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Optimal stopping problems for a family of continuous-time Markov processes*

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Abstract

In this paper we study the well-know optimal stopping problem applied to a general family of continuous-time Markov process. The approach to follow is merely analytic and it is based on the characterization of stopping problems through the study of a certain variational inequality; namely one solution of this inequality will coincide with the optimal value of the stopping problem. In addition, by means of this characterization, it is possible to find the so-named continuation region, and as a byproduct obtaining the optimal stopping time. The most of the material is based on the semigroup theory, infinitesimal generators and resolvents. The chapter is a complete version of the former presentation without detailed proofs in [27].

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1 Introduction

Optimal stopping problems are perhaps one of the most interesting and studied problems in the theory of stochastic processes. Succesfull methods have been devloped during decades to show the existence and several characterizations of optimal stopping times. The most studied methods to address these problems are definitely the theory of Snell envelopes and backward-reflected stochastic differential equations —see Bickel, et. al. [9], El Karoui et. al. [16, 17], Goran and Shyryaev [19], Hamedène and Jeanblanc [20]—, but on the other hand, there is also another useful method that tackles stopping problems from a merely analytical viewpoint —see Bensoussan and Lions [4, 6], Menaldi [24, 25], Oksendal and Sulem [33], Robin [36, 37], among others.

One of the main differences of the second method with respect to the former is the assumption of a markovian structure of the process, so in principle it could seem more restrictive. However, its analytical nature allows the use of sophisticated tools of functional analysis, set topology, or even more, the use of numerical approximations of the original (and theoretic) problem —see for instance Glowinski et.al.[18].

In this work we shall apply the analytical approach we have already mentioned, and extend several works on this line. But before to specify the details, we can depart to mentioning some pioneer works on this analytical direction, such as Bensoussan and Lions [4, 6] and Bensoussan [5]. All these works were focused on the study of both optimal stopping and impulsive control problems associated to non-degenerated diffusion processes. Based on these works, several authors followed the same line (with both/either theoretical and/or applied viewpoints) that have produced during decades a spread of knowledge on this field.

Other former but nor less important works were developed by Robin [36, 37] and later by Stettner [39] that also applied analytical tools for solving optimal stopping problems on general continuous-time Markov-Feller

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processes. Within the analysis of the aforementioned papers, we highlight the assumptions on the state space of either type: locally compact or compact.

Following with the description of the former literature, we can quote Menaldi's works [24, 25, 26] as well as the one by Menaldi and Sritharan [30], in which the authors analyzed two great families of Markov-Feller process: (1) degenerate stochastic differential equations with either jumps or without jumps, and (2) Navier-Stokes equations; in all these mentioned works the authors take advantage to the particularities of the model in order to explore the regularity of the optimal values. As for the discrete-time models there is a hand full of works such as Bensoussan [8] Rieder [35], Horiguchi [21] and Jasso-Fuentes et.al. [22].

In this work we use the same line that Robin's works [36, 37] but we drop the local-compactness assumption of the state space. It is important to say that our model is based on the existence of a Markov process that lives on a *fixed* probability space, wheras in the aforementioned references, this space is constructed through the canonical space. This implies that both works are not a special case of each other. Actually, we are somehow inspired from the ideas scattered in reference [27]. One difference of this reference with respect to this proposal, is the nature of the dynamical system and also the general details, since in this work we detail point by point all the arguments of the proofs.

The content of this paper is organized as follows: In Section 2, we describe the class of Markov process we are interesting with, and its associated semigroup. Due to a minimal set of assumptions imposed to this process, we will be forced to introduce a seminorm that measures the maximum value of functions along the trajectories (rather that over the whole space, that is the usual case of the supremum norm). This seminorm, produces some properties of the aforementioned semigroup such as a kind of Feller version that is measured through this seminorm. By the end of the section, we will define the corresponding infinitesimal generator and the resolvent operators that both together play a substantial role within the analysis of the optimal stopping problems. In Section 3, we will turn our attention to the study of the so-called penalized problem, whose main characteristic is the associated parametric family of functional equations that will be analyzed in this part; in particular, the existence and regularity of these functional equations are ensured. Later we will consider a certain variational inequality. This inequality satisfies the following two nice properties: (i) one of its subsolutions becomes the limit of the (unique) solutions of the aforementioned family of parametric equations and (ii) the maximal sub-solution of this inequality is just the minimal cost of our stopping problem; this last property will be proved later in Section 4, in which we will also provide a characterization of the optimal stopping time as a hitting time associated to a given set so-called continuation region or contact set. We finalize the manuscript with some concluding remarks on Section 5.

2 A family of Markov processes

In this section we introduce the dynamics of our stopping problem. This dynamics consists of a continuous-time Markov process that in turn defines a family of operators so-called the semigroup of the process. With these elements it is possible to introduce both an infinitesimal generator and a resolvent operator related to that semigroup. These later operators will play a substantial role for the analysis of the optimal stopping problem. The way to construct the above mentioned mathematical objects is not straightforward due to the generality of the state space.

2.1 Preliminaries

Let $\mathcal{E} := (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a fixed filtered probability space, satisfying the usual conditions (i.e., $\mathcal{F}_t$ is right-continuous and $\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets). Also consider an open subset $\mathcal{O}$ of a Banach space with norm $|\cdot|$, and let us write $\mathcal{B}(\mathcal{O})$ to denote the σ-algebra generated by $\mathcal{O}$.

Throughout this work we will be working with a generic *homogeneous* $\mathcal{O}$-valued stochastic process $\{y(t,x)\}_{t\geq 0}$, with initial condition $x \in \mathcal{O}$ (i.e. $\mathbb{P}(y(0,x) = x) = 1$), defined on $\mathcal{E}$, satisfying the following conditions:

(a) There exist constants $\alpha_0 > 0$, and $k \geq 1$, as well as a measurable function $w : \mathcal{O} \to [1, +\infty)$ satisfying $\lim_{|x|\to\infty} w(x) = \infty$, such that all together satisfy the following:

(a.1)
$$\mathrm{E}\left[\sup_{s\geq 0}\left\{e^{-\alpha_0 s}w(y(s,x))\right\}\right] \leq kw(x), \quad \forall\, x\in\mathcal{O}, \quad \text{and} \tag{2.1}$$

(a.2)
$$\mathrm{E}\left[e^{-\alpha_0 s}w(y(s,x))\right] \leq w(x), \quad \forall\, x\in\mathcal{O} \quad \text{and} \quad \forall\, s\geq 0, \tag{2.2}$$

where $\mathrm{E}[\cdot]$ is the expectation associated to $\mathbb{P}$.

(b) The Markov property:
$$\mathbb{P}(y(t+s,x)\in B|\mathcal{F}_s) = \mathbb{P}(y(t,y(s,x))\in B), \quad a.s. \quad \forall\, t,s\geq 0,\ B\in\mathcal{B}(\mathcal{O}). \tag{2.3}$$

The right-hand side of the above equality is understood as the evaluation of the mapping $z\mapsto \mathbb{P}(y(t,z)\in B)$ at $z=y(s,x)$.

(c) The following relation holds true for all $s,t\geq 0$, and $x\in\mathcal{O}$
$$\mathrm{E}[h(y(t,y(s,x)))] = \mathrm{E}[h(y(s,y(t,x)))]. \tag{2.4}$$

(d) For each $x\in\mathcal{O}$, $t\mapsto y(t,x)$ has not discontinuities of second kind. Moreover, all $x\in\mathcal{O}$ and $\varepsilon>0$ there is $\delta>0$ such that if $0\leq t\leq\delta$ then
$$\mathbb{P}\big(\sup_{0\leq s\leq\frac{1}{\varepsilon}} |y(t+s,x)-y(s,x)|\geq\varepsilon\big) < \varepsilon. \tag{2.5}$$

On the other hand, let $B(\mathcal{O})$ be the space consisting of measurable functions $h:\mathcal{O}\to\mathbb{R}$ such that
$$h(y(t,x))\in L_1(\Omega,\mathbb{R}), \quad \forall\, t\geq 0,\ x\in\mathcal{O}. \tag{2.6}$$

Note that every bounded measurable function satisfies the latter property.

• *Remark* 2.1. (a) It is worth to say that properties (2.1), (2.2), and (2.5) are common in special cases of Markov processes, such as those that come from solutions of both ordinary and partial stochastic differential equations —see Bensoussan and Lions[4, 6], Bensoussan [5], Menaldi [24, 25, 26], Menaldi and Sritharan [28, 29, 30].

(b) It is not difficult to prove that the Markov property (2.3) is equivalent to this one:
$$\mathrm{E}[h(y(t,y(s,x)))] = \mathrm{E}[h(y(t+s,x))|\mathcal{F}_s] \quad \forall\, t\geq s\geq 0,\ x\in\mathcal{O}, \quad \forall\, h\in B(\mathcal{O}). \tag{2.7}$$

(c) Condition (2.4) is a kind of uniqueness on the paths. This type of relation is satisfied for a big family of Markov processes $\{y(t,x)\}_{t\geq 0}$, for instance the well-known family of Ito's process (with or without jumps, of finite or infinite dimension) —see Bensoussan and Lions[4, 6], Bensoussan [5], Menaldi [24, 25, 26], Menaldi and Sritharan [28, 29, 30], Da Prato [12, 13], among others.

(d) By writing the set of right-discontinuities of $\{y(t,x)\}_{t\geq 0}$ as
$$\cup_{\varepsilon>0}\cap_{\delta>0}\{\sup_{0\leq t\leq\delta}|y(t+s,x)-y(s,x)|\geq\varepsilon\}.$$

we can deduce, using (2.5), that it has right-continuous paths. Hence as it neither has not second order discontinuities, we can conclude that $\{y(t,x)\}_{t\geq 0}$ is càdlàg.

We will also think over the space of functions $h\in B(\mathcal{O})$ with the property of
$$\sup_{x\in\mathcal{O}}\frac{|h(x)|}{w(x)} < \infty. \tag{2.8}$$

This space is denoted by $B_w(\mathcal{O})$ that will be endowed with the norm
$$\|h\|_w := \sup_{x\in\mathcal{O}}\frac{|h(x)|}{w(x)}. \tag{2.9}$$

It is common to say that every function in $B_w(\mathcal{O})$ satisfies a finite w-growth. In addition, it is not difficult to show that $(B_w(\mathcal{O}), \|\cdot\|_w)$ is a Banach space.

Finally, using the (fixed) constant $\alpha_0 > 0$ appearing in (2.1), we introduce the family of seminorms $\{p(\cdot, x)\}_{x\in\mathcal{O}}$ on $B(\mathcal{O})$ by

$$p(h,x) = \mathrm{E}\left[\sup_{s\geq 0}\left\{e^{-\alpha_0 s}\left|h(y(s,x))\right|\right\}\right], \quad \forall\, x \in \mathcal{O}. \tag{2.10}$$

Each element of the above family is in fact a seminorm because $p(h,x) \geq 0$, $p(ah,x) = |a|\, p(h,x)$ for all $a \in \mathbb{R}$ and $p(h+g,x) \leq p(h,x) + p(g,x)$, but if $p(h,x) = 0$ then $\{h(y(s,x))\}_{s\geq 0}$ is indistinguishable of the constant process equals to zero.

We shall denote by $B_p(\mathcal{O})$ the subspace of $B(\mathcal{O})$ consisting of functions h satisfying

$$p(h,x) < \infty, \quad \forall\, x \in \mathcal{O}. \tag{2.11}$$

Note that the definition of this later space, together with the definition of $B_w(\mathcal{O})$ in (2.9), and the assumption in (2.1), all together yield that $B_w(\mathcal{O}) \subseteq B_p(\mathcal{O}) \subseteq B(\mathcal{O})$.

Throughout this paper we will be assuming that our process $y(\cdot,\cdot)$ will hold the conditions (2.1) *to* (2.5)*. Thus, all the upcoming results will be intrinsically assuming all these properties, without to necessarily specify them within the statement of each result.*

2.2 The associated semigroup

For $\alpha \geq \alpha_0$, with α_0 as in (2.1), we define the family of operators $\{\Phi_\alpha(t)\}_{t\geq 0}$ on $B_p(\mathcal{O})$ by

$$\Phi_\alpha(t)h(x) = \mathrm{E}[e^{-\alpha t} h(y(t,x))], \quad \forall\, x \in \mathcal{O},\ h \in B_p(\mathcal{O}),\ t \geq 0. \tag{2.12}$$

In view of $\Phi_\alpha(t)$ is essentially an integral (with respect to the probability measure $\mathbb{P}$), we have that it is monotone, that is, $h \geq 0$ implies $\Phi_\alpha(t)h \geq 0$ for any $t \geq 0$. Besides, from the definition of $\Phi_\alpha(t)$ in (2.12), it is clear that $\Phi_\alpha(0)h = h$. This family of operators also satisfies the *semigroup property* $\Phi_\alpha(t)\Phi_\alpha(s) = \Phi_\alpha(t+s)$ that follows directly from the Markov property, namely for $h \in B_p(\mathcal{O})$,

$$\begin{aligned}\Phi_\alpha(t)\Phi_\alpha(s)h(x) =& \mathrm{E}[e^{-\alpha t}\Phi_\alpha(s)h(y(t,x))] = \mathrm{E}[e^{-\alpha t}\,\mathrm{E}[e^{-\alpha s}h(y(t,y(s,x)))]]\\ =& \mathrm{E}[e^{-\alpha(t+s)}\,\mathrm{E}[h(y(t+s,x))|\mathcal{F}_s]] = \mathrm{E}[e^{-\alpha(t+s)}h(y(t+s,x))]\\ =& \Phi_\alpha(t+s)h(x).\end{aligned}$$

If $h \in B_w(\mathcal{O})$ then, using the inequality (2.2) as well as the norm in (2.9), we get the following

$$\begin{aligned}|\Phi_\alpha(t)h(x)| \leq \mathrm{E}[e^{-\alpha t}\left|h(y(t,x))\right|] &= \mathrm{E}\left[e^{-\alpha t}\frac{|h(y(t,x))|}{w(y(t,x))}w(y(t,x))\right]\\ &\leq \|h\|_w\,\mathrm{E}[e^{-\alpha_0 t}w(y(t,x))] \leq \|h\|_w\, w(x), \quad \forall\, x \in \mathcal{O}.\end{aligned}$$

Hence,

$$\|\Phi_\alpha(t)h\|_w \leq \|h\|_w\,. \tag{2.13}$$

The semigroup property naturally arises when the operators $\Phi_\alpha(t)$ are defined as an integral with respect to a given transition probability kernel $p(x,t,\cdot) = \mathbb{P}[y(t,x) \in \cdot]$ that in turn satisfies the well-known Chapman-Kolmogorov equations. This last type of equations is very common in specific models, such as continuous-time Markov chains, Lévy Processes, partial stochastic differential equations, to mention a few. (See Anderson [1], Applebaum [2], Da Prato [12, 13], among others). The family of the operators Φ_α defined in (2.12) will be called throughout this work as the associated *semigroup* of the Markov process $\{y(t,x)\}_{t\geq 0}$.

As we will see in the following result, the semigroup Φ_α satisfies the contraction property with respect to the seminorm $p(\cdot,x)$, the details are as follows.

