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Genus 0, 1, 2 actions of some almost simple groups of lie rank 2

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GENUS 0, 1, 2 ACTIONS OF SOME ALMOST SIMPLE GROUPS OF LIE RANK 2

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DEDICATION

To my Father and Mother

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CHAPTER 1 Introduction

The concept of *genus* of a transitive group comes from an idea in complex analysis. Let ϕ be a nonconstant meromorphic function and $\phi : X \to P_z^1$ be a cover of the compact connected Riemann surface $P_z^1 = C \cup \infty$ with r branch points $z = (p_1, p_2, ..., p_r)$. Denote the degree of the cover by N. Choose an ordered r-tuple $(s_1, ..., s_r)$ of *classical generators* for $\pi_1(U_z, z_0)$ where U_z is the r-punctured sphere $P_z^1 \setminus z$. Then ϕ and an ordering of the points of X over z_0 determines the image of the entries of $(s_1, ..., s_r)$ in the monodromy group G of the cover: Each s_i maps to some $x_i \in G$. X is connected implies that G is transitive. Suppose that $g = g(X)$ is the genus of X. Then g, N, r are related by:

1. Transitive group G is generated by the r nonidentity elements $x_1, ..., x_r$: $G = \langle x_1, x_2, ..., x_r \rangle$;

2. The product of the r elements is the identity: $x_1 \cdots x_r = 1$;

3. $x_1, ..., x_r \in S_N$ satisfy Riemann-Hurwitz formula

$$2(N + g - 1) = \sum_{i=1}^{r} Ind(x_i)$$

   where $Ind(x_i)$ is the *permutation index* of x_i. The *permutation index* of a permutation is defined as the difference of degree and the number of disjoint cycles of the permutation.

Riemann existence theorem states the converse of the above theorem. If G is a transitive group of degree N with a generating system $\{x_1, x_2, ...x_r\}$ that satisfies 1, 2 and 3 then there is a compact and connected Riemann surface X of genus g, a non-constant meromorphic function ϕ to the Riemann sphere P_z^1 and $r-$element subset $\{p_1, p_2, ..., p_r\}$ of P_z^1 such that

$\{p_1, p_2, ..., p_r\}$ is the set of branch points of this cover and in addition G and the monodromy group $M(X, \phi)$ are conjugate in S_N.

Arising from the above topology idea we define some algebraic definitions. Suppose that G is a transitive group of degree N. A tuple of elements $(x_1, ..., x_r)$ satisfying condition 1, 2 and 3 is called a *genus g generating tuple* of G. If G has such a tuple we say that G is a *genus g group* and has a *genus g action*. We say that the generating tuple $(x_1, ..., x_r)$ is *of type* $(C_1, ..., C_r)$ where C_i is the conjugacy class of x_i in G.

A major contribution to the study of the finite group and its genus is paper [GT]. In their paper the authors gave three theorems that outline the structure of genus 0 primitive groups of affine type based on the classification of primitive groups given by Aschbacher. About how the condition of genus of a group influences its structure in general, they made the following Guralnick-Thompson conjecture:

Conjecture 1 (Guralnick-Thompson)**.** *Fix $g \geqslant 0$. There are only finitely many nonabelian simple groups except alternating groups that can occur as a composition factor of the monodromy group G of genus g.*

The proof of this conjecture was completed by Frohardt and Magaard in [FMAnn]. Later in [FGM2002] Frohardt , Guralnick and Magaard classified all genus 0 actions for groups with component of Lie rank one (a *component* of a finite group is a quasisimple subnormal subgroup). These types of groups include almost simple groups of Lie rank one. A group G is said to be *almost simple* if it contains some non-abelian simple group S and is contained within the automorphism group of S $Aut(S)$: $S \leq G \leq Aut(S)$.

A transitive permutation group G (or transitive action of G) on a set Ω with $|\Omega| > 1$ is *primitive* if there is no partition of Ω preserved by G except for the two trivial partitions (the partition with a single part, and the partition into singletons). Glauberman reference in [GT] proved

Lemma 2. *If S is a composition factor of a group of genus zero, then S is a composition factor of some primitive group of genus zero.*

Because of the above lemma we are interested in primitive actions in this paper.

This paper focuses on almost simple groups of type $PSL(3, q)$ acting on the set of projective points of 2-dimensional projective geometry $P\Omega(2, q)$ i.e. the set of 1-dimensional subspaces of 3-dimensional vector space $V(3, q)$. The complete list of genus 0, 1, 2 almost simple groups of type $PSL(3, q)$ and all of their genus 0, 1, 2 generating tuples on this special set is given. This work gets done by both the proof in group theory and calculations of GAP (Groups, Algorithms, Programming) software. Also genus 0, 1, 2 generating tuples for almost simple groups of type $PSL(3, q)$ on their other primitive actions and genus 0, 1, 2 generating tuples for almost simple groups of type $PSp(4, q)$ and $PSU(4, q)$ on their primitive actions for some q are given. The list is put in appendix B.

A feasible way to check if an action is a genus g action is to check if it has genus g generating tuples. It can be very hard to determine whether a set can generate the entire group in group theory. GAP software is a powerful implement to do it for some groups. GAP is a computer algebra system for computational discrete algebra with particular emphasis on computational group theory. So far the library of GAP supplies all primitive actions whose degree are less than or equal to 2499. In this paper we will use GAP to find all genus 0, 1, 2 generating tuples for groups with low degrees.

The constructure of this paper is like the following. Chapter 1 is the introduction. Chapter 2 is the main part of this paper. Section 2.1 introduces preliminaries. In section 2.2 we will prove that there is no genus 0, 1, 2 actions of $PSL(3, q)$ and $PGL(3, q)$ on the set of of projective points of 2-dimensional projective geometry $P\Omega(2, q)$ when $q \geqslant 31$. The proof of this result bases on an observation about the eigne spaces of linear transformations. For any element in $PGL(3, q)$, the fixed projective points in $P\Omega(2, q)$ is a 1-dimensional eigen space of the linear transformations in its preimage in $GL(3, q)$. Analyzing all possible numbers and dimensions of eigen spaces for a linear transformation, we can get that all possible numbers of fixed points of any element in $PGL(3,)$ are $0, 1, 2, 3, q+1$ or $q+2$. We consider elements in $PGL(3,q)$ as two types: one type fixes at most 3 points and other fixes $q+1$ or $q+2$ points. Cauchy-Frobenius formula induces that permutation index of a permutation actually is determined by the degree and the number of fixed points. Combining Cauchy-Frobenius formula with Riemann-Hurwitz formula, we can set up an inequality which helps bound q. Almost simple groups of type $PSL(3,q)$ containing field automorphisms do not have genus 0, 1, 2 actions on the set of projective points of 2-dimensional projective geometry $P\Omega(2,q)$ when $q \geqslant 27$. This is the main result in section 2.3. The key property to prove this result is the following: the number of fixed points of elements in $Aut(PSL(3,q)) \setminus PGL(3,q)$ is no more that the number of fixed points of the pure field automorphism associate with it. Since involutions and elements of order 3 are often discussed in the proof section 2.4 talks about the permutation indices of these elements. Chapter 3 states the result. Appendix A is the GAP code I used to find genus 0, 1, 2 generating tuples for groups. Appendix B is the list of generating tuples of some almost simple groups of Lie Rank 2.

CHAPTER 2 Action of Almost Simple Groups of Type $PSL(3,q)$ on Ω

Section 2.1 Preliminaries

First introduce some notations and definitions.

Suppose that G is a permutation group of degree N and $x \in G$.

$Ind(x)$ is the *permutation index* of x and defined as $N-$ the number of disjoint cycles of x_i.

$Fix(x)$ is the number of fixed points of x. $Fix(x)$ is a class function. Elements that are conjugate have the same amount of fixed points.

$fpr(x)$ is the *fixed point ratio* of x. It is defined as the quotient of $Fix(x)$ and the degree N: $fpr(x) = \frac{Fix(x)}{N}$.

$\underline{x} = (x_1, .., x_r)$ is a tuple of r elements in G. $\underline{d} = (d_1, ..., d_r)$ where d_i is the order of element x_i. Because of braiding actions on the tuple $\underline{x}$, we can assume that $d_i \leqslant d_j$ if $i \leqslant j$ for $\underline{d}$.

The important quantity

$$A(\underline{d}) = \sum_{i=1}^{r} \frac{d_i - 1}{d_i}.$$

Suppose that $\underline{d}$ and $\underline{d'}$ have the same length and $d_1 \leqslant d'_1, ..., d_i \leqslant d'_i$ then $A(\underline{d}) \leqslant A(\underline{d'})$. A theorem in [H] showed that when $A(\underline{d}) < A((2,3,7))$ the only non-solvable group that can be generated by $\underline{x}$ is Alt_5. In particular tuple $\underline{x}$ can not generate an almost simple group of type $PSL(3,q)$. So $\underline{d} \neq (2,2,d)$ for all $d \geqslant 2$, $(2,3,3), (2,3,4), (2,3,5), (2,3,6), (3,3,3), (2,2,2,2)$ and

$$A(\underline{d}) \geqslant \frac{85}{42}$$

in this paper.

$v(\tilde{x})$ denotes the codimension of the largest eigen space of a linear transformation $\tilde{x} \in GL(n,q)$.

Ω denotes the set of the projective points of projective geometry $PG(n-1,q)$ i.e. the set of 1-dimensional subspaces of vector space over finite field $GF(q)$. It will not cause ambiguity that $n = 3$ or n represents the dimension in general in this paper. $|\Omega| = \frac{q^n-1}{q-1}$. When $n = 3$, $|\Omega| = q^2 + q + 1$.

e_d is an element of order d in this paper.

Carter describes the structure of the automorphism group of a group of Lie type in [C].

Lemma 3. *Let $G = \mathcal{L}(K)$ when $\mathcal{L}$ is simple and $K = GF(q)$. Let θ be an automorphism* *of G. Then there exist inner, diagonal, graph and field automorphism i, d, g, f such that $\theta = idgf$.*

Graph automorphisms do not act on Ω. The biggest almost simple group of type $PSL(n,q)$ that acts on Ω is called *projective semilinear group* and denoted as $P\Gamma L(n,q)$. It is the extension of $PGL(n,q)$ by field automorphisms. I will introduce the action of $P\Gamma L(n,q)$ on Ω in details later.

This paper considers the action of groups G, with $PSL(n,q) \leq G \leq P\Gamma L(n,q)$, on Ω for $n = 3$.

In [FGM2010] the authors prove that :

Lemma 4. *If (G, Ω) is a primitive classical point action of degree at least 10^4, then the action has genus larger than 2.*

Section 2.2 Action of $PSL(3,q)$ and $PGL(3,q)$ on the Set of Projective Points

In this section we will prove that there are no genus 0, 1, 2 generating tuples for $PSL(3,q)$ and $PGL(3,q)$ when $q \geqslant 31$.

In fact $PSL(n,q)$ is isomorphic to a subgroup of $PGL(n,q)$. Here we identify the subgroup with $PSL(n,q)$.

$PSL(n,q)$ extending by diagonal automorphisms get $PGL(n,q)$. When $n = 3$, $|PGL(3,q) : PSL(3,q)| = 1$ or 3. All almost simple groups of type $PSL(3,q)$ that are contained in $PGL(3,q)$ are $PSL(3,q)$ and $PGL(3,q)$.

In general the general linear group $GL(n,q)$ acts on Ω. But this action is not faithful when $(n, q-1) > 1$. The kernel of this action is the center of $GL(n,q)$ Z i.e. the set of all scalars. The *projective general linear group* is defined as the quotient of $GL(n,q)$ and Z. $PGL(n,q) = GL(n,q)/Z$. $PGL(n,q)$ acts primitively on Ω. The *projective special* *linear group* $PSL(n,q)$ is defined as the quotient of $SL(n,q)$ and the center of $SL(n,q)$. $PSL(n,q) = SL(n,q) \diagup Z \cap SL(n,q)$. It also acts primitively on Ω.

Let $x \in PGL(n,q)$ and $\langle v \rangle \in \Omega$ be a fixed point of x. Suppose that $\tilde{x}$ is an element in the preimage of x in $GL(n,q)$. $\tilde{x}$ also fixes $\langle v \rangle$. In other words $\langle v \rangle$ is a $1-$dimensional eigenspace of $\tilde{x}$. The fixed points of $x \in PGL(n,q)$ are the $1-$spaces spanned by eigenvectors of of $\tilde{x} \in GL(n,q)$.

We classify elements in $PGL(3,q)$ by their fixed points. By the above discussion we can begin with the eigen spaces of the elements in the preimage.

For any nonidentity element here are all of the situations that can happen:

$v(\tilde{x})$	Type of Eigen Spaces of $\tilde{x} \in GL(3,q)$	Number of Fixed Points of $x \in PGL(3,q)$
3	no eigenspace	0
2	one 1-dimensional eigenspace	1
1	one 2-dimensional eigenspace	$q+1$
2	two 1-dimensional eigenspaces	2
1	one 1-dimensional and one 2-dimensional eigenspaces	$q+2$
2	three 1-dimensional eigenspaces	3

The Cauchy-Frobenius formula gives a connection between $Ind(x)$ and $Fix(x)$. A good reference about this lemma is [N].

Lemma 5. *(Cauchy-Frobenius formula) If x is a permutation of order d on a set of order N, then $Ind(x) = N - \frac{1}{d}\sum_{y \in \langle x \rangle} Fix(y)$ where $\langle x \rangle$ is the group generated by x .*

We take out the identity in the group $\langle x\rangle$ in this formula.

$$Ind(x) = N - \frac{N}{d} - \frac{1}{d}\sum_{y\in\langle x\rangle^{\#}} Fix(y)$$

where $\langle x\rangle^{\#}$ is the set of all elements in the group generated by x except the identity. Divide both sides by N, we get

$$\frac{Ind(x)}{N} = \frac{d-1}{d} - \frac{1}{d}\sum_{y\in\langle x\rangle^{\#}} fpr(y)$$

Thus knowledge about the fixed point ratio of element gives us valuable information about its index. Let $x \in PGL(3,q)$ and $\tilde{x}$ be an element in the preimage of x in $GL(3,q)$. There are two types of x : if $v(\tilde{x}) = 1$, x fixes $q+2$ or $q+1$ points; and if $v(\tilde{x}) > 1$, x fixes at most 3 points. Take advantage of this view, we will develop a method to show that there are no low genus generating tuples for $PGL(3,q)$ and $PSL(3,q)$ when $q \geqslant 31$.

Let $\mathcal{B}$ be the set of elements with $q+1$ or $q+2$ fixed points in $PGL(3,q)$. Notice that

$$fpr(y) \leqslant \begin{cases} \frac{q+2}{N} & , y \in \mathcal{B} \\ \frac{3}{N} & , y \notin \mathcal{B} \end{cases}$$

Set $\delta_1 = \frac{q+2}{N}$ and $\delta_2 = \frac{3}{N}$.

$$\sum_{y\in\langle x\rangle^{\#}} fpr(y) = \sum_{y\in\langle x\rangle^{\#}\cap\mathcal{B}} fpr(y) + \sum_{y\in\langle x\rangle^{\#}\setminus\mathcal{B}} fpr(y)$$

$$\sum_{y\in\langle x\rangle^{\#}} fpr(y) \leqslant \mid \langle x\rangle^{\#}\cap\mathcal{B} \mid \delta_1 + \mid \langle x\rangle^{\#}\setminus\mathcal{B} \mid \delta_2$$

$$\sum_{i=1}^{r}\sum_{y\in\langle x_i\rangle^{\#}} \frac{fpr(y)}{d_i} \leqslant \sum_{i=1}^{r} \frac{\mid \langle x_i\rangle^{\#}\cap\mathcal{B} \mid}{d_i}\delta_1 + \sum_{i=1}^{r} \frac{\mid \langle x_i\rangle^{\#}\setminus\mathcal{B} \mid}{d_i}\delta_2$$

$$\leqslant \sum_{i=1}^{r} \frac{\mid \langle x_i\rangle^{\#}\cap\mathcal{B} \mid}{d_i}(\delta_1-\delta_2) + \sum_{i=1}^{r} \frac{\mid \langle x_i\rangle^{\#} \mid}{d_i}\delta_2$$

$$\leqslant \sum_{i=1}^{r} \frac{| \langle x_i \rangle^{\#} \cap \mathcal{B} |}{d_i}(\delta_1 - \delta_2) + \sum_{i=1}^{r} \frac{d_i - 1}{d_i}\delta_2$$

$$\leqslant \sum_{i=1}^{r} \frac{| \langle x_i \rangle^{\#} \cap \mathcal{B} |}{d_i}(\delta_1 - \delta_2) + A(\underline{d})\delta_2$$

Set another important quantity :

$$\beta = \sum_{i=1}^{r} \frac{| \langle x_i \rangle^{\#} \cap \mathcal{B} |}{d_i}$$

Combining Riemann-Hurwitz formula with Cauchy-Frobenius formula

$$\frac{2(N+g-1)}{N} = \sum_{i=1}^{r} \frac{Ind(x_i)}{N} = A(\underline{d}) - \sum_{i=1}^{r} \sum_{y \in \langle x_i \rangle^{\#}} \frac{fpr(y)}{d_i}$$

$$2 + \frac{2(g-1)}{N} \geqslant A(\underline{d})(1-\delta_2) - \beta(\delta_1 - \delta_2)$$

Set $\epsilon_0 = \frac{2(g-1)}{N}$, we get the main formula:

$$A(\underline{d}) \leqslant \frac{(2+\epsilon_0) + \beta(\delta_1 - \delta_2)}{1 - \delta_2} \qquad (*)$$

If we set $\delta_1 = \delta_2$ in the above formula, i.e. we understand that $fpr(x) \leqslant \delta_1$ for all $x \in G^{\#}$ then we can get a rougher formula :

$$A(\underline{d}) \leqslant \frac{2+\epsilon_0}{1-\delta_1} \qquad (**)$$

Now bound β for the tuple $\underline{x} = (x_1, x_2, ..., x_r)$.

If $x_i \in \mathcal{B}$, then all of the powers of x_i that are nonidentity are also in $\mathcal{B}$. Because if a point is fixed by x_i, it is also fixed by x_i^k. Thus $Fix(x_i) \leqslant Fix(x_i^k)$. In this case there are $d_i - 1$ elements in $\mathcal{B}$ in group $\langle x_i \rangle$. If $x_i \notin \mathcal{B}$, there are $\phi(d_i)$ generators in group $\langle x_i \rangle$ where ϕ is the Euler's totient function. All of these generators are not in $\mathcal{B}$ either. So there are at most $d_i - \phi(d_i) - 1$ elements in $\mathcal{B}$ in group $\langle x_i \rangle$. We get

$$\beta \leqslant \sum_{x_i \in \mathcal{B}} \frac{d_i - 1}{d_i} + \sum_{x_i \notin \mathcal{B}} \frac{d_i - \phi(d_i) - 1}{d_i}$$

In $PGL(3,q)$ and $PSL(3,q)$, $\delta_1 = \frac{q+2}{q^2+q+1}$, $\delta_2 = \frac{3}{q^2+q+1}$ and $\epsilon_0 \leqslant \frac{2}{q^2+q+1}$ for all genus 0, 1, 2 tuples.

Assume that $q \geqslant 17$ then $\delta_1 \leqslant \frac{19}{307}$, $\delta_2 \leqslant \frac{3}{307}$ and $\epsilon_0 \leqslant \frac{2}{307}$. Using formula (**) we get $A(\underline{d}) \leqslant \frac{77}{36}$. If $r \geqslant 4$, $A(\underline{d}) \geqslant A((2,2,2,3)) = \frac{13}{6}$. But $\frac{13}{6} > \frac{77}{36}$. So we get $r = 3$. If $\frac{85}{42} \leqslant A(\underline{d}) \leqslant \frac{77}{36}$ then $\underline{d}$ only can be $(2,3,n)$ with $7 \leqslant n \leqslant 36$, $(2,4,n)$ with $5 \leqslant n \leqslant 9$, $(2,5,5)$, $(2,5,6)$, $(3,3,4)$ and $(3,3,5)$.

A corollary can be got from the main theorem in [Scott].

