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ON OPTIMAL ERGODIC CONTROL OF DIFFUSIONS WITH JUMPS*

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Abstract

Our purpose is to study an optimal ergodic control problem where the state of the system is given by a diffusion process with jumps in the whole space. The corresponding dynamic programming (or Hamilton-Jacobi-Bellman) equation is a quasi-linear integro-differential equation of second order. A key result is to prove the existence and uniqueness of an invariant density function for a jump diffusion, whose lower order coefficients are only locally bounded and Borel measurable. Based on this invariant probability, existence and uniqueness (up to an additive constant) of solutions to the ergodic HJB equation is established. **Key words and phrases:** Jump diffusion, interior Dirichlet problem, exterior Dirichlet problem, ergodic optimal control, Green function, Girsanov transformation.

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Introduction

We are interested in controlling a diffusion process with jumps in the whole space. The controller can access only the drift term of the state equation, so that the state of the system is defined via the Girsanov transformation. The programming equation is an ergodic quasi-linear integro-differential equation. Solving this equation an optimal feedback can be obtained.

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Most of the ergodic properties of diffusion processes and their relation with partial differential equations are well know in the classic literature. However, similar questions for diffusion processes with jumps are not so popular, only recently some attention was given, cf. [18], Garroni and Menaldi [8] and reference therein.

Due to applications in stochastic control (in particular the action of a feedback function), we have to be able to treat diffusions with jumps with only locally bounded and Borel measurable lower order coefficients (where the control is applied). Moreover, since we are interested in the whole space, an assumption related to the existence of a Liapunov's function is needed. We assume the drift is of linear growth at infinity, but the existence and regularity of the Green function or transition density function (even in the purely partial differential equations case) as proved in Garroni and Menaldi [8] does not applied.

To show the existence of an invariant measure, many arguments are based on the so-called *Doeblin's condition*, which in turn is based on the strict positivity of the Green functions deduced from the strong maximum principle. The technique, to solve an elliptic second (order linear) equation (without zero order term) in the whole space, is based on the construction of an ergodic operator. For instance, we refer to the books of Bensoussan [1] and Khasminskii [12]. Related discussions can be found in the books of Borkar [4] and Ethier and Kurtz [6].

Now we are going to describe, without all the technical assumptions, the ergodic problem we will consider. Let $v(x)$ be a Borel measurable function from $\mathbb{R}^d$ into a compact metric space V (i.e., a measurable feedback). The dynamic of the system (for a given feedback) follows a diffusion with jumps in $\mathbb{R}^d$, i.e. a (strong) Markov process $(\Omega, P, X_t, t \geq 0)$ with semigroup $(\Phi_v(t), t \geq 0)$ and infinitesimal generator A_v, as discussed in the next section. A long run average cost is associated to the controlled system by

$$J(v) = \liminf_{T\to\infty} \frac{1}{T}\int_0^T \Phi_v(t) f dt,$$

where $f = f(x, v(x))$ is the running cost of the system under the feedback v. Our purpose is to give a characterization of the optimal cost

$$\lambda = \inf\{J(v) : v(\cdot)\}$$

and to construct an optimal feedback control $\hat{v}$. Notice that the dynamic equation is given later by (4.1) and that we expect λ to be a constant.

A formal application of the dynamic programming principle (as described in Fleming and Soner [7]) yields the following Hamilton-Jacobi-Bellman equation

$$\inf_v\{A_v u(x)\} = \lambda \ \text{ in } \ \mathbb{R}^d,$$

where the infimum is calculated for each fixed x, and v in V. An optimal feedback control is obtained as the minimizer $\hat{v}(x)$ in the HJB equation.

In order to study this Hamilton-Jacobi-Bellman equation we need some previous discussion. In Section 1, we give some details on the construction of the diffusion with jumps in the whole space $\mathbb{R}^d$, under convenient assumptions. Next we recall some results (cf. [19]) related to the invariant probability measure μ_v, for any measurable feedback v. Finally, we consider the above HJB equation under appropriate assumptions

1. Ergodic Control Problem

Before setting the optimal ergodic control problem, we need to recall some facts about diffusions with jumps in the whole space. To that effect, we consider an integro-differential operator of the form

$$I_0\varphi(x) = \int_{\mathbb{R}^d_*} [\varphi(x+z) - \varphi(x) - z \cdot \nabla\varphi(x)] M_0(x, dz), \tag{1.1}$$

where ∇ is the gradient operator in x, the Levy kernel $M_0(x, dz)$ is a Radon measure on $\mathbb{R}^d_* = \mathbb{R}^d \setminus \{0\}$ for any fixed x, and satisfies

$$\int_{|z|<1} |z|^2 M_0(x, dz) + \int_{|z|\geq 1} |z| M_0(x, dz) < \infty, \quad \forall x \in \mathbb{R}^d. \tag{1.2}$$

It is clear that this operator is associated with a jump process in $\mathbb{R}^d$, e.g., Gikhman and Skorokhod [9].