Proposition 2.2. *For each $h \in B_p(\mathcal{O})$, $t, s \geq 0$ and $x \in \mathcal{O}$ we have that $p(\Phi_\alpha(t)h, x) \leq p(h,x)$.*

Proof. Fixed $h \in B_p(\mathcal{O})$, $t, s \geq 0$ and $x \in \mathcal{O}$, we have

$$\begin{aligned} p(\Phi(t)h, x) &= \mathrm{E}\left[\sup_{s\geq 0}\{e^{-\alpha_0 s}|\,\mathrm{E}[e^{-\alpha t}h(y(t, y(s, x)))]|\}\right] \\ &= \mathrm{E}\left[\sup_{s\geq 0}\{e^{-\alpha_0 s}|\,\mathrm{E}[e^{-\alpha t}h(y(s, y(t, x)))]|\}\right] \quad \text{(by (2.4))} \\ &\leq \mathrm{E}\left[\,\mathrm{E}[\sup_{s\geq 0}\{e^{-\alpha_0 s}e^{-\alpha t}|h(y(s, y(t, x)))|\}]\right] \end{aligned}$$

On the other hand, it is not difficult to prove that the Markov property in (2.6) implies the Markov property (see e.g. [40, Section 5.2.2.]) in the following sense

$$\mathrm{E}[\sup_{s\geq 0}\{e^{-\alpha_0 s}e^{-\alpha t}|h(y(s, y(t, x)))|\}] = \mathrm{E}[\sup_{s\geq 0}\{e^{-\alpha_0 s}e^{-\alpha t}|h(y(s+t, x))|\}\big|\mathcal{F}_t],$$

Then we can conclude that

$$\begin{aligned} p(\Phi(t)h, x) &\leq \mathrm{E}\left[\,\mathrm{E}[\sup_{s\geq 0}\{e^{-\alpha_0 s}e^{-\alpha t}|h(y(s+t, x))|\}\big|\mathcal{F}_t]\right] \\ &= \mathrm{E}[\sup_{s\geq 0}\{e^{-\alpha_0 s}e^{-\alpha t}|h(y(s+t, x))|\}] \\ &\leq \mathrm{E}[\sup_{s\geq 0}\{e^{-\alpha_0 s}|h(y(s, x))|\}] = p(h, x). \end{aligned}$$

□

We now impose the next assumption to our model.

Assumption 2.3. We assume that

$$(t, x) \mapsto \mathbb{P}(y(t, x) \in B) \quad \text{is measurable } \forall\, B \text{ in } \mathcal{B}(\mathcal{O}). \tag{2.14}$$

The measurability assumption is a well known fact, as it is established in Dellacherie and Meyer[14], Ethier and Kurtz [15], Rogers and Williams [38]. A clear consequence of the above property is that $(t, x) \mapsto \mathrm{E}[h(y(t, x))]$ is measurable for every simple function $h : \mathcal{O} \to \mathbb{R}$. An thus, a standard convergence procedure to every $h \in B(\mathcal{O})$ from sequences of simple functions yield that

$$(t, x) \mapsto \mathrm{E}[h(y(t, x))] \quad \text{is measurable } \forall\, h \in B(\mathcal{O}). \tag{2.15}$$

- *Remark* 2.4. (a) Assumption 2.3 together with (2.13), give us that $\Phi_\alpha(t)$ leaves invariant the space $B_w(\mathcal{O})$; actually, our family of operators $t \mapsto \Phi_\alpha(t)$ satisfies the properties of the so-called *monotone semigroup of contractions* defined on $B_w(\mathcal{O})$.

  (b) Even more, Assumption 2.3 and Proposition 2.2 also give the invariance of the semigroup Φ_α over the set $B_p(\mathcal{O})$.

Continuity of the semigroup. In many situations, the above semigroup satisfies the so-called *strong* continuity (see, e.g. Anderson [1], Applebaum [2], Da Prato [12], Böttcher et. al. [10], among others)

$$\|\Phi_\alpha(t)h - h\| \to 0, \quad \text{as } t \downarrow 0, \tag{2.16}$$

applied to a suitable space of functions h —for example, the set of continuous functions that vanish at infinity—. The above case is very common when the dimension of $\mathcal{O}$ is either finite-dimensional or locally compact. However, there exist situations when $\mathcal{O}$ does not hold the previous two properties —for example assume that $\mathcal{O}$ is a Hilbert space as in references [28, 29, 30])—, so convergence (2.16) is not longer valid. However, it is possible to obtain a sort of continuity type in the next weaker sense (see, for instance Böttcher et. al. [10], Menaldi [27], or Menaldi and Sritharan [30]).

$$\Phi_\alpha(t)h(x) - h(x) \to 0, \quad \text{as } t \downarrow 0 \quad \forall\, x \in \mathcal{O}, \tag{2.17}$$

where h is Borel measurable. One of the disadvantages of this later continuity is that it produces a lack of regularity of some sophisticated mathematical objects (i.e., infinitesimal generator, the resolvent operator, among others), whose definitions depend strongly from the convergence in (2.17).

Since our hypotheses of the state space $\mathcal{O}$ are not restricted to the cases of finite dimension nor local compactness, it is expected to not obtain convergence of type (2.16), even when we could use the norm $\|\cdot\|_w$. To avoid this inconvenient, we shall seek an intermediate convergence, weaker than (2.16) but a little stronger than (2.17) so that we are in conditions to achieve regularity properties for the infinitesimal generator and on the resolvent operator. The key point is to define a suitable functions set whose elements are continuous in certain sense but at the same time, the semigroup applied to this set can be continuous in seminorm (see Definition 2.6 below).

Let us now define the concept of convergence in seminorm that is crucial to define continuity in seminorm sense.

Definition 2.5. We say that a sequence h_n in $B_p(\mathcal{O})$ *converges in seminorm* to some h in $B_p(\mathcal{O})$ as $n \to \infty$, denoted by $\text{s-lim}_{n\to\infty} h_n = h$, if

$$\lim_{n\to\infty} p(h_n - h, x) = 0, \ \forall\, x \in \mathcal{O} \tag{2.18}$$

Moreover, if the elements of the above sequence are in $B_w(\mathcal{O})$ then we say that h_n *converges boundedly in seminorm* to h as $n \to \infty$, denoted by $\text{bs-lim}_{n\to\infty} h_n = h$, provided the following conditions are satisfied

$$\begin{cases} \sup_{n\in\mathbb{N}} \|h_n\|_w < \infty; \\ \text{s-lim}_{n\to\infty} h_n = h. \end{cases} \tag{2.19}$$

Note that for each $x \in \mathcal{O}$, $t \geq 0$, and $h \in B_w(\mathcal{O})$, a simple use of the bound (2.1) yields that

$$\begin{aligned} p(h,x) &= \mathrm{E}[\sup_{s\geq 0} e^{-\alpha_0 s}\, |h(y(s,x))|] \\ &\leq \mathrm{E}[\sup_{s\geq 0} e^{-\alpha_0 s}\, \|h\|_w\, w(y(s,x))] \leq k\, \|h\|_w\, w(x) < \infty. \end{aligned} \tag{2.20}$$

The above relation means that convergence in norm implies convergence in seminorm which, at the same time, implies pointwise convergence.

Definition 2.6. We define the subspace $C_p(\mathcal{O})$ of $B_p(\mathcal{O})$ that is conformed by the functions h such that:

(a) $\text{s-lim}_{t\downarrow 0} \Phi_\alpha(t)h = h$,

(b) for each $x \in \mathcal{O}$ we have that $\{h(y(s,x))\}_{s\geq 0}$ is a cad-lag process

We also denote the intersection $C_p(\mathcal{O}) \cap B_w(\mathcal{O})$ by $C_p^w(\mathcal{O})$.

The next proposition shows further properties of the sets $C_p(\mathcal{O})$ and $C_p^w(\mathcal{O})$

Proposition 2.7. *Under Assumption 2.3, we have*

(a) *The sets $C_p(\mathcal{O})$ and $C_p^w(\mathcal{O})$ are non-empty.*

(b) *For every $t \geq 0$:*

- **(b.1)** $\Phi_\alpha(t)h \in C_p(\mathcal{O})$ *when* $h \in C_p(\mathcal{O})$,
- **(b.2)** $\Phi_\alpha(t)h \in C_p^w(\mathcal{O})$ *when* $h \in C_p^w(\mathcal{O})$.

Proof. (a) Let $C_u(\mathcal{O})$ be the space of bounded uniform continuous functions and take $h \in C_u(\mathcal{O})$. Note that

$$p(\Phi_\alpha(t)h - h, x) \leq p(\Phi_\alpha(t)h - e^{-\alpha t}h, x) + p(e^{-\alpha t}h - h, x) \leq p(\Phi_\alpha(t)h - e^{-\alpha t}h, x) + (e^{-\alpha t} - 1)p(h,x),$$

where $(e^{-\alpha t} - 1) \to 0$ when $t \downarrow 0$. So, we aim to show $p(\Phi_\alpha(t)h - e^{-\alpha t}h, x) \to 0$ when $t \downarrow 0$. Namely, for any $t \geq 0$ and $x \in \mathcal{O}$, we have

$$\begin{aligned} p(\Phi_\alpha(t)h - e^{-\alpha t}h, x) &\leq \mathrm{E}\left[\sup_{s\geq 0} e^{-\alpha_0 s}\, |h(y(t+s,x)) - h(y(s,x))]|\right] \\ &\leq \mathrm{E}\left[\sup_{0\leq s\leq T} e^{-\alpha_0 s}\, |h(y(t+s,x)) - h(y(s,x))]|\right] \\ &\quad + \mathrm{E}\left[\sup_{s\geq T} e^{-\alpha_0 s}\, |h(y(t+s,x)) - h(y(s,x))]|\right] \end{aligned} \tag{2.21}$$

for any $T > 0$. In order to bound this expression, let us take $\varepsilon > 0$ and choose $0 < \delta_1 \leq \varepsilon$ such that $|x - \bar{x}| < \delta_1$ implies $|h(x) - h(\bar{x})| < \varepsilon$. In turn, in virtue of (2.5), let us choice $0 < \delta_0 \leq \delta_1$ such that for all $0 \leq t \leq \delta_0$ we have

$$\mathbb{P}(\sup_{0\leq s\leq \frac{1}{\delta_1}} |y(t+s,x) - y(s,x)| \geq \delta_1) < \delta_1.$$

Letting $T = \frac{1}{\delta_0}$ we get

$$\begin{aligned}
&\mathrm{E}\big[\sup_{0\leq s\leq \frac{1}{\delta_0}} e^{-\alpha_0 s}\left|e^{-\alpha t}h(y(t+s,x)) - h(y(s,x))]\right|\big] \\
&\leq \mathrm{E}\big[\sup_{0\leq s\leq \frac{1}{\delta_0}} e^{-\alpha_0 s}\left|e^{-\alpha t}h(y(t+s,x)) - h(y(s,x))]\right| 1_{\sup_{0\leq s\leq \frac{1}{\delta_0}}|y(t+s,x)-y(s,x)|<\delta_0}\big] \\
&\quad + \mathrm{E}\big[\sup_{0\leq s\leq \frac{1}{\delta_0}} e^{-\alpha_0 s}\left|e^{-\alpha t}h(y(t+s,x)) - h(y(s,x))]\right| 1_{\sup_{0\leq s\leq \frac{1}{\delta_0}}|y(t+s,x)-y(s,x)|\geq\delta_0}\big].
\end{aligned}$$

The fact that h is bounded (uniformly), gives us

$$\begin{aligned}
&\mathrm{E}\big[\sup_{0\leq s\leq \frac{1}{\delta_0}} e^{-\alpha_0 s}\left|e^{-\alpha t}h(y(t+s,x)) - h(y(s,x))]\right| 1_{\sup_{0\leq s\leq \frac{1}{\delta_0}}|y(t+s,x)-y(s,x)|\geq\delta_0}\big] \\
&\quad\leq 2\left\|h\right\|_\infty \mathbb{P}(\sup_{0\leq s\leq \frac{1}{\delta_0}} |y(t+s,x) - y(s,x)| \geq \delta_0) < 2\left\|h\right\|_\infty \varepsilon,
\end{aligned} \tag{2.22}$$

where we have denoted by $\|\cdot\|_\infty$ the supremum norm. On the other hand, the uniformly continuity of h gives us

$$\mathrm{E}\big[\sup_{0\leq s\leq \frac{1}{\delta_0}} e^{-\alpha_0 s}\left|e^{-\alpha t}h(y(t+s,x)) - h(y(s,x))]\right| 1_{\sup_{0\leq s\leq \frac{1}{\delta_0}}|y(t+s,x)-y(s,x)|<\delta_0}\big] < \varepsilon. \tag{2.23}$$

We have for the second term in the right-hand side of (2.21)

$$\mathrm{E}\big[\sup_{s\geq \frac{1}{\delta_0}} e^{-\alpha_0 s}e^{-\alpha t}\left|h(y(t+s,x)) - h(y(s,x))]\right|\big] \leq \sup_{s\geq \frac{1}{\delta_0}} 2e^{-\alpha_0 s}\left\|h\right\|_\infty \leq 2e^{-\alpha_0\frac{1}{\varepsilon}}\left\|h\right\|_\infty. \tag{2.24}$$

Using the estimations (2.22), (2.23) and (2.24) in (2.21) we get $p(\Phi_\alpha(t)h - e^{-\alpha t}h, x) \to 0$ as $t \downarrow 0$. This proves that h satisfies part (a) of Definition 2.6. But also note that h trivially satisfies Definition 2.6(b) because $\{y(t,x)\}_{t\geq 0}$ is càdlàg. Therefore we can easily conclude that $C_u(\mathcal{O}) \subset C_p^w(\mathcal{O}) \subset C_p(\mathcal{O})$, which proves part (a) of this proposition.