Lemma 6. *(Scott Bound) Suppose that $G \leq GL(n,q)$. If a tuple of elements $\underline{x} = (x_1, x_2, x_3)$ satisfies $G = \langle x_1, x_2, x_3 \rangle$ and $x_1 x_2 x_3 = 1$, then $v(x_i) + v(x_j) \geqslant n$ where $i \neq j$ and $i, j = 1, 2, 3$. In particular if $n \geqslant 3$ and $i \neq j$ then either $v(x_i) \geqslant 2$ or $v(x_j) \geqslant 2$.*

This lemma implies that for a tuple (x_1, x_2, x_3) there is at most one x_i belong to $\mathcal{B}$.

Now we count β for all tuples with $A(\underline{d}) \leqslant \frac{77}{36}$.

1. $\underline{d} = (2, d_2, d_3)$: In $PGL(3,q)$ and $PSL(3,q)$ all involutions are in $\mathcal{B}$. For $(2, d_2, d_3)$ tuples, the involution is in $\mathcal{B}$ then other two elements are not in $\mathcal{B}$. $\beta \leqslant \frac{1}{2} + \frac{d_2 - \phi(d_2) - 1}{d_2} + \frac{d_3 - \phi(d_3) - 1}{d_3}$.

2. $\underline{d} = (3,3,4)$: For (3,3,4) tuple, if one element of order 3 in $\mathcal{B}$ then other element of order 3 and the element of order 4 are not in $\mathcal{B}$, $\beta \leqslant \frac{2}{3} + 0 + \frac{1}{4} = \frac{11}{12}$; and if the element of order 4 in $\mathcal{B}$ then other 2 elements of order 3 are not in $\mathcal{B}$, $\beta \leqslant \frac{3}{4} + 0 + 0 = \frac{3}{4}$. In summary for (3,3,4) tuple $\beta \leqslant \frac{11}{12}$.

3. $\underline{d} = (3,3,5)$: For (3,3,5) tuple, if element of order 3 is in $\mathcal{B}$, then $\beta \leqslant \frac{2}{3} + 0 + 0 = \frac{2}{3}$, if element of order 5 is in $\mathcal{B}$, then $\beta \leqslant \frac{4}{5} + 0 + 0 = \frac{4}{5}$. So for (3,3,5) tuple $\beta \leqslant \frac{4}{5}$.

In summary

$\underline{d} =$	$\beta \leqslant$
$(2,3,n)$ with $7 \leqslant n \leqslant 36$	$\frac{6}{5}$
$(2,4,n)$ with $5 \leqslant n \leqslant 9$	$\frac{5}{4}$
$(2,5,5)$	$\frac{1}{2}$
$(2,5,6)$	1
$(3,3,4)$	$\frac{11}{12}$
$(3,3,5)$	$\frac{4}{5}$

The maximal β in the above table is $\frac{5}{4}$. Set $\beta \leqslant \frac{5}{4}$ and $q \geqslant 17$ in formula $(*)$ we get $A(\underline{d}) \leqslant \frac{159}{76}$.

All $A(\underline{d})$ that are less that $\frac{159}{76}$ and their the corresponding bounds for β are

$\underline{d}$	$\beta \leqslant$
$(2,3,n)$ with $7 \leqslant n \leqslant 13$	$\frac{13}{12}$
$(2,4,5)$	$\frac{3}{4}$
$(2,4,6)$	$\frac{5}{4}$
$(3,3,4)$	$\frac{11}{12}$

Consider $(2,4,6)$ tuple independently. $A(\underline{d}) = \frac{25}{12}$ and $\beta \leqslant \frac{5}{4}$. Solve inequality $(*)$ we get a bound for q. $q \leqslant 17$.

All other tuples have $\beta \leqslant \frac{13}{12}$. Set $q \geqslant 17$ in formula $(*)$ we get $A(\underline{d}) \leqslant \frac{25}{12}$. All tuples with $A(\underline{d}) \leqslant \frac{25}{12}$ are $(2,3,n)$ with $7 \leqslant n \leqslant 12$, $(2,4,5)$, $(2,4,6)$, $(3,3,4)$.

Now for each $\underline{d}$, we count its corresponding β and $A(\underline{d})$. Substitue $\delta_1 = \frac{q+2}{q^2+q+1}$, $\delta_2 = \frac{3}{q^2+q+1}$ and $\epsilon_0 = \frac{2}{q^2+q+1}$ in formula $(*)$:

$$A(\underline{d}) \leqslant \frac{2(q^2+q+2)+\beta(q-1)}{q^2+q-2}$$

We can solve the inequality to get a bound for q. Recall that $q \geqslant 17$.

Theorem 7. *If $PGL(3,q)$ or $PSL(3,q)$ has a generating system $\underline{x}$ of genus g where $g \leqslant 2$, then one of the following is true:*

(a) $q \leqslant 13$;

(b) $\underline{d}$ and q are in the following table

$\underline{d}$	β	$A(\underline{d})$	$q =$
$(2,3,7)$	$\frac{1}{2}$	$\frac{85}{42}$	$17, 19, 23, 25, 27, 29$
$(2,3,8)$	$\frac{7}{8}$	$\frac{49}{24}$	$17, 19, 23, 25$
$(2,3,9)$	$\frac{13}{18}$	$\frac{37}{18}$	$17, 19$
$(2,3,10)$	*1*	$\frac{31}{15}$	$17, 19$
$(2,3,12)$	$\frac{13}{12}$	$\frac{25}{12}$	17
$(2,4,5)$	$\frac{3}{4}$	$\frac{41}{20}$	$17, 19$
$(2,4,6)$	$\frac{5}{4}$	$\frac{25}{12}$	17

Section 2.3 Action of Subgroups of Projective Semilinear Group $P\Gamma L(3,q)$ on the Set of Projective Points

Let $V = V(n,F)$ be an n-dimensional vector space over a field F, and let $B = \{v_1, ..., v_n\}$ be a basis of V. Any automorphism σ of F induces a bijection from V to V by acting on the coordinates with respect to this basis; i.e. $(\sum x_i v_i)^\sigma = \sum x_i^\sigma v_i$. This bijection will also be referred to as σ. A *σ-semilinear map* T is the composition of this σ with an F-linear transformation $M \in GL(n,F)$; i.e. $T(v) = M(v^\sigma)$, for all $v \in V$. Note that $T(\lambda v) = \lambda^\sigma T(v)$. T acts on Ω. If $F = GF(q)$ is a finite field, any field automorphisms is some power of the *Frobenius automorphism* $a \mapsto a^p$, where $p = \text{char}F$. Suppose that the associate field automorphism with T is $a \mapsto a^{p^h}$ with $h \geqslant 1$, we call the corresponding σ-semilinear map a

p^h*-semilinear map*. It is easy to see that any p^h-semilinear map T on $V(n,q)$ extends to a p^h-semilinear map on $V(n,K)$, where K denotes the algebraic closure of $GF(q)$.

From now on we only use σ to denote the Frobenius map.

The group generated by $GL(n,q)$ and σ is called a *general semilinear group* and denoted as $\Gamma L(n,q)$. Furthermore $\Gamma L(n,q) = GL(n,q) \rtimes \langle\sigma\rangle$. The *projective semilinear group* $P\Gamma L(n,q) = PGL(n,q) \rtimes \langle\sigma\rangle$. All primitive almost simple groups of type $PSL(n,q)$ acting on Ω satisfy $PSL(n,q) \leq G \leq P\Gamma L(n,q)$.

$|P\Gamma L(n,q) : PGL(n,q)| = k$ when $q = p^k$.

The analysis about the eigen spaces in the above section must be refined for elements in $P\Gamma L(n,q) \setminus PGL(n,q)$.

The main theorem in [DFH] said that :

Lemma 8. *Let $T = M\sigma^h \in \Gamma L(n,q)$ where σ is the Frobenius map and Char(GF(q))= p. Let K be the algebraic closure of $GF(q)$. Then the p^h-semilinear extention map $\hat{T}$ on $V(n,K)$ fixes exactly $(p^h)^n$ vectors in $V(n,K)$ and all of these vectors constitute to a n-dimensional space over some subfield of size p^h of K (not necessarily in GF(q)). $\hat{T}$ on $V(n,K)$ fixes the vectors of a copy of $V(n,p^h)$.*

We will show that this lemma gives a bound of $Fix(x)$ for all $x \in P\Gamma L(n,q) \setminus PGL(n,q)$.

Corollary 9. *The number of fixed points of any element in $P\Gamma L(n,q) \setminus PGL(n,q)$ is no more than the number of fixed points of the field automorphism map associate with it.*

Proof: Suppose that $F \subseteq E$ is a field extension. It is easy to see that there is an injective mapping between the set of 1-dimensional subspaces of $V(n,F)$ to the set of 1-dimensional subspaces of $V(n,E)$. $f : PG(n-1,F) \to PG(n-1,E)$ defines as $f(\langle v\rangle_F) = \langle v\rangle_E$.

Let T be the same as lemma 2.4 and $\hat{T}$ be its extension map on $V(n,K)$. Let $V(n, GF(p^h)_{\hat{T}})$ be the n-dimensional space over a subfield $GF(p^h)_{\hat{T}}$ fixed by $\hat{T}$ in $V(n,K)$.

Consider injective mappings $f_1 : PG(n-1,q) \to PG(n-1,K)$ and $f_2 : PG(n-1, GF(p^h)_{\hat{T}}) \to PG(n-1,K)$.

In the proof of the main theorem in [4], the authors talked about this fact : if $T(v) = \lambda v$ then $\hat{T}$ fixes a vector that is a K-multiple of v. $\hat{T}(\lambda^{\frac{-1}{p^h-1}} v) = \lambda^{\frac{-p^h}{p^h-1}}(\lambda v) = \lambda^{\frac{-1}{p^h-1}} v$.

So if $\langle v \rangle$ is a fixed point of T in $PG(n-1,q)$ then $f_1(\langle v \rangle) = \langle v \rangle_K = \langle \lambda^{\frac{-1}{p^h-1}} v \rangle_K = f_2(\langle \lambda^{\frac{-1}{p^h-1}} v \rangle)$ in $V(n,K)$. This says that if $\langle v \rangle$ is a fixed point of T in $PG(n-1,q)$ then the image element of $\langle v \rangle$ under map f_1 lies in the image of map f_2. Since f_1 and f_2 are injective, we get

$$Fix(T) = \mid \text{Image}(f_1) \cap \text{Image}(f_2) \mid \leqslant |\text{Image}(f_2)| = |V(n, GF(p^h)_{\hat{T}})| = \frac{(p^h)^n - 1}{p^h - 1}$$

This shows that

$$Fix(M\sigma^h) \leqslant Fix(\sigma^h)$$

for all $M \in GL(n,q)$. In other words the number of fixed points of T in $\Gamma L(n,q) \setminus GL(n,q)$ is no more than the number of fixed points of the field automorphism map associate with T. Their projective image in $P\Gamma L(n,q)$ follow the same rules. ■

Corollary 10. *The fixed point ratio δ_1 satisfies:*

1. *If $G \subseteq PGL(3,q)$, $\delta_1 \leqslant \frac{q+2}{q^2+q+1}$.*

2. *Suppose that G contains field automorphism.*

   (a) *If G contains a field automorphism of order 2, then this field automorphism supplies the biggest fixed point ratio for G : $\delta_1 \leqslant \frac{q+q^{\frac{1}{2}}+1}{q^2+q+1}$.*

   (b) *If G does not contain field automorphism of order 2, the biggest fixed point ratio for elements in $G \setminus PGL(3,q)$: $\leqslant \frac{q^{\frac{2}{3}}+q^{\frac{1}{3}}+1}{q^2+q+1}$. It is less than $\frac{q+2}{q^2+q+1}$. In this case $\delta_1 \leqslant \frac{q+2}{q^2+q+1}$.*

In summary, if there are involutions in $G \setminus PGL(3,q)$ *then* $\delta_1 \leqslant \frac{q+q^{\frac{1}{2}}+1}{q^2+q+1}$ *for all elements in* G*; if there are no involutions in* $G \setminus PGL(3,q)$ *then* $\delta_1 \leqslant \frac{q+2}{q^2+q+1}$ *for all elements in* G*.*

The following result is well-known.

Lemma 11. *Let* G *be a group satisfy* $G = \langle x_1, x_2, ..., x_r \rangle$ *with* $x_1 x_2 \cdots x_r = 1$ *and the orders of* $x_i's$ *are pairwise coprime , then* $[G,G] = G$*.*

Proof: Let $[G,G]$ be the derived subgroup of G. Consider the group homomorphism $G \to G/[G,G]$. Let $\bar{x_i}$ be the image of x_i under this map for each i. $o(\bar{x_i})|o(x_i)$ implies that the orders of $\bar{x_i}'s$ are also pairwise coprime. For any j, $o(x_j^{-1}) = o(\bar{x_1} \cdots \bar{x_{j-1}} \bar{x_{j+1}} \cdots \bar{x_r})$ is the least common multiples of $o(\bar{x_1}), ..., o(\bar{x_{j-1}}), o(\bar{x_{j+1}}), ..., o(\bar{x_r})$ since $G/[G,G]$ is abelian. $o(x_j^{-1})$ must be coprime to itself. We get $o(x_j^{-1}) = 1$ and $x_j = 1$ for all j. So $G = [G,G]$. ■

By this lemma every almost simple group with (2,3,7) generating system must be simple.

Lemma 12. *There is only one conjugacy class of involutions in* $P\Gamma L(3,q^2) \setminus PGL(3,q^2)$*.*

I will give a proof of this result in section 2.4.

The following basic fact in group theory is very useful for our discussion. Suppose that $N \trianglelefteq G$ and $x \in G \setminus N$. Then $gcd(o(x), |G:N|) \neq 1$.

Next I will use inequality $A(\underline{d}) \leqslant \frac{2+\epsilon_0}{1-\delta_1}$ $(**)$ to give a bound to q.

Recall that $A(\underline{d}) \geqslant \frac{85}{42}$, $\epsilon_0 = \frac{2}{q^2+q+1}$ and δ_1 is the fixed point ration for all elements in G.

Substitute $\delta_1 \leqslant \frac{q+q^{\frac{1}{2}}+1}{q^2+q+1}$ in $(**)$ we get $q \leqslant 97$. All powers that are less that 97 are $\{4, 8, 9, 16, 25, 27, 32, 49, 64, 81\}$.

Suppose that $q = 81$. σ has order 4. We only consider $PSL(3,81) \rtimes \langle \sigma^2 \rangle$ and $P\Gamma L(3,81)$. Both $PSL(3,81) \rtimes \langle \sigma^2 \rangle$ and $P\Gamma L(3,81)$ have involutions outside $PSL(3,81)$. So $\delta_1 \leqslant \frac{81+81^{\frac{1}{2}}+1}{81^2+81+1} = \frac{91}{6643}$. Substitute in $(**)$ get $A(\underline{d}) \leqslant \frac{1661}{819}$. $\underline{d}$ only can be $(2,3,7)$. But (2,3,7) tuple can not generates G.

Suppose that $q = 64$. σ has order 6. We only consider $PSL(3,64) \rtimes \langle \sigma \rangle$, $PSL(3,64) \rtimes \langle \sigma^2 \rangle$, $PSL(3,64) \rtimes \langle \sigma^3 \rangle$, $PGL(3,64) \rtimes \langle \sigma^2 \rangle$, $PGL(3,64) \rtimes \langle \sigma^3 \rangle$ and $P\Gamma L(3,64)$. $\delta_1 \leqslant \frac{64+64^{\frac{1}{2}}+1}{64^2+64+1} = \frac{1}{57}$

then $A(\underline{d}) \leqslant \frac{2081}{1022}$. $\underline{d}$ only can be (2,3,7). But every almost simple group with (2,3,7) generating system must be simple.

Suppose that $q = 49$. σ has order 2. All groups we consider are $PGL(3,49)$, $PSL(3,49) \rtimes \langle\sigma\rangle$ and $P\Gamma L(3,49)$. $\delta_1 \leqslant \frac{49+49^{\frac{1}{2}}+1}{49^2+49+1}$ and $A(\underline{d}) \leqslant \frac{2452}{1197}$. $\underline{d}$ only can be $(2,3,7)$ and $(2,3,8)$. The above lemma eliminates $(2,3,7)$. Discuss about (2,3,8) tuples. In this tuple the involution and element of order 8 are in $G \setminus PGL(3,49)$ and the element of order 3 come from $PGL(3,49)$. There is only one conjugacy class of involutions in $PSL(3,49) \rtimes \langle\sigma\rangle \setminus PSL(3,49)$ and $P\Gamma L(3,49) \setminus PGL(3,49)$. σ is a representative. $Ind(\sigma) = \frac{1}{2}(N - Fix\sigma_0) = \frac{1}{2}(2451-(7^2+7+1)) = 1197$. Let e_3 be an element of order 3 in $PGL(3,49)$. $Fix(e_3) \in \{0,1,2,3,50,51\}$. Since e_3 has prime order, the number of fixed point of e_3 and the degree N arecongruent modulo 3: $N \equiv Fix(e_3) (\mod 3)$. This gives that $Fix(e_3) \in \{0,3,51\}$ and $Ind(e_3) \in \{1634,1632,1600\}$. We use GAP to get the indices of e_8. Since $P\Gamma L(3,49)$ is a big group, an available way is to find a Sylow 2-subgroup of $P\Gamma L(3,49)$ at first. Then filter out all of the elements of order 8 inside the Sylow 2-subgroup and find their indices. The result is that all elements of order 8 in $P\Gamma L(3,49) \setminus PGL(3,49)$ have index 2137. The sum of indices is at least 1197+1600+2137=4934 which is bigger than $2(N+1) = 4902$. So there is no $(2,3,8)$ tuples.

Suppose that $q = 32$. σ has order 5. All groups we consider are $P\Gamma L(3,32)$. $\delta_1 \leqslant \frac{32+2}{32^2+32+1}$ and $A(\underline{d}) \leqslant \frac{2116}{1023}$. But the possible smallest $A(\underline{d}) = A((2,5,5)) = \frac{21}{10} > \frac{2116}{1023}$. So there are no tuples for 32.

Suppose that $q = 27$. σ has order 3. All groups we consider are $P\Gamma L(3,27)$. $\delta_1 \leqslant \frac{27+2}{27^2+27+1}$ implies $A(\underline{d}) \leqslant \frac{379}{182}$. $\underline{d}$ can be (2,3,7), (2,3,8), (2,3,9), (2,3,10), (2,3,11), (2,4,5). Since a 3-tuple needs at least 2 elements outside $PSL(3,27)$ to generate $P\Gamma L(3,27)$ and all elements in $P\Gamma L(3,27) \setminus PSL(3,27)$ have order of multiple of 3, (2,3,9) is the only possible tuple.

Now we count the indices of elements for (2,3,9) tuples. Suppose that e_2 is an involution in $PSL(3,27)$, then $Fix(e_2) = 29$ and $Ind(e_2) = \frac{1}{2}(27^2 - 1) = 364$. Suppose that e_3 is an element of order 3 and e_9 is an element of order 9 in $P\Gamma L(3,27)\backslash PSL(3,27)$. Then $Fix(e_3) \leqslant Fix(\sigma) = 13$ and $Fix(e_9) \leqslant Fix(\sigma) = 13$. $e_9^3 \in PSL(3,27)$ has order 3. So $Fix(e_9^3) = 28$ or 1. We get $Ind(e_3) \geqslant \frac{2}{3}(757 - 13) = 496$ and $Ind(e_9) \geqslant 757 - \frac{1}{9}(757 + 2 \times 28 + 6 \times 13) = 658$. $Ind(e_2) + Ind(e_3) + Ind(e_9) \geqslant 364 + 496 + 658 = 1518 > 1516$. So there is no genus 0, 1, 2 (2,3,9) generating tuples.

This shows that all genus 0, 1, 2 groups are of $q \leqslant 25$.

