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SINGULAR ERGODIC CONTROL FOR MULTIDIMENSIONAL GAUSSIAN PROCESSES

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1. Introduction

The class of singular stochastic control problems, which has been extensively studied lately, deals with systems described by a linear stochastic differential equation with control functional being of an additive nature. The main feature of such problems is that the control functional need not to be absolutely continuous with respect to time. In fact, the optimal control functionals in these problems are singular.

More precisely, we assume that the fluctuation of the stochastic system under control is described by a n-dimensional Gaussian process $(y(t), t \geq 0)$ with a variable vector-drift and a constant diffusion matrix. The control is realized by a non-anticipating process of bounded variation $(\nu(t), t \geq 0)$, i.e. the state equation is the following stochastic differential equation in Itô's sense:

$$
\begin{aligned}
(1.1) \qquad & dy(t) = [g + fy(t)]dt + \sigma dw(t) + d\nu(t), t > 0, \\
& y(0) = x,
\end{aligned}
$$

where $(\Omega, \mathcal{F}, P, \mathcal{F}(t), w(t), t \geq 0)$ is a standard Brownian motion in $\Re^n$, g is a constant n-dimensional vector, f and σ are constant $n \times n$ matrices, and x is the initial position.

The cost associated with the position of the process is measured by a convex nonnegative function h, and the cost of controlling is proportional to the displacement induced by this control. We are interested in minimizing the limiting time-average expected (i.e., ergodic) cost, that is in finding

$$
(1.2) \qquad \inf_{\nu(\cdot)} \limsup_{T \to \infty} \frac{1}{T} E\{\int_0^T h(y(t))dt + c|\nu|(T)\}.
$$

Here c is a positive real number, and $|\nu|(T)$ denotes the total variation of ν on $[0, T]$. More precisely, if $(\nu(t), t \geq 0)$ is an adapted process with bounded variation then $|\nu|(T)$ is defined as

$$
(1.3) \; |\nu|(T) = \sup\{\sum_{i=1}^{k} |\nu(t_i) - \nu(t_{i-1})| : 0 = t_0 < t_1 < \ldots < t_k = T\},
$$

where $|\cdot|$ is the Euclidian norm in $\Re^n$.

Another class of infinite-horizon problems deals with the minimization of the total expected discounted cost

$$(1.4)\qquad u_\alpha(x) = \inf_{\nu(\cdot)} E\{\int_0^\infty e^{-\alpha t} h(y_x(t))dt + c\int_0^\infty e^{-\alpha t} d|\nu|(t)\}.$$

The Hamilton-Jacobi-Bellman (HJB) equation for the optimal cost function $u_\alpha(x)$ is given by

$$(1.5)\qquad \min\{Lu_\alpha(x) - \alpha u_\alpha(x) + h(x), c - |\nabla u_\alpha(x)|\} = 0 \text{ in } \Re^n,$$

where

$$\begin{aligned} \nabla &= (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}), \\ L &= \frac{1}{2}\sum_{i,j=1}^n (\sum_{k=1}^n \sigma_{ik}\sigma_{jk})\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n (g_i + \sum_{j=1}^n f_{ij}x_j)\frac{\partial}{\partial x_i}. \end{aligned}$$

Similar to the classical situation, we can write the HJB equation for the ergodic problem (1.2), namely

$$(1.6)\qquad \min\{Lv(x) - \lambda + h(x), c - |\nabla v(x)|\} = 0 \text{ in } \Re^n.$$

This last equation (1.6) contains two unknowns, the function v and the number λ. The constant λ represents the optimal ergodic cost (1.2), which is independent of the initial position x. The function v, however, does not have an explicit probabilistic interpretation, in contrast to the function u_α given by (1.4). Moreover, the function v is defined by (1.5) only up to an additive constant. The singular control problem with discounted criterion (1.4) and the corresponding equation (1.5) was recently investigated mainly in one dimension by various authors, e.g. Chow et al. [4], Karatzas and Shreve [13, 14], Menaldi and Robin [19, 20, 23], Sun and Menaldi [36], Taksar [37,39,40] and the references therein.

The analysis of ergodic control problems with objective cost (1.2) in one-dimension can be found in Karatzas [12], Menaldi and Robin [21], and Taksar [38], under several kind of assumptions. Specific features of the one-dimensional case allow to differentiate the HJB equation and reduce it to the solution of (Stefan) free boundary problem for a second order ordinary differential equation. This technique does not work for dimension higher than one. Then, to find the solution of (1.6) in the multidimensional case we need to start with the solution of (1.5) and to investigate the behavior of u_α when α converges to zero.

Let us mention that a variety of techniques used in ergodic control can be found in Bensoussan [1], Borkar and Ghosh [3], Garroni and Menaldi [8], Kushner [15], Lions and Perthame [16], Menaldi and Robin [22], Robin [29], Stettner [33], Sun [35], Tarres [41] and others.

We follow the notation in Menaldi and Taksar [25,26], which is the starting point of the current paper.

In Section 2 we formulate the assumptions and state the main results. A priori estimates are given in Section 3. Next, in Section 4 we prove the main results.

2. Statement of the problem and main results

The state of the system is $(y(t), t \geq 0)$, given by the Itô equation (1.1) where

$$\begin{aligned} &g = (g_i, i = 1, \ldots, n) \ \text{ is a vector,} \\ &f = (f_{ij}, i, j = 1, \ldots, n) \text{ and} \\ &\sigma = (\sigma_{ik}, i, k = 1, \ldots, n) \ \text{ are matrices.} \end{aligned} \tag{2.1}$$

The following conditions are supposed to be satisfied by the parameters of the model

$$\begin{aligned} &\alpha, c \ \text{ are positive numbers,} \\ &\sigma \ \text{ is an invertible matrix,} \\ &f \ \text{ is a stable matrix, i.e. } e^{tf} \ \text{ is bounded} \\ &\text{as } t \text{ goes to } +\infty. \end{aligned} \tag{2.2}$$

The set of control functional $\mathcal{V}$ consists of all right continuous processes $(\nu(t), t \geq 0)$ valued in $\Re^n$, progressively measurable w.r.t. the complete and right continuous filtration $(\mathcal{F}(t), t \geq 0)$ and such that the variation process $|\nu|(t)$ of (1.3) satisfies

$$E\{|\nu|(t)\} < \infty, \ \ \forall t \geq 0. \tag{2.3}$$

For technical reasons we adopt the convention $\nu(0-) = 0$, thus allowing $\nu(\cdot)$ to have discontinuity at 0. With this convention

$$\begin{aligned} dy(t) &= [g + fy(t)]dt + \sigma dw(t) + d\nu(t), \quad t \geq 0, \\ y(0-) &= x, \quad y(0) = x + \nu(0). \end{aligned} \tag{2.4}$$

The holding cost function satisfies the polynomial growth conditions below. There exist constants $p > 1, C_0, C_1, C_2 > 0$ such that for any $0 < \lambda < 1$, and any $x, \chi \in \Re^n, |\chi| = 1$ we have

$$0 \leq h(x) \leq C_0(1 + |x|)^p, \tag{2.5}$$

$$|h(x) - h(x + \lambda\chi)| \leq C_1\lambda(1 + h(x)), \tag{2.6}$$

$$0 < h(x + \lambda\chi) + h(x - \lambda\chi) - 2h(x) \leq C_2\lambda^2(1 + h(x)). \tag{2.7}$$

