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Relaxation and linear programs on a hybrid control model

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Abstract

Some optimality results on hybrid control problems are presented in this paper. The hybrid model under study consists of two sub-dynamics, one of a standard-type governed by an ordinary differential equation, and the other one of a special-type having a discrete evolution. We focus on the case when the interaction between the sub-dynamics takes place only when the state of the system reaches a given and fixed region of the state space. The controller is able to apply *two* controls, each of them is applied to each of the two sub-dynamics, whereas the state follows a composed evolution, of continuous-type and discrete-type. By means of the relaxation technique, we provide the existence of a pair of controls that minimizes an incurred (discounted) cost. We conclude the analysis by introducing an auxiliary infinite dimensional linear program to show the equivalence between the initial control problem and its associated relaxed counterpart.

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1 Introduction

Hybrid control systems can be considered as a subclass of controlled dynamical systems with the key property of that its associated dynamic may undergo structural modifications from-time-to-time, exerted by the own controller or by means of the location of the state of the system. The hybrid control system we are interested in is composed by two sub-dynamics: one of a standard type that runs under almost all situations, and another one of a special type that is activated under extreme circumstances. Any change of sub-dynamic may produce a structural modification in the system and, at the same time, an opportunity for an instantaneous (and sizeable) change in the state of the system. Naturally, at any given time, only one of the two sub-dynamics (standard/special) must be active.

The dynamic is a key feature in hybrid control models, however, the form of the state and control is also important: The state of the system does not only describe a "usual" description of the phenomenon, but also has a record keeping mechanism. Specifically, the state is represented as a pair, where the first entrance describes the standard evolution of the system (continuous-type variable) and the second one acts as a variable that records the structural changes (discrete-type variable). As for the control variable, the controller is able to apply two controls: one control acting only on the standard sub-dynamic and another control acting only on the special sub-dynamic.

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The interaction between the sub-dynamics is only possible when the state variable reaches a specific region of the state space. In this situation, the hybrid control model is said to satisfy the *automaton* property due that those changes between the sub-dynamics are triggered in an *automatic* way.

Under the previous components, the aim for the controller is to find a control policy, regarded as a pair (each component of this pair corresponds to a control policy of each of the two sub-dynamics), with the purpose of minimizing an infinite-horizon discounted cost criterion introduced later on. To accomplish this goal, we base ourselves on the relaxation technique, which is a useful and well-known tool to provide optimal control policies. We shall provide optimality results under this framework by applying two different type of hypotheses. In addition, an auxiliary infinite dimensional linear program is analyzed in order to show the equivalence between the control problem under study with its associated relaxed counterpart.

Related literature on hybrid control systems is vast and it covers both theoretical and practical results. To mention a handful of related works with a more theoretical point of view, we can quote Azhmyakov et al. [3], Barles et al. [4], Bensoussan and Menaldi [6], Branicky et al. [7], Dharmatti and Ramaswamy [8], Lygeros [12], Riedinger et. al. [14], Shaikh and Caines [15], Zhang & James [18], among others. All these references are based on the dynamic programming approach (through the analysis of some quasi-variational inequalities) or by means of the well-known maximum principle. From the point of view of applications, specifically in robotics, aircraft planning, and automata, we can mention, for instance, Posa et al. [13], Soler et al. [16], Tavernini [17], although there are many others in the literature.

This paper is, in some sense, a continuation of Bensoussan and Menaldi [6]. Indeed, this later reference provided conditions ensuring the existence of the optimal value $u(x, n)$, regarded as a continuous viscosity solution of certain quasi-variational inequalities (QVI). However, the existence of optimal controls were not studied in that paper; in fact, the study of such existence becomes delicate because there is not enough regularity to provide optimal control policies obtained as a straightforward consequence of the QVI. However, from the use of relaxation methods either on the control variable or on the state-control variables, it is possible to overcome the difficulty of finding optimal controls in this hybrid environment.

To the best of our knowledge there is not existing literature that have used the same methodology applied in this work. Let us mention that the pre-print Zhao et al. [19] is somehow related to ours. Indeed, Zhao et al. [19] analyzes the optimality of a *finite-horizon* cost through the use of linear programms and occupation measures. Optimal policies are obtained under convexity and affine properties on the cost function. In our paper, however, we tackle an optimal control problem of an *infinite-horizon discounted* cost criterion and the techniques used in here are in two directions: (1) we study a relaxed control approach when certain regularity in some parts of our model is satisfied, and (2) the use of occupation measures and linear programs when such regularity is not known in addvance. In both cases, the techniques used to find optimal control policies differ considerably to the arguments provided by Zhao et al. [19].

Our paper is divided in four sections. Indeed, section 2 presents the details of our dynamical system and introduces some elements of it, such as state, action, and interface spaces, the dynamic of the system, types of control policies, the payoff to be optimized, as well as our main assumptions. Section 3 provides optimality results of the control model under the so-called transversality assumption. To this end, the control is regarded as a "distribution" of controls, i.e., the concept of relaxed control and its correspondent optimality criterion are used. A continuity-type of the trajectories of the system with respect to the relaxed controls is necessary here, which is the key to find optimal results. In section 4, the same problem as in Section 3 is studied, but without assuming the transversality condition; this forces us to regard the state as a "distribution" of states. In this scenario, the control problem is rewritten as an infinite-dimensional linear problem, under which the control policies are replaced by measures with some characteristics. An important feature of the space where these measures live is its relative compactness property. Then, standard results on continuity-compactness are applied to show the existence of an optimal measure that optimizes our performance criterion. Section 5 is devoted to showing the equivalence between the control problem under study and the control problem associated to the relaxed policies. For this purpose, an auxiliary infinite dimensional linear program and its corresponding dual are discussed. By studying the restrictions of the dual problem, it is possible to deduce such equivalence under an additional hypothesis on the costs. Finally, Section 6 provides a discussion on the the transversality condition, which allows to give a detailed proof of Proposition 3.2.

A warning: For easy of notation, throughout this manuscript, we shall be using the notation (x, n) to represent the initial condition of the state of the system $(x(\cdot), n(\cdot))$ in (2.1) below, but sometimes it will be denoted as (x_0, n_0) or $(x(0), n(0))$ when the context is required.

2 Model definition

The controlled dynamic system we are interested in is composed: by the state space $S = \mathbb{R}^d \times \mathcal{N}$, with $\mathcal{N} \subset \mathbb{R}^l$, by the control spaces $V \subset \mathbb{R}^p$ and $K \subset \mathbb{R}^q$ and by the interface set $D \subset S$. We have two sub-dynamics, one of a standard type governed by an ordinary differential equation (ODE) and the other sub-dynamic with an (instantaneous) impulsive or transitional character. The state of the system is denoted by the pair $(x, n) \in S$, where x and n represent the continuous-type and discrete-type states, respectively.

Changes on the state (x, n) are conducted over time through the two sub-dynamics (standard and special) as well as interventions of the controller carried out by the selection over time of two control parameters $v \in V$ and $k \in K$ acting on the standard and special sub-dynamics respectively.

The activation of each sub-dynamic is decided automatically depending on the location of the variable state. To be more specific, when the state (x, n) belongs to $S \setminus D$, the standard dynamic is turned on and it is affected by the control v. When the state (x, n) touches D, then the special sub-dynamic takes place whose control variable is now k. Certainly, one and only one of the two sub-dynamics must be active along the time.

Formally speaking, the aforementioned dynamic is represented as follows:

$$\begin{aligned} &\underbrace{\big(\dot{x}(t), \dot{n}(t)\big) = \big(g(x(t), n(t_i), v(t)), 0\big)}_{\text{continuous sub-dynamic}}, \text{ for } \quad t \in [t_i, t_{i+1}[, \\ &\underbrace{(x(t_i), n(t_i)) = \big(X(x(t_i-), n(t_i-), k_i), N(x(t_i-), n(t_i-), k_i)\big)}_{\text{discrete sub-dynamic}}, \quad i = 0, 1, \cdots \\ &t_{i+1} := \inf\big\{t \geq t_i : (x(t-), n(t_i)) \in D\big\}, \quad \text{when } t_i < \infty. \end{aligned} \tag{2.1}$$

with initial condition $t_0 = 0$, $(x(0), n(0)) = (x_0, n_0) \in S$, and either:

$$\begin{cases} \big(X(x(t_0-), n(t_0-), k_0), N(x(t_0-), n(t_0-), k_0)\big) := \big(X(x_0, n_0, k_1), N(x_0, n_0, k_1)\big) \\ \text{when } (x_0, n_0) \in D, \text{ or} \\ \\ \big(X(x(t_0-), n(t_0-), k_0), N(x(t_0-), n(t_0-), k_0)\big) := (x_0, n_0) \\ \text{when } (x_0, n_0) \in S \setminus D. \end{cases} \tag{2.2}$$

In the above dynamic, $t-$ means the left limit of t, and the value $v(t)$ represents the control action used by the controller at time t, which is exerted in the standard sub-dynamic; in contrast, the values $\{k_i : i \geq 1\}$ are the control actions applied by the controller at each time t_i in the special sub-dynamic. For convenience, here, we have used an extra variable k_0. This variable becomes a fictitious element in our model and just makes sense when $(x_0, n_0) \in D$, whose value is actually k_1; in the other situation (i.e., when $(x_0, n_0) \in S \setminus D$) this variable is not used. Besides, the sequence $\{t_i\}$ is referred to as a *time-interface* set and it is obtained by means of the last line in (2.1).

In other words, the standard sub-dynamic evolves as an ordinary differential equation (ODE) in the continuous-type variable x, with drift (or vector field) denoted by the function $g : S \smallsetminus D \times V \mapsto \mathbb{R}^d$, while the discrete-type variable n remains constant. In contrast, the special sub-dynamic is composed by the *transition function* $(X, N) : D \times K \mapsto S$.

• *Remark* 2.1. **(a)** In previous works, (see, for instance, Branicky et al. [7]), the ODE in the hybrid dynamic (2.1) has been considered to change of dimension each time the special sub-dynamic is turned on. In this case, the sequence $\{d_k\}$, whose elements consist of the dimension of $x \in \mathbb{R}^{d_k}$ at the k-th activation of n, could be either bounded or unbounded. In the former situation, we can define $d := \max_k\{d_k\}$ and work with a single dimension; whereas in the later case, we can consider as the dimension of x the space of sequences $\mathbb{R}^\infty_F := \bigcup_{k\geq 1} \mathbb{R}^k$, i.e., sequences of real numbers with only a finite number of non-zero terms. The convergence of elements in $\mathbb{R}^\infty_F$ is given via the inductive topology, i.e, $x^{(n)} \to x$ if and only if (i) all the $x^{(n)}$ and x belong to the same $\mathbb{R}^k$ for some k (sufficiently large) and (ii) $x^{(n)} \to x$ in $\mathbb{R}^k$. Our analysis here is based on the former scenario; that is when the ODE is of a single finite (and likely large) dimension without any changes.

(b) As part of our hypotheses, we have assumed that $S_n := \{x \in \mathbb{R}^d : (x, n) \in S\} = \mathbb{R}^d$, for all $n \in \mathcal{N}$. Otherwise, if $S_n \subset \mathbb{R}^d$, a more detailed analysis would apply. Namely, by defining

$$D_n = \{x \in \mathbb{R}^d : (x, n) \in D\}, \tag{2.3}$$

the case $\partial S_n \subset D_n$ (imposed for instance as the assumption (A2) in Barles et al. [4]), does not generate any inconvenient, because when the state (x, n) reaches the boundary ∂S_n, it reaches also D_n, so a discrete transition is triggered according to the rule in (2.1) (see also(2.6) below). However, the case $\partial S_n \smallsetminus D_n \neq \emptyset$ is more delicate, since there might be situations when the standard sub-dynamic x gets away the set S_n in a *finite* period of time. In this case, some additional conditions must be imposed to the vector field g in order to ensure the state x stays inside or even in the border of S_n. We may then have the following conditions: if S_n is a closed set, then one needs the continuous dynamics cannot leave the set $S_n \smallsetminus D_n$. This condition can be achieved if the boundary ∂S_n is piecewise smooth and the vector field $g(\cdot, n, \cdot)$ is not pointing toward the exterior of S_n on points of the active boundary $\partial S_n \smallsetminus D_n$. Alternatively, if S_n is an open set, then on the active boundary $\partial S_n \smallsetminus D_n$ (if it is non-empty), the vector field $g(\cdot, n, \cdot)$ must be pointing strictly toward the interior of S_n. Other possibilities may be used; for instance, we can stop the system evolution when leaving the region of interest $S_n \smallsetminus D_n$. Note also that the case $\partial S_n \smallsetminus D_n \neq \emptyset$ does not necessarily force the boundary ∂D_n to satisfy a transversality condition (but likely such a condition must be imposed to the region $\partial S_n \smallsetminus D_n$).

(c) The dynamic (2.1) considers, in principle, the case of multiple (instantaneous) transitions triggered by the discrete sub-dynamic at some time t_i. As we will see later, we will impose assumptions to the model to avoid this possibility.

Control policies: We will refer to as an admissible control policy a pair $(v(\cdot), \{k_i\})$ consisting of:

- A continuous-type control that is a Borel measurable V-valued function $v(\cdot)$ on $[0, \infty[$. We denote by $\mathcal{V}$ the set of all continuous-type control policies.
- An impulse-type (or discrete-type) control that consists of a sequence $\{k_i\}$ such that $k_i \in K \subset \mathbb{R}^q$. We denote by $\mathcal{K}$ the set of all impulse-type controls.

Throughout this paper we will assume the following assumptions hold true.

(a) The interface and control sets satisfy

$$\begin{cases} \text{The set interface } D \text{ is closed.} \\ \text{The control spaces } V \text{ and } K \text{ are compact.} \end{cases} \tag{2.4}$$

(b) There exists a positive constant M, such that

$$\begin{cases} g : S \smallsetminus D \times V \to \mathbb{R}^d, \ \text{continuous}, \\ |g(x,n,v)| \leq M, \ \ \forall x,n,v \\ |g(x,n,v) - g(x',n,v)| \leq M|x-x'|, \ \ \forall x,x',n,v \end{cases} \tag{2.5}$$

(c)

The transition function $(X,N) : D \times K \mapsto S$ is uniformly continuous. (2.6)

• *Remark* 2.2. Since K is compact, by a simple use of Tychonoff's theorem, the space of impulse controls $\mathcal{K}$ is compact too with respect to the product topology.

• *Remark* 2.3. It is easy to verify that conditions in (2.4) and (2.5) ensure that, for any admissible policy $v(\cdot)$, the solution $x(t) = x(s) + \int_s^t g(x(r),n,v(r))\, dr$ of the ordinary differential equation (ODE) $\dot{x}(t) = g(x(t),n,v(t))$ exists and it is unique, $\forall\ t \geq s$ and each $n \in \mathcal{N}$ (see, for instance, Fleming and Rishel [9]).

Construction of the controlled paths: In the following lines, we present the construction of the state of the system along time (controlled paths) by means of an *algorithm*. The algorithm provides, in particular, the existence and uniqueness of the evolution of the state $t \mapsto (x(t),n(t))$. Furthermore, it also generates the sequence $\{t_i : i \geq 1\}$ of time-interfaces, the path $t \mapsto (x(t),n(t))$ defined on each interval of times $[t_i,t_{i+1}[$, and the sequence of transitions $\{(x(t_i),n(t_i)) : i \geq 1\}$, whenever $t_i < \infty$. For easy of notation, and when the context is required, we will rewrite $(x(t_i-),n(t_i-))$ by (x_i,n_i) or by (x_{i-},n_{i-}).

