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SPECTRAL COLLOCATION METHOD FOR COMPACT INTEGRAL OPERATORS

by

CAN HUANG

DISSERTATION

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DEDICATION

To My Parents

Siyong Huang and Yueping Zheng

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1 Introduction

The early history of integral equations goes back to the special integral equations studied by several mathematicians of late eighteenth century and early nineteenth century–Laplace, Fourier, Poisson, Abel and Liouville while the systematic study of integral equations begins with work of Volterra [41], where he transformed an integral equation $\int_0^t \mathcal{K}(t,s)y(s)ds = g(t),\ t \in [0,T]$ by differentiation with respect to t, into an integral equation of the form

$$y(t) = \int_0^t K(t,s)y(s)ds + f(t),\ t \in [0,T], \tag{1.1}$$

and Fredholm [20], where he gave necessary and sufficient conditions for solvability of integral equations of the form

$$y(t) = \int_0^T K(t,s)y(s)ds + f(t),\ t \in [0,T]. \tag{1.2}$$

Since then, integral equations and integro-differential equations are used as mathematical models for many and varied physical situations, for example in population dynamics [16] and financial mathematics [29, 30] and integral equations also occur as reformulations of other mathematical problems, see [22, 26, 31, 44]. Among these integral equations, singular kernel problems are especially attractive, see [2, 19, 28, 39].

The development of theory and application of integral equations stimulates the development of numerical treatment of integral equations. Some well-known monographs are [3, 4, 6, 15]. In the present thesis, we try to apply a spectral collocation method to solve integral or integro-differential equations and related eigenvalue problems. Here, compact kernels include weakly singular kernels, smooth or piecewise smooth kernels.

- Numerical solution to integral or integro-differential equations.

Specifically, we study the numerical solution of equations (1.1), (1.2) and integro-differential equations of the form

$$y'(t) = a(t)y(t) + \int_0^t K(t,s)y(s)ds + f(t),\ y(0) = y_0,\ t \in [0,T] \tag{1.3}$$

and

$$y'(t) = a(t)y(t) + \int_0^T K(t,s)y(s)ds + f(t),\ y(0) = y_0,\ t \in [0,T] \tag{1.4}$$

where the kernel $K(t,s)$ is the weakly singular kernel with $0 < \mu < 1$

$$K(t,s) = \begin{cases} \frac{1}{(t-s)^\mu} & \text{Volterra Equations} \\ \frac{1}{|t-s|^\mu} & \text{Fredholm Equations} \end{cases} \tag{1.5}$$

$a(t)$ is a smooth function and $f(t)$ is a given function and we assume that these equations possess a unique solution [21, 44].

- Numerical approximation to eigenvalues

We consider eigenvalue problem of

$$\int_0^1 K(t,s)y(s)ds = \lambda y(t),\ t \in [0,1]. \tag{1.6}$$

where $K(t,s)$ is compact with aforementioned forms. Properties of eigenvectors of weakly singular kernel is studied in [38].

In this thesis, we particularly focus on the equations with weakly singular kernels because this type of equations are related to fractional differential equations. Application of fractional differential equations include Fluid Flow, Rhology, Dynamical

Processes in Self-Similar and Porous Structures, Electrical Networks, Probability and Statisics and so on. It is well-known that Cauchy problem of fractional differential equations and the Volterra integral equations are equivalent under certain conditions, see [27]. For example,

$$\begin{cases} ({}^C D^{\alpha}_{a+} y)(x) = f[x, y(x)] \ (\alpha > 0; a \leq x \leq b) \\ y^{(k)}(a) = b_k, b_k \in R, k = 0, 1, \cdots, n-1; n = -[-\alpha] \end{cases} \tag{1.7}$$

where $({}^C D^{\alpha}_{a+} y)$ is the Caputo derivative defined as

$$({}^C D^{\alpha}_{a+} y) = \frac{1}{\Gamma(n-a)} \int_a^x \frac{y^{(n)}(t)}{(x-t)^{\alpha-n+1}} dt$$

and $[\cdot]$ is the integer part of a number. This equation can be reduced to the Volterra integral equation

$$y(x) = \sum_{j=0}^{n-1} \frac{b_j}{j!}(x-a)^j + \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f[t, y(t)]dt}{(x-t)^{1-\alpha}} \tag{1.8}$$

Moreover, if $f(x, y) \in C[a, b]$ and satisfies the Lipschitz condition, then there exists unique solutions to the Cauchy problem (1.7) in the space $C^{\alpha,n-1}[a, b]$:

$$C^{\alpha,n-1}[a, b] = \{y(x) \in C^{n-1}[a, b] :^C D^{\alpha}_{a+} y \in C[a, b]\}$$

and to the Cauchy problem (1.8) in the space $C^n[a, b]$, see [27, Theorem 3.25]. So, if $y(x) \in C^n[a, b]$, then (1.7) and (1.8) are equivalent. Furthermore, the condition can be relaxed to $y(x) \in L[a, b]$ for the equivalence of Volterra equation and Riemann-Liouville fractional differential equations, see also [27].

Thus, numerical approximation of integral equations with compact kernels is of large importance and thus, a very hot topic in the past fifty years. Big effort can be found in [3, 4, 6, 22] while numerical solutions of integro-differential equation are studied in [6, 33, 40] etc. However, in [3, 4], authors devoted to integral equations with smooth kernels and for weakly singular kernel problem, they used very basic numerical quadratures such as Simpson's Rule or Gauss quadrature to approximate the integration. The disadvantage is that the order of Simpson's Rule is low and Gauss quadrature does not yield a high accuracy for integration with weakly singular kernels. In [6], however, only Volterra equations (integro-differential) equations and related delay equations are studied. Many other numerical analysts used graded meshes to develop numerical schemes with an optimal order of convergence, see [21, 25, 33]. To use this method, certain properties of solutions should be known before implementation, for example, the pole and its order. Recently, in [13, 14] a spectral Jacobi-collocation method was proposed and analyzed to solve Volterra equations. The basic idea is to collocate equations at some Jacobi points and use a highly accurate quadrature to approximate the integration in (1.1). In this thesis, however, instead of numerical integration, we apply exact integration of the composition of the Legendre polynomials and the kernel to solve the Volterra equation (1.1), which leads to less error and computation cost of our method. It will be shown that a geometric (super-geometric) rate of convergence can be achieved by using our method if the true solution $y(t)$ satisfies Condition (R):

$$\|y^{(k)}\|_{L^\infty[0,T]} \leq ck!R^{-k}$$

or Condition (M):

$$\|y^{(k)}\|_{L^\infty[0,T]} \le cM^k, M > 1$$

not only for Volterra equations but also for Fredholm equations as well as their corresponding integro-differential equations. Here, R is sufficiently large. If R is small, hp-version of our method is necessary and for Volterra equations, if the solution is not smooth enough, we may take some function transformations as in [14] to change the equation into a new one so that the new equation possesses better regularity. As to super-geometric convergence of spectral collocation method for differential equations, readers are referred to [42, 45].

Regarding to the numerical approximation of the eigenvalue problems with compact kernels, various methods such as Galerkin method, Petrov-Galerkin method, collocation method, Nyström method, and degenerate kernel methods have been intensively studied. The results are well-documented in the literature. Here, we mention a few related to our work. In the mid-70's, Osborn established a general spectral approximation theory for compact operators T, when a sequence of $\{T_n\}$ approximates T in a collectively compact manner. The analysis of [32] covers many methods and provides a basis for the convergence analysis of our method. In [18], Dellwo and Friedman proposed a new approach by solving a polynomial eigenvalue problem of a higher degree, base on which, Alam etc. [1] obtained an accelerated spectral approximation for eigenelements. Kulkarni [24] introduced another method by involving a new approximation operator T_n and obtained a high-order convergence rate. In addition, a multiscale method was discussed in [12]. Comprehensive studies for eigenvalue problem can be found in [5, 15, 43].

Because the regularity of eigenfunctions directly affect the convergence rate of eigenvalue approximation, we approximate eigenfunctions by some appropriate orthogonal polynomial expansions. Different from previous methods in the literature, we use the exact integration when calculating the convolution of the singular kernel with the orthogonal polynomials as we did in solving integral equations. The key ingredients here are some special identities. It is worthy to mention that in one of our numerical experiments, we obtain a thirteen-digit of accuracy of the approximation of the first eigenvalue for a weakly singular kernel even if its eigenvector is not smooth.

To summarize, in this dissertation, we investigate the numerical solution of Volterra /Fredholm integral equation with weakly singular kernels, Volterra/Fredholm integro-differential equation with weakly singular kernels and eigenvalue problem for compact integral operators including weakly singular kernels, smooth or piecewise smooth kernels and SDE with jump diffusion. Our contributions include:

1. By using some identities, we avoid large numerical quadrature errors accumulated with the singular kernels and thereby obtain higher accuracy for numerical approximation of integral equations and eigenvalue approximations; we avoid product integration method and therefore reduce the computational cost.

2. We prove geometric or super-geometric rate of convergence for numerical approximation of integral or integro-differential equations which was observed in [13, 14]. Also, we observe and prove the geometric or super-geometric rate of convergence for eigenvalue approximation given a general integraql compact operator.

3. We observe and prove a refined convergence rate of eigenvalue approximation (compared to geometric or super-geometric rate of convergence), if the kernel is positive definite.

4. For the weakly singular kernel of the form $\frac{1}{|t-s|^{\mu}}, 0 < \mu < 1$, we can obtain a very high digit of accuracy even though the eigenvector is not smooth.

The rest of the dissertation is arranged as follows.

In Chapter 2, we will provide algorithms of spectral collocation method for integral equations, including some essential identities that we use in our algorithms. Numerical Experiments and the proof of geometric or super-geometric rate of convergence for these equations are also covered in this chapter.

Chapter 3 aims to provide algorithms, numerical experiments and convergence analysis for integro-differential equations.

In Chapter 4, we will study the algorithms and numerical experiments for eigenvalue problems, especially for the eigenvalue approximation of weakly singular kernel problem. We use two different methods, compare their accuracy and make a convergence analysis.

Finally, we end this dissertation with conclusions and further remarks in Chapter 5.

2 Spectral Collocation Method for Integral Equations

This chapter is devoted to illustrate the algorithms for solving integral equations with weakly singular kernels. For the integral operators of the form $\frac{1}{(t-s)^\mu}$ or $\frac{1}{|t-s|^\mu}$, $0 < \mu < 1$, we apply some existing identities to obtain a high digit of accuracy and also avoid product integration and therefore reduce the computational cost.

2.1 Preliminary knowledge

The class of Jacobi polynomials $P_k^{(\alpha,\beta)}(x)$ are essentially used in our algorithms. It is well-known that these polynomials are polynomials solutions to the Jacobi differential equation [9],

$$\frac{d}{dx}[(1-x)^{1+\alpha}(1+x)^{1+\beta}y'] + n(n+\alpha+\beta+1)(1-x)^\alpha(1+x)^\beta y = 0. \tag{2.1}$$

Under the normalization $P_k^{(\alpha,\beta)}(1) = \binom{k+\alpha}{k}$, one has the expression, namely,

$$P_k^{(\alpha,\beta)}(x) = \frac{1}{2^k}\sum_{l=0}^{k}\binom{k+\alpha}{k-l}\binom{k+\beta}{l}(x-1)^l(x+1)^{k-l}. \tag{2.2}$$

Jacobi polynomials satisfy the three-term recursive relations:

$$\begin{aligned} &P_0^{(\alpha,\beta)}(x) = 1, P_1^{(\alpha,\beta)}(x) = \frac{1}{2}[(\alpha-\beta) + (\alpha+\beta+2)x] \\ &a_{1,k}P_{k+1}^{(\alpha,\beta)}(x) = a_{2,k}P_k^{(\alpha,\beta)}(x) - a_{3,k}P_{k-1}^{(\alpha,\beta)}(x) \end{aligned} \tag{2.3}$$

where

$$a_{1,k} = 2(k+1)(k+\alpha+\beta+1)(2k+\alpha+\beta),$$

$$a_{2,k} = (2k+\alpha+\beta+1)(\alpha^2-\beta^2)+x\Gamma(2k+\alpha+\beta+3)/\Gamma(2k+\alpha+\beta),$$

$$a_{3,k} = 2(k+\alpha)(k+\beta)(2k+\alpha+\beta+2). \tag{2.4}$$

A useful formula that relates Jacobi polynomials and their derivatives is

$$\frac{d^m}{dx^m}P_k^{(\alpha,\beta)}(x) = 2^{-m}\frac{\Gamma(k+\alpha+\beta+1+m)}{\Gamma(k+\alpha+\beta+1)}P_{k-m}^{(\alpha+m,\beta+m)}(x); \tag{2.5}$$

in particular, one has

$$\frac{d}{dx}P_k^{(\alpha,\beta)}(x) = \frac{1}{2}(k+1+\alpha+\beta)P_{k-1}^{(\alpha+1,\beta+1)}(x). \tag{2.6}$$

Let $h(t)$ be a smooth function on $[0,T]$ and write

$$f(x) = h(\frac{T}{2}(1+x)), x \in [-1,1].$$

Let T_p be the Chebyshev polynomial of the first kind with degree p and $I_pf \in P_p[-1,1]$ interpolate f at $(p+1)$ Chebyshev points: $x_i = \cos\frac{2i+1}{2p+2}\pi, i = 0,\cdots,p$. Then the remainder of the interpolation is

$$f(x) - I_pf(x) = f[x_0,x_1,\cdots,x_p,x]\nu(x) \tag{2.7}$$

where $\nu(x) = (x-x_0)(x-x_1)\cdots(x-x_p)$.

Hence,

$$f(x) - I_pf(x) = \frac{f[x_0,x_1,\cdots,x_p,x]}{2^{p+1}}T_{p+1}(x), \tag{2.8}$$

since the leading coefficient of $T_{p+1}(x)$ is 2^{p+1}. Moreover, if $f \in C^{p+1}[-1,1]$, the divided difference

$$f[x_0, x_1, \cdots, x_p, x] = \frac{f^{(p+1)}(\xi_x)}{(p+1)!}, \quad \xi_x \in (-1,1). \tag{2.9}$$

Therefore, if h satisfies condition (R), we have

$$\|f - I_p f\|_{L^\infty[-1,1]} \le C\left(\frac{T}{4R}\right)^{p+1} \tag{2.10}$$

and if h satisfies condition (M), we obtain by the Stirling formula,

$$\|f - I_p f\|_{L^\infty[-1,1]} \le \frac{C}{\sqrt{p+1}}\left(\frac{eMT}{4(p+1)}\right)^{p+1} \tag{2.11}$$

Define a weighted L^2 norm by

$$\|v\|_{w^{\alpha,\beta}} = \left(\int_{-1}^{1}(1-x)^\alpha(1+x)^\beta|v(x)|^2dx\right)^{\frac{1}{2}}.$$

From (2.8), the rates of interpolation error under condition (R) and condition (M) in weighted L^2 norm are exactly the same as those in L^∞ norm, respectively.

Next, we introduce two identities, which will be essential in this paper.

Lemma 2.1.1 ([35]) Let a, b be positive constants and $L_n(x)$ be the Legendre polynomial with degree n on $[-1,1]$, then

$$\int_a^b (x-a)^{\alpha-1}L_n(\frac{x}{b})dx = \frac{n!}{(\alpha)_{n+1}}(b-a)^\alpha P_n^{(\alpha,-\alpha)}(\frac{a}{b}), \ -b < a < b; \alpha > 0, \tag{2.12}$$

$$\int_{-a}^b (b-x)^{\beta-1}L_n(\frac{x}{a})dx = \frac{n!}{(\beta)_{n+1}}(b+a)^\beta P_n^{(-\beta,\beta)}(\frac{b}{a}), \quad -a < b < a; \beta > 0, \tag{2.13}$$

where $(k)_{n+1} = k(k+1)\cdots(k+n)$.

