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Extended powers of manifolds and the Adams spectral sequence

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Abstract. The extended power construction can be used to create new framed manifolds out of old. We show here how to compute the effect of such operations in the Adams spectral sequence, extending partial results of Milgram and the author. This gives the simplest method of proving that Jones' 30-manifold has Kervaire invariant one, and allows the construction of manifolds representing Mahowald's classes η_4 and η_5, among others.

1. Introduction

Many interesting spectra R are S-algebras, or, more generally, H_∞-ring spectra. As such they come equipped with compatible maps $D_r(R) \longrightarrow R$ extending the r-fold product maps $R^{(r)} \longrightarrow R$. Here D_r is the r^{th} extended power of a spectrum ([**HRS**]), which is an extension to the category of spectra of the familiar space level construction, $D_r X = E\Sigma_r^+ \wedge_{\Sigma_r} X^{(r)}$. The extension of the product to the extended powers leads to operations in the homotopy of R and in the Adams spectral sequence converging to the homotopy of R. These were studied in [**HRS**], [**RJM**], [**RJM2**], and [**RJM72**]. When $R = S$, the sphere spectrum, we may interpret its homotopy groups as bordism classes of framed manifolds, and the operations derived from the extended powers assume a natural geometric form. Here we provide a dictionary for translating between these three contexts and use that translation to give a simple proof that Jones' 30-manifold ([**JJ**]) has Kervaire invariant one, and to construct manifolds representing a variety of homotopy classes. The key point is that the calculations are easiest in the Adams spectral sequence. For example, our proof shows that the natural framing of Jones' 30-manifold has Kervaire invariant one. Heretofore it was only known that twisting the framing would alter the Kervaire invariant, so that *some* framing had Kervaire invariant one.

We shall work exclusively at the prime 2. All homology or cohomology is to have mod 2 coefficients. Naturally, there are analogous results at odd primes.

The extended power operations which apply to $\pi_n S$ are parameterized by the homotopy groups $\pi_* D_r(S^n)$. *Therefore, the set of operations which apply to* π_n *depends on* n. We shall let n denote the degree upon which we are acting throughout this paper. This will be the degree of the homotopy class, the dimension of the framed manifold, or the total degree $t - s$ of an element in $\mathrm{Ext}^{s,t}$.

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Given $\alpha \in \pi_{rn+m} D_r S^n$ we define

$$\alpha^* : \pi_n S \longrightarrow \pi_{rn+m} S$$

to take $x \in \pi_n S$ to the composite

$$\alpha^*(x) := \xi D_r(x) \alpha : S^{rn+m} \longrightarrow D_r S^n \longrightarrow D_r S \longrightarrow S$$

where $\xi : D_r S \longrightarrow S$ is the H_∞ structure map of S. Now $D_r S^n = T(n\zeta_r)$, the Thom complex of the direct sum of n copies of the permutation bundle ζ_r induced by the evident permutation representation $\Sigma_r \longrightarrow O(r)$. The Thom-Pontrjagin construction allows us to interpret $\pi_* D_r S^n$ as $\Omega_*^{fr}(B\Sigma_r; n\zeta_r)$, the framed bordism of the classifying space of the symmetric group Σ_r with twisted coefficients in $n\zeta_r$. Thus, an extended power operation is specified by giving an m-manifold M, a Σ_r covering $\widetilde{M} \to M$ classified by $f : M \to B\Sigma_r$, and a stable isomorphism F of the normal bundle ν_M with $f^* n\zeta_r$. Such a triple $[M, f, F]$ defines an operation

$$M_{f,F} : \Omega_n^{fr}(*) \longrightarrow \Omega_{rn+m}^{fr}(*)$$

as follows. Given a framed n-manifold $[N, t]$, where t is a stable trivialization of ν_N, we set

$$M_{f,F}(N) = \widetilde{M} \times_{\Sigma_r} N^r,$$

where Σ_r acts on N^r by permuting factors, with a framing derived from F and t. Note that the dimension n of the manifolds upon which $M_{f,F}$ can act is determined by the multiple of the permutation bundle $f^*\zeta_r$ which is equal to ν_M. Since this equality need hold only in $J(M)$, which is generally finite, an operation applies to all n satisfying an appropriate congruence modulo the order of $f^*\zeta_r$ in $J(M)$. Details can be found in Milgram [**RJM**],[**RJM2**] and Jones [**JJ**], where it is shown that the Thom-Pontrjagin construction converts this operation into the extended power construction in homotopy.

Before turning to the operations in the Adams spectral sequence, let us describe the dictionary translating operations on framed manifolds to operations on homotopy. These and the analogous operations in the Adams spectral sequence are summarized in Table 1.

We have an especially simple basic operation, the 'cup-i' operation

$$\cup_i : \pi_n \longrightarrow \pi_{2n+i}$$

which exists whenever $n \equiv -(i+1)$ modulo the order of the canonical line bundle over $RP^i = S^i/\Sigma_2$. This condition is equivalent to the existence of an isomorphism $F : n\zeta_2 \longrightarrow \nu_{RP^i}$ with which we define the operation

$$N \mapsto S^i \times_{\Sigma_2} N \times N,$$

the framed manifold version of cup-i.

