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# GENERALIZED LAME-CLAPEYRON SOLUTION FOR A ONE-PHASE SOURCE STEFAN PROBLEM

José Luis Menaldi <sup>\*</sup>      Domingo Alberto Tarzia <sup>†</sup>

## ABSTRACT

In this paper we obtain a generalized Lamé-Clapeyron solution for a one-phase Stefan problem with a particular type of sources. Necessary and sufficient conditions are given in order to characterize the source term which provides a unique solution. Some estimates on the free boundary and the temperature are presented. In particular, asymptotic expansions are given for small Stefan number and source.

**KEY WORDS** Stefan problem, similarity variable, Lamé-Clapeyron solution, phase-change problem, free boundary problems, exact solutions, asymptotic expansions, quasi-steady-state method.

**1985 AMS Subject Classification** 35R35, 80A20, 35K05, 35C05.

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# I. Introduction

It is well-known the classic Lamé-Clapeyron solution for the one-phase Stefan problem corresponding to a semi-infinite material with constant thermal coefficients [1, 7, 13]. Without loss of generality, we suppose the phase-change temperature is  $0^\circ C$ . We consider the following fusion problem: the material is initially in solid phase at the melting temperature and for all possible instant we have a constant temperature  $B > 0$  on the fixed face  $x = 0$ . The problem consists in finding the liquid-solid interface (free boundary)  $x = s(t) > 0$ , defined for  $t > 0$  with  $s(0) = 0$ , and the liquid temperature  $\theta = \theta(x, t) > 0$ , defined for  $0 < x < s(t)$ ,  $t > 0$ , such that they satisfy the following conditions:

- (1) (i)  $\rho c \theta_t - k \theta_{xx} = 0, 0 < x < s(t), t > 0$ ,
- (ii)  $\theta(0, t) = B > 0, t > 0$ ,
- (iii)  $\theta(s(t), t) = 0, t > 0$ ,
- (iv)  $k \theta_x(s(t), t) = -\rho \ell \dot{s}(t), t > 0$ ,
- (v)  $s(0) = 0$ ,

where  $k > 0$  is the thermal conductivity,  $\rho > 0$  the mass density,  $c > 0$  the specific heat and  $\ell > 0$  the latent heat of fusion.

We denote by  $\alpha = \frac{k}{\rho c} > 0$  the diffusion coefficient and  $Ste = \frac{Bc}{\ell} > 0$  the Stefan number. We shall denote  $a = \sqrt{\alpha} = \sqrt{\frac{k}{\rho c}} > 0$ , for convenience in the notation.

The solution of problem (1) is given by [1, 2, 7, 9, 12]:

$$(2) \quad s_0(t) = 2a\xi_0\sqrt{t}, \quad \theta_0(x, t) = B\left[1 - \frac{1}{\operatorname{erf}(\xi_0)} \operatorname{erf}\left(\frac{x}{2a\sqrt{t}}\right)\right],$$

where  $\xi_0 > 0$  is the unique solution of the following equation:

$$(3) \quad F_0(x) = \frac{Ste}{\sqrt{\pi}}, \quad x > 0$$

with the notation

$$(4) \quad F_0(x) = x \exp(x^2) \operatorname{erf}(x),$$

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-u^2) du \quad (\text{error function}).$$

In the present paper, we consider a generalization of the equation (1-i) by

$$(1-i-bis) \quad \rho c \theta_t - k \theta_{xx} = g(x, t), \quad 0 < x < s(t), \quad t > 0,$$

where  $g = g(x, t)$  represents a source term for the heat equation. We shall denote with (P) the problem corresponding to conditions (1-i-bis) and (1-ii  $\dots$  v). In many physical applications a volumetric heating/ cooling terms is considered. For the singular case

$$(5) \quad g(x, t) = \rho \ell \frac{1}{t} \beta\left(\frac{x}{2a\sqrt{t}}\right),$$

where  $\beta = \beta(\eta)$  is any function with appropriate regularity properties (for instance,  $\beta \in$  either  $C^0(\mathfrak{R}^+)$  or  $C^1(\mathfrak{R}^+)$ ), we prove necessary and sufficient conditions for the function  $\beta$  to have an exact solution for problem (P). This exact solution can be considered as a generalized Lamé-Clapeyron solution for the one-phase Stefan problem (P) with  $\beta \neq 0$ . Because of the singularity of the source, (P) can be regarded as an ill-posed problem. For a general theory on one-phase Stefan problem we refer to [1, 4, 5, 9].

We shall denote by  $(N - n)$  the formula  $(n)$  of Section  $N$  and we shall omit  $N$  in the same paragraph. Idem for theorems, lemmas, corollaries, remarks and notes.

In §II, we characterize the set of function  $\beta$  for which a unique generalized Lamé-Clapeyron solution exists, given by (II-10), (II-11) and (II-12). For that we define a function  $Z$  by (II-30) or (II-33), which permit us to obtain a unique solution, as well as either several solutions or no solutions at all. Similar situations has been recently found in other similarity solutions for the Stefan problem [6].

In §III, we consider the problem for small Stefan number  $Ste \ll 1$  which is characterized for the quasi-steady-state solution when function  $\beta$  is constant and given by  $\beta(\eta) = \beta(0) < 1$ .

In §IV, we give several estimates for the temperature and free boundary.

## II. Generalized Lamé-Clapeyron Solution

We consider the following free boundary problem for the heat equation: Find the free boundary  $x = s(t) > 0$ , defined for  $t > 0$  and  $s(0) = 0$ , and the temperature  $\theta = \theta(x, t) > 0$ , defined for  $0 < x < s(t)$ ,  $t > 0$ , such that they satisfy the following conditions:

$$(P) \quad \rho c \theta_t - k \theta_{xx} = g(x, t), \quad 0 < x < s(t), \quad t > 0,$$

$$\theta(0, t) = B > 0, \quad t > 0,$$

$$\theta(s(t), t) = 0, \quad t > 0,$$

$$k\theta_x(s(t), t) = -\rho\ell\dot{s}(t), \quad t > 0, \quad s(0) = 0,$$

for a given source function  $g = g(x, t)$ , fixed face temperature  $B > 0$  and constant thermal coefficients  $k, \rho, c, \ell > 0$ .

Since our interest is finding solutions of the Lamé-Clapeyron type for problem **(P)**, we apply the immobilization domain method [3, 8, 12], that is, we are looking for solutions of the following type

$$(1) \quad \theta(x, t) = T(y)$$

where the new independent spatial variable  $y$  is defined by

$$(2) \quad y = \frac{x}{s(t)}.$$

In this case, the problem (P) is transformed as follows: *Find functions  $T = T(y)$ , defined for  $0 < y < 1$ , and  $s = s(t)$ , defined for  $t > 0$  such that*

$$(3) \quad s(t)\dot{s}(t) y T'(y) + a^2 T''(y) = -\frac{s^2(t)}{\rho c} g(y, s(t), t), \quad 0 < y < 1, \quad t > 0,$$

$$T(0) = B > 0, \quad T(1) = 0, \quad T'(1) = -\frac{\rho\ell}{k} s(t)\dot{s}(t), \quad t > 0, \quad s(0) = 0.$$

We must have necessarily that  $s(t)\dot{s}(t) = \text{Const.}$ , i.e.