(b.1) Let $h \in C_p(\mathcal{O})$. Now, in virtue of Proposition 2.2, we have that $p(\Phi_\alpha(t)h, x) \leq p(h, x)$, for each $x \in \mathcal{O}$ and $t \geq 0$. Hence

$$p(\Phi_\alpha(s)\Phi_\alpha(r)h - \Phi_\alpha(r)h, x) = p(\Phi_\alpha(r)(\Phi_\alpha(s)h - h), x) \leq p(\Phi_\alpha(s)h - h, x) \to 0, \quad s \downarrow 0.$$

This shows that $\Phi_\alpha(r)h \in C_p(\mathcal{O})$ for all $r \geq 0$, for every element $h \in C_p(\mathcal{O})$. It remains to prove that the process $\{\Phi_\alpha(t)h(y(s,x))\}_{s\geq 0}$ is cad-lag for each $x \in \mathcal{O}$ and $t \geq 0$. To do this, let $s_0 \geq 0$ and $\{s_n\}_{n\in\mathbb{N}}$ be a decreasing sequence in $[0,\infty)$ converging to s_0. Take $t \geq 0$ and $x \in \mathcal{O}$. We have that $\{h(y(s+t,x))\}_{s\geq 0}$ is a cad-lag process and $\sup_{s\geq 0} e^{-\alpha_0 s}|h(y(s+t,x))| \in L_1(\Omega)$ because $h \in C_p(\mathcal{O})$ and satisfies (2.11). Hence, applying Theorem 45 in [14], the right continuity of both the filtration and the process $h(y(s,x))$, as well as the Markov property, we deduce

$$\begin{aligned}
\lim_{n\to\infty} e^{-\alpha_0 s_n}e^{\alpha t}\Phi_\alpha(t)h(y(s_n,x)) &= \lim_{n\to\infty} \mathrm{E}[e^{-\alpha_0 s_n}h(y(s_n+t,x))|\mathcal{F}_{s_n}] \\
&= \mathrm{E}[e^{-\alpha_0 s_0}h(y(s_0+t,x))|\mathcal{F}_{s_0}] = e^{-\alpha_0 s_0}e^{\alpha t}\Phi_\alpha(t)h(y(s_0,x)), \quad \text{a.s.}
\end{aligned}$$

Due to the continuity of the exponential function, from the above we deduce that $\lim_{s\downarrow s_0}\Phi_\alpha(t)h(y(s,x)) = \Phi_\alpha(t)h(y(s_0,x))$, a.s. On the other hand, using again Theorem 45 in [14] and the existence of left-limits of the process $h(y(s,x))$ we get $\lim_{s\uparrow s_0}\Phi_\alpha(t)h(y(s,x)) = \mathrm{E}[e^{-\alpha t}h(y(t+s_0^-,x))|\mathcal{F}_{s_0^-}]$, a.s. Therefore, the process $\{\Phi_\alpha(t)h(y(s,x))\}_{s\geq 0}$ is cad-lag.

(b.2) If $h \in C_p^w(\mathcal{O})$, then we have that $\|\Phi_\alpha(t)h\|_w \leq \|h\|_w < \infty$ due to (2.13), yielding that $\Phi_\alpha(t)h \in C_p^w(\mathcal{O})$. □

Our next target is to describe a closedness properties of both $C_p(\mathcal{O})$ and $C_p^w(\mathcal{O})$ under (boundedly) seminorm-convergence. For this end, we will prove the next ancillary result.

Lemma 2.8. *Consider a sequence of functions $\{h_n\}_{n\in\mathbb{N}}$ together with a function h all contained in $B(\mathcal{O})$. For each $x \in \mathcal{O}$, suppose that $\lim_{n\downarrow 0} p(h_n - h, x) = 0$. Then, there exists a subsequence $\{n_k\}_{k\in\mathbb{N}}$ (dependent of x), such that*

$$\lim_{k\to\infty} \sup_{s\geq 0} \left\{ e^{-\alpha_0 s} \left| h_{n_k}(y(s,x)) - h(y(s,x)) \right| \right\} = 0, \qquad a.s.$$

Proof. We note that convergence in seminorm implies that

$$\sup_{s\geq 0} \left\{ e^{-\alpha_0 s} \left| h_n(y(s,x)) - h(y(s,x)) \right| \right\} \to 0 \quad \text{as } n \to \infty, \tag{2.25}$$

where the last convergence is of $L_1(\Omega, \mathbb{R})$ type. Then, the above sequence converges also in measure and this yields the existence of a subsequence which converges a.s. □

Theorem 2.9. *Let h and $\{h_n\}_{n\in\mathbb{N}}$ be functions all in $B(\mathcal{O})$. Then, under Assumption 2.3, the following assertions hold true.*

(a) *If $h_n \in B_p(\mathcal{O})$ and* $\text{s-lim}_{n\to\infty} h_n = h$ *then $h \in B_p(\mathcal{O})$.*

(b) *If $h_n \in C_p(\mathcal{O})$ and* $\text{s-lim}_{n\to\infty} h_n = h$ *then $h \in C_p(\mathcal{O})$.*

(c) *If $h_n \in C_p^w(\mathcal{O})$ and* $\text{bs-lim}_{n\to\infty} h_n = h$ *then $h \in C_p^w(\mathcal{O})$.*

Proof. *(a)* Given $x \in \mathcal{O}$, there exists $n \in \mathbb{N}$ such that $p(h - h_n, x) \leq 1$ and we have that $|h| \leq |h - h_n| + |h_n|$. Then due to the triangular inequality of the seminorm, we get $p(h, x) \leq p(h - h_n, x) + p(h_n, x) < \infty$ and therefore $h \in B_p(\mathcal{O})$.

(b) Let us suppose $h_n \in C_p(\mathcal{O})$ and $\text{s-lim}_{n\to\infty} h_n = h$. Then we have that

$$\begin{aligned} p(\Phi_\alpha(t)h - h, x) \leq & p(\Phi_\alpha(t)h - \Phi_\alpha(t)h_n, x) + p(\Phi_\alpha(t)h_n - h_n, x) + p(h_n - h, x) \\ \leq & 2p(h_n - h, x) + p(\Phi_\alpha(t)h_n - h_n, x). \end{aligned}$$

Letting $t \downarrow 0$ and hence $n \to \infty$ to the last expression, we get $\lim_{t\downarrow 0} p(\Phi_\alpha(t)h - h, x) = 0$, for each $x \in \mathcal{O}$. On the other hand, a simple use of Lemma 2.8 ensures the existence of a subsequence $\{n_k\}_{k\in\mathbb{N}}$ such that

$$\sup_{s\geq 0} \left\{ e^{-\alpha_0 s} \left| h_{n_k}(y(s,x)) - h(y(s,x)) \right| \right\} \to 0, \quad \text{a.s.} \tag{2.26}$$

when $k \to \infty$. Note that this subsequence depends on r and x. Let $t_0 \geq 0$, we have that

$$\begin{aligned} |h(y(t,x)) - h(y(t_0,x))| \leq & |h(y(t,x)) - h_{n_k}(y(t,x))| + |h_{n_k}(y(t,x)) - h_{n_k}(y(t_0,x))| \\ & + |h_{n_k}(y(t_0,x)) - h(y(t_0,x))| \\ \leq & (e^{\alpha_0 t} + e^{\alpha_0 t_0}) \sup_{s\geq 0} e^{-\alpha_0 s} \left| h_{n_k}(y(s,x)) - h(y(s,x)) \right| \\ & + \left| h_{n_k}(y(t,x)) - h_{n_k}(y(t_0,x)) \right|. \end{aligned} \tag{2.27}$$

Since $t \mapsto h_{n_k}(y(t,x))$ is right-continuous and considering the convergence (2.26), we then apply the limits $t \downarrow t_0$ and hence $k \to \infty$ on the last expression and obtain $\lim_{t\downarrow t_0} |h(y(t,x)) - h(y(t_0,x))| = 0$ a.s. On the other hand, in the same way as in (2.27), we get

$$\begin{aligned} \left| h(y(t,x)) - h(y(t_0^-,x)) \right| \leq & (e^{\alpha_0 t} + e^{\alpha_0 t_0}) \sup_{s\geq 0} e^{-\alpha_0 s} \left| h_{n_k}(y(s,x)) - h(y(s,x)) \right| \\ & + \left| h_{n_k}(y(t,x)) - h_{n_k}(y(t_0^-,x)) \right|. \end{aligned}$$

We apply the limits $t \uparrow t_0$ and hence $k \to \infty$ on the last expression and obtain $\lim_{t\uparrow t_0} \left| h(y(t,x)) - h(y(t_0^-,x)) \right| = 0$ a.s. due to the left-limits existence.

(c) If $h_n \in C_p^w(\mathcal{O})$ and $\text{bs-lim}_{n\to\infty} h_n = h$, then we have that $\text{s-lim}_{n\to\infty} h_n = h$ and $\sup_{n\in\mathbb{N}} \|h_n\|_w$, then due to part (b), we have that $h \in C_p(\mathcal{O})$ and we need to demonstrate $\|h\|_w < \infty$. Namely, we have that seminorm convergence implies pointwise convergence, hence $\frac{|h(x)|}{w(x)} = \lim_{n\to\infty} \frac{|h_n(x)|}{w(x)} \leq \sup_{n\in\mathbb{N}} \|h_n\|_w < \infty$ implying $\|h\|_w < \infty$ and therefore $h \in C_p^w(\mathcal{O})$. □

2.3 The Infinitesimal generator and the Resolvent

We define the infinitesimal generator $(D(\mathcal{A}_\alpha), \mathcal{A}_\alpha)$ associated to the semigroup Φ_α as follows

$$\begin{cases} D(\mathcal{A}_\alpha) & := \{h \in C_p^w(\mathcal{O}) : \exists \, \text{bs-lim}_{t\downarrow 0} \frac{h-\Phi_\alpha(t)h}{t}\}; \\ \mathcal{A}_\alpha \, h & := \text{bs-lim}_{t\downarrow 0} \frac{h-\Phi_\alpha(t)h}{t}. \end{cases} \tag{2.28}$$

• *Remark* 2.10. In virtue of Definition 2.6 and Theorem 2.9, every limit in (2.28) belongs to $C_p^w(\mathcal{O})$.

Recall from Assumption 2.3 that $t \mapsto \Phi_\alpha(t)h(x)$ is measurable for every $h \in B(\mathcal{O})$ and $x \in \mathcal{O}$, then we are in conditions to define the *Resolvent operator* $\{\mathcal{R}_\alpha\}_{\alpha>\alpha_0}$ by

$$\mathcal{R}_\alpha \, h(x) = \int_0^\infty \Phi_\alpha(t)h(x) \, dt, \; \forall x \in \mathcal{O}, \; h \in B(\mathcal{O}), \tag{2.29}$$

where the integral is taken in the Lebesgue sense for real valued functions. A direct consequence of this definition is that $\mathcal{R}_\alpha \, h$ is Borel measurable, for each fixed α. Also, if $h \in B_p(\mathcal{O})$ then Fubini's Theorem as long with Proposition 2.2 yield

$$\begin{aligned} p(\mathcal{R}_\alpha \, h, x) &\leq \mathrm{E}\big[\sup_{s\geq 0} e^{-\alpha_0 s} \int_0^\infty |\Phi_\alpha(t)h(y(s,x))| \; dt\big] \leq \mathrm{E}\big[\int_0^\infty \sup_{s\geq 0} e^{-\alpha_0 s} |\Phi_\alpha(t)h(y(s,x))| \; dt\big] \\ &= \int_0^\infty \mathrm{E}\big[\sup_{s\geq 0} e^{-\alpha_0 s} |\Phi_\alpha(t)h(y(s,x))|\big] \, dt = \int_0^\infty p(\Phi_\alpha(t)h, x) \, dt \\ &= \int_0^\infty e^{-(\alpha-\alpha_0)t} p(\Phi_{\alpha_0}(t)h, x) \, dt \leq \frac{1}{\alpha-\alpha_0} p(h,x), \end{aligned} \tag{2.30}$$

which implies $\mathcal{R}_\alpha \, h \in B_p(\mathcal{O})$. Moreover, if $h \in B_w(\mathcal{O})$ then we have

$$\|\mathcal{R}_\alpha \, h\|_w \leq \int_0^\infty e^{-(\alpha-\alpha_0)t} \, \|\Phi_{\alpha_0}(t)h\|_w \; dt \leq \frac{1}{\alpha-\alpha_0} \, \|h\|_w < \infty, \tag{2.31}$$

and so $\mathcal{R}_\alpha \, h \in B_w(\mathcal{O})$.

Our next goal is to prove the stronger fact that $\mathcal{R} \, h \in C_p^w(\mathcal{O})$ when $h \in C_p^w(\mathcal{O})$, that is, we will show that $\mathcal{R}_\alpha$ maps $C_p^w(\mathcal{O})$ into itself. Such result will be provided in Theorem 2.14 below. Before doing this, we will check some useful properties:

In the same way as in (2.30) it is easy to demonstrate that

$$p(\int_a^b \Phi_\alpha(t)h \; dt, x) \leq \int_a^b p(\Phi_\alpha(t)h, x) \; dt, \quad \text{for every } 0 \leq a \leq b \leq \infty. \tag{2.32}$$

Besides, we can interchange the semigroup and the resolvent; namely, for every $\beta > \alpha_0$ and $\alpha \geq \alpha_0$, using Fubini's Theorem we get

$$\begin{aligned} \mathcal{R}_\beta \, \Phi_\alpha(t)h(x) &= \int_0^\infty \Phi_\alpha(t)\Phi_\beta(s)h(x) \; ds = \int_0^\infty \mathrm{E}[e^{-\alpha t}\Phi_\beta(s)h(y(t,x))] \; ds \\ &= \mathrm{E}\left[e^{-\alpha t} \int_0^\infty \Phi_\beta(s)h(y(t,x)) \; ds\right] = \Phi_\alpha(t) \, \mathcal{R}_\beta \, h(x). \end{aligned} \tag{2.33}$$

The use of Fubini's Theorem is justified since

$$\int_0^\infty \mathrm{E}[e^{-\alpha t} \, |\Phi_\beta(s)h(y(t,x))|] \; ds \leq \Phi_\alpha(t) \, \mathcal{R}_\beta \, |h| \, (x) \leq \frac{1}{\beta-\alpha_0} \, \|h\|_w \, w(x).$$

Our next result uses the following notation:

$$u(t) = \Phi_\alpha(t)h, \; \text{ for a given } h \in C_p(\mathcal{O}) \text{ and } \alpha > \alpha_0. \tag{2.34}$$

Lemma 2.11. *Fix $x \in \mathcal{O}$. Then:*

(a) For all $t_0 \geq 0$ we have that $\lim_{t\to t_0} p(u(t) - u(t_0), x) = 0$.

(b) We have $\lim_{t\to\infty} p(u(t), x) = 0$.

(c) For all $\varepsilon > 0$ there exists $\delta = \delta(x, \varepsilon) > 0$ such that if $|t - s| \leq \delta$ then $p(u(t) - u(s), x) \leq \varepsilon$.

Proof. (a) The case $t_0 = 0$ is straightforward, so let us consider the case $t_0 > 0$. Namely, we have that

$$p\big(\Phi_\alpha(t + t_0)h - \Phi_\alpha(t_0)h, x\big) = p\big(\Phi_\alpha(t_0)(\Phi_\alpha(t)h - h), x\big) \leq p\big(\Phi_\alpha(t)h - h, x\big) \to 0 \quad \text{as } t \downarrow 0.$$

Also, we have that

$$p\big(\Phi_\alpha(t_0 - t)h - \Phi_\alpha(t_0)h, x\big) = p\big(\Phi_\alpha(t_0 - t)(h - \Phi_\alpha(t)h), x\big) \leq p\big(\Phi_\alpha(t)h - h, x\big) \to 0 \quad \text{as } t \downarrow 0.$$

Whence

$$\lim_{t\to t_0^+} p\big(\Phi_\alpha(t)h - \Phi_\alpha(t_0)h, x\big) = \lim_{t\to t_0^-} p\big(\Phi_\alpha(t)h - \Phi_\alpha(t_0)h, x\big) = 0,$$

which proves (a).

(b) By Proposition 2.2, we have that $p(u(t), x) = p(e^{-(\alpha-\alpha_0)t}\Phi_{\alpha_0}(t)h, x) \leq e^{-(\alpha-\alpha_0)t}p(h, x)$. Due to $\alpha - \alpha_0 > 0$, we can take $T > 0$ such that $e^{-(\alpha-\alpha_0)t}p(h, x) \leq \varepsilon$ for all $t \geq T$.