Similarly, let L_0 be a second order uniformly elliptic operator associated with a diffusion process in the whole space, i.e.

$$L_0 = \sum_{i,j=1}^{d} a_{ij}(x)\partial_{ij} + \sum_{i=1}^{d} b_i(x)\partial_i, \tag{1.3}$$

where the coefficients (a_{ij}) are bounded and Lipschitz continuous, i.e. for some $c_0, M > 0$

$$\begin{cases} c_0|\xi|^2 \leq \displaystyle\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \leq c_0^{-1}|\xi|^2 \quad \forall x, \xi \in \mathbb{R}^d, \\ |a_{ij}(x) - a_{ij}(x')| \leq M|x - x'|, \quad \forall x, x' \in \mathbb{R}^d, \end{cases} \tag{1.4}$$

$a_{ij} = a_{ji}$, and the first order coefficients (b_i) are Lipschitz continuous, i.e. for some $M > 0$,

$$\begin{cases} |b_i(x) - b_i(x')| \leq M|x - x'|, \quad \forall x, x' \in \mathbb{R}^d \\ b_i(0) = 0, \quad i = 1, \ldots, d. \end{cases} \tag{1.5}$$

The fact that $b = (b_i)$ vanishes on the origin and the additional assumption

$$-\sum_{i=1}^{d} b_i(x)x_i \geq c_1|x|^2, \quad \forall x \in \mathbb{R}^d, |x| \geq r_1, \tag{1.6}$$

for some constants $c_1, r_1 > 0$, will allow us to show some "stability" on the system.

The Levy kernel $M_0(x, dz)$ is assumed to have a particular structure, namely

$$M_0(x, A) = \int_{\{\zeta : j(x,\zeta)\in A\}} m_0(x, \zeta)\pi(d\zeta), \tag{1.7}$$

where $\pi(\cdot)$ is a σ-finite measure on the measurable space $(F, \mathcal{F})$, the functions $j(x, \zeta)$ and $m_0(x, \zeta)$ are measurable for (x, ζ) in $\mathbb{R}^d \times F$, and there exist a measurable and positive function $j_0(\zeta)$ and constants $C_0 > 0$, $1 \le \gamma \le 2$ [γ is referred to as the order of M_0 or I_0] such that for every x, ζ we have

$$\begin{cases} |j(x,\zeta)| \le j_0(\zeta), \qquad 0 \le m_0(x,\zeta) \le 1, \\ \int_F |j_0(\zeta)|^p (1 + j_0(\zeta))^{-1} \pi(d\zeta) \le C_0, \quad \forall p \in [\gamma, 2], \end{cases} \tag{1.8}$$

the function $j(x,\zeta)$ is continuously differentiable in x for any fixed ζ and there exists a constant $c_0 > 0$ such that for any (x,ζ) we have

$$c_0 \le \det(\mathbf{1} + \theta \nabla j(x,\zeta)) \le c_0^{-1}, \quad \forall \theta \in [0,1], \tag{1.9}$$

where $\mathbf{1}$ denotes the identity matrix in $\mathbb{R}^d, \nabla$ is the gradient operator in x, and $\det(\cdot)$ denotes the determinant of a matrix.

Depending on the assumptions on the coefficients of the operators L_0, I_0 and on the domain $\mathcal{O}$ of $\mathbb{R}^d$, we can construct the corresponding Markov-Feller process. The reader is referred to the books Bensoussan and Lions [3], Gikhman and Skorokhod [9] (among others) and references therein. Usually, more regularity on the coefficients $j(x,\zeta)$ and $m_0(x,\zeta)$ is needed, e.g.

$$\begin{cases} |m_0(x,\zeta) - m_0(x',\zeta)| \le M|x - x'|, \quad \forall x, x' \in \mathbb{R}^d, \\ |j(x,\zeta) - j(x',\zeta)| \le j_0(\zeta)|x - x'|, \quad \forall x, x' \in \mathbb{R}^d, \end{cases} \tag{1.10}$$

for some constant $M > 0$ and the same function $j_0(\zeta)$ as in assumption (1.8). Thus the integro-differential operator I_0 has the form

$$I_0\varphi(x) = \int_F [\varphi(x + j(x,\zeta)) - \varphi(x) - j(x,\zeta) \cdot \nabla\varphi(x)] m_0(x,\zeta)\pi(d\zeta). \tag{1.11}$$

It is possible to show that the Markov-Feller process associated with the infinitesimal generator $L_0 + I_0$ (which is referred to as the "diffusion with jumps") has a transition probability density function $G_0(x,t,y)$, which is smooth in some sense (cf. Garroni and Menaldi [8]).

Since our purpose is to treat control problems, we remark that (in general) the optimal feedback is not smooth. This forces us to consider some coefficients (e.g. those of first order) which are only measurable. To that effect, we will use the so-called Girsanov transformation.

Let $\Omega = D([0,+\infty), \mathbb{R}^d)$ be the canonical space of right continuous functions with left-hand limits ω from $[0,+\infty)$ into $\mathbb{R}^d$ endowed with the Skorokhod topology. Denote by either X_t or $X(t)$ the canonical process and by F_t the filtration generated by $\{X_s : s \le t\}$ (universally completed and right-continuous). Now let $(\Omega, P^0, F_t, X_t, t \ge 0)$ be the (homogeneous) Markov-Feller process with transition density function $G_0(x,t,y)$ associated with the integro-differential operator $L_0 + I_0$, i.e. the density w.r.t. the Lebesgue measure of $P^0\{X(t) \in dy \mid X(s) = x\}$ is equal to $G_0(x, t-s, x)$. For the sake of simplicity, we will refer to $(P_x^0, X(t), t \ge 0)$ as the above Markov-Feller process, where P_x^0 denote the conditional probability w.r.t. $\{X(0) = x\}$.