When $q = 25$, σ has order 2. The groups are $PSL(3,25)$, $PGL(3,25)$, $PSL(3,25) \rtimes \langle\sigma\rangle$,$P\Gamma L(3,25)$. Let H be $PSL(3,25)$ or $PGL(3,25)$. $\delta_1 \leqslant \frac{5^2+5+1}{25^2+25+1} = \frac{31}{651}$ and $A(\underline{d}) \leqslant \frac{1304}{620}$. $\underline{d}$ can be $(2,3,n)$ with $7 \leqslant n \leqslant 15$, $(2,4,5)$, $(2,4,6)$, $(2,5,5)$, $(3,3,4)$. It needs at least two elements of even order to generate $H \rtimes \langle\sigma\rangle$. Only $(2,3,8)$, $(2,3,10)$, $(2,3,12)$, $(2,3,14)$, $(2,4,5)$, $(2,4,6)$ tuples are possible.

Section 2.4 Indices of Elements of Order 2 and 3

Section 2.4.1 Conjugacy Class of Involutions in Almost Simple Groups of Type $PSL(3,q)$

The following basic results are known.

Lemma 13. *1. Suppose that $M \in GL(3,q)$ and $o(M)|q-1$then M is conjugate to a diagonal matrix in $GL(3,q)$.*

2. If char $GF(q)$=2 there is a single conjugacy class of involutions in $GL(3,q)$; if char $GF(q) \neq 2$, there are 3 conjugacy classes of involutions in $GL(3,q)$.

3. *There is a single conjugacy class of involutions in* $SL(3, q)$.

4. $PSL(3, q)$ *contains a single conjugacy class of involutions. This class is of size* $q^2(q^2 + q + 1)$ *for* q *odd and of size* $(q^2 - 1)(q^2 + q + 1)$ *for* q *even.*

5. *There is a single conjugacy class of involutions in* $PGL(3, q)$. *In particular, all involutions of* $PGL(3, q)$ *are in* $PSL(3, q)$.

Give a proof for the last result in the above lemma.

Proof: If $3 \nmid q-1$, $PSL(3, q) = PGL(3, q)$. There is a single conjugacy class of involutions in $PSL(3, q)$.

If $3 \mid q - 1$, $PSL(3, q) \trianglelefteq PGL(3, q)$ and $|PGL(3, q) : PSL(3, q)| = 3$. So the order of any element in $PGL(3, q) \setminus PSL(3, q)$ is a multiple of 3. All of the involutions in $PGL(3, q)$ lie in $PSL(3, q)$. And there is only one conjugacy class of involutions in $PSL(3, q)$. ■

Lemma 14. *Let* $H = PSL(3, q^2)$ *or* $PGL(3, q^2)$. *Let* $G = H \rtimes \langle \sigma^k \rangle$ *where* σ^k *is a power of Frobenius automorphism with even order, then there is a single conjugacy class of involutions in* $G \setminus H$. *Suppose that* $\tau : x \to x^q$ *is the unique involution in* $\langle \sigma \rangle$. *Then* τ *is a representative for this class.*

Proof: Suppose that $M\alpha$ is an involution in $G \setminus H$ with $M \in H$ and $\alpha \in \langle \sigma \rangle$. Then $1 = M\alpha M\alpha = M\alpha^2\alpha^{-1}M\alpha = M\alpha^2 M^\alpha$. So $M\alpha^2 = (M^\alpha)^{-1} \in H$, this implies that $\alpha^2 = 1$. So $\alpha = \tau$. This shows that all involutions $G \setminus H$ are of form $M\tau$.

Now both $M\tau$ and τ are involutions. The group generated by them $\langle M\tau, \tau \rangle$ is a dihedral group. The property of dihedral group says that if the order of M is odd then all of involutions in the dihedral group are conjugate. And if M has even order, there are two conjugacy classes of involutions in $\langle \tau, M\tau \rangle$: $[\tau]$ and $[N\tau]$. N is an involution in H and commutes with τ. $M\tau \in [N\tau]$. The centralizer of τ in H is $PSL(3, q)$ or $PGL(3, q)$. So $N \in PSL(3, q)$ or $N \in PGL(3, q)$.

Discuss in two cases: q is odd and q is even.

Suppose that q is odd. Let ξ be a primitive element in $GF(q^2)$. Then $-1 = \xi^{\frac{q^2-1}{2}}$ is the only involution in $GF(q^2)^{\#}$ and ξ^{q+1} is a primitive element in $GF(q)$. Since $GF(q)^{\#}$ is a cyclic subgroup of $GF(q^2)^{\#}$. -1 in $GF(q^2)$ is also -1 in $GF(q)$.

There is a single conjugacy class of involutions in subgroup of H: $PGL(3,q)$ or $PSL(3,q)$. There exists an element $A \in PGL(3,q)$ or $PSL(3,q)$ such that $N^A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

Notice that $PGL(3,q)$ and $PSL(3,q)$ are in the centralizer of τ hence A commutes with τ. We can get $(N\tau)^A = N^A\tau^A = N^A\tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\tau$. $N\tau$ and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\tau$ are conjugate.

Next I will construct an element $B \in H$ satisfying $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\tau = (B\tau)^{-1}\tau(B\tau)$ and thus $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\tau$ and τ are conjugate under G. Finally $M\tau$ and τ are conjugate in G. We finish the proof of the result.

Notice that $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\tau = (B\tau)^{-1}\tau(B\tau)$ implies that $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = (B\tau)^{-1}\tau B$ $= \tau^{-1}B^{-1}\tau B = (B^{-1})^{\tau}B$. Let $B = \begin{pmatrix} \xi^{q+1} & 0 & 0 \\ 0 & \xi^{\frac{q+1}{2}} & 0 \\ 0 & 0 & \xi^{\frac{q+1}{2}} \end{pmatrix}$. Then B satisfies the above equation.

Suppose that q is even. There exists an element $A \in PGL(3,q)$ or $PSL(3,q)$ such that $N^A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $(N\tau)^A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\tau$. Still construct matrix B such that $(\tau)^{B\tau} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\tau$. Thus $M\tau$ and τ are conjugate in G.

B satisfies that $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (B^{-1})^{\sigma}B$.

By Theorem 3 in [CH] polynomial $x^q + x + 1$ has x^q roots in $GF(q^2)$. Pick a be one of the roots of $x^q + x + 1$. Let $B^{-1} = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. This B satisfies the above equation. ■

Corollary 15. *Let $H = PSL(3, q^2)$ or $PGL(3, q^2)$. Let $G = H \rtimes \langle \sigma^k \rangle$ where σ^k is a power of Frobenius automorphism with even order, then all involutions in $G \setminus H$ have $q^2 + q + 1$ fixed points in Ω and their index is $\frac{1}{2}(q^4 - q)$.*

Section 2.4.2 Indices of Elements of Order 2 and 3 in $PSL(3, q)$ and $PGL(3, q)$

Now discuss the indices for involutions and elements of order 3 in $PSL(3, q)$ and $PGL(3, q)$. Use e_d denote an element of order d.

LEMMA 2.12 : *Let G be $PGL(3, q)$ or $PSL(3, q)$. If $2 \nmid q$, then $Fix(e_2) = q + 2$ and $Ind(e_2) = \frac{1}{2}(q^2 - 1)$; if $2 \mid q$, then $Fix(e_2) = q + 1$ and $Ind(e_2) = \frac{1}{2}(q^2)$.*

Proof: There is only one conjugacy class of involutions in G.

If char$GF(q) \neq 2$, then $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is an involution in $PSL(3,q)$. It has $q+2$ fixed points. So $Fix(e_2) = q+2$ and $Ind(e_2) = \frac{1}{2}(q^2-1)$. If char$GF(q) = 2$, then $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is an involution in $PSL(3,q)$. It has $q+1$ fixed points. So $Fix(e_2) = q+1$ and $Ind(e_2) = \frac{1}{2}q^2$.

■

For 3-element we have

LEMMA 2.13: In $PSL(3,q)$,

(1) if $q \equiv 2 (\mod 3)$, then $Fix(e_3) = 1$, $Ind(e_3) = \frac{2}{3}(q^2+q)$;

(2) if $q \equiv 0 (\mod 3)$, then $Fix(e_3) = 1$ or $q+1$, $Ind(e_3) = \frac{2}{3}(q^2+q)$ or $\frac{2}{3}(q^2)$;

(3) if $q \equiv 1 (\mod 3)$, and $9 \nmid q-1$ then $Fix(e_3) = 3$ and $Ind(e_3) = \frac{2}{3}(q^2+q-2)$;

(4) in general, if $q \equiv 1 (\mod 3)$, $Fix(e_3) \in \{0, 3, q+2\}$ and $Ind(e_3) \in \{\frac{2}{3}(q^2+q+1), \frac{2}{3}(q^2+q-2), \frac{2}{3}(q^2-1)\}$.

Proof: Suppose element e_p has prime order p. Then all powers of e_p except identity have the same fixed points. $Ind(e_p) = N - \frac{1}{p}[(p-1)Fix(e_p) + Fix(1)] = \frac{p-1}{p}[N - Fix(e_p)]$.

(1) $Ind(e_3) = \frac{2}{3}[q^2+q+1-Fix(e_3)]$ and $Ind(e_3)$ is an integer.

Hence $3 \big| [q^2+q+1-Fix(e_3)]$ this gives $Fix(e_3)$ only can be 1 or $q+2$. Next I will show $q+2$ can not exist.

Suppose that v is an eigen vector of e_3 then $e_3 v = \lambda v$ for some nonzero number in $GF(q)$. So $v = Iv = (e_3)^3 v = \lambda^3 v$. So $\lambda^3 = 1$. But $3 \nmid q-1$, there is no element of order 3 in $GF(q)$, we get $\lambda = 1$. So all eigen vector of e_3 belong to eigen value 1. Suppose that e_3 fixes $q+2$ points, e_3 has one 2-dimensional eigen space and one 1-dimensional eigen space. Both of them belong to 1. We get e_3 is the identity. This is a contradiction. So e_3 only can fix 1 point.

(2) When $3\big|q$, $|PSL(3,q)| = |SL(3,q)| = q^2(q^3-1)(q^3-q)$, the Sylow 3-subgroup has order q^3. Set $\left\{\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a,b,c \in GF(q)\right\}$ is a Sylow 3-subgroup of $PSL(3,q)$. In this subgroup all elements fix 1 or $q+1$ projective points.

(3) Hence $3\big|[q^2+q+1-Fix(e_3)]$ this gives $Fix(e_3)$ only can be 0, 3 or $q+2$.

If $3 \mid q-1$, then there are two elements of order 3 in $GF(q)$, say λ and λ^{-1}. Suppose that $9 \nmid q-1$, the Sylow 3-subgroup of $SL(3,q)$ has order 27 and the Sylow 3-subgroup of $PSL(3,q)$ has order 9.

$$\text{Group}H = \left(\left\langle\begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^{-1} \end{pmatrix}\right\rangle \times \left\langle\begin{pmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda^{-1} \end{pmatrix}\right\rangle\right) \rtimes \left\langle\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\right\rangle$$

is a Sylow 3-subgroup in $SL(3,q)$. The homomorphism image $\overline{H}$ of H in $PSL(3,q)$ has order 9. So $\overline{H}$ is a Sylow 3-subgroup of $PSL(3,q)$. All nonidentity elements in H have order 3. Thus all nonidentity elements in $\overline{H}$ also have order 3. Let M be one element of order 3 in the $SL(3,q)$. $o(M)|q-1$, M can be diagonalized over $GF(q)$. M fixes three 1-dimensional subspaces. Thus an element of order 3 in $PSL(3,q)$ fixes 3 projective points.

(4) If $3 \mid q-1$, $Ind(e_3) = \frac{2}{3}[q^2+q+1-Fix(e_3)]$ is an integer. $3\big|[q^2+q+1-Fix(e_3)]$ this gives $Fix(e_3) \in \{0, 3, q+2\}$ and $Ind(e_3) \in \{\frac{2}{3}(q^2+q+1), \frac{2}{3}(q^2+q-2), \frac{2}{3}(q^2-1)\}$. ■

Corollary 16. *In $PSL(3,17)$, there are only possible (2,3,8) generating tuples has genus less than or equal to 2.*

Proof: Since $7 \nmid |PSL(3,17)|$ there is no (2,3,7) tuple in $PSL(3,17)$. And $5 \nmid |PSL(3,17)|$ so there are no (2,3,10) and (2,4,5) tuples in $PSL(3,17)$.

In $PSL(3,17) = SL(3,17)$, $Ind(e_2) = \frac{1}{2}(17^2-1) = 144$, $Ind(e_3) = \frac{2}{3}(17^2+17) = 204$.

Now count the index of element of order 4. $4|17-1$, so e_4 can be diagonalized over $GF(17)$. $Fix(e_4) = 3$ or 19. There are exactly two elements of order 4 in $GF(17)^{\#}$, say $\{\zeta, \zeta^{-1}\}$. $det(e_4) = 1$ implies that the three diagonal elements in e_4 have to be $(1, \zeta, \zeta^{-1})$. So $Fix(e_4) = 3$ for all elements of order 4. $Ind(e_4) = N - \frac{1}{4}(N + Fix(e_4) + Fix(e_2) + Fix(e_4^{-1})) = 224$.

Discuss elements of order 6. Since $6 \nmid 17-1$ the eigenvalue of e_6 only can be 1. This implies that $Fix(e_6) \neq 3, 19$. Since $gcd(6, 17) = 1$, we get $Fix(e_6) \neq 18$. Maschke' theorem implies that $Fix(e_6) = 2$ then $Fix(e_6) = 3$. So $Fix(e_6) \neq 2$. Now count the index of $e_6 = N - \frac{1}{6}(N + 2Fix(e_6) + 2Fix(e_3) + Fix(e_2)) = 307 - \frac{1}{6}(307 + 2 + 19 + 2Fix(e_6))$ is an integer so $Fix(e_6) = 1$ and $Ind(e_6) = 252$.

Discuss elements of order 8. e_8 is 2-element. So $Fix(e_8) \equiv 307(\mod 2)$. We get $Fix(e_8) \in \{1, 3, 19\}$. $8|16$, so e_8 can be diagonalized over $GF(17)$. So $Fix(e_8) \in \{3, 19\}$ and $Ind(e_8) \in \{264, 256\}$.

Discuss elements of order 9. e_9 is 3-element. So $Fix(e_9) = 1$ or 19. $9 \nmid 16$, so the eigenvalue of e_9 is 1. This implies that $Fix(e_9) \neq 19$. So $Fix(e_9) = 1$ and $Ind(e_9) = 272$.

Discuss elements of order 12. $Ind(e_12) = N - \frac{1}{12}(N + 4Fix(e_12) + 2Fix(e_6) + 2Fix(e_4) + 2Fix(e_3) + Fix(e_2)) = 307 - \frac{1}{12}(307 + 4Fix(e_12) + 2 + 6 + 2 + 19)$ is an integer so $Fix(e_12) = 0$ or 3. But the only possible eigenvalue of e_12 is 1. So $Fix(e_12) = 0$. $Ind(e_{12}) = 279$.

By Riemann-Hurwitz formula, the only possible tuples are (2,3,8) tuples for $PSL(3, 17)$.

■

CHAPTER 3 Result

Theorem 17. *Genus 0, 1, 2 almost simple groups of type $PSL(3, q)$ acting on the projective points of $2-$dimensional projective geometry $PG(2, q)$ happen to $q \leqslant 13$. Here is the list.*

q	$PSL(3,q)$	$PGL(3,q)$	$P\Gamma L(3,q)$	$PSL(3,q) \rtimes \langle \sigma^k \rangle$
3	$0,1,2$			
4	$0,1,2$	$0,1,2$	$0,1,2$	$PSL(3,4).2$ $0,1,2$
5	$0,1,2$			
7	$0,1,2$	$0,1,2$		
8	$1,2$		2	
9	$1,2$		$1,2$	
11	2			
13	–	$1,2$		
$\geqslant 16$	–	–	–	–

Note:

- Blank means identical to another group listed.
- " – " means no low genus tuples.
- The last column is an extension group of $PSL(3,q)$ with a power of standard field automorphism but not $P\Gamma L(3,q)$.