(c) Using part (b) above, we can take $T > 0$ large enough such that for all $t \geq T$, $p(u(t), x) \leq \frac{\varepsilon}{2}$. For each $t \in [0, T]$ we choose $\delta_t > 0$ which satisfies that if $|t - s| \leq \delta_t$ then $p(u(t) - u(s), x) \leq \frac{\varepsilon}{2}$ because of part (a). On the other hand, by defining $B(t, \delta) := \{r \geq 0 | |r - t| \leq \delta\}$, we have that $[0, T] \subset \bigcup_{0\leq t\leq T} B(t, \frac{\delta_t}{2})$. Hence, there exist $t_1, \ldots, t_n$ such that $[0, T] \subset \bigcup_{k=1}^n B(t_k, \frac{\delta_{t_k}}{2})$ because of the compactness of $[0, T]$. We put $\delta = \min\{\frac{\delta_{t_k}}{2} : k = 1, \ldots, n\} > 0$. Take $t, s \geq 0$ such that $|t - s| \leq \delta$, and assume $k^* \in \{1, \ldots, n\}$ is the associated index of the neighborhood that covers t; i.e., $t \in B(t_{k^*}, \frac{\delta_{t_{k^*}}}{2})$. This implies that $|s - t_{k^*}| \leq |s - t| + |t - t_{k^*}| \leq \delta + \frac{\delta_{t_{k^*}}}{2} \leq \delta_{t_{k^*}}$, this last gives that $s \in B(t_{k^*}, \delta_{t_{k^*}})$, yielding that

$$p\big(u(t) - u(s), x\big) \leq p(u(t) - u(t_{k^*}), x) + p(u(t_{k^*}) - u(s), x) \leq \varepsilon.$$

Now, suppose that $t \in \bigcap_{k=1}^n B(t_k, \frac{\delta_{t_k}}{2})^c$ and $s \in B(t_j, \frac{\delta_{t_j}}{2})$ for some $j \in \{1, \ldots, n\}$. By the same argument as above, we can conclude that $t \in B(t_j, \delta_{t_j})$, that implies $p\big(u(t) - u(s), x\big) \leq \varepsilon$. Finally, if $t, s \geq T$ and remembering the choice of T at the beginning, we obtain $p(u(t) - u(s), x) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. □

Lemma 2.12. *For each u as in (2.34), there exists a sequence of functions $u_n : [0, \infty) \to C_p(\mathcal{O})$ such that*

$$\lim_{n\to\infty} \sup_{t\geq 0} p(u(t) - u_n(t), x) = 0. \tag{2.35}$$

Moreover, if $h \in C_p^w(\mathcal{O})$ then we can choose the above sequence such that $u_n(t) \in C_p^w(\mathcal{O})$, for all $n \in \mathbb{N}$ and $t \geq 0$.

Proof. For fixed $x \in \mathcal{O}$ and $n \in \mathbb{N}$, we define

$$E_{n,k} := [\frac{k-1}{n}, \frac{k}{n}), \quad k = 1, \ldots, n^2,$$
$$F_n := [n, \infty).$$

Define also the sequence of functions

$$u_n(t) := \sum_{k=1}^{n^2} u(t_k)\mathbf{1}_{E_{n,k}}(t) + u(n)\mathbf{1}_{F_n}(t), \tag{2.36}$$

with $t_k = \frac{k-1}{n}$. Note by Proposition 2.7, for all $n \in \mathbb{N}$ and $t \geq 0$, each $u_n(t)$ is in $C_p(\mathcal{O})$ because they are linear combination of functions in $C_p(\mathcal{O})$. In the same way, if $h \in C_p^w(\mathcal{O})$ in (2.34) then in virtue of this same proposition, $u \in C_p^w(\mathcal{O})$, yielding also $u_n(t) \in C_p^w(\mathcal{O})$.

Given $\varepsilon > 0$ we can choose $N \in \mathbb{N}$ such that $e^{-(\alpha-\alpha_0)N}p(h,x) \leq \frac{\varepsilon}{2}$ and if $|t-s| \leq \frac{1}{N}$ then $p(u(t) - u(s),x) \leq \varepsilon$. Now choose $n \geq N$ and take $t \geq 0$ as in Lemma 2.11. First suppose that there exists $k \in \{1,\dots,n^2\}$ such that $t \in [\frac{k-1}{n},\frac{k}{n})$ then, we have $\left|t - \frac{k-1}{n}\right| \leq \frac{1}{n}$ implying $u(t) - u_n(t) = u(t) - u(t_k)$. Thus, we get $p(u(t) - u_n(t),x) \leq \varepsilon$. Otherwise, if such k does not exist, then we have $t \geq n$ that gives us $u(t) - u_n(t) = u(t) - u(n)$, yielding that

$$\begin{aligned} p(u(t) - u_n(t),x) &= p(u(t) - u(n),x) \leq p(u(t),x) + p(u(n),x) \\ &\leq p(e^{-(\alpha-\alpha_0)t}\Phi_{\alpha_0}(t)h,x) + p(e^{-(\alpha-\alpha_0)n}\Phi_{\alpha_0}(n)h,x) \\ &\leq 2e^{-(\alpha-\alpha_0)N}p(h,x) \leq \varepsilon. \end{aligned}$$

So we conclude that $\sup_{t\geq 0} p(u(t) - u_n(t),x) \leq \varepsilon$ for all $n \geq N$. □

• *Remark* 2.13. We know that $u_n(t)$ belongs to $C_p(\mathcal{O})$ (resp. to $C_p^w(\mathcal{O})$) if $h \in C_p(\mathcal{O})$ (resp. $\in C_p^w(\mathcal{O})$). Also, because of the definition of u_n in (2.36) we have that for each $x \in \mathbb{R}$ the function $t \mapsto u_n(t)(x) = \sum_{k=1}^{n^2} u(t_k)(x)\mathbf{1}_{E_{n,k}}(t) + u(n)(x)\mathbf{1}_{F_n}(t)$ is simple and real valued. Hence, given $\beta > 0$ the function $t \mapsto e^{-\beta t}u_n(t)(x)$ is Lebesgue integrable with integral given by

$$\int_a^b e^{-\beta t}u_n(t)(x)\ dt = \sum_{k=1}^{n^2} u(t_k)(x)\int_{E_{n,k}\cap[a,b]} e^{-\beta t}\ dt + u(n)(x)\int_{F_n\cap[a,b]} e^{-\beta t}\ dt. \tag{2.37}$$

We note that the above integral, as function of x, belongs to $C_p(\mathcal{O})$ (resp. to $C_p^w(\mathcal{O})$), because it is a sum of functions in $C_p(\mathcal{O})$ (resp. $C_p^w(\mathcal{O})$). Then, we simply denote this integral by $\int_a^b e^{-\beta t}u_n(t)\ dt$.

We have arrived to our first main result regarding to the regularity of the resolvent $\mathcal{R}_\alpha$, when the integrand satisfies that regularity.

Theorem 2.14. *Assume that Assumption 2.3 is valid. Then, for all $0 \leq a \leq b \leq \infty$, and $\beta > 0$, the next relation holds true*

$$\text{s-lim}_{n\to\infty}\int_a^b e^{-\beta t}u_n(t)\,dt = \int_a^b e^{-\beta t}u(t)\,dt, \tag{2.38}$$

for the functions u and $\{u_n\}$ introduced in Lemma 2.12. In particular, we have that $\mathcal{R}_\alpha h$ is contained in $C_p(\mathcal{O})$. Analogously, we obtain the same result with $C_p^w(\mathcal{O})$ instead of $C_p(\mathcal{O})$ if $h \in C_p^w(\mathcal{O})$ with bs-lim *instead of* s-lim *in* (2.38).

Proof. By the inequality in (2.32) as long with Lemma 2.12, we get

$$\begin{aligned} p\Big(\int_a^b e^{-\beta t}u_n(t)\,dt - \int_a^b e^{-\beta t}u(t)\,dt,x\Big) &\leq \int_a^b e^{-\beta t}p\big(u_n(t) - u(t),x\big)\,dt \\ &\leq \sup_{t\in[a,b]} p\big(u_n(t) - u(t),x\big)\int_a^b e^{-\beta t}\,dt \\ &\leq \frac{1}{\beta}\sup_{t\geq 0} p\big(u_n(t) - u(t),x\big) \to 0, \end{aligned}$$

when $n \to \infty$. That is $\text{s-lim}_{n\to\infty}\int_a^b e^{-\beta t}u_n(t)\ dt = \int_a^b e^{-\beta t}u(t)\ dt$, that implies $\int_a^b e^{-\beta t}u(t)\ dt \in C_p(\mathcal{O})$ due to Theorem 2.9. Moreover, in the case of $h \in C_p^w(\mathcal{O})$ we have that $u(t) \in C_p^w(\mathcal{O})$ and $\|u(t)\|_w = \|\Phi_\alpha(t)h\|_w \leq \|h\|_w$ for all $t \geq 0$. Using this last inequality together with (2.37) we get

$$\begin{aligned} \left\|\int_a^b e^{-\beta t}u_n(t)\ dt\right\|_w &\leq \sum_{k=1}^{n^2}\|u(t_k)\|_w\int_{E_{n,k}\cap[a,b]} e^{-\beta t}\ dt + \|u(n)\|_w\int_{F_n\cap[a,b]} e^{-\beta t}\ dt \\ &\leq \sum_{k=1}^{n^2}\|h\|_w\int_{E_{n,k}\cap[a,b]} e^{-\beta t}\ dt + \|h\|_w\int_{F_n\cap[a,b]} e^{-\beta t}\ dt = \|h\|_w\int_a^b e^{-\beta t}\ dt < \infty. \end{aligned} \tag{2.39}$$

Hence, $\sup_{n\in\mathbb{N}} \left\| \int_a^b e^{-\beta t} u_n(t)\ dt \right\|_w < \infty$, and therefore $\text{bs-lim}_{n\to\infty} \int_a^b e^{-\beta t} u_n(t)\ dt = \int_a^b e^{-\beta t} u(t)\ dt$, that implies $\int_a^b e^{-\beta t} u(t)\ dt \in C_p^w(\mathcal{O})$, again due to Theorem 2.9. In particular, taking $\beta = \frac{\alpha-\alpha_0}{2} > 0$, $u(t) = \Phi_{\beta+\alpha_0}(t)h$, $a = 0$, and $b = \infty$, we obtain

$$\begin{aligned}\mathcal{R}_\alpha\, h(x) &= \int_0^\infty \Phi_\alpha(t)h(x)\,dt = \int_0^\infty e^{-(\alpha-\alpha_0)t}\Phi_{\alpha_0}(t)h(x)\,dt \\ &= \int_0^\infty e^{-\frac{\alpha-\alpha_0}{2}t}\Phi_{\frac{\alpha-\alpha_0}{2}+\alpha_0}(t)h(x)\,dt = \int_0^\infty e^{-\beta t}u(t)(x)\,dt.\end{aligned}$$

Thus, $\mathcal{R}_\alpha\, h$ is in $C_p(\mathcal{O})$ (resp. in $C_p^w(\mathcal{O})$ when $h \in C_p^w(\mathcal{O})$). □

The next result is a useful property of the integrals of semigroups that is very common in finite-dimensional spaces.

Lemma 2.15. *Let $h \in C_p^w(\mathcal{O})$. For any $t_0 \geq 0$ we have*

$$\text{bs-lim}_{t\downarrow 0} \frac{1}{t}\int_{t_0}^{t_0+t} \Phi_\alpha(s)h\ ds = \Phi_\alpha(t_0)h. \tag{2.40}$$

Proof. Let $t_0 \geq 0$ and fix $x \in \mathcal{O}$. By Theorem 2.9 (c), we get that $\frac{1}{t}\int_{t_0}^{t_0+t}\Phi_\alpha(s)h \in C_p^w(\mathcal{O})$. Since $t \mapsto \Phi_\alpha(t)h$ is continuous in seminorm, given $\varepsilon > 0$ we consider $\delta > 0$ such that $|t_0 - s| < \delta$ implies $p(\Phi_\alpha(s)h - \Phi_\alpha(t_0)h, x) < \varepsilon$. Hence, if $|t| \leq \delta$ then, by (2.32) we get

$$\begin{aligned}p\Big(\frac{1}{t}\int_{t_0}^{t_0+t}\Phi_\alpha(s)h\ ds - \Phi_\alpha(t_0)h, x\Big) =& p\Big(\frac{1}{t}\int_{t_0}^{t_0+t}[\Phi_\alpha(s)h - \Phi_\alpha(t_0)h]\ ds, x\Big) \\ \leq& \frac{1}{t}\int_{t_0}^{t_0+t} p(\Phi_\alpha(s)h - \Phi_\alpha(t_0)h, x)\ ds < \varepsilon.\end{aligned}$$

On the other hand, using (2.13) we get

$$\left\|\frac{1}{t}\int_{t_0}^{t_0+t}\Phi_\alpha(t)h\,ds\right\|_w \leq \frac{1}{t}\int_{t_0}^{t_0+t}\|\Phi_\alpha(t)h\|_w\ ds \leq \frac{1}{t}\int_{t_0}^{t_0+t}\|h\|_w\ ds = \|h\|_w\,. \tag{2.41}$$

Thus, we have proved $\text{bs-lim}_{t\downarrow 0}\frac{1}{t}\int_{t_0}^{t_0+t}\Phi_\alpha(t)h\,ds = \Phi_\alpha(t_0)h$. □

Our next definition has to do with the diferentiability of semigroups.

Definition 2.16. We say that $t \mapsto \Phi_\alpha(t)h$ is *boundedly differentiable in seminorm* in a fixed point $r \geq 0$ if the limit

$$\text{bs-lim}_{t\to 0}\frac{\Phi_\alpha(t+r)h - \Phi_\alpha(r)h}{t}$$

exists in $C_p^w(\mathcal{O})$.

- *Remark* 2.17. (a) If $h \in C_p^w(\mathcal{O})$ and the above limit exists, then Theorem 2.9(c) ensures that this limit belongst to $C_p^w(\mathcal{O})$.

  (b) The boundedly differentiablily in seminorm implies the pointwise differentiablity; i.e., for each $x \in \mathcal{O}$, $\lim_{t\downarrow 0}\frac{\Phi_\alpha(t+r)h(x)-\Phi_\alpha(t)h(x)}{t}$.

The next theorem shows a relation between the semigroup Φ_α and the infinitesimal generator $\mathcal{A}_\alpha$, among other important properties.

