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Journal of Modern Applied Statistical Methods

Volume 6 | Issue 1 Article 14

5-1-2007

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Journal of Modern Applied Statistical Methods
May, 2007, Vol. 6, No. 1, 141-152

1538 – 9472/07/$95.00

Approximate Bayesian Confidence Intervals for the Mean of an Exponential Distribution Versus Fisher Matrix Bounds Models

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The aim of this article is to obtain and compare confidence intervals for the mean of an exponential distribution. Considering respectively the square error and the Higgins-Tsokos loss functions, approximate Bayesian confidence intervals for parameters of exponential population are derived. Using exponential data, the obtained approximate Bayesian confidence intervals will then be compared to the ones obtained with Fisher Matrix bounds method. It is shown that the proposed approximate Bayesian approach relies only on the observations. The Fisher Matrix bounds method, that uses the z-table, does not always yield the best confidence intervals, and the proposed approach often performs better.

Key words: Estimation, loss functions, Monte Carlo simulation, statistical analysis.

Introduction

There is a significant amount of research in Bayesian analysis and modeling which has been published the last thirty-five years Harris B. 1976, Higgins J. J. Tsokos 1976, Shafer R. E. 1973. A Bayesian analysis implies the exploitation of suitable prior information and the choice of a loss function in association with Bayes' Theorem. It rests on the notion that a parameter within a model is not merely an unknown quantity, but rather behaves as a random variable, which follows some distribution. In the area of life testing, it is indeed realistic to assume that a life parameter is stochastically dynamic. This assertion is supported by the fact that the complexity of electronic and structural systems is likely to cause undetected component interactions resulting in an unpredictable fluctuation of the life parameter. Drake (1966) provided an account for the use of Bayesian statistics in reliability problems. He stated,

> He [a Bayesian] realizes… that his selection of a prior (distribution) to express his present state of knowledge will necessarily be somewhat arbitrary. But he greatly appreciates this opportunity to make his entire assumptive structure clear to the world…Why should an engineer not use his engineering judgment and prior knowledge about a parameter in the classical distribution he has picked? For example, if it is the mean time between failures (MTBF) of an exponential distribution that must be evaluated from some tests, he undoubtedly has some idea of what the value will turn out to be". (315-320)

Consider the exponential underlying model characterized by

$$f(x)=\theta e^{-\theta x};x \geq 0,\theta \succ 0 \qquad (1)$$

It is well known that once the underlying model is found to have an exponential distribution,

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Fisher Matrix bounds method (Nelson, 1982) uses the Z-table and considers the following confidence interval [] for θ.

$$L_\theta = \frac{\hat{\theta}}{e^{\frac{K_\alpha \sqrt{Var(\hat{\theta})}}{\hat{\theta}}}}$$

and

$$U_\theta = \hat{\theta} e^{\frac{K_\alpha \sqrt{Var(\hat{\theta})}}{\hat{\theta}}}, \qquad (2)$$

where K_α is defined by

$$\alpha = \frac{1}{\sqrt{2\pi}} \int_{K_\alpha}^{\infty} e^{-\frac{t^2}{2}} dt = 1 - \Phi(K_\alpha)$$

and

$$Var(\hat{\theta}) = \left(\frac{\partial^2 \Lambda}{\partial \theta^2} \right)^{-1}$$

Λ is the log-likelihood function of the exponential distribution (1).

Fisher Matrix bounds method considers large samples to ensure the use of the Z-table.. With some studies that have been conducted with small samples it has been found that the assumption of normal approximations for estimates based on small sample sizes reduces the accuracy of confidence bounds (Hartley, 2004).

For the above model (1), approximate Bayesian confidence bounds for the parameter θ and the population mean $\frac{1}{\theta}$ will be derived to challenge Fisher bounds method (2).

Although there is no specific analytical procedure that allows us to identify the appropriate loss function to be used, the most commonly used is the square error loss function. One of the reasons for selecting this loss function is because of its analytical tractability in Bayesian analysis. As it will be shown, selecting the square error loss does not always lead to the best approximate Bayesian confidence intervals. However, the obtained approximate Bayesian confidence intervals corresponding to the square error and the Higgins-Tsokos loss functions will be respectively used to challenge Fisher bounds method (2). The loss functions that will be used are given below, along with a statement of their key characteristics.