The sum of homotopy classes corresponds to the pointwise sum of the corresponding operations:

$$(\alpha_1 + \alpha_2)(x) = \alpha_1(x) + \alpha_2(x)$$

In framed bordism, this corresponds to the disjoint union $M_1 \amalg M_2$ or connected sum $M_1 \# M_2$ with the natural framing. Both come from the fold map of the domain.

There is also a pointwise product $\alpha_1 * \alpha_2$,

$$(\alpha_1 * \alpha_2)(x) = \alpha_1(x)\alpha_2(x),$$

Operation	Manifolds	Homotopy	Adams spectral sequence
basic	$S^i \times_{\Sigma_2} (\cdot)^{(2)}$	$\cup_i$	Sq^{n+i}
sum	$M_1 \coprod M_2$	$\alpha_1 + \alpha_2$	$a_1 + a_2$
product	$M_1 \times M_2$	$\alpha_1 * \alpha_2$	$a_1 * a_2$
composition	$\widetilde{M_1} \times_{\Sigma_r} (M_2)^r$	$\alpha_1\alpha_2 := \alpha_1 \circ \alpha_2$	$a_1 a_2 := a_1 \circ a_2$

TABLE 1. Translation between the different manifestations of the extended power operations on an element of degree n.

which is derived from block sum $B\Sigma_{r_1} \times B\Sigma_{r_2} \longrightarrow B\Sigma_{r_1+r_2}$. This corresponds to the product $M_1 \times M_2$. A special case is scalar multiplication $\theta * \alpha$, where $\theta \in \pi_* S$, which corresponds to the case in which M_1 is itself framed (with structure map $f : M_1 \longrightarrow B\Sigma_0 \simeq *$).

The composite $\alpha_1 \circ \alpha_2$ corresponds to the manifold $\widetilde{M_1} \times_{\Sigma_{r_1}} M_2^{r_1}$. These are derived from the natural map $E\Sigma_{r_1} \times_{\Sigma_{r_1}} (B\Sigma_{r_2})^{r_1} \simeq B(\Sigma_{r_1} \wr \Sigma_{r_2}) \longrightarrow B\Sigma_{r_1 r_2}$. As usual with composition of functions, we shall denote this by juxtaposition: $\alpha_1\alpha_2 := \alpha_1 \circ \alpha_2$ and therefore we must retain the $*$ to denote pointwise product.

The algebraic extended power construction ([**JPM**]) induces Steenrod operations in the Ext module which appears as the E_2-term of the Adams spectral sequence ([**RJM72**],[**HRS**]). Let $\mathcal{A}$ be the Steenrod algebra acting upon Ext of modules over a cocommutative Hopf algebra ([**JPM**] or [**HRS**]), and let $\mathcal{A}(n)$ be its quotient by operations of excess less than n.

The basic operation, corresponding to cup-i, is Sq^{n+i} in the indexing we have chosen (see section 2). The sum and composite operations are the usual pointwise sum and composite of Steenrod operations. The pointwise product, however, forces us to enlarge the set of operations we consider to the polynomial algebra on $\mathcal{A}$. We shall denote this operation by $*$ as above, so that, for example,

$$(Sq^i Sq^j)(x) = Sq^i(Sq^j(x))$$

while

$$(Sq^i * Sq^j)(x) = Sq^i(x) Sq^j(x)$$

using the product in $\mathrm{Ext}_{\mathcal{A}}(Z_2, Z_2)$ on the right hand side of the latter equation. Let $P[V]$ be the free commutative algebra generated by the vector space V. We will define, for each n, a natural ring isomorphism

$$\sigma_n : \bigoplus_r H_* B\Sigma_r \longrightarrow P[\mathcal{A}(n)].$$

THEOREM 2. *The operation $M_{f,F}$ is detected by $\sigma_n(f_*[M])$ in the Adams spectral sequence.*

This is the natural generalization to all r of the familiar fact that the cup-i construction is detected by Sq^{n+i}, since $\sigma_n(f_*[RP^i])) = \sigma_n(Q^i) = Sq^{n+i}$ in that case. This theorem is the key result in our proof that Jones' 30-manifold has Kervaire invariant 1.

If $f_*[M] = 0$, Theorem 2 tells us only that $M_{f,F}(N)$ will be detected in higher than its "natural" filtration. In order to detect such operations by nontrivial operations in Ext, we must "extend scalars" by means of a spectral sequence ([**RRB**])

$$\mathrm{Ext}_{\mathcal{A}}(Z_2, Z_2) \otimes H_* B\Sigma_r \Longrightarrow \Omega_*^{fr}(B\Sigma_r; n\zeta_r).$$

With this extension, the pointwise product by an element $\pi_* S$ corresponds to pointwise product by an element of $\mathrm{Ext}_{\mathcal{A}}(Z_2, Z_2)$.