APPENDIX A. Sample GAP Program

It is a record of the GAP software for computation of $P\Sigma L(3,4)$.

```
>A:=AllPrimitiveGroups(DegreeOperation,21);
[ PGL(2, 7), A(7), S(7), PSL(3, 4)=M(21), PSigmaL(3, 4), PGL(3, 4), PGammaL(3, 4),
A(21), S(21)]
>g:=A[5];;
>r:=List(ConjugacyClasses(g),x->Representative(x));;
>rr:=List([2..Size(r)],i->r[i]);;
>cn:=function(x)
return ClassNames(CharacterTable(g),"ATLAS")[Position(r,x)];
end;
>g1:=CommutatorSubgroup(g,g);;
>s4t:=UnorderedTuples(rr,4);;
>s3t:=UnorderedTuples(rr,3);;
>s5t:=UnorderedTuples(rr,5);;
>s6t:=UnorderedTuples(rr,6);;
>Fs3t:=Filtered(s3t,x->Size(Intersection(x,g1))<=1);;Size(last);
>M:=MaximalSubgroupClassReps(g); ;
>List(M,x->Index(g,x));
[280, 120, 56, 21, 21, 2]
> h280:=ActionHomomorphism(g,RightCosets(g,M[1]),OnRight);
> pindex280:=function(x) return 280-Length(Orbits(Group(x),[1..280]));end;
> h120:=ActionHomomorphism(g,RightCosets(g,M[2]),OnRight);
> pindex120:=function(x) return 120-Length(Orbits(Group(x),[1..120]));end;
```

```
> h56:=ActionHomomorphism(g,RightCosets(g,M[3]),OnRight);
> pindex56:=function(x) return 56-Length(Orbits(Group(x),[1..56]));end;
> h21A:=ActionHomomorphism(g,RightCosets(g,M[4]),OnRight);
> h21B:=ActionHomomorphism(g,RightCosets(g,M[5]),OnRight);
> pindex21:=function(x) return 21-Length(Orbits(Group(x),[1..21]));end;
> List(rr,x->[cn(x),Order(Centralizer(g,x)),[pindex21(x∧h21A),
pindex21(x∧h21B), pindex56(x∧h56),pindex120(x∧h120),pindex280(x∧h280)]]);
> g280:=function(a)
return 1/2*( Sum( List(a,x->pindex280(x∧h280))) )-280+1;end;
> g120:=function(a)
return 1/2*( Sum( List(a,x->pindex120(x∧h120))) )-120+1;end;
> g56:=function(a)
return 1/2*( Sum( List(a,x->pindex56(x∧h56))) )-56+1;end;
> g21A:=function(a)
return 1/2*( Sum( List(a,x->pindex21(x∧h21A))) )-21+1;end;
> 21B:=function(a)
return 1/2*( Sum( List(a,x->pindex21(x∧h21B))) )-21+1;end;

    # filtered 3-tuples
>lg:= [0, 1, 2];
>lgenus:=Filtered([1..Size(Fs3t)],i->Intersection([g21A(Fs3t[i]),g21B(Fs3t[i]),
g56(Fs3t[i]), g120(Fs3t[i]),g280(Fs3t[i])],lg)<>[ ] and g21A(Fs3t[i]) in NonnegativeIntegers
and g21B(Fs3t[i]) in NonnegativeIntegers and g56(Fs3t[i]) in NonnegativeIntegers and g120(Fs3t[i])
in NonnegativeIntegers and g280(Fs3t[i]) in NonnegativeIntegers );
> ltuples:=List(lgenus,i->Fs3t[i]);;
> for i in [1..Size(ltuples)]
```

```
do Sort(ltuples[i], function(v,w) return Order(v)¡Order(w);end);od;
> Sort(ltuples, function(v,w)
if Order(v[1])<Order(w[1]) then return Order(v[1])<Order(w[1]) ;fi; if Order(v[1])=Order(w[1])
then return Position(r,v[1])<Position(r,w[1]);fi; end);
>for t in ltuples do Print([cn(t[1]),cn(t[2]),cn(t[3])],"\ n");od;
>gt:=function(t)
local x, cx, ccly,cclz, orbsy, yrep, F1;
x:=t[1]; cx:=Centralizer(g,x);
ccly:=ConjugacyClass(g,t[2]);
cclz:=ConjugacyClass(g,t[3]);
orbsy:=OrbitsDomain(cx,ccly); yrep:=List(orbsy,y->y[1]);;
F1:=Filtered(yrep,y->g=Group(x,y) and (x*y)∧-1 in cclz);
return List(F1,y->[x,y,(x*y)∧-1]);end;
> L:=List(ltuples,t->gt(t));;
> Fltuples:=Filtered(L,x->x<>[ ]);;Size(last);
> for xx in Fltuples do Print("&",List ( List(xx[1], y ->Filtered(rr,r-> y in Conjugacy-
Class(g,r)) ),z -> cn(z[1])) , " & " , g21A(xx[1]), " & ",g21B(xx[1]), " & ", g56(xx[1]), " &
", g120(xx[1]), " & ", g280(xx[1]), "\ n"); od;
```

```
# filtered 4-tuples
>Fs4t:=Filtered(s4t,x->Size(Intersection(x,g1))<=2);;
>lgenus:=Filtered([1..Size(Fs4t)],i->Intersection([g21A(Fs4t[i]),g21B(Fs4t[i]),
g56(Fs4t[i]), g120(Fs4t[i]),g280(Fs4t[i])],lg)<>[ ] and g21A(Fs4t[i]) in NonnegativeIntegers
and g21B(Fs4t[i]) in NonnegativeIntegers and g56(Fs4t[i]) in NonnegativeIntegers and g120(Fs4t[i])
in NonnegativeIntegers and g280(Fs4t[i]) in NonnegativeIntegers );
>ltuples:=List(lgenus,i->Fs4t[i]);;
```

```
>for i in [1..Size(ltuples)]
do Sort(ltuples[i], function(v,w) return Order(v)<Order(w);end);od;
>Sort(ltuples, function(v,w)
if Order(v[1])<Order(w[1]) then return Order(v[1])<Order(w[1]) ;fi ; if Order(v[1])=Order(w[1])
then return Position(r,v[1])< Position(r,w[1]);fi; end);
>for t in ltuples do Print([cn(t[1]),cn(t[2]),cn(t[3]),cn(t[4])],"\ n");od;
>gt:=function(t)
local x, cx, ccly,cclz, ccla, orbsy, yrep,h, F1;
x:=t[1]; cx:=Centralizer(g,x);
ccly:=ConjugacyClass(g,t[2]);
cclz:=ConjugacyClass(g,t[3]);
ccla:=ConjugacyClass(g,t[4]);
orbsy:=OrbitsDomain(cx,ccly); yrep:=List(orbsy,y->y[1]);;
h:=function(i)
local cy,cxy,orbsz,zrep,F;
cy:=Centralizer(g,yrep[i]);
cxy:=Intersection(cx,cy);
orbsz:=OrbitsDomain(cxy,cclz);
zrep:=List(orbsz,z->z[1]);
F:=Filtered(zrep,z->g=Group(x,yrep[i],z) and (x*yrep[i]*z)∧-1 in ccla);
return List(F,z->[x,yrep[i],z,(x*yrep[i]*z)∧-1]);end;
return Union(List([1..Size(yrep)],h));end;
>L:=List(ltuples,t->gt(t));;
>Fltuples:=Filtered(L,x->x<>[ ]);;Size(last);
> for xx in Fltuples do Print("&",List ( List(xx[1], y ->Filtered(rr,r-> y in Conjugacy-
Class(g,r)) ),z -> cn(z[1])) , " & " , g21A(xx[1]), " & ",g21B(xx[1]), " & ", g56(xx[1]), " &
```

```
", g120(xx[1]), " & ", g280(xx[1]) , "\ n"); od;

    # filtered 5-tuples
>Fs5t:=Filtered(s5t,x->Size(Intersection(x,g1))<=3);;
>lgenus:=Filtered([1..Size(Fs5t)],i->Intersection([g21A(Fs5t[i]),g21B(Fs5t[i]),
g56(Fs5t[i]), g120(Fs5t[i]),g280(Fs5t[i])],lg)<>[ ] and g21A(Fs5t[i]) in NonnegativeIntegers
and g21B(Fs5t[i]) in NonnegativeIntegers and g56(Fs5t[i]) in NonnegativeIntegers and g120(Fs5t[i])
in NonnegativeIntegers and g280(Fs5t[i]) in NonnegativeIntegers );Size(last);
>ltuples:=List(lgenus,i->Fs5t[i]);;
>for i in [1..Size(ltuples)]
do Sort(ltuples[i], function(v,w) return Order(v)¡Order(w);end);od;
>Sort(ltuples, function(v,w)
if Order(v[1])<Order(w[1]) then return Order(v[1])<Order(w[1]) ;fi; if Order(v[1])=Order(w[1])
then return Position(r,v[1])<Position(r,w[1]);fi; end);
>for t in ltuples do Print([cn(t[1]),cn(t[2]),cn(t[3]),cn(t[4]),cn(t[5])],"\ n");od;
>gt:=function(t)
local x, cx, ccly,cclz, ccla, cclb,orbsy, yrep,h;
x:=t[1]; cx:=Centralizer(g,x);
ccly:=ConjugacyClass(g,t[2]);
cclz:=ConjugacyClass(g,t[3]);
ccla:=ConjugacyClass(g,t[4]);
cclb:=ConjugacyClass(g,t[5]);
orbsy:=OrbitsDomain(cx,ccly); yrep:=List(orbsy,y->y[1]);;
h:=function(i)
local cy,cxy,orbsz,zrep,j,h1;
cy:=Centralizer(g,yrep[i]);
```

```
cxy:=Intersection(cx,cy);
orbsz:=OrbitsDomain(cxy,cclz);
zrep:=List(orbsz,z->z[1]);
for j in [1..Size(zrep)] do
h1:=function(j)
local cz,cxyz,orbsa,arep,u,t;
cz:=Centralizer(g,zrep[j]);
cxyz:=Intersection(cxy,cz);
orbsa:=OrbitsDomain(cxyz,ccla);
arep:=List(orbsa,a->a[1]);
u:=Filtered(arep,a->g=Group(x,yrep[i],zrep[j],a) and (x*yrep[i]*zrep[j]*a)∧-1 in cclb);
t:=List(u,a->[x,yrep[i],zrep[j],a,(x*yrep[i]*zrep[j]*a)∧-1]);
return t; end;od;
return Union(List([1..Size(zrep)],h1));end;
return Union(List([1..Size(yrep)],h));end;
>L:=List(ltuples,t->gt(t));;Size(last);
>Fltuples:=Filtered(L,x->x<>[ ]);;Size(last);
>for xx in Fltuples do Print("&",List ( List(xx[1], y ->Filtered(rr,r-> y in Conjugacy-
Class(g,r)) ),z -> cn(z[1])) , " & " , g21A(xx[1]), " & ",g21B(xx[1]), " & ", g56(xx[1]), " &
", g120(xx[1]), " & ", g280(xx[1]), "\ n"); od;

# filtered 6-tuples
>s6t:=UnorderedTuples(rr,6);;
>Fs6t:=Filtered(s6t,x->Size(Intersection(x,g1))<=4);;
>lgenus:=Filtered([1..Size(Fs6t)],i->Intersection([g21A(Fs6t[i]),g21B(Fs6t[i]),
g56(Fs6t[i]), g120(Fs6t[i]),g280(Fs6t[i])],lg)<> [ ] and g21A(Fs6t[i]) in NonnegativeInte-
```

```
gers and g21B(Fs6t[i]) in NonnegativeIntegers and g56(Fs6t[i]) in NonnegativeIntegers and
g120(Fs6t[i]) in NonnegativeIntegers and g280(Fs6t[i]) in NonnegativeIntegers );
>ltuples:=List(lgenus,i->Fs6t[i]);;
>for i in [1..Size(ltuples)]
do Sort(ltuples[i], function(v,w) return Order(v)<Order(w);end);od;
>Sort(ltuples, function(v,w)
if Order(v[1])<Order(w[1]) then return Order(v[1])<Order(w[1]) ;fi; if Order(v[1])=Order(w[1])
then return Position(r,v[1])<Position(r,w[1]);fi; end);
for t in ltuples do Print([cn(t[1]),cn(t[2]),cn(t[3]),cn(t[4]),cn(t[5]),cn(t[6])],"\ n");od;
>gt:=function(t)
local x, cx, ccly,cclz, ccla, cclb,cclc,orbsy, yrep,h;
x:=t[1]; cx:=Centralizer(g,x);
ccly:=ConjugacyClass(g,t[2]);
cclz:=ConjugacyClass(g,t[3]);
ccla:=ConjugacyClass(g,t[4]);
cclb:=ConjugacyClass(g,t[5]);
cclc:=ConjugacyClass(g,t[6]);
orbsy:=OrbitsDomain(cx,ccly); yrep:=List(orbsy,y->y[1]);;
h:=function(i)
local cy,cxy,orbsz,zrep,j,h1;
cy:=Centralizer(g,yrep[i]);
cxy:=Intersection(cx,cy);
orbsz:=OrbitsDomain(cxy,cclz);
zrep:=List(orbsz,z->z[1]);
for j in [1..Size(zrep)] do
h1:=function(j)
```

```
local cz,cxyz,orbsa,arep,k,h11;
cz:=Centralizer(g,zrep[j]);
cxyz:=Intersection(cxy,cz);
orbsa:=OrbitsDomain(cxyz,ccla);
arep:=List(orbsa,a->a[1]);
for k in [1..Size(arep)] do h11:=function(k) local ca,cxyza,orbsb,brep,u,t;
ca:=Centralizer(g,arep[k]);
cxyza:=Intersection(cxyz,ca);
orbsb:=OrbitsDomain(cxyza,cclb);
brep:=List(orbsb,b->b[1]);
u:=Filtered(brep,b->(x*yrep[i]*zrep[j]*arep[k]*b)∧-1 in cclc
and g=Group(x,yrep[i],zrep[j],arep[k],b));
t:=List(u,b->[x,yrep[i],zrep[j],arep[k],b,(x*yrep[i]*zrep[j]*arep[k]*b)∧-1]);
return t;end;od;
return Union(List([1..Size(arep)],h11));end;od;
return Union(List([1..Size(zrep)],h1));end;
return Union(List([1..Size(yrep)],h));end;
>L:=List(ltuples,t->gt(t));;
>Fltuples:=Filtered(L,x->x<>[ ]);;Size(last);
>for xx in Fltuples do Print("&",List ( List(xx[1], y ->Filtered(rr,r-> y in Conjugacy-
Class(g,r)) ),z -> cn(z[1])) , " & " , g21A(xx[1]), " & ",g21B(xx[1]), " & ", g56(xx[1]), " &
", g120(xx[1]), " & ", g280(xx[1]), "\ n"); od;
```

When the order of the group is big, the above calculation is slow or does not work. Then filter by order of elements can be used. For example, check tuple of type [2A, 6F, 6G].

```
>x:=Random(2A);;cx:=Centralizer(g,x);;
```

```
>F1:=Filtered( 6,y->Order(x*y) = 6 );;
>orbsy:=OrbitsDomain(cx,F1);;
>yrep:=List(orbsy,y->y[1]);;
>F2:=Filtered(yrep,y->g=Group(x,y));;Size(last);
```

Another example for 4-tuple of type [2B, 2B, 2D, 4B] . Use Cartesian product.

```
>x:=Random(2B);;cx:=Centralizer(g,x);;
>orbsy:=OrbitsDomain(cx,2B);;
>yrep:=List(orbsy,y->y[1]);;
>CP:=Cartesian(yrep,2D);;
>F1:=Filtered(CP,y->Order(x*y[1]*y[2])= 4);;
>F2:=Filtered(F1,y->(x*y[1]*y[2])∧-1 in 4B);;
>F3:=Filtered(F2, y-> g=Group(x, y[1], y[2]));;Size(last);
```

APPENDIX B. List of Genus 0 ,1, 2 Generating Tuples

$PSL(3,3)$	Tuples	52	117	144	234
	[2A, 4A, 13A]	0	0	13	14
	[2A, 4A, 13B]	0	0	13	14
	[2A, 4A, 13C]	0	0	13	14
	[2A, 4A, 13D]	0	0	13	14
	[2A, 8A, 8B]	0	0	19	23
	[2A, 3B, 13A]	0	0	5	6
	[2A, 3B, 13B]	0	0	5	6
	[2A, 3B, 13C]	0	0	5	6
	[2A, 3B, 13D]	0	0	5	6
	[2A, 6A, 13A]	0	0	19	25
	[2A, 6A, 13B]	0	0	19	25
	[2A, 6A, 13C]	0	0	19	25
	[2A, 6A, 13D]	0	0	19	25
	[3B, 4A, 4A]	0	0	11	14
	[3B, 4A, 6A]	0	0	17	25
	[3B, 3B, 6A]	0	0	9	17
	[3B, 3B, 4A]	0	0	3	6
	[3B, 6A, 6A]	0	0	23	36
	[3A, 4A, 8A]	0	0	22	31
	[3A, 4A, 8B]	0	0	22	31
	[3A, 3B, 8A]	0	0	14	23
	[3A, 3B, 8B]	0	0	14	23
	[4A, 4A, 6A]	0	0	25	33
	[4A, 4A, 4A]	0	0	19	22
	[4A, 6A, 6A]	0	0	31	44
	[6A, 6A, 6A]	0	0	37	55
	[2A, 2A, 3A, 8A]	0	0	40	54

[2A, 2A, 3A, 8B]	0	0	40	54
[2A, 2A, 4A, 6A]	0	0	43	56
[2A, 2A, 6A, 6A]	0	0	49	67
[2A, 2A, 3B, 6A]	0	0	35	48
[2A, 2A, 4A, 4A]	0	0	37	45
[2A, 2A, 2A, 13A]	0	0	31	37
[2A, 2A, 2A, 13B]	0	0	31	37
[2A, 2A, 2A, 13C]	0	0	31	37
[2A, 2A, 2A, 13D]	0	0	31	37
[2A, 2A, 3B, 3B]	0	0	21	29
[2A, 2A, 3B, 4A]	0	0	29	37
[2A, 3A, 3A, 3B]	0	0	35	54
[2A, 3A, 3A, 4A]	0	0	43	62
[2A, 2A, 2A, 3A, 3A]	0	0	61	85
[2A, 2A, 2A, 2A, 6A]	0	0	61	79
[2A, 2A, 2A, 2A, 3B]	0	0	47	60
[2A, 2A, 2A, 2A, 4A]	0	0	55	68
[2A, 2A, 2A, 2A, 2A, 2A]	0	0	73	91
[2A, 8A, 13A]	1	1	22	30
[2A, 8A, 13B]	1	1	22	30
[2A, 8A, 13C]	1	1	22	30
[2A, 8A, 13D]	1	1	22	30
[2A, 8B, 13A]	1	1	22	30
[2A, 8B, 13B]	1	1	22	30
[2A, 8B, 13C]	1	1	22	30
[2A, 8B, 13D]	1	1	22	30
[3A, 4A, 13A]	1	1	25	38
[3A, 4A, 13B]	1	1	25	38
[3A, 4A, 13C]	1	1	25	38
[3A, 4A, 13D]	1	1	25	38
[3A, 3B, 13A]	1	1	17	30
[3A, 3B, 13B]	1	1	17	30

[3A, 3B, 13C]	1	1	17	30
[3A, 3B, 13D]	1	1	17	30
[3A, 6A, 13A]	1	1	31	49
[3A, 6A, 13B]	1	1	31	49
[3A, 6A, 13C]	1	1	31	49
[3A, 6A, 13D]	1	1	31	49
[3B, 4A, 8A]	1	1	20	30
[3B, 4A, 8B]	1	1	20	30
[3B, 3B, 8A]	1	1	12	22
[3B, 3B, 8B]	1	1	12	22
[3B, 6A, 8A]	1	1	26	41
[3B, 6A, 8B]	1	1	26	41
[4A, 4A, 8A]	1	1	28	38
[4A, 4A, 8B]	1	1	28	38
[4A, 6A, 8A]	1	1	34	49
[4A, 6A, 8B]	1	1	34	49
[6A, 6A, 8A]	1	1	40	60
[6A, 6A, 8B]	1	1	40	60
[2A, 3A, 6A, 6A]	1	1	61	91
[2A, 3A, 3B, 6A]	1	1	47	72
[2A, 3A, 3B, 3A]	1	1	33	53
[2A, 3A, 3B, 4A]	1	1	41	61
[2A, 3A, 4A, 6A]	1	1	55	80
[2A, 3A, 4A, 4A]	1	1	49	69
[2A, 2A, 3A, 13B]	1	1	43	61
[2A, 2A, 3A, 13A]	1	1	43	61
[2A, 2A, 3A, 13C]	1	1	43	61
[2A, 2A, 3A, 13D]	1	1	43	61
[2A, 2A, 6A, 8B]	1	1	52	72
[2A, 2A, 3B, 8A]	1	1	38	53
[2A, 2A, 3B, 8B]	1	1	38	53
[2A, 2A, 6A, 8A]	1	1	52	72

[2A, 2A, 4A, 8B]	1	1	46	61
[2A, 2A, 4A, 8A]	1	1	46	61
[2A, 2A, 2A, 3A, 6A]	1	1	73	103
[2A, 2A, 2A, 3A, 3B]	1	1	59	84
[2A, 2A, 2A, 3A, 4A]	1	1	67	92
[2A, 2A, 2A, 2A, 8A]	1	1	64	84
[2A, 2A, 2A, 2A, 8B]	1	1	64	84
[2A, 2A, 2A, 2A, 2A, 3A]	1	1	85	115
[2A, 13A, 13A]	2	2	25	37
[2A, 13A, 13B]	2	2	25	37
[2A, 13A, 13C]	2	2	25	37
[2A, 13A, 13D]	2	2	25	37
[2A, 13B, 13B]	2	2	25	37
[2A, 13B, 13C]	2	2	25	37
[2A, 13B, 13D]	2	2	25	37
[2A, 13C, 13C]	2	2	25	37
[2A, 13C, 13D]	2	2	25	37
[2A, 13D, 13D]	2	2	25	37
[3B, 4A, 13A]	2	2	23	37
[3B, 4A, 13B]	2	2	23	37
[3B, 4A, 13C]	2	2	23	37
[3B, 4A, 13D]	2	2	23	37
[3A, 8A, 13A]	2	2	34	54
[3A, 8A, 13B]	2	2	34	54
[3A, 8A, 13C]	2	2	34	54
[3A, 8A, 13D]	2	2	34	54
[3A, 8B, 13A]	2	2	34	54
[3A, 8B, 13B]	2	2	34	54
[3A, 8B, 13C]	2	2	34	54
[3A, 8B, 13D]	2	2	34	54
[3B, 8A, 8B]	2	2	29	46
[3B, 8A, 8A]	2	2	29	46

[3B, 8B, 8B]	2	2	29	46
[3B, 3B, 13A]	2	2	15	29
[3B, 3B, 13B]	2	2	15	29
[3B, 3B, 13C]	2	2	15	29
[3B, 3B, 13D]	2	2	15	29
[3B, 6A, 13A]	2	2	29	48
[3B, 6A, 13B]	2	2	29	48
[3B, 6A, 13C]	2	2	29	48
[3B, 6A, 13D]	2	2	29	48
[4A, 4A, 13A]	2	2	31	45
[4A, 4A, 13B]	2	2	31	45
[4A, 4A, 13C]	2	2	31	45
[4A, 4A, 13D]	2	2	31	45
[4A, 8A, 8A]	2	2	37	54
[4A, 8A, 8B]	2	2	37	54
[4A, 8B, 8B]	2	2	37	54
[4A, 6A, 13A]	2	2	37	56
[4A, 6A, 13B]	2	2	37	56
[4A, 6A, 13C]	2	2	37	56
[4A, 6A, 13D]	2	2	37	56
[6A, 8A, 8A]	2	2	43	65
[6A, 8A, 8B]	2	2	43	65
[6A, 8B, 8B]	2	2	43	65
[6A, 6A, 13A]	2	2	43	67
[6A, 6A, 13B]	2	2	43	67
[6A, 6A, 13C]	2	2	43	67
[6A, 6A, 13D]	2	2	43	67
[2A, 3A, 6A, 8B]	2	2	64	96
[2A, 3A, 6A, 8A]	2	2	64	96
[2A, 3A, 3B, 8A]	2	2	50	77
[2A, 3A, 3B, 8B]	2	2	50	77
[2A, 3A, 4A, 8A]	2	2	58	85