$$(4) \quad s(t) = 2a\xi\sqrt{t},$$

where the dimensionless parameter  $\xi > 0$  is unknown. Then, from (3) and (4), and for the source term  $g$ , given by expression (I-5), we obtain the following problem:

$$(5) \quad T''(y) + 2\xi^2 y T'(y) = -\frac{4\xi^2}{\rho c} \beta(\xi y), \quad 0 < y < 1,$$

$$T(0) = B, \quad T(1) = 0, \quad T'(1) = -\frac{2\ell}{c} \xi^2.$$

If we define

$$(6) \quad R(\eta) = T\left(\frac{\eta}{\xi}\right) \quad (\text{or } T(y) = R(\xi y)), \quad \eta = \xi y,$$

problem (5) is equivalent to

$$(7) \quad R''(\eta) + 2\eta R'(\eta) = -\frac{4\ell}{c}\beta(\eta), \quad 0 < \eta < \xi,$$

$$R(0) = B, \quad R(\xi) = 0, \quad R'(\xi) = -\frac{2\ell}{c}\xi.$$

After some elementary computations the solution  $R = R(\eta)$  and  $\xi > 0$  of problem (7) is given explicitly by

$$(8) \quad R(\eta) = B - \frac{\ell\sqrt{\pi}}{c}\xi \exp(\xi^2) \operatorname{erf}(\eta) +$$

$$+ \frac{4\ell}{c} \int_0^\eta \left[ \int_r^\xi \beta(y) \exp(y^2) dy \right] \exp(-r^2) dr, \quad 0 < \eta < \xi,$$

where  $\xi > 0$  must verify the condition

$$(9) \quad R(\xi) = 0.$$

Taking into account (I-4), we can do an integration into condition  $R(\xi) = 0$  and we obtain that the number  $\xi > 0$  must verify the equation:

$$(10) \quad F(x, \beta) = \frac{\operatorname{Ste}}{\sqrt{\pi}}, \quad x > 0,$$

where function  $F = F(x) = F(x, \beta)$  is defined for  $x > 0$  and  $\beta = \beta(r)$  by the expression

$$(11) \quad F(x, \beta) = F_0(x) - 2 \int_0^x \exp(r^2) \operatorname{erf}(r) \beta(r) dr.$$

**Note 1.** From now on we shall note with  $x > 0$  the dimensionless first variable of function  $F$  and the spatial variable for the temperature  $\theta$ .  $\square$

Therefore, we have obtained the following abstract theorem in the case where equation (10) has at least one solution  $\xi > 0$ .

**THEOREM 1.** An explicit solution of problem (P) with the source term  $g$ , defined by (I-5), as function of  $\beta$ , is given by:

$$(12) \quad \theta(x, t) = B \left\{ 1 - \frac{\sqrt{\pi}}{\operatorname{Ste}} \xi \exp(\xi^2) \operatorname{erf}(\eta) + \right.$$

$$\left. + \frac{4}{\operatorname{Ste}} \int_0^\eta \left[ \int_r^\xi \beta(y) \exp(y^2) dy \right] \exp(-r^2) dr \right\},$$

$$s(t) = 2a\xi\sqrt{t}, \quad \eta = \frac{x}{2a\sqrt{t}} \in (0, \xi),$$

where the number  $\xi > 0$  is a solution of equation (10).  $\square$

**Remark 1.** Function  $F$  satisfies the following properties:

$$(13) \quad F(0^+, \beta) = 0,$$

$$(14) \quad \frac{\partial F}{\partial x}(x, \beta) = \frac{2x}{\sqrt{\pi}} + \exp(x^2) \operatorname{erf}(x)[1 + 2x^2 - 2\beta(x)],$$

$$(15) \quad \frac{\partial^2 F}{\partial x^2}(x, \beta) = \frac{4}{\sqrt{\pi}}(1 + x^2) + 2F_0(x)(3 + 2x^2) - 4\left[\frac{1}{\sqrt{\pi}} + F_0(x)\right]\beta(x) - 2\exp(x^2) \operatorname{erf}(x)\beta'(x).$$

It is clear that (14) and (15) are meaningful when  $\beta$  is continuous and continuous differentiable, respectively. The space where  $\beta(\cdot)$  will be considered has not been yet defined.  $\square$

**LEMMA 2.** Function  $F$  satisfies the following properties:

$$(i) \quad \frac{\partial F}{\partial x}(x, \beta) > 0 \iff \beta(x) < \psi_0(x)$$

where function  $\psi_0 = \psi_0(x)$  is defined by

$$(16) \quad \psi_0(x) = \frac{1}{2} + x^2 + \frac{x}{\sqrt{\pi}}G(x), \quad G(x) = \frac{\exp(-x^2)}{\operatorname{erf}(x)}, \quad x > 0.$$

(ii)  $\frac{\partial^2 F}{\partial x^2}(x, \beta) \geq 0$ ,  $\forall x > 0$  if and only if  $\beta$  verifies the following differential inequality:

$$(17) \quad \beta'(x) + M'(x)\beta(x) \leq N(x), \quad \forall x > 0,$$

where functions  $M$  and  $N$  are defined by

$$(18) \quad \begin{aligned} M(x) &= x^2 + \log(\operatorname{erf}(x)), \quad x > 0, \\ N(x) &= x(3 + 2x^2) + \frac{2}{\sqrt{\pi}}(1 + x^2)G(x), \quad x > 0. \end{aligned}$$

(iii) The differential inequality (17) implies that function  $\beta$  necessarily satisfies the following inequality  $\beta(x) \leq \psi_{C_\beta}(x)$ ,  $\forall x > 0$ , where function  $\psi_C$  is defined by

$$(19) \quad \psi_C(x) = \psi_0(x) + CG(x), \quad x > 0 (C \in \mathfrak{R}),$$

and  $C_\beta$  is supposed to be a real number (finite) which is defined by  $C_\beta = \lim_{x \rightarrow 0^+} [\beta(x) \operatorname{erf}(x)]$ , as function of  $\beta$ .

**Proof.** By means of the following expressions.

$$\begin{aligned}
(20) \quad M(x) &= 2 \int [x + \frac{1}{\sqrt{\pi}} G(x)] dx, \exp(-M(x)) = G(x), \\
\int x \exp(x^2) \operatorname{erf}(x) dx &= \frac{1}{2} \exp(x^2) \operatorname{erf}(x) - \frac{x}{\sqrt{\pi}}, \\
\int x^3 \exp(x^2) \operatorname{erf}(x) dx &= \frac{x}{\sqrt{\pi}} - \frac{x^3}{3\sqrt{\pi}} + \frac{(x^2 - 1)}{2} \exp(x^2) \operatorname{erf}(x), \\
\int N(x) \exp(M(x)) dx &= \frac{x}{\sqrt{\pi}} + (\frac{1}{2} + x^2) \exp(x^2) \operatorname{erf}(x), \\
\exp(-M(x)) \int N(x) \exp(M(x)) dx &= \psi_0(x),
\end{aligned}$$

and some elementary computation the Lemma can be established.  $\square$

**LEMMA 3.** If function  $\beta$  verifies the inequality

$$(21) \quad \beta(x) \leq \psi_C(x), \quad \forall x > 0,$$

for some constant  $C \in \mathfrak{R}$ , then we have the following estimates:

$$\begin{aligned}
(22) \quad F(x, \beta) &\geq F(x, \psi_C) = -2Cx, \quad \forall x > 0, \\
\frac{\partial F}{\partial x}(x, \beta) &\geq \frac{\partial F}{\partial x}(x, \psi_C) = -2C, \quad \forall x > 0.
\end{aligned}$$