Square Error Loss Function

The popular square error loss function places a small weight on estimates near the true value and proportionately more weight on extreme deviation from the true value of the parameter. Its popularity is due to its analytical tractability in Bayesian modeling. The square error loss is defined as follows:

$$L_{SE}(\hat{\theta}, \theta) = \left(\hat{\theta} - \theta \right)^2 \qquad (3)$$

Higgins-Tsokos Loss Function:

The Higgins-Tsokos loss function places a heavy penalty on extreme over- or underestimation. That is, it places an exponential weight on extreme errors. The Higgins-Tsokos loss function is defined as follows:

$$L_{HT}(\hat{\theta}, \theta) = \frac{f_1 e^{f_2(\hat{\theta}-\theta)} + f_2 e^{-f_1(\hat{\theta}-\theta)}}{f_1 + f_2} - 1, f_1, f_2 \succ 0. \qquad (4)$$

Assume that θ behaves as a random variable that is being characterized by the Pareto probability density function given by

$$f_1(\theta) = \frac{a}{b} \left(\frac{b}{\theta} \right)^{a+1}; \theta \geq b \succ 0, a \succ 0. \qquad (5)$$

The Pareto prior has been selected because of its mathematical tractability. Using observations from exponential distributions, the Pareto will approximate prior (5) in such a way that good approximate Bayesian estimates of θ are obtained.

Preliminaries

Let x_1, x_2, ……., x_n denote the observations of a given system that are being characterized by the exponential distribution (1). The following posterior distribution is obtained:

$$h(\theta \setminus x)\frac{\theta^{n-a-1}e^{-\theta\sum_1^n x_i}}{\int_b^\infty \theta^{n-a-1}e^{-\theta\sum_1^n x_i}d\theta},\theta \succ b.. \quad (6)$$

Methodology

Approximate confidence bounds for θ

With respectively the following approximate priors for the square error and the Higgins-Tsokos loss functions, good approximate Bayesian estimates of θ are obtained.

Approximate prior for the square error loss:

$$f_1(\theta)=\frac{a_0}{b_0}\left(\frac{b_0}{\theta}\right)^{a_0+1};\theta \geq b \succ 0, a \succ 0. \quad (7)$$

$$a_0 = n, b_0 = \frac{n-1}{\sum_1^n x_i}$$

Approximate prior for the Higgins-Tsokos loss:

$$f_1(\theta)=\frac{a_1}{b_1}\left(\frac{b_1}{\theta}\right)^{a_1+1};\theta \geq b_1 \succ 0, a_1 \succ 0. \quad (8)$$

$$a_1 = n, b_1 = \frac{n}{\sum_1^n x_i} - \frac{1}{f_1+f_2}\mathrm{Ln}\left(\frac{\sum_1^n x_i + f_2}{\sum_1^n x_i - f1}\right)$$

with

$$f_1 \prec \sum_{i=1}^n x_i .$$

It's easily shown that the approximate Bayesian estimate of the parameter θ, subject to the square error loss; is the same as the Bayesian estimate of θ under the Higgins-Tsokos loss. They are equal to

$$\frac{n}{\sum_{i=1}^n x_i}.$$

Using respectively the approximate posterior distributions that correspond to (7) and (8), along with the equalities $P(\theta \succ L|\, x) = 1-\alpha/2$ and $P(\theta \succ U|\, x) = \alpha/2$, the following lower and upper confidence bounds for θ are obtained:

Approximate Bayesian confidence bounds of θ corresponding to the square error:

$$L_{\theta(SE)} = \frac{n-1-\ln(1-\alpha/2)}{\sum_1^n x_i}$$

and

$$U_{\theta(SE)} = \frac{n-1-\ln(\alpha/2)}{\sum_1^n x_i} \quad (9)$$

Approximate Bayesian confidence bounds of θ corresponding to the Higgins-Tsokos:

$$L_{\theta(HT)}\frac{n-\mathrm{Ln}(1-\alpha/2)}{\sum_{i=1}^n x_i} - \frac{1}{f_1+f_2}\mathrm{Ln}\left(\frac{\sum_{i=1}^n x_i + f_2}{\sum_{i=1}^n x_i - f_1}\right)$$

$$U_{\theta(HT)}\frac{n-\mathrm{Ln}(\alpha/2)}{\sum_{i=1}^n x_i} - \frac{1}{f_1+f_2}\mathrm{Ln}\left(\frac{\sum_{i=1}^n x_i + f_2}{\sum_{i=1}^n x_i - f_1}\right)$$