Suppose that a pair $(v(\cdot),\{k_i\}) \in \mathcal{V} \times \mathcal{K}$ is given. Then, the algorithm runs as follows:

(0) *Initialization.* Assume (x_0,n_0) is a given initial state at time $t_0 = 0$, and according to whether (x_0,n_0) belongs or not to D, the counter i is set:
If (x_0,n_0) belongs to $S \smallsetminus D$, set $i = 0$, $(x_i,n_i) = (x_0,n_0)$, and go to step (1); *else* if (x_0,n_0) belongs to D, set $i = 1$, $(x_{i-},n_{i-}) = (x_0,n_0)$, and go to step (2).

(1) *Continuous-type.* If (x_i,n_i) belongs to $S \smallsetminus D$, then the standard sub-dynamic is activated at time $t_i < \infty$ and the continuous-type state evolves as $x(t) = x_i + \int_{t_i}^t g(x(s),n_i,v(s))ds$ for any $t_i \leq t < t_{i+1}$ (with t_i as in (2.1)), whereas the discrete-type state remains constant, with value $n(t) = n_i$. Thus, $(x_{i+1-},n_{i+1-}) = (x(t_{i+1}-),n(t_{i+1}-))$ belongs to D, and $x(t)$ belongs to $S \smallsetminus D$, for every t in the period of time $[t_i,t_{i+1}[$. If $t_{i+1} = \infty$, then stop successfully.

(2) *Discrete-type.* If (x_{i-},n_{i-}) belongs to D, then the special sub-dynamic is activated at time $t_i < \infty$ and a new state $(X,N)(x_{i-},n_{i-},k_i) = (x,n)$ is produced. Now, if the state (x,n) belongs to $S \smallsetminus D$ then set $(x_i,n_i) = (x,n)$. Otherwise, i.e., if the state (x,n) belongs to D then set $(x_{i+1-},n_{i+1-}) = (x,n)$ and a discrete-type transition, either $(x_{i+1},n_{i+1}) = (X,N)(x_{i+1-},n_{i+1-},k_{i+1})$ with (x_{i+1},n_{i+1}) in $S \smallsetminus D$ or $(x_{i+1-},n_{i+1-}) = (X,N)(x_{i+1-},n_{i+1-},k_{i+1})$ with (x_{i+1-},n_{i+1-}) in D, is triggered again. This triggering is repeated until eventually the state (x_{i+j},n_{i+j}) belong to $S \smallsetminus D$. In this case, $t_i = t_{i+1} = \cdots = t_{i+j}$, each (x_{i-},n_{i-}), (x_{i+1-},n_{i+1-}), $\ldots$, (x_{i+j-},n_{i+j-}) belongs to D, and (x_{i+j},n_{i+j}) belongs to $S \smallsetminus D$. Note that $j+1$ discrete-type transitions occurred at the same instant of time t_i, and if a finite j as above is not found then this step never ends and this construction fails.

(3) *Iteration.* Now repeat (1) and (2) alternatively, i.e., after (1) go to (2) and after (2) go to (1).

(4) *Ending.* If this construction get trapped in (2), then the hybrid evolution exists only up to the time $t_i < \infty$. Otherwise this iteration may end only after step (1) is completed successfully with $t_{i+1} = \infty$, or it may keep repeating to generate an infinite sequence $t_0 \leq \cdots \leq t_i \leq t_{i+1} \leq \cdots$ of impulse/switching times. In any case, the hybrid trajectory $t \mapsto (x(t),n(t))$ is defined as a cad-lag function on $[t_0,t_i[$, for any $i \geq 0$, with t_i being either $< +\infty$ or $+\infty$.

In the above algorithm, we can remark that (i) each k_i is used only when $t_i < \infty$, $i \geq 1$; thus, the variable i counts the impulse/switching times, (ii) t_0=0 is the initial time, (iii) the first impulse/switching time t_1 may be equal to t_0.

As was pointed out earlier, some conditions on the data are necessary to ensure that (a) the procedure (0),...,(3) does not end with step (2), i.e., to ensure that after a finite number of instantaneous transitions, a state in $S \setminus D$ can be reached; and (b) to allow the evolution runs over time; i.e., the sequence $\{t_i : t \geq 1\}$ diverges to $+\infty$. A sufficient condition that overcomes these drawbacks is the following:

There exist constants c and C, satisfying

$$0 < c \leq \{|\xi - X(x,n,k)| + |\eta - N(x,n,k)|\} \leq C, \quad \forall\ (x,n),(\xi,\eta) \in D,\ k \in K. \tag{2.7}$$

The following result ensures that the trajectories in (2.1) are well defined in the following sense.

Proposition 2.4. *Under assumptions* (2.4)-(2.7)*, for any pair of controls* $(v(\cdot),\{k_i\}) \in \mathcal{V}\times\mathcal{K}$*, the trajectory* $t \mapsto (x(t),n(t))$ *obtained from the hybrid algorithm, exists, is unique, and it does not allow simultaneous jumps; i.e., there exists a constant* $\mathtt{h} := (M+1)^{-1}\log[1+c(M+1)/M] > 0$ *such that the sequence of impulse times* $\{t_i : i \geq 1\}$ *satisfies* $t_{i+1} \geq t_i + \mathtt{h}$*, for all* $i = 0,1,\cdots$*. As a consequence,* $t_i \to \infty$ *as* $i \to \infty$*, and the trajectories are continuous from the right with left limits on* $[0,\infty[$*.*

Proof. The existence and uniqueness follows by the definition of each transition $(X,N)(x(t_i-),n(t_i-),k_i)$ and from the existence and uniqueness of each trajectory $x(t)$ on $[t_i,t_{i+1}[$ both constructed via the hybrid algorithm. Namely, when the hybrid algorithm is at step (1), the continuous-type variable x evolves as an ODE with usual assumptions to guarantee existence and uniqueness (see Remark 2.3), whereas the discrete-type variable n is a constant. On the other hand, when the hybrid algorithm is at stage (2), the transition function (X,N) takes place, which of course it produces one and only one value from some (x_{i-},n_{i-}) into (x,n). Therefore, the existence and uniqueness of the whole evolution $t \mapsto (x(t),n(t))$ follows by linking together the trajectory in accordance with the hybrid algorithm steps. The last part of this theorem, follows from Bensoussan and Menaldi [6, Theorem 2.1]. □

Definition 2.5. Let $\mathcal{X}$ be a subset of $\mathbb{R}^j$, with $j \geq d+l+p+q$ (recall the dimension of the state-action spaces).

(a) We denote by $B_b(\mathcal{X})$ the space of all Borel measurable and bounded real-valued functions on $\mathcal{X}$, endowed with the supremum norm $\|\cdot\|$.

(b) The spaces $C_b(\mathcal{X})$ and $C_b^{\mathtt{u}}(\mathcal{X})$ will denote two subspaces of $B_b(\mathcal{X})$ consisting of all continuous and all uniform continuous functions, respectively.

(c) Consider the special case $\mathcal{X} \equiv S$. We denote by $C_b^{1,0}(S)$ the set of all real-valued functions defined on S, satisfying

$$C_b^{1,0}(S) := \{\varphi \in C_b(S),\ \partial_{x_i}\varphi \in C_b(S),\ i = 1,\cdots,d\}. \tag{2.8}$$

with norm defined by

$$\|\varphi\|_1 := \|\varphi\| + \sum_{i=1}^{d} \|\partial_{x_i}\varphi\|.$$

Note that in (2.8), the derivative of φ is only applied to the variable x but not to n.

For every pair $(v(\cdot), \{k_i\}) \in \mathcal{V} \times \mathcal{K}$, the dynamic (2.1) can be characterized as the following integration by parts formula (e.g., similar to Bensoussan and Lions [5, p. 87]): for each $\varphi \in C_b^{1,0}(S)$ and $t \geq 0$:

$$\begin{aligned} &e^{-\alpha t}\varphi(x(t), n(t)) - \varphi(x,n) = \\ &\int_{t_0}^{t} e^{-\alpha s}\left[g(x(s), n(s), v(s)) \cdot \nabla_x \varphi(x(s), n(s)) - \alpha\varphi(x(s), n(s))\right]\, ds + \\ &\quad + \sum_{i=0}^{\infty} e^{-\alpha t_i}\left[\varphi\left(X(x(t_i-), n(t_i-), k_i), N((x(t_i-), n(t_i-), k_i)\right) - \varphi\left(x(t_i-), n(t_i-)\right)\right] \mathbf{1}_{\{t_i \leq t\}}, \end{aligned} \tag{2.9}$$

where the time-interface set $\{t_i : i \geq 0\}$ is generated by means of the hybrid algorithm.

Performance index: We introduce the instantaneous and switching cost rates $f : S \smallsetminus D \times V \mapsto \mathbb{R}$ and $\ell : D \times K \mapsto \mathbb{R}$, respectively, satisfying the following conditions:

$$\begin{cases} f \geq 0 \quad \text{and} \quad f \in C_b^{\mathrm{u}}(S \smallsetminus D \times V). \\ \ell \geq 0 \quad \text{and} \quad \ell \in C_b^{\mathrm{u}}(D \times K). \end{cases} \tag{2.10}$$

With the above ingredients, if (x, n) denotes the initial state at time $t_0 = 0$, then, for each pair of controls $(v(\cdot),\{k_i\}) \in \mathcal{V} \times \mathcal{K}$, the total cost incurred by the controller is defined as

$$J\big(x, n; v(\cdot), \{k_i\}\big) = \int_0^{\infty} e^{-\alpha t}\; f(x(t), n(t), v(t))dt + \sum_{i=0}^{\infty} e^{-\alpha t_i}\; \ell(x(t_i-), n(t_i-), k_i), \tag{2.11}$$

where the set of impulse times $\{t_i : i \geq 1\}$ is generated by the set D through the hybrid algorithm as was explained earlier. Note that Proposition 2.4 and assumption (2.10) imply that the total cost (2.11) is *finite* for every pair of controls $(v(\cdot), \{k_i\}) \in \mathcal{V} \times \mathcal{K}$.

The *value function* or *optimal cost* is defined by the function $u(x, n)$ satisfying

$$u(x, n) = \inf_{\big(v(\cdot),\{k_i\}\big) \in \mathcal{V} \times \mathcal{K}} J\big(x, n; v(\cdot), \{k_i\}\big). \tag{2.12}$$

Moreover, if there exists a pair $\big(\hat{v}(\cdot), \{\hat{k}_i\}\big) \in \mathcal{V} \times \mathcal{K}$ satisfying $J\big(x, n; \hat{v}(\cdot), \{\hat{k}_i\}\big) = u(x, n)$, then we will refer to it as an *optimal pair*.

A direct consequence of the above paragraphs is that, in principle, the optimal cost u is an element of $B_b(S)$.

3 Regular case

The existence and characterization of the optimal cost (2.12), has been previously studied in Bensoussan and Menaldi [6], in which, the authors provided conditions ensuring the existence of the optimal value $u(x, n)$, regarded as a continuous viscosity solution of certain quasi-variational inequalities (QVI). However, the existence of optimal controls were not studied in that paper; in fact, the study of such existence becomes delicate because there is not enough regularity to provide optimal control policies obtained as a straightforward consequence of the QVI.

An effective method to find optimal controls is the relaxation technique. In this method, the set $\mathcal{V}$ is embedded into a bigger set $\mathtt{V}$ that has the property to be a compact and convex set (under an appropriate topology). Working in this new set, it is possible to show the existence of an element, say $\mathtt{v} \in \mathtt{V}$ together with a suitable impulse control $\{k_i\}$ such that both controls become optimal for the minimization problem (2.12).

In the rest of this section, we:

- introduce the concept of relaxed controls and show that this set of controls is compact under a suitable topology.
- define a new optimal control problem related to the set of relaxed controls.
- impose a transversality condition to the set-interface D that allows to ensure the continuity (in certain sense) of the trajectories $(x(\cdot), n(\cdot))$, with respect to the control variables $\mathtt{v} \in \mathtt{V}$ and $\{k_i\} \in \mathcal{K}$.
- prove the existence of a pair $\big(\hat{\mathtt{v}}(\cdot), \{\hat{k}_i\}\big) \in \mathtt{V} \times \mathcal{K}$ such that, under this pair, the total cost defined in (3.3) below equals the value function (2.12).

To begin with this analysis, we shall denote by $\mathcal{P}(V)$ the set of all probability measures on V. Let $\mathtt{V}$ be the set of functions $\mathtt{v} : [0, \infty[\mapsto \mathcal{P}(V)$. We can identify every $v \in \mathcal{V}$ as an element in $\mathtt{V}$ by the relation $v(t)$ "is isomorphic to" $\delta_{v(t)}(\cdot)$, where δ_a denotes the Dirac measure at point a. From this last relation, we can interpret $\mathcal{V}$ as a subset of $\mathtt{V}$. The later set is known as the set of *relaxed controls*.

The following proposition ensures an important properties on the set $\mathtt{V}$. (For a proof, see for instance, Gamkrelidze [10, Theorem 8.1]).

Proposition 3.1. *Under the compactness assumption on V given in* (2.4)*, the set of relaxed controls $\mathtt{V}$ is weakly sequentially compact; that is, for any sequence $\{\mathtt{v}^m(\cdot)\}$ of $\mathtt{V}$, there exists an element $\mathtt{v}(\cdot) \in \mathtt{V}$ and a subsequence of $\{\mathtt{v}^m(\cdot)\}$ (to be denoted again as $\{\mathtt{v}^m(\cdot)\}$) such that*

$$\int_0^T \int_V g(t,v)\mathtt{v}_t^m(dv)dt \to \int_0^T \int_V g(t,v)\mathtt{v}_t(dv)dt \quad \text{as } m \to \infty, \quad \forall\ g \in C_b([0,T] \times V) \quad \forall\ T > 0. \tag{3.1}$$

In this case, we denote the above convergence as $\mathtt{v}^m \overset{w}{\to} \mathtt{v}$ as $m \to \infty$.

Applying a relaxed control $\mathtt{v}(\cdot) \in \mathtt{V}$ (in lieu of $v(\cdot) \in \mathcal{V}$), together with an impulse-type control $\{k_i\}$ into the hybrid dynamic (2.1), the following *relaxed dynamic* is generated:

$$\begin{aligned} \big(\dot{x}(t), \dot{n}(t)\big) &= \left(\int_V g(x(t), n(t_i), v)\mathtt{v}_t(dv), 0\right), \qquad \text{for} \quad t \in [t_i, t_{i+1}[, \\ (x(t_i), n(t_i)) &= \big(X(x(t_i-), n(t_i-), k_i), N(x(t_i-), n(t_i-), k_i)\big), \quad i = 0, 1, \cdots \\ t_{i+1} &:= \inf\big\{t \geq t_i : (x(t-), n(t_i)) \in D\big\}, \qquad \text{when } t_i < \infty, \end{aligned} \tag{3.2}$$

with initial condition as in (2.2).

Following the steps (0) to (3) of the hybrid algorithm deducted in previous pages, we can formally deduce the existence and uniqueness of a solution $t \mapsto (x(t), n(t))$ in (3.2) by mimicking the arguments of section 2 as in the non-relaxed case. It is also clear that Proposition 2.4 is still valid in the framework of relaxed controls.