Hence, for any smooth function $\varphi(x)$ the process

$$Y_\varphi(t) = \varphi(X(t)) - \int_0^t (L_0 + I_0)\varphi(X(s))ds \tag{1.12}$$

is a P_x-martingale. This follows immediately from the representation

$$\begin{cases} E_x\{\varphi(X(t))\} & = & \int_{\mathbb{R}^d} G_0(x,t,y)\varphi(y)dy + \\ & & + \int_0^t ds \int_{\mathbb{R}^d} G_0(x,t-s,y)(L_0+I_0)\varphi(y)dy, \end{cases} \tag{1.13}$$

and the Markov property. Moreover, it is also possible to express the process X_t as follows

$$dX(t) = a^{1/2}(X(t))dw(t) + \int_{\mathbb{R}^d_*} z\mu_X(dt,dz) + b(X(t))dt, \tag{1.14}$$

where $(w(t), t \geq 0)$ is a standard Wiener process in $\mathbb{R}^d$, $a^{1/2}(x)$ is the positive square root of the matrix $(a_{ij}(x))$ and $b(x)$ is the vector $(b_i(x))$. The process μ_X is the martingale measure associate with the process $(X(t), t \geq 0)$, i.e. if $\eta_X(t,A)$ denotes the integer random measure defined as the number of jumps the process $X(\cdot)$ on $(0,t]$ with values in $A \subset \mathbb{R}^d_*$ (recall that $\mathbb{R}^d_* = \mathbb{R}^d \setminus \{0\}$) then

$$\mu_X(dt,A) + \pi_X(dt,A) = \eta_X(dt,A), \tag{1.15}$$

where $\mu_X(t,A)$ is a square integral (local) martingale quasi-left continuous and $\pi_X(t,A)$ is a predictable increasing process obtained via the Doob-Meyer decomposition, and

$$\pi_X(dt,dz) = M_0(X(t-),dz)dt, \tag{1.16}$$

where $M_0(x,dz)$ is the Levy kernel used to define the integro-differential operator I_0 given by (1.1), see e.g., Bensoussan and Lions [3].

Let $g(x) = (g_1(x), \ldots, g_d(x)$ and $c(x,z)$ be functions defined for x in $\mathbb{R}^d$, $z \in \mathbb{R}^d_\star$ such that

$$\begin{cases} g_i, c \text{ are bounded, measurable and,} \\ 0 \leq c(x,z) \leq C_0(1 \wedge |z|), \quad \forall x, z, \end{cases} \tag{1.17}$$

where C_0 is a constant.

Consider the exponential martingale $(e(t), t \geq 0)$ as the solution of the stochastic differential equation

$$\begin{cases} de(t) & = & e(t)[r_X(t)dw(t) + \int_{\mathbb{R}^d_\star} \gamma_X(t,z)\mu_X(dt,dz)], \\ e(0) & = & 1, \end{cases} \tag{1.18}$$

where

$$\begin{cases} r_X(t) & = & a^{-1/2}(X(t))g(X(t)), \\ \gamma_X(t,z) & = & zc(X(t),z), \end{cases} \tag{1.19}$$

i.e.,

$$\begin{cases} e(t) & = & \exp\{\int_0^t r_X(s)dw(s) - \int_0^t |r_X(s)|^2 ds + \\ & & + \int_0^t \int_{\mathbb{R}^d_\star} \gamma_X(s,z)\mu_X(ds,dz) - \\ & & - \int_0^t \int_{\mathbb{R}^d_\star} [\gamma_X(s,z) - \ln(1+\gamma_X(s,z))]\pi_X(ds,dz)\}. \end{cases} \tag{1.20}$$

If we denote by

$$L = L_0 + \sum_{i=1}^{d} g_i(x)\partial_i \tag{1.21}$$

and

$$I\varphi(x) = I_0\varphi(x) + \int_{\mathbb{R}^d_\star} [\varphi(x+z) - \varphi(x)]c(x,z)M_0(x,dz), \tag{1.22}$$

then, by means of Itô's formula one can prove that for any smooth function φ, the process

$$Z_\varphi(t) = \varphi(X(t)) - \int_0^t (L+I)\varphi(X(s))ds \tag{1.23}$$

is a P_x-martingale, where the new probability measure P_x on Ω is defined as

$$dP_x = e(t)dP_x^0 \ \text{ on } \ F_t. \tag{1.24}$$

Notice that the probability measures P_x^0 and P_x are absolutely continuous, one with respect to the other. Also, a representation of the form (1.14) is valid under the new probability measure P_x, i.e.

$$\begin{cases} dX(t) &= a^{1/2}(X(t))dw(t) + [b(X(t)) + g(X(t))]dt + \\ & \quad + \int_{\mathbb{R}^d_\star} z\mu(dt,dz), \end{cases} \tag{1.25}$$

where $(w(t), t \geq 0)$ is again a standard Wiener process and μ is the martingale measure associate with the (canonical) process $X(t)$ under the new measure P_x.

Remark 1.1 *Due to the linear growth of the coefficients $b_i(x), i = 1, \ldots, d$, we can not use directly the construction in Garroni and Menaldi [8] of the Green function (or transition density).* □

Now we are ready to formulate our optimal ergodic control problem. Let $f(x,v)$, $g(x,v) = (g_1(x,v), \ldots, g_d(x,v))$, and $c(x,v,z)$ be functions defined for (x,v) in $\mathbb{R}^d \times V$, z in $\mathbb{R}^d_\star$ such that

$$\begin{cases} f, g_i, c \ \text{ are bounded, measurable,} \\ \text{and continuous in the control variable } \ v, \\ 0 \leq c(x,v,z) \leq C_0(1 \wedge |z|), \quad \forall x, v, z, \end{cases} \tag{1.26}$$

where C_0 is a constant and V is a compact metric space.

We consider the Markov-Feller process $(P_x, X(t), t \geq 0)$ defined on the canonical space $D([0,\infty[, \mathbb{R}^d)$ described above, corresponding to the integro-differential operator $L_0 + I_0$ in the whole space.