**Proof.** From elementary computation and by taking into account the following expressions:

$$\begin{aligned}
(23) \quad \int \operatorname{erf}(x) dx &= x \operatorname{erf}(x) + \frac{1}{\sqrt{\pi}} \exp(-x^2), \\
\int (\frac{1}{2} + x^2) \exp(x^2) \operatorname{erf}(x) dx &= \frac{F_0(x)}{2} - \frac{x^2}{2\sqrt{\pi}},
\end{aligned}$$

we can prove the Lemma.  $\square$

**Remark 2.** Let be the following real functions

$$\begin{aligned}
(24) \quad U(x) &= \frac{1}{2} + x^2 + \frac{1}{2} \exp(-x^2), \quad x > 0, \\
V(x) &= \frac{1}{x} \exp(x^2) \operatorname{erf}(x) = \frac{1}{xG(x)}, \quad x > 0, \\
\varphi_C(x) &= \frac{1}{2} + x^2 + CG(x) = \psi_C(x) - \frac{xG(x)}{\sqrt{\pi}}, \quad x > 0, (C \in \mathfrak{R}).
\end{aligned}$$



In what follows, we will use the properties :

$$(25) \quad \begin{aligned} G(0^+) &= +\infty, \quad G(+\infty) = 0, \quad G' < 0 \text{ in } \mathfrak{R}^+, \\ V(0^+) &= \frac{2}{\sqrt{\pi}}, \quad V(+\infty) = +\infty, \quad V' > 0 \text{ in } \mathfrak{R}^+, \\ U(0^+) &= 1, \quad U(+\infty) = +\infty, \quad U'(0^+) = 0, \quad U' > 0 \text{ in } \mathfrak{R}^+, \\ \psi_0(0^+) &= 1, \quad \psi_0(+\infty) = +\infty, \quad \psi'_0(0^+) = 0, \quad \psi'_0 > 0 \text{ in } \mathfrak{R}^+, \\ \frac{1}{2} + x^2 &< U(x) < \psi_0(x), \quad \forall x > 0, \end{aligned}$$

$$(26) \quad \begin{aligned} \varphi_C(0^+) &= +\infty, \quad \varphi_C(+\infty) = +\infty, \\ \lim_{C \rightarrow +\infty} [\min_{x>0} \varphi_C(x)] &= +\infty \quad (C > 0), \\ \varphi_C(0^+) &= -\infty, \quad \varphi_C(+\infty) = +\infty, \quad \varphi'_C > 0 \text{ in } \mathfrak{R}^+ \quad (C < 0), \\ \frac{1}{2} + x^2 &< \varphi_C(x) < \psi_0(x), \quad \forall x > C\sqrt{\pi} \quad (C > 0), \\ \lim_{x \rightarrow +\infty} [\psi_0(x) - (\frac{1}{2} + x^2)] &= 0. \quad \square \end{aligned}$$

Now we can prove the following first existence theorem.

**THEOREM 4.** Let  $\beta$  be a real continuous function in  $\mathfrak{R}_0^+$ .

(i) If  $\beta$  verifies the inequality

$$(27) \quad \beta(x) < \psi_0(x) - \varepsilon, \quad \forall x \geq 0 \text{ for some } \varepsilon > 0,$$

then there exists a unique number  $\xi = \xi(\text{Ste}) > 0$  which is the solution of the equation (10) for each  $\text{Ste} > 0$ .

(ii) If  $\beta$  verifies the inequality  $\beta \leq \psi_C$  in  $\mathfrak{R}^+$  for some constant  $C < 0$  then we obtain the same conclusion of part (i).

(iii) If  $\beta$  verifies the inequality  $\beta \leq \varphi_C$  in  $\mathfrak{R}^+$  for some constant  $C > 0$  then there exist an number  $\xi = \xi(\text{Ste}) > 2C\sqrt{\pi}$  which is a solution of the equation (10) for all  $\text{Ste} > 0$ .

(iv) If  $\beta \equiv \varphi_C$  in part (iii) we can also deduce the uniqueness of the number  $\xi$ .

**Proof.** (i) It is enough to apply part (i) of Lemma 2, (13) and the fact that  $F(+\infty, \beta) = +\infty$  because

$$(28) \quad F(x, \beta) \geq F(x, \psi_0 - \varepsilon) = 2\varepsilon \int_0^x \exp(r^2) \operatorname{erf}(r) dr.$$

(ii) It is a consequence of Lemma 3.

(iii) We have

$$(29) \quad F(x, \beta) \geq \frac{1}{\sqrt{\pi}}x(x - 2C\sqrt{\pi}), \forall x > 0,$$

that is we obtain at least a solution  $\xi > 2C\sqrt{\pi}$  of the equation (10). This procedure can be applied for all  $\text{Ste} > 0$ .

(iv) Since we have the equality in (29), we can also obtain the uniqueness of  $\xi$ .  $\square$

**COROLLARY 5.** (i) The inequality  $\beta(x) < \frac{1}{2} + x^2, \forall x > 0$ , is a sufficient condition to have a unique number  $\xi > 0$ , solution of the equation (10).

(ii) If  $\beta$  is real function bounded from above by a constant  $M_0$ , then there exists at least an element  $\xi > 0$  solution of the equation (10).

**Proof.** In view of (26) it is sufficient to choose  $C > 0$  so large such that  $\min_{x>0} \varphi_C(x) \geq M_0$  and then we can apply part (iii) of Theorem 4.  $\square$

**COROLLARY 6.** For the particular and interesting case  $\beta \leq 0$  in  $\mathfrak{R}_0^+$  we have a unique element  $\xi > 0$ , solution of the equation (10).  $\square$

**Remark 3.** In parts (iii) of Theorem 4 and (ii) of Corollary 5 we can not affirm the uniqueness of the element  $\xi$ .  $\square$

**Remark 4.** If  $\beta$  verifies condition (27) we have that  $\beta(0^+) < 1$  and the existence and uniqueness of number  $\xi$ . We can also have  $\beta(0^+) \geq 1$  by choosing a suitable constant  $C > 0$  (part (iii) of Theorem 4) but in this case we can not assure the uniqueness of  $\xi$ .  $\square$

We give now a general theorem which give us sufficient conditions on function  $\beta$  to have a unique number  $\xi > 0$ , solution of the equation (10).

**THEOREM 7.** Let  $\beta$  be a continuous real function on  $\mathfrak{R}^+$  such that  $x\beta(x)$  is locally integrable on  $\mathfrak{R}^+$ . Define the function  $Z$  by

$$(30) \quad Z = Z_\beta(x) = \exp(x^2) \operatorname{erf}(x)[\psi_0(x) - \beta(x)], x > 0,$$

which is continuous and locally integrable on  $\mathfrak{R}^+$ . If the function  $Z$  satisfies the following conditions

$$(31) \quad Z > 0 \text{ on } (\nu, +\infty), \quad \int_0^{+\infty} Z(t)dt = +\infty,$$

where  $\nu \geq 0$  is defined by

$$(32) \quad \nu = \inf\{x \geq 0 / \int_0^x Z(t)dt > 0\} \quad (\equiv \nu_Z)$$

Then for any  $\text{Ste} > 0$ , there exists a unique number  $\xi = \xi(\text{Ste}) > 0$  which is the solution of the equation (10) for the given function  $\beta$ . Conversely, if for the given function  $\beta$ , the equation (10) a unique root  $\xi = \xi(\text{Ste}) > 0$  for any  $\text{Ste} > 0$  then there exists a continuous and locally integrable function  $Z$  on  $\mathfrak{R}^+$  satisfying (31) and (32) such that

$$(33) \quad \beta(x) = \psi_0(x) - Z(x)G(x) \quad (\equiv \beta_Z(x)), \quad x > 0,$$

where functions  $\psi_0$  and  $G$  are defined by (16). Moreover in any case the root  $\xi > \nu$ .