For each initial condition (x, n) and any pair $(\mathtt{v}, \{k_i\}) \in \mathtt{V} \times \mathcal{K}$, we define the relaxed cost function

$$\mathtt{J}\big(x, n; \mathtt{v}(\cdot), \{k_i\}\big) = \int_0^\infty e^{-\alpha t} \int_V\ f(x(t), n(t), v)\mathtt{v}_t(dv)\ dt + \sum_{i=0}^\infty e^{-\alpha t_i}\ \ell(x(t_i-), n(t_i-), k_i). \tag{3.3}$$

Furthermore, we define the optimal relaxed cost by

$$\mathtt{u}(x, n) = \inf_{\big(\mathtt{v}, \{k_i\}\big) \in \mathtt{V} \times \mathcal{K}} \mathtt{J}\big(x, n; \mathtt{v}(\cdot), \{k_i\}\big). \tag{3.4}$$

Recall the set D_n defined in (2.3). We denote by ∂D and ∂D_n the boundaries of D and D_n, respectively, we also write $\overset{\circ}{D}$ as the interior D.

In order to prove the existence of a pair $(\hat{\mathtt{v}}(\cdot), \{\hat{k}_i\}) \in \mathtt{V} \times \mathcal{K}$ so that $u(x,n) = \mathtt{J}\big(x, n; \hat{\mathtt{v}}(\cdot), \{\hat{k}_i\}\big)$, we will impose the following transversality condition on the boundary of D (and as a consequence on D_n). These conditions have been considered in previous works (see, e.g., Bensoussan and Menaldi [6] or Branicky et. al. [7]).

For all $(x,n) \in \partial D$, there exists $\eta(x,n)$, so-named unit inner normal to ∂D_n and a positive constant ρ_0 such that, for any n, the function $x \mapsto \eta(x,n)$ belongs to $C_b(\mathbb{R}^d)$, and

$$\begin{cases} \partial D_n \ \text{ is smooth,} & \\ |\eta(x,n)| = 1, \ \forall x \in \partial D_n, & \text{(unit normal vector)} \\ |\eta(x,n) \cdot g(x,n,v)| \ge \rho_0, \quad \forall\ (x,n,v) \in \partial D \times V & \text{(transversality condition).} \end{cases} \tag{3.5}$$

Our next result concerns the continuity of the trajectories $(x(\cdot), n(\cdot))$ with respect to pair of controls $(\mathtt{v}, \{k_i\}) \in \mathtt{V} \times \mathcal{K}$. The proof will be provided in section 6, which strongly uses the transversality condition (3.5).

Proposition 3.2. *Suppose that assumptions* (2.4)-(2.7),(2.10), *and* (3.5) *are satisfied. Consider a sequence of controls* $\{(\mathtt{v}^m, \{k_i^m\})\}$ *whose elements are in* $\mathtt{V} \times \mathcal{K}$, *and a pair* $(\mathtt{v}^\infty, \{k_i^\infty\}) \in \mathtt{V} \times \mathcal{K}$, *such that* $\mathtt{v}^m \overset{w}{\to} \mathtt{v}^\infty$ *and* $\{k_i^m\} \to \{k_i^\infty\}$ *(with respect to the product topology of* $\mathcal{K}$*), as* $m \to \infty$. *Denote by* $(x^m(\cdot), n^m(\cdot))$ *the trajectory* (3.2) *correspondent to the pair* $(\mathtt{v}^m, \{k_i^m\})$, *for* $m \ge 1$. *Then,* $(x^m(\cdot), n^m(\cdot)) \to (x^\infty(\cdot), n^\infty(\cdot))$ *locally uniformly in almost every point. Moreover, this limit trajectory satisfies* (3.2) *with controls* $(\mathtt{v}^\infty, \{k_i^\infty\})$.

Now we can establish one of our main theorems regarding the existence of relaxed controls that minimize the optimal cost u defined in (2.12).

Theorem 3.3. *Under the assumptions of Proposition 3.2, there exists a pair* $(\hat{\mathtt{v}}, \{\hat{k}_i\})$, *consisting in a relaxed control* $\hat{\mathtt{v}} \in \mathtt{V}$ *and an impulse-type control* $\{\hat{k}_i\} \in \mathcal{K}$ *such that* $\mathtt{J}(x, n, \hat{\mathtt{v}}(\cdot), \{\hat{k}_i\}) = u(x,n)$, *where* $\mathtt{J}$ *is the total cost defined in* (3.3).

Proof. By definition of infimum, we can select a minimizing sequence $\{(\delta_{v^m(\cdot)}(\cdot), \{k_i^m\})\} \in \mathtt{V} \times \mathcal{K}$ such that

$$J\big(x, n, v^m(\cdot), \{k_i^m\}\big) = \mathtt{J}\big(x, n, \delta_{v^m(\cdot)}(\cdot), \{k_i^m\}\big) \downarrow u(x,n) \quad \text{as } m \to \infty, \tag{3.6}$$

where

$$\mathtt{J}\big(x, n, \delta_{v^m(\cdot)}(\cdot), \{k_i^m\}\big) = \int_0^\infty e^{-\alpha t} \int_V \ f(x^m(t), n^m(t), v) \delta_{v^m(t)}(dv)\ dt + \sum_{i=0}^\infty e^{-\alpha t_i^m} \ \ell(x(t_i^m -), n(t_i^m -), k_i^m). \tag{3.7}$$

Since both sets $\mathtt{V}$ and $\mathcal{K}$ are compact (the former set in a weak sense), we claim that $\delta_{v^m(\cdot)} \overset{w}{\to} \hat{\mathtt{v}} \in \mathtt{V}$ and $\{k_i^m\} \to \{\hat{k}_i\} \in \mathcal{K}$ under a suitable subsequence. For easy of notation, will write this subsequence as the original one.

The former integral in (3.7) can be expressed as follows:

$$\begin{aligned} &\int_0^\infty e^{-\alpha t} \int_V \ f(x^m(t), n^m(t), v) \delta_{v^m(t)}(dv)\ dt = \sum_{i=0}^\infty \int_{t_i^\infty}^{t_{i+1}^\infty} e^{-\alpha t} \int_V \ f(x_i^m(t), n_i^m(t), v) \delta_{v^m(t)}(dv)\ dt = \\ &= \sum_{i=0}^\infty \ \int_{t_i^\infty}^{t_{i+1}^\infty} e^{-\alpha t} \int_V \ [f(x_i^m(t), n_i^m(t), v) - f(x_i^\infty(t), n_i^\infty(t), v)] \delta_{v^m(t)}(dv)\ dt + \\ &+ \sum_{i=0}^\infty \ \int_{t_i^\infty}^{t_{i+1}^\infty} e^{-\alpha t} \int_V f(x_i^\infty(t), n_i^\infty(t), v) \delta_{v^m(t)}(dv)\ dt, \end{aligned} \tag{3.8}$$

where the paths $t \mapsto (x_i^m(t), n_i^m(t))$, $i \geq 0$ are introduced by equations (6.1)–(6.4) in the Appendix 6, whereas the sequences $\{t_i^\infty\} \subset [0,T]$ is a fixed sequence (actually, it is generated according to the proof of Proposition 3.2 in the aforementioned appendix).

By the proof of Proposition 3.2, we claim that $(x_i^m(\cdot), n_i^m(\cdot)) \to (x_i^\infty(\cdot), n_i^\infty(\cdot))$ uniformly as $m \to \infty$. Hence, the property of $f \in C_b^{\mathrm{u}}(S \smallsetminus D \times V)$, implies that $f(x_i^m(\cdot), n_i^m(\cdot), v) \to f(x_i^\infty(\cdot), n_i^\infty(\cdot), v)$ on the interval $[t_i^\infty, t_{i+1}^\infty[$ as $m \to \infty$ for all $v \in V$. Using this later property, we can easily verify that

$$\int_V \big[f(x_i^m(t), n_i^m(t), v) - f(x_i^\infty(t), n_i^\infty(t), v)\big]\delta_{v^m(t)}(dv) \to 0 \quad \text{as } m \to \infty \quad \forall\, t \in [t_i^\infty, t_{i+1}^\infty].$$

Then, by the simple use of the dominated convergence theorem, we can conclude

$$\sum_{i=0}^{\infty} \int_{t_i^\infty}^{t_{i+1}^\infty} e^{-\alpha t} \int_V \big[f(x_i^m(t), n_i^m(t), v) - f(x_i^\infty(t), n_i^\infty(t), v)\big]\delta_{v^m(t)}(dv)\ dt \to 0 \quad \text{as } m \to \infty.$$

Moreover, based on the continuity of the trajectory $(x^\infty(\cdot), n^\infty(\cdot))$ on $[t_i^\infty, t_{i+1}^\infty[$, we see that the mapping $(t,v) \mapsto f(x^\infty(t), n^\infty(t), v)$ is continuous on $[t_i^\infty, t_{i+1}^\infty[$ too. Then, using the fact of $\delta_{v^m(\cdot)} \xrightarrow{w} \hat{\mathtt{v}}$, we deduce

$$\int_V f(x_i^\infty(t), n_i^\infty(t), v)]\delta_{v^m(t)}(dv)\ dt \to \int_V f(x_i^\infty(t), n_i^\infty(t), v)]\hat{\mathtt{v}}_t(dv)\ dt \quad \forall\, t \in [t_i^\infty, t_{i+1}^\infty[. \tag{3.9}$$

Hence, a simple use of dominated convergence theorem, leads to

$$\sum_{i=0}^{\infty} \int_{t_i^\infty}^{t_{i+1}^\infty} e^{-\alpha t} \int_V f(x_i^\infty(t), n_i^\infty(t), v)]\delta_{v^m(t)}(dv)\ dt \to$$
$$\to \sum_{i=0}^{\infty} \int_{t_i^\infty}^{t_{i+1}^\infty} e^{-\alpha t} \int_V f(x_i^\infty(t), n_i^\infty(t), v)]\hat{\mathtt{v}}_t(dv)\ dt \quad \text{as } m \to \infty.$$

Therefore, we have proved that

$$\int_0^\infty e^{-\alpha t} \int_V\ f(x^m(t), n^m(t), v)\delta_{v^m(t)}(dv)\ dt \to \int_0^\infty e^{-\alpha t} \int_V\ f(x^\infty(t), n^\infty(t), v)\hat{\mathtt{v}}_t(dv)\ dt \quad \text{as } m \to \infty. \tag{3.10}$$

On the other hand, the continuity of the mapping $(x,n,v) \mapsto \ell(x,n,k)$, the uniform convergence $(x_i^m(\cdot), n_i^m(\cdot)) \to (x_i^\infty(\cdot), n_i^\infty(\cdot))$ on $[t_i^\infty, t_{i+1}^\infty[$, and the convergences $t_i^m \to t_i^\infty$ and $\{k_i^m\} \to \{\hat{k}_i\}$ established in Proposition 3.2 and inside its proof, ensure that

$$e^{-\alpha t_i^m} \ell(x_i^m(t_i^m-), n_i^m(t_i^m-), k_i^m) \to e^{-\alpha t_i^\infty} \ell(x_i^\infty(t_i^\infty-), n_i^\infty(t_i^\infty-), \hat{k}_i),$$

Then, using again the dominated convergence theorem, we deduce

$$\sum_{i=0}^{\infty} e^{-\alpha t_i^m}\ \ell(x_i^m(t_i^m-), n_i^m(t_i^m-), k_i^m) \to \sum_{i=0}^{\infty} e^{-\alpha t_i^\infty}\ \ell(x_i^\infty(t_i^\infty-), n_i^\infty(t_i^\infty-), \hat{k}_i) \tag{3.11}$$

Thus, based on (3.10) and (3.11), we can conclude

$$\begin{aligned} \mathtt{J}\big(x, n, \delta_{v^m(\cdot)}(\cdot), \{k_i^m\}\big) &= \int_0^\infty \int_V e^{-\alpha t} f(x^m(t), n^m(t), v)\delta_{v^m(t)}(dv)\ dt + \\ &+ \sum_{i=0}^{\infty} e^{-\alpha t_i^m}\ \ell(x^m(t_i^m-), n^m(t_i^m-), k_i^m) \downarrow \\ &\to \int_0^\infty \int_V e^{-\alpha t} f(x^\infty(t), n^\infty(t), v)\hat{\mathtt{v}}_t(dv)\ dt + \sum_{i=0}^{\infty} e^{-\alpha t_i^\infty}\ \ell(x^\infty(t_i^\infty-), n^\infty(t_i^\infty-), \hat{k}_i) = \\ &= \mathtt{J}\big(x, n, \hat{\mathtt{v}}, \{\hat{k}_i\}\big) = u(x,n), \end{aligned} \tag{3.12}$$

which proves the result. □

4 General case

In this section we drop the transversality condition given in (3.5). This implies that the continuity of the trajectories established in Proposition 3.2 may not longer be valid. So different arguments must be applied.

In this section we establish the following:

- For each $t \geq 0$, the family of trajectories $\{(x^m(t), n^m(t))\}_m$ associated to a sequence of controls $(\mathtt{v}^m, \{k_i^m\}) \in \mathtt{V} \times \mathcal{K}$, is bounded for all $m \geq 1$.
- The definition of occupation measures associated to the trajectories $(x(\cdot), n(\cdot))$
- The pre-compactness of the occupation measures with respect to a large set of finite measures satisfying suitable properties.
- The existence of an optimal measure under which $u(x, n)$ attains the minimum.

To this purpose, similar to (3.2), the expression (2.9) can be regarded as follows: for each $\varphi \in C_b^{1,0}(S)$, $t \geq 0$, and $\mathtt{v} \in \mathtt{V}$:

$$
\begin{aligned}
&e^{-\alpha t}\varphi(x(t), n(t)) - \varphi(x, n) = \\
&\int_0^t e^{-\alpha s}\left[\int_V g(x(s), n(s), v)\mathtt{v}_s(dv) \cdot \nabla_x \varphi(x(s), n(s)) - \alpha\varphi(x(s), n(s))\right] ds + \\
&\qquad + \sum_{i=0}^{\infty} e^{-\alpha t_i}\left[\varphi\left(X(x(t_i-), n(t_i-), k_i), N((x(t_i-), n(t_i-), k_i)\right) - \varphi\left(x(t_i-), n(t_i-)\right)\right] \mathbf{1}_{\{t_i \leq t\}},
\end{aligned}
\tag{4.1}
$$

For $\lambda \geq 1$, we define

$$
c_0 \equiv c_0(\lambda) := \sup\left\{\frac{x \cdot g(x, n, v)}{\lambda + |x|^2 + |n|^2} : (x, n) \in S \smallsetminus D,\ v \in V\right\},
\tag{4.2}
$$

where g is the vector field in (3.2). Since g is bounded by the constant M (see (2.5)),

$$
\frac{x \cdot g(x, n, v)}{\lambda + |x|^2 + |n|^2} \leq \frac{|x|M}{\lambda + |x|^2 + |n|^2} \leq \Big(\frac{|x|}{\sqrt{\lambda + |x|^2 + |n|^2}}\Big)\Big(\frac{M}{\sqrt{\lambda + |x|^2 + |n|^2}}\Big),
$$

i.e., $c_0 \leq M\lambda^{-1/2}$. This implies that given the discount factor $\alpha > 0$ in (2.11) or in (3.3), there exists $\lambda > 0$ large enough so that $c_0 < \alpha$, and this is our choice of $\lambda > 0$ for all what follows.