Also we suppose that h is strictly convex and

$$|x|^{-1}h(x) \to \infty \;\text{ as }\; |x| \to \infty. \tag{2.8}$$

We set

$$\mathcal{J}(x, \nu, \alpha) = \int_0^\infty e^{-\alpha t} h(y_x(t))dt + c\int_0^\infty e^{-\alpha t} d|\nu|(t), \tag{2.9}$$

$$J(x, \nu, \alpha) = E\{\mathcal{J}(x, \nu, \alpha)\} \tag{2.10}$$

and

$$K(x, \nu) = \limsup_{T\to\infty} \frac{1}{T} E\{\int_0^T h(y_x(t))dt + c|\nu|(T)\}. \tag{2.11}$$

Thus

$$u_\alpha(x) = \inf\{J(x, \nu, \alpha) : \nu \in \mathcal{V}\}, \tag{2.12}$$

$$\lambda = \inf\{K(x, \nu) : \nu \in \mathcal{V}\}. \tag{2.13}$$

Our main results are the following:

Theorem 2.1

The optimal ergodic cost λ is independent of the initial state x, and

$$\alpha u_\alpha(x) \to \lambda \ \text{ as } \ \alpha \to 0, \tag{2.14}$$

where the convergence is locally uniform in x belonging to $\Re^n$. □

Theorem 2.2

There exist a convex and Lipschitz continuous function v and a bounded, open and nonempty region D in $\Re^n$ such that

$$\begin{aligned} &Lv + h \geq \lambda \ \text{ in } \ \mathcal{D}'(\Re^n) \\ &|\nabla v| \leq c \ \text{ a.e. in } \ \Re^n, v(0) = 0, \end{aligned} \tag{2.15}$$

$$\begin{aligned} &v \ \text{ belongs to } \ W^{2,\infty}(D) \ \text{ and} \\ &Lv + h = \lambda \ \text{ a.e. in } \ D \\ &|\nabla v| = c \ \text{ on } \ \partial D. \end{aligned} \tag{2.16}$$

Moreover, if ∂D is of class C^3, v is three times continuously differentiable on $\bar{D} = D \cup \partial D$, and ∇v is never tangent to ∂D, then there exists $\nu_x^\star$ in $\mathcal{V}$ such that

$$K(x, \nu_x^\star) = \lambda, \ \ \forall x \in \Re^n, \tag{2.17}$$

i.e., $\nu^\star$ is an optimal ergodic (or stationary) policy. □

Remark that $\mathcal{D}'(\Re^n)$ denotes the space of Schwartz distributions in $\Re^n$ and $W^{2,\infty}(D)$ is the Sobolev space of functions with Lipschitz continuous first derivatives in D. More precise conditions on the boundary ∂D and the gradient direction ∇v are given in the last section.

3. A priori estimates

Denote by

$$\beta(t) = gt + \sigma w(t), \ \ t \geq 0. \tag{3.1}$$

Then the state $(y_x(t), t \geq 0)$ given by (2.5) satisfies

$$y_x(t) = e^{tf}x + \int_0^t e^{(t-s)f} d\beta(s) + \int_0^t e^{(t-s)f} d\nu(s), t \geq 0. \tag{3.2}$$

Each control $(\nu(t), t \geq 0)$ can be decomposed into a continuous component $(\nu^c(t), t \geq 0)$ and a purely jump component $(\nu^j(t), t \geq 0)$, i.e.

$$\begin{aligned} &\nu(t) = \nu^c(t) + \nu^j(t), \forall t \geq 0, \\ &\nu^c(\cdot) \ \text{ is continuous and } \ \nu^c(0) = 0, \\ &\nu^j(\cdot) \ \text{ is singular and } \ \nu^j(0-) = 0. \end{aligned} \tag{3.3}$$

Then, the cost of controlling is

$$c\int_0^\infty e^{-\alpha t} d|\nu|(t) = c\int_0^\infty e^{-\alpha t} d|\nu^c|(t) + \sum_{t\geq 0} ce^{-\alpha t}|\nu^j(t) - \nu^j(t-)|.$$

Notice that $\nu^c(\cdot)$ and $\nu^j(\cdot)$ have locally bounded variation, $\nu^j(\cdot)$ is right continuous with countably many discontinuities.

Based on Menaldi and Robin [23], Menaldi and Taksar [25], we obtain

Proposition 3.1

Let the assumptions (2.1) ,..., (2.8) hold. Then there exists a constant $K_0 > 1$ such that for any $0 < \lambda < 1$, any x, χ in $\Re^n$, $|\chi| = 1, \alpha > 0$ we have

$$0 \leq u_\alpha(x) \leq c|x| + (K_0 - 1)\alpha^{-1}, \tag{3.4}$$

$$|u_\alpha(x) - u_\alpha(x + \lambda\chi)| \leq C_1\lambda(c|x| + K_0\alpha^{-1}), \tag{3.5}$$

$$\begin{aligned} 0 &\leq u_\alpha(x + \lambda\chi) + u_\alpha(x - \lambda\chi) - 2u_\alpha(x) \\ &\leq C_2\lambda^2(c|x| + K_0\alpha^{-1}), \end{aligned} \tag{3.6}$$

where c is the constant of (2.2) that appears in the cost (2.9), and C_1, C_2 are the constants of assumptions (2.6), (2.7).

Proof

The convexity of u_α follows from the convexity of the holding cost h, the linearity in ν of the dynamics (2.4) and the fact that the set of control $\mathcal{V}$ is convex.

To prove (3.4) we consider the reflected diffusion process $(y_0(t), t \geq 0)$ satisfying

$$
\begin{aligned}
& dy_0(t) = [g + f y_0(t)]dt + \sigma dw(t) - y_0(t) d\xi_0(t), \quad t > 0, \\
& y_0(0) = 0, \quad \xi_0(0) = 0, \\
& |y_0(t)| \leq 1, \forall t \geq 0, \text{ and } d\xi_0(t) \neq 0 \text{ only if } |y_0(t)| = 1,
\end{aligned} \tag{3.7}
$$

where the process $(\xi_0(t), t \geq 0)$ is continuous and increasing. Now, Itô's formula applied to the function

$$(y, t) \longmapsto |y|^2 e^{-\alpha t}$$

gives

$$
\begin{aligned}
& |y_0(T)|^2 e^{-\alpha T} + \alpha \int_0^T |y_0(t)|^2 e^{-\alpha t} dt + 2 \int_0^T e^{-\alpha t} d\xi_0(t) = \\
& = 2 \int_0^T y_0(t) \cdot [g + f y_0(t)] e^{-\alpha t} dt + \int_0^T tr(\sigma\sigma^\star) e^{-\alpha t} dt + \\
& \qquad +2 \int_0^T y_0(t) \cdot \sigma e^{-\alpha t} dw(t).
\end{aligned}
$$

Hence

$$E\{\int_0^\infty e^{-\alpha t} d\xi_0(t)\} \leq [|g| + |f| + \frac{1}{2} tr\ (\sigma\sigma^\star)]\alpha^{-1}. \tag{3.8}$$

Thus, for any x in $\Re^n$ we define

$$
\begin{aligned}
\nu_x(t) &= -x - \int_0^t y_0(t)\xi_0(t) dt, \quad \forall t \geq 0, \\
y_x(t) &= y_0(t), \quad \forall t \geq 0,
\end{aligned} \tag{3.9}
$$

which satisfy the stochastic equation (2.4). We have

$$u_\alpha(x) \leq J(x, \nu_x, \alpha) = c|x| + J(0, \nu_0, \alpha).$$

In view of (3.8) we obtain (3.4) for

$$K_0 = 1 + c[|g| + |f| + \frac{1}{2}\ tr\ (\sigma\sigma^\star)] + \sup\{c|h(x)| : |x| \leq 1\}. \tag{3.10}$$

To show (3.5) we start with

$$|u_\alpha(x) - u_\alpha(x+\lambda\chi)| \le \sup\{|J(x,\nu,\alpha) - J(x+\lambda\chi,\nu,\alpha)|\}.$$