Recall that the homology of the symmetric groups can be described in terms of polynomials q in the Dyer-Lashof operations Q^I (see section 2).

THEOREM 8. *If $[M, f, F]$ is detected by $\Sigma a_q \otimes q$ in this spectral sequence, then the operation $M_{f,F}$ is detected by $\Sigma a_q \sigma_n(q)$ in the Adams spectral sequence.*

The edge homomorphism in this spectral sequence is $[M, f, F] \mapsto 1 \otimes f_*[M]$, so that Theorem 8 is a generalization of Theorem 2. We give examples of this result which produce framed manifolds representing Mahowald's elements η_4 and η_5, in dimensions 16 and 32, respectively.

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2. Results

In the Adams spectral sequence

$$\mathrm{Ext}_{\mathcal{A}}^{s,t}(Z_2, Z_2) \Longrightarrow \pi_{t-s}(S),$$

the topologically significant total degree is $t - s$. Let us index the generators of $\mathcal{A}$ so that Sq^i raises this by i. Thus

$$Sq^i : \mathrm{Ext}^{s,t} \longrightarrow \mathrm{Ext}^{s+t-i,2t}$$

This is a homological, rather than cohomological, indexing, and as a result the Adem relations and admissability conditions for the Sq^i are the same as those for the Dyer-Lashof operations. If $I = (i_k, \ldots, i_1, i_0)$, let $Sq^I = Sq^{i_k} \ldots Sq^{i_1} Sq^{i_0}$. We say that I and Sq^I are *admissable* if each $i_{j+1} \leq 2i_j$. Recall that Sq^I acts trivially upon $\mathrm{Ext}^{s,t}$ if $\mathrm{excess}(I) < t - s$, where the excess of an admissable operation Sq^I is defined by $\mathrm{excess}(I) = i_k - i_{k-1} - \ldots - i_0$. Let $\mathcal{A}(n)$ be the quotient of $\mathcal{A}$ by those operations of excess less than n, so that $P[\mathcal{A}(n)]$ (redundantly) parameterizes the nonzero operations on $\mathrm{Ext}^{s,s+n}$. This is redundant since an operation of excess n has the same effect as the square of another operation. Precisely, if $\mathrm{excess}(I) = n$, and $x \in \mathrm{Ext}^{s,s+n}$, then $Sq^I(x) = (Sq^{I'}(x))^2$ where $I' = (i_{k-1}, \ldots, i_0)$. This requires that we let Sq^ϕ be the identity operation of $\mathcal{A}$, where ϕ is the empty sequence.

Recall that $\bigoplus_r H_* B\Sigma_r$ is the free commutative algebra on the admissable Dyer-Lashof operations, with $H_* B\Sigma_r$ being the summand of weight r operations, where

$$\begin{array}{rrcl} & \mathrm{weight}([1]) & = & 1 \\ & \mathrm{weight}(xy) & = & \mathrm{weight}(x) + \mathrm{weight}(y) \\ \textit{and} & \mathrm{weight}(Q^i x) & = & 2\,\mathrm{weight}(x). \end{array}$$

Here $[1] \in H_0 B\Sigma_1$ is the 'fundamental class', the generator $[r] \in H_0 B\Sigma_r$ is the product of [1] with itself r times, and we identify the operation $Q^I = Q^{(i_k, \cdots, i_0)}$ with the class $Q^I[1] = Q^{i_k} \cdots Q^{i_1} Q^{i_0}[1]$. Note that $Q^0[r] = [2r]$. This algebra is the homology of the little cubes construction applied to S^0, $H_* CS^0$, and passes to the familiar description of H_*QS^0 under the group completion map $CS^0 \to QS^0$ ([CLM]).

There is a similar description of $H_*D_rS^n$, where $D_rX = E\Sigma_r^+ \wedge_{\Sigma_r} X^{(r)}$ is the r^{th} extended power of a space X. Namely, $\bigoplus_r H_*D_rS^n = H_*QS^n$ is the free commutative algebra on the $Q^I\iota_n$, where Q^I is admissable with excess greater than n, and $H_*D_rS^n$ consists of the weight r summand, where the weight of ι_n is 1.

Definition 1. *Let*

$$\sigma_n : \bigoplus_r H_*B\Sigma_r \to P[\mathcal{A}(n)]$$

be the ring homomorphism with respect to sum and pointwise product defined on generators by

$$\sigma_n(Q^I) = \sigma_n(Q^I[1]) = Sq^{d(I,n)},$$

where $d(I,n) = (i_k + 2^k n, \dots, i_1 + 2n, i_0 + n)$ *if* $I = (i_k, \dots, i_1, i_0)$.

To be clear, we note a few examples. Clearly $\sigma_n([1]) = Sq^{\phi}$, which is the identity operation, and $\sigma_n([2]) = \sigma_n(Q^0[1]) = Sq^n$, which is the squaring operation in degree n, consistent with the relation $[2] = [1] * [1]$. In general, $\sigma_n([r])$ is the r^{th} power operation, the lowest degree operation of weight r. Increasing degree rather than weight, the operations of weight 2 are the $\sigma_n(Q^i) = Sq^{n+i}$.