We also define

$$
\begin{aligned}
c_1 \equiv c_1(\lambda) := \sup\Big\{&\big[\big(\lambda + |X(x, n, k)|^2 + |N(x, n, k)|^2\big)^{1/2} - \big(\lambda + |x|^2 + |n|^2\big)^{1/2}\big] \times \\
&\times \big(|X(x, n, k) - x|^2 + |N(x, n, k) - n|^2\big)^{-1/2} : (x, n) \in D,\ k \in K\Big\}.
\end{aligned}
\tag{4.3}
$$

From the estimate

$$
\left|(\lambda + |x|^2 + |n|^2)^{1/2} - (\lambda + |x'|^2 + |n'|^2)^{1/2}\right| \leq (|x - x'|^2 + |n - n'|^2)^{1/2},
\tag{4.4}
$$

we can deduce that c_1 belongs to $[-1, 1]$.

Now take an arbitrary pair of controls $(\mathtt{v}, \{k_i\}) \in \mathtt{V} \times \mathcal{K}$ together with its corresponding trajectory $(x(\cdot), n(\cdot))$ satisfying (3.2).

Proposition 4.1. *Assume the conditions* (2.4)-(2.7)*, and* (2.10)*. Fix* $\lambda \geq 1$ *sufficiently large so that* $c_0(\lambda) < \alpha$*, and use any sequence* $\{(\mathtt{v}^m, \{k_i^m)\}$ *of elements of* $\mathtt{V} \times \mathcal{K}$*. Then, for each* $t \geq 0$*, the solution* $(x^m(t), n^m(t))$ *in* (3.2) *correspondent to the* m*-th element of* $\{(\mathtt{v}^m, \{k_i^m)\}$*, satisfies:*

$$\big(\lambda + |(x^m(t)|^2 + |n^m(t))|^2\big)^{1/2} \leq C_t \qquad \forall\ m \geq 1, \tag{4.5}$$

with $C_t = e^{\alpha t}\big(\lambda + |x_0|^2 + |n_0)|^2\big)^{1/2} + Cc_1 e^{\alpha t}/(1 - e^{-\alpha \mathtt{h}})$.

Proof. Replacing $\varphi(\cdot)$ in (4.1) by the function $(\lambda + |x|^2 + |n|^2)^{1/2}$ and using the estimates (4.2)–(4.4), we obtain

$$\begin{aligned}
&\big(\lambda + |(x^m(t)|^2 + |n^m(t))|^2\big)^{1/2} e^{-\alpha t} \leq \\
&\quad \leq \big(\lambda + |(x_0|^2 + |n_0)|^2\big)^{1/2} + \int_0^t (c_0 - \alpha) e^{-\alpha s}\big[\lambda + |(x^m(s)|^2 + |n^m(s))|^2\big]^{1/2} \mathrm{d}s + \\
&\quad + c_1 \Big\{ \sum_{i=1}^{\infty} e^{-\alpha t_i^m} \big(|X(x^m(t_i^m-), n^m(t_i^m-), k_i^m) - x^m(t_i^m-)|^2 + \\
&\qquad\qquad + |N(x^m(t_i^m-), n^m(t_i^m-), k_i^m) - n^m(t_i^m-)|^2\big)^{1/2} \Big\} \mathbf{1}_{\{t_i^m \leq t\}}, \\
&t_{i+1}^m = \inf\big\{t > t_i^m : (x^m(t-), n^m(t_i^m) \in D\big\}, \quad i \geq 0, \quad \forall t > 0.
\end{aligned}$$

Therefore, the result follows by using the upper bond in condition (2.7) as well as the fact of that $c_0 < \alpha$. ☐

Let us go back to expression (4.1). Since g is bounded (see (2.5)) and using Proposition 2.4 (specifically the fact of $\sum_i e^{-\alpha t_i} \leq \sum_i e^{-\alpha i \mathtt{h}}$), we see that for every $\varphi \in C_b^{1,0}(S)$, we can let $t \to \infty$ in both sides of (4.1) to obtain

$$\begin{aligned}
\varphi(x,n) &= -\int_0^{\infty} e^{-\alpha t} \left[\int_V g(x(t), n(t), v)\mathtt{v}_t(dv) \cdot \nabla_x \varphi(x(t), n(t)) - \alpha\varphi(x(t), n(t))\right] dt - \\
&- \sum_{i=0}^{\infty} e^{-\alpha t_i} \left[\varphi\left(X(x(t_i-), n(t_i-), k_i), N((x(t_i-), n(t_i-), k_i)\right) - \varphi\left(x(t_i-), n(t_i-)\right)\right] \\
&= \int_0^{\infty} e^{-\alpha t} \int_V A^v \varphi(x(t), n(t))\ \mathtt{v}_t(dv)\ dt + \sum_{i=0}^{\infty} e^{-\alpha t_i} L^{k_i} \varphi\left(x(t_i-), n(t_i-)\right),
\end{aligned} \tag{4.6}$$

where for all $(x,n) \in S$, $v \in V$, and $k \in K$,

$$\begin{aligned}
A^v \varphi(x,n) &:= -g(x,n,v) \cdot \nabla_x \varphi(x,n) + \alpha\varphi(x,n), \text{and} \\
L^k \varphi(x,n) &:= \varphi(x,n) - \varphi\left(X(x,n,k), N(x,n,k)\right).
\end{aligned} \tag{4.7}$$

It is easy to see that under assumptions (2.5) and (2.6), the mappings $(x,n,v) \mapsto A^v\varphi(x,n) \in C_b(S \setminus D \times V)$ and $(x,n,k) \mapsto L^k \varphi(x,n) \in C_b(D \times K)$, for all $\varphi \in C_b^{1,0}(S)$.

Our next definition regards to the introduction of the so-named occupation measures. For easy of notation, any sequence $\{k_i\} \in \mathcal{K}$ will be denoted by $\mathtt{k}$.

Occupation measures: For each control pair $(\mathtt{v}, \mathtt{k}) \in \mathtt{V} \times \mathcal{K}$ and each initial condition (x,n), we define the *occupation* measures

$$\begin{aligned}
\mu_{(x,n)}^{\mathtt{v},\mathtt{k}}(\mathtt{A} \times \mathtt{B}) &:= \int_0^{\infty} \int_V e^{-\alpha t} \mathbf{1}_{\mathtt{A}}(x(t), n(t)) \mathbf{1}_{\mathtt{B}}(v) \mathtt{v}_t(dv) dt, \quad \forall\ \mathtt{A} \times \mathtt{B} \subseteq S \setminus D \times V \\
\nu_{(x,n)}^{\mathtt{v},\mathtt{k}}(\mathtt{A} \times \mathtt{B}) &:= \sum_{i=0}^{\infty} e^{-\alpha t_i} \mathbf{1}_{\mathtt{A}}(x(t_i-), n(t_i-)) \mathbf{1}_{\mathtt{B}}(k_i), \quad \forall\ \mathtt{A} \times \mathtt{B} \subseteq D \times K,
\end{aligned} \tag{4.8}$$

We denote the set of all occupation measures by

$$M_{(x,n)}\big(S\smallsetminus D\times V\big) := \big\{\mu^{\mathtt{v},\mathtt{k}}_{(x,n)},\ \forall\,(\mathtt{v},\mathtt{k})\in \mathtt{V}\times\mathcal{K}\big\} \quad\text{and}\quad M_{(x,n)}(D\times K) := \big\{\nu^{\mathtt{v},\mathtt{k}}_{(x,n)},\ \forall\,(\mathtt{v},\mathtt{k})\in \mathtt{V}\times\mathcal{K}\big\}$$

Observe that each of the measures $\mu^{\mathtt{v},\mathtt{k}}_{(x,n)}$ and $\nu^{\mathtt{v},\mathtt{k}}_{(x,n)}$ satisfies

$$0\le \mu^{\mathtt{v},\mathtt{k}}_{(x,n)} \le \frac{1}{\alpha} \quad\text{and}\quad 0\le \nu^{\mathtt{v},\mathtt{k}}(x,n) \le \frac{1}{1-e^{-\alpha\mathtt{h}}} \quad \forall\,(\mathtt{v},\mathtt{k})\in \mathtt{V}\times\mathcal{K}. \tag{4.9}$$

where $\mathtt{h}$ is the constant in Proposition 2.4. In fact, the normalized measures

$$\tilde{\mu}^{\mathtt{v},\mathtt{k}}_{(x,n)} := \alpha\ \mu^{\mathtt{v},\mathtt{k}}_{(x,n)} \quad\text{and}\quad \tilde{\nu}^{\mathtt{v},\mathtt{k}}_{(x,n)} := \hat{S}_{\mathtt{v},\mathtt{k}}\ \nu^{\mathtt{v},\mathtt{k}}_{(x,n)}, \tag{4.10}$$

with $\hat{S}_{\mathtt{v},\mathtt{k}} := [1/\sum_{i=0}^{\infty} e^{-\alpha t_i^{(\mathtt{v},\mathtt{k})}}]$, turn out to be probability measures on $S\smallsetminus D\times V$ and $D\times K$, respectively.

Definition 4.2. **(a)** Let $M_b(S\smallsetminus D\times V)$ (resp. $M_b(D\times K)$) be the space of all signed finite measures on $S\smallsetminus D\times V$ (resp. on $D\times K$).

(b) Denote by $M_b^+(S\smallsetminus D\times V)$ (resp. $M_b^+(D\times K)$) the subset of all nonnegative elements of $M_b(S\smallsetminus D\times V)$ (resp. of $M_b(D\times K)$).

Consider now the following problem:

$$\begin{aligned}
&(W)\ \text{minimize}\ \langle(\mu,\nu),(f,\ell)\rangle,\\
&\text{subject to}\\
&\langle\delta_{(x,n)},\varphi\rangle = \langle(\mu,\nu),(A^v\varphi, L^k\varphi)\rangle, \quad \forall\varphi\in C_b^{1,0}(S);\\
&(\mu,\nu)\le\left(\frac{1}{\alpha}\ ,\ \frac{1}{1-e^{-\alpha\mathtt{h}}}\right)\\
&(\mu,\nu)\in M_b^+\big(S\smallsetminus D\times V\big)\times M_b^+(D\times K).
\end{aligned} \tag{4.11}$$

Note that the relation (4.6) can be rewritten in terms of the occupation measures as follows:

$$\langle\delta_{(x,n)},\varphi\rangle = \langle\mu^{\mathtt{v},\mathtt{k}}_{(x,n)}, A^v\varphi\rangle + \langle\nu^{\mathtt{v},\mathtt{k}}_{(x,n)}, L^k\varphi\rangle = \langle(\mu^{\mathtt{v},\mathtt{k}}_{(x,n)},\nu^{\mathtt{v},\mathtt{k}}_{(x,n)}),(A^v\varphi, L^k\varphi)\rangle, \quad \forall\varphi\in C_b^{1,0}(S). \tag{4.12}$$

From (4.9) and (4.12), every occupation measure $(\mathtt{v},\mathtt{k})\mapsto(\mu^{\mathtt{v},\mathtt{k}}_{(x,n)},\nu^{\mathtt{v},\mathtt{k}}_{(x,n)})$ satisfies the restrictions of the weak problem (W). The use of occupation measures also ensures that the total cost (3.3) can be expressed as

$$\mathtt{J}(x,n,\mathtt{v}(\cdot),\{k_i\}) = \langle\mu^{\mathtt{v},\mathtt{k}}_{(x,n)}, f\rangle + \langle\nu^{\mathtt{v},\mathtt{k}}_{(x,n)}, \ell\rangle = \langle(\mu^{\mathtt{v},\mathtt{k}}_{(x,n)},\nu^{\mathtt{v},\mathtt{k}}_{(x,n)}),(f,\ell)\rangle, \quad \forall(\mathtt{v},\mathtt{k})\in\mathtt{V}\times\mathcal{K}. \tag{4.13}$$

We shall endow the standard (Prohorov's) weak convergence to the spaces $M_b(S\smallsetminus D\times V)$ and $M_b(D\times K)$; i.e., we say that a sequence of measures $\{\mu_m\}$ of elements in $M_b(S\smallsetminus D\times V)$ converges to some $\mu\in M_b(S\smallsetminus D\times V)$ and denote such convergence by $\mu_m \stackrel{w}{\to} \mu$ if and only if

$$\int_{S\smallsetminus D\times V} h(x,n,v)\ \mu_m(d(x,n,v)) \underset{m\to\infty}{\to} \int_{S\smallsetminus D\times V} h(x,n,v)\ \mu(d(x,n,v)) \quad \forall\ h\in C_b(S\smallsetminus D\times V).$$

Similarly, we can define the above convergence on the set $M_b(D\times K)$.

The following proposition ensures that the sets of occupation measures $M_{(x,n)}\big(S\smallsetminus D\times V\big)$ and $M_{(x,n)}\big(D\times K\big)$ are pre-compact (in a weak sense) relative to the spaces $M_b(S\smallsetminus D\times V)$ and $M_b(D\times K)$.

Proposition 4.3. *The sets $M_{(x,n)}\big(S\setminus D\times V\big)$ and $M_{(x,n)}\big(D\times K\big)$ are weakly pre-compact; that is, for any sequence $\{\mu^{\mathtt{v},\mathtt{k},m}_{(x,n)}\}_m\in M_{(x,n)}\big(S\setminus D\times V\big)$, there exists an element $\mu\in M_b(S\setminus D\times V)$ such that*

$$\mu^{\mathtt{v},\mathtt{k},m}_{(x,n)}\stackrel{w}{\to}\mu,\quad \text{as } m\to\infty$$

along a subsequence. Similarly, for any sequence $\{\nu^{\mathtt{v},\mathtt{k},m}_{(x,n)}\}_m\in M_{(x,n)}\big(D\times K\big)$, there exists an element $\nu\in M_b(D\times K)$ such that, under a suitable subsequence,

$$\nu^{\mathtt{v},\mathtt{k},m}_{(x,n)}\stackrel{w}{\to}\nu,\quad \text{as } m\to\infty.$$

Proof. We proceed by showing that the family of normalized measures (4.10) is tight, so this will imply tightness on the original family (4.8). To this end, fix $T>0$ and define the compact set

$$\bar B_{C_T}(x,n):=\big\{(z,r)\in S: |z-x|+|r-n|\le C_T\big\},\quad \text{with } C_T \text{ as in (4.5) at value } t=T,$$

and where (x,n) denotes the initial state of the dynamic (3.2). Now take a sequence $\{\mu^{\mathtt{v},\mathtt{k},m}_{(x,n)}\}_m\in M_{(x,n)}\big((S\setminus D)\times V\big)$ and denote by $(x^m(\cdot),n^m(\cdot))$ the corresponding trajectory. Observe that by Proposition 4.1, the path $s\mapsto(x^m(s),n^m(s))$ belongs to $\bar B_{C_T}(x,n)$, for all $m\ge1$, with $s\in[0,T]$; in fact, the triplet $s\mapsto(x^m(s),n^m(s),v)\in\bar B_{C_T}(x,n)\times V$. Then, for every $\varepsilon>0$, we can chose T sufficiently large, such that

$$\begin{aligned}\tilde\mu^{\mathtt{v},\mathtt{k},m}_{(x,n)}\big(\bar B_{C_T}(x,n)\times V\big)&=\alpha\int_0^\infty\int_V e^{-\alpha s}\mathbf{1}_{\bar B_{C_T}(x,n)}(x^m(s),n^m(s))\mathbf{1}_V(v)\ \mathtt{v}_s(dv)\ ds=\\&=\alpha\int_0^T e^{-\alpha s}ds=(1-e^{-\alpha T})>1-\varepsilon,\quad\forall\ m\ge1.\end{aligned}$$

Similarly,

$$\begin{aligned}\tilde\nu^{\mathtt{v},\mathtt{k},m}_{(x,n)}\big(\bar B_{C_T}(x,n)\times K\big)&=\hat S_{\mathtt{v}_m,\mathtt{k}_m}\sum_{i=0}^\infty e^{-\alpha t_i^m}\mathbf{1}_{\bar B_{C_T}(x,n)}(x^m(t_i-),n^m(t_i-))\mathbf{1}_K(k_i)=\\&=\hat S_{\mathtt{v}_m,\mathtt{k}_m}\sum_{i=0}^\infty e^{-\alpha t_i^m}\mathbf{1}_{\{t_i\le T\}}>1-\varepsilon,\quad\forall\ m\ge1,\end{aligned}$$

where the last inequality is due to the fact of $\hat S_{\mathtt{v}_m,\mathtt{k}_m}\sum_{i=0}^\infty e^{-\alpha t_i^m}\mathbf{1}_{\{t_i\le T\}}\uparrow 1$ as $T\to\infty$. Then, by definition both sequences $\{\tilde\mu^{\mathtt{v},\mathtt{k},m}_{(x,n)}\}$ and $\{\tilde\nu^{\mathtt{v},\mathtt{k},m}_{(x,n)}\}$ are tight, and thus relatively compact by Prohorov's theorem. Of course this property on the normalized measures leads to the same property when using occupation measures, which proves the result. □

The next theorem gives a characterization of the value function $u(x,u)$ in terms of a pair of measures, say (μ,ν) that satisfies the restrictions of problem (W).