An admissible control is a stochastic process $(v(t), t \geq 0)$ with values in V, adapted to the filtration F_t. For any admissible control $(v(t), t \geq 0)$ we can use the Girsanov transformation (1.18),(1.24) to define an exponential martingale $e(t) = e_v(t)$ and a new probability measure denoted by $P_x = P_x^v$ such that $(P_x^v, X(t), t \geq 0)$ represents the state of the system. Notice that in this case, $e_v(t)$ is given by (1.20) with

$$\left\{ \begin{array}{lcl} r_X(t) & = & a^{-1/2}(X(t))g(X(t), v(t)), \\ \gamma_X(t,z) & = & z\, c(X(t), v(t), z). \end{array} \right. \tag{1.27}$$

A cost is associated with the controlled system $(P_x^v, X(t), t \geq 0)$ by

$$J_x(v) = \lim_{T\to\infty} E_x^v\{\frac{1}{T}\int_0^T f(X(t), v(t))dt\}. \tag{1.28}$$

Our purpose is to give a characterization of the optimal cost

$$\lambda = \inf\{J_x(v) : v(\cdot)\} \tag{1.29}$$

and to construct an optimal control $\hat{v}(t)$.

It is useful to remark that we expect to obtain an optimal Markovian control, i.e.

$$\hat{v}(t) = \hat{v}(X(t)), \quad \forall t \geq 0, \tag{1.30}$$

for some feedback function $\hat{v}(x)$ and to prove that the optimal cost λ is constant, i.e., independent of the initial condition $X(0) = x$.

For a given feedback control $v = v(x)$, the controlled state of the system $(P_x^v, X(t), t \geq 0)$ is a Markov-Feller process with infinitesimal generator of the form $L + I$, as in (1.21) and (1.22), with

$$\left\{ \begin{array}{lcl} g(x) & = & g(x, v(x)), \\ c(x,z) & = & c(x, v(x), z). \end{array} \right. \tag{1.31}$$

2. Dirichlet Problem

Denote by $v(x)$ any measurable feedback. Let $L = L_v$ and $I = I_v$ be the second order differential operator (1.21) and the integro-differential operator (1.22) as before. Also we set $f(x) = f(x, v(x))$ given by (1.26). For a given bounded and smooth domain $\mathcal{O}$, we consider first the interior Dirichlet problem

$$\left\{ \begin{array}{rcll} -(L+I)u & = & f & \text{in } \mathcal{O}, \\ u & = & h & \text{in } \mathbb{R}^d \setminus \mathcal{O}, \end{array} \right. \tag{2.1}$$

and next the exterior Dirichlet problem

$$\left\{ \begin{array}{rcll} -(L+I)u & = & f & \text{in } \mathbb{R}^d \setminus \overline{\mathcal{O}}, \\ u & = & h & \text{in } \overline{\mathcal{O}}, \end{array} \right. \tag{2.2}$$

where f and h are given measurable and bounded functions.

Notice the fact that there is not zero-order coefficient and the non-local character of the integro-differential operator I. So that for the interior problem [exterior problem] we need the solution u to be defined in a neighborhood of the closure $\overline{\mathcal{O}}$ [$\mathbb{R}^d \setminus \mathcal{O}$, respectively]. Thus, we seek the solution as defined in the whole space $\mathbb{R}^d$.

A natural way to handle the non-homogeneous boundary conditions is to pursue the following two-steps. First we suitably extend the boundary (or exterior) data h to the whole space, for instance if h is defined in $\mathbb{R}^d \setminus \mathcal{O}$ then we extend h to the whole $\mathbb{R}^d$ preserving its regularity properties. Next, we solve an homogeneous problem (interior Dirichlet problem with $h = 0$) for $u - h$, where we use the zero-extension to define the non-local operator I. With this in mind, we can re-consider the interior Dirichlet problem [or the exterior Dirichlet problem] as

$$\begin{cases} -(L+I)u = f \text{ in } \mathcal{O}, \\ u = h \text{ in } \partial\mathcal{O}. \end{cases} \tag{2.3}$$

Notice that if we modify the function h outside of $\overline{\mathcal{O}}$ then the value Iu may change (since it uses the values of u outside of $\overline{\mathcal{O}}$). Then, we understand the solution u of the above equation as $u = v + w$ where v solves a non-homogeneous Dirichlet boundary conditions second-order differential equation

$$\begin{cases} -Lv = 0 \text{ in } \mathcal{O}, \\ v = h \text{ in } \partial\mathcal{O}, \end{cases} \tag{2.4}$$

and w solves an homogeneous (interior) Dirichlet problem

$$\begin{cases} -(L+I)w = f + Iv \text{ in } \mathcal{O}, \\ w = 0 \text{ in } \mathbb{R}^d \setminus \mathcal{O}, \end{cases} \tag{2.5}$$

for the whole integro-differential operator $L + I$. Sufficient conditions to solve the PDE (2.4) are well known (cf. Gilbarg and Trudinger [10], Ladyzhenskaya and Uraltseva [14]) so we will state results concerning the existence, uniqueness and regularity for the solutions of the homogeneous interior Dirichlet problem (i.e., with $h = 0$) with an integro-differential operator of the above form. Actually, we will need a weak formulation of this problem. An excellent treatment of the interior Dirichlet problem for this class of integro-differential operators can be found in Gimbert and Lions [11] and, based on Garroni and Menaldi [8], some natural extensions to the exterior Dirichlet problem are detailed in [19].