Theorem 2.18. *Suppose that Assumption 2.3 is valid. Then, for each $h \in D(\mathcal{A}_\alpha)$, we have that $\Phi_\alpha(t)h \in D(\mathcal{A}_\alpha)$ for all $t > 0$. Furthermore, the function $t \mapsto \Phi_\alpha(t)h$ is boundedly differentiable in seminorm on $(0,\infty)$, and the following relation holds*

$$-\frac{d}{dt}(\Phi_\alpha(t)h) = \mathcal{A}_\alpha\,\Phi_\alpha(t)h = \Phi_\alpha(t)\,\mathcal{A}_\alpha\,h, \quad \forall\ t > 0. \tag{2.42}$$

Proof. First note that

$$\frac{1}{s}(\Phi_\alpha(t)h - \Phi_\alpha(t+s)h) = \Phi_\alpha(t)\frac{1}{s}(h - \Phi_\alpha(s)h). \tag{2.43}$$

Next, by using the fact of $\text{s-lim}_{s\downarrow 0}\frac{1}{s}(h - \Phi_\alpha(s)h) = \mathcal{A}_\alpha\, h$ as long with Proposition 2.2, we have that

$$-\frac{d^+}{dt}\Phi_\alpha(t)h = \text{s-lim}_{s\downarrow 0}\frac{1}{s}(\Phi_\alpha(t)h - \Phi_\alpha(t+s)h) = \Phi_\alpha(t)\,\mathcal{A}_\alpha\, h.$$

On the other hand, taking into account (2.43) we get

$$\left\|\frac{1}{s}(\Phi_\alpha(t)h - \Phi_\alpha(t+s)h)\right\|_w \leq \frac{1}{s}\left\|\Phi_\alpha(t)(h - \Phi_\alpha(s)h)\right\|_w \leq \frac{1}{s}\left\|h - \Phi_\alpha(s)h\right\|_w \leq \sup_{s\geq 0}\frac{1}{s}\left\|h - \Phi_\alpha(s)h\right\|_w < \infty.$$

The last inequality is due to the boundedly convergence in seminorm $\text{bs-lim}_{s\downarrow 0}\frac{1}{s}(h - \Phi_\alpha(s)h)$ in (2.19) applied to the definition of $\mathcal{A}_\alpha$. Hence $\Phi_\alpha(t)h \in D(\mathcal{A}_\alpha)$ and $\mathcal{A}_\alpha\,\Phi_\alpha(t)h = \Phi_\alpha(t)\,\mathcal{A}_\alpha\, h$. In the same way, given $0 \leq s \leq t$ we have

$$\frac{1}{-s}(\Phi_\alpha(t)h - \Phi_\alpha(t-s)h) = \Phi_\alpha(t-s)\frac{1}{s}(h - \Phi_\alpha(s)h), \tag{2.44}$$

again, by definition of $\mathcal{A}_\alpha$ and Proposition 2.2 we get

$$-\frac{d^-}{dt}\Phi_\alpha(t)h = \text{s-lim}_{s\downarrow 0}\frac{1}{-s}(\Phi_\alpha(t)h - \Phi_\alpha(t-s)h = \Phi_\alpha(t)\,\mathcal{A}_\alpha\, h.$$

By (2.44), we note that

$$\left\|\frac{1}{-s}(\Phi_\alpha(t)h - \Phi_\alpha(t-s)h)\right\|_w \leq \frac{1}{s}\left\|\Phi_\alpha(t-s)(h - \Phi_\alpha(s))\right\|_w \leq \frac{1}{s}\left\|h - \Phi_\alpha(s)\right\|_w \leq \sup_{s\geq 0}\frac{1}{s}\left\|h - \Phi_\alpha(s)h\right\|_w < \infty.$$

where the last inequality is new again due to $\text{bs-lim}_{s\downarrow 0}\frac{1}{s}(h - \Phi_\alpha(s)h)$. Yielding that $-\frac{d}{dt}\Phi_\alpha(t)h = \mathcal{A}_\alpha\,\Phi_\alpha(t)h$, which proves (2.42). □

Our next two results are crucial for our analysis: the first one shows the density of the domain $\mathcal{D}(\mathcal{A}_\alpha)$ into the space $C_p^w(\mathcal{O})$, whereas the second proves that the resolvent is the inverse operator of the generator; that is, $\mathcal{A}_\alpha^{-1} = \mathcal{R}_\alpha$.

Theorem 2.19. *Under the Assumption of Theorem 2.18, the domain $D(\mathcal{A}_\alpha)$ is dense in $C_p^w(\mathcal{O})$ in the sense of the boundedly seminorm-convergence.*

Proof. Take $h \in C_p^w(\mathcal{O})$ and define $h_n := n\int_0^{\frac{1}{n}}\Phi_\alpha(s)h\; ds$. By the proof of Lemma 2.15, we know that $h_n \in C_p^w(\mathcal{O})$ and $\text{bs-lim}_{n\to\infty} h_n = h$, so it is sufficient to show that $h_n \in D(\mathcal{A}_\alpha)$. Indeed, using Fubini's Theorem, we have that

$$\begin{aligned}
\Phi_\alpha(t)h_n(x) =& \mathrm{E}\left[e^{-\alpha t} n\int_0^{\frac{1}{n}}\Phi_\alpha(s)h(y(t,x))\; ds\right]\\
=& n\int_0^{\frac{1}{n}}\mathrm{E}[e^{-\alpha t}\Phi_\alpha(s)h(y(t,x))]\; ds\\
=& n\int_0^{\frac{1}{n}}\Phi_\alpha(s+t)h(x)\; ds = n\int_t^{t+\frac{1}{n}}\Phi_\alpha(s)h(x)\; ds.
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
\frac{1}{t}(h_n - \Phi_\alpha(t)h_n) =& n\Big(\frac{1}{t}\int_0^{\frac{1}{n}}\Phi_\alpha(s)h\; ds - \frac{1}{t}\int_t^{t+\frac{1}{n}}\Phi_\alpha(s)h\; ds\Big)\\
=& n\Big(\frac{1}{t}\int_0^{t}\Phi_\alpha(s)h\; ds - \frac{1}{t}\int_{\frac{1}{n}}^{t+\frac{1}{n}}\Phi_\alpha(s)h\; ds\Big).
\end{aligned}$$

Using this last fact together with Lemma 2.15, we get s-lim$_{t\downarrow 0}\frac{1}{t}(h_n - \Phi_\alpha(t)h_n) = n(h - \Phi_\alpha(\frac{1}{n})h)$. We have also the relation

$$\begin{aligned}\left\|\frac{1}{t}(h_n - \Phi_\alpha(t)h_n)\right\|_w \leq & \frac{n}{t}\left\|\int_0^t \Phi_\alpha(s)h\, ds\right\|_w + \frac{n}{t}\left\|\int_{\frac{1}{n}}^{t+\frac{1}{n}} \Phi_\alpha(s)h\, ds\right\|_w \\ \leq & \frac{n}{t}\int_0^t \left\|\Phi_\alpha(s)h\right\|_w\, ds + \frac{n}{t}\int_{\frac{1}{n}}^{t+\frac{1}{n}} \left\|\Phi_\alpha(s)h\right\|_w\, ds \leq 2n\left\|h\right\|_w .\end{aligned}$$

Hence, $h_n \in D(\mathcal{A}_\alpha)$. ☐

Theorem 2.20. *Let Assumption 2.3 holds true. Then, for each $\alpha > 0$, the operator $\mathcal{A}_\alpha$ from $D(\mathcal{A}_\alpha)$ to $C_p^w(\mathcal{O})$ is biyective. Besides, the following identity is satisfied*

$$\mathcal{A}_\alpha^{-1} = \mathcal{R}_\alpha .$$

Proof. Let us show first that $\mathcal{A}_\alpha$ is surjective. Let $h \in C_p^w(\mathcal{O})$ and $s \geq 0$, we have that

$$\begin{aligned}\Phi_\alpha(s)\,\mathcal{R}_\alpha\, h(x) = & \mathrm{E}\left[e^{-\alpha s}\int_0^\infty \Phi_\alpha(t)h(y(s,x))\, dt\right] \\ = & \int_0^\infty \mathrm{E}[e^{-\alpha s}\Phi_\alpha(t)h(y(s,x))]\, dt = \int_0^\infty \Phi_\alpha(t+s)h(x)\, dt \\ = & \int_s^\infty \Phi_\alpha(t)h(x)\, dt.\end{aligned}$$

Then,

$$\begin{aligned}\frac{1}{s}(\mathcal{R}_\alpha\, h - \Phi_\alpha(s)\,\mathcal{R}_\alpha\, h) = & \frac{1}{s}\int_0^\infty \Phi_\alpha(t)h(x)\, dt - \frac{1}{s}\int_s^\infty \Phi_\alpha(t)h(x)\, dt \\ = & \frac{1}{s}\int_0^s \Phi_\alpha(t)h(x)\, dt.\end{aligned}$$

By Lemma 2.15, we deduce that bs-lim$_{s\downarrow 0}\frac{1}{s}(\mathcal{R}_\alpha\, h - \Phi_\alpha(s)\,\mathcal{R}_\alpha\, h) = h$ which implies $\mathcal{R}_\alpha\, h \in D(\mathcal{A}_\alpha)$ and $\mathcal{A}_\alpha\,\mathcal{R}_\alpha\, h = h$, and therefore $\mathcal{A}_\alpha$ is surjective. Now, let us show that $\mathcal{A}_\alpha$ is injective. Take $h \in D(\mathcal{A}_\alpha)$ such that $\mathcal{A}_\alpha\, h = 0$. By Theorem 2.18, we have that

$$\frac{d}{dt}\Phi_\alpha(t)h(x) = -\Phi_\alpha(t)\,\mathcal{A}_\alpha\, h(x) = 0, \quad \forall\, x \in \mathcal{O},$$

which implies $t \mapsto \Phi_\alpha(t)h(x)$ is a real constant. But,

$$|\Phi_\alpha(t)h(x)| \leq e^{-(\alpha-\alpha_0)t}\left\|h\right\|_w w(x),$$

so, $\lim_{t\to\infty}\Phi_\alpha(t)h(x) = 0$. Moreover, we have $\Phi_\alpha(0)h(x) = h(x)$ and then, $h(x) = 0$ for all $x \in \mathcal{O}$. Thus, we have concluded that $\mathcal{A}_\alpha$ is invertible with inverse given by $\mathcal{R}_\alpha$. ☐

As a direct consequence of both Theorems 2.20 and 2.18 we can get, for all $h \in C_p^w(\mathcal{O})$, the relation

$$\mathcal{R}_\alpha\, h - \Phi_\alpha(t)\,\mathcal{R}_\alpha\, h = \int_0^t \Phi_\alpha(s)h\, ds = \int_0^t \Phi_\alpha(s)\,\mathcal{A}_\alpha\,\mathcal{R}_\alpha\, h\, ds = -\int_0^t \frac{d}{ds}\Phi_\alpha(s)\,\mathcal{R}_\alpha\, h\, ds. \tag{2.45}$$

We conclude this section by providing some properties of the operators $\mathcal{A}_\alpha$ and $\mathcal{R}_\alpha$.

Proposition 2.21. *For all $h \in C_p^w(\mathcal{O})$ and $\beta > 0$, we have the next relation in seminorm*

$$\text{bs-lim}_{\alpha\to\infty}\,\alpha\,\mathcal{R}_{\alpha+\beta}\, h = h. \tag{2.46}$$

Proof. First, let us prove the resolvent equation given in (2.47) below. Let $\alpha, \beta > \alpha_0$ with $\alpha \neq \beta$ and take $h \in C_p^w(\mathcal{O})$, then

$$\begin{aligned}
\mathcal{R}_\alpha \mathcal{R}_\beta h &= \int_0^\infty \Phi_\alpha(t)\mathcal{R}_\beta h\,dt = \int_0^\infty \Phi_\alpha(t)\int_0^\infty \Phi_\beta(s)h\,ds\,dt \\
&= \int_0^\infty e^{-(\alpha-\beta)t}\int_0^\infty \Phi_\beta(s+t)h\,ds\,dt = \int_0^\infty e^{-(\alpha-\beta)t}\int_t^\infty \Phi_\beta(r)h\,dr\,dt \\
&= \int_0^\infty \Phi_\beta(r)h\int_0^r e^{-(\alpha-\beta)t}\,dt\,dr = \int_0^\infty \Big(\frac{1}{\alpha-\beta} - \frac{e^{-(\alpha-\beta)r}}{\alpha-\beta}\Big)\Phi_\beta(r)hdr \\
&= \frac{1}{\alpha-\beta}\Big(\int_0^\infty \Phi_\beta(r)h\,dr - \int_0^\infty e^{-(\alpha-\beta)r}\Phi_\beta(r)h\,dr\Big) \\
&= \frac{1}{\alpha-\beta}(\mathcal{R}_\beta h - \mathcal{R}_\alpha h). \qquad (2.47)
\end{aligned}$$

Next, we will prove $\lim_{\alpha\to\infty} p(\alpha\mathcal{R}_\alpha h - h, x) = 0$ for all $h \in C_p^w(\mathcal{O})$. Let us assume first that $h \in D(\mathcal{A}_\alpha)$ and let us take $g \in C_p^w(\mathcal{O})$ such that $h = \mathcal{R}_\beta g$. We have

$$\alpha\mathcal{R}_\alpha h = \alpha\mathcal{R}_\alpha\mathcal{R}_\beta g = \frac{\alpha}{\alpha-\beta}(\mathcal{R}_\beta g - \mathcal{R}_\alpha g) = \frac{\alpha}{\alpha-\beta}h - \frac{\alpha}{\alpha-\beta}\mathcal{R}_\alpha g.$$

It is easy to see that $\lim_{\alpha\to\infty}\left\|\frac{\alpha}{\alpha-\beta}h - h\right\|_w = 0$ and $\lim_{\alpha\to\infty}\left\|\frac{\alpha}{\alpha-\beta}\mathcal{R}_\alpha g\right\|_w = 0$, where the last limit is due to (2.31). Therefore,

$$\lim_{\alpha\to\infty}\|\alpha\mathcal{R}_\alpha h - h\|_w = 0.$$

By (2.20) we see that the above convergence in norm implies the convergence in seminorm: $\text{s-lim}_{\alpha\to\infty}\alpha\mathcal{R}_\alpha h = h$.

Now, consider the general case $h \in C_p^w(\mathcal{O})$. Let h_n be a sequence in $D(\mathcal{A}_\alpha)$ such that $\text{bs-lim}_{n\to\infty} h_n = h$. We have

$$|\alpha\mathcal{R}_\alpha h - h| \le |\alpha\mathcal{R}_\alpha h - \alpha\mathcal{R}_\alpha h_n| + |\alpha\mathcal{R}_\alpha h_n - h_n| + |h_n - h|,$$

applying (2.30) to the above inequality we get

$$0 \le p(\alpha\mathcal{R}_\alpha h - h, x) \le \frac{\alpha}{\alpha-\alpha_0}p(h-h_n, x) + p(\alpha\mathcal{R}_\alpha h_n - h_n, x) + p(h_n - h, x).$$

Letting $\alpha \to \infty$ and hence $n \to \infty$ in the last inequality, we easily deduce that

$$\lim_{\alpha\to\infty} p(\alpha\mathcal{R}_\alpha h - h, x) = 0;$$

in other words $\text{s-lim}_{\alpha\to\infty}\alpha\mathcal{R}_\alpha h = h$. Moreover, by (2.31) we get $\|\alpha\mathcal{R}_\alpha h\|_w \le \alpha/(\alpha-\alpha_0)\|h\|_w$, and so $\text{bs-lim}_{\alpha\to\infty}\alpha\mathcal{R}_\alpha h = h$. It remains to show (2.46). For this purpose, let $\beta > 0$ and note that $\alpha\mathcal{R}_{\alpha+\beta} = (\alpha+\beta)\mathcal{R}_{\alpha+\beta} - \beta\mathcal{R}_{\alpha+\beta}$, we know that $\text{bs-lim}_{\alpha\to\infty}(\alpha+\beta)\mathcal{R}_{\alpha+\beta} h = h$ and $\text{bs-lim}_{\alpha\to\infty}\beta\mathcal{R}_{\alpha+\beta} = 0$, hence $\text{bs-lim}_{\alpha\to\infty}\alpha\mathcal{R}_{\alpha+\beta} h = h$. □

• *Remark* 2.22. Note by (2.47), the resolvent family of operators $\alpha \mapsto \mathcal{R}_\alpha$ is commutative.