Theorem 4.4. *Under the assumptions* (2.4)-(2.7)*, and* (2.10)*, there exists a pair $(\hat\mu,\hat\nu)\in M_b\big(S\setminus D\times V\big)\times M_b(D\times K)$ satisfying the restrictions* (4.11) *of problem (W), and such that the value function $u(x,n)$ in* (2.12) *becomes $u(x,n)=\langle(\hat\mu,\hat\nu),(f,\ell)\rangle$.*

Proof. Choose a minimizing sequence of controls $(\{\delta_{v_m(\cdot)}(\cdot)\},\{k_i^m\})\in\mathcal V\times\mathcal K$, so that

$$J\big(x,n,v_m(\cdot),\{k_i^m\}\big)=\mathtt{J}\big(x,n,\delta_{v_m(\cdot)}(\cdot),\{k_i^m\}\big)\downarrow u(x,n)\quad\text{as } m\to\infty. \tag{4.14}$$

By (4.13), we can relate to $\mathtt{J}((x,n,\delta_{v_m(\cdot)}(\cdot),\{k_i^m\})$ its correspondent occupation measures $(\{(\mu_m,\nu_m)\}_m \in M_b(S\smallsetminus D\times V)\times M_b(D\times K)$ defined in (4.8). Moreover, every pair $(\delta_{v_m(\cdot)}(\cdot)),\{k_i^m\})\in\mathcal{V}\times\mathcal{K}$ produces a trajectory given in (2.1), which in turns it is equivalent to (2.9). Hence, in virtue of (4.12) together with the fact of (μ_m,ν_m) are in fact measures (and therefore are non negative), we claim that each element of the above sequence satisfies the restrictions of problem (4.11).

By compactness of the control spaces, we know that $\delta_{v_m(\cdot)}\overset{w}{\to}\hat{\mathtt{v}}$ and $\{k_i^m\}\to\{\hat{k}_i\}$, for some $\hat{\mathtt{v}}\in\mathtt{V}$ and $\{\hat{k}_i\}\in\mathcal{K}$, so we can invoke Proposition 4.3 to deduce the existence of a pair $(\hat{\mu},\hat{\nu})\in M_b\big(S\smallsetminus D\times V\big)\times M_b(D\times K)$ such that $(\mu_m,\nu_m)\overset{w}{\mapsto}(\hat{\mu},\hat{\nu})$ as $m\to\infty$. From (4.14), this limit pair is the one associated with the value function $u(x,n)$; i.e., $u(x,n)=\langle(\hat{\mu},\hat{\nu}),(f,\ell)\rangle$.

It only remains to prove that $(\hat{\mu},\hat{\nu})$ satisfies the restrictions in (4.11). To do that, note again that for every $m\geq 1$,

$$\langle\delta_{(x,n)},\varphi\rangle=\langle(\mu_m,\nu_m),(A^v\varphi,L^k\varphi)\rangle,\ \forall\varphi\in C_b^{1,0}(S). \tag{4.15}$$

Since the mappings $(x,n,v)\mapsto A^v\varphi(x,n)\in C_b(S\smallsetminus D\times V)$ and $(x,n,k)\mapsto L^k\varphi(x,n)\in C_b(D\times K)$, for all $\varphi\in C_b^{1,0}(S)$, and due to the fact of $(\mu_m,\nu_m)\overset{w}{\mapsto}(\hat{\mu},\hat{\nu})$ as $m\to\infty$, we deduce

$$\langle\delta_{(x,n)},\varphi\rangle=\lim_{m\to\infty}\langle(\mu_m,\nu_m),(A^v\varphi,L^k\varphi)\rangle=\langle(\hat{\mu},\hat{\nu}),(A^v\varphi,L^k\varphi)\rangle,\quad\forall\varphi\in C_b^{1,0}(S),$$

and

$$0\leq\lim_m\mu_m=\hat{\mu}\leq\frac{1}{\alpha}\quad\text{and}\quad 0\leq\lim_m\nu_m=\hat{\nu}\leq\frac{1}{1-e^{-\alpha\mathtt{h}}}$$

This proves the result. □

5 Linear programs

This section is devoted to show that the control problems (2.12) and (3.4) are equivalent for the regular case of Section (3). Furthermore, we also provide a second equivalence of the former problem with the minimization of an ancillary infinite dimensional linear program; in particular, the restrictions of the correspondent dual couterpart of this linear program will be of great importance. This material could be also interesting from the point of view of approximations since infinite-dimensional linear programs can be analyzed through approximations of finite-dimensional linear programs—see, for instance, Lasserre [11]. In this way, our results in this section could provide a guide to approximate the value function u in (2.12) by means of finite-dimensional linear programs; such later analysis is left out of the scope of this paper.

It is also important to mention that along this section we shall use a regularity-type condition on the value function u that becomes true under the transversality condition (3.5).

We summarize this section into the next four main facts:

- Problem (W), see (4.11), is embeded into a linear program (P), see (5.3).
- Problem (P) has an associated dual counterpart (P^*), see (5.4).
- Under an extra assumption on the value function u —see (5.13), we prove $\inf(P)\geq u(x,n)$. As a consequence, problems (2.12) and (3.4) are equivalent; i.e., $u(x,n)=\mathtt{u}(x,n)=\min(W)=\inf(P)$.
- The transversality condition (3.5) implies that assumption (5.13) is satisfied.

Dual pairs: General results in duality theory show that, for some subset $\mathcal{X}$ of $\mathbb{R}^j$, with j as in Definition 2.5, the topological (strong) dual of $C_b(\mathcal{X})$ turns out to be $M_b(\mathcal{X})$, with C_b and M_b corresponding to the sets in Definitions 2.5(b) and 4.2, respectively. It follows that these spaces define a dual pair under the duality

$$\big\langle M_b(\mathcal{X}), C_b(\mathcal{X})\big\rangle_{\mathcal{X}} = \int_{\mathcal{X}} g\ d\eta, \quad \forall g \in C_b(\mathcal{X}),\ \eta \in M_b(\mathcal{X}). \tag{5.1}$$

The above spaces are Banach spaces under their associated norms; nevertheless, hereafter, convergence in the space $M_b(\mathcal{X})$ is assumed to hold under the weak topology $\sigma(C_b(\mathcal{X}), M_b(\mathcal{X}))$; i.e,. $\mu_n \xrightarrow{w} \mu$ implies $\langle \mu_n, g\rangle \to \langle \mu, g\rangle$, for all $g \in C_b(\mathcal{X})$.

On the other hand, for the case $\mathcal{X} \equiv S$, recall the space $C_b^{1,0}(S)$ introduced in Definition 2.5(c). We shall denote its (algebraic) dual as $\mathcal{D}_b(S)$. These spaces endowed with the weak topology $\sigma(C_b^{1,0}(S), \mathcal{D}_b(S))$ and $\sigma(\mathcal{D}_b(S), C_b^{1,0}(S))$ become dual pairs under the duality

$$\big\langle \mathcal{D}_b(S), C_b^{1,0}(S)\big\rangle_S = \int_S h\ d\varrho, \quad \forall h \in C_b^{1,0}(S)),\ \varrho \in \mathcal{D}_b(S).$$

Observe that $\delta_{(x,n)}(\cdot) \in \mathcal{D}_b(S)$ for all $(x, n) \in S$.

Now let us go back to the definition of the pair (A^v, L^k) introduced in (4.7). Namely, this pair regarded as a single operator, maps $C_b^{1,0}(S)$ into $C_b(S \smallsetminus D \times V) \times C_b(D \times K)$. A useful property of this operator is:

Proposition 5.1. *The operator (A^v, L^k) is continuous with respect to the norm induced by $C_b(S \smallsetminus D \times V) \times C_b(D \times K)$, defined as:*

$$\|(g_1, g_2)\|_* := \max\{\|g_1\|, \|g_2\|\} \quad \forall g_1 \in C_b(S \smallsetminus D \times V), \quad g_2 \in C_b(D \times K).$$

Proof. It is easy to verify that

$$\begin{aligned} &\|A^v\varphi\| \le (\|g\| + \alpha)\,\|\varphi\|_1, \quad \text{and} \\ &\|L^k\varphi\| \le 2\|\varphi\|_1, \quad \forall \varphi \in C_b^{1,0}(S). \end{aligned}$$

Then, we have

$$\begin{aligned} \|(A^v, L^k)\varphi\|_* &= \max\Big\{\|A^v\varphi\|,\ \|L^k\varphi\|\Big\} \\ &\le (\|g\| + \alpha + 2)\cdot\|\varphi\|_1 \end{aligned}$$

This proves the result. □

Consider the product space $M_b((S \smallsetminus D) \times V) \times M_b(D \times K)$. For each pair (μ, η) in the above product space, define the functional $\Phi_{(\mu,\eta)}$ on $C_b^{1,0}(S)$ as follows:

$$\Phi_{(\mu,\eta)}\varphi := \langle(\mu, \eta), (A^v, L^k)\varphi\rangle = \langle \mu, A^v\varphi\rangle + \langle \eta, L^k\varphi\rangle \qquad \forall \varphi \in C_b^{1,0}(S).$$

Since (A^v, L^k) is continuous, so is $\Phi_{(\mu,\eta)}$. This implies the existence of an element $\nu_{(\mu,\eta)} \in \mathcal{D}_b(S)$ such that

$$\Phi_{(\mu,\eta)}\varphi = \langle \nu_{(\mu,\eta)}, \varphi\rangle = \langle(\mu, \eta)(A^v, L^k)\varphi\rangle \qquad \forall \varphi \in C_b^{1,0}(S).$$

Since this holds for every $(\mu, \eta) \in M_b((S \smallsetminus D) \times V) \times M_b(D \times K)$, we can define the operator $(A^v, L^k)^* : M_b((S \smallsetminus D) \times V) \times M_b(D \times K) \mapsto \mathcal{D}_b(S)$ as follows

$$(A^v, L^k)^*(\mu, \eta) := \nu_{(\mu,\eta)}, \quad \forall (\mu, \eta) \in M_b((S \smallsetminus D) \times V) \times M_b(D \times K).$$

It is clear that $(A^v, L^k)^*$ is the adjunct of (A^v, L^k), because

$$\langle (A^v, L^k)^*(\mu, \eta), \varphi\rangle = \langle(\mu, \eta), (A^v, L^k)\varphi\rangle, \tag{5.2}$$

for all $(\mu,\eta) \in M_b(S \setminus D \times V) \times M_b(D \times K)$ and $\varphi \in C_b^{1,0}(S)$.

Since $\mathcal{D}_b(S)$ is the algebraic dual of $C_b^{1,0}(S)$, by construction of the adjunct operator $(A^v, L^k)^*$ we see that it maps the product space $M_b(S \setminus D \times V) \times M_b(D \times K)$ into the space $\mathcal{D}_b(S)$. A characterization of this last assertion is given in the following result (e.g., a proof can be found in Aliprantis and Border [1, Theorem 6.43]).

Proposition 5.2. *The operator $(A^v, L^k)^*$ is weakly continuous; i.e., it is continuous with respect to the weak topologies $\sigma\big(M_b(S \setminus D \times V) \times M_b(D \times K), C_b(S \setminus D \times V) \times C_b(D \times K)\big)$ and $\sigma\big(\mathcal{D}_b(S), C_b^{1,0}(S)\big)$.*

Cones: Now define the natural cones of the space $C_b(S \setminus D \times V) \times C_b(S \times K)$ as follows:

$$\big[C_b(S \setminus D \times V) \times C_b(S \times K)\big]^+ := \big\{(\varphi_1,\varphi_2) \in C_b(S \setminus D \times V) \times C_b(D \times K) \mid \varphi_1 \geq 0,\ \varphi_2 \geq 0\big\},$$

We also define its correspondent dual cone as

$$\begin{aligned}\big[M_b(S \setminus D \times V) \times M_b(D \times K)\big]^+ := \big\{&(\mu_1,\mu_2) \in M_b(S \setminus D \times V) \times M_b(D \times K) \\ &\mid \langle(\mu_1,\mu_2),(\varphi_1,\varphi_2)\rangle \geq 0 \quad \forall(\varphi_1,\varphi_2) \in \big[C_b(S \setminus D \times V) \times C_b(S \times K)\big]^+\big\}.\end{aligned}$$

With all the previous definitions, we define the linear program

$$\begin{aligned}&(P) \text{ minimize } \langle(\mu,\nu),(f,\ell)\rangle,\\ &\text{subject to}\\ &\delta_{(x,n)} = (A^v, L^k)^*(\mu,\nu);\\ &(\mu,\nu) \in \big[M_b(S \setminus D \times V) \times M_b(D \times K)\big]^+.\end{aligned} \tag{5.3}$$

Note that the above duality $\langle(\mu,\nu),(f,\ell)\rangle$ is intended as $\langle\mu,f\rangle_{S\setminus D\times V} + \langle\nu,\ell\rangle_{D\times K}$ (see (5.1)).

The dual problem of (P) turns out to be

$$\begin{aligned}&(P^*) \text{ maximize } \langle\delta_{(x,n)},\varphi\rangle,\\ &\text{subject to}\\ &(f,\ell) - (A^v, L^k)\varphi \in \big[C_b(S \setminus D \times V) \times C_b(S \times K)\big]^+;\\ &\varphi \in C_b^{1,0}(S).\end{aligned} \tag{5.4}$$

The later restriction can be seen as

$$A^v\varphi \leq f, \quad L^k\varphi \leq \ell. \tag{5.5}$$

Consistency: It is obvious that the dual problem is feasible (i.e., the restrictions are nonempty). Indeed, use the constant function $0 \in C_b^{1,0}$ and since the costs f and g are nonnegative, the assertion is true. On the other hand, we have already verified that the weak problem (W) is feasible, so does (P). This implies that the problems (P) and (P^*) are both consistent. Then we define the value of primal problem (P)

$$\inf(P) := \inf\big\{\big\langle(\mu,\nu),(f,\ell)\big\rangle \mid (\mu,\nu) \text{ is feasible for } (P)\big\}, \tag{5.6}$$

In a similar manner, the value of the dual problem (P^*) is defined by

$$\sup(P^*) := \sup\big\{\langle\delta_{(x,n)},\varphi\rangle \mid \varphi \text{ is feasible for } (P^*)\big\}, \tag{5.7}$$

Since the problem (W) satisfies the restrictions (5.3), we have $\inf(W) \geq \inf(P)$.