By means of (3.4) we can consider only controls $\nu(\cdot)$ which satisfy

$$J(x,\nu,\alpha) \le c|x| + (K_0 - 1)\alpha^{-1}.$$

Since

$$|h(y_x(t)) - h(y_{x+\lambda\chi}(t))| \le C_1\lambda|e^{tf}\chi|(1 + h(y_x(t))),$$

where C_1 is the constant in the hypothesis (2.6), we deduce (3.5) after noticing that

$$|e^{tf}\chi| \le 1.$$

In order to prove (3.6) we start with

$$\begin{aligned} u_\alpha(x+\lambda\chi) \;&+\; u_\alpha(x-\lambda\chi) - 2u_\alpha(x) \le \\ &\le \sup_\nu\{J(x+\lambda\chi,\nu,\alpha) + J(x-\lambda\chi,\nu,\alpha) - 2J(x,\nu,\alpha)\}. \end{aligned}$$

As before, because of

$$\begin{aligned} h(y_{x+\lambda\chi}(t)) \;&+\; h(y_{x-\lambda\chi}(t)) - 2h(y_x(t)) \le \\ &\le C_2\lambda^2|e^{tf}\chi|^2(1 + h(y_x(t))), \end{aligned}$$

where C_2 is the constant of assumption (2.7), we obtain (3.6). □

Corollary 3.2

Assume the hypotheses of Theorem 3.1 and

(3.11) $$\frac{\partial^2 h}{\partial x_i \partial x_j} \text{ is bounded in } \Re^n, \forall\, i,j = 1,\dots,n,$$

(3.12) all eigenvalues of f are strictly negative.

Then

(3.13) $$\frac{\partial^2 u_\alpha}{\partial x_i \partial x_j} \text{ is equi-bounded (in } \alpha > 0) \text{ in } \Re^n.$$

Proof.

In view of (3.11) we have

$$h(x + \lambda\chi) + h(x - \lambda\chi) - 2h(x) \le C_3\lambda^2$$

for some constant C_3. The hypothesis (3.12) implies that there is a constant $\delta > 0$ such that

$$|e^{tf}\chi|^2 \le e^{-\delta t}, \quad \forall t \ge 0.$$

Therefore

$$u_\alpha(x + \lambda\chi) + u_\alpha(x - \lambda\chi) - 2u_\alpha(x) \le C_3(\alpha + \delta)^{-1}\lambda^2, \tag{3.14}$$

which gives (3.13). □

Following Chow et al. [4], Menaldi and Robin [19], Menaldi and Taksar [25] we can show that the optimal cost (2.12) satisfies

$$\begin{aligned} & u_\alpha \in W^{2,\infty}_{loc}(\Re^n) \ \text{(locally Lipschitz first derivatives)} \\ & Lu_\alpha - \alpha u_\alpha + h \ge 0, \ \text{a.e. in } \Re^n, \\ & |\nabla u_\alpha| \le c \ \text{in } \Re^n, \\ & Lu_\alpha - \alpha u_\alpha + h = 0 \ \text{a.e. in } [|\nabla u_\alpha| < c], \end{aligned} \tag{3.15}$$

where $[|\nabla u_\alpha| < c]$ denotes the set of points x in $\Re^n$ satisfying $|\nabla u_\alpha(x)| < c$. Actually, u_α is the maximum subsolution, i.e. if u satisfies the first three conditions of (3.15) then $u \le u_\alpha$ in $\Re^n$. Since h is at least Lipschitz continuous and σ invertible the last equality in (3.15) holds pointwise and u_α is smooth in that region.

Define the open set

$$D_\alpha = \{x \in \Re^n : |\nabla u_\alpha(x)| < c\}, \tag{3.16}$$

and the sets

$$\begin{aligned} D = \{ & x \in \Re^n : \ \text{there are } r = r(x) \text{ and sequences} \\ & x_k \to x, \alpha_k \to 0 \ \text{as } k \to \infty \ \text{such that} \\ & B(x_k, r) \subset D_{\alpha_k}, \forall k = 1, 2, \ldots\}, \end{aligned} \tag{3.17}$$

$$\begin{aligned} S = \{ & x \in \Re^n : \ \text{there are sequences } x_k \to x, \\ & \alpha_k \to 0 \ \text{as } k \to \infty \ \text{such that } x_k \notin D_{\alpha_k}, \\ & \forall k = 1, 2, \ldots\}, \end{aligned} \tag{3.18}$$

where $B(x,r)$ is the open ball of radius r and center x.

Proposition 3.3

Let the assumptions (2.1), ..., (2.8) hold. Then D is bounded, open and S is closed, and

$$D \cup S = \Re^n. \tag{3.19}$$

Proof

First, we are going to prove that there exists a ball of radius $K_1 > 0$, independent of $\alpha > 0$, such that

$$D_\alpha \subset B(0, K_1), \quad \forall\, 0 < \alpha < 1. \tag{3.20}$$

Indeed, on D_α we have

$$Lu_\alpha - \alpha u_\alpha + h = 0.$$

Since u_α is convex,

$$Lu_\alpha(x) \geq (g + fx).\nabla u_\alpha(x)$$

and because

$$|\nabla u_\alpha(x)| \leq c, \quad \forall x \in \Re^n$$

we obtain

$$-Lu_\alpha(x) \leq c(|g| + |f||x|), \quad \text{a.e. in } \Re^n.$$

Thus, in view of the estimate (3.4) we have

$$h(x) \leq c|g| + (K_0 - 1) + c(\alpha + |f|)|x|, \quad \forall x \in D_\alpha. \tag{3.21}$$

By means of the hypothesis (2.8) on h we can define

$$\begin{aligned} K_1 &= \sup\{x \in \Re^d : h(x) \leq a + b|x|\}, \\ a &= (c+1)|g| + |f| + \frac{1}{2}\, tr\ (\sigma\sigma^\star) + \sup\{h(x) : |x| \leq 1\}, \\ b &= c(|f| + 1), \end{aligned} \tag{3.22}$$

to get (3.20). Hence D is bounded.

To show (3.19) we are going to establish that

$$\text{if } \ x \notin D \ \text{ then } \ x \in S. \tag{3.23}$$

Indeed, let $x \notin D$. Then for every $r > 0$, every sequences $x_k \to x$, $\alpha_k \to 0$ we can not have

$$B(x_k, r) \subset D_{\alpha_k}, \quad \forall k = 1, 2, \ldots$$

Thus, we can construct sequences $r_k \to 0, x_k \to x$, $\alpha_k \to 0$ as $k \to \infty$ such that

$$B(x_k, r_k) \cap (\Re^n \setminus D_{\alpha_k}) \neq \phi, \quad \forall k = 1, 2, \ldots$$

So, there exists a sequence y_k such that

$$y_k \in B(x_k, r_k) \setminus D_{\alpha_k}, \quad \forall k = 1, 2, \ldots$$

Therefore $y_k \to x$ as $k \to \infty$ and

$$y_k \notin D_{\alpha_k}, \quad \forall k = 1, 2, \ldots,$$

i.e. x belongs to S, by definition.