Notice that if u denotes the solution of the non-homogeneous interior Dirichlet problem (2.1) with $f(x) = f(x, v(x))$ then we expect to have the following stochastic representation

$$u(x) = E_x^v\{\int_0^\tau f(X(t), v(X(t)))dt + h(X(\tau))\}, \tag{2.6}$$

where $\tau = \tau_v$ is the first exit time of the process $X(t)$ from the closed set $\overline{\mathcal{O}}$, i.e.

$$\tau = \inf\{t \geq 0 \,:\, X(t) \notin \overline{\mathcal{O}}\}, \tag{2.7}$$

$E_x^v\{\cdot\}$ is the mathematical expectation with respect to the measure P_x^v, and $(P_x^v, X(t), t \geq 0)$ is the diffusion with jumps corresponding to $L_v + I_v$. Sometimes it is convenient to call a *probabilistic solution* of the interior (exterior) Dirichlet problem a measurable and bounded (locally bounded, for the exterior) function u satisfying:

$$\begin{cases} u(X(t))\mathbf{1}_{(t<\tau)} + \displaystyle\int_0^{\tau\wedge t} f(X(s))ds + h(X(\tau))\mathbf{1}_{(t\geq\tau)} \\ \qquad \text{is a } F_t - \text{(local) martingale,} \end{cases} \tag{2.8}$$

with the exit time $\tau = \tau(\overline{\mathcal{O}})$ for the interior problem and $\tau = \tau(\mathbb{R}^d \setminus \mathcal{O})$, where

$$\tau(D) = \inf\{t \geq 0 : X(t) \notin D\}, \tag{2.9}$$

where $\tau = \infty$ if $X(t) \in D$, $\forall t \geq 0$. Certainly, for a probability solution, the above stochastic representation is valid.

Let us turn our attention to the variational formulation of the (homogeneous) interior Dirichlet problem (2.1), i.e. a solution in $W_0^{1,p}(\mathcal{O})$. The key point here is to establish that $L + I$ preserves Sobolev spaces, i.e. it maps $W_0^{1,p}(\mathcal{O})$ into $W^{-1,p}(\mathcal{O})$, and that a weak version of the maximum principle holds. We need to assume that $j(x, \zeta)$ has a bounded second derivative in x, i.e. there exist $\delta > 0$ such that for some constant $C > 0$, and with $F_\delta = \{\zeta \in F \ : \ j_0(\zeta) < \delta\}$ we have

$$\|\nabla_x^2 j(\cdot, \zeta)\|_{L^\infty(\mathbb{R}^d)} \leq C, \quad \forall \zeta \in F_\delta \tag{2.10}$$

as well as assumption (1.10) to make sense of $I_0\varphi$ for a test function φ not necessary smooth. Next, a version of the Maximum Principle is necessary to show that the bilinear form

$$a(\varphi, \psi) = -\langle L\varphi, \psi\rangle - \langle I\varphi, \psi\rangle \tag{2.11}$$

is continuous and coercive in $H_0^1(\mathcal{O})$. We state the main results in this direction.

Theorem 2.1 (Interior) *Let the assumptions* (1.4), (1.5), (1.8), (1.9), (1.10), (1.26) *and* (2.10) *hold. Then there exists a unique probability solution u of the (homogeneous, i.e. $h = 0$) interior Dirichlet problem* (2.1) *in $W_0^{1,p}(\mathcal{O})$, for any $1 < p < \infty$. Moreover, u belongs to $W_{loc}^{2,p}(\mathcal{O})$ and for some constant C (independent of the feedback) we have*

$$\|u\|_{L^\infty} \leq C\|f\|_{L^\infty} . \quad \square \tag{2.12}$$

The above results for $j(x, \zeta) = j(\zeta)$, independent of x, have been proved in Bensoussan and Lions [3] and extended later to W_0^p in Gimbert and Lions [11]. We refer to [19] for details. The general case with $h \neq 0$, can be treated by means of the PDE problem (2.4).

The exterior Dirichlet problem presents some extra difficulties and it is not easily found in the literature. Notice that in our setting, the first order coefficient $b(x)$ has a linear growth, so that standard arguments do not apply and the meaning of the boundary conditions becomes an issue. Here we adopt the probability solution sense and the variational formulation with a weight (Liapunov's type) function of the form

$$\psi_q(x) = (2 + |x|^2)^{q/2}, \quad q > 0. \tag{2.13}$$

Under the assumption (1.6), for any $q > 0$ there exist α_q, $c_q > 0$ and a ball B_q such that

$$\begin{cases} L\psi_q(x) + I\psi_q(x) \leq -\alpha_q \psi_q(x), & \forall x \in \mathbb{R}^d \setminus B_q, \\ |L\psi_q(x)| + |I\psi_q(x)| \leq c_q \psi_q(x), & \forall x \in \mathbb{R}^d \setminus B_q, \end{cases} \tag{2.14}$$

provided we suppose that the function $j_0(\zeta)$ of (1.8) satisfies

$$\int_F [j_0(\zeta)]^q (1 + j_0(\zeta))^{-1} \pi(d\zeta) < \infty, \tag{2.15}$$

which is always true if $q \leq 1$.

Theorem 2.2 (Exterior) *Let the assumptions* (1.4), ..., (1.10), (1.26) *and* (2.10) *hold. Suppose* $\mathcal{O}$ *a smooth bounded domain containing the ball* B_q *given by* (2.14). *Then there exists one and only one probability solution* u *of the (homogeneous, i.e.* $h = 0$*) exterior Dirichlet problem* (2.2) *in* $W^{1,p}_{loc}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, *for any* $1 < p < \infty$. *Moreover,* u *belongs to* $W^{2,p}_{loc}(\mathbb{R}^d \setminus \mathcal{O})$ *and*

$$\|u\psi_{-q}\|_{L^\infty} \le \frac{1}{\alpha_q}\|f\psi_{-q}\|_{L^\infty}, \tag{2.16}$$