Specifically, if we choose $a = 1, b = x, \beta = 1 - \mu$ in (2.13), then we obtain

$$\int_{-1}^{x} \frac{L_n(t)}{(x-t)^{\mu}} dt = \frac{n!}{(1-\mu)_{n+1}} (1+x)^{1-\mu} P_n^{(\mu-1,1-\mu)}(x), \tag{2.14}$$

and $a = x, b = 1, \alpha = 1 - \mu$ in (2.12), we achieve

$$\int_{x}^{1} \frac{L_n(t)}{(t-x)^{\mu}} dt = \frac{n!}{(1-\mu)_{n+1}} (1-x)^{1-\mu} P_n^{(1-\mu,\mu-1)}(x). \tag{2.15}$$

2.2 Algorithm

2.2.1 Algorithm for Volterra integral equation with weakly singular kernel

The equation is of the form:

$$y(t) - \int_0^t \frac{y(s)}{(t-s)^{\mu}} ds = f(t), \ t \in [0, T]. \tag{2.16}$$

After a change of variable

$$t = \frac{T}{2}(1+x),$$

(2.16) can be written as

$$u(x) - \int_0^{T(1+x)/2} (\frac{T}{2}(1+x) - s)^{-\mu} y(s) ds = g(x), \tag{2.17}$$

where $x \in [-1, 1]$ and

$$u(x) = y\left(\frac{T}{2}(1+x)\right), \ g(x) = b\left(\frac{T}{2}(1+x)\right). \tag{2.18}$$

Then, we make another change of variable

$$s = \frac{T}{2}(1+\tau), \ \tau \in [-1, x], \tag{2.19}$$

we arrive at

$$u(x) - \left(\frac{T}{2}\right)^{1-\mu} \int_{-1}^{x} \frac{u(\tau)}{(x-\tau)^{\mu}} d\tau = g(x). \tag{2.20}$$

Let $u_p(x) = \sum_{j=0}^{p} c_j L_j(x)$, which is the approximation of $u(x)$. Then we require that c_j's satisfy the equations at the collocation points,

$$\sum_{j=0}^{p} c_j L_j(x_i) - \left(\frac{T}{2}\right)^{1-\mu} \sum_{j=0}^{p} c_j \int_{-1}^{x_i} \frac{L_j(\tau)}{(x_i-\tau)^{\mu}} d\tau = g(x_i), \quad i = 0, \cdots, p. \tag{2.21}$$

By virtue of (2.14), we have

$$\sum_{j=0}^{p} c_j \left(L_j(x_i) - \left(\frac{T}{2}\right)^{1-\mu} \frac{j!}{(1-\mu)_{j+1}} (1+x_i)^{1-\mu} P_j^{(\mu-1,1-\mu)}(x_i) \right) = g(x_i), \quad i = 0, \cdots, p. \tag{2.22}$$

2.2.2 Algorithm for Fredholm integral equations

We solve a class of Fredholm integral equation of form:

$$y(t) - \int_{0}^{T} \frac{y(s)}{|t-s|^{\mu}} ds = f(t), \ t \in [0, T]. \tag{2.23}$$

Make a change of variable

$$t = \frac{T}{2}(1+x),$$

and we obtain

$$u(x) - \int_{0}^{T(1+x)/2} \left(\frac{T(1+x)}{2} - s \right)^{-\mu} y(s) ds - \int_{T(1+x)/2}^{T} \left(s - \frac{T(1+x)}{2} \right)^{-\mu} y(s) ds = g(x) \tag{2.24}$$

where $u(x)$ and $g(x)$ are defined the same as those in (2.18).

Again, let

$$s = \frac{T}{2}(1+\tau), \tau \in [-1, 1]. \tag{2.25}$$

We reach

$$u(x) - \left(\frac{T}{2}\right)^{1-\mu} \int_{-1}^{x} \frac{u(\tau)}{(x-\tau)^{\mu}} d\tau - \left(\frac{T}{2}\right)^{1-\mu} \int_{x}^{1} \frac{u(\tau)}{(x-\tau)^{\mu}} d\tau = g(x),$$

$$x \in [-1,1] \quad (2.26)$$

Let $u_p(x) = \sum_{j=0}^{p} c_j L_j(x)$ be the approximation of $u(x)$. Then c_j's satisfy the equations

$$\sum_{j=0}^{p} c_j L_j(x_i) - \left(\frac{T}{2}\right)^{1-\mu} \sum_{j=0}^{p} c_j \int_{-1}^{x_i} \frac{L_j(\tau)}{(x_i-\tau)^{\mu}} d\tau - \left(\frac{T}{2}\right)^{1-\mu} \sum_{j=0}^{p} c_j \int_{x_i}^{1} \frac{L_j(\tau)}{(\tau-x_i)^{\mu}} d\tau$$
$$= g(x_i), \quad i = 0, \cdots, p. \quad (2.27)$$

Applying (2.14) and (2.15), we obtain

$$\sum_{j=0}^{p} c_j \Bigg(L_j(x_i) - \left(\frac{T}{2}\right)^{1-\mu} \frac{j!}{(1-\mu)_{j+1}} (1+x_i)^{1-\mu} P_j^{(\mu-1,1-\mu)}(x_i)$$
$$- \left(\frac{T}{2}\right)^{1-\mu} \frac{j!}{(1-\mu)_{j+1}} (1-x_i)^{1-\mu} P_j^{(1-\mu,\mu-1)}(x_i) \Bigg) = g(x_i), \quad i = 0, \cdots, p.$$
$$(2.28)$$

Remark. Note that (2.21) and (2.22), (2.27) and (2.28) are equivalent. However, in numerical application, we use (2.22) and (2.28) and for the convenience of theoretical analysis, we use (2.21) and (2.27).

For the sake of analysis, we define the Lagrange interpolation at $(p+1)$ Chebyshev points, i.e

$$I_p u(x_i) = u(x_i), \; 0 \le i \le p. \quad (2.29)$$

It is clear that

$$I_p u(x_i) = \sum_{i=0}^{p} u(x_i) l_i(x), \quad (2.30)$$

where $l_i(x)$ is the Lagrange interpolation basis function at Chebyshev points.

It is obvious that

$$y(t) = y\left(\frac{T}{2}(1+x)\right) = u(x), \quad t \in [0,T] \text{ and } x \in [-1,1]. \tag{2.31}$$

Hence,

$$y_p(t) = y_p\left(\frac{T}{2}(1+x)\right) = u_p(x), \quad t \in [0,T] \text{ and } x \in [-1,1]. \tag{2.32}$$

Therefore,

$$(y - y_p)(t) = (u - u_p)(x) := e(x). \tag{2.33}$$

2.3 Theoretical Analysis

This section is devoted to the convergence analysis of our algorithms to Volterra (2.16) or Fredholm (2.23) integral equations.

2.3.1 Volterra integral equations

We first present some useful lemmas.

Lemma 2.3.1. [4] Let $\{l_j(x)\}_{j=0}^p$ be the Lagrange interpolation polynomials at the Chebyshev points, then

$$\|I_p\|_\infty := \max_{x\in[-1,1]} \sum_{j=0}^{p} |l_j(x)| = \mathcal{O}(\log p). \tag{2.34}$$

In our analysis, we shall apply the following generalization of Gronwall's leamma, see [23].

Lemma 2.3.2. Suppose $L \geq 0, 0 < \mu < 1$ and $v(t)$ is a non-negative, locally integrable function defined on $[0, T]$ satisfying

$$u(t) \leq v(t) + L \int_0^t (t-s)^{-\mu} u(s) ds. \tag{2.35}$$

Then there exists a constants $C = C(\mu)$ such that

$$u(t) \leq v(t) + CL \int_0^t (t-s)^{-\mu} v(s) ds, \quad 0 \leq t < T. \tag{2.36}$$

Let $C^{r,\kappa}([0, T])$ denote the space of function whose r-th derivatives are Hólder continuous with exponent κ, endowed with the usual norm $\| \cdot \|_{r,\kappa}$. Then from the result of [36, 37], we have the following lemma

Lemma 2.3.3. Let r be a non-negative integer and $\kappa \in (0, 1)$. Then for any $v \in C^{r,\kappa}([-1, 1])$, there exists a polynomial function $\mathcal{T}_N v \in P_N$ such that

$$\|v - \mathcal{T}_N v\|_\infty \leq CN^{-(r+\kappa)} \|v\|_{r,\kappa} \tag{2.37}$$

Here, $\mathcal{T}_N$ is a linear operator from $C^{r,\kappa}$ to P_N.

We now define a linear integral operators $\mathcal{M}_1$ by

$$\mathcal{M}_1 v(x) = \int_{-1}^{x} \frac{v(\tau)}{(x-\tau)^\mu} d\tau. \tag{2.38}$$

Lemma 2.3.4. [13] Let $\kappa \in (0, 1-\mu)$ and $\mathcal{M}_1$ be defined by (2.38) under the condition $0 < \kappa < 1 - \mu$. Then for any function $v \in C([0, 1])$, there exists a positive constant C such that

$$\|\mathcal{M}_1 v\|_{0,\kappa} \leq C \|v\|_\infty. \tag{2.39}$$

Theorem 2.3.5. Let $y(t)$ and $u_p(x)$ be the exact solution and spectral approximation of (2.16), respectively.

1) If y satisfies the condition (R), then, for sufficiently large p,

$$\|u - u_p\|_\infty \leq C\left(\frac{T}{4R}\right)^{p+1}, \tag{2.40}$$

2) If y satisfies the condition (M), then, for sufficiently large p,

$$\|u - u_p\|_\infty \leq \frac{C}{\sqrt{p+1}}\left(\frac{eMT}{4(p+1)}\right)^{p+1}. \tag{2.41}$$

Proof: Using (2.21), we have

$$u_i = \left(\frac{T}{2}\right)^{1-\mu} \int_{-1}^{x_i} \frac{u_p(\tau)}{(x_i - \tau)^\mu} d\tau + g(x_i), \tag{2.42}$$

where

$$u_i = \sum_{j=0}^{p} c_j L_j(x_i).$$

Note that the true solution at Chebyshev points satisfies

$$u(x_i) = \left(\frac{T}{2}\right)^{1-\mu} \int_{-1}^{x_i} \frac{u(\tau)}{(x_i - \tau)^\mu} d\tau + g(x_i). \tag{2.43}$$

Thus, let $e(x) = u(x) - u_p(x)$,

$$u(x_i) - u_i = \left(\frac{T}{2}\right)^{1-\mu} \int_{-1}^{x_i} \frac{e(\tau)}{(x_i - \tau)^\mu} d\tau. \tag{2.44}$$

Multiplying both sides by $l_i(x)$ and summing up from 0 to p and applying the fact that $u_p(x) = \sum_{j=0}^{p} c_j L_j(x) = \sum_{j=0}^{p} u_j l_j(x)$ give

$$I_p u - u_p = \left(\frac{T}{2}\right)^{1-\mu} I_p\left[\int_{-1}^{x} \frac{e(\tau)}{(x - \tau)^\mu} d\tau\right], \tag{2.45}$$

We write

$$e(x) = u(x) - I_p u(x) + I_p u(x) - u_p(x). \tag{2.46}$$

Obviously,

$$\begin{aligned} e(x) &= \left(\frac{T}{2}\right)^{1-\mu} \int_{-1}^{x} \frac{e(\tau)}{(x-\tau)^{\mu}} d\tau + \underbrace{u(x) - I_p u(x)}_{I_1} \\ &\quad \underbrace{+\left(\frac{T}{2}\right)^{1-\mu} \left\{ (I_p - I)\left[\int_{-1}^{x} \frac{e(\tau)}{(x-\tau)^{\mu}} d\tau \right] \right\}}_{I_2}, \end{aligned} \tag{2.47}$$

where I is the identity operator.

By the Gronwall's inequality, namely, Lemma 2.3.2, we have

$$e(x) \leq |I_1 + I_2| + C \int_{-1}^{x} (x-\tau)^{-\mu} |I_1 + I_2| d\tau. \tag{2.48}$$

Therefore,

$$\|e\|_{\infty} \leq C(\|I_1\|_{\infty} + \|I_2\|_{\infty}). \tag{2.49}$$

It follows from (2.10) and (2.11) that if $y(t)$ satisfies condition (R), then

$$\|I_1\|_{\infty} \leq C \left(\frac{T}{4R}\right)^{p+1}, \tag{2.50}$$

and if $y(t)$ satisfies condition (M), then

$$\|I_1\|_{\infty} \leq \frac{C}{\sqrt{p+1}} \left(\frac{eMT}{4(p+1)}\right)^{p+1}. \tag{2.51}$$

Now let's estimate I_2. We derive from Lemma 2.3.3 and Lemma 2.3.4 that

$$\|\mathcal{M}_1 e - \mathcal{T}_p \mathcal{M}_1 e\|_{\infty} \leq C p^{-\kappa} \|\mathcal{M}_1 e\|_{0,\kappa}, \quad \kappa \in (0, 1-\mu). \tag{2.52}$$

Hence,

$$\|I_2\|_\infty = \left(\frac{T}{2}\right)^{1-\mu} \|(I_p - I)\mathcal{M}_1 e\|_\infty \tag{2.53}$$
$$= \left(\frac{T}{2}\right)^{1-\mu} \|(I_p - I)(\mathcal{M}_1 e - \mathcal{T}_p \mathcal{M}_1 e)\|_\infty \tag{2.54}$$
$$\leq \left(\frac{T}{2}\right)^{1-\mu} (1 + \|I_p\|_\infty)\|\mathcal{M}_1 e - \mathcal{T}_p \mathcal{M}_1 e\|_\infty \tag{2.55}$$
$$\leq Cp^{-\kappa} \log p \|e\|_\infty. \tag{2.56}$$

Therefore, for sufficiently large p, we obtain

$$\|I_2\|_\infty \leq \frac{1}{2C}\|e\|_\infty, \tag{2.57}$$

which implies

$$\|e\|_\infty \leq C\left(\frac{T}{4R}\right)^{p+1} \quad \text{(y satisfies condition (R));} \tag{2.58}$$

$$\|e\|_\infty \leq \frac{C}{\sqrt{p+1}}\left(\frac{eMT}{4(p+1)}\right)^{p+1} \quad \text{(y satisfies condition (M)).} \tag{2.59}$$

2.3.2 Fredholm integral equations

For our method, the error analysis of the L^∞ norm for Fredholm equations is similar to that of Volterra equations. However, the analysis for Fredholm type equations is not included in [13, 14]. Let us start with a property of weakly singular integral operators.

Lemma 2.3.6 ([39] Corollary 2.1.) Weakly singular integral operators are compact from L^∞ to $C[0,T]$(and hence from L^∞ to L^∞) and from $C[0,T]$ to $C[0,T]$.

We now define a linear integral operators $\mathcal{M}_2$ by

$$\mathcal{M}_2 v(x) = \int_x^1 \frac{v(\tau)}{(\tau - x)^\mu} d\tau. \tag{2.60}$$

Lemma 2.3.7 Let $\kappa \in (0, 1-\mu)$ and $\mathcal{M}_2$ be defined by (2.60) under the condition $0 < \kappa < 1 - \mu$. Then for any function $v \in C[0,1]$, there exists a positive constant C such that

$$\|\mathcal{M}_2 v\|_{0,\kappa} \leq C\|v\|_\infty. \tag{2.61}$$

Proof: We need to prove

$$\frac{|\mathcal{M}_2 v(t_2) - \mathcal{M}_2 v(t_1)|}{|t_2 - t_1|^\kappa} \leq C\|v\|_\infty. \tag{2.62}$$

Without loss of generality, we assume $t_1 < t_2$. Then, we have

$$\begin{aligned}
&\frac{|\mathcal{M}_2 v(t_2) - \mathcal{M}_2 v(t_1)|}{|t_2 - t_1|^\kappa} \\
= \;& (t_2 - t_1)^{-\kappa} \left| \int_{t_2}^1 \frac{v(s)}{(s - t_2)^\mu} ds - \int_{t_1}^1 \frac{v(s)}{(s - t_1)^\mu} ds \right| \\
= \;& (t_2 - t_1)^{-\kappa} \left| \int_{t_2}^1 \left[\frac{1}{(s - t_2)^\mu} - \frac{1}{(s - t_1)^\mu} \right] v(s) ds - \int_{t_1}^{t_2} \frac{v(s)}{(s - t_1)^\mu} ds \right| \\
\leq \;& I_1 + I_2,
\end{aligned} \tag{2.63}$$

where

$$\begin{aligned}
I_1 &= (t_2 - t_1)^{-\kappa} \left| \int_{t_2}^1 \left[\frac{1}{(s - t_2)^\mu} - \frac{1}{(s - t_1)^\mu} \right] v(s) ds \right| \\
I_2 &= (t_2 - t_1)^{-\kappa} \left| \int_{t_1}^{t_2} \frac{v(s)}{(s - t_1)^\mu} ds \right|.
\end{aligned}$$

We now estimate these two terms one by one. By simple calculation and the condition $\kappa \in (0, 1 - \mu)$, we derive

1) If $t_2 - t_1 \leq 1 - t_2$,

$$\begin{aligned}
I_1 &\leq \frac{(t_2-t_1)^{-\kappa}}{1-\mu}\left|(1-t_2)^{1-\mu} - (1-t_1)^{1-\mu} + (t_2-t_1)^{1-\mu}\right|\|v\|_\infty \\
&\leq \frac{(t_2-t_1)^{-\kappa}}{1-\mu}\left|(1-\mu)(1-\xi)^{-\mu}(t_2-t_1) + (t_2-t_1)^{1-\mu}\right|\|v\|_\infty, \ \xi \in (t_1, t_2) \\
&\leq \frac{(t_2-t_1)^{-\kappa}}{1-\mu}\left|(1-\mu)(1-t_2)^{-\mu}(t_2-t_1) + (t_2-t_1)^{1-\mu}\right|\|v\|_\infty \\
&\leq C(t_2-t_1)^{1-\mu-\kappa}\|v\|_\infty \\
&\leq C\|v\|_\infty. \qquad (2.64)
\end{aligned}$$

due to $t_2 - t_1 < 1$.

2) If $t_2 - t_1 > 1 - t_2$, then

$$\begin{aligned}
I_1 &\leq \frac{(t_2-t_1)^{-\kappa}}{1-\mu}\left|(1-t_2)^{1-\mu} - (1-t_1)^{1-\mu} + (t_2-t_1)^{1-\mu}\right|\|v\|_\infty \\
&= \frac{(t_2-t_1)^{1-\mu-\kappa}}{1-\mu}\left|1 + \left(\frac{1-t_2}{t_2-t_1}\right)^{1-\mu} - \left(1 + \frac{1-t_2}{t_2-t_1}\right)^{1-\mu}\right|\|v\|_\infty \qquad (2.65)
\end{aligned}$$

Noting that the function

$$k(x) = 1 + x^{1-\mu} - (1+x)^{1-\mu}, \quad x \in [0,1]$$

is non- negative and achieves its maximum at $x = 1$, we have

$$I_1 \leq C(t_2-t_1)^{1-\mu-\kappa}\|v\|_\infty \leq C\|v\|_\infty. \qquad (2.66)$$

As for I_2,

$$\begin{aligned} I_2 &= (t_2-t_1)^{-\kappa}\left|\int_{t_1}^{t_2}\frac{1}{(s-t_1)^{\mu}}v(s)ds\right| \\ &\le (t_2-t_1)^{-\kappa}\|v\|_\infty\left|\int_{t_1}^{t_2}\frac{1}{(s-t_1)^{\mu}}ds\right| \\ &\le \|v\|_\infty\frac{(t_2-t_1)^{-\kappa}}{1-\mu}(t_2-t_1)^{1-\mu} \\ &\le C\|v\|_\infty. \end{aligned} \tag{2.67}$$

Then, the desired result is followed by combining (2.64), (2.66) and (2.67).