Approximate Bayesian confidence bounds for the exponential population mean

Thus, we respectively obtain the following $100(1-\alpha)\%$ empirical Bayes confidence bounds for the mean b of the exponential failure model, when the squared error and the Higgins-Tsokos loss functions are considered:

$$L_{b(SE)} = \frac{\sum_{i=1}^{n} x_i}{n-1-\ln(\alpha/2)}$$

and

$$U_{b(SE)} = \frac{\sum_{i=1}^{n} x_i}{n-1-\ln(1-\alpha/2)}, \qquad (10)$$

and

$$L_{b(HT)} = \frac{1}{\dfrac{n-Ln(\alpha/2)}{\sum_{i=1}^{n} x_i} - \dfrac{1}{f_1+f_2} Ln\left(\dfrac{\sum_{i=1}^{n} x_i + f_2}{\sum_{i=1}^{n} x_i - f_1}\right)} \qquad (11)$$

$$U_{b(HT)} = \frac{1}{\dfrac{n-Ln(1-\alpha/2)}{\sum_{i=1}^{n} x_i} - \dfrac{1}{f_1+f_2} Ln\left(\dfrac{\sum_{i=1}^{n} x_i + f_2}{\sum_{i=1}^{n} x_i - f_1}\right)}.$$

Numerical Results

In order to compare the proposed approximate Bayesian approach to the Fisher Matrix bounds method, samples that have been obtained from exponentially distributed populations will be considered. For the Higgins-Tsokos loss function, consider $f_1 = 1, f_2 = 1$. The lengths of the Fisher Matrix bounds and approximate Bayesian confidence intervals are respectively denoted by l_F, l_{SE} and l_{HT}.

Example 1

Monte Carlo simulation has been used to generate the following 30 observations from the exponential distribution with mean equal to 1.

0.9549716 ,	0.09670773 ,	0.09107758,
2.6951610 ,	1.47495800 ,	0.56762340
1.2636410,	1.60653000 ,	0 94337030,
0.5499995 ,	0.64000010 ,	0.62536590
1.4492260 ,	0.78403890 ,	1.08172600,
0.3108478,	1.47283200,	0.47580980
3.1378870 ,	0.11715670 ,	0.92341850,
0.5124997	0.22012280	3.81572700
0.5791140 ,	0.50421350 ,	0.14532570 ,
0.7749708	1.07792000	1.08156300.

Table 1: Fisher Matrix Bounds and Approximate Bayesian Confidence Intervals of the Exponential Population Mean When the Population Mean is Equal to 1.

Confidence level	Fisher Matrix bounds	*Approx.Bayesian bounds (SE)*	*Approx.Bayesian bounds (HT)*
80%	0.7909 – 1.2621	0.9575 – 1.0298	0.9575 – 1.0298
90%	0.7392 – 1.3503	0.9368 – 1.0317	0.9368 – 1.0317
95%	0.6985 – 1.4289	0.9169 – 1.0326	0.9169 – 1.0326
99%	0.6238 – 1.6002	0.8739 – 1.0334	0.8739 – 1.0334

Confidence level	$(l_F) \div (l_{SE})$	$(l_F) \div (l_{HT})$
80%	6.5172	6.5172
90%	6.4394	6.4394
95%	6.3128	6.3128
99%	6.1216	6.1216

Example 2

Monte Carlo simulation has been used to generate the following 30 observations from the exponential distribution with mean equal to 9

2.0270,	4.0103,	30.0421,	0.1189,	2.7558.
13.7441,	13.3840,	27.0930,	7.3750,	3.7323,
23.4171,	0.06310.	5.6839,	8.7473,	10.2778,
25.2331,	10.1903,	0.3761,	3.3068,	3.4954,
6.9136,	1.8234,	16.3160,	2.4359,	19.9108,
2.5285,	3.9314,	3.4645,	6.9229,	10.4509.

Table 2: Fisher Matrix Bounds and Approximate Bayesian Confidence Intervals of the Exponential Population Mean When the Population Mean is Equal to 9.

