D. T., & Nalawade, K. T. (2003). Estimation of the parameter of binomial distribution in presence of prior point information. *Journal of the Indian Statistical Association*, 41(1), 117-128.

Singh, D. C., Prakash, G., & Singh, P. (2007). Shrinkage testimators for the shape parameter of Pareto distribution using the LINEX loss function. *Communication in Statistics Theory and Methods*, 36(4), 741-753.

Thompson, J. R. (1968). Some shrinkage techniques for estimating the mean. *Journal of the American Statistical Association*, 63, 113-122.

Varian, H. R. (1975). A Bayesian approach to real estate assessment. In *Studies in Bayesian Econometrics and Statistics in Honor of L. J. Savage*, Eds. S. E. Feinberge, and A. Zellner, 195-208. Amsterdam: North Holland.

Table 5: Risk and Bayes Risk of the Bayes Shrinkage Estimators

| r=06 :: $\theta=04$ :: $\alpha=8.50$ :: $\beta=32.00$ :: a=0.50 |        |        |        |        |        |        |        |
|---|--------|--------|--------|--------|--------|--------|--------|
| $\delta$  | 0.40   | 0.50   | 0.75   | 1.00   | 1.25   | 1.50   | 1.60   |
| $\theta_0$  | 1.60   | 2.00   | 3.00   | 4.00   | 5.00   | 6.00   | 6.40   |
| $RE_{(S)}(\bar{\theta}_1, U_r)$                                 | 1.1571 | 1.5140 | 3.1921 | 5.0625 | 3.1921 | 1.5140 | 1.1571 |
| $RE_{(L)}(\bar{\theta}_1, U_r)$                                 | 1.3339 | 1.7305 | 3.5411 | 5.2767 | 3.1873 | 1.4909 | 1.1333 |
| $RE_{(S)}(\bar{\theta}_2, U_r)$                                 | 1.0278 | 1.3889 | 3.4623 | 6.8918 | 3.4623 | 1.3889 | 1.0278 |
| $RE_{(L)}(\bar{\theta}_2, U_r)$                                 | 1.1872 | 1.5902 | 3.8596 | 7.2137 | 3.4783 | 1.3748 | 1.0101 |
| $R_{(BS)}(\bar{\theta}_1)$                                      | 3.7507 | 3.1416 | 2.0511 | 1.5778 | 1.7219 | 2.4832 | 2.9606 |
| $R_{(BL)}(\bar{\theta}_1)$                                      | 0.0164 | 0.0130 | 0.0103 | 0.0098 | 0.0180 | 0.0336 | 0.0421 |
| $R_{(BS)}(\bar{\theta}_2)$                                      | 4.3068 | 3.5505 | 2.1963 | 1.6086 | 1.7875 | 2.7328 | 3.3256 |
| $R_{(BL)}(\bar{\theta}_2)$                                      | 0.0184 | 0.0141 | 0.0183 | 0.0100 | 0.0202 | 0.0398 | 0.0505 |

Table 6: Risk and Bayes Risk of the Reliability Function and Hazard Rates

| r=06 :: $\theta=04$ :: t=250 :: $\alpha=8.50$ :: $\beta=32.00$ |        |        |        |                    |        |        |        |
|--|--------|--------|--------|--------------------|--------|--------|--------|
| a →  | 0.50   | 1.00   | 1.50   | a →                | 0.50   | 1.00   | 1.50   |
| $\Psi_1$   | 0.0097 | 0.0097 | 0.0097 | $\Psi_3$           | 97.860 | 48.930 | 32.620 |
| $R_{(S)}(\Psi_1)$  | 16.000 | 16.000 | 16.000 | $R_{(S)}(\Psi_3)$  | 7.1480 | 9.0530 | 10.810 |
| $R_{(BS)}(\Psi_1)$   | 35.192 | 35.192 | 35.192 | $R_{(BS)}(\Psi_3)$ | 37.160 | 38.380 | 42.950 |
| $R_{(L)}(\Psi_1)$  | 1.1353 | 3.0183 | 5.0025 | $R_{(L)}(\Psi_3)$  | 1.1550 | 1.9160 | 3.8300 |
| $R_{(BL)}(\Psi_1)$   | 1.7744 | 4.1825 | 6.6077 | $R_{(BL)}(\Psi_3)$ | 1.9700 | 4.3600 | 6.7790 |
| a →  | 0.50   | 1.00   | 1.50   | a →                | 0.50   | 1.00   | 1.50   |
| $\rho_1$   | 0.0218 | 0.0218 | 0.0218 | $\rho_3$           | 53.270 | 26.704 | 17.849 |
| $R_{(S)}(\rho_1)$  | 13.941 | 13.941 | 13.941 | $R_{(S)}(\rho_3)$  | 6.7758 | 8.2750 | 9.9006 |
| $R_{(BS)}(\rho_1)$   | 33.257 | 33.257 | 33.257 | $R_{(BS)}(\rho_3)$ | 35.803 | 37.005 | 41.536 |
| $R_{(L)}(\rho_1)$  | 1.0214 | 2.7575 | 4.6040 | $R_{(L)}(\rho_3)$  | 1.1513 | 1.7864 | 3.6102 |
| $R_{(BL)}(\rho_1)$   | 1.7085 | 4.0468 | 6.4038 | $R_{(BL)}(\rho_3)$ | 1.9298 | 4.2714 | 6.6433 |