[2A, 3A, 4A, 8B]	2	2	58	85
[2A, 3A, 3A, 13A]	2	2	55	85
[2A, 3A, 3A, 13B]	2	2	55	85
[2A, 3A, 3A, 13C]	2	2	55	85
[2A, 3A, 3A, 13D]	2	2	55	85
[2A, 3B, 6A, 6A]	2	2	59	90
[2A, 3B, 3B, 3B]	2	2	31	52
[2A, 4A, 6A, 6A]	2	2	67	98
[2A, 6A, 6A, 6A]	2	2	73	109
[2A, 2A, 6A, 13B]	2	2	55	79
[2A, 2A, 6A, 13C]	2	2	55	79
[2A, 2A, 6A, 13D]	2	2	55	79
[2A, 2A, 6A, 13A]	2	2	55	79
[2A, 2A, 3B, 13A]	2	2	41	60
[2A, 2A, 3B, 13B]	2	2	41	60
[2A, 2A, 3B, 13C]	2	2	41	60
[2A, 2A, 3B, 13D]	2	2	41	60
[2A, 2A, 4A, 13A]	2	2	49	68
[2A, 2A, 4A, 13B]	2	2	49	68
[2A, 2A, 4A, 13C]	2	2	49	68
[2A, 2A, 4A, 13D]	2	2	49	68
[2A, 2A, 8A, 8A]	2	2	55	77
[2A, 2A, 8A, 8B]	2	2	55	77
[2A, 2A, 8B, 8B]	2	2	55	77
[2A, 3B, 4A, 4A]	2	2	47	68
[2A, 3B,3B, 6A]	2	2	45	71
[2A, 3B, 4A, 6A]	2	2	53	79
[2A, 3B, 3B, 4A]	2	2	39	60
[2A, 4A, 4A, 6A]	2	2	61	87
[2A, 4A, 4A, 4A]	2	2	55	76
[3A, 3A, 4A, 6A]	2	2	67	104
[3A, 3A, 3B, 3B]	2	2	45	77

	[3A, 3A, 3B, 4A]	2	2	53	85
	[3A, 3A, 4A, 4A]	2	2	61	93
	[3A, 3A, 3A, 8A]	2	2	64	102
	[3A, 3A, 3A, 8B]	2	2	64	102
	[3A, 3A, 6A, 6A]	2	2	73	115
	[3A, 3A, 3B, 6A]	2	2	59	96
	[2A, 3A, 3A, 3A, 3A]	2	2	85	133
	[2A, 2A, 3A, 3A, 6A]	2	2	85	127
	[2A, 2A, 3A, 3A, 3B]	2	2	71	108
	[2A, 2A, 3A, 3A, 4A]	2	2	79	116
	[2A, 2A, 2A, 3A, 8A]	2	2	76	108
	[2A, 2A, 2A, 3A, 8B]	2	2	76	108
	[2A, 2A, 2A, 6A, 6A]	2	2	85	121
	[2A, 2A, 2A, 3B, 6A]	2	2	71	102
	[2A, 2A, 2A, 4A, 6A]	2	2	79	110
	[2A, 2A, 2A, 3B, 3B]	2	2	57	83
	[2A, 2A, 2A, 3B, 4A]	2	2	65	91
	[2A, 2A, 2A, 4A, 4A]	2	2	73	99
	[2A, 2A, 2A, 2A, 13A]	2	2	67	91
	[2A, 2A, 2A, 2A, 13B]	2	2	67	91
	[2A, 2A, 2A, 2A, 13C]	2	2	67	91
	[2A, 2A, 2A, 2A, 13D]	2	2	67	91
	[2A, 2A, 2A, 2A, 3A, 3A]	2	2	97	139
	[2A, 2A, 2A, 2A, 2A, 6A]	2	2	97	133
	[2A, 2A, 2A, 2A, 2A, 3B]	2	2	83	114
	[2A, 2A, 2A, 2A, 2A, 4A]	2	2	91	122
	[2A, 2A, 2A, 2A, 2A, 2A, 2A]	2	2	109	145

$PSL(3,3).2$	Tuples	52	117	144	234
	[2B, 4B, 8A]	0	3	8	8
	[2B, 3B, 8C]	0	0	0	1

[2B, 3B, 12A]	0	2	3	6
[2B, 3B, 12B]	0	2	3	6
[2B, 4A, 6B]	0	1	3	4
[2A, 2B, 2B, 3B]	0	3	7	10
[2A, 6B, 6B]	1	6	9	13
[2B, 4A, 8C]	1	3	8	9
[2B, 6A, 6B]	1	5	9	15
[2B, 4A, 12A]	1	5	11	14
[2B, 4A, 12B]	1	5	11	14
[4B, 4B, 4A]	1	10	19	20
[2A, 2B, 2B, 4A]	1	6	15	18
[2A, 6B, 8C]	2	8	14	18
[2A, 6B, 12A]	2	10	17	23
[2A, 6B, 12B]	2	10	17	23
[2B, 4B, 13B]	2	7	11	15
[2B, 4B, 13A]	2	7	11	15
[2B, 6A, 8C]	2	7	14	20
[2B, 6A, 12A]	2	9	17	25
[2B, 6A, 12B]	2	9	17	25
[2A, 2B, 2B, 6A]	2	10	21	29
[2A, 2A, 2B, 6B]	2	11	21	27
[2B, 2B, 2B, 4B]	2	7	13	19

$PSL(3,4)$ Tuples	21A	21B	56A	56B	56C	120A	120B	120C	280
[2A, 4A, 7A]	0	0	0	1	1	4	3	4	12
[2A, 4A, 7B]	0	0	0	1	1	4	3	4	12
[2A, 4B, 7A]	0	0	1	0	1	3	4	4	12
[2A, 4B, 7B]	0	0	1	0	1	3	4	4	12
[2A, 4C, 7A]	0	0	1	1	0	4	4	3	12
[2A, 4C, 7B]	0	0	1	1	0	4	4	3	12
[2A, 5A, 5B]	0	0	1	1	1	5	5	5	13

[3A, 4A, 4B]	0	0	2	2	3	7	7	8	20
[3A, 4A, 4C]	0	0	2	3	2	8	7	7	20
[3A, 4B, 4C]	0	0	3	2	2	7	8	7	20
[3A, 3A, 5A]	0	0	3	3	3	7	7	7	19
[3A, 3A, 5B]	0	0	3	3	3	7	7	7	19
[2A, 2A, 2A, 5A]	0	0	3	3	3	13	13	13	37
[2A, 2A, 2A, 5B]	0	0	3	3	3	13	13	13	37
[2A, 2A, 3A, 3A]	0	0	5	5	5	15	15	15	43
[2A, 2A, 2A, 2A, 2A]	0	0	5	5	5	21	21	21	61
[2A, 5A, 7A]	1	1	3	3	3	8	8	8	21
[2A, 5A, 7B]	1	1	3	3	3	8	8	8	21
[2A, 5B, 7A]	1	1	3	3	3	8	8	8	21
[2A, 5B, 7B]	1	1	3	3	3	8	8	8	21
[3A, 4A, 5A]	1	1	4	5	5	12	11	12	29
[3A, 4A, 5B]	1	1	4	5	5	12	11	12	29
[3A, 4B, 5A]	1	1	5	4	5	11	12	12	29
[3A, 4B, 5B]	1	1	5	4	5	11	12	12	29
[3A, 4C, 5A]	1	1	5	5	4	12	12	11	29
[3A, 4C, 5B]	1	1	5	5	4	12	12	11	29
[3A, 3A, 7A]	1	1	5	5	5	10	10	10	27
[3A, 3A, 7B]	1	1	5	5	5	10	10	10	27
[4A, 4A, 4B]	1	1	3	4	5	12	11	13	30
[4A, 4A, 4C]	1	1	3	5	4	13	11	12	30
[4A, 4B, 4B]	1	1	4	3	5	11	12	13	30
[4A, 4B, 4C]	1	1	4	4	4	12	12	12	30
[4A, 4C, 4C]	1	1	4	5	3	13	12	11	30
[4B, 4B, 4C]	1	1	5	3	4	11	13	12	30
[4B, 4C, 4C]	1	1	5	4	3	12	13	11	30
[2A, 2A, 2A, 7A]	1	1	5	5	5	16	16	16	45
[2A, 2A, 2A, 7B]	1	1	5	5	5	16	16	16	45
[2A, 2A, 3A, 4A]	1	1	6	7	7	20	19	20	53
[2A, 2A, 3A, 4B]	1	1	7	6	7	19	20	20	53

[2A, 2A, 3A, 4C]	1	1	7	7	6	20	20	19	53
[2A, 7A, 7A]	2	2	5	5	5	11	11	11	29
[2A, 7A, 7B]	2	2	5	5	5	11	11	11	29
[2A, 7B, 7B]	2	2	5	5	5	11	11	11	29
[3A, 4A, 7A]	2	2	6	7	7	15	14	15	37
[3A, 4A, 7B]	2	2	6	7	7	15	14	15	37
[3A, 4B, 7A]	2	2	7	6	7	14	15	15	37
[3A, 4B, 7B]	2	2	7	6	7	14	15	15	37
[3A, 4C, 7A]	2	2	7	7	6	15	15	14	37
[3A, 4C, 7B]	2	2	7	7	6	15	15	14	37
[3A, 5A, 5A]	2	2	7	7	7	16	16	16	38
[3A, 5A, 5B]	2	2	7	7	7	16	16	16	38
[3A, 5B, 5B]	2	2	7	7	7	16	16	16	38
[4A, 4A, 5A]	2	2	5	7	7	17	15	17	39
[4A, 4A, 5B]	2	2	5	7	7	17	15	17	39
[4A, 4B, 5A]	2	2	6	6	7	16	16	17	39
[4A, 4B, 5B]	2	2	6	6	7	16	16	17	39
[4A, 4C, 5A]	2	2	6	7	6	17	16	16	39
[4A, 4C, 5B]	2	2	6	7	6	17	16	16	39
[4B, 4B, 5A]	2	2	7	5	7	15	17	17	39
[4B, 4B, 5B]	2	2	7	5	7	15	17	17	39
[4B, 4C, 5A]	2	2	7	6	6	16	17	16	39
[4B, 4C, 5B]	2	2	7	6	6	16	17	16	39
[4C, 4C, 5A]	2	2	7	7	5	17	17	15	39
[4C, 4C, 5B]	2	2	7	7	5	17	17	15	39
[2A, 2A, 3A, 5A]	2	2	9	9	9	24	24	24	62
[2A, 2A, 3A, 5B]	2	2	9	9	9	24	24	24	62
[2A, 2A, 4A, 4A]	2	2	7	9	9	25	23	25	63
[2A, 2A, 4A, 4B]	2	2	8	8	9	24	24	25	63
[2A, 2A, 4B, 4B]	2	2	9	7	9	23	25	25	63
[2A, 2A, 4B, 4C]	2	2	9	8	8	24	25	24	63
[2A, 2A, 4C, 4C]	2	2	9	9	7	25	25	23	63

[2A, 2A, 4A, 4C]	2	2	8	9	8	25	24	24	63
[2A, 3A, 3A, 3A]	2	2	11	11	11	26	26	26	68
[2A, 2A, 2A, 2A, 3A]	2	2	11	11	11	32	32	32	86

$PSL(3,4).2_1$ Tuples	56A	56B	56C	105	120A	120B	120C	280	336
[2B, 4B, 5A]	0	0	0	0	0	0	0	3	3
[2B, 4D, 8C]	3	2	2	3	4	3	3	13	16
[2B, 4A, 8C]	2	2	3	3	3	4	3	13	16
[2B, 4C, 8A]	3	2	2	3	4	3	3	13	16
[2B, 4A, 8A]	2	3	2	3	3	3	4	13	16
[2B, 4C, 8B]	2	2	3	3	3	4	3	13	16
[2B, 4D, 8B]	2	3	2	3	3	3	4	13	16
[2B, 4B, 7A]	2	2	2	3	3	3	3	11	13
[4B, 4B, 4A]	2	3	3	10	12	13	13	29	35
[4C, 4B, 4B]	3	2	3	10	13	13	12	29	35
[4D, 4B, 4B]	3	3	2	10	13	12	13	29	35
[2A, 2A, 2B, 4B]	2	2	2	6	8	8	8	27	29

$PSL(3,4).2_2$ Tuples	21A	21B	56	120	280
[2B, 5A, 8A]	0	0	1	5	17
[2B, 4A, 14B]	0	0	1	4	16
[2B, 4A, 14A]	0	0	1	4	16
[2B, 6A, 7B]	0	0	1	4	18
[2B, 6A, 7A]	0	0	1	4	18
[2A, 6A, 8A]	0	0	2	8	24
[2A, 4C, 14B]	0	0	2	7	22
[2A, 4C, 14A]	0	0	2	7	22
[3A, 4C, 6A]	0	0	4	11	32
[4C, 4C, 4A]	0	0	4	12	32
[2B, 2B, 3A, 4A]	0	0	4	13	43

[2A, 2B, 3A, 4C]	0	0	5	16	49
[2A, 2A, 2B, 8A]	0	0	3	13	41
[2A, 2B, 2B, 7A]	0	0	2	9	35
[2A, 2B, 2B, 7B]	0	0	2	9	35
[2B, 7A, 8A]	1	1	3	8	25
[2B, 5A, 14A]	1	1	3	8	25
[2B, 7B, 8A]	1	1	3	8	25
[2B, 5A, 14B]	1	1	3	8	25
[2A, 6A, 14B]	1	1	4	11	32
[2A, 8A, 8A]	1	1	4	12	31
[2A, 6A, 14A]	1	1	4	11	32
[3A, 4C, 8A]	1	1	6	15	39
[3A, 6A, 6A]	1	1	6	15	42
[4C, 4C, 5A]	1	1	6	16	41
[4B, 4C, 6A]	1	1	5	15	42
[4C, 4A, 6A]	1	1	6	16	42
[2A, 2B, 3A, 6A]	1	1	7	20	59
[2B, 2B, 3A, 5A]	1	1	6	17	52
[2A, 2A, 2B, 14B]	1	1	5	16	49
[2B, 2B, 4C, 6A]	1	1	7	21	65
[2B, 2B, 4A, 4A]	1	1	6	18	53
[2B, 2B, 4A, 4B]	1	1	5	17	53
[2A, 2B, 4C, 4A]	1	1	7	21	59
[2A, 2B, 4B, 4C]	1	1	6	20	59
[2A, 2A, 2B, 14A]	1	1	5	16	49
[2A, 2A, 4C, 4C]	1	1	8	24	65
[2A, 2A, 2B, 2B, 3A]	1	1	8	25	76
[2B, 2B, 2B, 2B, 4A]	1	1	7	23	76
[2A, 2B, 2B, 2B, 4C]	1	1	8	26	82
[2B, 7B, 14B]	2	2	5	11	33
[2B, 7A, 14B]	2	2	5	11	33
[2B, 7B, 14A]	2	2	5	11	33

[2B, 7A, 14A]	2	2	5	11	33
[2A, 8A, 14B]	2	2	6	15	39
[2A, 8A, 14A]	2	2	6	15	39
[3A, 6A, 8A]	2	2	8	19	49
[3A, 4C, 14B]	2	2	8	18	47
[3A, 4C, 14A]	2	2	8	18	47
[4C, 4C, 7A]	2	2	8	19	49
[4C, 4C, 7B]	2	2	8	19	49
[4C, 5A, 6A]	2	2	8	20	51
[4B, 4C, 8A]	2	2	7	19	49
[4B, 6A, 6A]	2	2	7	19	52
[4C, 4A, 8A]	2	2	8	20	49
[4A, 6A, 6A]	2	2	8	20	52
[2B, 3A, 3A, 4C]	2	2	11	27	74
[2A, 2B, 3A, 8A]	2	2	9	24	66
[2B, 2B, 3A, 7B]	2	2	8	20	60
[2B, 2B, 3A, 7A]	2	2	8	20	60
[2B, 2B, 4C, 8A]	2	2	9	25	72
[2A, 2B, 4B, 6A]	2	2	8	24	69
[2B, 2B, 6A, 6A]	2	2	9	25	75
[2A, 2B, 4A, 6A]	2	2	9	25	69
[2B, 2B, 4A, 5A]	2	2	8	22	62
[2B, 2B, 4B, 5A]	2	2	7	21	62
[2A, 2B, 4C, 5A]	2	2	9	25	68
[2A, 2A, 4C, 6A]	2	2	10	28	75
[2A, 2B, 2B, 2B, 6A]	2	2	10	30	92
[2B, 2B, 2B, 2B, 5A]	2	2	9	27	85
[2A, 2A, 2B, 2B, 4A]	2	2	10	30	86
[2A, 2A, 2B, 2B, 4B]	2	2	9	29	86
[2A, 2A, 2A, 2B, 4C]	2	2	11	33	92
[2A,2A,2B,2B,2B,2B]	2	2	11	35	109

$PSL(3,4).2_3$	Tuples	56	105	120	280	336
	[2B, 3A, 10B]	0	1	1	7	6
	[2B, 3A, 10A]	0	1	1	7	6
	[2B, 4B, 6A]	0	1	1	7	7
	[2A, 2B, 2B, 3A]	0	3	3	17	16
	[2B, 4A, 10A]	1	6	5	17	19
	[2B, 4A, 10B]	1	6	5	17	19
	[2B, 4B, 8B]	1	4	3	13	15
	[2A, 2B, 2B, 4A]	1	8	7	27	29
	[2A, 6A, 8A]	2	7	9	25	29
	[2A, 6A, 6A]	2	4	6	20	21
	[2B, 5B, 6A]	2	5	5	16	17
	[2B, 5B, 8A]	2	8	8	21	25
	[2B, 5A, 8A]	2	8	8	21	25
	[2B, 4B, 10A]	2	6	6	17	19
	[2B, 4B, 10B]	2	6	6	17	19
	[2B, 5A, 6A]	2	5	5	16	17
	[2A, 2B, 2B, 4B]	2	8	8	27	29

$PGL(3,4)$	Tuples	21A	21B	280	960
	[2A, 6A, 15A]	0	0	33	129
	[3B, 5A, 6A]	0	0	37	145
	[2A, 6A, 15C]	0	0	33	129
	[2A, 3E, 15D]	0	0	10	44
	[2A, 3D, 15C]	0	0	10	44
	[2A, 6B, 15D]	0	0	33	129
	[2A, 3E, 15B]	0	0	10	44
	[3C, 3E, 5A]	0	0	14	60
	[3B, 5B, 6A]	0	0	37	145
	[2A, 3D, 15A]	0	0	10	44
	[3C, 5B, 6B]	0	0	37	145

[3C, 3E, 5B]	0	0	14	60
[3C, 5A, 6B]	0	0	37	145
[2A, 6B, 15B]	0	0	33	129
[3A, 3D, 6B]	0	0	19	74
[3B, 3D, 5A]	0	0	14	60
[3A, 6A, 6B]	0	0	42	159
[3A, 3E, 6A]	0	0	19	74
[3B, 3D, 5B]	0	0	14	60
[2A, 2A, 3B, 6A]	0	0	61	241
[2A, 2A, 3C, 6B]	0	0	61	241
[2A, 2A, 3B, 3D]	0	0	38	156
[2A, 2A, 3C, 3E]	0	0	38	156
[2A, 3D, 21C]	1	1	13	52
[2A, 6A, 21C]	1	1	36	137
[3C, 6A, 15D]	1	1	55	209
[2A, 3D, 21B]	1	1	13	52
[3D, 6A, 6A]	1	1	40	156
[3C, 3D, 15D]	1	1	32	124
[3C, 6B, 7B]	1	1	45	172
[2A, 6A, 21B]	1	1	36	137
[3B, 3E, 15A]	1	1	32	124
[2A, 6B, 21D]	1	1	36	137
[2A, 6B, 21A]	1	1	36	137
[3C, 3E, 7A]	1	1	22	87
[2A, 3E, 21D]	1	1	13	52
[3A, 3B, 21D]	1	1	37	135
[3E, 6B, 6B]	1	1	40	156
[3C, 3D, 15B]	1	1	32	124
[3C, 4A, 15A]	1	1	44	169
[3C, 4A, 15C]	1	1	44	169
[3C, 6A, 15B]	1	1	55	209
[6A, 6A, 6A]	1	1	63	241