We can now make the first main theorem precise.

Theorem 2. *The operation* $M_{f,F}$ *is detected in the Adams spectral sequence by the operation* $\sigma_n(f_*[M])$.

Example 3. *The "cup-i" construction,* $RP^i_{f,F} : N \mapsto S^i \times_{\Sigma_2} N \times N$*, discussed in the Introduction, is detected by* $Sq^{n+i} = \sigma_n(Q^i)$ *in Ext.*

This example is well known ([**RJM**],[**RJM2**]). It exists whenever $n \equiv -(i+1)$ modulo the order of the canonical line bundle over $RP^i = S^i/\Sigma_2$. The bordism class in $\Omega_i(B\Sigma_2; n\zeta_2)$ is that of $[RP^i, f, F]$, where f is the inclusion of the i-skeleton in $RP^\infty = B\Sigma_2$. Thus $f_*[RP^i]$ is the nonzero class $Q^i \in H_i(B\Sigma_2)$. Theorem 2 implies that *any* stable isomorphism F of ν_{RP^i} with $-(i+1)f^*\zeta_2$ will yield an operation detected by Sq^{n+i}. (See also the comments following Observation 6.)

Some instances of the cup-i operation are

1. $S^1 \times_{\Sigma_2} (S^1 \times S^7)^2$ represents $\eta\eta_4$, if $(S^1 \times S^7)$ is given the product of the η (i.e., complex) and σ (i.e., Cayley number) framings. This follows from the calculation $Sq^9(h_1h_3) = h_1^2h_4 + h_2h_3^2 = h_1^2h_4$.
2. $S^1 \times_{\Sigma_2} N^2$ represents h_0e_0 if N represents c_0 since $Sq^9(c_0) = h_0e_0$.
3. $S^3 \times_{\Sigma_2} N^2$ represents c_1, with the same N, since $Sq^{11}(c_0) = c_1$.
4. If N_j is a 2^j manifold representing Mahowald's η_j then $S^1 \times_{\Sigma_2} N_j \times N_j$ is detected by $Sq^{2^j+1}(h_1h_j) = h_2h_j^2 + h_1^2h_{j+1}$. Now $h_1^2h_{j+1}$ can be constructed as $S^1 \times N_{j+1}$ with the nontrivial framing of S^1. It follows that the connected sum
$$V_j = (S^1 \times_{\Sigma_2} N_j \times N_j) \# -(S^1 \times N_{j+1})$$
is detected by $h_2h_j^2$. If there exists a Kervaire invariant one manifold θ_j, we must have a framed cobordism from V_j to $S^3 \times \theta_j$ where S^3 has the quaternionic framing, at least modulo higher filtration in the Adams spectral sequence.

In our next example, we compute the effect in the Adams spectral sequence of the operation constructed by John Jones [**JJ**].

EXAMPLE 4. *Let* $X = RP^2 \# S^1 \times S^1$ *and let* $f : X \to B\Sigma_4$ *be*

$$X \xrightarrow{c} RP^2 \vee S^1 \times S^1 \longrightarrow B\Sigma_2 \vee B(\Sigma_2 \times \Sigma_2) \xrightarrow{Bi \vee Br} B\Sigma_4,$$

where c *is the collapse map, the middle map is made from inclusions of skeleta,* i *is the natural inclusion and* r *is the regular representation. Let* $n \equiv 3 \bmod (4)$ *so that there is a stable isomorphism* $F : \nu_X \longrightarrow f^* n\zeta_4$*. The operation* $X_{f,F}$ *is detected by* $Sq^{2n+1}Sq^{n+1} + Sq^n * Sq^{n+2} = Sq^{n+1} * Sq^{n+1} + Sq^n * Sq^{n+2}$.

In particular, we obtain Kervaire invariant 1 manifolds in dimensions 14 and 30 from this operation.

COROLLARY 5. *If* S^n *has the Hopf invariant one framing,* $n = 3$ *or* 7*, then* $X_{f,F}(S^n)$ *has Kervaire invariant 1.*

Of course, $X_{f,F}(S^3)$ is not the simplest 14-manifold of Kervaire invariant 1 we could construct. The product $S^7 \times S^7$, with the Cayley framing on each factor, is another. Theorem 8 will allow us to show that they are framed-cobordant.

The corollary follows immediately from the fact that $(Sq^{n+1} * Sq^{n+1} + Sq^n * Sq^{n+2})(h_i) = h_{i+1} * h_{i+1} + h_i^2 * 0 = h_{i+1}^2$. We will show that $f_*[X] = (Q^1)^2 + Q^0Q^2$ in the next section.

This is the first operation which Theorem 2 allows us to detect which is not decomposable into cup-i operations and products with $\pi_* S$. The homotopy operation was used by Milgram ([**RJM72**]) to show that θ_4 exists. The manifold operation was found by Jones [**JJ**] in his study of the Kervaire invariant of extended powers. His proof that it produces a manifold of Kervaire invariant 1 relied on a formula for the Kervaire invariant of $M_{f,F}(S^n)$ after a change of framing. I first gave this elementary proof in the topology seminar at MIT in 1979.