where α_q *is given by* (2.14). $\square$

3. Invariant Measure

As in the previous section, we assume given a Borel measurable feedback control and we denote by $L+I$ the corresponding integro-differential operator. Now, let $\mathcal{O}$ be a sufficiently large smooth and bounded domain (e.g. a ball) so that the following non-homogeneous exterior Dirichlet problem

$$\begin{cases} (L+I)u = 0 & \text{in } \mathbb{R}^d \setminus \overline{\mathcal{O}}, \\ u = \varphi & \text{in } \overline{\mathcal{O}}, \end{cases} \tag{3.1}$$

can be solved in $W^{2,p}_{loc}(\mathbb{R}^d \setminus \overline{\mathcal{O}}) \cap W^{1,p}_{loc}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ for non-negative φ in $W^{1,p}(\mathcal{O}) \cap L^\infty(\mathbb{R}^d)$. Now, consider the non-homogeneous interior Dirichlet problem in a larger domain (ball) $B \supset \overline{\mathcal{O}}$,

$$\begin{cases} (L+I)v = 0 & \text{in } B, \\ v = u & \text{in } \mathbb{R}^d \setminus B, \end{cases} \tag{3.2}$$

which can be solved in $W^{2,p}_{loc}(B) \cap W^{1,p}(B) \cap L^\infty(\mathbb{R}^d)$, for any v in $W^{1,p}_{loc}(\mathbb{R}^d \setminus \overline{\mathcal{O}}) \cap L^\infty(\mathbb{R}^d)$. Therefore we can define the linear operator

$$\begin{cases} P : W^{1,p}(\mathcal{O}) \cap L^\infty(\mathbb{R}^d) \to W^{1,p}(\mathcal{O}) \cap L^\infty(\mathbb{R}^d), \\ P\varphi = v, \end{cases} \tag{3.3}$$

where the solution u of (3.2) has been restricted to the domain $\overline{\mathcal{O}}$. Notice that we are using *weak solutions* of problems (3.1) and (3.12). Strong solutions, i.e. u in $W^{2,p}(\mathbb{R}^d \setminus \overline{\mathcal{O}})$ and v in $W^{2,p}(B)$, require some extra assumptions on the integro-differential opearator I, c.f. Gimbert and Lions [11] and [19].

By means of the weak maximum principle, we can prove that

$$\varphi \ge 0 \quad \text{implies} \quad P\varphi \ge 0. \tag{3.4}$$

Since $P\varphi = 1$ for $\varphi = 1$, the operator P can be identified with a (one-step) transition probability measure on $(\overline{\mathcal{O}}, \mathcal{B})$, so that

$$\begin{cases} P : B(\overline{\mathcal{O}}) \to B(\overline{\mathcal{O}}), \\ P\varphi(x) = \displaystyle\int_{\overline{\mathcal{O}}} \varphi(y) P(x, dy), \end{cases} \tag{3.5}$$

where $B(\overline{\mathcal{O}})$ is the space of bounded Borel measurable functions on $\overline{\mathcal{O}}$. Moreover, it can be proved (cf. [19]) that P is an ergodic operator, i.e. defining

$$\lambda(x,y,F) = P\mathbf{1}_F(x) - P\mathbf{1}_F(y), \tag{3.6}$$

for any x, y in X and any Borel subset F of $\overline{\mathcal{O}}$, where $\mathbf{1}_F$ is the characteristic function of the set B, we have

$$\exists\delta > 0 \ / \ \lambda(x,y,F) \leq 1-\delta, \quad \forall x,y \in \overline{\mathcal{O}},\ \forall F \in \mathcal{B}. \tag{3.7}$$

Hence, based on Doob's ergodicity theorem, there exists a unique probability measure on $(\overline{\mathcal{O}}, \mathcal{B})$, denoted by ν, such that

$$|P^n\varphi(x) - \int_{\overline{\mathcal{O}}} \varphi d\nu| \leq Ke^{-\rho n}\|\varphi\|, \qquad \forall n = 1,2,\ldots \tag{3.8}$$

where $\rho = -\ln(1-\delta)$, $K = 2/(1-\delta)$. The measure ν is the unique invariant probability for $(P, \overline{\mathcal{O}}, \mathcal{B})$, i.e. the unique probability on $\overline{\mathcal{O}}$ such that

$$\int_{\overline{\mathcal{O}}} \varphi d\nu = \int_{\overline{\mathcal{O}}} P\varphi d\nu, \quad \forall\varphi \in B(\overline{\mathcal{O}}). \tag{3.9}$$

At this point we consider the interior and exterior Dirichlet problems

$$\begin{cases} -(L+I)u_0 &= f \quad \text{in } \mathbb{R}^d \setminus \overline{\mathcal{O}}, \\ u_0 &= 0 \quad \text{in } \overline{\mathcal{O}}, \end{cases} \tag{3.10}$$

and

$$\begin{cases} -(L+I)v &= f \quad \text{in } B, \\ v &= u_0 \quad \text{in } \mathbb{R}^d \setminus B. \end{cases} \tag{3.11}$$

Based on the results of the previous section, we can define the operator

$$T : L^\infty_q(\mathbb{R}^d) \longrightarrow W^{1,p}_{loc}(\mathbb{R}^d) \cap L^\infty_q(\mathbb{R}^d), \qquad Tf = v_0 \tag{3.12}$$

with possesses the property (3.4). Thus we can define a positive measure $\tilde{\mu}$ on $\mathbb{R}^d$ (unnormalized) by

$$\int_{\mathbb{R}^d} f(x) d\tilde{\mu}(dx) = \int_{\mathcal{O}} Tf(x)\nu(dx) \tag{3.13}$$

