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A YONEDA DESCRIPTION OF THE STEENROD OPERATIONS

R. R. BRUNER

Let $k = \mathbb{F}_p$, the prime field of characteristic $p > 0$, and let A be a cocommutative Hopf algebra over k. Products can be defined in $\mathrm{Ext}_A(k, k)$ in two fundamentally different ways.

First, if $\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow k \longrightarrow 0$ is a projective resolution of k, then, given cocycles $x : P_n \longrightarrow k$ and $y : P_m \longrightarrow k$, we can lift x to a chain map $\tilde{x} : P \longrightarrow P$ and define xy to be represented by the *Yoneda composite* $y\tilde{x}_m : P_{n+m} \longrightarrow P_m \longrightarrow k$. This method applies to any augmented algebra A over k, and, in general, yields a non-commutative product.

Second, the Hopf algebra structure on A allows us to consider $P \otimes P$ as an A-module by pullback along the diagonal $\Delta : A \longrightarrow A \otimes A$. The isomorphism $k \longrightarrow k \otimes k$ lifts to a chain map $\Delta : P \longrightarrow P \otimes P$ and, given cocycles x and y as above, we may define xy to be represented by the cocycle

$$(x \otimes y)\pi_{n,m}\Delta : P_{n+m} \longrightarrow (P \otimes P)_{n+m} \longrightarrow P_n \otimes P_m \longrightarrow k.$$

It is easily shown that this produces the same product as the first definition.

When A is cocommutative the chain map $\Delta : P \longrightarrow P \otimes P$ is chain homotopy cocommutative and the chain homotopies give rise to Steenrod operations denoted

$$Sq^i : \mathrm{Ext}_A^{s,t}(k, k) \longrightarrow \mathrm{Ext}_A^{s+i,2t}(k, k)$$

when $p = 2$, and similarly for odd p.

Problem: Find a description of the Steenrod operations in terms of Yoneda composites, avoiding any use of $\Delta : P \longrightarrow P \otimes P$.

The motivation for this is computational efficiency. In mechanical calculations of Ext ([1], [2]), when P is large enough to strain available memory, $P \otimes P$ is hopelessly out of reach. Worse, even when P is not overly large, the map Δ is so redundant that it is useless. For example, in my calculations of the cohomology of the mod 2 Steenrod algebra, the calculation of Ext out to internal degree $t = 60$ can be done in a few hours on a small workstation. However, the calculation of Δ on the class $h_3^3 h_5 \in \mathrm{Ext}^{4,44}$ required 14 days and contained 25,000,000 terms. This happens because Δ must capture all possible decompositions of an element.

In contrast, the calculation of the product structure by computing chain maps which lift cocycles is quite efficient because we need only lift those cocycles which are indecomposable, and we need only keep track of their values on an A-basis for the resolution. Using this approach, I have calculated the entire product structure of the cohomology of the Steenrod algebra through internal degree $t = 140$. The largest of the vector spaces $P_{s,t}$ involved are roughly 50,000 dimensional over $\mathbb{F}_2$, so that any attempt to do this calculation using Δ would be pointless.

There are two objections which have been raised to the possibility of such a description. First, if A is not a Hopf algebra, then its cohomology does not support Steenrod operations, and in general is not even commutative. This simply means that the coproduct of A must enter in some way into any definition of the Steenrod operations. This is also clear from the fact that the Steenrod operations detect the difference between different coproducts. For example, in $A = \mathbb{F}_2[x, y]$ there are no nontrivial Steenrod operations when both x and y are primitive, but if we let $\Delta(y) = 1 \otimes y + x \otimes x + y \otimes 1$ then we get $Sq^0(\bar{x}) = \bar{y}$, where $\bar{x}$ and $\bar{y}$ are dual to x and y.

One way in which the coproduct of A could enter would be through dualization. The category of A-modules has an internal Hom dual to the internal tensor product. A cocycle $x \in \mathrm{Ext}^n_A(k, k)$ corresponds to an n-extension of k by k, whose dual is another n-extension of k by k representing $\pm x$. Their composite (representing $Sq^n x = x^2$) has a symmetric character which might be exploited to produce shorter extensions representing $Sq^{n-1}x, \ldots, Sq^0 x$. Alternatively, an isomorphism between the n-extension represented by x and its dual might be a source of cocycles representing the Steenrod operations on x. The isomorphism $\mathrm{Hom}(P \otimes P, k) \cong \mathrm{Hom}(P, P^*)$, under which composition with the transposition becomes dualization may be useful here as well.

The second objection is a related one. This is the observation that the Yoneda composite in $\mathrm{Ext}_A(M, M)$ is not in general commutative, even when A is a Hopf algebra. However, Steenrod operations can be defined in $\mathrm{Ext}_A(M, N)$ only when we assume that N is a commutative algebra and M a cocommutative coalgebra in the category of A-modules ([3]). In this situation, $\mathrm{Ext}_A(M, N)$ will be commutative using the evident external product internalized along the product of N and the coproduct of M. If we assume that $M = N$ is a bicommutative bialgebra in the category of A-modules, it is likely that the Yoneda composite product defined in $\mathrm{Ext}_A(M, M)$ agrees with this commutative product, as when $M = k$.

References

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