The following result ensures the so-named weak duality between the above linear programs (P) and (P^*). The proof of this fact is proved for general infinite linear spaces by Anderson and Nash [2].

Proposition 5.3 (Weak duality). *The values of* (P) *and* (P^*) *are finite and they satisfy*

$$\sup(P^*) \le \inf(P). \tag{5.8}$$

• *Remark* 5.4. It is possible to get the equality in (5.8) under the extra condition on the cost functions. This condition must guarantee a lower bound imposed to the cost rates $f \ge c_0$ and $\ell \ge c_0$ for some positive constant c_0.

Relation of (P) and the value function u: To begin this part, let us first consider all the functions $\hat{f}(x,n,v) \in C_b^{\mathrm{u}}(S \smallsetminus D \times V)$ and $\hat{\ell}(x,n,k) \in C_b^{\mathrm{u}}(D \times K)$ with the additional property to be Lipschitz continuous at the variable x uniformly with respect to the others; i.e., there is a positive constant $M_{\hat{f},\hat{\ell}}$, such that

$$\sup_{n,v} \big|\hat{f}(x,n,v) - \hat{f}(y,n,v)\big| + \sup_{n,k} \big|\hat{\ell}(x,n,k) - \hat{\ell}(y,n,k)\big| \le M_{\hat{f},\hat{\ell}}|x-y| \quad \forall x,y \in \mathbb{R}^d. \tag{5.9}$$

For any initial condition $(x,n) \in S$, and each pair of controls $(v(\cdot),\{k_i\}) \in \mathcal{V} \times \mathcal{K}$, let us associate to the functions $\hat{f}$ and $\hat{\ell}$ the total cost

$$\hat{J}\big(x,n;v(\cdot),\{k_i\}\big) = \int_0^\infty e^{-\alpha t}\ \hat{f}(x(t),n(t),v(t))dt + \sum_{i=0}^\infty e^{-\alpha t_i}\ \hat{\ell}(x(t_i-),n(t_i-),k_i), \tag{5.10}$$

as well as its corresponding value function

$$\hat{u}(x,n) = \inf_{\big(v(\cdot),\{k_i\}\big) \in \mathcal{V}\times\mathcal{K}} \hat{J}\big(x,n;v(\cdot),\{k_i\}\big). \tag{5.11}$$

Now let us go back to the original costs (f,ℓ) defined in (2.10). By the properties of these functions, it is easy to see that the mappings $x \mapsto f(x,n,v)$ and $x \mapsto \ell(x,n,k)$ can be approximated by Lipschitz continuous functions uniformly on compact sets.

Then, let us denote by $\{(f^m,\ell^m)\}$ a given sequence of Lipschitz continuous functions satisfying

$$\begin{cases} \sup\limits_{x\in\hat{X}} \sup\limits_{(n,v)\in\mathcal{N}\times V} \big|f^m(x,n,v) - f(x,n,v)\big| \to 0 \quad \text{and} \quad \sup\limits_{x\in\hat{X}} \sup\limits_{(n,k)\in\mathcal{N}\times\mathcal{K}} \big|\ell^m(x,n,k) - \ell(x,n,k)\big| \to 0, \\ \text{as } m \to \infty, \text{ for every compact set } \hat{X} \subset \mathbb{R}^d. \end{cases} \tag{5.12}$$

Let us impose the following condition to the value functions $u^m(x,n)$ associated to the elements of above convergent sequence:

$$\begin{cases} \text{There exists a sequence of functions } \{(f^m,\ell^m)\} \text{ satisfying (5.9) and (5.12) under which} \\ \text{the corresponding value function } u^m(x,n) \text{ in (5.11) is Lipschitz continuous in the variable } x, \\ \text{uniformly in } n \in \mathcal{N}, \text{ with constant Lipschitz } M_u^m. \end{cases} \tag{5.13}$$

• *Remark* 5.5. Assumption (5.13) may seem to be a little strong, however, as we will see later, there are situations (such as those when the boundary of the set-interface D is regular), where this condition turns out to be a consequence of our present assumptions.

We now present an ancillary result that is based on the dynamic programming principle. For a proof, we can quote Bensoussan and Menaldi [6, Corollary 3.8].

Lemma 5.6. *For any pair of functions* f^m, ℓ^m *satisfying the assumption* (5.13), *its corresponding value function* u^m *in* (5.11) *satisfies the following system of quasi-variational inequalities (QVI).*

$$\begin{aligned} &0 \le f^m(x,n,v) + \langle g(x,n,v), \partial_x u^m(x,n)\rangle - \alpha u^m(x,n) \\ &0 \le \ell^m(x,n,k) + u^m(X(x,n,k),N(x,n,k)) - u^m(x,n) \quad \textit{for almost all } x \in \mathbb{R}^d \textit{ and } \forall\ n,\ v,\ k. \end{aligned} \tag{5.14}$$

Given some $\varepsilon > 0$, we define the function $\varrho_\varepsilon : \mathbb{R}^m \mapsto \mathbb{R}$ satisfying the following

$$(i)\ \varrho_\varepsilon \geq 0, \qquad (ii)\ \varrho_\varepsilon(x) = 0 \quad \forall\ |x| \geq \varepsilon, \qquad (iii)\ \int_{B_\varepsilon(x)} \varrho_\varepsilon(x)\ dx = 1,$$

$$(iv)\ \varrho_\varepsilon \text{ is infinitely differentiable.}$$

Now consider the following function, which is defined in terms of the convolution between u^m and ϱ_ε

$$u^m_\varepsilon(x,n) = \varrho_\varepsilon * u^m(x,n) := \int_{B_\varepsilon(x)} \varrho_\varepsilon(x-y) u^m(y,n)\ dy = \int_{B_\varepsilon(0)} \varrho_\varepsilon(y) u^m(x-y,n)\ dy. \tag{5.15}$$

As a direct consequence of the definition of u^m_ε, we easily see that $\|u^m_\varepsilon\| \leq \|u^m\|$. The next result regards a regularity property of u^m_ε

Lemma 5.7. *The function u^m_ε belongs to the set $C^{1,0}_b(S)$.*

Proof. From the definition of u^m_ε, it is evident that this function is differentiable with continuous derivatives at $x \in \mathbb{R}^d$, and also the mapping $n \mapsto u^m_\varepsilon(x,n)$ is continuous.

It remains to prove that such derivatives are bounded. Then,

$$\begin{aligned}
\left|u^m_\varepsilon(x,n) - u^m_\varepsilon(y,n)\right| &= \left| \int_{B_\varepsilon(0)} \varrho_\varepsilon(z) u^m(x-z,n)\ dz - \int_{B_\varepsilon(0)} \varrho_\varepsilon(z) u^m(y-z,n)\ dz \right| = \\
&= \left| \int_{B_\varepsilon(0)} \varrho_\varepsilon(z)[u^m(x-z,n) - u^m(y-z,n)]\ dz \right| \leq \\
&\leq \int_{B_\varepsilon(0)} \varrho_\varepsilon(z) M^m_u |x-y|\ dz = M^m_u |x-y|.
\end{aligned} \tag{5.16}$$

Hence

$$\frac{\left|u^m_\varepsilon(x,n) - u^m_\varepsilon(y,n)\right|}{|x-y|} \leq M^m_u \quad \text{implying} \quad \left|\partial_x u^m_\varepsilon(x,n)\right| \leq M^m_u.$$

□

Lemma 5.8. *The value function u^m can be approximated uniformly by means of a sequence of functions u^m_ε as $\varepsilon \to 0$; i.e.,*

$$\|u^m_\varepsilon - u^m\| \to 0, \quad \text{as } \varepsilon \to 0.$$

Proof. Observe that

$$\begin{aligned}
\left|u^m_\varepsilon(x,n) - u^m(x,n)\right| &= \left| \int_{B_\varepsilon(0)} \varrho_\varepsilon(y) u^m(x-y,n)\ dy - u^m(x,n) \right| \leq \\
&\leq \int_{B_\varepsilon(0)} \varrho_\varepsilon(y) \left|u^m(x-y,n) - u^m(x,n)\right|\ dy \leq \\
&\int_{B_\varepsilon(0)} \varrho_\varepsilon(y) M^m_u |y|\ dy \leq M^m_u \varepsilon.
\end{aligned}$$

Thus,

$$\|u^m_\varepsilon - u^m\| = \sup_{(x,n)\in S} |u^m_\varepsilon(x,n) - u^m(x,n)| \leq M^m_u \varepsilon,$$

which gives $\|u^m_\varepsilon - u^m\| \to 0$ as $\varepsilon \to 0$. □

Lemma 5.9. *The function u^m_ε satisfies*

$$\begin{aligned}\gamma^m_\varepsilon &\le f^m(x,n,v) + \langle g(x,n,v), \partial_x u^m_\varepsilon(x,n)\rangle - \alpha u^m_\varepsilon(x,n) \quad \forall (x,n,v) \in S \times V,\\ \beta^m_\varepsilon &\le \ell^m(x,n,k) + u^m_\varepsilon(X(x,n,k), N(x,n,k)) - u^m_\varepsilon(x,n) \quad \forall (x,n,k) \in D \times K.\end{aligned} \tag{5.17}$$

where γ_ε and β_ε are two constants with the property of $\gamma_\varepsilon \to 0$ and $\beta_\varepsilon \to 0$ as $\varepsilon \to 0$.

Proof. First recall that the value function u^m satisfies (5.14). Then, applying to these QVI the convolution with the function ϱ_ε, we obtain

$$\begin{aligned}&0 \le f^m(x,n,v) * \varrho_\varepsilon + \langle g(x,n,v), \partial_x u^m(x,n) * \varrho_\varepsilon\rangle - \alpha u^m(x,n) * \varrho_\varepsilon \quad \text{for almost all } (x,n) \in S \text{ and } \forall\, v \in V\\ &0 \le \ell^m(x,n,k) * \varrho_\varepsilon + u^m(X(x,n,k), N(x,n,k)) * \varrho_\varepsilon - u^m(x,n) * \varrho_\varepsilon \quad \forall (x,n,k) \in D \times K.\end{aligned}$$

Then, the proof reduces to show that

$$(i)\ \|f^m * \varrho_\varepsilon - f^m\| \to 0, \qquad (ii)\ \|\langle g, \partial_x u^m\rangle * \varrho_\varepsilon - \langle g, \partial_x u^m_\varepsilon\rangle\| \to 0, \qquad (iii)\ \|\ell^m * \varrho_\varepsilon - \ell^m\| \to 0 \quad \text{as } \varepsilon \to 0.$$

Let us prove (i):

$$\begin{aligned}&\left|f^m * \varrho_\varepsilon(x,n,v) - f^m(x,n,v)\right| = \left|\int_{B_\varepsilon(0)} \varrho_\varepsilon(y) f^m(x-y,n,v)\, dy - f^m(x,n,v)\right| \le\\ &\le M^m_{f,\ell} \int_{B_\varepsilon(0)} \varrho_\varepsilon(y)|y|\, dy \le M^m_{f,\ell} \cdot \varepsilon.\end{aligned} \tag{5.18}$$

Then, the result follows by applying supremum over all $(x,n,v) \in S \setminus D \times V$. In the same way, we can prove (iii).

It remains to prove (ii): To this end, observe that

$$\begin{aligned}&\left|\langle g(x,n,v), \partial_x u^m(x,n)\rangle * \varrho_\varepsilon(x) - \langle g(x,n,v), \partial_x u^m_\varepsilon(x,n)\rangle\right| =\\ &= \left|\sum_{j=1}^d \int_{B_\varepsilon(0)} [g_j(x-y,n,v) - g_j(x,n,v)]\partial_{x_j} u^m(x-y,n)\varrho_\varepsilon(y)\, dy\right| \le\\ &\le \int_{B_\varepsilon(0)} \left\|\partial_x u^m\right\| M|y|\varrho_\varepsilon(y)\, dy \le M\, \varepsilon\, \left\|\partial_x u^m\right\|,\end{aligned} \tag{5.19}$$

where M is the constant introduced in (2.5). Hencefort (5.17) becomes true. □

We now establish the relations between the values of the linear program $\inf(P)$ with the value function $u(x,n)$ in (2.12). To begin with this, we note that all of our previous results have implied the relations

$$u(x,n) \ge \mathtt{u}(x,n) \ge \inf(W) \ge \inf(P), \tag{5.20}$$

Next theorem claims that $u(x,n) \le \inf(P)$, yielding the equality in the above relation. This result uses, in some sense, the restrictions of the dual problem (P^*).

Theorem 5.10. *Suppose that assumptions* (2.4)-(2.7),(2.10), *and* (5.13) *are satisfied. Then, for each $(x,n) \in S$, we have*

$$\inf(P) = u(x,n) \tag{5.21}$$

Proof. Let $u^m_\varepsilon(x,n)$ be the approximation of $u^m(x,n)$ established in (5.15). By (5.17), it satisfies

$$\gamma^m_\varepsilon \leq f^m(x,n,v) + \langle g(x,n,v), \partial_x u^m_\varepsilon(x,n)\rangle - \alpha u^m_\varepsilon(x,n) \quad \forall (x,n,v) \in S \times V,$$
$$\beta^m_\varepsilon \leq \ell^m(x,n,k) + u^m_\varepsilon(X(x,n,k), N(x,n,k)) - u^m_\varepsilon(x,n) \quad \forall (x,n,k) \in D \times K.$$

Recalling the definitions

$$\begin{aligned} A^v\varphi(x,n) &:= -\langle g(x,n,v), \partial_x \varphi(x,n)\rangle + \alpha\varphi(x,n), \text{and} \\ L^k\varphi(x,n) &:= \varphi(x,n) - \varphi\left(X(x,n,k), N(x,n,k)\right). \end{aligned} \tag{5.22}$$

we can rewrite

$$\begin{aligned} &\gamma^m_\varepsilon + A^v u^m_\varepsilon(x,n) \leq f^m(x,n,v) \qquad \forall (x,n,v) \in S \setminus D \times V, \text{ and} \\ &\beta^m_\varepsilon + L^k u^m_\varepsilon(x,n) \leq \ell^m(x,n,k) \quad \forall (x,n,k) \in D \times K. \end{aligned} \tag{5.23}$$

On the other hand, consider a feasible pair (μ_1, μ_2) for (P). Then, we have the following relations

$$\begin{aligned} &\langle(\mu_1,\mu_2),(f^m,\ell^m)\rangle = \langle \mu_1, f^m\rangle + \langle \mu_2, \ell^m\rangle \\ &\geq \langle \mu_1, A^v u^m_\varepsilon\rangle + \langle \mu_1, \gamma^m_\varepsilon\rangle + \langle \mu_2, L^k u^m_\varepsilon\rangle + \langle \mu_2, \beta^m_\varepsilon\rangle \quad \text{(by (5.23))} \\ &= \big\langle(\mu_1,\mu_2),(A^v, L^k)u^m_\varepsilon\big\rangle + \langle(\mu_1,\mu_2),(\gamma^m_\varepsilon, \beta^m_\varepsilon\rangle \\ &= \big\langle(A^v, L^k)^*(\mu_1,\mu_2), u^m_\varepsilon\big\rangle + \langle(\mu_1,\mu_2),(\gamma^m_\varepsilon, \beta^m_\varepsilon\rangle \qquad \text{(by (5.2))} \\ &= u^m_\varepsilon(x,n) + \langle(\mu_1,\mu_2),(\gamma^m_\varepsilon, \beta^m_\varepsilon\rangle \qquad \text{(by (5.3))} \end{aligned} \tag{5.24}$$

Letting $\varepsilon \to 0$ in this last expression, we can deduce

$$\langle(\mu_1,\mu_2),(f^m,\ell^m)\rangle \geq u^m(x,n) \tag{5.25}$$

On the other hand,

$$\begin{aligned} &|u^m(x,n) - u(x,n)| \leq \sup_{v(\cdot),\{k_i\}} \left|J^m(x,n,v(\cdot),\{k_i\}) - J(x,n,v(\cdot),\{k_i\})\right| \leq \\ &\leq \sup_{v(\cdot)} \int_0^\infty e^{-\alpha t}\left|f^m(x(t),n(t),v(t)) - f(x(t),n(t),v(t))\right| + \\ &+ \sup_{\{k_i\}} \sum_{i=0}^\infty e^{-\alpha t_i}\left|\ell^m(x(t_i-),n(t_i-),\{k_i\}) - \ell(x(t_i-),n(t_i-),\{k_i\})\right| = \\ &\to 0, \end{aligned}$$

where the last convergence follows from (5.12). Finally, by letting $m \to \infty$ in (5.25), we get

$$\langle(\mu_1,\mu_2),(f,\ell)\rangle \geq u(x,n) \qquad \forall (x,n) \in S.$$

Since (μ_1,μ_2) were chosen arbitrary, we assert $\inf(P) \geq u(x,n)$. This last inequality together with (5.20), yield (5.21). □

As a consequence of the above result, we can deduce the following.