Theorem 2.3.8 Let $y(t)$ and $u_p(x)$ be the exact solution and spectral approximation of (2.23), respectively.

1) If y satisfies the condition (R), then, for sufficiently large p,

$$\|u-u_p\|_\infty \le C\left(\frac{T}{4R}\right)^{p+1}, \tag{2.68}$$

2) If y satisfies the condition (M), then for sufficiently large p,

$$\|u-u_p\|_\infty \le \frac{C}{\sqrt{p+1}}\left(\frac{eMT}{4(p+1)}\right)^{p+1}. \tag{2.69}$$

Proof: Using (2.27), we have

$$u_i = \left(\frac{T}{2}\right)^{1-\mu}\int_{-1}^{x_i}(x_i-\tau)^{-\mu}u_p(\tau)d\tau + \left(\frac{T}{2}\right)^{1-\mu}\int_{x_i}^{1}(\tau-x_i)^{-\mu}u_p(\tau)d\tau + g(x_i), \tag{2.70}$$

where

$$u_i = \sum_{j=0}^{p} c_j L_j(x_i).$$

The true solution at Chebyshev points satisfies

$$u(x_i) = \left(\frac{T}{2}\right)^{1-\mu}\int_{-1}^{x_i}(x_i-\tau)^{-\mu}u(\tau)d\tau + \left(\frac{T}{2}\right)^{1-\mu}\int_{x_i}^{1}(\tau-x_i)^{-\mu}u(\tau)d\tau + g(x_i), \tag{2.71}$$

Thus, let $e(x) = u(x) - u_p(x)$,

$$u(x_i) - u_i = \left(\frac{T}{2}\right)^{1-\mu} \int_{-1}^{x_i} (x_i - \tau)^{-\mu} e(\tau) d\tau + \left(\frac{T}{2}\right)^{1-\mu} \int_{x_i}^{1} (\tau - x_i)^{-\mu} e(\tau) d\tau. \quad (2.72)$$

Multiplying both sides by $l_i(x)$ and summing up from 0 to p, and applying the fact that $u_p(x) = \sum_{j=0}^{p} c_j L_j(x) = \sum_{j=0}^{p} u_j l_j(x)$ give

$$I_p u - u_p = \left(\frac{T}{2}\right)^{1-\mu} I_p\left[\int_{-1}^{x} (x - \tau)^{-\mu} e(\tau) d\tau\right] + \left(\frac{T}{2}\right)^{1-\mu} I_p\left[\int_{x}^{1} (\tau - x)^{-\mu} e(\tau) d\tau\right]. \quad (2.73)$$

We write

$$e(x) = u(x) - I_p u(x) + I_p u(x) - u_p(x). \quad (2.74)$$

Obviously,

$$\begin{aligned} e(x) \quad &= \quad \left(\frac{T}{2}\right)^{1-\mu} \int_{-1}^{1} |x - \tau|^{-\mu} e(\tau) d\tau + \underbrace{u(x) - I_p u(x)}_{I_1} \\ &\quad + \underbrace{\left(\frac{T}{2}\right)^{1-\mu} \left\{ (I_p - I)\left[\int_{-1}^{1} |x - \tau|^{-\mu} e(\tau) d\tau\right]\right\}}_{I_2}, \end{aligned} \quad (2.75)$$

where I is the identity operator.

By Lemma 2.3.6 and the Fredholm Alternative, we have

$$\|e\|_\infty \quad \leq \quad C(\|I_1\|_\infty + \|I_2\|_\infty). \quad (2.76)$$

It follows from (2.10) and (2.11) that if $y(t)$ satisfies condition (R), then

$$\|I_1\|_\infty \leq C\left(\frac{T}{4R}\right)^{p+1}, \quad (2.77)$$

and if $y(t)$ satisfies condition (M), then

$$\|I_1\|_\infty \leq \frac{C}{\sqrt{p+1}}\left(\frac{eMT}{4(p+1)}\right)^{p+1}. \quad (2.78)$$

We derive from Lemma 2.3.3, 2.3.4 and 2.3.7 that

$$\|\mathcal{M}_i e - \mathcal{T}_p \mathcal{M}_i e\|_\infty \le C p^{-\kappa} \|\mathcal{M}_i e\|_{0,\kappa}, \quad \kappa \in (0, 1-\mu),\ i = 1, 2. \tag{2.79}$$

Hence,

$$\begin{aligned}
\|I_2\|_\infty &= \left(\frac{T}{2}\right)^{1-\mu} \|(I_p - I)\mathcal{M}_1 e + (I_p - I)\mathcal{M}_2 e\|_\infty \\
&= \left(\frac{T}{2}\right)^{1-\mu} \Big(\|(I_p - I)(\mathcal{M}_1 e - \mathcal{T}_p \mathcal{M}_1 e)\|_\infty + \|(I_p - I)(\mathcal{M}_2 e - \mathcal{T}_p \mathcal{M}_2 e)\|_\infty \Big) \\
&\le \left(\frac{T}{2}\right)^{1-\mu} (1 + \|I_p\|_\infty)(\|\mathcal{M}_1 e - \mathcal{T}_p \mathcal{M}_1 e\|_\infty + \|\mathcal{M}_1 e - \mathcal{T}_p \mathcal{M}_2 e\|_\infty) \\
&\le C p^{-\kappa} \log p \|e\|_\infty.
\end{aligned} \tag{2.80}$$

The desired results (2.68) and (2.69) follow from (2.77), (2.78) and (2.80) for sufficiently large p.

2.4 Numerical Results

2.4.1 Volterra Integral Equation

In this subsection, we will find numerical approximation to solutions of two examples to demonstrate our Theorem 2.3.5. Unlike the numerical scheme in [13, 14] which they called spectral Jacobi method, our scheme is of the form

$$LC_p - AC_p = G_p, \tag{2.81}$$

where $C_p = [c_0, \cdots, c_p]^T$ and $G_p = [g(x_0), \cdots, g(x_p)]^T$ and the elements of the matrix $A = (a_{ij})$ and $L = (l_{ij})$ are given by

$$\begin{aligned} a_{ij} &= \left(\frac{T}{2}\right)^{1-\mu} \frac{j!}{(1-\mu)_{j+1}} (1+x_i)^{1-\mu} P_j^{(\mu-1,1-\mu)}(x_i), \\ l_{ij} &= L_j(x_i), \end{aligned}$$

which is obtained from (2.22).

In this subsection, reference curve 1 is the graph of function $f(p) = \frac{1}{2^{p+1}}$ associating with condition (R) and reference curve 2 is the graph of $f(p) = \frac{1}{\sqrt{p+1}}(\frac{e^{1.5}T}{4(p+1)})^{p+1}$ corresponding to condition (M).

Example 2.4.1 Consider a Volterra integral equation of the form (1.1) on [0, 6] with $\alpha = \frac{1}{2}$ and

$$b(t) = (t+2)^{2/3} - \frac{3\pi}{8}(t+2)^2 + \frac{3}{4}(t+2)^2 \arctan\left(\sqrt{\frac{2}{t}}\right) - \frac{\sqrt{2t}}{4}(3t+10).$$

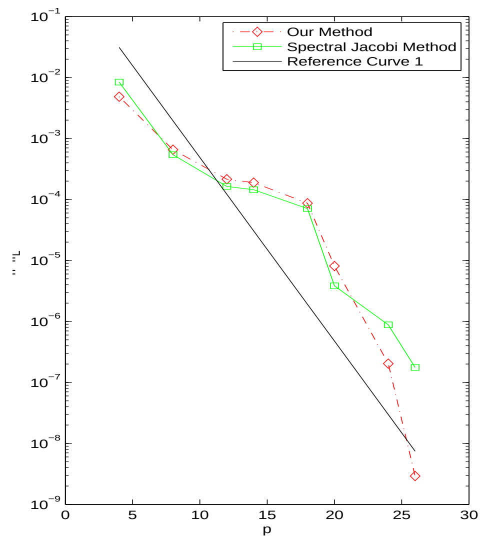
The exact solution for this example is $y(t) = (2+t)^{3/2}$. Obviously, this solution satisfies condition (R). Hence, we expect a geometric rate of convergence. Numerical errors for our method and the spectral Jacobi method are presented in Table 1 and Figure 1. We see that our method outperforms the spectral Jacobi method for larger $p > 0$.

Example 2.4.2 Consider a Volterra equation of the form (1.1) on [0, 4] with $\mu = \frac{2}{3}$ and $b(t) = e^t(1-\gamma(\frac{1}{3}, t))$, where $\gamma(a, x)$ is the lower incomplete gamma function defined by:

$$\gamma(a, x) = \int_0^x t^{a-1} e^{-t} dt.$$

Table 1: Example 2.4.1: L^∞ error and weighted L^2 error for $t \in [0,6]$

N		4	8	12	14
$\|\cdot\|_{L^\infty}$	Our Method	4.8408e-03	6.5600e-04	2.1539e-04	1.8947e-04
	Spectral Jacobi Mehtod	8.4195e-03	5.4146e-04	1.6376e-04	1.4467e-04
$\|\cdot\|_{w^{-1/2,-1/2}}$	Our method	7.0517e-03	7.0605e-04	2.1543e-04	1.8630e-04
$\|\cdot\|_{w^{-\mu,0}}$	Spectral Jacobi Method	1.3429e-02	6.8552e-04	1.9362e-04	1.6830e-04
N		18	20	24	26
$\|\cdot\|_{L^\infty}$	Our method	8.7019e-05	8.1105e-06	2.0336e-07	2.9167e-09
	Spectral Jacobi Method	7.0872e-05	3.8413e-06	8.7916e-07	1.7642e-07
$\|\cdot\|_{w^{-1/2,-1/2}}$	Our method	8.3915e-05	7.7757e-06	1.9346e-07	2.7674e-09
$\|\cdot\|_{w^{-\mu,0}}$	Spectral Jacobi Method	8.0931e-05	4.3619e-06	9.9085e-07	1.9832e-07



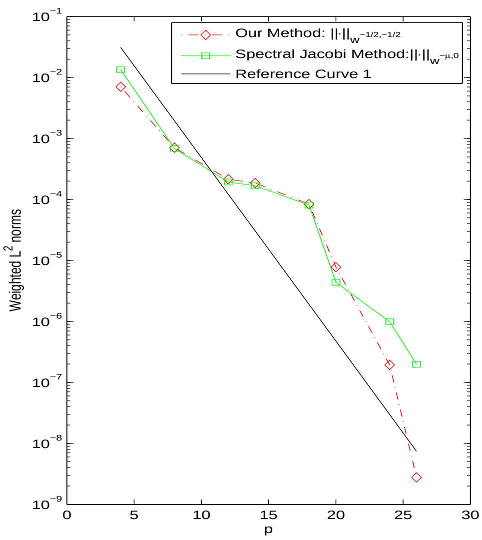
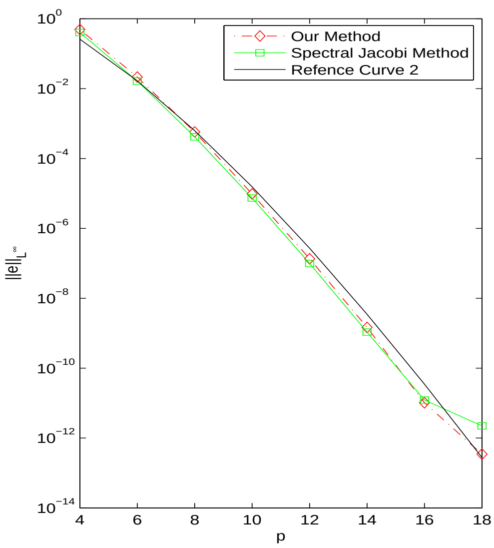


Figure 1. L^∞ error for both methods Weighted L^2 error for both methods

Table 2: Example 2.4.2: L^∞ error and weighted L^2 error for $t \in [0,4]$

N		4	6	8	10
$\|\cdot\|_{L^\infty}$	Our Method	4.8974e-01	2.1410e-02	5.7430e-04	9.7091e-06
	Spectral Jacobi Mehtod	4.0381e-01	1.6403e-02	4.1529e-04	7.3809e-06
$\|\cdot\|_{w^{-1/2,-1/2}}$	Our method	5.8315e-01	2.1210e-02	4.9030e-04	7.3712e-06
$\|\cdot\|_{w^{-\mu,0}}$	Spectral Jacobi Method	7.5536e-01	2.7139e-02	6.2278e-04	1.0286e-05
N		12	14	16	18
$\|\cdot\|_{L^\infty}$	Our method	1.3511e-07	1.4702e-09	1.0267e-11	3.4817e-13
	Spectral Jacobi Method	9.9530e-08	1.0766e-09	1.2221e-11	2.2382e-12
$\|\cdot\|_{w^{-1/2,-1/2}}$	Our method	9.4324e-08	9.6632e-10	6.4585e-12	2.3305e-13
$\|\cdot\|_{w^{-\mu,0}}$	Spectral Jacobi Method	1.3159e-07	1.3714e-09	1.5158e-11	2.7214e-12



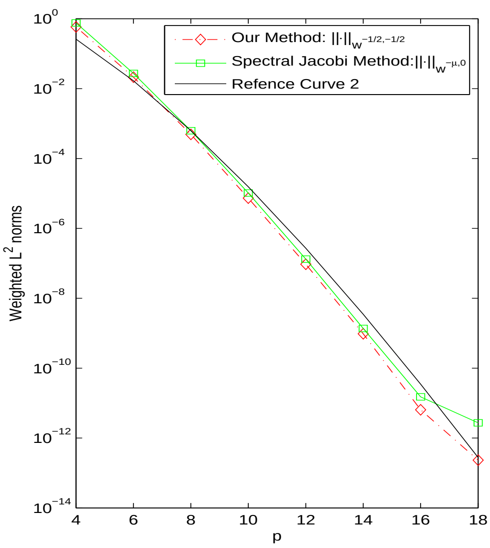


Figure 2. L^∞ error for both methods   Weighted L^2 error for both methods

The exact solution of this equation is $y(t) = e^t$, which satisfies condition (M). We expect a supergeometric rate of convergence for numerical approximations, see Table 2 and Figure 2.

2.4.2 Fredholm Integral Equation

From algorithm for (2.23), we can obtain the scheme

$$LC_p - AC_p = G_p,$$

where $C_p = [c_0, \cdots, c_p]^T$, $G_p = [g(x_0), \cdots, g(x_p)]^T$, and the elements of the matrix $A = (a_{ij})$ and $L = (l_{ij})$ are given by

$$\begin{aligned} a_{ij} &= \left(\frac{T}{2}\right)^{1-\mu} \frac{j!}{(1-\mu)_{j+1}} (1+x_i)^{1-\mu} P_j^{(\mu-1,1-\mu)}(x_i) \\ &\quad + \left(\frac{T}{2}\right)^{1-\mu} \frac{j!}{(1-\mu)_{j+1}} (1-x_i)^{1-\mu} P_j^{(1-\mu,\mu-1)}(x_i), \\ l_{ij} &= L_j(x_i). \end{aligned}$$

In this subsection, reference curve 1 is the graph of function $f(p) = \frac{1}{4^{p+1}}$ associating with condition (R) and reference curve 2 is the graph of $f(p) = \frac{1}{\sqrt{p+1}}(\frac{eT}{4(p+1)})^{p+1}$ corresponding to condition (M).

Example 2.4.3 Consider a Fredholm equation of the form

$$y(t) = \int_0^6 \frac{y(s)}{|t-s|^\mu} ds + b(t). \quad t \in [0, 6] \tag{2.82}$$

Specifically, we choose $\mu = 1/2$ and

$$\begin{aligned} b(t) &= \sqrt{t+2} - \frac{\pi}{2}(t+2) + (t+2)\arctan\left(\sqrt{\frac{2}{t}}\right) - \sqrt{2t} - 2\sqrt{12-2t} \\ &\quad -(t+2)\log(2\sqrt{6-t} + 4\sqrt{2}) + (t+2)\log(2\sqrt{t+2}) \end{aligned}$$

Table 3: Example 2.4.3: errors for $t \in [0,6]$

N	4	6	8	10
$\|\cdot\|_{L^\infty}$	4.1805e-04	1.2503e-04	4.7910e-06	1.5799e-07
$\|\cdot\|_{w^{-1/2,-1/2}}$	6.5631e-04	1.6064e-04	5.7565e-06	1.8530e-07
N	12	14	18	22
$\|\cdot\|_{L^\infty}$	2.4987e-08	6.1532e-10	5.0531e-11	6.8390e-14
$\|\cdot\|_{w^{-1/2,-1/2}}$	3.4284e-08	8.4890e-10	1.2349e-10	1.3079e-13

so that the true solution is $y(t) = \sqrt{t+2}$. Clearly, the solution satisfies condition (R). Numerical errors are reported in Table 3 and Figure 3.