[3B, 4A, 15D]	1	1	44	169
[3B, 6B, 15A]	1	1	55	209
[2A, 3E, 21A]	1	1	13	52
[3C, 3A, 21B]	1	1	37	135
[3B, 4A, 15B]	1	1	44	169
[3B, 3D, 7A]	1	1	22	87
[3B, 3E, 15C]	1	1	32	124
[3B, 3D, 7B]	1	1	22	87
[3E, 4A, 6A]	1	1	29	116
[3C, 3E, 7B]	1	1	22	87
[3D, 3D, 6A]	1	1	17	71
[3B, 6A, 7B]	1	1	45	172
[3C, 6B, 7A]	1	1	45	172
[3E, 3E, 6B]	1	1	17	71
[3B, 6A, 7A]	1	1	45	172
[4A, 6A, 6B]	1	1	52	201
[3A, 3B, 21A]	1	1	37	135
[3B, 6B, 15C]	1	1	55	209
[3D, 4A, 6B]	1	1	29	116
[6B, 6B, 6B]	1	1	63	241
[3E, 3D, 4A]	1	1	6	31
[3C, 3A, 21C]	1	1	37	135
[2A, 3C, 3C, 3D]	1	1	60	236
[2A, 3B, 3B, 6B]	1	1	83	321
[2A, 3B, 3B, 3E]	1	1	60	236
[2A, 3B, 3C, 4A]	1	1	72	281
[2A, 3C, 3C, 6A]	1	1	83	321
[2A, 15D, 15C]	2	2	49	177
[3C, 4A, 21B]	2	2	47	177
[2A, 15A, 15B]	2	2	49	177
[3B, 5A, 15D]	2	2	53	193
[3C, 6A, 21A]	2	2	58	217

[2A, 15C, 15B]	2	2	49	177
[3C, 3D, 21A]	2	2	35	132
[3C, 5B, 15A]	2	2	53	193
[3C, 5B, 15C]	2	2	53	193
[3B, 5B, 15B]	2	2	53	193
[3C, 4A, 21C]	2	2	47	177
[3B, 5B, 15D]	2	2	53	193
[2A, 15D, 15A]	2	2	49	177
[3C, 5A, 15C]	2	2	53	193
[3B, 3E, 21C]	2	2	35	132
[3B, 6B, 21B]	2	2	58	217
[3B, 6B, 21C]	2	2	58	217
[3B, 5A, 15B]	2	2	53	193
[3B, 3E, 21B]	2	2	35	132
[3B, 4A, 21D]	2	2	47	177
[3C, 3D, 21D]	2	2	35	132
[3C, 5A, 15A]	2	2	53	193
[3A, 6A, 15A]	2	2	58	207
[3A, 6B, 15B]	2	2	58	207
[3A, 6B, 15D]	2	2	58	207
[3A, 3D, 15C]	2	2	35	122
[3A, 3D, 15A]	2	2	35	122
[3E, 5B, 6A]	2	2	38	140
[3E, 3D, 5A]	2	2	15	55
[3B, 4A, 21A]	2	2	47	177
[3C, 6A, 21D]	2	2	58	217
[3E, 5A, 6A]	2	2	38	140
[3E, 3D, 5B]	2	2	15	55
[3A, 6A, 15C]	2	2	58	207
[3A, 3E, 15D]	2	2	35	122
[3D, 5B, 6B]	2	2	38	140
[5A, 6A, 6B]	2	2	61	225

	[5B, 6A, 6B]	2	2	61	225
	[3D, 5A, 6B]	2	2	38	140
	[3A, 3E, 15B]	2	2	35	122
	[2A, 2A, 3B, 15B]	2	2	77	289
	[2A, 2A, 3C, 15C]	2	2	77	289
	[2A, 2A, 3E, 3D]	2	2	39	151
	[2A, 2A, 3C, 15A]	2	2	77	289
	[2A, 2A, 3B, 15D]	2	2	77	289
	[2A, 2A, 3D, 6B]	2	2	62	236
	[2A, 3C, 3A, 3E]	2	2	63	234
	[2A, 2A, 6A, 6B]	2	2	85	321
	[2A, 2A, 3E, 6A]	2	2	62	236
	[2A, 3C, 3A, 6B]	2	2	86	319
	[2A, 3B, 3C, 5A]	2	2	81	305
	[2A, 3A, 3B, 6A]	2	2	86	319
	[2A, 3A, 3B, 3D]	2	2	63	234
	[2A, 3B, 3C, 5B]	2	2	81	305
	[3C, 3C, 3C, 4A]	2	2	94	361
	[3B, 3B, 3C, 3D]	2	2	82	316
	[3B, 3B, 3B, 4A]	2	2	94	361
	[3B, 3B, 3C, 6A]	2	2	105	401
	[3B, 3C, 3C, 6B]	2	2	105	401
	[3B, 3C, 3C, 3E]	2	2	82	316
	[3A, 3A, 3B, 3C]	2	2	87	317
	[2A, 2A, 2A, 3B, 3C]	2	2	105	401

$PSL(3,4).6$	Tuples	105	280	336	960
	[2B, 3B, 12A]	1	6	9	30
	[2B, 3C, 12B]	1	6	9	30

$PSL(3,4).2^2$	Tuples	56	105	120	280	336
	[2D, 4C, 8A]	0	4	4	12	14
	[2D, 4D, 8E]	1	3	4	14	14
	[2C, 6C, 8E]	2	4	6	21	26
	[2C, 8E, 8D]	2	7	9	26	34
	[2D, 4C, 14A]	2	7	7	20	23
	[2D, 6A, 8B]	2	7	7	24	28
	[2D, 6A, 6B]	2	4	4	18	20
	[2D, 2D, 2C, 6A]	2	8	8	33	39
	[2D, 2B, 2C, 4C]	2	4	4	22	27

$P\Gamma L(3,4)$	Tuples	21A	21B	280	960
	[2B, 3C, 14B]	0	0	4	39
	[2B, 3C, 14A]	0	0	4	39
	[2B, 6A, 15B]	0	0	28	125
	[3B, 6A, 6A]	0	0	39	157
	[2B, 6A, 15A]	0	0	28	125
	[2B, 4B, 21A]	0	0	21	95
	[2B, 4B, 21B]	0	0	21	95
	[4B, 4B, 6B]	0	0	43	161
	[2B, 6B, 14B]	0	0	27	124
	[2B, 6B, 14A]	0	0	27	124
	[3C, 4B, 4B]	0	0	20	76
	[2B, 2B, 2B, 14B]	0	0	39	200
	[2B, 2B, 2B, 14A]	0	0	39	200
	[2A, 2B, 2B, 15B]	0	0	45	205
	[2B, 2B, 3B, 5A]	0	0	49	221
	[2A, 2B, 2B, 15A]	0	0	45	205
	[2B, 2B, 4B, 4B]	0	0	55	237
	[2A, 2B, 3B, 6A]	0	0	56	237
	[2B, 2B, 3A, 3C]	0	0	31	150

[2B, 2B, 3A, 6B]	0	0	54	235
[2B, 2B, 2B, 2B, 3A]	0	0	66	311
[2A, 2A, 2B, 2B, 3B]	0	0	73	317
[2B, 6A, 21A]	1	1	31	133
[2B, 8A, 15A]	1	1	35	147
[2B, 8A, 15B]	1	1	35	147
[3B, 4B, 14A]	1	1	44	166
[2B, 6A, 21B]	1	1	31	133
[3B, 4B, 14B]	1	1	44	166
[4B, 6A, 6B]	1	1	53	199
[3B, 6A, 8A]	1	1	46	179
[3C, 4B, 6A]	1	1	30	114
[2A, 2B, 2B, 21A]	1	1	48	213
[2A, 2B, 2B, 21B]	1	1	48	213
[2B, 2B, 3B, 15B]	1	1	67	285
[2B, 2B, 3C, 3C]	1	1	29	147
[2B, 2B, 3C, 4A]	1	1	41	192
[2B, 2B, 6B, 6B]	1	1	75	317
[2B, 2B, 3C, 6B]	1	1	52	232
[2B, 2B, 4A, 6B]	1	1	64	277
[2A, 2B, 3B, 8A]	1	1	63	259
[2B, 2B, 3B, 15A]	1	1	67	285
[2B, 2B, 3B, 7A]	1	1	57	248
[2A, 2B, 3C, 4B]	1	1	47	194
[2B, 2B, 3B, 7B]	1	1	57	248
[2A, 2B, 4B, 6B]	1	1	70	279
[2B, 2B, 4B, 6A]	1	1	65	275
[2B, 3A, 3B, 4B]	1	1	71	277
[2B, 3B, 3B, 6A]	1	1	78	317
[2B, 2B, 2B, 2B, 4A]	1	1	76	353
[2B, 2B, 2B, 2B, 6B]	1	1	87	393
[2B, 2B, 2B, 2B, 3C]	1	1	64	308

[2A, 2B, 2B, 3B, 3B]	1	1	95	397
[2A, 2B, 2B, 2B, 4B]	1	1	82	355
[2B, 2B, 2B, 2B, 2B, 2B]	1	1	99	469
[2B, 8A, 21A]	2	2	38	155
[2B, 14B, 15B]	2	2	43	172
[3B, 6A, 14A]	2	2	54	204
[2B, 14A, 15B]	2	2	43	172
[2B, 8A, 21B]	2	2	38	155
[2B, 14A, 15A]	2	2	43	172
[3C, 6A, 6A]	2	2	40	152
[2B, 14B, 15A]	2	2	43	172
[3B, 8A, 8A]	2	2	53	201
[3B, 6A, 14B]	2	2	54	204
[4B, 4B, 15A]	2	2	59	209
[3C, 4B, 8A]	2	2	37	136
[4B, 6B, 8A]	2	2	60	221
[4B, 4B, 15B]	2	2	59	209
[6A, 6A, 6B]	2	2	63	237
[2B, 2B, 6A, 6A]	2	2	75	313
[2B, 2B, 3A, 15A]	2	2	70	283
[2B, 2B, 4B, 8A]	2	2	72	297
[2A, 2B, 3C, 6A]	2	2	57	232
[2B, 2B, 5A, 6B]	2	2	73	301
[2A, 2B, 3B, 14B]	2	2	71	284
[2B, 2B, 3A, 15B]	2	2	70	283
[2A, 2B, 3B, 14A]	2	2	71	284
[2B, 2B, 3B, 21A]	2	2	70	293
[2A, 2B, 6A, 6B]	2	2	80	317
[2B, 2B, 3B, 21B]	2	2	70	293
[2B, 3A, 3B, 6A]	2	2	81	315
[2B, 3C, 3B, 4B]	2	2	69	274

	[2B, 3B, 3B, 8A]	2	2	85	339
	[2B, 2B, 3C, 5A]	2	2	50	216
	[2B, 3B, 4B, 4A]	2	2	81	319
	[2B, 3B, 4B, 6B]	2	2	92	359
	[2A, 3B, 4B, 4B]	2	2	87	321
	[2B, 2B, 2B, 2B, 5A]	2	2	85	377
	[2A, 2B, 2B, 2B, 6A]	2	2	92	393
	[2B, 2B, 2B, 3B, 4B]	2	2	104	435
	[2A, 2A, 2B, 2B, 6B]	2	2	97	397
	[2A, 2B, 2B, 3A, 3B]	2	2	98	395
	[2A, 2A, 2B, 2B, 3C]	2	2	74	312
	[2B, 2B, 3B, 3B, 3B]	2	2	117	477
	[2A, 2A, 2B, 2B, 2B, 2B]	2	2	469	473

$PSL(3,4).3.2_3$	Tuples	105	280	336	960
	[2B, 3C, 8A]	1	1	3	11

$PSL(3,4).D_{12}$	Tuples	105	280	336	960
	[2C, 4C, 6A]	1	7	7	31
	[2B, 2C, 2D, 3B]	1	10	15	63
	[2B, 6A, 6F]	2	14	19	71
	[2C, 6D, 6B]	2	16	17	72

$PSL(3,5)$	Tuples	31A	31B	3100	3875	4000
	[2A, 3A, 24A]	0	0	166	219	245
	[2A, 3A, 24B]	0	0	166	219	245
	[2A, 3A, 24C]	0	0	166	219	245

[2A, 3A, 24D]	0	0	166	219	245
[2A, 6A, 8A]	0	0	283	371	415
[2A, 6A, 8B]	0	0	283	371	415
[2A, 8A, 10A]	0	0	391	504	551
[2A, 8B, 10A]	0	0	391	504	551
[2A, 5B, 8B]	0	0	239	314	351
[2A, 5B, 8A]	0	0	239	314	351
[3A, 4C, 4C]	0	0	231	301	329
[3A, 3A, 4C]	0	0	114	147	157
[4C, 4C, 4C]	0	0	348	455	501
[2A, 2A, 2A, 8A]	0	0	501	664	751
[2A, 2A, 2A, 8B]	0	0	501	664	751
[2A, 3A, 31A]	1	1	183	240	264
[2A, 3A, 31B]	1	1	183	240	264
[2A, 3A, 31C]	1	1	183	240	264
[2A, 3A, 31D]	1	1	183	240	264
[2A, 3A, 31E]	1	1	183	240	264
[2A, 3A, 31F]	1	1	183	240	264
[2A, 3A, 31G]	1	1	183	240	264
[2A, 3A, 31H]	1	1	183	240	264
[2A, 3A, 31I]	1	1	183	240	264
[2A, 3A, 31J]	1	1	183	240	264
[2A, 4C, 31A]	1	1	300	394	436
[2A, 4C, 31B]	1	1	300	394	436
[2A, 4C, 31C]	1	1	300	394	436
[2A, 4C, 31D]	1	1	300	394	436
[2A, 4C, 31E]	1	1	300	394	436
[2A, 4C, 31F]	1	1	300	394	436
[2A, 4C, 31G]	1	1	300	394	436
[2A, 4C, 31H]	1	1	300	394	436
[2A, 4C, 31I]	1	1	300	394	436
[2A, 4C, 31J]	1	1	300	394	436

[2A, 8A, 12A]	1	1	417	535	583
[2A, 8A, 12B]	1	1	417	535	583
[2A, 8B, 12A]	1	1	417	535	583
[2A, 8B, 12B]	1	1	417	535	583
[2A, 8A, 20B]	1	1	471	602	651
[2A, 8B, 20A]	1	1	471	602	651
[3A, 4A, 6A]	1	1	365	462	493
[3A, 4A, 8A]	1	1	433	543	579
[3A, 4A, 5B]	1	1	321	405	429
[3A, 4B, 5B]	1	1	321	405	429
[3A, 4A, 10A]	1	1	473	595	629
[3A, 4B, 10A]	1	1	473	595	629
[3A, 4B, 8B]	1	1	433	543	579
[3A, 4B, 6A]	1	1	365	462	493
[4A, 4C, 8A]	1	1	550	697	751
[4B, 4C, 8B]	1	1	550	697	751
[2A, 2A, 4B, 5A]	1	1	791	1016	1101
[2A, 2A, 4A, 5A]	1	1	791	1016	1101
[2A, 2A, 3A, 4A]	1	1	583	755	829
[2A, 2A, 3A, 4B]	1	1	583	755	829
[2A, 6A, 24A]	2	2	416	537	581
[2A, 6A, 24B]	2	2	416	537	581
[2A, 6A, 24C]	2	2	416	537	581
[2A, 6A, 24D]	2	2	416	537	581
[2A, 8A, 24A]	2	2	484	618	667
[2A, 8A, 24C]	2	2	484	618	667
[2A, 8B, 24D]	2	2	484	618	667
[2A, 8B, 24B]	2	2	484	618	667
[2A, 5B, 24A]	2	2	372	480	517
[2A, 5B, 24B]	2	2	372	480	517
[2A, 5B, 24C]	2	2	372	480	517
[2A, 5B, 24D]	2	2	372	480	517

[2A, 10A, 24A]	2	2	524	670	717
[2A, 10A, 24D]	2	2	524	670	717
[2A, 10A, 24C]	2	2	524	670	717
[2A, 10A, 24B]	2	2	524	670	717
[2A, 12A, 20A]	2	2	537	685	733
[2A, 12A, 20B]	2	2	537	685	733
[2A, 12B, 20B]	2	2	537	685	733
[2A, 12B, 20A]	2	2	537	685	733
[2A, 12B, 12A]	2	2	483	618	665
[2A, 20A, 20B]	2	2	591	752	801
[3A, 3A, 10A]	2	2	355	444	457
[3A, 3A, 5B]	2	2	203	254	257
[3A, 3A, 8A]	2	2	315	392	407
[3A, 3A, 8B]	2	2	315	392	407
[3A, 4C, 6A]	2	2	364	465	493
[3A, 4C, 8B]	2	2	432	546	579
[3A, 4A, 20A]	2	2	553	693	729
[3A, 3A, 6A]	2	2	247	311	321
[3A, 4C, 8A]	2	2	432	546	579
[3A, 4B, 20B]	2	2	553	693	729
[3A, 4A, 12B]	2	2	499	626	661
[3A, 5A, 8A]	2	2	523	653	679
[3A, 4B, 12A]	2	2	499	626	661
[3A, 5A, 8B]	2	2	523	653	679
[3A, 4C, 5B]	2	2	320	408	429
[3A, 4C, 10A]	2	2	472	598	629
[4C, 4C, 8A]	2	2	549	700	751
[4C, 4C, 5B]	2	2	437	562	601
[4C, 4C, 6A]	2	2	481	619	665
[4C, 4C, 8B]	2	2	549	700	751
[4C, 4C, 10A]	2	2	589	752	801
[2A, 2A, 2A, 24A]	2	2	634	830	917

	[2A, 2A, 2A, 24B]	2	2	634	830	917
	[2A, 2A, 2A, 24C]	2	2	634	830	917
	[2A, 2A, 2A, 24D]	2	2	634	830	917
	[2A, 2A, 4C, 4C]	2	2	699	912	1001
	[2A, 2A, 3A, 4C]	2	2	582	758	829
	[2A, 2A, 3A, 3A]	2	2	465	604	657

$PSL(3,5).2$	Tuples	186	775	3100	3875	4000
	[2B, 3A, 8A]	1	8	51	65	69

$PSL(3,7)$ 57	0 (8)	1 (7)	2 (6)
	[2A, 4A, 7B]	[2A, 3A, 19A]	[2A, 4A, 16A]
	[2A, 4A, 7C]	[2A, 3A, 19B]	[2A, 4A, 16B]
	[2A, 4A, 7D]	[2A, 3A, 19C]	[2A, 4A, 16C]
	[2A, 4A, 8A]	[2A, 3A, 19D]	[2A, 4A, 16D]
	[2A, 4A, 8B]	[2A, 3A, 19E]	[3A, 3A, 6A]
	[2A, 4A, 14A]	[2A, 3A, 19F]	[3A, 4A, 4A]
	[3A, 3A, 4A]	[2A, 2A, 2A, 7A]	
	[2A, 2A, 2A, 4A]		

$PGL(3,7)$ 57	0 (2)	1 (12)	2 (11)
	[2A, 3A, 21B]	[2A, 3A, 24B]	[2A, 3A, 48A]
	[2A, 3B, 21A]	[2A, 3A, 24D]	[2A, 3A, 48D]
		[2A, 3A, 42B]	[2A, 3A, 48F]
		[2A, 3B, 24A]	[2A, 3A, 48G]
		[2A, 3B, 24C]	[2A, 3B, 48B]
		[2A, 3B, 42A]	[2A, 3B, 48C]
		[2A, 6C, 12A]	[2A, 3B, 48E]
		[2A, 6D, 12B]	[2A, 3B, 48H]