The framed cobordism class $[X, f, F]$ does not depend upon the particular choice of framing F in Example 4 by the following observation.

OBSERVATION 6. *Given a manifold* M*, a map* $f : M \to X$*, and two stable isomorphisms* $F, F' : \nu_M \to f^*\xi$*, the difference*

$$[M, f, F] - [M, f, F'] \in \Omega_*(X; \xi)$$

lies in the kernel of the Hurewicz homomorphism.

Proof: The composite

$$\Omega_m(X;\xi) = \pi_{m+|\xi|} T\xi \xrightarrow{\text{h}} H_{m+|\xi|} T\xi \xrightarrow{\Phi^{-1}} H_m X$$

of the Hurewicz homomorphism and the inverse of the Thom isomorphism is given by $[M, f, F] \mapsto f_*[M]$, which is independent of F. □

In the case at hand, $n \equiv 3 \bmod (4)$, the homomorphism $\Phi^{-1}h$

$$\begin{array}{ccc} \Omega_2^{fr}(B\Sigma_4; n\zeta_4) & = & Z/2 \\ \downarrow & & \\ H_2(B\Sigma_4) & = & Z/2 \times Z/2 \end{array}$$

is monic. Hence all framings F yield the same framed cobordism class $[X, f, F]$.

For the cup-i operations, different choices of framing will usually yield different operations, since the kernel of the Hurewicz homomorphism is not usually zero.

However, since they are all detected by Sq^{n+i} in the Adams spectral sequence, they all have the same image under $\Phi^{-1}h$ and hence the differences between them will all lie in higher filtration than the cup-i operation.

Theorem 2 allows us to detect an operation α^* if α has nontrivial Hurewicz image, or equivalently, if $f_*[M]$ is nonzero. In order to detect all operations, we note that the Hurewicz homomorphism is the edge homomorphism in the Adams spectral sequence. Better, when it is nonzero, it is the map 'is detected by' from $\pi_* D_r S^n$ to an 'Adams-Atiyah-Hirzebruch spectral sequence' which mixes the cellular filtration of $D_r S^n$ with the Adams filtration of homotopy. (See Milgram [**RJM72**] for prime r, and Bruner [**RRB**] for general r and related spectral sequences). The essential point is to delay the Adams resolution of a homology class until the filtration to which it will map in the Adams resolution of S^0 under the map $\xi \circ D_r x$. We accomplish this by smashing an Adams resolution of S^0 with the cellular filtration and totalizing the resulting bifiltered spectrum. In general, the resulting E_2 term is the target of a bicomplex spectral sequence which collapses in this case because we can arrange that the attaching maps will all be 0 in homology. Further, since $\xi \circ D_r x$ maps $D_r S^n$ into S so that the cellular filtration is mapped to the Adams filtration ([**HRS**]), the natural extension of $\xi \circ D_r x$ to the bifiltered spectrum maps the total filtration to the Adams filtration ([**RRB**]). Together with Theorem 2 these facts imply the following result.

THEOREM 7. *Given $x \in \pi_n S^0$ detected by $\bar{x} \in \mathrm{Ext}_{\mathcal{A}}^{s,n+s}(Z_2, Z_2)$, there exists an $\mathrm{Ext}_{\mathcal{A}}(Z_2, Z_2)$ - linear map from the Adams-Atiyah-Hirzebruch spectral sequence to the Adams spectral sequence,*

$$\begin{array}{ccccc} \widetilde{E}_2 & = & \mathrm{Ext}_{\mathcal{A}}(Z_2,Z_2)\otimes H_*B\Sigma_r & \Rightarrow & \pi_* D_r^{rs} S^n \\ & & \downarrow & & \downarrow \\ E_2 & = & \mathrm{Ext}_{\mathcal{A}}(Z_2,Z_2) & \Rightarrow & \pi_* S \end{array}$$

which

1. *converges to the map $\xi \circ D_r x$ which sends α to $\alpha^*(x)$, and*
2. *sends $Q^I \in H_* B\Sigma_r$ to $\sigma_n(Q^I)(\bar{x})$.*

THEOREM 8. *The operation $M_{f,F}$ is detected in the Adams spectral sequence by the operation $\Sigma_q\, a_q \sigma_n(q)$ if $f_*(\alpha_M) \in \pi_* D_r S^n$ is detected by $\Sigma_q\, a_q \otimes q$ in the Adams-Atiyah-Hirzebruch spectral sequence.*

EXAMPLE 9. *Let K be the Klein bottle, and let $f = ip : K \longrightarrow S^1 \hookrightarrow B\Sigma_2$ be the usual fibering p, followed by the inclusion i of the 1-skeleton. Let $n \equiv 3 \bmod 4$, and let $F : \nu_K \longrightarrow f^* n\zeta_2$ be a stable isomorphism which restricts to the η framing of $S^1 = p^{-1}(pt)$. The operation $K_{f,F}$ is detected by $h_1 Sq^{n+1}$ in the Adams spectral sequence.*

Since $\widetilde{K} = S^1 \times S^1$ with Σ_2 action $(\theta_1, \theta_2) \mapsto (\theta_1 + \pi, -\theta_2)$, we get $K_{f,F}(N) = S^1 \times S^1 \times_{\Sigma_2} N \times N$.