In order to prove that D is open, we use (3.19) and we establish that S is closed. Indeed, let $x_k \to x$ as $k \to \infty$ with

$$x_k \in S, \quad \forall k = 1, 2, \ldots$$

By definition, there exist sequences $x_{k,n} \to x_k$, $\alpha_{k,n} \to 0$ as $n \to \infty$ such that

$$x_{k,n} \notin D_{\alpha_{k,n}}, \quad \forall n, k = 1, 2, \ldots$$

So, we can choose $n = n(k)$ such that $x_{k,n(k)} \to x$, $\alpha_{k,n(k)} \to 0$ as $k \to \infty$ and

$$x_{k,n(k)} \notin D_{\alpha_{k,n(k)}}, \quad \forall k = 1, 2, \ldots,$$

i.e. x belongs to S. Hence S is closed and D is open. $\Box$

Theorem 3.4

Under the assumptions (2.1), ..., (2.8) the set D defined by (3.17) is nonempty. Moreover, for every $0 < \alpha < 1$ we have

$$|\nabla u_\alpha(x) - \nabla u_\alpha(x')| \leq K_2 |x - x'|, \quad \forall x, x' \in D_\alpha, \tag{3.24}$$

for some constant K_2 independent of α, and

$$u_\alpha(x + \theta \nabla u_\alpha(x)) = u_\alpha(x) + c^2\theta, \quad \forall x \notin D_\alpha, \ \forall \theta > 0, \tag{3.25}$$

where u_α is the discounted optimal cost function (2.12), and c is the constant that appears in the cost (2.9).

Proof

Since u_α is convex and continuously differentiable we have for every x in $\Re^n$,

$$u_\alpha(x+\theta\nabla u_\alpha(x)) - u_\alpha(x)] \geq \theta|\nabla u_\alpha(x)|^2, \ \ \forall\theta > 0.$$

On the other hand, the inequality

$$|\nabla u_\alpha(x)| \leq c, \ \ \forall x \in \Re^n$$

implies, for any x in $\Re^n$,

$$|u_\alpha(x+\theta\nabla u_\alpha(x)) - u_\alpha(x)| \leq c|\nabla u_\alpha(x)|\theta, \ \ \forall\theta > 0.$$

Because $|\nabla u_\alpha(x)| = c$ whenever x is not in D_α, we conclude (3.25).

Let us recall that the Schauder local estimates on elliptic partial differential equations imply that u_α has smooth second derivative on D_α and

$$Lu_\alpha(x) - \alpha u_\alpha(x) + h(x) = 0, \ \ \forall x \in D_\alpha.$$

Thus, because u_α is convex we need only to show that for some set of n independent direction $\{\chi_1, \chi_2, \dots, \chi_n\}$ in $\Re^n$,

$$\sum_{k=1}^{n} \frac{\partial^2 u_\alpha}{\partial \chi_k^2}(x) \leq K_2, \ \ \forall x \in D_\alpha, \tag{3.26}$$

for some constant K_2 independent of $0 < \alpha < 1$.

Now, to establish (3.26) we take $\chi_k = \sigma_k|\sigma_k|^{-1}$, where σ_k is the k column of the matrix σ. Then

$$\frac{\partial^2 u_\alpha}{\partial \chi_k^2} = |\sigma_k|^{-2} \sum_{i,j=1}^{n} \sigma_{ik}\sigma_{jk} \frac{\partial^2 u_\alpha}{\partial x_i \partial x_j}$$

and for x in D_α we get

$$\sum_{k=1}^{n} \frac{\partial^2 u_\alpha}{\partial \chi_k^2}(x) \leq (\min_k |\sigma_k|)^{-2}[\alpha u_\alpha(x) - (g + fx)\cdot\nabla u_\alpha(x) - h(x)],$$

in view of the inequalities (3.15) satisfied by u_α. Thus, we deduce (3.26) with

$$K_2 = 2(\min_k |\sigma_k|)^{-2}[K_0 - 1 + c|g| + c(1+|f|)K_1], \tag{3.27}$$

where K_0 and K_1 are given (3.10) and (3.22). Here, we have used the estimate (3.4) and the inclusion (3.20).

The remaining part is to show that D is nonempty. To that effect, let x_α be a point in $\Re^n$ where $u_\alpha(\cdot)$ attains its absolute minimum. Then $\nabla u_\alpha(x_\alpha) = 0$ and x_α belongs to D_α. By means of the estimate (3.24) we deduce that

$$B(x_\alpha, \varepsilon) \subset D_\alpha, \ \ \forall 0 < \alpha < 1, \ \ \forall 0 < \varepsilon \le c K_2^{-1},$$

where K_2 is the constant given by (3.27) that appears in (3.24). Therefore, any limit point of the family $\{x_\alpha, 0 < \alpha < 1\}$ belongs to D. Notice that at least one limit point exists in view of the bound (3.20). □

Let $\rho(\cdot)$ be a smooth and positive convolution kernel, i.e. $\rho(\cdot)$ is an infinite differentiable function such that

$$\rho(x) \ge 0, \ \forall x, \ \ \rho(x) = 0 \text{ if } |x| \ge 1, \ \ \int_{\Re^n} \rho(x) dx = 1.$$

Define

$$u_\alpha^\varepsilon(x) = \int_{\Re^n} u_\alpha(x - \varepsilon y) \rho(y) dy, \ \ \varepsilon > 0, \tag{3.28}$$

and

$$h^\varepsilon(x) = \int_{\Re^n} [h(x - \varepsilon y) - \varepsilon \sum_{i,j=1}^{n} f_{ij} y_j \frac{\partial u_\alpha}{\partial x_i}(x - \varepsilon y)] \rho(y) dy. \tag{3.29}$$

The inequalities (3.15) satisfied by u_α imply

$$\begin{aligned} L u_\alpha^\varepsilon - \alpha u_\alpha^\varepsilon + h^\varepsilon \ge 0 \ \text{ in } \ \Re^n, \\ |\nabla u_\alpha^\varepsilon| \le c \ \text{ in } \ \Re^n \end{aligned} \tag{3.30}$$

for any $\varepsilon, \alpha > 0$.

Consider the set

$$D_\alpha^{\delta,\varepsilon} = \{x \in \Re^n : L u_\alpha^\varepsilon(x) - \alpha u_\alpha^\varepsilon(x) + h^\varepsilon(x) < \delta\}, \tag{3.31}$$

for any $\alpha, \varepsilon, \delta > 0$. As in the proof of (3.20) in Proposition 3.3, the fact that $u_\alpha^\varepsilon(\cdot)$ is convex gives the estimate

$$D_\alpha^{\delta,\varepsilon} \subset B(0, K_1) \ \ \forall 0 < \alpha, \varepsilon, \delta < 1, \tag{3.32}$$

where the radius of the ball is now

$$
\begin{aligned}
(3.33) \qquad K_1 &= \sup\{x \in \Re^n : h^\star(x) \le a + b|x|\}, \text{ with} \\
h^\star(x) &= \inf\{h(y) : y \in \Re^n, |x-y| \le 1\}\ , \\
a &= (c+1)|g| + 2|f| + \frac{1}{2}\, tr\ (\sigma\sigma_\star) + \sup\{h(x) : |x| \le 1\}, \\
b &= c(|f|+1),
\end{aligned}
$$

which is a finite number in view of the hypothesis (2.8).

Define the set

$$
(3.34) \qquad \begin{aligned}
\tilde{D}_\alpha^\delta = \{x \in \Re^n :&\ \text{ There exist sequences} \\
x_k \to x,\ \ \varepsilon_k \to 0 \ &\text{ as } \ k \to \infty \ \text{ such that} \\
&x_k \in D_\alpha^{\delta,\varepsilon_k},\ \ \forall k = 1, 2, \ldots\}.
\end{aligned}
$$

As in Proposition 3.3 we can prove that $\tilde{D}_\alpha^\delta$ is bounded, closed and

$$
(3.35) \qquad D_\alpha \subset \tilde{D}_\alpha^\delta \subset \bar{B}(0, K_1),\ \ \forall 0 < \alpha, \delta < 1.
$$

Since D_α^δ is increasing in δ we have

$$
(3.36) \qquad D_\alpha \subset \tilde{D}_\alpha = \bigcap_{\delta > 0} \tilde{D}_\alpha^\delta \subset \bar{B}(0, K_1),\ \ \forall 0 < \alpha < 1,
$$

with $\tilde{D}_\alpha$ being a closed subset of $\Re^n$ and $\bar{B}(0, K_1)$ the closed ball of center 0 and radius K_1.