Next define the probability measure μ by

$$\mu(F) = \frac{\tilde{\mu}(F)}{\tilde{\mu}(\mathbb{R}^d)}, \quad \forall F \in \mathcal{B}(\mathbb{R}^d). \tag{3.14}$$

Theorem 3.1 (Invariant Measure) *Let the assumptions* (1.4)—(1.10), (1.26) *and* (2.10) *hold. Then* μ*, given by* (3.14)*, is an invariant probability measure for the diffusion with jumps in* $\mathbb{R}^d$*, i.e. for any bounded and Borel measurable function* f *we have*

$$\int_{\mathbb{R}^d} E_x\{f(X(t))\}\mu(dx) = \int_{\mathbb{R}^d} f(x)\mu(dx). \tag{3.15}$$

Moreover, the invariant probability measure μ is unique and

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T E_x\{f(X(t))\}dt = \int_{\mathbb{R}^d} f(x)\mu(dx), \tag{3.16}$$

for any bounded and Borel measurable function f. Furthermore, the measure μ is absolutely continuous w.r.t. the Lebesgue measure, i.e. we can write

$$\int_{\mathbb{R}^d} f(x)\mu(dx) = \int_{\mathbb{R}^d} f(x)m(x)dx, \tag{3.17}$$

where the invariant density $m(x)$ satisfies

$$m \geq 0, \qquad \int_{\mathbb{R}^d} m(x)dx = 1. \quad \square \tag{3.18}$$

Now we can discuss the ergodic linear equation. Consider the space

$$L_q^\infty(\mathbb{R}^d) = \{\varphi \ : \ \varphi\psi_{-q} \in L^\infty(\mathbb{R}^d)\}, \tag{3.19}$$

for $q > 0$ and $\psi_{-q}(x) = (2 + |x|^2)^{-q/2}$. The linear equation is then

$$\begin{cases} u \in W_{loc}^{2,p}(\mathbb{R}^d) \cap L_q^\infty(\mathbb{R}^d), \quad p \geq d, \ q > 0, \\ -(L+I)u = f \ \text{ a.e. in } \ \mathbb{R}^d. \end{cases} \tag{3.20}$$

Theorem 3.2 (Linear Equation) *Let assumptions* (1.4)—(1.10), (1.26) *and* (2.10) *hold. Then the linear integro-differential equation* (3.20) *has a solution u (unique up to an additive constant) if and only f has a zero-mean, i.e.*

$$\mu(f) \doteq \int_{\mathbb{R}^d} f(x)\mu(dx) = 0, \tag{3.21}$$

where $\mu(dx)$ is the unique invariant probability measure defined by (3.14)*. Moreover, under the above zero-mean conditon, there exists a solution of* (3.20) *for which we have the a priori estimate*

$$\|u\psi_{-q}\|_{L^\infty(\mathbb{R}^d)} \leq C_q \|f\psi_{-q}\|_{L^\infty(\mathbb{R}^d)} \tag{3.22}$$

for some positive constant C_q depending only on q, d and the bounds imposed by the assumptions on the coefficients of the operators L and I. $\square$

Remark 3.3 *Notice that the discounted linear equation*

$$\begin{cases} u \in W_{loc}^{2,p}(\mathbb{R}^d) \cap L_q^\infty(\mathbb{R}^d), \quad p \geq d, \ q > 0, \\ -(L+I)u_\alpha + \alpha u_\alpha = f \ \text{ a.e. in } \ \mathbb{R}^d. \end{cases} \tag{3.23}$$

with $\alpha > 0$ has a unique solution under the assumption of the previous theorem. Moreover, based the estimate (3.22)*, one can prove that ∇u_α and $\nabla^2 u_\alpha$ remain uniformly bounded in $L_{loc}^p(\mathbb{R}^d)$ as $\alpha \to 0$, for any finite p.* $\square$

4. Programming Equation

Let v be a given Borel measurable function from $\mathbb{R}^d$ into V, referred to as a measurable feedback. The dynamic of the system follows the stochastic integro-differential equation

$$\left\{ \begin{array}{rcl} dX(t) & = & a^{1/2}(X(t))dw(t) + [b(X(t)) + g(X(t), v(X(t)))]dt + \\ & & + \int_{\mathbb{R}^d_\star} z\mu(dt, dz), \end{array} \right. \tag{4.1}$$

on the canonical probability space $\Omega = D([0, \infty), \mathbb{R}^d)$, with the probability $P = P^v_x$ satisfying

$$P\{X(0) = x\} = 1, \tag{4.2}$$

and where $w(t) = w_v(t)$ is a standard Wiener process in $\mathbb{R}^d$ and $\mu(dt, dx) = \mu_v(dt, dx)$ is an integer random (martingale) measure associated with a Poisson measure with characteristic Levy kernels

$$M_0(x, dz),\ c(x, v(x), z)M_0(x, dz). \tag{4.3}$$

Then (Ω, P^x, X^v) defines a Markov-Feller process on $\mathbb{R}^d$ (so-called diffusion with jumps) with infinitesimal generator A_v, which is an extension of the integro-differential operator $L_0 + I_0 + \mathcal{L}_v$, cf. (1.3), (1.11), and where $\mathcal{L}_v$ is given by

$$\left\{ \begin{array}{rcl} \mathcal{L}_v\varphi(x) & = & \int_F [\varphi(x + j(x, \zeta)) - \varphi(x)]c(x, v(x), j(x, \zeta)) \times \\ & & \times m_0(x, \zeta)\pi(d\zeta) + \sum_{i=1}^{d} g_i(x, v(x))\partial_i\varphi(x) \end{array} \right. \tag{4.4}$$

At this point, we can re-formulate our optimal ergodic problem as in the introduction i.e., for a given measurable feedback control $v(x)$ there exists a unique invariant probability measure $\mu_v(dx)$ in $\mathbb{R}^d$ of the Markov-Feller process (Ω, P^x, X^v) as above. The long run average cost associated with the controlled system is given by

$$J(v) = \int_{\mathbb{R}^d} f(x, v(x))\mu_v(dx). \tag{4.5}$$

Recall our assumptions on the data (1.4)—(1.10), (1.26) and (2.10). Our purpose is to give a characterization of the optimal cost

$$\lambda = \inf\{J(v) : v(\cdot)\} \tag{4.6}$$

and to construct an optimal feedback control $\hat{v}$.