The fact that $[K, f, F]$ is detected by η on the 1-cell of $B\Sigma_2$ means that the restriction of F to $S^1 = p^{-1}(pt)$ must be the η framing of S^1.

Note that the differential $d_2 Sq^{n+1} x = h_0 x^2$, for n odd, implies that, in general, $Sq^{n+1} x$ by itself does not detect a homotopy class. For example, if $x = h_3$ then $Sq^8 h_3 = h_4$ does not survive, but Theorem 8 implies that $h_1 h_4$ does. Thus, we obtain a manifold representing η_4.

Corollary 10. *If S^7 has the Cayley number framing, then $K_{f,F}(S^7) = S^1 \times S^1 \times_{\Sigma_2} (S^7)^2$, with its natural framing, is detected by h_1h_4 in the Adams spectral sequence.*

Another class to which this operation applies is c_1 in the 19-stem, which we obtained earlier as $Sq^{11}(c_0)$. Since $Sq^{20}(c_1) = h_1e_1$, we find that $h_1^2e_1$ is represented by $S^1 \times S^1 \times_{\Sigma_2} (S^3 \times_{\Sigma_2} N^2)^2$, where N represents c_0.

Our last example involves Σ_4 again. Let T^n be the n-torus and let

$$\begin{array}{rcl} W & = & T^4/(\theta_0,\theta_1,\theta_2,\theta_3) \sim (\theta_0+\pi,\theta_2,\theta_1,-\theta_3) \\ p \quad \downarrow & & \\ Y & = & T^3/(\theta_0,\theta_1,\theta_2) \sim (\theta_0+\pi,\theta_2,\theta_1) \end{array}$$

be the obvious S^1 bundle. The manifold $Y = BG$, where G is the semidirect product $Z^2 \rtimes Z$ in which the generator of Z exchanges the two factors of Z^2. There is an obvious homomorphism from G to $D_8 = Z_2^2 \rtimes Z_2$, the 2-Sylow subgroup of Σ_4, and the map $j : Y \longrightarrow B\Sigma_4$ induced by it and inclusion of the 2-Sylow satisfies $j_*[Y] = Q^2Q^1\iota \in H_3B\Sigma_4$.

If $f = jp : W \longrightarrow Y \longrightarrow B\Sigma_4$, then, when $n \equiv 7 \bmod 8$, there is a framing $F : \nu_W \longrightarrow f^*n\zeta_4$, which restricts to the η framing of $S^1 = p^{-1}(pt)$.

Example 11. *$W_{f,F}$ is detected by $h_1Sq^{2n+2}Sq^{n+1}$ when $n \equiv 7 \bmod 8$.*

Since $Sq^{16}Sq^8(h_3) = h_5$, the operation takes σ to η_5. Thus, we obtain a manifold representing η_5.

Corollary 12. *If S^7 has the Cayley number framing, then $W_{f,F}(S^7)$, with its natural framing, is detected by h_1h_5 in the Adams spectral sequence.*

3. Proofs

Note that σ_n does not preserve composition. We define a related homomorphism

$$\rho_n : H_*QS^n \longrightarrow P[\mathcal{A}(n)]$$

by letting

$$\rho_n(Q^I) = Sq^I$$

on generators, and extending similarly. This function preserves all three operations. The relation between ρ_n and σ_n is simple.

Lemma 13. *$\sigma_n = \rho_n\Phi_n$, where $\Phi_n : H_*B\Sigma_r \to H_*D_rS^n$ is the Thom isomorphism of the bundle $n\zeta_r$.*

Proof: We must show that $\Phi_n(Q^I) = Q^{d(I,n)}$. This is immediate from the cellular structure of D_rS^n and $B\Sigma_r^+ = D_rS^0$ since

$$Q^i(x) = \theta_*(e_{i-deg(x)} \otimes x \otimes x)$$

and

$$\Phi_n(e_i \otimes x \otimes x) = e_i \otimes \Phi_n(x) \otimes \Phi_n(x)$$

and $\Phi_n([1]) = \iota_n$. □

Proof of Theorem 2: First, recall the relation between $[M,f,F]$ and the corresponding homotopy class α. The Thom-Pontrjagin construction associates to M a stable homotopy class $\alpha_M \in \pi_{m+|\nu_M|}T\nu_M$. The stable map $F:\nu_M \to n\zeta_r$ induces $TF: T\nu_M \to T(n\zeta_r) = D_rS^n$ and α is the composite