Theorem 3.5

Let the assumptions (2.1),...,(2.8) hold and $(\nu(t), t \ge 0)$ be an optimal control for the discounted cost (2.10) with a fix $\alpha > 0$ and some x in $\Re^d$. Then

$$
(3.37) \qquad P\{y(t) \in \tilde{D}_\alpha\} = 1,\ \ \forall t \ge 0,
$$

$$
(3.38) \qquad \begin{aligned}
|\nu^c|(t) &= \int_0^t \chi(y(s)) \notin D_\alpha) d|\nu^c|(s),\ \ \forall t \ge 0, \\
\nu^j(t) &= \nu^j(t-) \ \text{ if } \ y(t-) \in D_\alpha,\ \ \forall t \ge 0,
\end{aligned}
$$

where $(y(t), t \ge 0)$ is the state of the system corresponding to the control $(\nu(t), t \ge 0)$ through (3.2), and $(\nu^c(t), t \ge 0)$ (resp. $(\nu^j(t), t \ge 0)$ is the

continuous (resp. jump) component given by (3.3).
Proof

First, we apply Itô's formula for the semimartingale (cfr. Meyer [27]) to the function $u_\alpha^\varepsilon(\cdot)$, as defined by (3.28), and the process $(y(t), t \geq 0)$ given by (3.2) to obtain

$$
\begin{aligned}
u_\alpha^\varepsilon(x) &= E\{\int_0^\infty e^{-\alpha t}[\alpha u_\alpha^\varepsilon(y(t)) - L u_\alpha^\varepsilon(y(t))]dt - \\
&- \sum_{t\geq 0} e^{-\alpha t}[u_\alpha^\varepsilon(y(t)) - u_\alpha^\varepsilon(y(t-))] - \\
&- \int_0^\infty e^{-\alpha t}\nabla u_\alpha^\varepsilon(y(t)) \cdot d\nu^c(t)\}, \ \ \forall \varepsilon > 0,
\end{aligned} \tag{3.39}
$$

Notice that u_α^ε is a smooth function with polynomial growth, and the jumps of the state of the system satisfy

$$
y(t) - y(t-) = \nu^j(t) - \nu^j(t-), \ \ \forall t \geq 0. \tag{3.40}
$$

Since, $(\nu(t), t \geq 0)$ is optimal we have

$$
u_\alpha(x) = J(x, \nu, \alpha),
$$

which together with (3.39) prove

$$
\begin{aligned}
& E\{\int_0^\infty e^{-\alpha t}[h^\varepsilon - \alpha u_\alpha^\varepsilon + L u_\alpha^\varepsilon](y(t))dt\} + \\
& + E\{\sum_{t\geq 0} e^{-\alpha t}[c|\nu^j(t) - \nu^j(t-)| + u_\alpha^\varepsilon(y(t)) - u_\alpha^\varepsilon(y(t-))] + \\
& + \int_0^\infty e^{-\alpha t}[c\, d|\nu^c|(t) + \nabla u_\alpha^\varepsilon(y(t)) \cdot d\nu^c(t)]\} = \\
& = [u_\alpha(x) - u_\alpha^\varepsilon(x)] + E\{\int_0^\infty e^{-\alpha t}[h^\varepsilon - h](y(t))dt.
\end{aligned} \tag{3.41}
$$

By virtue of the inequalities (3.30), each of the two terms on the left-hand side is nonnegative. As ε goes to zero we deduce

$$
\begin{aligned}
E\{\sum_{t\geq 0} e^{\alpha t}[c|\nu^j(t) - \nu^j(t-)| + u_\alpha(y(t)) - u_\alpha(y(t-))] + \\
+ \int_0^\infty e^{-\alpha t}[cd|\nu^c|(t) + \nabla u_\alpha(y(t)) \cdot d\nu^c(t)]\} = 0,
\end{aligned}
$$

which implies (3.38), after using (3.40) and the fact that

$$|\nabla u_\alpha| \leq c \;\text{ in }\; \Re^n, \quad and|\nabla u_\alpha| < c \;\text{ in }\; D_\alpha.$$

On the other hand, we have

$$\delta \int_0^\infty e^{-\alpha t} P\{y(t) \notin D_\alpha^{\delta,\varepsilon}\} dt \leq$$

$$\leq E\{\int_0^\infty e^{-\alpha t}[h^\varepsilon - \alpha u_\alpha^\varepsilon + L u_\delta^\varepsilon](y(t))dt\} \leq r(x,\alpha,\varepsilon),$$

with

$$r(x,\alpha,\varepsilon) \to 0 \;\text{ as }\; \varepsilon \to 0.$$

Now, notice that according to the definition (3.34) $\tilde{D}_\alpha^\delta$ is the superior limit of the family of sets $\{D_\alpha^{\delta,\varepsilon}, 0 < \varepsilon \leq 1\}$. Therefore

$$P\{y(t) \notin \tilde{D}_\alpha^\delta\} \leq \liminf_{\varepsilon \to 0} P\{y(t) \notin D_\alpha^{\delta,\varepsilon}\}.$$

Summing up, we conclude that

$$\int_0^\infty e^{-\alpha t} P\{y(t) \notin \tilde{D}_\alpha^\delta\} dt = 0.$$

Hence, we deduce (3.37) after using the right continuity of the process $(y(t), t \geq 0)$ and (3.36). □

Remark 3.1

The property (3.35) in Theorem 3.5 expresses the fact that "it is not optimal to let the system exits the region $\tilde{D}_\alpha$". Also, the property (3.38) says that "it is not optimal to control the system inside the region D_α". By the way, because $\tilde{D}_\alpha$ is bounded, we have shown that any optimal control for the discounted cost, keeps the system on a bounded set, uniformly w.r.t. α in (0,1]. □

Corollary 3.6

Under the assumptions (2.1),...,(2.8) and

$$\text{(3.42)} \qquad \text{all eigenvalues of } f \text{ are strictly negative}$$

we have the estimate

$$\text{(3.43)} \qquad |\nabla u_\alpha(x) - \nabla u_\alpha(x')| \leq K_2|x - x'|, \;\; \forall x, x' \in \Re^n,$$

for some constant K_2 independent of α.

Proof

By means of the technique of Menaldi and Robin [19], Menaldi and Taksar [26], Taksar [40], we can prove that for each fix discount factor $\alpha > 0$ and any initial state x, there exists an optimal control $(\nu_x^\alpha(t), t \geq 0)$, i.e.

$$u_\alpha(x) = J(x, \nu_x^\alpha, \alpha), \ \ \forall x \in \Re^n, \ \alpha > 0.$$

Then, Theorem 3.5 implies that for some K_1 and any $0 < \alpha < 1$

$$P\{|y_x^\alpha(t)| \leq K_1\} = 1, \ \ \forall t \geq 0. \tag{3.44}$$

As in Proposition 3.1 and Corollary 3.2 we start with

$$\begin{aligned} u_\alpha(x + \lambda\chi) + u_\alpha(x - \lambda\chi) - 2u_\alpha(x) \leq \\ \leq J(x + \lambda\chi, \nu_x^\alpha, \alpha) + J(x - \lambda\chi, \nu_x^\alpha, \alpha) - 2J(x, \nu_x^\alpha, \alpha), \end{aligned} \tag{3.45}$$

where χ is any direction. In view of (3.44) and the hypothesis (2.7) we have

$$h(y_{x+\lambda\chi}^\alpha(t)) + h(y_{x-\lambda\chi}^\alpha(t)) - 2h(y_x^\alpha(t)) \leq C_2\lambda^2|e^{tf}\chi|^2 \sup\{(1 + h(y))) : |y| \leq K_1\}.$$