**Proof.** Owing to the definition of functions  $\psi_0$  and  $G$ , and the second equality in (23), we obtain

$$(34) \quad \begin{aligned} F(x, \beta_Z) &= F_0(x) - 2 \int_0^x \exp(r^2) \operatorname{erf}(r) [\psi_0(r) - Z(r)G(r)] dr = \\ &= 2 \int_0^x Z(r) dr, \quad \forall x \geq 0. \end{aligned}$$

From elementary consideration on function  $Z$ , the proof is achieved.

Conversely, if we define

$$(35) \quad \nu = \inf\{x \geq 0 / F(x, \beta) > 0\}, \quad Z(x) = \frac{1}{2} \frac{\partial F}{\partial x}(x, \beta), \quad x \geq 0,$$

we get the thesis because the relation (14),  $F$  is strictly increasing function over  $(\nu, +\infty)$  and verifies  $F(+\infty, \beta) = +\infty$  (by uniqueness of number  $\xi$ , solution of equation (10), for all  $\text{Ste} > 0$ ). By definition of number  $\nu$  in (35) we have  $\xi > \nu$ .  $\square$

**Remark 5.** Let  $Z$  be the function in Theorem 7. If  $Z(0^+) > 0$  (or  $Z(0^+) < 0$ ) then  $\beta_Z(0^+) = -\infty$  (or  $\beta_Z(0^+) = +\infty$ ). On the other hand, if  $Z(0^+) = 0$  then

$$(36) \quad \beta_Z(0^+) = 1 - \frac{\sqrt{\pi}}{2} \lim_{x \rightarrow 0^+} \frac{Z(x)}{x} = 1 - \frac{\sqrt{\pi}}{2} Z'(0^+),$$

when  $Z'(0^+)$  exists.  $\square$

**Remark 6.** We can improve Remark 4 by using appropriate conditions for the function  $Z$  in Theorem 7. We have  $\beta(0^+) = +\infty$  by taking  $Z(0^+) < 0$  and  $\beta(0^+) = 1 - \frac{\sqrt{\pi}}{2} Z'(0^+) > 0$  by taking  $Z(0^+) = 0$  and  $Z'(0^+) < 0$ . We have also improved condition (27) to have a unique

$\xi > 0$  because it is possible to have  $\beta > \psi_0$  in the interval  $(0, \nu)$  by choosing an appropriate function  $Z$  such that  $Z < 0$  in  $(0, \nu)$ .  $\square$

**Remark 7.(i)** If we choose function  $Z$  like  $Z(x) = Z_C(x) = \frac{x}{\sqrt{\pi}} - C$  ( $C > 0$ ), which satisfies the hypotheses of Theorem 7, then we obtain  $\beta_{Z_C} \equiv \varphi_C$  and  $\nu = 2C\sqrt{\pi}$  (c.f. Theorem 4).

**(ii)** On the other hand, if  $Z(x) = \varepsilon > 0$ ,  $\forall x > 0$ , then we get  $\beta_Z(x) = \psi_0(x) - \varepsilon G(x)$ ,  $\forall x > 0$ , and  $\nu = 0$ .

**(iii)** Similarly, if  $Z(x) = \varepsilon \exp(x^2) \operatorname{erf}(x)$ ,  $\forall x > 0$ , for some constant  $\varepsilon > 0$ , then we have  $\beta_Z(x) = \psi_0(x) - \varepsilon$ ,  $\forall x > 0$ , and  $\nu = 0$  (c.f. Theorem 4).  $\square$

**Remark 8.** If  $Z \leq 0$  ( $Z > 0$ ) in a right neighborhood of  $x = 0$  under the hypotheses of Theorem 7 then we have  $\nu \geq 0$  ( $\nu = 0$ ).  $\square$

**Remark 9.** Under the relations (30) and (33) we can deduce the following equivalences:

$$(37) \quad \text{(i)} \quad \beta < \psi_0 \text{ in } \mathfrak{R}^+ \iff Z > 0 \text{ in } \mathfrak{R}^+.$$

$$\text{(ii)} \quad \beta(x) < \psi_0(x) - \varepsilon, \forall x > 0 \iff Z(x) > \varepsilon \exp(x^2) \operatorname{erf}(x), \forall x > 0$$

where  $\varepsilon$  is any positive real number.  $\square$

**Remark 10.** (Example of nonexistence solution of the equation (10)) We consider the case  $Z(x) = \exp(-x)$ ,  $x > 0$ , which gives

$$(38) \quad \beta(x) = \frac{1}{2} + x^2 + \left[ \frac{x}{\sqrt{\pi}} - \exp(-x) \right] \frac{\exp(-x^2)}{\operatorname{erf}(x)}, \quad x > 0,$$

$$\nu = 0, \text{ and } F(x, \beta) = 2[1 - \exp(-x)], \quad x > 0.$$

Therefore, for the physical situation  $\frac{\operatorname{Ste}}{\sqrt{\pi}} = \frac{Bc}{\ell\sqrt{\pi}} \geq 2$ , we have not an explicit solution of the type (12). This occurs because function  $Z$  does not verify the limit condition (31). Moreover, others examples of nonexistence solution of the equation (10) can be constructed by choosing functions  $Z$  with the property  $\int_0^{+\infty} Z(x)dx < +\infty$ .  $\square$

**Remark 11.** (Example of multiple solutions of equation (10)) We consider the case  $Z(x) = 3(x-1)(x-3)$ ,  $x > 0$ , which gives

$$(39) \quad \beta(x) = \frac{1}{2} + x^2 + \frac{\exp(-x^2)}{\operatorname{erf}(x)} \left( -3x^2 + \left( 12 + \frac{1}{\sqrt{\pi}} \right) x - 9 \right),$$

$$F(x, \beta) = 2x(x-3)^2, \quad x > 0, \text{ and } \nu = 0.$$

Because the function  $F$  has in  $x = 1$  a relative maximum  $F(1) = 8$ , we deduce that the equation (10) has: three Solutions if  $0 < \text{Ste} < 8\sqrt{\pi}$ , two Solutions if  $\text{Ste} = 8\sqrt{\pi}$ , one Solution if  $\text{Ste} > 8\sqrt{\pi}$ .