Example 2.4.4 Consider a Fredholm equation of the form

$$y(t) = \int_0^{10} \frac{y(s)}{|t-s|^\mu} ds + b(t). \quad t \in [0,10] \tag{2.83}$$

We choose $\mu = 1/2$ and

$$\begin{aligned} b(t) \;=\; & \sin(t) + \sqrt{2\pi}\left[\cos(t)S\left(\sqrt{\frac{2t}{\pi}}\right) - \sin(t)C\left(\sqrt{\frac{2t}{\pi}}\right)\right] \\ & -\sqrt{2\pi}\left[\sin(t)C\left(\sqrt{\frac{2(10-t)}{\pi}}\right) + \cos(t)S\left(\sqrt{\frac{2(10-t)}{\pi}}\right)\right] \end{aligned}$$

Here, $C(u)$ and $S(u)$ are Fresnel integrals defined by

$$C(u) = \int_0^u \cos(\frac{\pi x^2}{2})dx, \quad S(u) = \int_0^u \sin(\frac{\pi x^2}{2})dx$$

Then, the true solution is $y(t) = \sin(t)$. It is obvious that the solution satisfy condition (M). Numerical results are given in Table 4 and Figure 4.

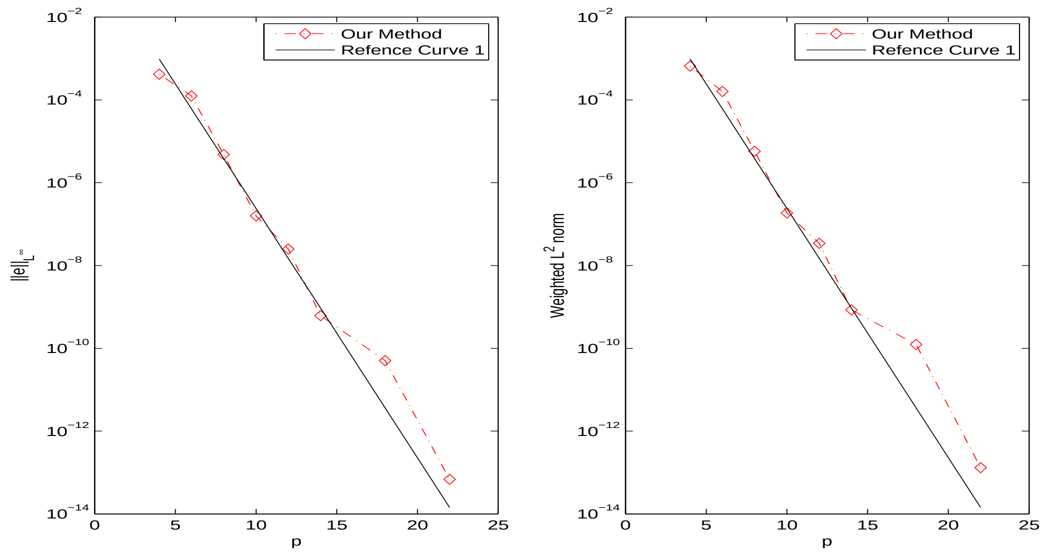


Figure 3. $\|\cdot\|_{L^\infty}$ error for our method $\|\cdot\|_{w^{-1/2,-1/2}}$ error for our method

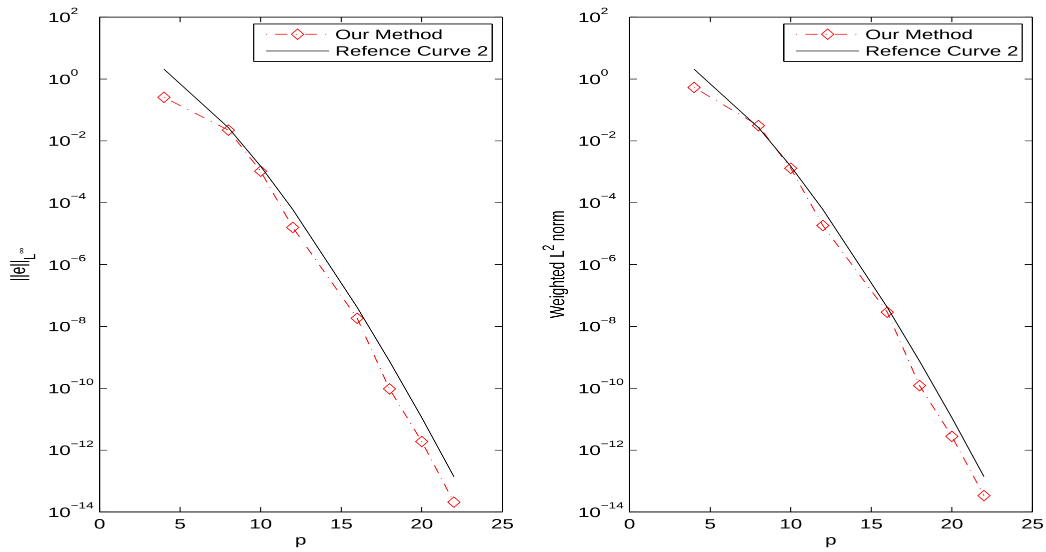


Figure 4. $\|\cdot\|_{L^\infty}$ error for our method $\|\cdot\|_{w^{-1/2,-1/2}}$ error for our method

Table 4: Example 2.4.4: errors for $t \in [0, 10]$

N	4	8	10	12
$\|\cdot\|_{L^\infty}$	2.5522e-01	2.2144e-02	1.0467e-03	1.5876e-05
$\|\cdot\|_{w^{-1/2,-1/2}}$	5.3307e-01	3.0980e-02	1.2953e-03	1.8215e-05
N	16	18	20	22
$\|\cdot\|_{L^\infty}$	1.8332e-08	9.5780e-11	1.8910e-12	2.0761e-14
$\|\cdot\|_{w^{-1/2,-1/2}}$	2.8859e-08	1.2175e-10	2.7897e-12	3.3669e-14

3 Spectral Collocation Method for Integro-Differential Equation

In this chapter, we focus on the spectral collocation method for integro-differential equations of first order whereas our algorithms can be generalized to arbitrary order of integro-differential equations with weakly singular kernels.

3.1 Algorithms

3.1.1 Algorithm for Volterra integro-differential equation

We consider the following equation

$$y'(t) = a(t)y(t) + \int_0^t \frac{y(s)}{(t-s)^\mu} ds + f(t),\ y(0) = y_0,\ t \in [0,T] \tag{3.1}$$

We take the same notations and variable transformation as in algorithm for (2.16), then we obtain

$$\frac{2}{T}u'(x) = f(x)u(x) + \left(\frac{T}{2}\right)^{1-\mu} \int_{-1}^x \frac{u(s)}{(t-s)^\mu} ds + g(x), \tag{3.2}$$

where $f(x) = a(\frac{T}{2}(1+x))$. Let

$$u_p(x) = y_0 + \sum_{j=1}^p c_j(L_j(x) + L_{j-1}(x))$$

and note that from (2.6)

$$\frac{d}{dx}L_n(x) = \frac{n+1}{2}P_{n-1}^{(1,1)}(x). \tag{3.3}$$

We have

$$\frac{1}{T}\sum_{j=1}^{p} c_j\left[(j+1)P_{j-1}^{(1,1)}(x_i) + jP_{j-2}^{(1,1)}(x_i)\right] = f(x_i)\left(y_0 + \sum_{j=1}^{p} c_j(L_j(x_i) + L_{j-1}(x_i))\right)$$

$$+\left(\frac{T}{2}\right)^{1-\mu}\int_{-1}^{x_i}\frac{y_0 + \sum\limits_{j=1}^{p} c_j(L_j(\tau) + L_{j-1}(\tau))}{(x_i-\tau)^{\mu}}d\tau + g(x_i), \quad i = 1,\cdots,p. \quad (3.4)$$

By virtue of (2.14) again, we obtain

$$\sum_{j=1}^{p} c_j\left[\frac{j+1}{T}P_{j-1}^{(1,1)}(x_i) + \frac{j}{T}P_{j-2}^{(1,1)}(x_i) - (L_j(x_i) + L_{j-1}(x_i))f(x_i)\right. \quad (3.5)$$

$$-\left(\frac{T}{2}\right)^{1-\mu}\frac{j!}{(1-\mu)_{j+1}}(1+x_i)^{1-\mu}P_j^{(\mu-1,1-\mu)}(x_i) - \left(\frac{T}{2}\right)^{1-\mu}\frac{(j-1)!}{(1-\mu)_j}(1+x_i)^{1-\mu}$$

$$\left.P_{j-1}^{(\mu-1,1-\mu)}(x_i)\right] = g(x_i) + f(x_i)y_0 + \left(\frac{T}{2}\right)^{1-\mu}\frac{y_0}{1-\mu}(1+x_i)^{1-\mu}, \quad i = 1,\cdots,p.$$

3.1.2 Algorithm for Fredholm integro-differential equation

We investigate algorithm for the equation

$$y'(t) = a(t)y(t) + \int_0^T \frac{y(s)}{|t-s|^{\mu}}ds + f(t),\ y(0) = y_0,\ t \in [0,T]. \quad (3.6)$$

We take the same notation and variable transformation as in algorithm for (2.23) and we obtain

$$\frac{2}{T}u'(x) = f(x)u(x) + \left(\frac{T}{2}\right)^{1-\mu}\int_{-1}^{x}\frac{u(\tau)}{(x-\tau)^{\mu}}d\tau + \left(\frac{T}{2}\right)^{1-\mu}\int_{x}^{1}\frac{u(\tau)}{(\tau-x)^{\mu}}d\tau + g(x),$$

$$x \in [-1,1]. \quad (3.7)$$

Again, $f(x)$ has the same definition as in the previous algorithm.

Let $u_p(x) = y_0 + \sum\limits_{j=1}^{p} c_j(L_j(x) + L_{j-1}(x))$ be the approximation of $u(x)$. Then c_j

must satisfy the equation

$$g(x_i) = \sum_{j=1}^{p} c_j \left(\frac{2}{T}(L_j^{'}(x_i) + L_{j-1}^{'}(x_i)) \right) - f(x_i)(y_0 + \sum_{j=1}^{p} c_j(L_j(x_i) + L_{j-1}(x_i)))$$
$$-\left(\frac{T}{2}\right)^{1-\mu} \int_{-1}^{x_i} \frac{y_0 + \sum_{j=0}^{p} c_j(L_j(\tau) + L_{j-1}(\tau))}{(x_i - \tau)^{\mu}} d\tau$$
$$-\left(\frac{T}{2}\right)^{1-\mu} \int_{x_i}^{1} \frac{y_0 + \sum_{j=0}^{p} c_j(L_j(\tau) + L_{j-1}(\tau))}{(\tau - x_i)^{\mu}} d\tau. \tag{3.8}$$

Apply (2.15) and (3.3), we obtain

$$\sum_{j=1}^{p} c_j \left[\frac{j+1}{T} P_{j-1}^{(1,1)}(x_i) + \frac{j}{T} P_{j-2}^{(1,1)}(x_i) - (L_j(x_i) + L_{j-1}(x_i)) f(x_i) \right. \tag{3.9}$$
$$-\left(\frac{T}{2}\right)^{1-\mu} \left(\frac{j!}{(1-\mu)_{j+1}} (1+x_i)^{1-\mu} P_j^{(\mu-1,1-\mu)}(x_i) + \frac{(j-1)!}{(1-\mu)_j} (1+x_i)^{1-\mu} P_{j-1}^{(\mu-1,1-\mu)}(x_i) \right.$$
$$\left.\left. + \frac{j!}{(1-\mu)_{j+1}} (1-x_i)^{1-\mu} P_j^{(1-\mu,\mu-1)}(x_i) + \frac{(j-1)!}{(1-\mu)_j} (1-x_i)^{1-\mu} P_{j-1}^{(\mu-1,1-\mu)}(x_i) \right) \right]$$
$$= g(x_i) + f(x_i) y_0 + \left(\frac{T}{2}\right)^{1-\mu} \left(\frac{y_0}{1-\mu}(1+x_i)^{1-\mu} + \frac{y_0}{1-\mu}(1-x_i)^{1-\mu} \right),$$
$$i = 1, \cdots, p. \tag{3.10}$$

3.2 Convergence Analysis

This section concerns the L^{∞} norm of errors for both type of equations. We will apply the Fredholm Alternative in the proof of this subsection.

Lemma 3.2.1 (The Fredholm Alternative) Let X be a Banach space, and let $A \in \mathcal{L}(X, X)$ be a compact operator. Then the equation $x = Ax + g, g \in X$ has a unique solution $x \in X$ if and only if the homogeneous equation $z = Az$ has only the trivial solution $z = 0$. In such a case, the operator $I - A$ has a bounded inverse

$(I - A)^{-1} \in \mathcal{L}(X, X)$.

3.2.1 Volterra Integro differential equations

Theorem 3.2.2 Let y and y_p be the exact solution and spectral approximation of (3.1), respectively.

1) If $y(t)$ satisfies the condition (R), then, for sufficiently large p,

$$\|y - y_p\|_\infty \leq \frac{C(p+2)}{R}\left(\frac{T}{4R}\right)^{p+1}, \tag{3.11}$$

2) If y satisfies the condition (M), then, for sufficiently large p,

$$\|y - y_p\|_\infty \leq \frac{C\log p}{\sqrt{p+1}}\left(\frac{eMT}{4(p+1)}\right)^{p+1}. \tag{3.12}$$

Proof: Using (3.6), we have

$$\left.\frac{du_p}{dx}\right|_{x_i} = f(x_i)u_i + \left(\frac{T}{2}\right)^{1-\mu}\int_{-1}^{x_i}(x_i - \tau)^{-\mu}u_p(\tau)d\tau + g(x_i), \tag{3.13}$$

where

$$u_i = y_0 + \sum_{j=1}^{p} c_j(L_j(x_i) + L_{j-1}(x_i))$$

Note that the true solution at Chebyshev points satisfies

$$\left.\frac{du}{dx}\right|_{x_i} = f(x_i)u(x_i) + \left(\frac{T}{2}\right)^{1-\mu}\int_{-1}^{x_i}(x_i - \tau)^{-\mu}u(\tau)d\tau + g(x_i), \tag{3.14}$$

Thus, let $e(x) = u(x) - u_p(x)$, we have

$$\left.\left(\frac{du}{dx} - \frac{du_p}{dx}\right)\right|_{x_i} = f(x_i)e(x_i) + \left(\frac{T}{2}\right)^{1-\mu}\int_{-1}^{x_i}(x_i - \tau)^{-\mu}e(\tau)d\tau. \tag{3.15}$$

Multiplying both sides by $l_i(x)$ and summing up from 1 to p and applying the fact that $u_p(x) = y_0 + \sum_{j=1}^{p} c_j(L_j(x) + L_{j-1}(x)) = y_0 l_0(x) + \sum_{j=1}^{p} u_j l_j(x)$, we obtain

$$\begin{aligned} & I_p u'(x) - I_p u_p'(x) - I_p\Big[f(x)(I_p u(x) - u_p(x))\Big] \\ = & \left(\frac{T}{2}\right)^{1-\mu} I_p\left[\int_{-1}^{x}(x-\tau)^{-\mu}e(\tau)d\tau\right] + I_p\Big[f(x)(u(x) - I_p u(x))\Big]. \quad (3.16) \end{aligned}$$

We write

$$e(x) = u(x) - I_p u(x) + I_p u(x) - u_p(x),\ e'(x) = u'(x) - I_p u'(x) + I_p u'(x) - u_p'(x). \quad (3.17)$$

Thus, by (3.16) and simple calculation,

$$\begin{aligned} e'(x) - f(x)e(x) &= u'(x) - I_p u'(x) - f(x)(u(x) - I_p u(x)) - (I - I_p)[f(x)(I_p u(x) - u_p(x))] \\ &\quad + I_p u'(x) - u_p'(x) - I_p[f(x)(I_p u(x) - u_p(x))] \\ &= \underbrace{u'(x) - I_p u'(x)}_{I_1} - \underbrace{f(x)(u(x) - I_p u(x))}_{I_2} - \underbrace{(I - I_p)[f(x)(I_p u(x) - u_p(x))]}_{I_3} \\ &\quad + \underbrace{\left(\frac{T}{2}\right)^{1-\mu}\left\{(I_p - I)\left[\int_{-1}^{x}(x-\tau)^{-\mu}e(\tau)d\tau\right]\right\}}_{I_4} + \underbrace{I_p[f(x)(u(x) - I_p u(x))]}_{I_5} \\ &\quad + \left(\frac{T}{2}\right)^{1-\mu}\int_{-1}^{x}(x-\tau)^{-\mu}e(\tau)d\tau. \quad (3.18) \end{aligned}$$

Denoting $I(x) = \sum_{k=1}^{5} I_k(x)$ and integrating both sides from -1 to x, we derive

$$\begin{aligned} e(x) &= \int_{-1}^{x} I(s)ds + \int_{-1}^{x}\left(f(s) + (\frac{T}{2})^{1-\mu}\int_{s}^{x}(v-s)^{-\mu}dv\right)e(s)ds \\ &= \int_{-1}^{x} I(s)ds + \int_{-1}^{x}\left(f(s) + (\frac{T}{2})^{1-\mu}\frac{(x-s)^{1-\mu}}{1-\mu}\right)e(s)ds \\ &\leq \tilde{I}(x) + L\int_{-1}^{x} e(s)ds, \quad (3.19) \end{aligned}$$

where $\tilde{I}(x) = \int_{-1}^{x} I(s)ds$, $L = \frac{T^{1-\mu}}{2^{2-2\mu}} \frac{1}{1-\mu} + \max_{x\in[-1,1]} |f(x)|$.