Since the assumption (3.42) implies that there is a constant $\delta > 0$ such that

$$|e^{tf}\chi|^2 \leq e^{-\delta t}, \ \ \forall t \geq 0,$$

we deduce the estimate (3.43) with

$$K_2 = C_2\delta^{-1} \sup\{(1 + h(y)) : |y| \leq K_1\},$$

where K_1 is the constant used in (3.44) and given by (3.33). □

Corollary 3.7

Under the hypotheses of Theorem 3.5 we have

$$\begin{aligned} u_\alpha(x) = E\{\int_0^T [h(y_x(t)) - \alpha u_\alpha(y_x(t))]dt + c|\nu|(T)\} + \\ +E\{u_\alpha(y_x(T))\}, \ \ \forall T \geq 0, \end{aligned} \tag{3.46}$$

where $u_\alpha(x)$ is the optimal cost (2.13) and $(y_x(t), t \geq 0)$ is the state process (3.2) associated with the optimal control $(\nu(t), t \geq 0)$.

Proof

As in Theorem 3.5, we apply Itô's formula for the semimartingale to get for every $\varepsilon > 0, T \geq 0$

$$u_\alpha^\varepsilon(x) = E\{u_\alpha^\varepsilon(y(T))\} - E\{\int_0^T Lu_\alpha^\varepsilon(y(t))dt +$$
$$+ \sum_{0 \leq t \leq T} [u_\alpha^\varepsilon(y(t)) - u_\alpha^\varepsilon(y(t-))] + \int_0^T \nabla u_\alpha^\varepsilon(y(t)) \cdot d\nu^c(t)\}.$$

The delicate point is to pass to the limit in the above equality. We proceed as follows

$$E\{\int_0^T [h(y(t)) - \alpha u_\alpha(y(t)) + Lu_\alpha^\varepsilon(y(t))]dt\} =$$
$$= E\{\int_0^T [h(y(t)) - h^\varepsilon(y(t))]dt + \int_0^T [\alpha u_\alpha^\varepsilon(y(t)) - \alpha u_\alpha(y(t))]dt\} +$$
$$+ E\{\int_0^T [h^\varepsilon(y(t)) - \alpha u_\alpha^\varepsilon(y(t)) + Lu_\alpha^\varepsilon(y(t))]dt\} = I + II.$$

Because $h^\varepsilon \to h$, $u_\alpha^\varepsilon \to u_\alpha$ locally uniformly on $\Re^n$ as $\varepsilon \to 0$ we obtain $I \to 0$ as $\varepsilon \to 0$. On the other hand

$$0 \leq II \leq e^{\alpha T} E\{\int_0^\infty e^{-\alpha t}[h^\varepsilon - \alpha u_\alpha^\varepsilon + Lu_\alpha^\varepsilon](y(t))dt\},$$

and in view of equality (3.41), the right-hand limit goes to zero as $\varepsilon \to 0$. Hence $II \to 0$ as $\varepsilon \to 0$, i.e.

$$-E\{\int_0^T Lu_\alpha^\varepsilon(y(t))dt\} \to E\{\int_0^T [h(y(t)) - \alpha u_\alpha(y(t))]dt$$

as $\varepsilon \to 0$.

Similarly, from (3.41) we deduce

$$-E\{\sum_{0 \leq t \leq T} [u_\alpha^\varepsilon(y(t)) - u_\alpha^\varepsilon(y(t-))] + \int_0^T \nabla u_\alpha^\varepsilon(y(t)) \cdot d\nu^c(t)\}$$

converges to

$$cE\{|\nu|(T)\},$$

i.e. (3.46) is valid. □

4. The ergodic value and potential function

We will study the convergence of the optimal discounted cost (2.12) to the optimal ergodic cost (2.13).

Proof of Theorem 2.1

First, in view of the estimate (3.4) in Proposition 3.1, the family $\{\alpha u_\alpha(\cdot), 0 < \alpha \leq 1\}$ is locally equibounded in $\Re^n$, i.e.

$$0 \leq \alpha u_\alpha(x) \leq \alpha c|x| + K_0, \quad \forall x \in \Re^n, \ \forall 0 < \alpha \leq 1. \tag{4.1}$$

From the inequalities (3.15) we have

$$|\nabla u_\alpha(x)| \leq c, \quad \forall x \in \Re^n, \quad \forall 0 < \alpha \leq 1. \tag{4.2}$$

Hence, there exist a number $\lambda_0 \geq 0$ and a sequence $\alpha_k \to 0$ as $k \to \infty$ such that

$$\alpha_k u_{\alpha_k}(x) \to \lambda_0 \ \text{ locally uniformly as } \ k \to \infty. \tag{4.3}$$

Next, we are going to prove that for any control $(\nu(t), t \geq 0)$ such that the ergodic cost (2.11) is finite, i.e. $K(x,\nu) < \infty$, we have

$$J(x,\nu,\alpha) < \infty, \quad \forall \alpha > 0, \tag{4.4}$$

where $J(x,\nu,\alpha)$ is the discounted cost (2.10), and also for every $\varepsilon > 0$ there exists $T_0 = T_0(\varepsilon, x, \nu)$ such that

$$E\{\int_0^T h(y(t))dt + c|\nu|(T)\} \leq [K(x,\nu) + \varepsilon]T, \quad \forall T \geq T_0. \tag{4.5}$$

Indeed, the condition (4.5) follows from the definition of the superior limit (2.11). In order to establish (4.4) we denote by

$$q(t) = E\{\int_0^t h(y(s))ds + c|\nu|(t)\}, \quad \forall t \geq 0.$$

A simple integration by parts shows that

$$J(x,\nu,\alpha) = \lim_{T\to\infty} [e^{-\alpha T} q(T) + \alpha \int_0^T e^{-\alpha t} q(t)dt].$$

By virtue of (4.5) we deduce that the right-hand side does not exceed

$$\alpha \int_0^{T_0} e^{-\alpha t}[K(x,\nu)+\varepsilon]T_0 dt + \alpha \int_{T_0}^{\infty} e^{-\alpha t}[K(x,\nu)+\varepsilon]t dt,$$

i.e.

$$J(x,\nu,\alpha) \leq [K(x,\nu)+\varepsilon](T_0+\alpha^{-1})e^{-\alpha T_0}, \tag{4.6}$$

which gives (4.4).

Now, we will show that the limit of αu_α does not exceed the optimal ergodic cost (2.13), i.e.

$$\limsup_{\alpha\to 0} \alpha u_\alpha(x) \leq \lambda, \quad \forall x \in \Re^n. \tag{4.7}$$

Indeed, for every $\varepsilon > 0$ there is control ν such that

$$K(x,\nu) \leq \lambda + \varepsilon,$$

where $K(x,\nu)$ is the ergodic cost (2.11). In view of the estimate (4.6) we get

$$\alpha u_\alpha(x) \leq \alpha J(x,\nu,\alpha) \leq (\lambda+2\varepsilon)(\alpha T_0+1)e^{-\alpha T_0}.$$

Since $\varepsilon > 0$ is arbitrary, this implies (4.7).