This occurs because function  $Z$  is not always positive over the interval  $(\nu, +\infty)$  ( $Z$  is negative over the interval  $(1,3)$ ). Moreover, others examples of multiple solutions of the equation (10) can be constructed by choosing suitable functions  $Z$ .  $\square$

**Remark 12.** (Chaotic example) We consider the case

$$(40) \quad \begin{aligned} Z(x) &= \cos x, \quad F(x, \beta) = 2 \sin x, \quad x > 0, \\ \beta(x) &= \frac{1}{2} + x^2 + \left[ \frac{x}{\sqrt{\pi}} - \cos x \right] \frac{\exp(-x^2)}{\text{erf}(x)}, \quad \text{and } \nu = 0, \end{aligned}$$

which gives for the equation (10): infinity countable set of solutions if  $0 < \text{Ste} \leq 2\sqrt{\pi}$ , nonexistence of solutions if  $\text{Ste} > 2\sqrt{\pi}$ .

**Note 2.** It is important to remark that the Theorem 7 is a constructive theorem to obtain a large family of functions  $\beta$ , that is a family of source term  $g$ , which has an exact solution of the Lamé-Clapeyron type (12) (c.f. Theorem 1). Moreover, the hypotheses for the function  $Z$  in Theorem 7 are optimal in order to have a unique solution  $\xi$  of the equation (10) (c.f. the counterexamples given in the last remarks). On the other hand, the relation (30) can be useful in several problems.  $\square$

### III. Solution For Small Stefan Number

We shall consider the behavior of the solution (II-10), (II-12) when the Stefan number is small (classic approximation [3, 10, 11]), i.e.  $0 < \text{Ste} = \frac{Bc}{\ell} \ll 1$ . If we suppose that function  $\beta$  verifies the conditions (II-27) and (II-37i) then we have  $Z_\beta > 0$  in  $\mathfrak{R}^+$ , i.e.  $\nu = 0$ . Therefore, we can use the following first order approximations:  $\exp(x^2) \approx 1$ ,  $\text{erf}(x) \approx \frac{2}{\sqrt{\pi}}x$ ,  $\beta(x) \approx \beta(0) = \beta_0 < 1 - \varepsilon < 1$ , in the definition (II-11) of function  $F$  to obtain the first order approximation of the solution  $\xi$  of the equation (II-10), i.e.

$$(1) \quad \xi_{ap} = \sqrt{\frac{\text{Ste}}{2(1 - \beta_0)}}.$$

**Remark 1.** It is important to remark that the first order approximation is only true when the number  $\nu$  is zero. Moreover, we can also suppose that  $\beta_0 < 1$  to obtain (1).  $\square$

We shall interpret the meaning of the formula (1).

**THEOREM 1.** The solution of the quasi-steady state free boundary problem (P) where  $g(x, t) = \frac{\rho\ell\beta_0}{t}$  with  $\beta_0 < 1$ , i.e.

$$(2) \quad \begin{aligned} -k\theta_{xx} &= \frac{1}{t}\rho\ell\beta_0, \quad 0 < x < s(t), \quad t > 0, \\ \theta(0, t) &= B > 0, \quad t > 0, \\ \theta(s(t), t) &= 0, \quad k\theta_x(s(t), t) = -\rho\ell\dot{s}(t), \quad t > 0, \\ s(0) &= 0 \end{aligned}$$

is given by

$$(3) \quad \theta(x, t) = \frac{B}{1-\beta_0} \left(1 - \frac{x}{s(t)}\right) \left[1 - \beta_0 \left(1 - \frac{x}{s(t)}\right)\right], \quad s(t) = 2a\xi_{ap}\sqrt{t}$$

where  $\xi_{ap}$  is given by (1).

**Proof.** We propose a quadratic functions in variable  $x$  for the temperature, that is  $\theta(x, t) = C_1\left(1 - \frac{x}{s(t)}\right) + C_2\left(1 - \frac{x}{s(t)}\right)^2$ , where  $C_1$  and  $C_2$  are two unknown constants. Solving all conditions, given in (2), we obtain an ordinary differential equation for  $s = s(t)$ , whose solution is given by (3) and (1). Moreover, we have  $C_1 = \frac{B}{1-\beta_0}$ ,  $C_2 = \frac{-B\beta_0}{1-\beta_0}$ .  $\square$

We suppose that function  $\beta$  (or its corresponding function  $Z$ ) verifies the hypotheses (e.g. Theorem II-7) to have a unique solution  $\xi$  of the equation (II-10). We shall give a general method to obtain  $\xi_{ap}$  as function of  $Z = Z(x)$  and  $\nu > 0$  when  $Z(\nu) > 0$ . Indeed, in this case, we have  $F(\nu, \beta) = 0$ ,  $\frac{\partial F}{\partial x}(\nu, \beta) = 2Z(\nu) > 0$ . Therefore, if we consider the first order approximation in equation (II-10) then we obtain that  $\xi > \nu > 0$  is the solution of the following linear equation

$$(4) \quad 2Z(\nu)(x - \nu) = \frac{\text{Ste}}{\sqrt{\pi}}, \quad x > \nu,$$

which is given by

$$(5) \quad \xi_{ap} = \nu + \frac{\text{Ste}}{2\sqrt{\pi}Z(\nu)}.$$

Thus, we have obtained the following lemma.

**LEMMA 2.** If function  $Z$  verifies the hypotheses of Theorem II-7 and  $Z(\nu) > 0$  then we obtain that the first order approximation  $\xi_{ap}$  of the solution  $\xi$  of the equation (II-10) for small Stefan number Ste is given by the expression (5).  $\square$

**Remark 2** If we consider the case  $Z = Z_C$ , where  $C > 0$  is a given constant, then we get (c.f. also Remark II-7)

$$(6) \quad F(x, \beta) = \frac{x}{\sqrt{\pi}}(x - 2C\sqrt{\pi}), \quad \beta(x) = \varphi_C(x), \quad x > 0, \quad \nu = 2C\sqrt{\pi} > 0,$$

and therefore, the exact solution of the equation (II-10) is given by

$$(7) \quad \xi_C = C\sqrt{\pi}\left[1 + \sqrt{1 + \frac{\text{Ste}}{\pi C^2}}\right]. \square$$

**Remark 3.** If we use the approximation  $\sqrt{1+x} \approx 1 + \frac{x}{2}$ ,  $|x| \ll 1$ , in (7) when  $\text{Ste} \ll 1$ , we obtain that  $\xi_C \approx \xi_{ap} = 2C\sqrt{\pi} + \frac{\text{Ste}}{2C\sqrt{\pi}}$ , where we can verify that  $\xi_{ap}$  coincides with the expression given by (5) because  $\nu = 2C\sqrt{\pi} > 0$  and  $Z(\nu) = C > 0$ .  $\square$

**Remark 4.** When  $\nu > 0$  and  $Z(\nu) = 0$  we replace the linear equation (4) by

$$(8) \quad 2 Z^{(n)}(\nu) \frac{(x - \nu)^n}{n!} = \frac{\text{Ste}}{\sqrt{\pi}}, \quad x > \nu,$$

where  $n > 1$  is the first order such that the  $n$ -derivative of function  $Z$  at the point  $\nu$  is different from 0.  $\square$

## IV. Temperature and Free Boundary Estimates

In this paragraph we shall give some estimates on the temperature  $\theta$  and the free boundary  $s(t)$ , given by (II-12), in particular for the coefficient  $\xi$  (solution of equation (II-10))

Firstly we can obtain the following result for the particular case  $\beta \leq 0$  over  $\mathfrak{R}_0^+$ .