Denote by $H(x, \varphi(x))$ the Hamiltonian

$$H(x, \varphi(x)) = \inf\{\mathcal{L}_v\varphi(x) + f(x, v) : v \in V\}, \tag{4.7}$$

where the operator $\mathcal{L}_v$ is given by (4.4), f satisfies (1.26) and φ belongs to the Sobolev space $W^{1,p}_{loc}(\mathbb{R}^d) \cap L^\infty_q(\mathbb{R}^d)$, cf. (2.13) and (3.17). It can be proved that $H(x, \varphi(x)$ belongs to $L^\infty_q(\mathbb{R}^d)$ for every φ in $W^{1,p}_{loc}(\mathbb{R}^d) \cap L^\infty_q(\mathbb{R}^d)$.

As mentioned in Remark 3.3, we can adapt the techniques used in Theorem 3.2 (cf. [19]) to show that the discounted nonlinear equation

$$\begin{cases} u \in W^{2,p}_{loc}(\mathbb{R}^d) \cap L^\infty_q(\mathbb{R}^d), \quad p \geq d, \ q > 0, \\ (L_0 + I_0)u_\alpha + H(\cdot, u_\alpha) = \alpha u_\alpha \ \text{ a.e. in } \ \mathbb{R}^d. \end{cases} \tag{4.8}$$

with $\alpha > 0$ possesses a unique solution and that ∇u_α and $\nabla^2 u_\alpha$ remain uniformly bounded in $L^p_{loc}(\mathbb{R}^d)$ as $\alpha \to 0$, for any finite p.

The limiting Hamilton-Jacobi-Bellman equation can be expressed as

$$\begin{cases} u \in W^{2,p}_{loc}(\mathbb{R}^d) \cap L^\infty_q(\mathbb{R}^d), \quad p \geq d, \ q > 0, \\ (L_0 + I_0)u + H(\cdot, u) = \lambda \ \text{ a.e. in } \ \mathbb{R}^d. \end{cases} \tag{4.9}$$

where the unknowns are the function u and the constant λ.

Essentially based on results for the linear equation, as in Bensoussan [1], we select a discounted optimal feedback, i.e.

$$v_\alpha(x) \in \text{Argmin } H(x, u_\alpha(x)), \quad \forall x \in \mathbb{R}^d, \tag{4.10}$$

where u_α is the solution of the nonlinear equation (4.8).

Theorem 4.1 (HJB equation) *Let the assumptions* (1.4)—(1.10), (1.26) *and* (2.10) *hold. Define*

$$\tilde{u}_\alpha = u_\alpha - \mu_\alpha(u_\alpha), \quad \alpha > 0, \tag{4.11}$$

where μ_α is the invariant probability measure corresponding to the integro-differential operator $L_0 + I_0 + \mathcal{L}_v$, with $v = v_\alpha$ as given by (4.11). *Then there exist a constant λ and a function u in $W^{2,p}_{loc}(\mathbb{R}^d) \cap L^\infty_q(\mathbb{R}^d)$ such that*

$$\alpha u_\alpha \to \lambda, \qquad \tilde{u}_\alpha \rightharpoonup u \ \text{ weakly*} \tag{4.12}$$

as α goes to zero. The couple (λ, u) is a solution of the HJB equation (4.9). *Moreover, λ is equal to the optimal cost* (4.6) *and any stationary feedback $\hat{v}$ satisfying*

$$\hat{v} \in \text{Argmin } H(x, u(x)), \quad \forall x \in \mathbb{R}^d, \tag{4.13}$$

produces the optimal cost, i.e. $\lambda = J(\hat{v})$. □

Notice that if the couple (λ, u) is a solution of the HJB equation (4.9) then λ is the optimal cost (4.6), and if v is a stationary optimal feedback (i.e. (4.13) holds for v), then u solves the linear equation

$$-(L_0 + I_0 + \mathcal{L}_v)u + \lambda = f_v \ \text{ a.e. in } \ \mathbb{R}^d, \tag{4.14}$$

where $f_v(x) = f(x, v(x))$. By means of the Itô's formula we get for every $x \in \mathbb{R}^d$, $T > 0$

$$u(x) = E^v_x\{\int_0^T [f(X(t), v(X(t))) - \lambda]dt\} + E^v_x\{u(X(T))\}. \tag{4.15}$$

Now, if the feedback and its invariant measure are such that

$$E_x^v\{u(X(T))\} \to \mu_v(u) \quad \text{as} \quad T \to \infty \tag{4.16}$$

then we deduce for every $x \in \mathbb{R}^d$, $T > 0$,

$$u(x) = E_x^v\{\int_0^\infty [f(X(t), v(X(t))) - \lambda]dt\} + \mu_v(u). \tag{4.17}$$

This is a representation of u, and gives uniqueness (up to a constant) for the potential function u. Actually, we conjecture that the transition density is strictly positive (for any stationary feedback), and therefore the strong mixing property holds, which in turn implies the convergence (4.16). To the best of our knowledge this has not been proved so far.

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