Confidence level	Fisher Matrix bounds	Approx.Bayesian bounds (SE)	Approx.Bayesian bounds (HT)
80%	7.1184 – 11.3598	8.6182 – 9.2688	8.6182 – 9.2688
90%	6.6534 – 12.1537	8.4315 – 9.2861	8.4315 – 9.2861
95%	6.2873 – 12.8614	8.2527 – 9.2944	8.2527 – 9.2944
99%	5.6144 – 14.4028	7.8655 – 9.3009	7.8655 – 9.3009

Confidence level	$(l_F) \div (l_{SE})$	$(l_F) \div (l_{HT})$
80%	6.5192	6.5192
90%	6.4361	6.4361
95%	6.3109	6.3109
99%	6.1226	6.1226

Example 3

Monte Carlo simulation has been used to generate the following 40 observations from the exponential distribution with mean equal to 20

4.5046,	8.9119,	66.7603,	0.2643,	6.1241,
30.5425,	29.7423,	60.2067,	16.3891,	8.2941,
52.0380,	0.1402,	12.6309,	19.4385,	22.8395,
52.3378,	3.4389,	19.3268,	8.2350,	3.4737,
56.0736,	22.6451,	0.8359,	7.3484,	7.7675,
15.3635,	4.05222,	36.2578,	5.6189,	8.7365,
7.6990,	15.3844,	23.2242,	11.8542,	63.6975,
14.8772,	32.9585,	2.2127,	5,4132,	44.2462

Table 3: Fisher Matrix Bounds and Approximate Bayesian Confidence Intervals of the Exponential Population Mean When the Population Mean is Equal to 20.

Confidence level	Fisher Matrix bounds	Approx.Bayesian bounds (SE)	Approx.Bayesian bounds (HT)
80%	16.5786 – 24.8507	19.6574 – 20.7619	19.6574 – 20.7619
90%	15.6366 – 26.3479	19.3330 – 20.7907	19.3330 – 20.7907
95%	14.8886 – 27.6715	19.0191 – 20.8045	19.0191 – 20.8045
99%	13.4983 – 30.5216	18.3281 – 20.8153	18.3281 – 20.8153

Confidence level	$(l_F) \div (l_{SE})$	$(l_F) \div (l_{HT})$
80%	7.4894	7.4894
90%	7.3480	7.3480
95%	7.1596	7.1596
99%	6.8443	6.8443

Example 4

The following exponential data and results were obtained by Washington State Department of Ecology while conducting research on the amount of lead concentration in certain types of fish found in the Spokane River.

Lead (Pb) Concentrations in 1999 Spokane River Fish Source: WA State Dept. of Ecology report concentrations in parts per million (ppm)			
Filets	trout	whitefish	sucker
	0.480	0.020	0.088
	0.071	0.020	0.210
	0.110	0.020	0.280
	0.320	0.020	0.030
	0.120	0.020	0.036
	0.220	0.065	0.047
	0.055	0.020	0.077
	0.320	0.037	0.069
	0.077	0.020	0.160
	0.081	0.036	0.088
	0.170		0.120
	0.130		0.054
	0.110		0.080
	0.081		0.059
	0.098		0.094
	0.180		0.059
	0.230		0.068
	0.082		0.020
	0.210		0.090
	0.200		0.046
	0.025		
	0.038		
Mean	**0.155**	**0.028**	**0.089**
std dev	**0.110**	**0.015**	**0.063**

Table 4: Fisher Matrix Bounds and Approximate Bayesian Confidence Intervals of the Mean Lead Concentration in Trout.

Confidence level	Fisher Matrix bounds	Approx.Bayesian bounds (SE)	Approx.Bayesian bounds (HT)
80%	0.11791 - 0.20351	0.15280 – 0.169507	0.15301 – 0.16976
90%	0.10896 – 0.22021	0.14820 – 0.16996	0.14839 – 0.17022
95%	0.10199 – 0.23526	0.14386 – 0.17018	0.14404 – 0.17044
99%	0.08936 – 0.26851	0.13471 – 0.17035	0.13487 – 0.17061

Confidence level	$(l_F) \div (l_{SE})$	$(l_F) \div (l_{HT})$
80%	5.1236	5.1104
90%	5.1125	5.0961
95%	5.0634	5.0481
99%	5.0266	5.0125

Table 5. Fisher Matrix Bounds and Approximate Bayesian Confidence Intervals of the Mean Lead Concentration in Whitefish.