**LEMMA 1.** If  $\beta \leq 0$  over  $\mathfrak{R}_0^+$  then we have

$$(1) \quad \begin{aligned} \text{(i)} \quad & 0 < \xi \leq \xi_0 \quad \text{and} \quad 0 < s(t) \leq s_0(t), t > 0, \\ \text{(ii)} \quad & \theta(x, t) \leq \theta_0(x, t), 0 \leq x \leq s(t), t > 0, \end{aligned}$$

where  $\theta_0, s_0, \xi_0$  represents the solution (I-2)-(I-3) corresponding to the case  $\beta \equiv 0$ .

**Proof.** By means of Corollary (II-6) and the fact that  $F(x, \beta) \geq F(x, 0) = F_0(x)$  we obtain  $\xi \leq \xi_0$ , that is (1-i). Therefore, in the domain  $0 < x < s(t), t > 0$  we can apply the maximum principle in order to obtain (1-ii).

**Remark 1.** We can not obtain the inequality (1-ii) by using directly the definition of  $\theta$ , given by the expression (II-12) (c.f. Corollary 4 for an improvement of (1-ii)).  $\square$

Let  $\theta_{\beta_i} = \theta_i, s_{\beta_i} = s_i, Z_{\beta_i} = Z_i, \nu_{Z_i} = \nu_i \geq 0$  and  $\xi(\beta_i) \geq 0$  be the temperature (defined by (II-12)), the free boundary (defined by (II-12)),  $Z$ -function (defined by (II-30)), the numbers  $\nu$  (defined by (II-32)) and  $\xi$  (solution of equation (II-10)) corresponding to the data  $\beta_i$  for  $i = 1, 2$ . We have the following comparison result:

**THEOREM 2.** (a) We have the following equivalence:

$$(2) \quad \beta_2 \leq \beta_1 \text{ over } \mathfrak{R}^+ \iff Z_1 \leq Z_2 \text{ over } \mathfrak{R}^+.$$

(b) If  $\beta_2 \leq \beta_1$  over  $\mathfrak{R}^+$  then we obtain the following properties.

$$(3) \quad \text{(i)} \quad \nu_2 \leq \nu_1, \quad \text{(ii)} \quad \nu_1 = 0 \Rightarrow \nu_2 = 0.$$

On the other hand, if  $Z_1$  verifies the hypotheses of Theorem (II-7) and  $Z_2 > 0$  on  $(\nu_2, \nu_1]$  then  $Z_2$  also verifies the hypotheses of Theorem (II-7). Moreover, we obtain that there exists a unique  $\xi(\beta_2)$  which satisfies the inequalities

$$(4) \quad \xi(\beta_2) \leq \xi(\beta_1), \quad s_2(t) \leq s_1(t), \quad t \geq 0.$$

(c) If in addition to the hypotheses of part (b) we have  $\beta_1 \geq 0$ , over  $\mathfrak{R}^+$ , then we get

$$(5) \quad \theta_2(x, t) \leq \theta_1(x, t), \quad 0 \leq x \leq s_2(t), \quad t > 0.$$

**Proof.** The equivalence (2) follows from the definitions (II-30) and (II-33). We use the fact  $\int_0^x Z_2(t)dt \geq \int_0^x Z_1(t)dt > 0, \forall x > \nu_1$ , to get that  $\nu_2 \leq \nu_1$ , i.e. (3). By using (a) we obtain  $\int_0^{+\infty} Z_2(t)dt = +\infty, \nu_2 \leq \nu_1, Z_2 > 0$  over  $(\nu_1, +\infty)$  and  $F(x, \beta_1) \leq F(x, \beta_2)$  over  $\mathfrak{R}^+$ , that is (4). We obtain (5) by means of  $\beta_1 \geq 0$  and the maximum principle in the domain  $0 < x < s_2(t), t > 0$ .  $\square$

**Remark 2.** The same consequence (4) of part (b) of Theorem 2 holds true if we replace the condition “ $Z_2 > 0$  over  $(\nu_2, \nu_1]$ ” by “ $\nu_1 = 0$ ”. This assertion makes use of (3-ii).  $\square$

In order to obtain some others estimates on the temperature we can modify the third term on the right hand size in (II-12) to have the following results.

**THEOREM 3.** Let  $\beta$  be a real continuous function over  $\mathfrak{R}^+$  which satisfies the hypotheses of Theorem II-7 relative to the unique solution  $\xi > 0$  of equation (II-10). Then we obtain :



(i) The temperature  $\theta$  can be expressed by

$$(6) \quad \theta(x, t) = \theta(\eta) = B\left\{1 - \frac{\sqrt{\pi}}{\text{Ste}} \xi \exp(\xi^2) \operatorname{erf}(\eta) + \frac{2\sqrt{\pi}}{\text{Ste}} \left[ \int_0^\eta \beta(r) \exp(r^2) \operatorname{erf}(r) dr + \operatorname{erf}(\eta) \int_\eta^\xi \beta(r) \exp(r^2) dr \right]\right\}, \text{ for } \eta = \frac{x}{2a\sqrt{t}} \in (0, \xi).$$

(ii) If  $\eta = \eta^*$  is a critical point of the temperature  $\theta = \theta(\eta)$  (that is  $\frac{d\theta}{d\eta}(\eta^*) = 0$ , i.e.  $\eta^*$  is a maximum, minimum or inflection point) then  $\eta = \eta^*$  must satisfy the following equation:

$$(7) \quad \int_\eta^\xi \beta(r) \exp(r^2) dr = \frac{1}{2} \xi \exp(\xi^2), \quad 0 < \eta < \xi.$$

Moreover, in this case, we have

$$(8) \quad \begin{aligned} \theta(\eta^*) &= B\left[1 + \frac{2\sqrt{\pi}}{\text{Ste}} \int_0^{\eta^*} \beta(r) \exp(r^2) \operatorname{erf}(r) dr\right] = \\ &= B\left\{1 + \frac{\sqrt{\pi}}{\text{Ste}} [F_0(\eta^*) - F(\eta^*, \beta)]\right\}, \end{aligned}$$

$$(9) \quad \frac{d^2\theta}{d\eta^2}(\eta^*) = \frac{-4B}{\text{Ste}} \beta(\eta^*).$$

**Proof.** (i) The Third term on the right hand side for the temperature  $\theta$  in (II-14) is a double integral over a domain which can be expressed by the union of two subdomains as follows  $(0 < \eta < \xi)$ :  $\{(r, y)/r < y < \xi, 0 < r < \xi\} = \{(r, y)/0 < r < \eta, \eta < y < \xi\} \cup \{(r, y)/0 < r < \eta, 0 < y < \eta\}$ . Therefore, we can deduce (6) after exchanging the order of integration in the triangle.