		[3C, 4A, 6C]	[3A, 3B, 4A]
		[3D, 4A, 6D]	[3A, 3E, 6D]
		[3C, 3A, 6E]	[3B, 3E, 6C]
		[3D, 3B, 6E]	
$PSL(3,8)$ 73	0 (0)	1 (12)	2 (1)
		[2A, 3A, 21A]	[3A, 3A, 4A]
		[2A, 3A, 21B]	
		[2A, 3A, 21C]	
		[2A, 3A, 21D]	
		[2A, 3A, 21E]	
		[2A, 3A, 21F]	
		[2A, 4A, 14A]	
		[2A, 4A, 14B]	
		[2A, 4A, 14C]	
		[2A, 4A, 14D]	
		[2A, 4A, 14E]	
		[2A, 4A, 14F]	
$P\Gamma L(3,8)$ 73	0 (0)	1 (0)	2 (2)
			[3B, 3C, 6B]
			[3A, 3C, 6A]
$PSL(3,9)$ 91	0 (0)	1 (4)	2 (5)
		[2A, 3B, 24A]	[2A, 4C, 16A]
		[2A, 3B, 24B]	[2A, 4C, 16B]
		[2A, 3B, 24C]	[2A, 4C, 16C]
		[2A, 3B, 24D]	[2A, 4C, 16D]

			[3B, 3B, 4C]

$P\Gamma L(3,9)$ 91	0 (0)	1 (2)	2 (1)
		[2B, 3B, 16A]	[2B, 5A, 6B]
		[2B, 3B, 16B]	

$PSL(3,11)$ 133	0 (0)	1 (0)	2 (8)
			[2A, 3A, 15A]
			[2A, 3A, 15B]
			[2A, 3A, 15C]
			[2A, 3A, 15D]
			[2A, 4A, 10E]
			[2A, 4A, 10F]
			[2A, 4A, 10G]
			[2A, 4A, 10H]

$PGL(3,13)$ 133	0 (0)	1 (4)	2 (4)
		[2A, 3D, 12H]	[2A, 3D, 12N]
		[2A, 3D, 12J]	[2A, 3D, 12P]
		[2A, 3E, 12G]	[2A, 3E, 12M]
		[2A, 3E, 12I]	[2A, 3E, 12O]

$PSp(4,3)$	Tuples	27	36	40A	40B	45
	[2A,6F,9A]	0	1	3	3	2
	[2A,6F,9B]	0	1	3	3	2
	[2A,6F,12A]	0	1	3	3	2
	[2A,6F,12B]	0	1	3	3	2

[2A,4B,9B]	0	0	0	0	0
[2A,4B,9A]	0	0	0	0	0
[2A,4A,12A]	0	0	2	1	1
[2A,4A,12B]	0	0	2	1	1
[2A,6B,9B]	0	1	2	2	1
[2A,6A,12A]	0	1	2	2	1
[2A,4A,9B]	0	0	2	1	1
[2A,4A,9A]	0	0	2	1	1
[2A,6B,12B]	0	1	2	2	1
[2A,5A,6D]	0	0	0	0	0
[2A,6A,9A]	0	1	2	2	1
[2A,5A,6C]	0	0	0	0	0
[2A,2A,2A,6D]	0	0	4	2	2
[2A,2A,2A,6C]	0	0	4	2	2
[2A,5A,9A]	1	2	3	3	3
[2A,5A,9B]	1	2	3	3	3
[2A,5A,12A]	1	2	3	3	3
[2A,5A,12B]	1	2	3	3	3
[3C,4A,9A]	1	3	4	5	4
[3C,4A,9B]	1	3	4	5	4
[3C,4A,12A]	1	3	4	5	4
[3C,4A,12B]	1	3	4	5	4
[3C,6F,12B]	1	4	5	7	5
[3C,6F,12A]	1	4	5	7	5
[3C,6F,9A]	1	4	5	7	5
[3C,6F,9B]	1	4	5	7	5
[3C,6B,9A]	1	4	4	6	4
[3C,6A,9B]	1	4	4	6	4
[3C,5A,6D]	1	3	2	4	3
[3C,5A,6C]	1	3	2	4	3
[3D,4B,6F]	1	1	2	1	2
[4A,4A,4B]	1	1	4	2	3

[4A,6B,6B]	1	3	6	5	4
[4A,6A,6A]	1	3	6	5	4
[4A,6F,6F]	1	3	8	7	6
[4A,4B,6F]	1	2	5	4	4
[4A,4A,4A]	1	1	6	3	4
[4A,6B,6F]	1	3	7	6	5
[4A,4A,6F]	1	2	7	5	5
[4A,4A,6B]	1	2	6	4	4
[4A,4A,6A]	1	2	6	4	4
[4A,4B,6A]	1	2	4	3	3
[4A,4B,6B]	1	2	4	3	3
[4A,6A,6F]	1	3	7	6	5
[4B,6F,6F]	1	3	6	6	5
[6F,6F,6A]	1	4	8	8	6
[6F,6F,6B]	1	4	8	8	6
[6F,6B,6A]	1	4	7	7	5
[6F,6F,6F]	1	4	9	9	7
[2A,2A,3C,6D]	1	3	6	6	5
[2A,2A,3C,6C]	1	3	6	6	5
[2A,2A,2A,12A]	1	2	7	5	5
[2A,2A,2A,12B]	1	2	7	5	5
[2A,2A,2A,9A]	1	2	7	5	5
[2A,2A,2A,9B]	1	2	7	5	5
[2A,6E,12B]	2	1	2	1	2
[2A,6E,12A]	2	1	2	1	2
[2A,6E,9A]	2	1	2	1	2
[2A,6E,9B]	2	1	2	1	2
[2A,6D,9A]	2	2	0	1	1
[2A,6C,9B]	2	2	0	1	1
[2B,5A,9A]	2	1	1	0	1
[2B,5A,9B]	2	1	1	0	1
[2B,5A,12A]	2	1	1	0	1

[2B,5A,12B]	2	1	1	0	1
[3A,4A,5A]	2	3	1	3	2
[3B,4A,5A]	2	3	1	3	2
[3A,5A,6F]	2	4	2	5	3
[3B,5A,6F]	2	4	2	5	3
[3C,5A,9A]	2	5	5	7	6
[3C,5A,9B]	2	5	5	7	6
[3C,5A,12A]	2	5	5	7	6
[3C,5A,12B]	2	5	5	7	6
[3D,5A,6A]	2	3	4	3	4
[3D,5A,6B]	2	3	4	3	4
[3D,4A,5A]	2	2	4	2	4
[3D,4B,5A]	2	2	2	1	3
[3D,5A,6F]	2	3	5	4	5
[4A,4A,5A]	2	3	7	5	6
[4A,5A,6A]	2	4	7	6	6
[4A,5A,6B]	2	4	7	6	6
[4A,4B,5A]	2	3	5	4	5
[4A,5A,6F]	2	4	8	7	7
[4B,5A,6A]	2	4	5	5	5
[4B,5A,6B]	2	4	5	5	5
[4B,5A,6F]	2	4	6	6	6
[4B,4B,5A]	2	3	3	3	4
[5A,6F,6B]	2	5	8	8	7
[5A,6F,6A]	2	5	8	8	7
[5A,6F,6F]	2	5	9	9	8
[5A,6B,6B]	2	5	7	7	6
[5A,6B,6A]	2	5	7	7	6
[5A,6A,6A]	2	5	7	7	6
[2A,2A,6F,6F]	2	5	13	11	10
[2A,2A,3A,6F]	2	4	6	7	5
[2A,2A,3B,6F]	2	4	6	7	5

[2A,2A,4A,6F]	2	4	12	9	9
[2A,2A,4B,6F]	2	4	10	8	8
[2A,2A,3D,6F]	2	3	9	6	7
[2A,2A,6B,6F]	2	5	12	10	9
[2A,2A,6F,6A]	2	5	12	10	9
[2A,2A,3C,12A]	2	5	9	9	8
[2A,2A,3C,12A]	2	5	9	9	8
[2A,2A,3A,4A]	2	3	5	5	4
[2A,2A,3A,6B]	2	4	5	6	4
[2A,2A,3B,4A]	2	3	5	5	4
[2A,2A,3B,6A]	2	4	5	6	4
[2A,2A,4A,4A]	2	3	11	7	8
[2A,2A,4A,4B]	2	3	9	6	7
[2A,2A,3D,4A]	2	2	8	4	6
[2A,2A,4A,6B]	2	4	11	8	8
[2A,2A,4A,6A]	2	4	11	8	8
[2A,2A,4B,4B]	2	3	7	5	6
[2A,2A,3D,4B]	2	2	6	3	5
[2A,2A,4B,6B]	2	4	9	7	7
[2A,2A,4B,6A]	2	4	9	7	7
[2A,2A,3C,12B]	2	5	9	9	8
[2A,2A,3C,9A]	2	5	9	9	8
[2A,2A,6B,6A]	2	5	11	9	8
[2A,2A,3D,6B]	2	3	8	5	6
[2A,2A,3D,6A]	2	3	8	5	6
[2A,2A,6B,6B]	2	5	11	9	8
[2A,2A,6A,6A]	2	5	11	9	8
[2A,3C,3C,6C]	2	6	8	10	8
[2A,3C,3C,6D]	2	6	8	10	8
[2A,2A,2B,12B]	2	1	5	2	3
[2A,2A,2B,12A]	2	1	5	2	3
[2A,2A,2B,9A]	2	1	5	2	3

[2A,2A,2B,9B]	2	1	5	2	3
[2B,9A,9A]	4	3	1	1	2
[2B,9A,12A]	4	3	1	1	2
[2B,9B,9B]	4	3	1	1	2
[2B,9B,12B]	4	3	1	1	2
[3A,4A,12B]	4	5	1	4	3
[3A,4A,9B]	4	5	1	4	3
[3A,5A,6E]	4	4	1	3	3
[3B,4A,12A]	4	5	1	4	3
[3B,4A,9A]	4	5	1	4	3
[3B,5A,6E]	4	4	1	3	3
[2B,2A,3B,4B]	3	2	1	1	1
[2B,2A,3A,4B]	3	2	1	1	1
[3A,9B,12A]	7	9	2	7	6
[3A,5A,9A]	5	7	2	6	5
[3A,5A,9B]	5	7	2	6	5
[3A,5A,12A]	5	7	2	6	5
[3A,5A,12B]	5	7	2	6	5
[3A,6F,9A]	4	6	2	6	4
[3A,9A,9A]	7	9	2	7	6
[3A,5A,5A]	3	5	2	5	4
[3B,5A,5A]	3	5	2	5	4
[3B,5A,9A]	5	7	2	6	5
[3B,5A,9B]	5	7	2	6	5
[3B,5A,12A]	5	7	2	6	5
[3B,5A,12B]	5	7	2	6	5
[3B,9B,9B]	7	9	2	7	6
[3B,6F,9B]	4	6	2	6	4
[3B,9A,12B]	7	9	2	7	6
[3C,6D,9B]	3	5	2	5	4
[3C,6C,9A]	3	5	2	5	4
[3D,6C,12B]	6	6	2	3	5

	[3D,6D,12A]	6	6	2	3	5
	[3D,4B,9A]	4	4	2	2	4
	[3D,4B,9B]	4	4	2	2	4
	[3D,5A,6C]	4	4	2	2	4
	[3D,5A,6D]	4	4	2	2	4
	[4A,4B,6C]	3	3	2	2	3
	[4A,4B,6D]	3	3	2	2	3
	[2A,3A,3A,6D]	8	10	2	8	6
	[2A,3B,3B,6C]	8	10	2	8	6
	[4A,4B,6E]	3	2	4	2	4
	[4A,4A,6E]	3	2	6	3	5
	[2A,2B,4A,4A]	3	2	9	4	6
	[2A,2B,4A,4B]	3	2	7	3	5
	[2B,2B,4B,6A]	4	2	5	1	3
	[2B,2B,4B,6B]	4	2	5	1	3
	[2B,2B,4B,6F]	4	2	6	2	4
	[3D,5A,6E]	4	3	4	2	5
	[4B,6E,6E]	5	3	4	2	5
	[2B,2B,4B,5A]	5	3	6	2	5
	[2B,3D,3D,3C]	4	3	5	2	5

$PSp(4,3).2$	Tuples	27	36	40A	40B	45
	[2A , 8A , 12A]	0	1	2	3	1
	[2A , 8A , 9A]	0	1	2	3	1
	[2A , 9A , 10A]	0	1	3	5	2
	[2A , 9A , 12B]	0	1	3	5	2
	[2A , 10A , 12A]	0	1	3	5	2
	[2B , 6E , 10A]	0	0	2	5	2
	[2B , 6E , 8A]	0	0	1	3	1
	[2D , 4B , 10A]	0	0	1	2	1
	[2D , 6D , 10A]	0	1	2	2	1
	[2D , 6C , 8A]	0	1	2	1	1

[2D , 4C , 10A]	0	0	0	0	0
[2D , 4B , 8A]	0	0	0	0	0
[2D , 4B , 12B]	0	0	1	2	1
[2D , 6C , 12B]	0	1	3	3	2
[2D , 6C , 10A]	0	1	3	3	2
[2D , 6F , 9A]	0	1	2	1	1
[2D , 4A , 9A]	0	1	1	0	0
[2D , 6F , 12A]	0	1	2	1	1
[2D , 4A , 12A]	0	1	1	0	0
[2D , 6D , 12B]	0	1	2	2	1
[2D , 5A , 6E]	0	0	0	1	0
[2A , 2A , 4C , 6A]	0	1	3	7	2
[2D , 2D , 2A , 10A]	0	1	3	5	2
[2D , 2B , 2A , 12A]	0	1	3	5	2
[2D , 2B , 2B , 6E]	0	0	2	5	2
[2D , 2B , 2A , 9A]	0	1	3	5	2
[2D , 2D , 2A , 8A]	0	1	2	3	1
[2D , 2D , 2D , 6F]	0	1	2	1	1
[2D , 2D , 2B , 6C]	0	1	3	3	2
[2D , 2D , 2B , 6D]	0	1	2	2	1
[2D , 2D , 2A , 12B]	0	1	3	5	2
[2D , 2D , 2D , 4A]	0	1	1	0	0
[2D , 2D , 2B , 4C]	0	0	0	0	0
[2D , 2D , 2B , 4B]	0	0	1	2	1
[2D, 2D, 2D, 2B, 2A]	0	1	3	5	2
[2B, 8A, 12B]	1	2	4	5	4
[2B, 8A, 8A]	1	2	3	3	3
[2B, 8A, 10A]	1	2	4	5	4
[2B, 12B, 12B]	1	2	5	7	5
[2B, 6B, 10A]	1	1	2	4	3
[2B, 6B, 12B]	1	1	2	4	3
[2B, 6B, 8A]	1	1	1	2	2

[2B, 10A, 12B]	1	2	5	7	5
[2B, 10A, 10A]	1	2	5	7	5
[2D, 5A, 6B]	1	1	0	0	1
[2D, 5A, 10A]	1	2	3	3	3
[2D, 5A, 12B]	1	2	3	3	3
[2D, 5A, 8A]	1	2	2	1	2
[3C, 6E, 10A]	1	3	6	7	5
[3C, 6E, 8A]	1	3	5	5	4
[4A, 4A, 6A]	1	3	4	4	3
[4A, 6F, 6A]	1	3	5	5	4
[4A, 6C, 6E]	1	3	6	8	5
[4B, 6F, 6E]	1	2	5	8	5
[4B, 4A, 6E]	1	2	4	7	4
[6A, 6F, 6F]	1	3	6	6	5
[6C, 6F, 6E]	1	3	7	9	6
[6D, 6F, 6E]	1	3	6	8	5
[2A, 2A, 4B, 12A]	1	3	7	12	6
[2A, 2D, 3C, 12A]	1	4	7	7	5
[2A, 2D, 4B, 6D]	1	3	6	9	5
[2A, 2D, 4B, 6C]	1	3	7	10	6
[2A, 2D, 4B, 4C]	1	2	4	7	4
[2A, 2D, 3C, 9A]	1	4	7	7	5
[2A, 2D, 6E, 6F]	1	3	7	11	6
[2A, 2D, 4A, 6E]	1	3	6	10	5
[2A, 2D, 4B, 4B]	1	2	5	9	5
[2A, 3A, 3C, 4D]	1	3	7	7	4
[2A, 2A, 6C, 12A]	1	4	9	13	7
[2A, 2A, 6E, 10A]	1	3	8	15	7
[2A, 2A, 6E, 8A]	1	3	7	13	6
[2A, 2A, 4B, 9A]	1	3	7	12	6
[2A, 2D, 6D, 6D]	1	4	7	9	5
[2A, 2D, 6C, 6D]	1	4	8	10	6

[2A, 2D, 4C, 6D]	1	3	5	7	4
[2A, 2D, 4C, 6C]	1	3	6	8	5
[2A, 2B, 4B, 6E]	1	2	6	12	6
[2A, 2B, 6D, 6E]	1	3	7	12	6
[2A, 2B, 4A, 6A]	1	3	6	9	5
[2A, 2B, 6C, 6E]	1	3	8	13	7
[2A, 2A, 6D, 12A]	1	4	8	12	6
[2A, 2A, 6D, 9A]	1	4	8	12	6
[2A, 2A, 6C, 9A]	1	4	9	13	7
[2A, 2D, 3B, 4C]	1	1	1	4	2
[2A, 2D, 6C, 6C]	1	4	9	11	7
[2A, 2B, 6A, 6F]	1	3	7	10	6
[2A, 2A, 5A, 6A]	1	3	6	10	5
[2A, 2A, 4C, 9A]	1	3	6	10	5
[2B, 2B, 2D, 8A]	1	2	4	5	4
[2B, 2B, 2D, 10A]	1	2	5	7	5
[2B, 2B, 2D, 12B]	1	2	5	7	5
[2B, 2D, 3C, 6E]	1	3	6	7	5
[2B, 2D, 2D, 5A]	1	2	3	3	3
[2B, 2B, 2D, 6B]	1	1	2	4	3
[2D, 2D, 3C, 6D]	1	4	6	4	4
[2D, 2D, 3C, 6C]	1	4	7	5	5
[2D, 2D, 3C, 4B]	1	3	5	4	4
[2D, 2D, 3C, 4C]	1	3	4	2	3
[2A, 2A, 2C, 3A, 3C]	1	3	7	10	4
[2A, 2A, 2D, 2D, 4B]	1	3	7	12	6
[2A, 2A, 2A, 2D, 9A]	1	4	9	15	7
[2A, 2A, 2A, 2D, 12A]	1	4	9	15	7
[2A, 2D, 2D, 2D, 3C]	1	4	7	7	5
[2A, 2A, 2D, 2D, 6D]	1	4	8	12	6
[2A, 2A, 2D, 2D, 6C]	1	4	9	13	7

[2A, 2A, 2D, 2D, 4C]	1	3	6	10	5
[2A, 2A, 2B, 2D, 6E]	1	3	8	15	7
[2A, 2A, 2A, 3A, 4D]	1	3	9	15	6
[2A, 2A, 2B, 2B, 6A]	1	3	8	14	7
[2B, 2B, 2B, 2D, 2D]	1	2	5	7	5
[2A,2A,2A,2A,2C,3A]	1	3	9	18	6
[2A,2A,2A,2D,2D,2D]	1	4	9	15	7
[2C, 8A, 10A]	2	1	1	3	2
[2C, 8A, 12B]	2	1	1	3	2
[2C, 10A, 10A]	2	1	2	5	3
[2C, 10A, 12B]	2	1	2	5	3
[2D, 6A, 10A]	2	2	1	0	1
[2D, 6A, 12B]	2	2	1	0	1
[2D, 6G, 10A]	2	1	1	2	2
[2D, 6E, 12A]	2	2	1	1	1
[2D, 6G, 8A]	2	1	0	0	1
[2D, 6G, 12B]	2	1	1	2	2
[2D, 6E, 9A]	2	2	1	1	1
[3A, 4A, 10A]	2	4	5	3	3
[3A, 6F, 8A]	2	4	5	2	3
[3A, 6F, 10A]	2	4	6	4	4
[3B, 6F, 10A]	2	3	5	7	6
[3B, 6F, 12B]	2	3	5	7	6
[3B, 4A, 12B]	2	3	4	6	5
[3B, 4A, 8A]	2	3	3	4	4
[3B, 4A, 10A]	2	3	4	6	5
[3B, 6F, 8A]	2	3	4	5	5
[3C, 8A, 10A]	2	5	8	7	7
[3C, 8A, 12B]	2	5	8	7	7
[3C, 8A, 8A]	2	5	7	5	6
[3C, 10A, 12B]	2	5	9	9	8
[3C, 10A, 10A]	2	5	9	9	8