Proposition 2.23. *Given $\alpha > \alpha_0$ and $\beta \ge 0$, we have*

$$\mathcal{A}_{\alpha+\beta} = \mathcal{A}_\alpha + \beta I. \qquad (2.48)$$

Proof. Let $h \in D(\mathcal{A}_\alpha)$. Then,

$$h - \Phi_{\alpha+\beta}(t)h = h - e^{-\beta t}\Phi_\alpha(t)h = h - \Phi_\alpha(t)h + (1-e^{-\beta t})\Phi_\alpha(t)h.$$

Multiplying by $\frac{1}{t}$ the last expression, and hence letting $t \downarrow 0$, we get $\mathcal{A}_{\alpha+\beta} h := \text{bs-lim}_{t\downarrow 0}\frac{1}{t}(h - \Phi_{\alpha+\beta}(t)h) = \mathcal{A}_\alpha h + \beta h$. □

3 The optimal stopping problem

This section deals with an optimal stopping control problem whose dynamical system is of Markov type studied in Section 2. The total cost consists of both a running cost that is paid when the dynamic is still running and a stopping cost that must to be paid once the dynamic is stopped. The way to tackle this problem is through a characterization of the optimal cost (value function) regarded as the maximal subsolution of a variational inequality defined later. In addition, by means of this characterization, it is also possible to find the well-known continuation region that in turn provides the associated optimal stopping time viewed as the fist hitting time of that region.

3.1 The statement of the problem

In this subsection we start our analysis recalling some mathematical objects introduced in Section 2. Namely, we recall the underlying stochastic process, consisting of the homogeneous Markov process $\{y(t,x)\}_{t\geq 0}$, $x \in \mathcal{O}$ defined on the probability space $\mathcal{E} := (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, with state space $(\mathcal{O}, |\cdot|)$, and satisfying $\mathbb{P}(y(0,x) = x) = 1$ as well as the properties (2.1)–(2.5).

We bring to mind that a stopping time is a random variable τ with values in the no-negative real numbers set such that the event $\{\tau \leq t\}$ is $\mathcal{F}_t$ measurable for every $t \geq 0$, with $\mathcal{F}_t$ the associated filtration to the space $\mathcal{E}$.

Let $\mathcal{T}$ be the set consisting of all stopping times introduced in the above paragraph. With this in mind, for $x \in \mathcal{O}$, $f, \varphi \in C_p^w(\mathcal{O})$, $\tau \in \mathcal{T}$, and $\alpha > \alpha_0 > 0$, we define the following cost function

$$J(x,\tau) \coloneqq \mathrm{E}\left[\int_0^\tau f(y(t,x))e^{-\alpha t}dt + \varphi(y(\tau,x))e^{-\alpha\tau}\right], \tag{3.1}$$

where as mentioned above, f and φ represent the running and stopping cost per unit of time respectively, and $e^{-\alpha\cdot}$ denotes the discount factor at each instant of time.

The optimal cost, also known as the *value function*, is then defined as

$$\hat{u}(x) = \inf_{\tau\in\mathcal{T}} J(x,\tau). \tag{3.2}$$

We will say that the random variable $\hat{\tau} \in \mathcal{T}$ is an *optimal stopping time* if it minimizes the cost (3.1) in the following way

$$\hat{u}(x) = J(x,\hat{\tau}). \tag{3.3}$$

One of the goals of this section will consist to showing that the value function $\hat{u}$ defined in (3.2) does exist in $C_p^w(\mathcal{O})$. Furthermore, this function satisfies the next variational inequality (VI) in the integral (or weak) form:

$$\hat{u} \leq \varphi, \quad \hat{u} \leq \int_0^t \Phi_\alpha(s) f\, ds + \Phi_\alpha(t)\hat{u}, \quad \forall\, t \geq 0. \tag{3.4}$$

3.2 Penalized method

We start our analysis by studying an ancillary problem so-called *penalized problem*. This problem consists of searching for a unique solution of the following *penalized equations*

$$\mathcal{A}_\alpha\, u_\varepsilon + \frac{1}{\varepsilon}(u_\varepsilon - \varphi)^+ = f, \quad \text{for each } \varepsilon > 0, \tag{3.5}$$

with

$$(u_\varepsilon - \varphi)^+ = \begin{cases} u_\varepsilon - \varphi, & \text{if } u_\varepsilon - \varphi \geq 0; \\ 0, & \text{if } u_\varepsilon - \varphi \leq 0, \end{cases}$$

Our goal is to prove that one subsolution of the inequality (3.4) can be characterized as the limit as $\varepsilon \downarrow 0$ of the sequence of solutions u_ε associated to (3.5). This limit function will be the "good one" for us.

Note that $(u_\varepsilon - \varphi)^+ = u_\varepsilon - (u_\varepsilon \wedge \varphi)$. Hence, Proposition 2.23 together with (3.5), imply

$$\mathcal{A}_{\alpha+\frac{1}{\varepsilon}}\, u_\varepsilon = f + \frac{1}{\varepsilon}(u_\varepsilon \wedge \varphi). \tag{3.6}$$

Applying $\mathcal{R}_{\alpha+\frac{1}{\varepsilon}}$ to the last equation we get

$$u_\varepsilon = \mathcal{R}_{\alpha+\frac{1}{\varepsilon}}(f + \frac{1}{\varepsilon}(u_\varepsilon \wedge \varphi)). \tag{3.7}$$

As mentioned earlier, we will prove that $u_0 := \text{s-lim}_{\varepsilon\downarrow 0} u_\varepsilon$ verifies the VI (3.4) as well as its corresponding regularity. To this end, we need the following technical result.

Lemma 3.1. *The following inequality holds for every measurable functions f, g, h from $\mathcal{O}$ to $\mathbb{R}$:*

$$|f \wedge h - g \wedge h| \leq |f - g|.$$

Proof. We have both $-|f-g| + g\wedge h \leq f - g + g = f$ and $-|f-g| + g\wedge h \leq h$ that imply $-|f-g| + g\wedge h \leq f \wedge h$. Analogously, we have $-|f-g| + f\wedge h \leq g \wedge h$, and joining the two obtained inequalities we get $|f\wedge h - g\wedge h| \leq |f-g|$. ☐

Theorem 3.2. *Assume that $f, \varphi \in C_p^w(\mathcal{O})$. Then, Assumption 2.3, we have the following.*

(a) *There exists a unique solution $u_\varepsilon \in D(\mathcal{A}_\alpha)$ of the penalized equation (3.5) for each $\varepsilon > 0$.*

(b) *For all $0 < \varepsilon' < \varepsilon$ we have that*

$$0 \leq u_\varepsilon - u_{\varepsilon'} \leq (u_\varepsilon - \varphi)^+ \leq \left|\mathcal{R}_{\alpha+\frac{1}{\varepsilon}} f + \mathcal{R}_{\alpha+\frac{1}{\varepsilon}} \varphi - \varphi\right|. \tag{3.8}$$

Furthermore, there exists the limit $u_0 := \text{s-lim}_{\varepsilon\downarrow 0} u_\varepsilon$ and therefore, $u_0 \in C_p(\mathcal{O})$.

Proof. First, we will show the existence of a unique solution u_ε of the penalized problem. Namely, based on (3.7), we define the non linear operator $T_\varepsilon : B_w(\mathcal{O}) \to B_w(\mathcal{O})$ given by $T_\varepsilon h := \mathcal{R}_{\alpha+1/\varepsilon}(f + \frac{1}{\varepsilon}(h \wedge \varphi))$. We will prove that T_ε is a contraction map. Indeed, as $h, g \in B_w(\mathcal{O})$, we have

$$T_\varepsilon h - T_\varepsilon g = \frac{1}{\varepsilon} \mathcal{R}_{\alpha+\frac{1}{\varepsilon}}(h\wedge\varphi - g\wedge\varphi).$$

Using the monotony of the resolvent together with Lemma 3.1 we get

$$\frac{1}{\varepsilon}\left|\mathcal{R}_{\alpha+\frac{1}{\varepsilon}}(h\wedge\varphi - g\wedge\varphi)\right| \leq \frac{1}{\varepsilon}\mathcal{R}_{\alpha+\frac{1}{\varepsilon}}|h\wedge\varphi - g\wedge\varphi| \leq \frac{1}{\varepsilon}\mathcal{R}_{\alpha+\frac{1}{\varepsilon}}|h-g|.$$

Now use (2.31) to obtain

$$\|T_\varepsilon h - T_\varepsilon g\|_w \leq \frac{\frac{1}{\varepsilon}}{\alpha - \alpha_0 + \frac{1}{\varepsilon}} \|h - g\|_w.$$

We know that $\frac{\frac{1}{\varepsilon}}{\alpha-\alpha_0+\frac{1}{\varepsilon}} < 1$ then T_ε is a contraction map on the Banach space $B_w(\mathcal{O})$, so there exist a unique u_ε in $B_w(\mathcal{O})$ such that $T_\varepsilon u_\varepsilon = u_\varepsilon$, this implies that u_ε solves (3.5). Moreover, we have that $\lim_{n\to\infty}\|T_\varepsilon^n h - u_\varepsilon\|_w = 0$ that implies convergence in seminorm.

On the other hand, using the fact that $f, \varphi \in C_p^w(\mathcal{O})$, and taking $h \in C_p^w(\mathcal{O})$, all together allow us to apply Theorem 2.20 to claim that $T_\varepsilon h = \mathcal{R}_{\alpha+\frac{1}{\varepsilon}}(f + \frac{1}{\varepsilon}(h\wedge\varphi)) \in D(\mathcal{A}_\alpha)$. Iterating n-times the operator T_ε, it is easy to see that $T_\varepsilon^n h \in D(\mathcal{A}_\alpha)$ for all $n \in \mathbb{N}$. Hence, un virtue of Theorem 2.9 we have $u_\varepsilon \in C_p^w(\mathcal{O})$, yielding that $u_\varepsilon = \mathcal{R}_{\alpha+\frac{1}{\varepsilon}}(f + \frac{1}{\varepsilon}(u_\varepsilon\wedge\varphi)) \in D(\mathcal{A}_\alpha)$.

Let us prove now the inequalities (3.8). Namely, let $0 < \varepsilon' < \varepsilon$, then from (3.6) we obtain

$$\begin{aligned}
\mathcal{A}_{\alpha+\frac{1}{\varepsilon}} u_{\varepsilon'} =& \mathcal{A}_{\alpha+\frac{1}{\varepsilon'}} u_{\varepsilon'} + (\frac{1}{\varepsilon} - \frac{1}{\varepsilon'})u_{\varepsilon'} = f + \frac{1}{\varepsilon'}(u_{\varepsilon'}\wedge\varphi) + (\frac{1}{\varepsilon} - \frac{1}{\varepsilon'})u_{\varepsilon'} \\
=& f + \frac{1}{\varepsilon'}u_{\varepsilon'} - \frac{1}{\varepsilon'}(u_{\varepsilon'} - \varphi)^+ + (\frac{1}{\varepsilon} - \frac{1}{\varepsilon'})u_{\varepsilon'} = f - \frac{1}{\varepsilon'}(u_{\varepsilon'} - \varphi)^+ + \frac{1}{\varepsilon}u_{\varepsilon'} \\
\leq& f - \frac{1}{\varepsilon}(u_{\varepsilon'} - \varphi)^+ + \frac{1}{\varepsilon}u_{\varepsilon'} = f + \frac{1}{\varepsilon}(u_{\varepsilon'}\wedge\varphi).
\end{aligned}$$

Applying $\mathcal{R}_{\alpha+\frac{1}{\varepsilon}}$ to the last inequality we obtain

$$u_{\varepsilon'} \leq T_\varepsilon u_{\varepsilon'}.$$

Iterating, we get $u_{\varepsilon'} \leq T_\varepsilon^n u_{\varepsilon'}$. Therefore, letting $n \to \infty$ we obtain $u_{\varepsilon'} \leq u_\varepsilon$.

Next, we will show that $u_\varepsilon - u_{\varepsilon'} \leq (u_\varepsilon - \varphi)^+ \leq \Big|\mathcal{R}_{\alpha+\frac{1}{\varepsilon}} f + \mathcal{R}_{\alpha+\frac{1}{\varepsilon}} \varphi - \varphi\Big|$. Namely, assuming $u_{\varepsilon'} \geq \varphi$ we get $u_\varepsilon - u_{\varepsilon'} \leq u_\varepsilon - \varphi \leq (u_\varepsilon - \varphi)^+$. Otherwise, if $\varphi \geq u_{\varepsilon'}$ then from (3.5) we obtain $\mathcal{A}_\alpha(u_\varepsilon - u_{\varepsilon'}) = -\frac{1}{\varepsilon}(u_\varepsilon - \varphi)^+ \leq 0$, and applying $\mathcal{R}_\alpha$ to the last inequality we get $u_\varepsilon - u_{\varepsilon'} \leq 0 \leq (u_\varepsilon - \varphi)^+$. Moreover, from (3.7) we obtain

$$\begin{aligned}
u_\varepsilon - \varphi &= \mathcal{R}_{\alpha+\frac{1}{\varepsilon}}(f + \frac{1}{\varepsilon}(u_\varepsilon \wedge \varphi)) - \varphi \\
&= \mathcal{R}_{\alpha+\frac{1}{\varepsilon}}(f + \frac{1}{\varepsilon}(u_\varepsilon \wedge \varphi) - \frac{1}{\varepsilon}\varphi) + \frac{1}{\varepsilon}\mathcal{R}_{\alpha+\frac{1}{\varepsilon}} \varphi - \varphi \\
&= \mathcal{R}_{\alpha+\frac{1}{\varepsilon}}(f - \frac{1}{\varepsilon}(\varphi - u_\varepsilon)^+) + \frac{1}{\varepsilon}\mathcal{R}_{\alpha+\frac{1}{\varepsilon}} \varphi - \varphi \\
&\leq \mathcal{R}_{\alpha+\frac{1}{\varepsilon}} f + \frac{1}{\varepsilon}\mathcal{R}_{\alpha+\frac{1}{\varepsilon}} \varphi - \varphi.
\end{aligned} \tag{3.9}$$

Hence,

$$0 \leq u_\varepsilon - u_{\varepsilon'} \leq (u_\varepsilon - \varphi)^+ \leq \big|\mathcal{R}_{\alpha+\frac{1}{\varepsilon}} f + \frac{1}{\varepsilon}\mathcal{R}_{\alpha+\frac{1}{\varepsilon}} \varphi - \varphi\big|. \tag{3.10}$$

Let $\varepsilon > \varepsilon' > 0$. Using $u_{\varepsilon'} \leq u_\varepsilon$ and $0 \leq u_\varepsilon - u_{\varepsilon'} \leq (u_\varepsilon - \varphi)^+$, we obtain that there exists the pointwise monotone limit $u_0 := \lim_{\varepsilon \downarrow 0} u_\varepsilon$ and $u_0 > -\infty$. Letting $\varepsilon' \downarrow 0$ in (3.10), we get $0 \leq u_\varepsilon - u_0 \leq \big|\mathcal{R}_{\alpha+\frac{1}{\varepsilon}} f + \frac{1}{\varepsilon}\mathcal{R}_{\alpha+\frac{1}{\varepsilon}} \varphi - \varphi\big|$. Thus, in virtue of the relations (2.30) and (2.46) we get

$$p(u_\varepsilon - u_0, x) \leq \frac{1}{\alpha + \frac{1}{\varepsilon} - \alpha_0} p(f, x) + p\big(\frac{1}{\varepsilon}\mathcal{R}_{\alpha+\frac{1}{\varepsilon}} \varphi - \varphi, x\big) \to 0, \quad \text{as } \varepsilon \downarrow 0,$$

and so $\text{s-}\lim_{\varepsilon \downarrow 0} u_\varepsilon = u_0$; this implies that $u_0 \in C_p(\mathcal{O})$ after using Theorem 2.9(b). □

3.3 Variational inequalities

Let $f, \varphi \in C_p^w(\mathcal{O})$. We say that $u \in C_p^w(\mathcal{O})$ satisfies the variational inequalities (VI) if:

$$\begin{cases} u \leq \int_0^t \Phi_\alpha(s) f\, ds + \Phi_\alpha(t) u, & \forall\, t \geq 0; \\ u \leq \varphi. \end{cases} \tag{3.11}$$

Any function $u \in C_p^w(\mathcal{O})$ that satisfies the VI above, will be referred to as a *subsolution*.