(ii) It follows from the expressions  $(0 < \eta < \xi)$ :

$$(10) \quad \frac{d\theta}{d\eta}(\eta) = \frac{2B}{\text{Ste}} \exp(-\eta^2) \left[ 2 \int_\eta^\xi \beta(r) \exp(r^2) dr - \xi \exp(\xi^2) \right],$$

$$(11) \quad \frac{d^2\theta}{d\eta^2}(\eta) = -2\eta \frac{d\theta}{d\eta}(\eta) - \frac{4B}{\text{Ste}} \beta(\eta). \quad \square$$

**Remark 3.** If  $\eta = \eta^*$  is a critical point of  $\theta = \theta(\eta)$  then we have a relative

$$(12) \quad \text{maximum if } \beta(\eta^*) > 0, \quad \text{minimum if } \beta(\eta^*) < 0. \quad \square$$

As a direct consequence of the above Theorem we have:

**COROLLARY 4.** (i) If  $\beta \leq 0$  over  $\mathfrak{R}^+$  then the temperature  $\theta$  can not have a critical point. Moreover, we can improve (1-ii) by

$$(13) \quad 0 \leq \theta(x, t) \leq \theta_0(x, t), \quad 0 \leq x \leq s(t), \quad t > 0.$$

(ii) Suppose  $\beta > 0$  over  $\mathfrak{R}^+$  and satisfies the hypotheses in order that equation (II-10) has a unique solution. If the equation (7) has at least a solution then the critical point  $\eta = \eta^*$  is unique and we have  $0 \leq \theta(\eta) \leq \theta(\eta^*)$ ,  $0 < \eta < \xi$ , with  $\theta(\eta^*) > B$ , where  $\theta(\eta^*)$  is given by (8). Moreover, if only  $\beta \geq 0$  over  $\mathfrak{R}^+$  then we may have an whole interval of critical points, but in any case we have  $\theta(\eta) \geq 0$ ,  $0 \leq \eta \leq \xi$ .  $\square$

**LEMMA 5.** If  $\beta \geq 0$  over  $\mathfrak{R}^+$  and  $Z$  satisfies the hypotheses of Theorem II-7, then we have

$$(14) \quad \begin{aligned} \xi &\geq \xi_0, \quad s(t) \geq s_0(t), \quad t \geq 0, \\ \theta(x, t) &\geq \theta_0(x, t), \quad 0 \leq x \leq s_0(t), \quad t \geq 0. \end{aligned}$$

**Proof.** It follows from Theorem 2 and the maximum principle.  $\square$

**COROLLARY 6.**(i) If  $\beta_2 \leq \beta_1$  over  $\mathfrak{R}^+$  and  $Z_1$  and  $Z_2$  verify the hypotheses of Theorem II-7 (We use the same notation of Theorem 2) then we have

$$(15) \quad 0 \leq \xi(\beta_2) \leq \xi(\beta_1), \quad 0 \leq s_2(t) \leq s_1(t), \quad t \geq 0.$$

(ii) If  $\beta(x) \leq \psi_0(x) - \varepsilon$  for some real constant  $\varepsilon > 0$  then  $0 < \xi(\beta) \leq H_0^{-1}(\frac{\text{Ste}}{2\varepsilon\sqrt{\pi}})$ , where  $H_0^{-1}$  is the inverse function of  $H_0$  which is defined by  $H_0(x) = \int_0^x \exp(r^2) \operatorname{erf}(r) dr$ ,  $x > 0$ , and satisfies  $H_0(0^+) = 0$ ,  $H_0(+\infty) = +\infty$ ,  $H_0' > 0$  over  $\mathfrak{R}^+$ .

(iii) If the function  $\beta$  satisfies the hypotheses of Theorem II-7 and  $\beta \leq \varphi_C$  over  $\mathfrak{R}^+$ , for some real constant  $C > 0$ , then we have

$$(16) \quad \begin{aligned} 2C\sqrt{\pi} < \xi(\beta) &\leq \xi_C = C\sqrt{\pi}[1 + \sqrt{1 + \frac{\text{Ste}}{\pi C^2}}], \\ \theta(\eta) &\leq \frac{B\sqrt{\pi}}{\text{Ste}} F_0(\xi_C), \quad 0 \leq \eta \leq \xi. \square \end{aligned}$$

**Remark 4.** It is necessary to suppose that  $\beta$  satisfies the hypotheses of Theorem II-7 (i.e.  $Z_\beta > 0$  over  $(\nu(Z_\beta), 2C\sqrt{\pi})$ ) in Corollary 6 part (iii) because  $\nu(Z(\varphi_C)) = \nu(Z_C) = 2C\sqrt{\pi} > 0$ , that is we can not apply Remark 2. Moreover, if  $\beta$  does not satisfy the hypotheses

of Theorem II-7 then in general we can not affirm the uniqueness of  $\xi(\beta)$ , but all these numbers satisfy the inequality (16) from part (ii) of Theorem II-4 and the fact that  $\frac{\text{Ste}}{\sqrt{\pi}} = F(\xi(\beta), \beta) \geq F(\xi(\beta), \varphi_C) = \xi(\beta)(\frac{\xi(\beta)}{\sqrt{\pi}} - 2C)$ , that is  $\xi(\beta) \leq \xi_C$ .  $\square$

**THEOREM 7.** Let  $Z$  and  $Z_*$  be two continuous real functions over  $\mathfrak{R}_0^+$  which satisfy the hypotheses of Theorem II-7. Let  $\beta$  and  $\beta_*$  be their corresponding  $\beta$ -functions. Let  $H_*$  be the real function defined by

$$(17) \quad H_*(x) = \frac{\text{Ste}}{2\sqrt{\pi}} + \int_0^x Z_*(r)dr, \quad x \geq 0.$$

Then we have the following results:

(i) The number  $\xi(\beta) > 0$  satisfies the inequality:

$$(18) \quad H_*(\xi(\beta)) \leq F_0(\xi(\beta)).$$

(ii) If, in addition, function  $Z_*$  is such that there exists at least a solution to the following equation

$$(19) \quad H_*(x) = F_0(x), \quad x > 0$$

then we have  $\xi(\beta) \geq \eta_0 > 0$ , where  $x = \eta_0 > 0$  is the first positive root of equation (19).

(iii) If the function  $Z_*$  satisfies the inequality

$$(20) \quad H_*(x) \leq F_0(x), \quad \text{for some } x > 0,$$

then there exists at least one number  $\eta_0 > 0$  which solves the equation (19).

(iv) If the function  $Z_*$  satisfies the inequality

$$(21) \quad Z_*(x) \leq \frac{x}{\sqrt{\pi}} + \left(\frac{1}{2} + x^2\right) \exp(x^2) \operatorname{erf}(x), \quad x \geq 0,$$

then the condition (20) holds true. Moreover, in this case we have  $\beta_* \geq 0$  over  $\mathfrak{R}^+$  and  $Z_*(0) \leq 0$ .

**Proof.** (i) From (II-11) and (II-23), we get

$$(22) \quad \frac{\text{Ste}}{\sqrt{\pi}} = F(\xi(\beta), \beta) \leq F(\xi(\beta), -\beta_*) = 2F_0(\xi(\beta)) - 2 \int_0^{\xi(\beta)} Z_*(r)dr,$$

that is (18).

(ii) It follows from (18), (19) and the definition of  $\eta_0 > 0$ .

(iii) This assertion follows from (19) and the fact that  $F_0(0^+) = 0$ ,  $H_*(0^+) = \frac{\text{Ste}}{2\sqrt{\pi}} > 0$ .