• *Remark* 5.11. The optimal control problem (2.12) is equivalent to its relaxed counterpart defined in (3.4).

Transversality case: We conclude this section by showing that, under the transversality condition (3.5), it is possible to deduce the assumption (5.13).

Firstly, we assert that the transversality condition provides a regularity of the trajectory $t \mapsto (x(t), n(t))$ with respect to the initial data (x, n). To be more specific, condition (3.5) allows the construction of a function, $\psi : S \smallsetminus D \mapsto \mathbb{R}$, satisfying, in some appropriate case (i.e., viscosity, distribution or semigroup sense) the inequality

$$\begin{cases} \langle g(x,n,v), \partial_x \psi(x,n)\rangle - \alpha\psi(x,n) \leq -1 & \forall\ (x,n) \in S \smallsetminus D, \\ \psi(x,n) = 0 & \forall\ (x,n) \in D. \end{cases}$$

Then, denoting by t_i and t'_i the i-th exit from the region $S \smallsetminus D$ subject to the trajectory has as initial conditions (x, n) and (x', n), respectively, we can obtain a continuity of these times in the following sense (see Bensoussan and Menaldi [6, p. 415])

$$|e^{-\alpha t_i} - e^{-\alpha t'_i}| \leq \tilde{C}\big[|x - x'| + |n - n|\big], \quad \text{for some positive constant } \tilde{C}. \tag{5.26}$$

By using the continuity (5.26), under a suitable induction procedure, similar to that given in the proof of Proposition 3.2 —see Section 6, it is possible to get a Lipschitz continuity applied to the continuous-type variable $x(\cdot)$ of either (6.2) or (6.4); i.e.,

$$\big|x(t) - x'(t)\big| \leq \tilde{M}|x - x'|, \quad \text{for some positive constant } \tilde{M}, \quad \forall\ t \geq 0. \tag{5.27}$$

As a consequence of the above result, we have:

Proposition 5.12. *For every pair of functions* $(\hat{f}, \hat{\ell})$*, satisfying the Lipschitz condition* (5.9)*, its corresponding value function* $\hat{u}(x, n)$ *is Lipschitz continuous, at the variable* x*, with associate constant* $M_{\hat{u}} := \tilde{M}M_{f^m,\hat{\ell}}[1/\alpha + 1/(1 - e^{-\alpha \mathrm{h}})]$.

Proof. To prove this result, let us use the notation (x^z, n^p) to emphasize that the trajectory $x(\cdot), n(\cdot)$ has begun at state $(x(0), n(0)) = (z, p)$. With this in mind, we let the estimation

$$\begin{aligned} |\hat{J}(z,p,v(\cdot),\{k_i\}) - \hat{J}(y,p,v(\cdot),\{k_i\})| &= \bigg| \int_0^\infty e^{-\alpha t} \hat{f}(x^z(t), n^p(t), v(t))dt + \sum_{i=0}^\infty e^{-\alpha t_i}\hat{\ell}(x^z(t_i-), n^p(t_i-), k_i) - \\ &\quad - \int_0^\infty e^{-\alpha t} \hat{f}(x^y(t), n^p(t), v(t))dt + \sum_{i=0}^\infty e^{-\alpha t_i}\hat{\ell}(x^y(t_i-), n^p(t_i-), k_i)\bigg| \leq \\ \leq \int_0^\infty e^{-\alpha t}\big|\hat{f}(x^z(t), n^p(t), v(t)) - \hat{f}(x^y(t), n^p(t), v(t))\big|dt + &\\ + \sum_{i=0}^\infty e^{-\alpha t_i}\big|\hat{\ell}(x^z(t_i-), n^p(t_i-), k_i) - \hat{\ell}(x^y(t_i), n^p(t_i), k_i)\big| &\\ \leq \int_0^\infty e^{-\alpha t} M_{\hat{f},\hat{\ell}}\big|x^z(t) - x^y(t)\big|dt + \sum_{i=0}^\infty e^{-\alpha t_i} M_{\hat{f},\hat{\ell}}\big|x^z(t_i-) - x^y(t_i-)\big| & \end{aligned} \tag{5.28}$$

By using the estimation (5.27), we know that

$$|x^z(t) - x^y(t)| \leq \tilde{M}|z - y|, \quad \text{for some constant } \tilde{M},$$

so we can conclude

$$|\hat{J}(z,p,v(\cdot),\{k_i\}) - \hat{J}(y,p,v(\cdot),\{k_i\})| \leq \tilde{M}M_{\hat{f},\hat{\ell}}\left[\frac{1}{\alpha} + \frac{1}{1 - e^{-\alpha \mathrm{h}}}\right]|x - y|. \tag{5.29}$$

Thus,

$$|\hat{u}(z,p)-\hat{u}(y,p)| \leq \sup_{v(\cdot),\{k_i\}} \left\{ |\hat{J}(z,p,v(\cdot),\{k_i\}) - \hat{J}(y,p,v(\cdot),\{k_i\})| \right\} \leq M_{\hat{u}}|x-y|,$$

with $M_{\hat{u}} := \tilde{M} M_{\hat{f},\hat{\ell}}[1/\alpha + 1/(1-e^{-\alpha \mathtt{h}})]$. □

6 Appendix: Proof of Proposition 3.2

Our first step is to rewrite the dynamic (3.2) as follows: For any fixed $(\mathtt{v},\{k_i\}) \in \mathtt{V} \times \mathcal{K}$, and a fixed initial condition (x_0,n_0), we define either:

Case 1: $(x_0,n_0) \in S \setminus D$.

$$\begin{aligned} &\big(x_i(t),n_i(t)\big) = \left(x_i(t_i) + \int_{t_i}^{t}\int_V g(x_i(s),n_i(s),v)\mathtt{v}_s(dv)ds \ , \ n_i(t_i)\right) \quad \text{for} \quad t \geq 0, \\ &(x_i(t_i),n_i(t_i)) = \big(X(x_{i-1}(t_i-),n_{i-1}(t_i-),k_i), N(x_{i-1}(t_i-),n_{i-1}(t_i-),k_i)\big), \\ &t_{i+1} := \inf\big\{t \geq t_i : (x_i(t-),n_i(t_i)) \in D\big\}, \quad \text{when } t_i < \infty, \quad \forall\ i = 0,1,\cdots, \\ &\big(X(x_{-1}(t_0-),n_{-1}(t_0-),k_0), N(x_{-1}(t_0-),n_{-1}(t_0-),k_0)\big) := (x_0,n_0), \quad t_0 = 0. \end{aligned} \tag{6.1}$$

In this case, (3.2) is equivalent to

$$(x(t),n(t)) = \sum_{i=0}^{\infty} (x_i(t),n_i(t))\mathbf{1}_{[t_i,t_{i+1}[}(t), \quad (x_0(t_0),n_0(t_0)) = (x_0,n_0) \quad \text{or} \tag{6.2}$$

Case 2: $(x_0,n_0) \in D$.

$$\begin{aligned} &\big(x_i(t),n_i(t)\big) = \left(x_i(t_i) + \int_{t_i}^{t}\int_V g(x_i(s),n_i(s),v)\mathtt{v}_s(dv)ds \ , \ n_i(t_i)\right) \quad \text{for} \quad t \geq 0, \\ &(x_i(t_i),n_i(t_i)) = \big(X(x_{i-1}(t_i-),n_{i-1}(t_i-),k_i), N(x_{i-1}(t_i-),n_{i-1}(t_i-),k_i)\big), \\ &t_{i+1} := \inf\big\{t \geq t_i : (x_i(t-),n_i(t_i)) \in D\big\}, \quad \text{when } t_i < \infty, \quad \forall\ i = 1,2,\cdots, \\ &\big(X(x_0(t_1-),n_0(t_1-),k_1), N(x_0(t_1-),n_0(t_1-),k_1)\big) := \big(X(x_0,n_0,k_1), N(x_0,n_0,k_1)\big), \ t_1 = 0, \end{aligned} \tag{6.3}$$

for which (3.2) turns to

$$(x(t),n(t)) = \sum_{i=1}^{\infty} (x_i(t),n_i(t))\mathbf{1}_{[t_i,t_{i+1}[}(t), \quad (x_1(t_1),n_1(t_1)) = (X(x_0,n_0,k_1), N(x_0,n_0,k_1)). \tag{6.4}$$

Now take a sequence of controls $\{(\mathtt{v}^m,\{k_i^m\})\}_m$ and denote by $(x_i^m(\cdot),n_i^m(\cdot))$ the trajectory in either (6.1) or (6.3) when the control pair $(\mathtt{v}^m,\{k_i^m\})$ is applied. Our first step consists in proving that, the convergence $(\mathtt{v}^m,\{k_i^m\}) \to (\mathtt{v}^\infty,\{k_i^\infty\})$ implies the existence of a trajectory $(x_0^\infty(\cdot),n_0^\infty(\cdot))$ satisfying again, either (6.1) or (6.3), respectively and such that $(x_0^m(\cdot),n_0^m(\cdot)) \to (x_0^\infty(\cdot),n_0^\infty(\cdot))$ locally uniformly as $m \to \infty$. This last trajectory governed by the pair $(\mathtt{v}^\infty,\{k_i^\infty\})$.

Let us proceed to prove case 1: To begin, we define the sequence

$$\big\{\big(X(x_{-1}^m(t_0-),n_{-1}^m(t_0-),k_0^m), N(x_{-1}^m(t_0-),n_{-1}^m(t_0-),k_0^m)\big)\big\},$$

with elements

$$\big(X(x_{-1}^m(t_0-),n_{-1}^m(t_0-),k_0^m), N(x_{-1}^m(t_0-),n_{-1}^m(t_0-),k_0^m)\big) := (x_0,n_0) \quad \forall\ m \geq 1.$$

This implies

$$\lim_{m\to\infty}\big(X(x_{-1}^m(t_0-),n_{-1}^m(t_0-),k_0^m),N(x_{-1}^m(t_0-),n_{-1}^m(t_0-),k_0^m)\big)=(x_0,n_0). \tag{6.5}$$

so, we define

$$\big(X(x_{-1}^\infty(t_0-),n_{-1}^\infty(t_0-),k_0),N(x_{-1}^\infty(t_0-),n_{-1}^\infty(t_0-),k_0)\big):=\\ \lim_{m\to\infty}\big(X(x_{-1}^m(t_0-),n_{-1}^m(t_0-),k_0^m),N(x_{-1}^m(t_0-),n_{-1}^m(t_0-),k_0)\big)$$

Now use (6.1) only for the case $i=0$; i.e., for every $t\geq 0$ and $t_0=0$,

$$\big(x_0^m(t),n_0^m(t)\big)=\\ =\bigg(X(x_{-1}^m(t_0-),n_{-1}^m(t_0-),k_0^m)+\int_0^t\int_V g(x_0^m(s),n_0^m(s),v)\mathrm{v}_s^m(dv)ds\ ,\ N(x_{-1}^m(t_0-),n_{-1}^m(t_0-),k_0^m)\bigg). \tag{6.6}$$

It is clear that

$$n_0^m(t)=N(x_{-1}^m(t_0-),n_{-1}^m(t_0-),k_0^m)=n_0\quad\forall\ m\geq 1\ \forall\ t\geq 0, \tag{6.7}$$

which implies that $\lim_m n_0^m(t)=n_0$, for all $t\geq 0$. Thus, we define $n_0^\infty(t):=\lim_m n_0^m(t)=n_0$.

On the other hand, since by (2.5) the vector field $g(x,n,v)$ is bounded uniformly on $S\times V$, then for all $m\geq 1$, $\int_V g(x,n,v)\mathrm{v}_t^m(dv)$ is bounded too. Now fix some $T>0$. According to (6.6), for all $0\leq s<t\leq T$,

$$|x_0^m(t)-x_0^m(s)|=\left|\int_s^t\int_V g(x_0^m(r),n_0^m(r),v)\mathrm{v}_r^m(dv)dr\right|\leq M(t-s)\quad\forall m\geq 1, \tag{6.8}$$

where M is the constant defined in (2.5). Also observe that for all $m\geq 1$ and all $0\leq t\leq T$,

$$|x_0^m(t)|\leq Mt+|x_0^m(0)|\leq MT+|X(x_{-1}^m(t_0-),n_{-1}^m(t_0-),k_0^m)|=MT+x_0<+\infty. \tag{6.9}$$

From (6.8) and (6.9), the family $\{x_0^m(\cdot)\}$ is equicontinuous and bounded, then by Arzelà-Ascoli theorem, it is relatively uniformly compact on $[0,T]$; that is, there exists a (uniform) convergent subsequence of $\{x_0^m(\cdot)\}$ (denoted again as the original sequence) such that $x_0^m(\cdot)\to x_0^\infty(\cdot)\in C_b([0,T])$, uniformly.