By the Gronwall inequality, we have

$$e(x) \leq |\tilde{I}(x)| + C \int_{-1}^{x} |\tilde{I}(\tau)| d\tau. \tag{3.20}$$

Therefore,

$$\|e\|_{\infty} \leq C \|\tilde{I}\|\|_{\infty} \leq 2C \|I\|_{\infty}. \tag{3.21}$$

Now, let us estimate terms from I_1 to I_5.

1) Estimates of I_1:

If y satisfies condition (R), then

$$\|I_1\|_{\infty} \leq \frac{C(p+2)}{R} \left(\frac{T}{4R} \right)^{p+1}, \tag{3.22}$$

and if y satisfies condition (M), then

$$\|I_1\|_{\infty} \leq \frac{C}{\sqrt{p+1}} \left(\frac{eMT}{4(p+1)} \right)^{p+1}. \tag{3.23}$$

2) Estimates of I_2:

The result follows directly from (2.10) and (2.11) and the fact that $|f(x)| \leq M$ on $[0, T]$.

Hence, If y satisfies condition (R), then

$$\|I_2\|_{\infty} \leq C \left(\frac{T}{4R} \right)^{p+1}, \tag{3.24}$$

and if y satisfies condition (M), then

$$\|I_2\|_{\infty} \leq \frac{C}{\sqrt{p+1}} \left(\frac{eMT}{4(p+1)}. \right)^{p+1}. \tag{3.25}$$

3) Estimates of I_3:

Since $f(x)$ is analytic and $I_p u(x) - u_p(x)$ are polynomials, the product of them

is

analytic also.

$$\|I_3\|_\infty \le \frac{C}{\sqrt{p+1}} \left(\frac{eT}{4(p+1)} \right)^{p+1}. \tag{3.26}$$

4) Estimates of I_4

It is exactly the same as I_2 in the proof of Theorem 2.3.5. Therefore, if p is sufficiently large, from the estimate of I_2 in Theorem 2.3.5, we obtain

$$\|I_4\|_\infty \le \frac{1}{4C} \|e\|_\infty. \tag{3.27}$$

5) Estimates of I_5

$$\|I_5\|_\infty \le C \log p \|u(x) - I_p u(x)\|_\infty. \tag{3.28}$$

Therefore,

$$\|I_5\|_\infty \le C \log p \left(\frac{T}{4R} \right)^{p+1}, \qquad (y \text{ satisfies condition (R)}); \tag{3.29}$$

$$\|I_5\|_\infty \le \frac{C \log p}{\sqrt{p+1}} \left(\frac{eMT}{4(p+1)} \right)^{p+1} \qquad (y \text{ satisfies condition (M)}). \tag{3.30}$$

Combine all estimates above, we derive

$$\|u - u_p\|_\infty \le \frac{C(p+2)}{R} \left(\frac{T}{4R} \right)^{p+1}, \qquad (y \text{ satisfies condition (R)}); \tag{3.31}$$

$$\|u - u_p\|_\infty \le \frac{C \log p}{\sqrt{p+1}} \left(\frac{eMT}{4(p+1)} \right)^{p+1} \qquad (y \text{ satisfies condition (M)}). \tag{3.32}$$

3.2.2 Fredholm integro-differential equation

Theorem 3.2.3 Let y and y_p be the exact solution and spectral approximation of (3.6), respectively. Then

1) If y satisfies the condition (R), then, for sufficiently large p,

$$\|y - y_p\|_\infty \leq \frac{C(p+2)}{R}\left(\frac{T}{4R}\right)^{p+1}, \tag{3.33}$$

2)If y satisfies the condition (M), then, for sufficiently large p,

$$\|y - y_p\|_\infty \leq \frac{C\log p}{\sqrt{p+1}}\left(\frac{eMT}{4(p+1)}\right)^{p+1}. \tag{3.34}$$

Proof: Follow exactly the same technique as in the proof of Theorem 3.2.2, we obtain

$$\begin{aligned} & I_p u'(x) - I_p u_p'(x) - I_p\Big[f(x)(I_p u(x) - u_p(x))\Big] \\ = & \left(\frac{T}{2}\right)^{1-\mu} I_p\left[\int_{-1}^{1} |x-\tau|^{-\mu} e(\tau)d\tau\right] + I_p\Big[f(x)(u(x) - I_p u(x))\Big]. \end{aligned} \tag{3.35}$$

Again, we write

$$e(x) = u(x) - I_p u(x) + I_p u(x) - u_p(x), \; e'(x) = u'(x) - I_p u'(x) + I_p u'(x) - u_p'(x). \tag{3.36}$$

Thus, by (3.10) and simple calculation,

$$\begin{aligned} e'(x) - f(x)e(x) &= u'(x) - I_p u'(x) - f(x)(u(x) - I_p u(x)) - (I - I_p)[f(x)(I_p u(x) - u_p(x))] \\ &\quad + I_p u'(x) - u_p'(x) - I_p[f(x)(I_p u(x) - u_p(x))] \\ &= \underbrace{u'(x) - I_p u'(x)}_{I_1} - \underbrace{f(x)(u(x) - I_p u(x))}_{I_2} - \underbrace{(I - I_p)[f(x)(I_p u(x) - u_p(x))]}_{I_3} \\ &\quad \underbrace{+\left(\frac{T}{2}\right)^{1-\mu}\left\{(I_p - I)\left[\int_{-1}^{1}|x-\tau|^{-\mu}e(\tau)d\tau\right]\right\}}_{I_4} + \underbrace{I_p[f(x)(u(x) - I_p u(x))]}_{I_5} \end{aligned}$$

$$+\left(\frac{T}{2}\right)^{1-\mu}\int_{-1}^{1}|x-\tau|^{-\mu}e(\tau)d\tau. \tag{3.37}$$

Denoting $\tilde{I}(x)=\sum_{k=1}^{5}I_k(x)$ and adopting the idea from Theorem 4.2.1 in [33], we let $z(x)=e'(x)$. From our algorithm, it is clear that $z(-1)=0$. Then, by change the order of integration,

$$\begin{aligned} z(x) &= \tilde{I}(x)+f(x)\int_{-1}^{x}z(s)ds+\left(\frac{T}{2}\right)^{1-\mu}\int_{-1}^{1}\int_{-1}^{s}|x-s|^{-\mu}dsz(u)du \\ &= \tilde{I}(x)+f(x)\int_{-1}^{x}z(s)ds+\left(\frac{T}{2}\right)^{1-\mu}\frac{1}{1-\mu}\int_{-1}^{x}(x-u)^{1-\mu}z(u)du \\ &\quad +\left(\frac{T}{2}\right)^{1-\mu}\frac{1}{1-\mu}\int_{x}^{1}(1-x)^{1-\mu}z(u)du. \end{aligned} \tag{3.38}$$

Define

$$Az=f(x)\int_{-1}^{x}z(s)ds+\left(\frac{T}{2}\right)^{1-\mu}\frac{1}{1-\mu}\int_{-1}^{x}(x-u)^{1-\mu}z(u)du+\left(\frac{T}{2}\right)^{1-\mu}\frac{1}{1-\mu}\int_{x}^{1}(1-x)^{1-\mu}z(u)du.$$

Clearly, A is compact by the Arzelá-Ascoli theory.

Hence, (3.38) can be written as

$$z=Az+\tilde{I}.$$

From our assumption and the Fredholm Alternative, $(I-A)$ has a bounded inverse. Thus,

$$\|z\|_\infty \le C\|\tilde{I}\|_\infty. \tag{3.39}$$

But $e(x)=\int_{-1}^{x}z(s)ds, x\in[-1,1]$, then we have

$$\|e\|_\infty \le 2\|z\|_\infty \le C\|\tilde{I}\|_\infty. \tag{3.40}$$

Clearly, I_1, I_2, I_3 and I_5 are the same as those of Theorem 3.2.2 and I_4 is the same as I_2 in Theorem 2.3.8, we obtain the same estimates of these terms. Combine all estimates above, we derive

$$\|I\|_\infty \leq \frac{C(p+2)}{R}\left(\frac{T}{4R}\right)^{p+1}, \qquad \text{(y satisfies condition (R));} \tag{3.41}$$

$$\|I\|_\infty \leq \frac{C\log p}{\sqrt{p+1}}\left(\frac{eMT}{4(p+1)}\right)^{p+1} \quad \text{(y satisfies condition (M)).} \tag{3.42}$$

The result follows.

3.3 Numerical Experiments

Our scheme for integro differential equations has the form

$$DC_p + LC_p - AC_p = G_p,$$

where $C_p = [c_0, \cdots, c_p]^T$ and $G_p = [g(x_0), \cdots, g(x_p)]^T$ and the elements of the matrix $A = (a_{ij})$, $L = (l_{ij})$ and $D = (d_{ij})$ are given by

Scheme for (3.1):

$$\begin{aligned}
a_{ij} &= \left(\frac{T}{2}\right)^{1-\mu}\frac{j!}{(1-\mu)_{j+1}}(1+x_i)^{1-\mu}P_j^{(\mu-1,1-\mu)}(x_i) \\
&\quad + \left(\frac{T}{2}\right)^{1-\mu}\frac{(j-1)!}{(1-\mu)_j}(1+x_i)^{1-\mu}P_{j-1}^{(\mu-1,1-\mu)}(x_i), \\
l_{ij} &= -(L_j(x_i) + L_{j-1}(x_i))f(x_i), \\
d_{ij} &= \frac{j+1}{T}P_{j-1}^{(1,1)}(x_i) + \frac{j}{T}P_{j-2}^{(1,1)}(x_i)
\end{aligned}$$

Algorithm for (3.6):

$$\begin{aligned}
a_{ij} &= \left(\frac{T}{2}\right)^{1-\mu} \frac{j!}{(1-\mu)_{j+1}}(1+x_i)^{1-\mu}P_j^{(\mu-1,1-\mu)}(x_i) \\
&+\left(\frac{T}{2}\right)^{1-\mu} \frac{(j-1)!}{(1-\mu)_j}(1+x_i)^{1-\mu}P_{j-1}^{(\mu-1,1-\mu)}(x_i) \\
&+\left(\frac{T}{2}\right)^{1-\mu} \frac{j!}{(1-\mu)_{j+1}}(1-x_i)^{1-\mu}P_j^{(1-\mu,\mu-1)}(x_i) \\
&+\left(\frac{T}{2}\right)^{1-\mu} \frac{(j-1)!}{(1-\mu)_j}(1-x_i)^{1-\mu}P_{j-1}^{(1-\mu,\mu-1)}(x_i) \\
l_{ij} &= -(L_j(x_i)+L_{j-1}(x_i))f(x_i), \\
d_{ij} &= \frac{j+1}{T}P_{j-1}^{(1,1)}(x_i) + \frac{j}{T}P_{j-2}^{(1,1)}(x_i)
\end{aligned}$$

In this whole section, reference curve 1 associate with Example 3.3.1 and Example 3.3.2 is the graph of $f(p) = \frac{p+2}{3^{p+1}}$ and reference curve 1 associate with Example 3.3.3 and Example 3.3.4 is the graph of $f(p) = \frac{p+2}{(\frac{17}{5})^{p+1}}$. Reference curve 2 is the graph of $f(p) = \frac{\log p}{\sqrt{p+1}}(\frac{eM}{4(p+1)})^{p+1}$ for all four examples.

Example 3.3.1 Now let us consider integro-differential equations. First, we consider an equation of the form (3.1) on [0, 6] with $\mu = \frac{1}{2}$, $a(t) = e^t$ and

$$b(t) = \frac{3}{2}\sqrt{t+2} - e^t(t+2)^{2/3} - \frac{3\pi}{8}(t+2)^2 + \frac{3}{4}(t+2)^2\arctan\left(\sqrt{\frac{2}{t}}\right) - \frac{\sqrt{2t}}{4}(3t+10).$$

The exact solution for this example is $y(t) = (2+t)^{3/2}$. Numerical errors of are reported in Table 5 and left part of Figure 5.

Example 3.3.2 Consider an integro-differential equation of the form (3.1) on [0, 6] with $\mu = \frac{1}{4}, a(t) = \sin(t)$ and $b(t) = e^t - e^t(\sin(t) - \gamma(\frac{1}{3}, t))$, where $\gamma(a, x)$ is the lower incomplete gamma function defined above. The exact solution of this equation

Table 5: Example 3.3.1: errors for $t \in [0,6]$

N	4	8	12	14
$\|\cdot\|_{L^\infty}$	2.5788e-04	4.2887e-06	3.3610e-08	4.6499e-09
N	18	20	22	24
$\|\cdot\|_{L^\infty}$	7.8877e-11	1.0431e-11	1.4531e-12	5.1514e-13

Table 6: Example 3.3.2: errors for $t \in [0,6]$

N	4	8	12	14
$\|\cdot\|_{L^\infty}$	4.9325e+01	7.9231e-02	2.9523e-05	3.0818e-07
N	18	20	22	24
$\|\cdot\|_{L^\infty}$	1.7735e-11	2.6148e-12	4.4338e-12	3.4106e-13

is $y(t) = e^t$. Numerical results are listed in Table 6 and right part of Figure 5.

Example 3.3.3 Consider a Fredholm integro-differential equation of the form (3.6) on [0,6]. We choose $\mu = 1/2$, $a(t) = e^t$ and

$$\begin{aligned} b(t) &= \frac{1}{2\sqrt{t+2}} - e^t\sqrt{t+2} - \frac{\pi}{2}(t+2) + (t+2)\arctan\left(\sqrt{\frac{2}{t}}\right) - \sqrt{2t} - 2\sqrt{12-2t} \\ &\quad -(t+2)\log(2\sqrt{6-t}+4\sqrt{2}) + (t+2)\log(2\sqrt{t+2}) \end{aligned}$$

so that the true solution is $y(t) = \sqrt{t+2}$. Readers are referred to Table 7 and left part of Figure 6 for numerical report.

Example 3.3.4 Consider an integro-differential equation of the form (3.6) on [0,

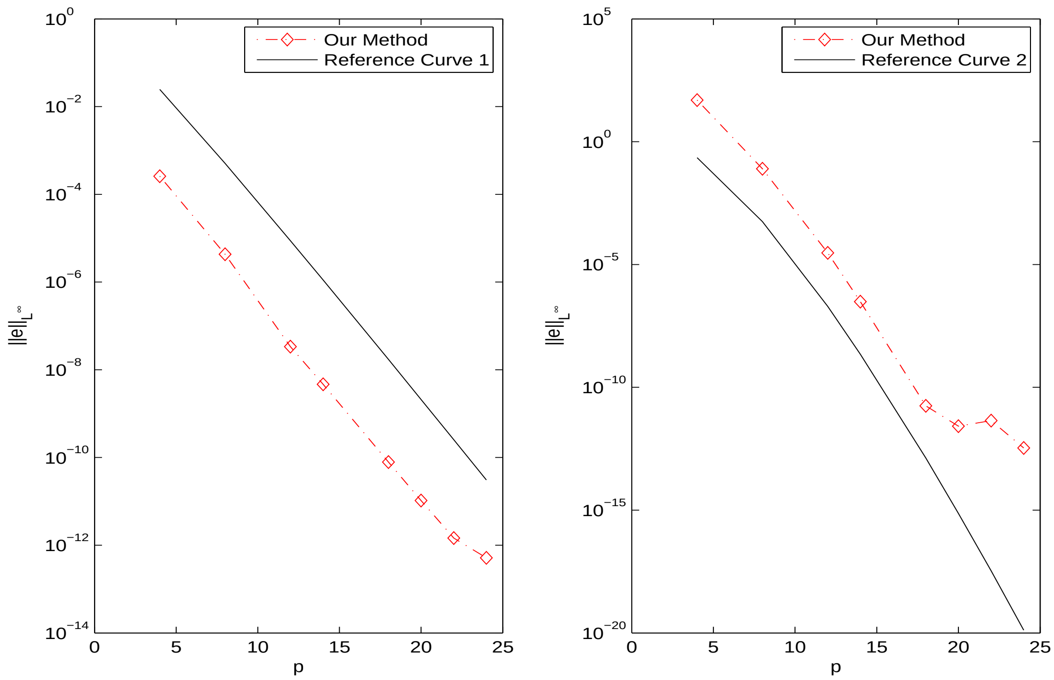


Figure 5. Left: $\|\cdot\|_{L^\infty}$ error for Example 3.3.1 Right: $\|\cdot\|_{L^\infty}$ error for Example 3.3.2