On the other hand, by definition of w in (2.1), it is obvious that $w \in B_w(\mathcal{O})$. For the later purposes, we need the next assumption in order to guarantee that the subsolution of interest associated to (3.11) is regular enough.

Assumption 3.3. We suppose that w defined in (2.1), belongs to $C_p^w(\mathcal{O})$.

• *Remark* 3.4. The above assumption is verified in particular models, see for instance, Menaldi [26, 27] or Menaldi and Sritharan [30], where the authors use a polynomial function of type $w(x) = k_1(k_2 + |x|^2)^p$, for some constants $k_1 \geq 1$, $k_2 \geq 0$.

Now let $u := \mathcal{R}_\alpha f - (\|\varphi\|_w + \frac{1}{\alpha - \alpha_0}\|f\|_w) w \in C_p^w(\mathcal{O})$. Note that $\mathcal{R}_\alpha f - \frac{1}{\alpha-\alpha_0}\|f\|_w w \leq 0$ because of (2.31), then $u \leq -\|\varphi\|_w w \leq \varphi$. We also have that

$$\Phi_\alpha(t) u \geq \Phi_\alpha(t)\mathcal{R}_\alpha f - (\|\varphi\|_w + \frac{1}{\alpha - \alpha_0}\|f\|_w) w.$$

Using the first equality in (2.45), we obtain

$$\int_0^t \Phi_\alpha(s) f\, ds + \Phi_\alpha(t) u = \mathcal{R}_\alpha f - \Phi_\alpha(t)\mathcal{R}_\alpha f + \Phi_\alpha(t) u \geq \mathcal{R}_\alpha f - (\|\varphi\|_w + \frac{1}{\alpha - \alpha_0}\|f\|_w) w = u.$$

Therefore, we have proved that $u \in C_p^w(\mathcal{O})$ defined in the previous paragraph satisfies the VI (3.11).

We will see next that the limit function u_0 obtained in the past subsection, is the maximal subsolution on $C_p^w(\mathcal{O})$ of the VI (3.11) and $\|u_0\|_w < \infty$ as it is established in the following theorem.

Theorem 3.5. *Under Assumptions 2.3 and 3.3, the limit function u_0 introduced in Theorem 3.2 verifies the VI (3.11). Moreover, every $u \in C_p^w(\mathcal{O})$ that is also subsolution of (3.11) satisfies $u \le u_0$; as a consequence $u_0 \in C_p^w(\mathcal{O})$.*

Proof. From (3.5) and the first equality in (2.45), we obtain

$$\begin{aligned} u_\varepsilon &= \mathcal{R}_\alpha(f - \frac{1}{\varepsilon}(u_\varepsilon - \varphi)^+) \\ &= \int_0^t \Phi_\alpha(s)(f - \frac{1}{\varepsilon}(u_\varepsilon - \varphi)^+)\, ds + \Phi_\alpha(t) u_\varepsilon \\ &\le \int_0^t \Phi_\alpha(s) f\, ds + \Phi_\alpha(t) u_\varepsilon. \end{aligned}$$

Moreover, for each $t \ge 0$, we have that $\Phi_\alpha(t)u_\varepsilon \to \Phi_\alpha(t)u_0$ pointwise as $\varepsilon \downarrow 0$, because $p(\Phi_\alpha(t)u_\varepsilon - \Phi_\alpha(t)u_0, x) \le p(u_\varepsilon - u_0, x) \to 0$, as $\varepsilon \downarrow 0$. So, letting $\varepsilon \downarrow 0$ in the last inequality we get

$$u_0 \le \int_0^t \Phi_\alpha(s) f\, ds + \Phi_\alpha(t) u_0.$$

On the other hand from (3.7) we have

$$u_\varepsilon = \mathcal{R}_{\alpha+\frac{1}{\varepsilon}}(f + \frac{1}{\varepsilon}(u_\varepsilon \wedge \varphi)) \le \mathcal{R}_{\alpha+\frac{1}{\varepsilon}}(f + \frac{1}{\varepsilon}\varphi). \tag{3.12}$$

In virtue of (2.30) and (2.46), we have

$$p\big(\mathcal{R}_{\alpha+\frac{1}{\varepsilon}}(f + \frac{1}{\varepsilon}\varphi) - \varphi, x\big) \le \frac{1}{\alpha + \frac{1}{\varepsilon} - \alpha_0} p(f, x) + p\big(\frac{1}{\varepsilon}\mathcal{R}_{\alpha+\frac{1}{\varepsilon}}\varphi - \varphi, x\big) \to 0, \quad \text{as } \varepsilon \downarrow 0.$$

The last relation implies in particular that $\mathcal{R}_{\alpha+\frac{1}{\varepsilon}}(f + \frac{1}{\varepsilon}\varphi) \to \varphi$ pointwise, as $\varepsilon \downarrow 0$. Hence, letting $\varepsilon \downarrow 0$ in (3.12) we get

$$u_0 \le \varphi,$$

which implies that u_0 satisfies (3.11).

It only remains to show that u_0 the maximal subsolution. Indeed, take $u \in C_p^w(\mathcal{O})$ that satisfies (3.11). Then, u satisfies: $u - \Phi_\alpha(t) \le \int_0^t \Phi_\alpha(s) f\, ds$. Apply then $\mathcal{R}_{\alpha+\frac{1}{\varepsilon}}$ to both sides of the last inequality and hence multiply by $\frac{1}{t}$, so that

$$\frac{1}{t}(\mathcal{R}_{\alpha+\frac{1}{\varepsilon}} u - \Phi_\alpha(t)\,\mathcal{R}_{\alpha+\frac{1}{\varepsilon}} u) \le \mathcal{R}_{\alpha+\frac{1}{\varepsilon}} \frac{1}{t}\int_0^t \Phi_\alpha(s) f\, ds.$$

The commutative property between $\Phi_\alpha(t)$ and $\mathcal{R}_{\alpha+\frac{1}{\varepsilon}}$ is due to (2.33). Using (2.45) again, the fact that $\alpha \mapsto \mathcal{R}_\alpha$ is a family of commutative operators given in (2.47), as well as the relation (2.33), we deduce

$$\frac{1}{t}(\mathcal{R}_{\alpha+\frac{1}{\varepsilon}} u - \Phi_\alpha(t)\,\mathcal{R}_{\alpha+\frac{1}{\varepsilon}} u) \le \frac{1}{t}(\mathcal{R}_\alpha\,\mathcal{R}_{\alpha+\frac{1}{\varepsilon}} f - \Phi_\alpha(t)\,\mathcal{R}_\alpha\,\mathcal{R}_{\alpha+\frac{1}{\varepsilon}} f),$$

thus letting $t \downarrow 0$ we get

$$\mathcal{A}_\alpha\,\mathcal{R}_{\alpha+\frac{1}{\varepsilon}} u \le \mathcal{A}_\alpha\,\mathcal{R}_\alpha\,\mathcal{R}_{\alpha+\frac{1}{\varepsilon}} f = \mathcal{R}_{\alpha+\frac{1}{\varepsilon}} f. \tag{3.13}$$

In virtue of Proposition 2.23, we know that $(\frac{1}{\varepsilon}I + \mathcal{A}_\alpha)\,\mathcal{R}_{\alpha+\frac{1}{\varepsilon}} = \mathcal{A}_{\alpha+\frac{1}{\varepsilon}}\,\mathcal{R}_{\alpha+\frac{1}{\varepsilon}} = I$, then

$$\mathcal{A}_\alpha\,\mathcal{R}_{\alpha+\frac{1}{\varepsilon}} = I - \frac{1}{\varepsilon}\mathcal{R}_{\alpha+\frac{1}{\varepsilon}}. \tag{3.14}$$

This last fact, together with the relation $u = u \wedge \varphi$ (recall that $u \le \varphi$), and (3.13) yield that

$$u \le \mathcal{R}_{\alpha+\frac{1}{\varepsilon}} f + \frac{1}{\varepsilon}\mathcal{R}_{\alpha+\frac{1}{\varepsilon}} u = \mathcal{R}_{\alpha+\frac{1}{\varepsilon}} f + \frac{1}{\varepsilon}\mathcal{R}_{\alpha+\frac{1}{\varepsilon}}(u \wedge \varphi) = T_\varepsilon u.$$

Iterating the last expression, we obtain that $u \le T_\varepsilon^n u$, implying that $u \le u_\varepsilon$. Finally, letting $\varepsilon \downarrow 0$, we obtain $u \le u_0$.

Finally, take $u \in C_p^w(\mathcal{O})$ that satisfies the VI (3.11) (we know that there exist at least a function in $C_p^w(\mathcal{O})$ satisfying the VI), then we have $u \le u_0 \le \varphi$ that implies $|u_0| \le |u_0 - u| + |u| \le |\varphi - u| + |u|$. Since $\varphi - u$ and u belongs to $C_p^w(\mathcal{O})$ we get that $\|u_0\|_w \le \|\varphi - u\|_w + \|u\|_w < \infty$. So, we conclude that $u_0 \in C_p^w(\mathcal{O})$ is the maximal subsolution on $C_p^w(\mathcal{O})$ of the VI (3.11). □

4 Solution of the stopping problem

In this section we will analyze the optimal control problem through the solution of the VI (3.11). In addition, we provide the way to find an optimal stopping time in terms of so-named *continuation region* or *contact set.*

To begin with, we will ask an additional property to the process $y(t,x)$

Proposition 4.1. *The Markov process $\{y(t,x)\}_{t\geq 0}$ satisfies the strong Markov property in the following sense: for all stopping time $\tau \in \mathcal{T}$ and $h \in B(\mathcal{O})$ we have*

$$\mathrm{E}[h(y(s+\tau,x))|\mathcal{F}_\tau] = \mathrm{E}[h(y(s,y(\tau,x)))], \tag{4.1}$$

where $\mathcal{F}_\tau$ is the σ-algebra generated of events $A \in \mathcal{F}$ for which $A \cap \{\tau \leq t\} \in \mathcal{F}_t$ for every $t \geq 0$.

Proof. First, let us suppose that τ has a numerable state space D in $\bar{\mathbb{R}}$. Then, we have that

$$\begin{aligned}\mathrm{E}[h(y(t+\tau,x))|\mathcal{F}_\tau] &= \sum_{s\in D} 1_{\tau=s}\,\mathrm{E}[h(y(t+\tau,x))|\mathcal{F}_\tau] = \sum_{s\in D} 1_{\tau=s}\,\mathrm{E}[h(y(t+s,x))|\mathcal{F}_s]\\ &= \sum_{s\in D} 1_{\tau=s}\,\mathrm{E}[h(y(t,y(s,x)))] = \mathrm{E}[h(y(t,y(\tau,x)))].\end{aligned}$$

In the general case, by Lemma 7.4 in [23] we can take a sequence of stopping times τ_n with numerable state space such that $\tau_n \downarrow \tau$. So, we have that

$$\mathrm{E}[h(y(t+\tau_n,x))|\mathcal{F}_{\tau_n}] = \mathrm{E}[h(y(t,y(\tau_n,x)))]$$

which implies

$$\mathrm{E}[e^{-\tau_n} h(y(t+\tau_n,x))|\mathcal{F}_{\tau_n}] = e^{-\alpha\tau_n}\,\mathrm{E}[h(y(t,y(\tau_n,x)))] = e^{-\alpha(\tau_n-t)}\Phi_\alpha(t)h(y(\tau_n,x)). \tag{4.2}$$

By the right continuity of $s \mapsto \Phi_\alpha(t)h(y(s,x))$ and the fact that $\tau_n \downarrow \tau$ we get $e^{-\alpha(\tau_n-t)}\Phi_\alpha(t)h(y(\tau_n,x)) \to e^{-\alpha(\tau-t)}\Phi_\alpha(t)h(y(\tau,x))$ when $n \to \infty$. By Lemma 7.3 in [23], we have $\mathcal{F}_\tau = \cap_{n\in\mathbb{N}}\mathcal{F}_{\tau_n}$ which together with Theorem 45 in [14] give us

$$\mathrm{E}[e^{-\tau_n} h(y(t+\tau_n,x))|\mathcal{F}_{\tau_n}] \to e^{-\alpha\tau}\,\mathrm{E}[h(y(t+\tau,x))|\mathcal{F}_\tau]$$

when $n \to \infty$. Using this last fact as long with (4.2) we conclude that

$$\mathrm{E}[h(y(t+\tau,x))|\mathcal{F}_\tau] = \mathrm{E}[h(y(t,y(\tau,x)))].$$

□

In order to characterize the optimal stopping time as the hitting time of certain region of the state space, we will also need the following property of our process $\{y(t,x)\}_{t\geq 0}$.

Assumption 4.2. The process $\{y(t,x)\}_{t\geq 0}$ is quasi-left continuous, that is, for every stopping time τ and any sequence of stopping times $\tau_1, \tau_2, \ldots$ such that $\tau_n \uparrow \tau$ we have that $y(\tau_n,x) \to y(\tau,x)$ $\mathbb{P}$-a.s. on $\{\tau < \infty\}$.

• *Remark* 4.3. (a) Assumption 4.2 is a little variation of the Hunt process definition.

(b) It is well-known that a Markov process associated to a strong Feller semigroup is a Hunt process —for further details, see Chug [11], Chapter 3.

Let us now establish the main result of this section.