$$\alpha = TF_*(\alpha_M) \in \pi_{rn+m}T(n\zeta_r) = \pi_{rn+m}D_rS^n.$$

Consider the homotopy operation α^* constructed from $[M,f,F]$. By [**HRS**, IV.5.4], it induces, in the Adams spectral sequence, the Steenrod operation determined by $ijqh(\alpha) \in H_*(B\Sigma_r^m, B\Sigma_r^{m-1})$

$$\begin{array}{ccccc}
\alpha \in & \pi_{rn+m}D_r^mS^n & \xrightarrow{h} & H_{rn+m}D_r^mS^n \\
 & & & \downarrow q \\
 & & & H_{rn+m}(D_r^mS^n, D_r^{m-1}S^n) \\
 & & & \downarrow j \\
ijqh(\alpha) \in & H_m(B\Sigma_r^m, B\Sigma_r^{m-1}) & \xleftarrow{i} & H_m(B\Sigma_r^m, B\Sigma_r^{m-1}) \otimes H_{nr}S^{n(r)}
\end{array}$$

where h is the Hurewicz homomorphism, q is induced by the quotient map, j is induced by the natural equivalence

$$D_r^mS^n/D_r^{m-1}S^n \longrightarrow B\Sigma_r^m/B\Sigma_r^{m-1} \wedge S^{n(r)}$$

and i is the Kunneth isomorphism $i(x \otimes \iota_n^r) = x$.

In the following commutative diagram, the maps Φ are Thom isomorphisms.

$$\begin{array}{ccccc}
\pi_mT\nu_M & \xrightarrow{TF_*} & \pi_{rn+m}D_r^mS^n & & \\
h\downarrow & & h\downarrow & & \\
H_mT\nu_M & \xrightarrow{TF_*} & H_{rn+m}D_r^mS^n & \xrightarrow{q} & H_{rn+m}(D_r^mS^n, D_r^{m-1}S^n) \\
\Phi\uparrow & & \Phi\uparrow & & \Phi\uparrow \\
H_mM & \xrightarrow{f_*} & H_mB\Sigma_r^m & \xrightarrow{q} & H_{rn+m}(B\Sigma_r^m, B\Sigma_r^{m-1})
\end{array}$$

Since $\alpha = TF_*(\alpha_M)$ and $h(\alpha_M) = \Phi[M]$ and $ij\Phi(x) = x$, a simple diagram chase shows that $ijqh(\alpha) = qf_*[M]$. Finally, to show that this operation is $\sigma_nf_*[M] = \rho_n\Phi f_*[M]$ it is sufficient to check it on generators $Q^i \in H_iB\Sigma_2$, since ρ_n preserves sums, products, and composites. For Q^i it is true by the definition of the Steenrod operation Sq^{n+i} [**HRS**, IV.2.4 and IV.5.4]. $\square$

Proof of Example 4: We wish to show that $f: X \to B\Sigma_4$ satisfies $f_*[X] = (Q^1\iota)^2 + \iota^2Q^2\iota \in H_2B\Sigma_4$ and $f^*(n\zeta_4) = \nu_X$. Since $\iota^2Q^2\iota$ is the image of $Q^2\iota$ under the natural inclusion $\Sigma_2 \to \Sigma_4$, and since $RP^2 \hookrightarrow RP^\infty = B\Sigma_2$ sends $[RP^2]$ to $Q^2\iota$, the composite $f_1: RP^2 \hookrightarrow RP^\infty = B\Sigma_2 \to B\Sigma_4$ will satisfy $f_{1*}[RP^2] = \iota^2Q^2\iota$. Similarly, since $S^1 = RP^1 \hookrightarrow RP^\infty = B\Sigma_2$ sends $[S^1]$ to $Q^1\iota$, we can realize $Q^1\iota Q^1\iota$ as the image of $[S^1 \times S^1]$ under an appropriate $f_2: S^1 \times S^1 = RP^1 \times RP^1 \hookrightarrow B\Sigma_2 \times B\Sigma_2 \to B\Sigma_4$.

Bundles over surfaces are entirely determined by their Stiefel-Whitney classes, so we will calculate w_1 and w_2. Let $H^*B\Sigma_2 = P[x]$, $H^*RP^2 = P[x]/(x^3)$, $H^*B\Sigma_2 \times B\Sigma_2 = P[x_1,x_2]$, and $H^*S^1 \times S^1 = E[x_1,x_2]$. Then we have

$$\begin{array}{rll}
 & H^1X = Z_2^3 & \text{generated by } x,\ x_1, \text{ and } x_2 \\
\textit{and} & H^2X = Z_2 & \text{generated by } x^2 = x_1x_2.
\end{array}$$

The tangent bundle of $S^1 \times S^1$ is trivial, so

$$w(\tau_X) = w(\tau_{RP^2}) = (1+x)^3 = 1 + x + x^2.$$

Thus $4\tau_X$ is trivial and $\nu_X = n\tau_X$ for any $n \equiv 3 \bmod 4$.

It is easy to see that $f_1^*\zeta_4 = 2 + \zeta_2$, so that $w(f_1^*\zeta_4) = 1 + x$.