In order to conclude, we need only to show that the limit value λ_0 in (4.3) coincides with the optimal ergodic cost λ in (2.13). To that purpose, we are going to prove that for every $\varepsilon > 0$ and any x in $\Re^n$ there exists a control $\nu_{\varepsilon,x}$ such that

$$K(x,\nu_{\varepsilon,x}) \leq \lambda_0 + \varepsilon. \tag{4.8}$$

Indeed, let ν_x^α be an optimal control for the discounted cost with $\alpha > 0$ to be selected later. By means of (3.46) in Corollary 3.7 we have

$$E\{\int_0^T \alpha u_\alpha(y(t))dt\} = E\{\int_0^T h(y(t))dt + c|\nu_x^\alpha|(T)\} +$$
$$+E\{u_\alpha(y(T))\} - u_\alpha(x), \quad \forall T \geq 0.$$

Since the state process $(y(t), t \geq 0)$ remains in a bounded set a.s., uniformly in x and $0 < \alpha < 1$, we obtain

$$K(x,\nu_x^\alpha) = \limsup_{T\to\infty} \frac{1}{T} E\{\int_0^T \alpha u_\alpha(y(t))dt\}.$$

Hence, if we choose $\nu_{\varepsilon,x} = \nu_x^\alpha$ with $\alpha = \alpha_k$,

$$|\alpha_k u_{\alpha_k}(y) - \lambda_0| \le \varepsilon, \ \ \forall\, |y| \le K_1,$$

where K_1 the constant given by (3.33), then we deduce (4.8). □

Now we will study the potential function $v(x)$. Let us define

$$v_\alpha(x) = u_\alpha(x) - u_\alpha(0), \ \ \forall x \in \Re^n, \ \ \forall 0 < \alpha < 1. \tag{4.9}$$

In view of the condition (3.15) we have

$$\begin{aligned} & v_\alpha \in W^{2,\infty}_{loc}(\Re^n), \\ & Lv_\alpha + h \ge \alpha u_\alpha \ \text{ a.e. in } \ \Re^n, \\ & |\nabla v_\alpha| \le c \ \text{ in } \ \Re^n, \\ & Lv_\alpha + h = \alpha u_\alpha \ \text{ a.e. in } \ D_\alpha, \end{aligned} \tag{4.10}$$

where the open set D_α is given by (3.16).

Proof of Theorem 2.2

First, because the gradients $(\nabla u_\alpha, 0 < \alpha \le 1)$ are bounded, there exist a Lipschitz continuous function v in $\Re^n$ and a subset Λ of (0,1] having 0 as limiting point such that as $\alpha \to 0$, α in Λ we have

$$\begin{aligned} & v_\alpha \to v \ \text{ locally uniformly in } \ \Re^n, \\ & \text{at each point } \ x \ \text{ with rational coordinates} \\ & \text{the gradient } \ \nabla v_\alpha(x) \ \text{ is convergent.} \end{aligned} \tag{4.11}$$

Since v_α is convex for any α, the limiting function v is also convex.

Thus, in the Schwartz' distribution sense we have

$$\begin{aligned} & Lv + h \ge \lambda \ \text{ in } \ \mathcal{D}'(\Re^n), \\ & |\nabla v| \le c \ \text{ a.e. in } \ \Re^n. \end{aligned} \tag{4.12}$$

Actually, the fact that v is convex implies that Lv is a Radon measure, so the first inequality in (4.12) holds as measures.

Let us slightly modify the definition (3.17) of the set D. We say that a point x_0 belongs to the set $D = D_\Lambda$ if and only if there exist a number $r_0 > 0$ and sequences $x_k \to x_0$, $\alpha_k \to 0$ as $k \to \infty$ such that for every k

$$B(x_k, r_0) \subset D_{\alpha_k}, \ \ \alpha_k \in \Lambda. \tag{4.13}$$

It is clear that as in Proposition 3.3 and Theorem 3.4 we can show that D is open, bounded and nonempty. Thus, we are going to prove that

$$Lv + h = \lambda \text{ in } \mathcal{D}'(D). \tag{4.14}$$

Indeed, let x_0 be any point D and let φ be any test function with support in $B(x_0, r_0)$ where $r_0 > 0$ is given in (4.13). Then, to establish (4.14) it suffices to show that

$$\int_{\Re^n} [v(x) L^\star \varphi(x) + h(x)\varphi(x)] dx = \lambda, \tag{4.15}$$

where $L^\star$ is the adjoint operator associated with L. To that purpose, we notice that the test function

$$\varphi_k(x) = \varphi(x - x_0 + x_k)$$

has support in $B(x_k, r_o)$. Therefore in view of (4.13) and the fact that

$$Lv_\alpha + h = \alpha u_\alpha \text{ in } \mathcal{D}'(D_\alpha)$$

we deduce

$$\int_{\Re^n} [v_{\alpha_k}(x) L^\star \varphi_k(x) + h(x)\varphi_k(x)] dx = \int_{\Re^n} \alpha_k u_{\alpha_k}(x)\varphi_k(x) dx$$

for every k. By means of (4.11) and the facts that

$$\alpha u_\alpha \to \lambda \text{ as } \alpha \to 0, \ \alpha \in \Lambda, \text{ locally uniformly,}$$

we obtain (4.15), after taking limit in k.

Since σ is invertible, the local Schauder estimate on (4.14) implies that v is smooth on D. Because v is convex, the technique of the Theorem 3.4 applies to the function v, i.e. there exists a constant $K_2 > 0$ such that

$$|\nabla v(x) - \nabla v(x')| \leq K_2 |x - x'|, \ \forall x, x' \in D, \tag{4.16}$$

and

(4.17) at each point x in $\Re^n$ where the gradient of v exists and $|\nabla v(x)| = c$ we have $v(x + \theta \nabla v(x)) = v(x) + c^2\theta, \ \forall \theta > 0,$

Hence, we have established that v belongs to $W^{2,\infty}(D)$. Also because the minimum of v is attained in D, the function v is bounded from below in the whole $\Re^n$.

Next, we will prove that each point x in $\Re^n \setminus D$ where the gradient of v exists we have $|\nabla v(x)| = c$, i.e.

$$|\nabla v| = c \text{ a.e. in } \Re^n \setminus D. \tag{4.18}$$

Indeed, if x_0 belongs to $\Re^n \setminus D$ then it suffices to show that the subdifferential of v at x_0 contains a vector of length equal to c, i.e.

$$\begin{gathered} \text{there is } p \text{ in } \Re^n \text{ such that } |p| = c \text{ and} \\ v(x_0 + \Delta x) - v(x_0) \geq p \cdot \Delta x, \quad \forall \Delta x \in \Re^n. \end{gathered} \tag{4.19}$$

To that purpose, because x_0 is in $\Re^n \setminus D$ there exist sequences $x_k \to x_0$, $\alpha_k \to 0$ as $k \to \infty$ such that (x_k, α_k) belongs to $(\Re^n \setminus D_{\alpha_k}, \Lambda)$ for every k. Thus

$$\begin{gathered} v_\alpha(x_k + \Delta x) - v_\alpha(x_k) \geq \nabla v_\alpha(x_k) \cdot \Delta x, \quad \forall \Delta x \in \Re^n, \\ |\nabla v_\alpha(x_k)| = c, \quad \forall k = 1, 2, \ldots, \forall 0 < \alpha \leq 1. \end{gathered} \tag{4.20}$$

Hence, we can find a subsequence of $\{\alpha_k, k = 1, 2, \ldots\}$, denoted by $\{\alpha_{k(n)}, n = 1, 2, \ldots\}$, and a vector p such that

$$\nabla v_{\alpha_{k(n)}}(x_{k(n)}) \to p \text{ as } n \to \infty, \quad |p| = c.$$

In view the convergence (4.11), we can take $\alpha = \alpha_{k(n)}$ and $k = k(n)$ in (4.20). As n goes to ∞ we get (4.19).