Theorem 4.4. *Under Assumptions 2.3, 3.3, and 4.2, the following statements hold true.*
(a) The optimal cost $\hat{u}$ in (3.2) is equal to the limit function u_0.
(b) The optimal stopping time can be regarded as the first hitting time of the so-called continuation region (a.k.a. contact set). That is, for all $x \in \mathcal{O}$,

$$\hat{\tau}(x) := \inf\{t \geq 0 : \hat{u}(y(t,x)) = \varphi(y(t,x))\} \quad (\textit{continuation region}), \tag{4.3}$$

satisfying $\hat{u}(x) = J(x,\hat{\tau}(x))$.
(c) If the stopping cost $\varphi \in D(\mathcal{A}_\alpha)$, then

$$\mathcal{R}_\alpha(f \wedge \mathcal{A}_\alpha\,\varphi) \leq \hat{u} \leq \mathcal{R}_\alpha\, f \wedge \varphi. \tag{4.4}$$

Proof. (a) Take $\tau \in \mathcal{T}$, where $\mathcal{T}$ is the set of stopping times defined at the beginning of the section. Moreover, define $u := f - \frac{1}{\varepsilon}(u_\varepsilon - \varphi)^+$. Then, from (3.5) we have

$$\begin{aligned} u_\varepsilon(x) = &\mathcal{R}_\alpha u(x) = \int_0^\infty \Phi_\alpha(s)u(x)ds = \int_0^\infty \mathrm{E}[e^{-\alpha s}u(y(s,x))]ds \\ &= \mathrm{E}[\int_0^\infty e^{-\alpha s}u(y(s,x))ds] = \mathrm{E}[\int_0^\tau e^{-\alpha s}u(y(s,x))ds] \\ &\quad + \mathrm{E}[\int_\tau^\infty e^{-\alpha s}u(y(s,x))ds]. \end{aligned} \tag{4.5}$$

Let us analyze the last term of (4.5). Using the strong Markov property (4.1) we get

$$\begin{aligned} &\mathrm{E}[\int_\tau^\infty e^{-\alpha s}u(y(s,x))ds] = \mathrm{E}[\int_0^\infty e^{-\alpha(s+\tau)}u(y(s+\tau,x))ds] \\ &= \int_0^\infty \mathrm{E}[e^{-\alpha(s+\tau)}u(y(s+\tau,x))]ds = \int_0^\infty \mathrm{E}[\mathrm{E}[e^{-\alpha(s+\tau)}u(y(s+\tau,x))|\mathcal{F}_\tau]]ds \\ &= \int_0^\infty \mathrm{E}[e^{-\alpha\tau}\,\mathrm{E}[e^{-\alpha s}u(y(s,y(\tau,x)))]]ds = \int_0^\infty \mathrm{E}[e^{-\alpha\tau}\Phi_\alpha(s)u(y(\tau,x))]ds \\ &= \mathrm{E}[e^{-\alpha\tau}\int_0^\infty \Phi_\alpha(s)u(y(\tau,x))ds] = \mathrm{E}[e^{-\alpha\tau}\mathcal{R}_\alpha u(y(\tau,x))] = \mathrm{E}[e^{-\alpha\tau}u_\varepsilon(y(\tau,x))] \end{aligned} \tag{4.6}$$

Hence, in virtue of (4.5) and (4.6), we have that

$$\begin{aligned} u_\varepsilon(x) &= \mathrm{E}[\int_0^\tau e^{-\alpha s}u(y(s,x))ds] + \mathrm{E}[\int_\tau^\infty e^{-\alpha s}u(y(s,x))ds] \\ &= \mathrm{E}[\int_0^\tau e^{-\alpha s}u(y(s,x))ds] + \mathrm{E}[e^{-\alpha\tau}u_\varepsilon(y(\tau,x))] \\ &= \mathrm{E}\Big[\int_0^\tau e^{-\alpha s}[f - \frac{1}{\varepsilon}(u_\varepsilon - \varphi)^+](y(s,x))ds + e^{-\alpha\tau}u_\varepsilon(y(\tau,x))\Big]. \end{aligned} \tag{4.7}$$

On the other hand, from the definition of the seminorm p, it is evident that $\mathrm{E}\left[e^{-\alpha\tau}|u_\varepsilon(y(\tau,x)) - u_0(y(\tau,x))|\right] \le p(u_\varepsilon - u_0, x) \to 0$ when $\varepsilon \downarrow 0$, where the last convergence is due to Theorem 3.2. Then, using this last fact along with (4.7) and Theorem 3.5, we obtain

$$\begin{aligned} u_0(x) = \lim_{\varepsilon\downarrow 0} u_\varepsilon(x) &\le \lim_{\varepsilon\downarrow 0}\mathrm{E}[\int_0^\tau e^{-\alpha s}f(y(s,x))ds + e^{-\alpha\tau}u_\varepsilon(y(\tau,x))] \\ &= \mathrm{E}[\int_0^\tau e^{-\alpha s}f(y(s,x))ds + e^{-\alpha\tau}\lim_{\varepsilon\downarrow 0}u_\varepsilon(y(\tau,x))] \\ &= \mathrm{E}[\int_0^\tau e^{-\alpha s}f(y(s,x))ds + e^{-\alpha\tau}u_0(y(\tau,x))] \\ &\le \mathrm{E}[\int_0^\tau e^{-\alpha s}f(y(s,x))ds + e^{-\alpha\tau}\varphi(y(\tau,x))] = J(x,\tau). \end{aligned}$$

Therefore, $u_0 \le \hat{u}$, after applying the infimum over all τ in last rightmost term.

On the other hand, for each $\varepsilon > 0$, let us consider the stopping time

$$\tau_\varepsilon(x) := \inf\{t \ge 0 : u_\varepsilon(y(t,x)) \ge \varphi(y(t,x))\}.$$

Now take a sequence $\{t_n\}_{n\in\mathbb{N}}$ in $[0,\infty)$ such that $t_n \downarrow \tau_\varepsilon(x)$ (pointwise w.r.t. $\omega \in \Omega$) and $u_\varepsilon(y(t_n,x)) \ge \varphi(y(t_n,x))$. Since $t \mapsto u_\varepsilon(y(t,x)) - \varphi(y(t,x))$ is continuous a.s., we obtain $u_\varepsilon(y(\tau_\varepsilon(x),x)) \ge \varphi(y(\tau_\varepsilon(x),x))$ when $t_n \downarrow \tau_\varepsilon(x)$. Then by (4.7), we deduce

$$\begin{aligned} u_\varepsilon(x) = &\mathrm{E}[\int_0^{\tau_\varepsilon} e^{-\alpha s}[f - \frac{1}{\varepsilon}(u_\varepsilon - \varphi)^+](y(s,x))ds + e^{-\alpha\tau_\varepsilon}u_\varepsilon(y(\tau_\varepsilon,x))] \\ = &\mathrm{E}[\int_0^{\tau_\varepsilon} e^{-\alpha s}f(y(s,x))ds + e^{-\alpha\tau_\varepsilon}\varphi(y(\tau_\varepsilon,x))] = J(\tau_\varepsilon,x) \ge \hat{u}(x). \end{aligned}$$

This shows that $u_0 = \lim_{\varepsilon\downarrow 0} u_\varepsilon \geq \hat{u}$. Joining the pieces, we conclude that $u_0 = \hat{u}$.

(b) Given $\varepsilon > \varepsilon'$, we know by the proof of Theorem 3.2 that $u_\varepsilon \geq u_{\varepsilon'}$ then we have the expression

$$\{s \geq 0 : u_{\varepsilon'}(y(s,x)) \geq \varphi(y(s,x))\} \subseteq \{s \geq 0 : u_\varepsilon(y(s,x)) \geq \varphi(y(s,x))\},$$

implying $\tau_\varepsilon \leq \tau_{\varepsilon'}$. So, there exists the monotone limit $\tau_\varepsilon \uparrow \tau_0$, as $\varepsilon \downarrow 0$. Also, because of the continuity of $s \mapsto u_0(y(s,x))$ on $[0,\infty)$ a.s., we have that $\varphi(y(\hat{\tau},x)) = u_0(y(\hat{\tau},x)) \leq u_\varepsilon(y(\hat{\tau},x))$, where $\hat{\tau}$ was defined in (4.3). Hence, we obtain $\tau_\varepsilon \leq \hat{\tau}$ that implies $\tau_0 \leq \hat{\tau}$.

On the other hand, the fact s-lim $u_\varepsilon = u_0$ gives us the existence of a sequence ε_n, $n \in \mathbb{N}$, such that $\varepsilon_n \downarrow 0$ and

$$\lim_{n\to\infty} \sup_{s\geq 0} e^{-\alpha_0 s} \left| u_{\varepsilon_n}(y(s,x)) - u_0(y(s,x)) \right| = 0, \quad a.s., \tag{4.8}$$

where this last asertion is due to Lemma 2.8. Also, because of $u_{\varepsilon_n} \geq u_0$, we have that

$$0 \leq u_{\varepsilon_n}(y(\tau_{\varepsilon_n},x)) - u_0(y(\tau_{\varepsilon_n},x)) \leq e^{\alpha_0 \tau_{\varepsilon_n}} \sup_{s\geq 0} e^{-\alpha_0 s} \left| u_{\varepsilon_n}(y(s,x)) - u_0(y(s,x)) \right|. \tag{4.9}$$

If $\tau_0 = \infty$ then $\infty = \tau_0 \leq \hat{\tau}$, so $\tau_0 = \hat{\tau}$. Now, suppose $\tau_0 < \infty$ a.s., then we have that $e^{\alpha_0 \tau_{\varepsilon_n}} \to e^{\alpha_0 \tau_0}$, when $n \to \infty$. Hence, using (4.8), the right hand side of inequality (4.9) converges to 0 when $n \to \infty$. Using 4.2 we deduce

$$\varphi(y(\tau_0,x)) = \lim_{n\to\infty} \varphi(y(\tau_{\varepsilon_n},x)) \leq \lim_{n\to\infty} u_{\varepsilon_n}(y(\tau_{\varepsilon_n},x)) = u_0(y(\tau_0,x)), \quad a.s.$$

Thus, the definition of $\hat{\tau}$ yields to $\hat{\tau} \leq \tau_0$ and so, $\hat{\tau} = \tau_0$. It remains to show that $u_0(x) = J(\hat{\tau}(x),x)$. Namely, consider $\varepsilon_0 > 0$ fixed. Given $0 < \varepsilon \leq \varepsilon_0$ and $t \leq \tau_{\varepsilon_0}$ we know that $t \leq \tau_\varepsilon$ and $u_\varepsilon(y(t,x)) < \varphi(y(t,x))$. Then the relation (4.7) leads to

$$u_\varepsilon(x) = \mathrm{E}[\int_0^{\tau_{\varepsilon_0}} e^{-\alpha s} f(y(s,x))\, ds + e^{-\alpha \tau_{\varepsilon_0}} u_\varepsilon(y(\tau_{\varepsilon_0},x))].$$

By monotone convergence and left quasi-left continuity of $y(s,x)$, letting $\varepsilon \downarrow 0$ and hence $\varepsilon_0 \downarrow 0$, we obtain

$$u_0(x) = \mathrm{E}[\int_0^{\tau_0} e^{-\alpha s} f(y(s,x))\, ds + e^{-\alpha \tau_0} \varphi(y(\tau_0,x))] = J(\tau_0,x).$$

Thus, we conclude that $\hat{\tau}$ is the optimal stopping time for the problem (3.1)–(3.3).

(c) Suppose $\varphi \in D(\mathcal{A}_\alpha)$ and let $v_\varepsilon := \frac{1}{\varepsilon} \mathcal{R}_{\alpha+\frac{1}{\varepsilon}} (f - \mathcal{A}_\alpha \varphi)^+$. In virtue of (3.9) and a variation of (3.14) we obtain

$$u_\varepsilon - \varphi \leq \mathcal{R}_{\alpha+\frac{1}{\varepsilon}} f + \frac{1}{\varepsilon} \mathcal{R}_{\alpha+\frac{1}{\varepsilon}} \varphi - \varphi = \mathcal{R}_{\alpha+\frac{1}{\varepsilon}} f - \mathcal{R}_{\alpha+\frac{1}{\varepsilon}} \mathcal{A}_\alpha \varphi, \tag{4.10}$$

which in turn gives $(u_\varepsilon - \varphi)^+ \leq \mathcal{R}_{\alpha+\frac{1}{\varepsilon}} (f - \mathcal{A}_\alpha \varphi)^+$. Using this last inequality together with (3.5), we get

$$f - \mathcal{A}_\alpha u_\varepsilon = \frac{1}{\varepsilon}(u_\varepsilon - \varphi)^+ \leq \frac{1}{\varepsilon} \mathcal{R}_{\alpha+\frac{1}{\varepsilon}} (f - \mathcal{A}_\alpha \varphi)^+ = v_\varepsilon, \tag{4.11}$$

or equivalently, $f - v_\varepsilon \leq \mathcal{A}_\alpha u_\varepsilon$, yielding that

$$\mathcal{R}_\alpha (f - v_\varepsilon) \leq u_\varepsilon, \tag{4.12}$$

after applying the resolvent operator in both sides of this later expression. Also note that by (2.46), we know that bs-$\lim_{\varepsilon\downarrow 0} v_\varepsilon = (f - \mathcal{A}_\alpha \varphi)^+$. Using this last property, we can let $\varepsilon \downarrow 0$ at (4.12) to deduce $\mathcal{R}_\alpha(f - (f - \mathcal{A}_\alpha \varphi)^+) = \mathcal{R}_\alpha(f \wedge \mathcal{A}_\alpha \varphi) \leq u_0$.

On the other hand, using (3.5) again we have that $\mathcal{A}_\alpha u_\varepsilon \leq f$, or equivalently, $u_\varepsilon \leq \mathcal{R}_\alpha f$, letting $\varepsilon \downarrow 0$, we obtain $u_0 \leq \mathcal{R}_\alpha f$ but also we have that $u_0 \leq \varphi$ because it is a subsolution of (3.11), then $\hat{u} = u_0 \leq \mathcal{R}_\alpha f \wedge \varphi$. Hence, we conclude that

$$\mathcal{R}_\alpha(f \wedge \mathcal{A}_\alpha \varphi) \leq u_0 \leq \mathcal{R}_\alpha f \wedge \varphi.$$

□

5 Concluding remarks

In this paper we have analyzed an optimal stopping problem for a general family of continuous-time Markov process. Such analysis was carried out through the study of the solutions of a certain variational inequality. The optimal stopping time was also obtained from the aid of the continuation region, which in turn depends on one solution of this inequality. To the best of our knowledge, the approach to follow is new as it tackles optimal stopping time problems on a family of Markov processes whose state space is not necessarily compact nor locally compact. We think that the generalization of stopping time problems when (1) we use a general family of continuous-time Markov process and (2) we consider a more general state space, give rise to a satisfactory advance on this field.

Finally, it should be noted that this work can open other types of control with stopping with the same assumptions than ours; for example a recursive multi-stopping time problems with controlled discontinuities (e.g. impulsive and switching control problems) —see, for instance, Bayraktar et. al., [3], Bensoussan and Lions [7], Menaldi [25, 26], Oksendal and Sulem [33], to mention just a few. Other extensions could be optimal stopping times with restrictions in which the stopping is conditioned to wait for a signal —see Menaldi and Robin [31, 32].

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