[3C, 6B, 12B]	2	4	6	6	6
[3C, 6B, 8A]	2	4	5	4	5
[3C, 6B, 10A]	2	4	6	6	6
[3C, 12B, 12B]	2	5	9	9	8
[4A, 6D, 8A]	2	5	7	7	6
[4A, 6D, 12B]	2	5	8	9	7
[4A, 6D, 10A]	2	5	8	9	7
[4A, 6B, 6C]	2	4	6	7	6
[4A, 6B, 6D]	2	4	5	6	5
[4A, 6F, 12A]	2	5	8	8	7
[4A, 6F, 9A]	2	5	8	8	7
[4A, 4A, 12A]	2	5	7	7	6
[4A, 4A, 9A]	2	5	7	7	6
[4A, 6C, 8A]	2	5	8	8	7
[4A, 6C, 12B]	2	5	9	10	8
[4A, 6C, 10A]	2	5	9	10	8
[4A, 5A, 6E]	2	4	6	8	6
[4B, 6F, 8A]	2	4	7	8	7
[4B, 6F, 12B]	2	4	8	10	8
[4B, 6F, 10A]	2	4	8	10	8
[4B, 4A, 12B]	2	4	7	9	7
[4B, 4A, 6B]	2	3	4	6	5
[4B, 4A, 8A]	2	4	6	7	6
[4B, 4A, 10A]	2	4	7	9	7
[4B, 6B, 6F]	2	3	5	7	6
[4C, 4A, 12B]	2	4	6	7	6
[4C, 4A, 10A]	2	4	6	7	6
[4C, 4A, 6B]	2	3	3	4	4
[4C, 4A, 8A]	2	4	5	5	5
[4C, 6F, 12B]	2	4	7	8	7
[4C, 6F, 10A]	2	4	7	8	7
[4C, 6F, 8A]	2	4	6	6	6

[4C, 6B, 6F]	2	3	4	5	5
[4D, 4C, 6E]	2	1	2	5	3
[5A, 6F, 6E]	2	4	7	9	7
[6B, 6C, 6F]	2	4	7	8	7
[6B, 6D, 6F]	2	4	6	7	6
[6C, 6F, 10A]	2	5	10	11	9
[6C, 6F, 8A]	2	5	9	9	8
[6C, 6F, 12B]	2	5	10	11	9
[6D, 6F, 10A]	2	5	9	10	8
[6D, 6F, 8A]	2	5	8	8	7
[6D, 6F, 12B]	2	5	9	10	8
[6F, 6F, 12A]	2	5	9	9	8
[6F, 6F, 9A]	2	5	9	9	8
[2A, 3C, 6A, 6F]	2	6	11	12	9
[2A, 3C, 4A, 6A]	2	6	10	11	8
[2A, 3C, 4B, 6E]	2	5	10	14	9
[2A, 3C, 6D, 6E]	2	6	11	14	9
[2A, 3C, 6C, 6E]	2	6	12	15	10
[2A, 2A, 10A, 10A]	2	5	11	17	10
[2A, 2A, 6B, 10A]	2	4	8	14	8
[2A, 2D, 6F, 10A]	2	5	10	13	9
[2A, 2B, 4B, 10A]	2	4	9	14	9
[2A, 2B, 6B, 6D]	2	4	7	11	7
[2A, 2B, 6B, 6C]	2	4	8	12	8
[2A, 2B, 3A, 10A]	2	4	7	8	5
[2A, 2B, 6D, 10A]	2	5	10	14	9
[2A, 2B, 6C, 10A]	2	5	11	15	10
[2A, 2B, 4C, 10A]	2	4	8	12	8
[2A, 2B, 3B, 10A]	2	3	6	11	7
[2A, 2A, 10A, 12B]	2	5	11	17	10
[2A, 2A, 6B, 8A]	2	4	7	12	7
[2A, 2A, 6B, 12B]	2	4	8	14	8

[2A, 2A, 8A, 10A]	2	5	10	15	9
[2A, 2A, 8A, 8A]	2	5	9	13	8
[2A, 2A, 5A, 12A]	2	5	9	13	8
[2A, 2B, 4B, 6B]	2	3	6	11	7
[2A, 2B, 4B, 12B]	2	4	9	14	9
[2A, 2B, 6F, 12A]	2	5	10	13	9
[2A, 2B, 4A, 12A]	2	5	9	12	8
[2A, 2B, 4C, 8A]	2	4	7	10	7
[2A, 2B, 3B, 8A]	2	3	5	9	6
[2A, 2D, 4A, 6B]	2	4	6	9	6
[2A, 2D, 4A, 10A]	2	5	9	12	8
[2A, 2D, 6F, 8A]	2	5	9	11	8
[2A, 2D, 4A, 8A]	2	5	8	10	7
[2A, 2D, 6B, 6F]	2	4	7	10	7
[2A, 2A, 5A, 9A]	2	5	9	13	8
[2A, 2A, 8A, 12B]	2	5	10	15	9
[2A, 2D, 3A, 5A]	2	4	5	4	3
[2A, 2D, 4B, 5A]	2	4	7	10	7
[2A, 2D, 6F, 12B]	2	5	10	13	9
[2A, 2D, 4A, 12B]	2	5	9	12	8
[2A, 2D, 3B, 5A]	2	3	4	7	5
[2A, 2B, 4B, 8A]	2	4	8	12	8
[2A, 2B, 6D, 12B]	2	5	10	14	9
[2A, 2B, 6C, 12B]	2	5	11	15	10
[2A, 2B, 6F, 9A]	2	5	10	13	9
[2A, 2B, 6C, 8A]	2	5	10	13	9
[2A, 2B, 4C, 6B]	2	3	5	9	6
[2A, 2B, 3A, 8A]	2	4	6	6	4
[2A, 2B, 6D, 8A]	2	5	9	12	8
[2A, 2B, 5A, 6E]	2	4	8	13	8
[2A, 2C, 4C, 6E]	2	1	2	8	3
[2A, 2B, 4C, 12B]	2	4	8	12	8

[2A, 2B, 3B, 12B]	2	3	6	11	7
[2A, 2B, 4A, 9A]	2	5	9	12	8
[2A, 2D, 5A, 6D]	2	5	8	10	7
[2A, 2D, 4C, 5A]	2	4	6	8	6
[2A, 2D, 5A, 6C]	2	5	9	11	8
[2A, 2A, 12B, 12B]	2	5	11	17	10
[2B, 2D, 3C, 10A]	2	5	9	9	8
[2B, 2C, 2D, 12B]	2	1	2	5	3
[2B, 2C, 2D, 10A]	2	1	2	5	3
[2B, 2C, 2D, 8A]	2	1	1	3	2
[2B, 2D, 3A, 4A]	2	4	5	3	3
[2B, 2D, 3B, 4A]	2	3	4	6	5
[2B, 2D, 4A, 4B]	2	4	7	9	7
[2B, 2D, 6C, 6F]	2	5	10	11	9
[2B, 2D, 4A, 6C]	2	5	9	10	8
[2B, 2D, 4A, 4C]	2	4	6	7	6
[2B, 2D, 3B, 6F]	2	3	5	7	6
[2B, 2D, 2D, 6A]	2	2	1	0	1
[2B, 2D, 3C, 12B]	2	5	9	9	8
[2B, 2D, 3A, 6F]	2	4	6	4	4
[2B, 2D, 4C, 6F]	2	4	7	8	7
[2B, 2D, 4B, 6F]	2	4	8	10	8
[2B, 2D, 6D, 6F]	2	5	9	10	8
[2B, 2D, 4A, 6D]	2	5	8	9	7
[2B, 2D, 2D, 6G]	2	1	1	2	2
[2B, 2B, 6E, 6F]	2	4	9	13	9
[2B, 2B, 4A, 6E]	2	4	8	12	8
[2B, 2D, 3C, 8A]	2	5	8	7	7
[2B, 2D, 3C, 6B]	2	4	6	6	6
[2C, 2D, 2D, 5A]	2	1	0	1	1
[2D, 2D, 4A, 6F]	2	5	8	8	7
[2D, 2D, 3C, 5A]	2	5	7	5	6

[2D, 2D, 6F, 6F]	2	5	9	9	8
[2D, 2D, 2D, 6E]	2	2	1	1	1
[2D, 2D, 4A, 4A]	2	5	7	7	6
[2D, 3C, 3C, 6E]	2	6	10	9	8
[2A, 2A, 2D, 3C, 6E]	2	6	12	17	10
[2A, 2A, 2B, 3C, 6A]	2	6	12	16	10
[2A, 2A, 2B, 2D,10A]	2	5	11	17	10
[2A, 2B, 2B, 2D, 4B]	2	4	9	14	9
[2A, 2B, 2B, 2D, 6D]	2	5	10	14	9
[2A, 2B, 2B, 2D, 6C]	2	5	11	15	10
[2A, 2B, 2D, 2D, 6F]	2	5	10	13	9
[2A, 2B, 2D, 2D, 4A]	2	5	9	12	8
[2A, 2B, 2B, 2D, 3A]	2	4	7	8	5
[2A, 2A, 2A, 4B, 6E]	2	5	12	22	11
[2A, 2A, 2B, 2D, 6B]	2	4	8	14	8
[2A, 2A, 2B,2B,12A]	2	5	11	17	10
[2A, 2A, 2D, 2D, 5A]	2	5	9	13	8
[2A, 2A, 2B, 2D, 8A]	2	5	10	15	9
[2A, 2B, 2B, 2D, 4C]	2	4	8	12	8
[2A, 2B, 2B, 2D, 3B]	2	3	6	11	7
[2A, 2B, 2B, 2B, 6E]	2	4	10	17	10
[2A, 2A, 2B, 2D, 12B]	2	5	11	17	10
[2A, 2A, 2A, 6C, 6E]	2	6	14	23	12
[2A, 2A, 2B, 2B, 9A]	2	5	11	17	10
[2A, 2A, 2A, 6D, 6E]	2	6	13	22	11
[2A, 2A, 2A, 6A, 6F]	2	6	13	20	11
[2A, 2A, 2A, 4A, 6A]	2	6	12	19	10
[2B, 2B, 2C, 2D, 2D]	2	1	2	5	3
[2B, 2B, 2D, 2D, 3C]	2	5	9	9	8
[2A,2A,2A,2A,2D,6E]	2	6	14	25	12
[2A,2A,2A,2A,2B,6A]	2	6	14	24	12

[2A,2A,2B,2B,2D,2D]	2	5	11	17	10
[4C, 4D, 6B]	3	2	2	4	4
[2A, 2C, 4C, 6B]	3	2	2	7	4
[2C, 2D, 4C, 4D]	4	2	2	5	4
[2A, 2C, 2C, 2D, 4C]	4	2	2	8	4
[2D, 6B, 9A]	3	3	1	0	2
[2D, 6B, 12A]	3	3	1	0	2
[2D, 2D, 2D, 6B]	3	3	1	0	2
[2D, 8A, 12A]	3	4	3	1	3
[2D, 8A, 9A]	3	4	3	1	3
[2C, 2D, 2D, 9A]	4	3	1	1	2
[2C, 2D, 2D, 12A]	4	3	1	1	2
[2D, 2D, 3A, 6G]	6	6	4	1	4
[2D, 2D, 3A, 4B]	4	5	4	1	3
[2D, 2D, 2D, 8A]	3	4	3	1	3
[2C, 2D, 2D, 2D, 2D]	4	3	1	1	2
[2D, 2D, 3A, 6C]	4	6	6	2	4
[2D, 2D, 3A, 12A]	7	9	7	2	6
[2D, 2D, 3A, 9A]	7	9	7	2	6
[2D, 2D, 3A, 5A]	5	7	6	2	5
[2D, 2D, 3C, 6A]	3	5	5	2	4
[2D, 2D, 3B, 6A]	6	6	3	2	5
[2D, 2D, 3B, 4C]	4	4	2	2	4
[2D, 2D, 2D, 2D, 3A]	7	9	7	2	6

$PSp(4,4)$	Tuples	85A	85B	120A	120B	136A	136B	1360
	[2C, 6A, 6B]	1	1	5	5	4	4	93
	[2C, 4B, 5E]	1	1	0	0	1	1	27
	[2C, 5E, 6A]	3	2	7	3	3	6	77
	[2C, 5E, 6B]	2	3	3	7	6	3	77

$PSp(4,4).2$	$85A$	0 (1)	1 (12)	2 (8)

	[2D, 4D, 15A]	[2A, 4D, 12A]	[2D, 4A, 12A]
		[2A, 4D, 10B]	[2D, 4A, 10B]
		[2B, 6D, 8A]	[2D, 6B, 12B]
		[2B, 6D, 8B]	[2D, 6B, 8A]
		[2D, 4B, 8B]	[2D, 6B, 8B]
		[2D, 4B, 8A]	[2D, 4C, 15B]
		[2D, 4D, 17B]	[3A, 4D, 6D]
		[2D, 4D, 17A]	[2A, 2D, 2D, 4A]
		[2D, 4A, 12B]	
		[2D, 4A, 8A]	
		[2D, 4A, 8B]	
		[2A, 2A, 2D, 4D]	

$PSp(4,4).2$ $85B$	0 (1)	1 (12)	2 (8)
	[2D,4C,15B]	[2A, 4C, 12B]	[2D, 4A, 12B]
		[2A, 4C, 10B]	[2D, 4A, 10B]
		[2C, 6C, 8B]	[2D, 4D, 15A]
		[2C, 6C, 8A]	[2D, 6A, 12A]
		[2D, 4C, 17B]	[2D, 6A, 8A]
		[2D, 4C, 17A]	[2D, 6A, 8B]
		[2D, 4B, 8A]	[3B, 4C, 6C]
		[2D, 4B, 8B]	[2A,2D,2D,4A]
		[2D, 4A, 12A]	
		[2D, 4A, 8A]	
		[2D, 4A, 8B]	
		[2A,2A,2D,4C]	

$PSp(4,4).2$ $120A$	0 (0)	1 (0)	2 (1)
			[2A,4C,10B]

$PSp(4,4).2$	$120B$	0 (0)	1 (0)	2 (1)
				[2A,4D,10B]
$PSp(4,5)$	$156A$	0 (2)	1 (0)	2 (1)
		[2B, 4B, 5E]		[2B, 3B,12A]
		[2B, 4B, 5F]		
$PSp(4,5)$	$156B$	0 (0)	1 (2)	2 (0)
			[2B, 4B, 5E]	
			[2B, 4B, 5F]	
$PSp(4,5)$	325	0 (0)	1 (0)	2 (2)
				[2B, 4B, 5E]
				[2B, 4B, 5F]
$PSp(4,5).2$	$156A$	0 (0)	1 (1)	2 (1)
			[2C,4D,6E]	[2B, 4D, 5D]
$PSp(4,5).2$	$156B$	0 (0)	1 (1)	2 (0)
			[2B, 4D, 5D]	

$PSp(4,5).2$ 300	0 (0)	1 (0)	2 (1)
			[2B, 4D, 5D]

$PSU(4,3).2_1$ 112	0 (0)	1 (0)	2 (6)
			[2C, 4B, 7A]
			[2C, 4D, 8A]
			[2C, 3C, 18B]
			[2C, 3C, 18A]
			[2C, 4D, 10A]
			[2C, 4B, 8B]

$PSU(4,3).2_2$ 112	0 (0)	1 (0)	2 (4)
			[2A, 3C, 14B]
			[2A, 3C, 14A]
			[2A, 4C, 7B]
			[2A, 4C, 7A]

$PSU(4,3).2_1^2$ 112	0 (0)	1 (0)	2 (5)
			[2B, 6P, 6J]
			[2C, 6P, 6D]
			[2D, 4G, 10C]
			[2D, 4H, 10A]
			[2D, 2C, 2B,3C]

$PSU(4,3).2_2^2$ 112	0 (0)	1 (0)	2 (2)
			[2C, 4B,10B]
			[2C, 4A,10C]

$PSU(4,3).D_8$112	0 (0)	1 (1)	2 (2)
		[2F, 4F, 6K]	[2A, 4F, 6J]
			[2F, 4F, 8E]

$PSU(4,3).2_1$ 126	0 (0)	1 (0)	2 (3)
			[2C, 4D, 8A]
			[2C, 4B, 8B]
			[2C, 4B, 7A]

$PSU(4,3).2_2$ 126	0 (0)	1 (2)	2 (1)
		[2A, 4C, 7A]	[2A, 5A, 6C]
		[2A, 4C, 7B]	

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ABSTRACT

GENUS 0, 1, 2 ACTIONS OF SOME ALMOST SIMPLE GROUPS OF LIE RANK 2

by

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Advisor: Dr. Daniel Frohardt

Major: Mathematics

Degree: Doctor of Philosophy

A transitive permutation group G is said to be a *genus g group* and *have a genus g action* if it has a tuple of nonidentity elements $(x_1, x_2, ...x_r)$ satisfying the following 3 conditions:

(1) G is generated by $x_1, ..., x_r$;

(2) The product of the r elements is identity: $x_1 \cdots x_r = 1$;

(3) $x_1, ..., x_r$ satisfy the Riemann-Hurwitz formula

$$2(N + g - 1) = \sum_{i=1}^{r} Ind(x_i)$$

To find all genus g groups, $g = 0, 1, 2$, it is enough to find all tuples of elements that satisfy the above 3 conditions.

The Cauchy-Frobenius Formula says that permutation index of an element x actually is defined by the fixed points of x :

$$Ind(x) = N - \frac{1}{d} \sum_{y \in \langle x \rangle} Fix(y)$$

A group G is said to be *almost simple* if it contains a non-abelian simple group S and is contained within the automorphism group of that simple group $Aut(S)$: $S \leq G \leq Aut(S)$.

This paper works on almost simple groups of type *projective special linear group* $PSL(3, q)$. Let group G be $PSL(3, q) \leq G \leq P\Gamma L(3, q)$ where $P\Gamma L(3, q)$is the *projective semilinear group.* G acts on points in the natural module, that is the set of projective points of 2−dimensional projective geometry $PG(2, q)$. There are two types of elements in $GL(3, q)^{\#}$: one type has a 2−dimensional eigenspace and the other only has 1−dimensional eigenspaces or no eigenspace. Map to *projective general linear group* $PGL(3, q)^{\#}$, this means that : one type fixes $q + 2$ or $q + 1$ points and other fixes at most 3 points. Take advantage of this fact we give an inequality about the orders of elements in the tuple relate to q. For $PSL(3, q)$ and $PGL(3, q)$ we have shown that all possible genus 0,1, 2 groups have $q \leqslant 29$. When G contains field automorphism, the number of fixed points of elements in $P\Gamma L(3, q) \setminus PGL(3, q)$ is no more than the one of field automorphisms. By this we have shown that all possible genus 0,1, 2 groups of $P\Gamma L(3, q)$ are of $q \leqslant 81$. Group theory is used to eliminate a few possible tuples for $P\Gamma L(3, q)$ with $32 \leqslant q \leqslant 81$.

When $q \leqslant 29$, we use GAP software to find all of the low genus groups. Permutation index is a class function. Use this we can easily find all tuples of conjugacy classes with sum of indices satisfying Riemann-Hurwitz formula. Then GAP supplies a powerful tool to check if these tuples can really generate the entire group.

The main results are the following:

Theorem *Genus 0, 1, 2 almost simple groups of type* $PSL(3, q)$ *acting on the projective* *points of* $2-$*dimensional projective geometry* $PG(2,q)$ *happen to* $q \leqslant 13$ *. Here is the list.*

q	$PSL(3,q)$	$PGL(3,q)$	$P\Gamma L(3,q)$	$PSL(3,q) \rtimes \langle \sigma^k \rangle$
3	$0,1,2$			
4	$0,1,2$	$0,1,2$	$0,1,2$	$PSL(3,4).2$ $0,1,2$
5	$0,1,2$			
7	$0,1,2$	$0,1,2$		
8	$1,2$		2	
9	$1,2$		$1,2$	
11	2			
13	$-$	$1,2$		
$\geqslant 16$	$-$	$-$	$-$	$-$

Note:

- Blank means identical to another group listed.
- " $-$ " means no low genus tuples.
- The last column is an extension group of $PSL(3,q)$ with a power of standard field automorphism but not $P\Gamma L(3,q)$.

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