Let us check now that this limit term satisfies the first sub-dynamic in (6.6) for the case $m=+\infty$. Namely, we write

$$\begin{aligned} x_0^m(t)&=x_0^m(0)+\int_0^t\int_V g(x_0^m(s),n_0^m(s),v)\mathrm{v}_s^m(dv)\ ds=\\ &=x_0^m(0)+\int_0^t\int_V\big[g(x_0^m(s),n_0^m(s),v)-g(x_0^\infty(s),n_0^\infty(s),v)\big]\mathrm{v}_s^m(dv)\ ds+\\ &+\int_0^t\int_V g(x_0^\infty(s),n_0^\infty(s),v)\mathrm{v}_s^m(dv)\ ds.\quad\forall\ 0\leq t\leq T. \end{aligned} \tag{6.10}$$

The continuity of both mappings $t\mapsto x_0^\infty(t)$ and $(x,n,v)\mapsto g(x,n,v)$ yield the continuity of $(t,v)\mapsto g(x_0^\infty(t),n_0^\infty(t),v)$. Then, based on Proposition 3.1, we deduce

$$\int_0^t\int_V g(x_0^\infty(s),n_0^\infty,v)\mathrm{v}_s^m(dv)\ ds\to\int_0^t\int_V g(x_0^\infty(s),n_0^\infty,v)\mathrm{v}_s^\infty(dv)\ ds\quad\text{as } m\to\infty\quad\forall\ 0\leq t\leq T. \tag{6.11}$$

Using the Lipschitz property of the vector field g, we can easily verify that the second term in the right-hand side of (6.10) goes to zero as $m\to\infty$. This last fact together with (6.11) and (6.7), yield

$$\big(x_0^m(t),n_0^m(t)\big)\to\big(x_0^\infty(t),n_0^\infty(t)\big)=\bigg(x_0+\int_0^t\int_V g(x_0^\infty(s),n_0^\infty(s),v)\mathrm{v}_s^\infty(dv)ds\ ,\ n_0\bigg). \tag{6.12}$$

Let us now analyze the first times when the sequence of process $\{(x_0^m(\cdot), n_0^m(\cdot))\}$ reaches the set interface D, for each $m \geq 0$ (a.k.a. exit times from the region $S \setminus D$), and study the convergence of such times. To this purpose and noting that $n_0^m(t) = n_0$ for all $t \geq 0$ and $m \geq 1$, we define

$$t_1^m := \inf\{t \geq 0 : (x_0^m(t-), n_0) \in D\} \quad \text{and} \quad t_1^\infty := \inf\{t \geq 0 : (x_0^\infty(t-), n_0) \in D\}. \tag{6.13}$$

Recalling the definition of D_{n_0} in (2.3), the above times can be regarded by

$$t_1^m = \inf\{t \geq 0 : x_0^m(t-) \in D_{n_0}\} \quad \text{and} \quad t_1^\infty := \inf\{t \geq 0 : x_0^\infty(t-) \in D_{n_0}\} \tag{6.14}$$

Since $(x_0, n_0) \in S \setminus D$, (which implies $x_0 \in \mathbb{R}^d \setminus D_{n_0}$), then $t_1^\infty > 0$. Therefore, there exists a positive constant $\hat{s}$ with the property of $0 < \hat{s} < t_1^\infty$. By the properties of $x_0^\infty(\cdot)$, we see that the trajectory $t \mapsto x_0^\infty(t)$ belongs to $\mathbb{R}^d \setminus D_{n_0}$ on $[0, \hat{s}]$. Observe that $\{x_0^\infty(s) : s \in [0, \hat{s}]\}$ is a compact set and it is contained in the open set $\mathbb{R}^d \setminus D_{n_0}$. Then, by using the uniform convergence of $x_0^m(\cdot) \to x_0^\infty(\cdot)$, we can ensure the existence of some natural number M such that for all $m \geq M$, the set $\{x_0^\infty(t), x_0^m(t) : 0 \leq t \leq \hat{s}\}$ belongs also to $\mathbb{R}^d \setminus D_{n_0}$, yielding that $t_1^m \geq \hat{s}$, for $m \geq M$. Hence, $\liminf_m t_1^m \geq \hat{s}$. Since $\hat{s}$ was chosen arbitrary, we can take this constant close enough to t_1^∞. Implying that $\liminf_m t_1^m \geq t_1^\infty$.

Note that if $t_1^\infty = +\infty$, then $\liminf_m t_1^m = +\infty$ and thus the proof would follow by applying the convergence (6.12) on the interval $[0, T]$, for every $T > 0$. From the above reason, we will focus now to the case $t_1^\infty < +\infty$; namely, using the assumptions in (3.5), for any $t > t_1^\infty$, there exists $\hat{t} < t$ such that $x_0^\infty(\hat{t}) \in \mathring{D}_{n_0}$. In virtue that $\mathring{D}_{n_0}$ is open, the convergence $x_0^m(\cdot) \to x_0^\infty(\cdot)$ (in this case we must take the constant T in Arzelà-Ascoli theorem greater or equal to $\hat{t}$) ensures the existence of some constant M large enough such that for all $m \geq M$, we can guarantee $x_0^m(\hat{t}) \in \mathring{D}_{n_0}$. Then, $t_1^m \leq \hat{t}$ and so $\limsup_m t_1^m \leq \hat{t}$. Finally, taking $\hat{t}$ close to t_1^∞, we can deduce that $\limsup_m t_1^m \leq t_1^\infty$. Combining the previous arguments, we affirm

$$\lim_{m\to\infty} t_1^m \to t_1^\infty. \tag{6.15}$$

The convergence in (6.15) together with the previous uniform convergence of $(x_0(\cdot), n_0(\cdot))$, imply that

$$(x_0^\infty(t_1^\infty-), n_0^\infty(t_1^\infty-)) = \lim_{m\to\infty} (x_0^m(t_1^m-), n_0^m(t_1^m-)). \tag{6.16}$$

Next, in virtue of the continuity of the mappings X, N and the convergences in (6.16) and $\{k_i^m\} \to \{k_i^\infty\}$, we deduce

$$(X, N)\big(x_0^m(t_1^m-), n_0^m(t_1^m-), k_1^m\big) \to (X, N)\big(x_0^\infty(t_1^\infty-), n_0^\infty(t_1^\infty-), k_1^\infty\big) \quad \text{as } m \to \infty. \tag{6.17}$$

In general, for the case $i \geq 1$, we firstly apply similar arguments as in (6.6) to obtain

$$\begin{aligned} \big(x_i^m(t), n_i^m(t)\big) = \Bigg(& X(x_{i-1}^m(t_i^m-), n_{i-1}^m(t_i^m-), k_i^m) + \\ + \int_{t_i^m}^t \int_V g(x_i^m(s), n_i^m(s), v) \mathrm{v}_s^m(dv) ds \ , \ & N(x_{i-1}^m(t_i^m-), n_{i-1}^m(t_i^m-), k_i^m) \Bigg) \quad \forall\, t \geq t_i^m \ \forall\, m \geq 1. \end{aligned} \tag{6.18}$$

Furthermore, the process $n(\cdot)$ behaves as

$$n_i^m(t) = N(x_{i-1}^m(t_i^m-), n_{i-1}^m(t_i^m-), k_i^m) \quad \forall\, t \geq t_i^m \ \forall\, m \geq 1; \tag{6.19}$$

Since the sequence $\{t_i^m\}$ is convergent, we can define $\bar{t}_i := \inf_m t_i^m$. Using this number, we can define a continuous extension of the trajectory $(x_i^m(t), n_i^m(t))$ on $[\bar{t}_i, +\infty[$, by letting

$$(x_i^m(t), n_i^m(t)) := \begin{cases} (x_i^m(t_i^m), n_i^m(t_i^m)) & \text{on } [\bar{t}_i, t_i^m[\\ (x_i^m(t), n_i^m(t_i^m)) & \text{on } [t_i^m, +\infty[. \end{cases} \tag{6.20}$$

Similar to the convergence (6.17), and the convergence $k_i^m \to k_i^\infty$, we can deduce

$$n_i^m(t) = N(x_{i-1}^m(t_i^m-), n_{i-1}^m(t_i^m-), k_i^m) \to N(x_{i-1}^\infty(t_i^\infty-), n_{i-1}^\infty(t_i^\infty-), k_i^\infty) = n_i^\infty(t), \quad \forall\, t \geq \bar{t}_i, \tag{6.21}$$

and this convergence is uniform.

On the other hand, using again (2.5), we can claim that the dynamic $x(\cdot)$ has the following properties. For all $T > \bar{t}_i$, we have

$$|x_i^m(t) - x_i^m(s)| \leq \int_s^t \int_V |g(x_i^m(r), n_i^m(r), v)| \mathrm{v}_r^m(dv)dr \leq M(t-s), \quad \forall\, \bar{t}_i \leq s < t \leq T, \tag{6.22}$$

and

$$\begin{aligned} |x_i^m(t)| \leq M(T-\bar{t}_i) + |X(x_{i-1}^m(\bar{t}_i-), n_{i-1}^m(\bar{t}_i-), k_i^m)| \leq \\ \leq MT + \sup_{m\geq 1} |X(x_{i-1}^m(\bar{t}_i-), n_{i-1}^m(\bar{t}_i-), k_i^m)| < +\infty, \quad \forall\, \bar{t}_i \leq t \leq T. \end{aligned} \tag{6.23}$$

where the last term is bounded due to the convergences

$$(x_{i-1}^\infty(t_i^\infty-), n_{i-1}^\infty(t_i^\infty-)) = \lim_{m\to\infty} (x_{i-1}^m(t_i^m-), n_{i-1}^m(t_i^m-)).$$

and

$$(X,N)\big(x_{i-1}^m(t_i^m-), n_{i-1}^m(t_i^m-), k_i^m\big) \to (X,N)\big(x_{i-1}^\infty(t_i^\infty-), n_{i-1}^\infty(t_i^\infty-), k_i^\infty\big) \quad \text{as } m \to \infty, \tag{6.24}$$

Hence, by the simple use of Arzelà- Ascoli theorem, there exists a uniformly convergent subsequence $\{x_i^m(\cdot)\}$ (denoted again as the original sequence) such that $x_i^m(\cdot) \to x_i^\infty(\cdot)$ on the interval on $[\bar{t}_i, T]$, in particular on $[t_i^\infty, T]$.

To verify that this limit term satisfies the first sub-dynamic in (6.18) for the case $m = +\infty$, we proceed as follows: From (6.18) and the continuous extension (6.20) we know that

$$x_i^m(t) = X(x_{i-1}^m(t_i^m-), n_{i-1}^m(t_i^m-), k_i^m) + \int_{t_i^m}^t \int_V g(x_i^m(s), n_i^m(s), v)\mathrm{v}_s^m(dv)ds. \tag{6.25}$$

By following the same steps as in the 0-th case, we can prove that

$$\int_{t_i^m}^t \int_V g(x_i^m(s), n_i^m(s), v)\mathrm{v}_s^m(dv)ds \to \int_{t_i^\infty}^t \int_V g(x_i^\infty(s), n_i^\infty(s), v)\mathrm{v}_s^\infty(dv)ds \quad \text{as } m \to \infty. \tag{6.26}$$

Then, by letting $m \to \infty$, the convergence in (6.26) together with (6.24), yield

$$\begin{aligned} &\big(x_i^m(t), n_i^m(t)\big) \to \big(x_i^\infty(t), n_i^\infty(t)\big) = \Bigg(X\big(x_{i-1}^\infty(t_i^\infty-), n_{i-1}^\infty(t_i^\infty-), k_i^\infty\big) + \\ &\int_{t_i^\infty}^t \int_V g(x_i^\infty(s), n_i^\infty(s), v)\mathrm{v}_s(dv)ds \ , \ N\big(x_{i-1}^\infty(t_i^\infty-), n_{i-1}^\infty(t_i^\infty-), k_i^\infty\big)\Bigg), \quad \forall\, t_i^\infty \leq t \leq T. \end{aligned} \tag{6.27}$$

Let us now analyze the first times (exit times) when the sequence of process $\{(x_i^m(\cdot), n_i^m(\cdot))\}$ is outside the set $S \smallsetminus D$, for each $m \geq 0$ and study the convergence these times. To this purpose, we define these exit times as

$$t_{i+1}^m := \inf\big\{t \geq t_i^m : \big(x_i^m(t-), n_i^m(t_i^m)\big) \in D\big\}, \quad \text{and} \quad t_{i+1}^\infty := \inf\big\{t \geq t_i^\infty : \big(x_i^\infty(t-), n_i^\infty(t_i^\infty)\big) \in D\big\}. \tag{6.28}$$

Our aim is to prove the convergence (6.29) below. Indeed, it is clear that $(x_i^\infty(t_i^\infty), n_i^\infty(t_i^\infty)) \in S \setminus D$, then $t_{i+1}^\infty > 0$. As a consequence, there exists a positive constant $\hat{s}$ satisfying $0 < \hat{s} < t_{i+1}^\infty$.

Using the continuous extension (6.20) it is easy to see that $(x_i^\infty(t), n_i^\infty(t)) \in S \setminus D$, for all $t \in [\hat{t}_i, \hat{s}]$. Furthermore, $\{(x_i^\infty(s), n_i^\infty(s) : s \in [\bar{t}_i, \hat{s}]\} \in S \setminus D$. Since $S \setminus D$ is open, we can use the uniform convergence of $(x_i^m(\cdot), n_i^m(\cdot)) \to (x_i^\infty(\cdot), n_i^\infty(\cdot))$ on $[\bar{t}_i, \hat{s}]$ to deduce the existence of some natural number M such that for all $m \geq M$, the set $\{(x_i^m(t), n_i^m(t)), (x_i^\infty(t), n_i^\infty(t)) : \bar{t}_i \leq t \leq \hat{s}\}$ is all contained in $S \setminus D$. This implies that $\hat{t}_{i+1}^m \geq \hat{s}$, for $m \geq M$, yielding that $\liminf_m \hat{t}_{i+1}^m \geq \hat{s}$. Since $\hat{s}$ was taken arbitrary, we can take this constant close enough to t_{i+1}^∞. Implying that $\liminf_m \hat{t}_{i+1}^m \geq t_{i+1}^\infty$.

The proof for the converse inequality $\limsup_m \hat{t}_{i+1}^m \leq t_{i+1}^\infty$ is identical to the 0-th case, so we shall omit it.

If $t_{i+1}^\infty = +\infty$, then $\liminf_m t_{i+1}^m = +\infty$ and thus the proof would follow by applying the convergence (6.27) on the interval $[t_i^\infty, T]$, for every $T > 0$.

Combining the previous arguments, we can deduce

$$\lim_{m\to\infty} t_{i+1}^m \to t_{i+1}^\infty, \tag{6.29}$$

Again, the convergence in (6.29) together with the uniform convergence of $(x_i^m(\cdot), n_i^m(\cdot)) \to (x_i^\infty(\cdot), n_i^\infty(\cdot))$ both imply that

$$\big(x_i^\infty(t_{i+1}^\infty-), n_i^\infty(t_{i+1}^\infty-)\big) = \lim_{m\to\infty} \big(x_i^m(t_{i+1}^m-), n_i^m(t_{i+1}^m-)\big), \tag{6.30}$$

yielding to the following convergence

$$(X,N)\big(x_i^m(t_{i+1}^m-), n_i^m(t_{i+1}^m-), k_{i+1}^m\big) \to (X,N)\big(x_i^\infty(t_{i+1}^\infty-), n_i^\infty(t_{i+1}^\infty-), k_{i+1}^\infty\big) \quad \text{as } m \to \infty, \tag{6.31}$$

and so on...

Now take a sequence of processes as in (6.2); i.e.,

$$(x^m(t), n^m(t)) = \sum_{i=0}^{\infty} (x_i^m(t), n_i^m(t)) \mathbf{1}_{[t_i^m, t_{i+1}^m[}(t), \quad (x_0^m(t_0), n_0^m(t_0)) = (x_0, n_0) \tag{6.32}$$

and define the limit trajectory

$$(x^\infty(t), n^\infty(t)) = \sum_{i=0}^{\infty} (x_i^\infty(t), n_i^\infty(t)) \mathbf{1}_{[t_i^\infty, t_{i+1}^\infty[}(t), \quad (x_0^\infty(t_0), n_0^\infty(t_0)) = (x_0, n_0) \tag{6.33}$$

By construction, $(x^m(t), n^m(t)) \to (x^\infty(t), n^\infty(t))$ uniformly on each interval $[t_i^\infty, t_{i+1}^\infty[$; in other words, the above convergence is locally uniformly in almost every point of $[0, \infty[$.

Finally, to prove Case 2; i.e., the case when $(x_0, n_0) \in D$, we use the previous steps but starting the analysis from step (6.17) and then follow the rest of the proof of Case 1. □

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