Define f_2 to be

$$f_2 : S^1 \times S^1 \hookrightarrow B\Sigma_2 \times B\Sigma_2 \xrightarrow{Br} B\Sigma_4$$

where the first map is the inclusion of the product of the 1-skeleta and r is the regular representation. Then $r^*\zeta_4$ is the sum of the four linear representations, so that $w(f_2^*\zeta_4) = (1+x_1)(1+x_2)(1+x_1+x_2) = 1 + x_1x_2$. Since $w_1(\zeta_4)$ is dual to $\iota^2 Q^1\iota$ and $w_2(\zeta_4)$ is dual to $Q^1\iota Q^1\iota$, we conclude that $f_{2*}[S^1 \times S^1] = Q^1\iota Q^1\iota$.

Therefore, $w(f^*\zeta_4) = 1 + x + x^2 = w(\tau_X)$ and hence $f^*n\zeta_4 = \nu_X$ for $n \equiv 3 \bmod (4)$. Finally, $f_*[X] = f_{1*}[RP^2] + f_{2*}[S^1 \times S^1] = \iota^2 Q^2\iota + Q^1\iota Q^1\iota$. □

Remarks:

1. If we had replaced the regular representation by the block sum $\Sigma_2 \times \Sigma_2 \to \Sigma_4$ in the definition of f_2, we would have produced an $f : X \to B\Sigma_4$ whose fundamental class maps correctly, but for which no multiple of ζ_4 would pull back to ν_X.
2. Jones [JJ] described the construction in terms of the dihedral group $D_8 \hookrightarrow \Sigma_4$ instead of Σ_4. It is easy to see that either construction produces the same operation $X_{f,F}$. The calculations based on D_8 are a bit messier because the extended power based upon the group D_8 is the composite D_2D_2, so that it carries the 2-fold composite operations before imposition of the Adem relations. (The natural map $D_2D_2 \to D_4$ amounts to imposition of the Adem relations.)

Proof of Theorems 7 and 8: Theorem 7 is the generalization of [**HRS**, IV.6] to all extended powers. ([**HRS**, IV.6] used an ad hoc construction of the spectral sequence, and was restricted to prime r. [**RRB**] allows us to remove this restriction.) The spectral sequence $\widetilde{E}_r$ is the spectral sequence of [**RRB**] defined by smashing the skeletal filtration of D_rS^n with an Adams resolution of S^0 and totalizing. Then $\widetilde{E}_1 = Tot(\mathcal{W} \otimes \mathcal{C})$ where $\mathcal{C}$ is the cellular chains of D_rS^n and $H(\mathcal{W}) = \mathrm{Ext}(Z_2, Z_2)$. The bicomplex spectral sequence for $\widetilde{E}_2 = H(Tot(\mathcal{W} \otimes \mathcal{C}))$ is

$$\begin{aligned} H(\mathcal{W}) \otimes H(\mathcal{C}) &= \mathrm{Ext}_A(Z_2, Z_2) \otimes H_*D_rS^n \\ &= \mathrm{Ext}_A(Z_2, Z_2) \otimes H_*B\Sigma_r^{rs} \quad \Rightarrow \quad \widetilde{E}_2 \end{aligned}$$

where we use the Thom isomorphism to replace $H_*D_rS^n$ by $H_*B\Sigma_r^{rs}$. If we choose a skeletal filtration for which the differential in $\mathcal{C}$ is trivial then the bicomplex spectral sequence collapses.

In [**HRS**, IV.5 and IV.7] it is shown that $\xi \circ D_rx$ maps the skeletal filtration of D_rS^n to the Adams filtration of S^0. By [**RRB**, Cor. 7] we have an induced map of spectral sequences. By [**HRS**, IV.5] and Theorem 2, the class q is mapped to $\sigma_n(q)(x)$. By [**RRB**, Prop. 2], the map is $\mathrm{Ext}_A(Z_2, Z_2)$ linear, completing the proof of Theorem 7. Theorem 8 is an immediate consequence. □

Proof of Example 9: For $r = 2$ and $n \equiv 3 \bmod 4$, the class $h_1 \otimes Q^1$ survives the spectral sequence of Theorem 7 to generate $\pi_{2n+2} D_2 S^n = Z/2$. The resulting homotopy operation is therefore detected by $h_1 Sq^{n+1}$. We wish to construct the geometric version of this operation. Thus we need a surface M and a map $f : M \to B\Sigma_2$ such that $\alpha = f_* \alpha_M \in \pi_{2n+2} D_2 S^n$ is detected by $h_1 \otimes Q^1$. Since α maps to η on the 1-cell, we take $M = K$, the Klein bottle, and let f be the fibering $K \to S^1$ followed by $S^1 = RP^1 \hookrightarrow B\Sigma_2$. The key point is that the natural map $D_2 S^n = T(n\zeta_2) \longrightarrow T(n\zeta_2 \oplus \lambda)$ sends the class Q^1 to the Thom class, so that $h_1 \otimes Q^1$ maps to η on the bottom class. □

Proof of Example 11: Similar to Example 9. □

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