Confidence level	Fisher Matrix bounds	Approx.Bayesian bounds (SE)	Approx.Bayesian bounds (HT)
80%	0.01854 – 0.04167	0.02698 - 0.03429	0.02556 – 0.03204
90%	0.01649 – 0.04684	0.02528 – 0.03452	0.02403 – 0.03224
95%	0.01495 – 0.05166	0.02378 – 0.03464	0.02267 – 0.03234
99%	0.01229 – 0.06285	0.02090 – 0.03472	0.02004 - 0.03241

Confidence level	$(l_F) \div (l_{SE})$	$(l_F) \div (l_{HT})$
80%	3.1641	3.5694
90%	3.2846	3.6967
95%	3.3802	3.7962
99%	3.6584	4.0873

Table 6. Fisher Matrix Bounds and Approximate Bayesian Confidence Intervals of the Mean Lead Concentration in Sucker.

Confidence level	Fisher Matrix bounds	Approx.Bayesian bounds (SE)	Approx.Bayesian bounds (HT)
80%	0.06666 – 0.11816	0.08742 – 0.09803	0.08799 – 0.09875
90%	0.06136 – 0.12835	0.08454 – 0.09833	0.08507 – 0.09905
95%	0.05725 – 0.13756	0.08183 – 0.09847	0.08234 – 0.09919
99%	0.04984 – 0.15802	0.07618 – 0.09858	0.07662 – 0.09931

Confidence level	$(l_F) \div (l_{SE})$	$(l_F) \div (l_{HT})$
80%	4.8539	4.7862
90%	4.8578	4.7918
95%	4.8263	4.7661
99%	4.8294	4.7677

Example 5

The following exponential data represent a random sample of cycles to failure in ten-thousands for twenty heater switches subject to an overload voltage.

0.01,	0.034,	0.194,	0.567,	0.601,
0.712,	1.291,	1.367,	1.949,	2.37,
2.411,	2.875,	3.162,	3.28,	3.491,
3.686,	3.854,	4.211,	4.397,	6.473.

Elfessi and Raineke (2001) conducted some studies on the above data and obtained the following the following maximum likelihood estimate and 95% confidence interval for the parameter θ: 0.4261 and (0.2603, 0.6322).

Table 1, Table 2 and Table 3 show that, in the first three examples, the proposed approximate Bayesian confidence intervals perform better than confidence interval obtained with Fisher Matrix bounds method. All seven Tables show that the proposed approximate Bayesian confidences intervals perform well.

Conclusion

Approximate Bayesian confidence intervals for parameters of exponential populations under two different loss functions have been derived. The loss functions that are employed are the square error and the Higgins-Tsokos loss functions. Based on the above numerical results, the following may be concluded:

Table 7: Fisher Matrix Bounds and Approximate Bayesian Confidence Intervals of θ

Confidence level	Fisher Matrix bounds	Approx.Bayesian bounds (SE)	Approx.Bayesian bounds (HT)
80%	0.32005 – 0.56732	0.38575 – 0.43256	0.38575 – 0.43256
90%	0.29464 – 0.61626	0.38460 – 0.44733	0.38459 – 0.44733
95%	0.27491 – 0.66049	0.38404 – 0.46210	0.38404 – 0.46210
99%	0.23932 – 0.75871	0.38361 – 0.49639	0.38361 – 0.49639

Confidence level	$(l_F) \div (l_{SE})$	$(l_F) \div (l_{HT})$
80%	5.2824	5.2824
90%	5.1270	5.1262
95%	4.9353	4.9395
99%	4.6053	4.6053

1. When representative samples are considered, the Fisher Matrix bounds method used to construct confidence intervals for exponential parameters does not always yield the best coverage accuracy.

2. The Fisher Matrix bounds method used to construct confidence intervals for the mean of an exponential population does not always yield the best coverage accuracy. In fact, in Table 1, Table 2 and Table 3, each of the obtained approximate Bayesian confidence intervals contains the population mean and is strictly included in the corresponding confidence interval obtained with Fisher Matrix bounds method.

3. Contrary to Fisher Matrix bounds method that uses the Z-table, the proposed approach relies only on the observations.

4. With the proposed approach, approximate Bayesian confidence intervals for exponential population means are easily computed for any level of significance.

5. Bayesian analysis contributes to reinforcing well-known statistical theories such as the estimation theory.

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