(iv) If the function  $Z_*$  satisfies the inequality (21) then, by taking into account (II-23), we obtain

$$(23) \quad H_*(x) \leq \frac{\text{Ste}}{2\sqrt{\pi}} + \frac{1}{2}F_0(x), \quad x \geq 0,$$

that is, we get (20) for all  $x \geq \xi_0 > 0$ . Moreover, by means of (II-33) we obtain  $\beta_* \geq 0$  over  $\mathfrak{R}^+$ .  $\square$

**COROLLARY 8.** If the function  $Z_*$  in Theorem 7 is given by  $Z_*(x) = \frac{x}{\sqrt{\pi}} - C$ ,  $x \geq 0$  ( $C > 0$ ), then we obtain

$$(24) \quad \begin{aligned} \beta_*(x) &= \varphi_C(x), \quad x > 0 \quad (\text{c.f. Remark II-9}), \\ H_*(x) &= \frac{x^2}{2\sqrt{\pi}} - Cx + \frac{\text{Ste}}{2\sqrt{\pi}}, \quad x \geq 0, \\ H_*(0) &= \frac{\text{Ste}}{2\sqrt{\pi}} > 0, \quad H'_*(0) = -C < 0, \quad H_*(+\infty) = +\infty, \\ H_*(x) &\geq H_*(C\sqrt{\pi}) = \frac{\sqrt{\pi}}{2} \left( \frac{\text{Ste}}{\pi} - C^2 \right), \quad x \geq 0. \end{aligned}$$

Moreover, there exists at least one solution of the equation (19) whose first positive root  $\eta_0$  satisfies the inequalities

$$(25) \quad \eta_1 < \eta_0 < \xi_0$$

where  $\xi_0$  is the unique solution of equation (I-3) and  $\eta_1$  is given by

$$(26) \quad \eta_1 = C\sqrt{\pi} + \frac{\text{Ste}}{\xi_0} - \sqrt{\left(C\sqrt{\pi} + \frac{\text{Ste}}{\xi_0}\right)^2 - \text{Ste}} > 0.$$

In the particular case where  $C > \sqrt{\frac{\text{Ste}}{\pi}}$  we also have

$$(27) \quad \eta_0 < \eta_2 = \pi \left( C - \sqrt{C^2 - \frac{\text{Ste}}{\pi}} \right).$$

**Proof.** From elementary computations we get (24). Taking into account that function  $H_*$  is a parabola with properties (24), and  $F_0$  is a exponential type function and satisfies  $F_0(0) = 0$ , we deduce that there exists at least one solution of the equation (19). The number  $\xi_0 > 0$  (unique solution of equation (I-3)) is also the unique solution of the following equation

$$(28) \quad F_0(x) = \frac{\text{Ste}}{\xi_0\sqrt{\pi}}x, \quad x > 0.$$

The equation

$$(29) \quad H_*(x) = \frac{\text{Ste}}{\xi_0 \sqrt{\pi}} x, \quad x > 0,$$

has two positive solutions. The first positive root of (29) is  $\eta_1 > 0$ , given by (26), which is well defined for all  $C > 0$  because  $\xi_0^2 = \frac{\text{Ste}}{\sqrt{\pi}} \frac{1}{V(\xi_0)} \leq \frac{\text{Ste}}{2}$ , after using (I-3) and (II-25). Then we can deduce the inequalities (25). In the particular case where  $C > \sqrt{\frac{\text{Ste}}{\pi}}$  the function  $H_*$  has two positive roots, whose first positive root is given by  $\eta_2$ , that is (27).  $\square$

**Remark 5.** Let  $A$  be the positive real constant defined by  $A = A(\text{Ste}, \xi_0) = \frac{1}{2} + \xi_0^2(1 + \frac{1}{\text{Ste}}) > 0$ , where  $\xi_0$  is the unique solution of equation (I-3). Then we have the following property  $\frac{\partial F}{\partial x}(\xi_0, \beta) > 0$ , for all continuous real functions  $\beta$  over  $\mathfrak{R}^+$  such that  $\beta(x) < A(\text{Ste}, \xi_0), x > 0$ . This follows from (I-3) and (II-14). Moreover, we get  $A(\text{Ste}, \xi_0) \leq 1 + \frac{\text{Ste}}{2}$ .  $\square$

Now, let us mention some results concerning to the non-uniqueness of the solution to the free boundary problem (P).

**LEMMA 9.** Assume that  $\beta \geq 0$  over  $\mathfrak{R}^+$  and that there are at least two roots  $\xi_1 < \xi_2$  of the equation (II-10) for the same function  $\beta$ . Then we have

$$(30) \quad \begin{aligned} s_1(t) &\leq s_2(t), \quad t \geq 0, \\ 0 &\leq \theta_1(x, t) \leq \theta_2(x, t), \quad 0 \leq x \leq s_1(t), \quad t \geq 0, \end{aligned}$$

where  $\theta_i = \theta_i(x, t)$  denotes the temperature corresponding to  $\xi_i$  ( $i = 1, 2$ ) and  $\beta$ . Moreover, we have

$$(31) \quad \theta_2(x, t) \geq 0, \quad 0 \leq x \leq s_2(t), \quad t \geq 0,$$

**Proof.** We use twice the maximum principle for the functions  $\theta_2$  and  $\theta_2 - \theta_1$ .  $\square$

To conclude, for a function  $\beta \approx 0$  we want to obtain a development of  $\xi(\beta)$  in a neighborhood of  $\xi_0$  given by the equation (I-3).

**LEMMA 10.** Let  $\beta_1$  be a continuously differentiable real function over  $\mathfrak{R}^+$ . Let  $\beta_\lambda$  be a function defined by  $\beta_\lambda(x) = \lambda\beta_1(x), x > 0$  and we suppose there exists two positive constants  $\lambda_0$  and  $\varepsilon_0$  such that

$$(32) \quad \beta_\lambda(x) = \lambda\beta_1(x) < \psi_0(x) - \varepsilon_0, \quad \text{for all } x > 0 \text{ and } 0 < \lambda < \lambda_0.$$

Then the unique number  $\xi(\beta_\lambda) = \xi_\lambda$ , solution of equation (II-10) for function  $\beta_\lambda$ , can be expressed as

$$(33) \quad \xi_\lambda = \xi(\beta_\lambda) = \xi_0 + \xi_1\lambda + \xi_2\lambda^2 + o(\lambda^2),$$

with

$$(34) \quad \xi_1 = \frac{\xi_0\sqrt{\pi}}{\text{Ste}A} \int_0^{\xi_0} \exp(r^2) \operatorname{erf}(r)\beta_1(r)dr,$$

$$(35) \quad \xi_2 = \frac{\xi_1}{A}[\beta_1(\xi_0) - \xi_0\xi_1(1 + \frac{1}{\text{Ste}} + A)],$$

where  $\xi_0$  is the unique solution of equation (I-3) and  $A = A(\text{Ste}, \xi_0)$  is given in Remark 5.

**Proof.** By the assumption (32) we have a unique number  $\xi(\beta_\lambda) = \xi_\lambda > 0$  which is the solution of equation (II-10). Let  $I = I(x, \lambda)$  be the real function defined by

$$(36) \quad I(x, \lambda) = F(x, \beta_\lambda) - \frac{\text{Ste}}{\sqrt{\pi}}, \quad x > 0, \quad 0 < \lambda < \lambda_0.$$

Taking into account that  $I(\xi_0, 0) = 0$ ,  $\frac{\partial I}{\partial x}(\xi_0, 0) = \frac{2\text{Ste}}{\xi_0\sqrt{\pi}}A > 0$ , we can apply the Dini implicit function Theorem to deduce the existence of a real function  $\xi_\lambda = \xi(\lambda)$ , defined in a right neighborhood of  $\lambda = 0$  which satisfies  $\xi(0) = \xi_0$  and (33).  $\square$

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