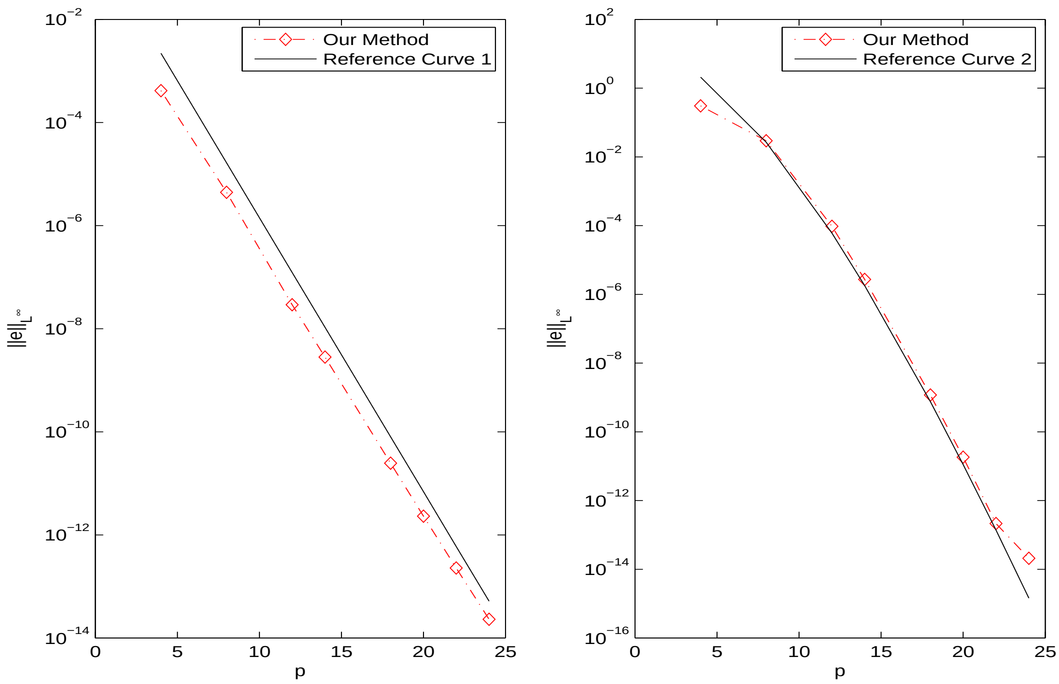


Figure 6. Left: $\|\cdot\|_{L^\infty}$ error for Example 3.3.3 Right: $\|\cdot\|_{L^\infty}$ error for Example 3.3.4

Table 7: Example 3.3.3: errors for $t \in [0, 6]$

N	4	8	12	14
$\|\cdot\|_{L^\infty}$	4.1506e-04	4.4205e-06	2.9091e-08	2.8076e-09
N	18	20	22	24
$\|\cdot\|_{L^\infty}$	2.4562e-11	2.3024e-12	2.2715e-13	2.3093e-14

Table 8: Example 3.3.4: errors for $t \in [0, 10]$

N	4	8	12	14
$\|\cdot\|_{L^\infty}$	3.0272e-01	2.9460e-02	9.5393e-05	2.7053e-06
N	18	20	22	24
$\|\cdot\|_{L^\infty}$	1.1763e-09	1.8301e-11	2.1344e-13	2.0872e-14

10]. We choose $\mu = 1/2$ and

$$\begin{aligned} b(t) &= \cos(t) - a(t)\sin(t) + \sqrt{2\pi}\left[\cos(t) S\left(\sqrt{\frac{2t}{\pi}}\right) - \sin(t) C\left(\sqrt{\frac{2t}{\pi}}\right)\right] \\ &\quad -\sqrt{2\pi}\left[\sin(t) C\left(\sqrt{\frac{2(10-t)}{\pi}}\right) + \cos(t) S\left(\sqrt{\frac{2(10-t)}{\pi}}\right)\right] \end{aligned}$$

so that the true solution is $y(t) = \sin(t)$. Here, $C(u)$ and $S(u)$ are Fresnel integrals defined as in Example 2.4.4 and

$$a(t) = \begin{cases} 0, & t \leq 5; \\ 1, & t > 5. \end{cases} \tag{3.43}$$

Numerical results are given in Table 8 and right part of Figure 6.

From our numerical experiments, we see the condition that we put on $a(t)$ can be violated as long as the solution $y(t)$ is sufficiently smooth.

4 Eigenvalue approximation for compact integral operators

In this chapter, we focus on problems of the form

$$\int_0^1 k(t,s)u(s)ds = \lambda u(t), \quad t \in [0,1], \tag{4.1}$$

where $k(t,s) = |t-s|^{-\mu}$ for $0 < \mu < 1$ and $k(t,s)$ is piecewisely smooth or smooth. We will develop algorithms for all these types of kernels and prove a convergence rate for eigenvalue approximation respectively.

4.1 Preliminary

Hypergeometric function, as a generalized function of standard function will be used in our algorithms. It is defined as follows:

$${}_pF_q(z) = \sum_{r=0}^{\infty} \frac{(a_1)_r \cdots (a_p)_r}{(b_1)_r \cdots (b_q)_r} \frac{z^r}{r!} \tag{4.2}$$

where $(a_j)_r$ and $(b_j)_r$ are the Pochhammer symbols defined as

$$(k)_r = k(k+1)\cdots(k+r-1),\ (k)_0 = 1. \tag{4.3}$$

The above series is defined when none of $b_j's, j = 1, \cdots, q$ is a negative integer or zero. A b_j can be zero provided there is a numerator parameter a_k such that $(a_k)_r$ becomes zero first before $(b_j)_r$ becomes zero. If any numerator parameter a_j is a negative integer or zero then the series terminates and becomes a polynomial in z. From the ratio test, it is evident that the series is convergent for all z if $q \geq p$, it is convergent for $|z| \leq 1$ if $p = q+1$ and diverge if $p > q+1$. Note that the library function "hypergeom" in Matlab is very slow in computation, we call "hypergeom" in Maple from Matlab to save time and preserve a high digit of accuracy.

Let $T : X \to X$ be a compact linear operator on a Banach space X and $\sigma(T)$ and $\rho(T)$ be the spectrum and resolvent of T respectively. Let λ be a nonzero eigenvalue of T with multiplicity m and let Γ be a circle centered at λ which lies in $\rho(T)$ and which encloses no other points in $\sigma(T)$. Then, the spectral projection associated with T and λ is defined by

$$E = -\frac{1}{2\pi i}\int_\Gamma (T - zI)^{-1} dz$$

and

$$\max_{z\in\Gamma} \|(T - zI)^{-1}\| \leq C$$

Let T_n be a sequence of operator in $\mathcal{B}(X)$ that converges to T in a collectively way, i.e., the set $\{T_n x : \|x\| \leq 1, n = 1, 2, \cdots\}$ is sequentially compact. For n large enough, $\Gamma \in \rho(T_n)$ and the associated projection,

$$E_n = -\frac{1}{2\pi i}\int_\Gamma (T_n - zI)^{-1} dz$$

exists and $\max_{z\in\Gamma} \|(T_n - zI)^{-1}\| \leq C$. Clearly, $\dim(E) = \dim(E_n) = m$ and $T_n E_n = E_n T_n$. Furthermore, the spectrum of T_n inside Γ, contains m approximations of λ, i.e. $\lambda_{n,1}, \lambda_{n,2}, \cdots, \lambda_{n,m}$, counted according to their algebraic multiplicities [15, 32].Let

$$\hat{\lambda}_n = \frac{\lambda_{n,1} + \lambda_{n,2} + \cdots + \lambda_{n,m}}{m}$$

Then we have the following theorem.

Theorem 4.1.1[32] For all n sufficiently large,

$$|\lambda - \hat{\lambda}_n| \le C\|(T - T_n)|_{R(E)}\|,$$

where $R(E)$ is the range of the projection E.

This is a rather general result. We may refine the result if the kernel is positive definite. Let

$$a(u,v) = \int_0^1 \int_0^1 k(t,s)u(s)v(t)dsdt, \quad b(u,v) = \int_0^1 u(t)v(t)dt, \tag{4.4}$$

where v is a test function in L^2 space V. If the bilinear operator $a(u,v)$ is coercive, then we can list eigenvalues of T by

$$\lambda_1 \ge \lambda_2 \ge \lambda_3 \ge \cdots \ge 0$$

with zero the only possible cluster point.

Let us consider a numerical approximation of the first eigen-pair (λ, u). Let (λ_p, v_p) be their Galerkin approximation, and let u_p be the Legendre expansion of u. We have

$$\lambda = \frac{a(u,u)}{b(u,u)} = \sup_{v \in V} \frac{a(v,v)}{b(v,v)}, \quad \lambda_p = \frac{a(v_p,v_p)}{b(v_p,v_p)} = \max_{v \in \mathcal{P}_p} \frac{a(v,v)}{b(v,v)}. \tag{4.5}$$

Here $\mathcal{P}_p$ is the polynomial space with degree no more than p. Denote $\tilde{\lambda}_p = \frac{a(u_p,u_p)}{b(u_p,u_p)}$, then we have the following lemma.

Lemma 4.1.2 Let λ, λ_p and $\tilde{\lambda}_p$ be defined as above and $a(u,v)$ be coercive, then

$$0 \le \lambda - \lambda_p \le \lambda - \tilde{\lambda}_p = \lambda \frac{\|u - u_p\|_b^2}{\|u\|_b^2} - \frac{\|u - u_p\|_a^2}{\|u\|_b^2}. \tag{4.6}$$

Proof: From Lemma 9.1 of [5] on page 701, we have

$$0 \le \nu_p - \nu \le \tilde{\nu}_p - \nu \le \frac{\|u - u_p\|_b^2}{\|u\|_a^2} - \nu \frac{\|u - u_p\|_a^2}{\|u\|_a^2}, \tag{4.7}$$

where $\nu = \frac{1}{\lambda}, \nu_p = \frac{1}{\lambda_p}$ and $\tilde{\nu}_p = \frac{1}{\tilde{\lambda}_p}$. Hence,

$$0 \leq \frac{\lambda - \lambda_p}{\lambda} \leq \lambda_p \frac{\|u - u_p\|_b^2}{\|u\|_a^2} - \frac{\lambda_p}{\lambda} \frac{\|u - u_p\|_a^2}{\|u\|_a^2}. \tag{4.8}$$

Using the fact that

$$a(u_p, u_p) = \lambda_p b(u_p, u_p),$$

we derive (4.6) from (4.8).

Next, we introduce some identities, which will be essential in this paper.

Lemma 4.1.3[34] Let $\alpha > -1, \beta > -1$ and $0 < \nu < 1$, then for $-1 < x < 1$ holds

$$\int_{-1}^{1} \frac{(1-t)^{\alpha}(1+t)^{\beta} P_m^{(\alpha,\beta)}(t)dt}{|x-t|^{\nu}} = \frac{\cos\frac{\pi\nu}{2}\Phi_1(x) + \cos\pi(\frac{\nu}{2} - \beta)\Phi_2(x)}{\Gamma(\nu)\cos\frac{\pi\nu}{2}}, \quad m = 0, 1, 2, \cdots \tag{4.9}$$

where

$$\begin{aligned} \Phi_1(x) &= \frac{\Gamma(m+\alpha+1)\Gamma(m+\nu)\Gamma(\beta-\nu+1)(-1)^m}{2^{-\alpha-\beta+\nu-1}\Gamma(m+\alpha+\beta-\nu+2)m!} \\ &\quad \times {}_2F_1(m+\nu, \nu-m-\alpha-\beta-1; -\beta+\nu; \frac{1+x}{2}), \qquad (4.10) \\ \Phi_2(x) &= \frac{\Gamma(m+\beta+1)\Gamma(\nu-\beta-1)(-1)^{m+1}}{2^{-\alpha}(1+x)^{\nu-\beta-1}m!} \\ &\quad \times {}_2F_1(m+\beta+1, -m-\alpha; \beta-\nu+2; \frac{1+x}{2}). \qquad (4.11) \end{aligned}$$

Here, ${}_2F_1(a, b; c; z)$ is known as Gauss' hypergeometric functions.

For the sake of convergence analysis, we need to introduce the error estimate of Gauss quadrature and some knowledge of matrix analysis.

Lemma 4.1.4 [17] Let $f \in C^{(2n)}$ and x_i and w_i are Gauss points and their corresponding weights on the interval $[a, b]$. Then

$$\int_a^b f(x)dx - \sum_{i=0}^{n} w_i f(x_i) = \frac{(b-a)^{2n+1}(n!)^4}{(2n+1)[(2n)!]^3} f^{(2n)}(\xi), \ \xi \in (a, b) \tag{4.12}$$

4.2 Algorithms

In this section, we develop algorithms for eigen-problem with all three kinds of kernels. Models that we consider in this section are

(1) Weakly singular kernels:

$$\lambda y(t) = \int_0^1 \frac{y(s)}{|t-s|^\mu} ds, \quad 0 < \mu < 1, \ t \in [0,1], \tag{4.13}$$

(2) Piecewise smooth kernel:

$$\lambda y(t) = \int_0^1 k(t,s) y(s) ds, \qquad t \in [0,1], \tag{4.14}$$

$$\text{where } k(t,s) = \begin{cases} t - s/2, & \text{if } 0 \le t \le s \le 1 \\ s/2, & \text{if } 0 \le s < t \le 1, \end{cases}$$

(3) Smooth kernel

$$\lambda y(t) = \int_0^1 e^{st} y(s) ds, \qquad t \in [0,1]. \tag{4.15}$$

4.2.1 The first algorithm for Weakly Singular Kernel

It is clear that (4.13) is equivalent to

$$\lambda y(t) = \int_0^t \frac{y(s)}{(t-s)^\mu} ds + \int_t^1 \frac{y(s)}{(s-t)^\mu} ds. \tag{4.16}$$

We make a change of variable $t = \frac{1+x}{2}$ and obtain

$$\int_0^{(1+x)/2} \left(\frac{1+x}{2} - s \right)^{-\mu} y(s) ds + \int_{(1+x)/2}^1 \left(s - \frac{1+x}{2} \right)^{-\mu} y(s) ds = \lambda u(x), \tag{4.17}$$

where $x \in [-1,1]$ and $u(x) = y\left(\frac{1+x}{2}\right)$. Next, we make another change of variable, $s = \frac{1+\tau}{2}$. and reach

$$\left(\frac{1}{2}\right)^{1-\mu} \int_{-1}^{x} (x-\tau)^{-\mu} u(\tau) d\tau + \left(\frac{1}{2}\right)^{1-\mu} \int_{x}^{1} (\tau - x)^{-\mu} u(\tau) d\tau = \lambda u(x),$$

$$x \in [-1,1] \quad (4.18)$$

Let $u_p(x) = \sum_{j=0}^{p} c_j L_j(x)$ be the approximation of $u(x)$. Obviously, c_j's satisfy the equation

$$\left(\frac{1}{2}\right)^{1-\mu} \sum_{j=0}^{p} c_j \int_{-1}^{x_i} \frac{L_j(\tau)}{(x_i-\tau)^{\mu}} d\tau + \left(\frac{1}{2}\right)^{1-\mu} \sum_{j=0}^{p} c_j \int_{x_i}^{1} \frac{L_j(\tau)}{(\tau - x_i)^{\mu}} d\tau = \lambda_p \sum_{j=0}^{p} c_j L_j(x_i). \quad (4.19)$$

Substituting (2.14) and (2.15), we obtain

$$\sum_{j=0}^{p} c_j \left[\left(\frac{1}{2}\right)^{1-\mu} \frac{j!}{(1-\mu)_{j+1}} (1+x_i)^{1-\mu} P_j^{(\mu-1,1-\mu)}(x_i) + \left(\frac{1}{2}\right)^{1-\mu} \frac{j!}{(1-\mu)_{j+1}} (1-x_i)^{1-\mu} P_j^{(1-\mu,\mu-1)}(x_i) \right] = \lambda_p \sum_{j=0}^{p} c_j L_j(x_i), \quad i = 0, \cdots, p. \quad (4.20)$$

If we write

$$\begin{aligned} a_{ij} &= \left(\frac{1}{2}\right)^{1-\mu} \frac{j!}{(1-\mu)_{j+1}} (1+x_i)^{1-\mu} P_j^{(\mu-1,1-\mu)}(x_i) \\ &\quad + \left(\frac{1}{2}\right)^{1-\mu} \frac{j!}{(1-\mu)_{j+1}} (1-x_i)^{1-\mu} P_j^{(1-\mu,\mu-1)}(x_i) \\ b_{ij} &= L_j(x_i). \end{aligned}$$

then we have $AC_p = \lambda_p B C_p$, where $A = (a_{ij}), B = b_{ij}, C_p = (c_0, c_1, \cdots, c_p)^T$.

4.2.2 The Second Algorithm for the Weakly Singular Kernel

From Theorem 3 of [38], we derive that the first true eigenvector is of the form

$$y(t) = \hat{d}_1 t^{1-\mu} + \hat{d}_2 (1-t)^{1-\mu} + \text{a smoother function } \phi(t). \tag{4.21}$$

Hence, we approximate the eigenvector by $u_p(t) = d_1 t^{1-\mu} + d_2(1-t)^{1-\mu} + \sum_{j=0}^{p} c_j P_j(t)$, where $P_j(t)$ is the shifted Legendre polynomials on $[0,1]$, $j = 0, 1, \cdots, p$. Substituting it into (4.13) and taking the same change of variable as the previous algorithm, we obtain

$$\begin{aligned}
&\left(\frac{1}{2}\right)^{2-2\mu}\left(d_1 \int_{-1}^{1} \frac{(1+\tau)^{1-\mu}}{|x-\tau|^{\mu}} d\tau + d_2 \int_{-1}^{1} \frac{(1-\tau)^{1-\mu}}{|x-\tau|^{\mu}} d\tau\right) \\
&+\left(\frac{1}{2}\right)^{1-\mu} \sum_{j=0}^{p} c_j \left(\int_{-1}^{x} (x-\tau)^{-\mu} L_j(\tau) d\tau + \int_{x}^{1} (\tau-x)^{-\mu} L_j(\tau) d\tau\right) \\
&= d_1 \left(\frac{1+x}{2}\right)^{1-\mu} + d_2 \left(\frac{1-x}{2}\right)^{1-\mu} + \sum_{j=0}^{p} c_j L_j(x), \quad x \in [-1,1].
\end{aligned} \tag{4.22}$$

From lemma 2.1.1 and lemma 4.1.3, and (4.22), we obtain

$$\begin{aligned}
&\left(\frac{1}{2}\right)^{2-2\mu} \frac{\Gamma(2-2\mu)}{2^{2\mu-2}\Gamma(3-2\mu)} {}_2F_1(\mu, 2\mu-2; 2\mu-1; \frac{1+x_i}{2}) d_1 + \left(\frac{1}{2}\right)^{2-2\mu} \frac{1}{\Gamma(\mu)} \\
&\Bigg(\frac{\Gamma(2-\mu)\Gamma(\mu)\Gamma(1-\mu)}{2^{2\mu-2}\Gamma(3-2\mu)} {}_2F_1(\mu, 2\mu-2; \mu; \frac{1+x_i}{2}) - \frac{\Gamma(\mu-1)}{2^{\mu-1}(1+x_i)^{\mu-1}} \\
&{}_2F_1(1, \mu-1; 2-\mu; \frac{1+x_i}{2})\Bigg) d_2 + \sum_{j=0}^{p} c_j \left[\left(\frac{1}{2}\right)^{1-\mu} \frac{j!}{(1-\mu)_{j+1}} (1+x_i)^{1-\mu} P_j^{(\mu-1,1-\mu)}(x_i)\right. \\
&\left.+\left(\frac{1}{2}\right)^{1-\mu} \frac{j!}{(1-\mu)_{j+1}} (1-x_i)^{1-\mu} P_j^{(1-\mu,\mu-1)}(x_i)\right] = \lambda_p \Bigg(\sum_{j=0}^{p} c_j L_j(x_i) + d_1 (\frac{1+x_i}{2})^{1-\mu} \\
&+ d_2 (\frac{1-x_i}{2})^{1-\mu}\Bigg), \quad i = 0, \cdots, p+2.
\end{aligned} \tag{4.23}$$

Note that the first hypergeometric function is not well-defined when $\mu = 1/2$. However, the integration of the two singular terms with the kernel are simpler, in which case, the linear system is

$$\Big(\frac{\pi(1+x_i)}{4}+\sqrt{\frac{1-x_i}{2}}+\frac{1+x_i}{2}\log(1+\sqrt{\frac{1-x_i}{2}})-\frac{1+x_i}{4}\log(\frac{1+x_i}{2})\Big)d_1$$
$$+\Big(\sqrt{\frac{1+x_i}{2}}-\frac{x_i-1}{2}\tanh^{-1}(\sqrt{\frac{1+x_i}{2}})-\frac{\pi(x_i-1)}{4}\Big)d_2+\sum_{j=0}^{p}c_j$$
$$\Big[\Big(\frac{1}{2}\Big)^{1-\mu}\frac{j!}{(1-\mu)_{j+1}}(1+x_i)^{1-\mu}P_j^{(\mu-1,1-\mu)}(x_i)+\Big(\frac{1}{2}\Big)^{1-\mu}\frac{j!}{(1-\mu)_{j+1}}(1-x_i)^{1-\mu}$$
$$P_j^{(1-\mu,\mu-1)}(x_i)\Big]=\lambda_p(d_1\sqrt{\frac{1+x_i}{2}}+d_2\sqrt{\frac{1-x_i}{2}}+\sum_{j=0}^{p}c_jL_j(x_i)),\quad i=0,\cdots,p+2. \tag{4.24}$$

4.2.3 Algorithm for the Piecewise Smooth Kernel

We make change of variables as before $t=\frac{1+x}{2}$, $\quad s=\frac{1+\tau}{2}$ and let $u(x)=y\big(\frac{1+x}{2}\big)$, we have

$$\lambda u(x)=\int_{-1}^{1}\frac{1+\tau}{8}u(\tau)d\tau+\frac{1}{4}\int_{x}^{1}(x-\tau)u(\tau)d\tau. \tag{4.25}$$

Let $u_p(x)=\sum_{j=0}^{p}c_jL_j(x)$ be the approximation of $u(x)$. Then c_j's satisfy

$$\lambda_p\sum_{j=0}^{p}c_jL_j(x_i)=\sum_{j=0}^{p}c_j\int_{-1}^{1}\frac{1+\tau}{8}L_j(\tau)d\tau+\sum_{j=0}^{p}c_j\int_{x_i}^{1}(x_i-\tau)L_j(\tau)d\tau$$
$$i=0,\cdots,p. \tag{4.26}$$

i.e.

$$\lambda_p\sum_{j=0}^{p}c_jL_j(x_i)=(\frac{c_0}{4}+\frac{c_1}{12})+\frac{1-x_i}{8}\sum_{j=0}^{p}c_j\sum_{k=0}^{p}w_k(x_i-x_k)L_j(\frac{1+x_i}{2}+\frac{1-x_i}{2}x_k)$$
$$i=0,\cdots,p. \tag{4.27}$$

Here, the numerical integration is exact. The scheme is of the form

$$AC_p=\lambda_pBC_p,$$

where

$$
\begin{aligned}
b_{ij} &= L_j(x_i) \\
a_{ij} &= \begin{cases}
\frac{1-x_i}{8}\sum\limits_{k=0}^{p} w_k(x_i - x_k)L_j(\frac{1+x_i}{2} + \frac{1-x_i}{2}x_k), & \text{if} \quad j \neq 0, 1; \\
\frac{1-x_i}{8}\sum\limits_{k=0}^{p} w_k(x_i - x_k)L_j(\frac{1+x_i}{2} + \frac{1-x_i}{2}x_k) + \frac{1}{4}, & \text{if} \quad j = 0; \\
\frac{1-x_i}{8}\sum\limits_{k=0}^{p} w_k(x_i - x_k)L_j(\frac{1+x_i}{2} + \frac{1-x_i}{2}x_k) + \frac{1}{12}, & \text{if} \quad j = 1.
\end{cases}
\end{aligned}
$$

4.2.4 Algorithm for the Smooth Kernel

Substitute the Legendre expansion $y(t) = \sum\limits_{j=0}^{p} y_j L_j(t)$ into (4.15) and collocating at n Gaussian points, we have

$$\sum_{j=0}^{p} y_j \int_0^1 e^{st_i} L_j(s)ds = \lambda \sum_{j=0}^{p} y_j L_j(t_i), \quad i = 1, 2, \cdots, n. \tag{4.28}$$

The matrix form of (4.28) is

$$K\mathbf{y} = \lambda L\mathbf{y}, \tag{4.29}$$

where

$$K_{ij} = \int_0^1 e^{st_i} L_j(s)ds, \; L_{ij} = L_j(t_i), \; \mathbf{y} = (y_1, y_2, \cdots, y_n)^T. \tag{4.30}$$

K_{ij} can be calculated by the n-point Gaussian quadrature

$$\int_0^1 e^{st_i} L_j(S)ds \approx \sum_{l=0}^{p} e^{s_l t_i} L_j(s_l) w_l, \quad s_k = t_k. \tag{4.31}$$

4.3 Convergence Analysis

Let L_k be the standard Legendre polynomial of degree k and $\pi_p f \in P_p[-1, 1]$ interpolate a smooth function f at $p+1$ Gauss points: $-1 < x_0 < \cdots < x_p < 1$. Let

T_k be the first kind Chebyshev polynomial of degree k. Then the remainder of the interpolation is

$$f(x) - \pi_p f(x) = f[x_0, x_1, \cdots, x_p, x]\nu(x), \tag{4.32}$$

where $\nu(x) = (x - x_0)(x - x_1)\cdots(x - x_p)$.

Note that

$$\begin{aligned} L_n(x) &= \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n \\ &= \frac{1}{2^n n!}\frac{d^n}{dx^n}\sum_{j=0}^{n}\binom{n}{j}x^{2(n-j)}(-1)^j \\ &= \frac{1}{2^n n!}\sum_{j=0}^{n}\binom{n}{j}(2n-2j)(2n-2j-1)\cdots(2n-2j-n+1)x^{n-2j}(-1)^j \end{aligned} \tag{4.33}$$

From the term with $j = 0$ we get the leading coefficient

$$\frac{1}{2^n n!}\binom{n}{0}(2n)(2n-1)\cdots(2n-n+1)(-1)^0 = \frac{(2n)!}{2^n(n!)^2} \tag{4.34}$$

By the Stirling formula,

$$\frac{(2n)!}{2^n(n!)^2} \approx \frac{(\frac{2n}{e})^{2n}\sqrt{4\pi n}}{2^n[(\frac{n}{e})^n\sqrt{2\pi n}]^2} = 2^n. \tag{4.35}$$

Hence,

$$f(x) - \pi_p f(x) \approx \frac{f[x_0, x_1, \cdots, x_p, x]}{2^{p+1}}L_{p+1}(x), \tag{4.36}$$

If $f \in C^{p+1}[-1, 1]$, the divided difference

$$f[x_0, x_1, \cdots, x_p, x] = \frac{f^{(p+1)}(\xi_x)}{(p+1)!}, \quad \xi_x \in (-1, 1). \tag{4.37}$$

The result can be concluded as the following theory.

Theorem 4.3.1 (1) If $y(t)$ satisfies condition (R): $\|y^{(k)}\|_{L^\infty[0,1]} \leq Ck!R^{-k}$, then

$$\|y - \pi_p y\|_{L^\infty[0,1]} \leq \frac{C}{(4R)^{p+1}}; \tag{4.38}$$

(2) If $y(t)$ satisfies condition (M): $\|y^{(k)}\|_{L^\infty[0,1]} \leq CM^k$, we have

$$\|y - \pi_p y\|_{L^\infty[0,1]} \leq \frac{C}{\sqrt{p+1}} \left(\frac{eM}{4(p+1)} \right)^{p+1}. \tag{4.39}$$

Proof: We make change of variables

$$t = \frac{1+x}{2}, \quad s = \frac{1+\tau}{2}, \qquad x, \tau \in [-1, 1]$$

and let $u(x) = y\left(\frac{1+x}{2}\right)$, then the result for y under condition (R) follows directly from (4.36), (4.37) and the fact that $dt = \frac{1}{2}dx$.

If y satisfies condition (M), by applying the Stirling's formula,

$$\begin{aligned} \|y - \pi_p y\|_{L^\infty[0,1]} &= \|u - \pi_p u\|_{L^\infty[-1,1]} \\ &\leq \frac{CM^{p+1}}{4^{p+1}(p+1)!} \\ &\approx \frac{CM^{p+1}}{\sqrt{2\pi(p+1)}(\frac{4(p+1)}{e})^{p+1}} \\ &= \frac{C}{\sqrt{p+1}} \left(\frac{eM}{4(p+1)} \right)^{p+1}. \end{aligned} \tag{4.40}$$

For non-smooth functions, we need some other estimates.

Theorem 4.3.2 [9] (i) For any $f \in H^k(-1,1)$,

$$\|f - \pi_p f\|_{L^2(-1,1)} \leq Cp^{-k} |f|_{H^{k,p}(-1,1)}. \tag{4.41}$$

(ii) For any $f \in H^k_w(-1,1)$,

$$\|f - \pi^c_p f\|_{L^2(-1,1)} \leq Cp^{-k} |f|_{H^{k,p}_w(-1,1)}, \tag{4.42}$$

where two seminorms are defined by

$$|f|_{H^{k,p}(-1,1)} = \left(\sum_{s=\min(k,p+1)}^{k} \|f^{(s)}\|^2_{L^2(-1,1)} \right)^{1/2},$$

$$|f|_{H_w^{k,p}(-1,1)} = \left(\sum_{s=\min(k,p+1)}^{k} \|f^{(s)}\|^2_{L^2_w(-1,1)} \right)^{1/2},$$

and the weight $w(x) = (1-x)^{-1/2}(1+x)^{-1/2}$ and π_p^c is the interpolatory operator on Chebyshev points.

If we write $(\pi_p x)(t) = \sum_{j=0}^{p} \xi_j L_j(t)$, which is an interpolatory projection from $R(E)$ to $R(E_p)$ and ξ_j is determined by

$$\sum_{j=0}^{p} \xi_j L_j(t_i) = x(t_i), \quad i = 0, \cdots, p,$$

then, our algorithms can be written as

$$T_p u_p = \lambda_p u_p, \quad \text{where} \quad T_p = \pi_p T. \tag{4.43}$$

Theorem 4.3.3 Let y be the exact first eigenvector and T be a compact operator in this chapter and T_p be defined as above, then

(1) If $y \in H^k(0,1)$,

$$|\lambda - \hat{\lambda}_p| \leq \frac{C}{(2p)^k}; \tag{4.44}$$

(2) Furthermore, if y satisfies condition (R),

$$|\lambda - \hat{\lambda}_p| \leq \frac{C}{(4R)^{p+1}}; \tag{4.45}$$

(3) Furthermore, if y satisfies condition (M),

$$|\lambda - \hat{\lambda}_p| \leq \frac{C}{\sqrt{p+1}} \left(\frac{eM}{4(p+1)} \right)^{p+1}. \tag{4.46}$$

Proof: The result follows directly from Theorem 4.3.1 ,4.3.2, and Theorem 2.2 of [24]. To make the paper self-contained, we put the proof here.

Let $\hat{E}_p = E_p|_{R(E)} : R(E) \to R(E_p)$. Then for large p, $\hat{E}_p$ is bijective and $\|\hat{E}_p^{-1}\| \le 2$ [32]. Define $\hat{T} = T_{R(E)}$, and $\hat{T}_p := \hat{E}_p^{-1} T_p \hat{E}_p$. Then

$$\begin{aligned} |\lambda - \hat{\lambda}_p| &= \frac{1}{m}|\text{trace}(\hat{T} - \hat{T}_p)| \le \|\hat{T} - \hat{T}_p\| \\ &= \|\hat{E}_n^{-1}(\hat{E}_p T - \hat{E} T_p)\| \\ &\le C\|(T - T_p)|_{R(E)}\| \\ &= C\|(I - \pi_p)T|_{R(E)}\| \end{aligned} \tag{4.47}$$

Since Tu is smoother than u, see [13], then the result follows.

Theorem 4.3.4 Let λ and λ_p be the exact eigenvalue and its numerical approximation of a positive definite operator T whose kernel is a piecewise smooth function, respectively. Then

(1) If u satisfies condition (K),

$$|\lambda - \lambda_p| \le C\left(\frac{1}{(4R)^{2p+2}} + \frac{e^{2p}}{p^{2p-3/2}2^{6p}}\right), \tag{4.48}$$

(2) If u satisfies condition (M),

$$|\lambda - \lambda_p| \le C\left(\frac{1}{p+1}(\frac{eM}{4(p+1)})^{2p+2} + \frac{e^{2p}}{p^{2p-3/2}2^{6p}}\right), \tag{4.49}$$

(3) If $u \in H^k[0,1]$,

$$|\lambda - \lambda_p| \le C\left(\frac{1}{(2p)^{2k}} + \frac{e^{2p}}{p^{2p-3/2}2^{6p}}\right), \tag{4.50}$$

Proof: By our algorithms, we have

$$\int_0^1 k(t_i, s)u_p(s)ds = \lambda_p u_p(t_i), \tag{4.51}$$

where t_i are $(p+1)$-Gauss points on [0,1].

Multiplying both sides by $L_j(t_i)w_i$ and summing up from 0 to p, we obtain

$$\sum_{j=0}^{p}\int_0^1 k(t_i,s)u_p(s)L_j(t_i)w_i ds = \lambda_p \sum_{j=0}^{p} u_p(t_i)L_j(t_i)w_i. \tag{4.52}$$

Here, w_i are weights of the Gauss quadrature.

If we write $\tilde{A} = (\int_0^1\int_0^1 k(t,s)L_j(s)L_i(t)dsdt)_{ij}$, $\tilde{B} = (\int_0^1 L_j(t)L_i(t)dt)_{ij}$ and recall that $u_p(x) = \sum_{i=0}^{p} \tilde{u}_i L_i(x)$, we obtain

$$\tilde{A}\tilde{u} = \tilde{\lambda}_p \tilde{B}\tilde{u},$$

where $\tilde{u} = [\tilde{u}_0, \tilde{u}_2, \cdots, \tilde{u}_p]^T$.

However, for most cases, we can only apply numerical quadrature to find elements of $\tilde{A}$ and $\tilde{B}$. If the kernel is piecewise smooth, we apply the Gauss quadrature piece by piece. Therefore, the system that we actually solve is

$$Au = \lambda_p Bu. \tag{4.53}$$

Now we are ready to analyze errors of eigenvalue approximations.

First, we analyze the case when the kernel is a linear piecewise polynomial. Noting the fact that $(p+1)$-Gauss quadrature is exact for all polynomials of degree less than or equal $2p+1$, we have

$$A = \tilde{A}, \quad \text{and} \quad B = \tilde{B}. \tag{4.54}$$

Here, the integration is piecewise, so is the numerical integration.

Denoting the arithmetic mean of the approximation of λ by λ_p again, if it is a multiple eigenvalue. We derive from Lemma 4.1.2 that

$$\begin{aligned}|\lambda-\lambda_p| &= |\lambda-\tilde{\lambda}_p| \\ &\le C\begin{cases}\frac{1}{4^{2p+2}}, & \text{if u satisfies condition (K);} \\ \frac{1}{p+1}(\frac{eM}{4(p+1)})^{2p+2}, & \text{if u satisfies condition (M);} \\ \frac{1}{(2p)^{2k}}, & \text{if } u\in H^k[0,1].\end{cases}\end{aligned} \tag{4.55}$$

If the kernel is piecewise smooth or smooth, from the analysis of previous case, we only need to estimate $A-\tilde{A}$ since $B=\tilde{B}$. If we write the remainder of Gaussian quadrature as ϵ, then

$$A-\tilde{A}\le C\epsilon\begin{bmatrix}1 & 1 & \cdots & 1\\ 1 & 1 & \cdots & 1\\ \vdots & \vdots & \ddots & \vdots\\ 1 & 1 & \cdots & 1\\ 1 & 1 & \cdots & 1\end{bmatrix}. \tag{4.56}$$

Here, we define $E<F$ if and only if $|(E)_{ij}|<|(F)_{ij}|$.

By the error estimate of the Gauss quadrature and the Stirling' formula, we have

$$\epsilon\le C\left(\frac{[p!]^4}{(2p+1)[(2p)!]^3}\right)\approx C\left(\frac{[\sqrt{2\pi p}(\frac{p}{e})^p]^4}{(2p+1)[\sqrt{2\pi(2p)}(\frac{2p}{e})^{2p}]^3}\right)\le C\left(\frac{e^{2p}}{p^{2p+1/2}2^{6p}}\right). \tag{4.57}$$

Hence,

$$\|A-\tilde{A}\|_n\le C\left(\frac{pe^{2p}}{p^{2p+1/2}2^{6p}}\right),\ n=1,\infty. \tag{4.58}$$

Clearly, (4.53) is equivalent to

$$B^{-1}Au=\lambda_p u. \tag{4.59}$$

Thus,

$$\|\tilde{B}^{-1}\tilde{A} - B^{-1}A\|_n \le \|B^{-1}\|_n\|\tilde{A} - A\|_n \le C\left(\frac{p^2e^{2p}}{p^{2p+1/2}2^{6p}}\right), n = 1, \infty, \tag{4.60}$$

by noting that $B = \tilde{B} = \mathrm{diag}(1, 1/3, \cdots, 1/(2p+1))$. Therefore, by a perturbation theory, see [15](Page 30), we have

$$|\lambda_p - \tilde{\lambda}_p| \le C\left(\frac{e^{2p}}{p^{2p-3/2}2^{6p}}\right). \tag{4.61}$$

Denoting the arithmetic mean of the approximation of λ by λ_p again, if it is a multiple eigenvalue. We derive from Lemma 4.1.2 and (4.61) that

$$\begin{aligned} |\lambda - \lambda_p| &\le |\lambda - \tilde{\lambda}_p| + |\tilde{\lambda}_p - \lambda_p| \\ &\le C \begin{cases} \left(\frac{1}{4^{2p+2}} + \frac{e^{2p}}{p^{2p-1/2}2^{6p}}\right), & \text{if u satisfies condition (K);} \\ \left(\frac{1}{p+1}\left(\frac{eM}{4(p+1)}\right)^{2p+2} + \frac{e^{2p}}{p^{2p-1/2}2^{6p}}\right), & \text{if u satisfies condition (M);} \\ \left(\frac{1}{(2p)^{2k}} + \frac{e^{2p}}{p^{2p-1/2}2^{6p}}\right), & \text{if } u \in H^k[0,1]. \end{cases} \end{aligned}$$

Remark Theorem 4.3.4 shows that though numerical integration contributes the error of eigenvalue approximation, it is trivial compared with truncation error for our method. Hence, in our numerical experiments, we ignore it for reference curves.

4.4 Numerical Results

In this section, we will find numerical approximation to solutions of some examples to demonstrate our theory.

Example 4.4.1[1] We consider a problem with exactly the form (4.14).Then each $\lambda_j = \frac{1}{(2j-1)^2\pi^2}$, $j = 1, 2, \cdots$ is an eigenvalue of T of algebraic multiplicity $m = 2$. Let

Table 9: Example 4.4.1: $\lambda - \lambda_p$

p	3	4	5	6
error	5.2298e-06	1.1826e-07	1.8490e-09	2.1241e-11
p	7	8	9	10
error	1.8759e-13	1.3045e-15		

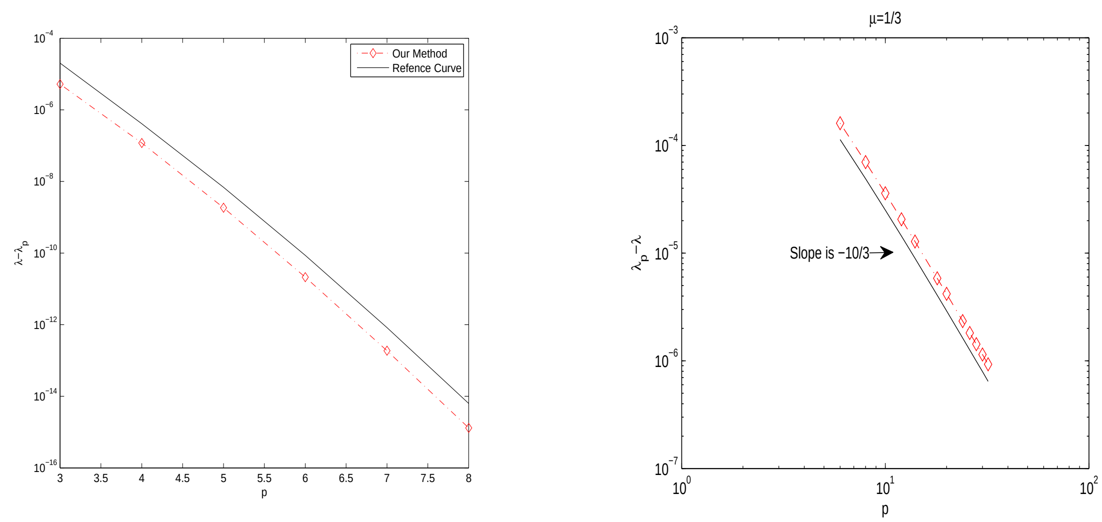


Figure 7: Linear piecewise kernel $|t-s|^{1/3}$ (The first algorithm)

$\hat{\lambda}$ denote the arithmetic mean of the two eigenvalue of T_p to the largest eigenvalue $\lambda = 1/\pi^2$. Numerical errors are presented in Table 9 and left part of Figure 7, from which, we see the error decays super-geometrically. Here Reference Curve is the graph of $f(p) = \frac{1}{100(p+1)}(\frac{e\pi}{4(p+1)})^{2p+2} + \frac{e^{2p}}{p^{2p-1/2}2^{6p}}$.

Example 4.4.2 Now let us consider an eigen-problem of the form (4.13) with $\mu = 1/3$. We use both algorithms to solve the problem. From [38], eigenvectors belong to $H^{\frac{7}{6}-\epsilon}$, where ϵ is a sufficiently small positive number and we expect to obtain

Table 10: Example 4.4.2: λ_p (The first algorithm)

p	10	20	30	40
	1.805741190980	1.805772959954	1.805776000579	1.805776693513
p	50	55	60	65
	1.805776926254	1.805776984244	1.805777024037	1.805777051585

Table 11: Example 4.4.2: λ_p (The second algorithm)

p	10	20	30	40
	1.805777162409	1.805777143959	1.80577714315	1.805777143861
p	50	55	60	65
	1.805777143840	1.805777143837	1.805777143834	1.805777143833

a convergence rate of $\mathcal{O}(p^{-7/3})$ based on Theorem 4.3.4 for the first algorithm. Unfortunately, we do not know the exact eigenvalues for such type of kernels. However, we list some of our numerical approximations in Table 6 and we use the numerical approximation of the second algorithm for $p = 70$ as our "exact" value to obtain right graph of figure 7. It is easy to see that we can only obtain a seven-digit of accuracy for this algorithm. For the second algorithm, we have Table 10 and the left part of Figure 8. We obtain an eleven-digit of accuracy and a convergence rate of $\mathcal{O}(p^{-14/3})$ for this algorithm.

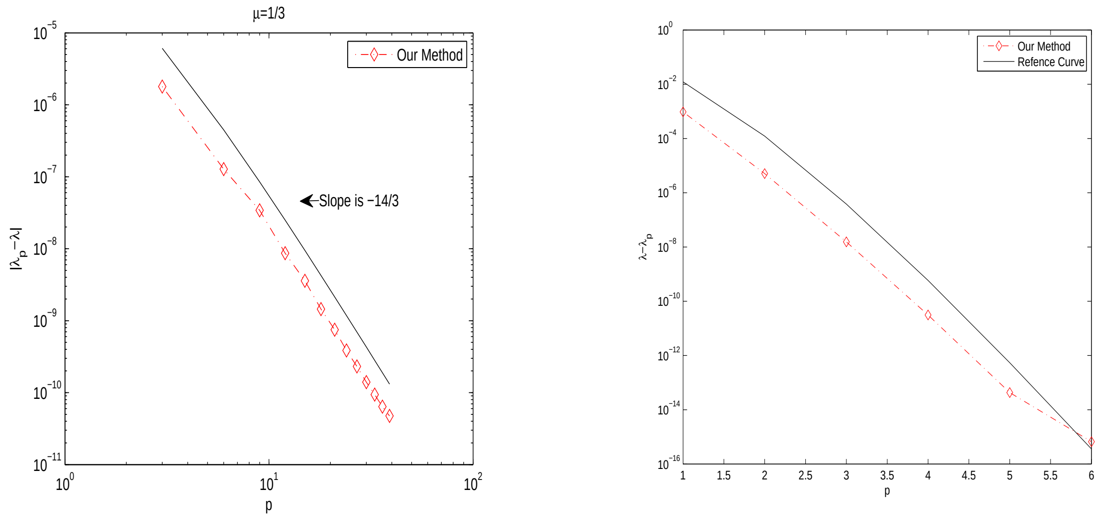


Figure 8. Kernel: $|t-s|^{1/3}$ (The second algorithm) Kernel: e^{st}

Table 12: Example 4.4.3: λ_p (The first algorithm)

p	10	20	30	40
	2.682832453413	2.682906259562	2.682914644141	2.682916773709
p	50	55	60	65
	2.682917548869	2.682917752298	2.682917896142	2.682917998252

Example 4.4.3 We consider an eigen-problem of the form (4.13) with $\mu = 1/2$. In this case, eigenvectors belongs to H^1. Again, we use both algorithms to solve it and consider the numerical approximation of the second algorithm for $p = 70$ as the "exact" first eigenvalue. Numerical results are shown in Table 12 ,Table 13 and figure 9.

Example 4.4.4 Consider the eigenvalue problem of the form (4.16). We apply the algorithm in section 4.2. Since the kernel is smooth, the first eigenvalue converge

Table 13: Example 4.4.3: λ_p (The second algorithm)

p	10	20	30	40
	2.682918574502	2.682918399599	2.682918386165	2.682918383536
p	50	55	60	65
	2.682918382751	2.682918382579	2.682918382452	2.682918382377

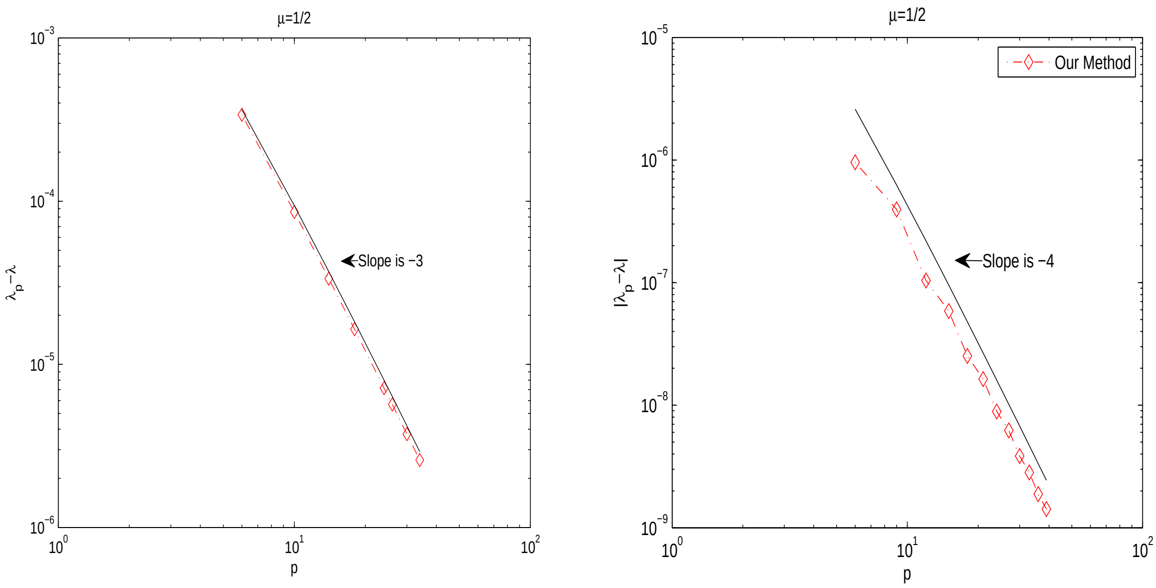


Figure 9. Kernel: $|t-s|^{1/2}$ (1st) Kernel: $|t-s|^{1/2}$ (2nd)

Table 14: Example 4.4.4: $\lambda - \lambda_p$

p	1	2	3	4
error	9.4969e-04	5.0595e-06	1.5456e-08	3.1190e-11
p	5	6		
error	4.3077e-14	6.6613e-16		

very fast, see Table 14 and right part of Figure 9. In this case, Reference Curve is the graph of $f(p) = \frac{1}{10(p+1)}(\frac{e}{2(p+1)})^{2p+2} + \frac{1}{10}\frac{e^{2p}}{p^{2p-1/2}2^{6p}}$.

5 Concluding Remarks

In response to the increasing needs of numerical approximation of integral/integro-differential equations with compact operators, especially with weakly singular kernels, this work is devoted to spectral collocation method for these equations and its eigenvalue problems.

By applying exact integration of the composition of the Legendre polynomials and the weakly singular kernel, we solve the Volterra/Fredholm integral equation. This method leads to less error and less computation cost. We also prove that a geometric (super-geometric) rate of convergence can be achieved by using our method if the true solution $y(t)$ satisfies Condition (R):

$$\|y^{(k)}\|_{L^\infty[0,T]} \leq ck!R^{-k}$$

or Condition (M):

$$\|y^{(k)}\|_{L^\infty[0,T]} \leq cM^k, M > 1$$

not only for Volterra equations but also for Fredholm equations as well as their corresponding integro-differential equations.

As for eigenvalue problems, we prove that the convergence rate of eigenvalue depends on the smoothness of the corresponding eigenfunction. If the eigenfunction satisfies condition (R) or condition (M), the convergence rate for eigenvalue is geometric or supergeometric. Furthermore, if the kernel is positive definite, the convergence rate is doubled. By applying known knowledge, we can obtain a very accurate approximation of the first eigenvalue for the weakly singular kernel of the form $1/|t-s|^{\mu},\ 0<\mu<1.$

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ABSTRACT

SPECTRAL COLLOCATION METHOD FOR COMPACT INTEGRAL OPERATORS

by

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December 2011

Advisor: Dr. Zhimin Zhang

Major: Mathematics

Degree: Doctor of Philosophy

We propose and analyze a spectral collocation method for integral equations with compact kernels, e.g. piecewise smooth kernels and weakly singular kernels of the form $\frac{1}{|t-s|^{\mu}}$, $0 < \mu < 1$. We prove that 1) for integral equations, the convergence rate depends on the smoothness of true solutions $y(t)$. If $y(t)$ satisfies condition (R): $\|y^{(k)}\|_{L^{\infty}[0,T]} \leq ck!R^{-k}$, we obtain a geometric rate of convergence; if $y(t)$ satisfies condition (M): $\|y^{(k)}\|_{L^{\infty}[0,T]} \leq cM^{k}$, we obtain supergeometric rate of convergence for both Volterra equations and Fredholm equations and related integro differential equations; 2) for eigenvalue problems, the convergence rate depends on the smoothness of eigenfunctions. The same convergence rate for the largest modulus eigenvalue approximation can be obtained. Moreover, the convergence rate doubles for positive compact operators. Our numerical experiments confirm our theoretical results.

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2. *Can Huang and Zhimin Zhang,* Stability of Stochastic Delay Differential Equations Driven by Lévy Noise with Markovian switch*, Submitted to Stochastic Analysis and Applications*
3. *Can Huang, Tang Tao and Zhimin Zhang,* Geometric/Supergeometric Convergence of Spectral Collocation Methods for Weakly Singular Volterra/Fredholm Integral Equations with Smooth Solutions*, Under Revision*
4. *Can Huang and Zhimin Zhang,* A Spectral Collocation Method for Eigenvalue Problems of Compact Integral Operators*, Preprint*
5. *Can Huang, Jing Shi, and Zhimin Zhang,* Two-Stage Runge-Kutta Methods for Stochastic Differential Equations with Jump Diffusion*, Under revision*