Now, in order to prove (2.16) we need only to prove that for every x in ∂D and any sequence $x_n \to x$ as $n \to \infty$ with x_n in D for every $n = 1, 2, \ldots$, we have

$$|\nabla v(x_n)| \to c \text{ as } n \to \infty. \tag{4.21}$$

Hence, because x_n belongs to D there exist $r_n > 0$ and sequences $x_{n,k} \to x_n$, $\alpha_{n,k} \to 0$ as $k \to \infty$ such that for every $n, k = 1, 2, \ldots$,

$$B(x_{n,k}, r_n) \subset D_{\alpha_{n,k}} \text{ and } \alpha_{n,k} \in \Lambda.$$

Let us define

$$\varepsilon_{n,k} = dist(x_{n,k}, \partial D_{\alpha,k}).$$

Notice that

$$\inf\{\varepsilon_{n,k} : k = 1, 2, \ldots\} \geq r_n > 0, \ \ \forall n.$$

However, the fact that x does not belong to D implies

$$\inf\{\varepsilon_{n,k} : n, k = 1, 2, \ldots\} = 0,$$

i.e.

$$r_n \to 0 \ \text{ as } \ n \to \infty.$$

Hence, we can construct sequences $x'_n \to x$, $\alpha_n \to 0$, $\varepsilon_n \to 0$ as $n \to \infty$ such that for every n,

$$x_n \in B(x'_n, \varepsilon_n) \subset D_{\alpha_n}, \ \ \alpha_n \in \Lambda,$$

$$\partial B(x'_n, \varepsilon_n) \cap \partial D_{\alpha_n} \neq \phi.$$

So, by taking points $(y_n, n = 1, 2, \ldots)$ in the above interception we get

$$|x'_n - y_n| = \varepsilon_n, \ \ y_n \in \partial D_{\alpha_n}.$$

Summing up, we have sequences $x_n \to x$, $y_n \to x$, $\alpha_n \to 0$ as $n \to \infty$ such that for every $n = 1, 2, \ldots$ the number α_n is in Λ, and the points x_n belongs to $D_{\alpha_n} \cap D$ and y_n belongs to ∂D_{α_n}. Moreover, we can choose a sequence $z_n \to x$ as $n \to \infty$ such that z_n belongs to $D_{\alpha_n} \cap D$ and has rational coordinates.

Therefore

$$|\nabla v(x_n)| \leq |\nabla v_{\alpha_n}(y_n)| + |\nabla v_{\alpha_n}(y_n) - \nabla v_{\alpha_n}(z_m)| +$$

$$+ |\nabla v_{\alpha_n}(z_m) - \nabla v(z_m)| + |\nabla v(z_m) - \nabla v(x_n))|.$$

Since $y_n \in \partial D_{\alpha_n}$ we have

$$|\nabla v_{\alpha_n}(y_n)| = c,$$

and in view of the estimate (3.24) of Theorem 3.4 (which is actually valid for x, x' in $D_\alpha \cup \partial D_\alpha$) and the inequality (4.16) we deduce

$$0 \leq c - |\nabla v(x_n)| \leq K_2(|y_n - z_m| + |z_m - x_n|) +$$

$$+ |\nabla v_{\alpha_n}(z_m) - \nabla v(z_m)|, \ \ \forall n, m.$$

Hence, by virtue of the convergence (4.11) we obtain

$$\limsup_{n \to \infty} [c - |\nabla v(x_n)| \leq 2K_2 |z_m - x|, \ \ \forall m,$$

which proves (4.21).

5. The Optimal Control

Finally, it remains to construct an optimal ergodic (or stationary) control. To that end, we assume that the domain D and the potential value function v satisfy:

(5.1) there exists a twice differentiable function ρ such that

$$D = \{x \in \Re^n : \rho(x) < 0\},$$
$$\partial D = \{x \in \Re^n : \rho(x) = 0\},$$
$$|\nabla \rho(x)| \geq 1, \quad \forall x \in \partial D$$

and

(5.2) there exists a function $M(x)$ from a neighborhood of ∂D into the set of symmetric matrices $n \times n$, which is twice-continuously differentiable and

$$z \cdot M(x) z > 0, \quad \forall z \in \Re^n, z \neq 0, \forall x,$$
$$-\nabla v(x) = M(x) \nabla \rho(x), \quad \forall x \in \partial D,$$

i.e., the free boundary ∂D and the potential v are smooth, and ∇v is never tangent to ∂D. Under these assumptions we can build the reflected diffusion process on $\bar{D}$ (e.g. Freidlin [7], Lions and Sznitman [17], McKean, Jr. [18], Menaldi and Robin [24], Meyer [27], Nakao [28], Saisho[ ], Sato and Ueno [30], Skorokhod [31], Stroock and Varadham [33], Venttsel [42], Watanabe [43], and the recent books Bensoussan and Lions [2], Chung and Williams [5], Ethier and Kurtz [6], Harrison [9], Ikeda and Watanabe [10] and others). Precisely, for each x in $\bar{D}$ there exist a continuous process $(y_x(t), t \geq 0)$ and a continuous and nondecreasing process $(\xi_x(t), t \geq 0)$ which are adapted to the Wiener process $(\Omega, \mathcal{F}, P, \mathcal{F}(t), w(t), t \geq 0)$ such that

(5.3)
$$\begin{aligned} & dy_x(t) = [g + f y_x(t)]dt + \sigma dw(t) - \nabla v(y_x(t)) d\xi_x(t), \\ & y_x(0) = x, \quad \xi_x(0) = 0, \\ & y_x(t) \in \bar{D}, \quad \forall t \geq 0, \\ & \xi_x(t) = \int_0^t \chi(y_x(t) \in \partial D) d\xi_x(t), \ \forall t \geq 0. \end{aligned}$$

Then, we define for each x in $\bar{D}$

(5.4)
$$\nu_x(t) = -\int_0^t \nabla v(y_x(s)) d\xi_x(s), \quad \forall t \geq 0.$$

Hence, Itô's formula gives

$$E\{v(y_x(T))\} = v(x) + E\{\int_0^T Lv(y_x(t))dt -$$
$$- \int_0^T |\nabla v(y_x(t))|^2 d\xi_x(t)\}, \ \ \forall T \geq 0.$$

Since

$$\begin{aligned} |\nabla v(y_x(t))|^2 &= c^2 \text{ if } y_x(t) \in \partial D, \\ Lv(y_x(t)) &= \lambda - h(y_x(t)), \ \ \forall t \geq 0, \\ d\xi_x(t) &= 0 \text{ if } y_x(t) \in D, \\ |\nu_x|(t) &= c\xi_x(t), \ \ \forall t \geq 0, \end{aligned}$$

we deduce

$$\lambda = \frac{1}{T} E\{\int_0^T h(y_x(t))dt + c|\nu_x|(T)\} +$$
$$+ \frac{1}{T} E\{v(y_x(T)) - v(x)\}, \ \ \forall T > 0.$$

So, as T goes to ∞ we obtain

$$\lambda = K(x, \nu_x), \ \ \forall x \in \bar{D}. \tag{5.5}$$

Because v is at least continuously differentiable in the whole $\Re^n$, for each x in $\Re^n$ we may consider the ordinary differential equation

$$\begin{aligned} \dot{\eta}_x(t) &= -\nabla v(\eta_x(t)), \ \ \forall t \geq 0, \\ \eta_x(0) &= x, \end{aligned} \tag{5.6}$$

and the first entry time in $\bar{D}$, i.e.

$$\tau_x = \inf\{t \geq 0 : \eta_x(t) \in \bar{D}\}. \tag{5.7}$$

By virtue of the equality

$$v(\eta_x(\tau_x)) = v(x) - \int_0^{\tau_x} |\nabla v(\eta_x(t))|^2 dt$$

and the fact that v is bounded from below, we deduce that

$$0 \leq \tau_x < \infty, \ \ p(x) = \eta_x(\tau_x) \in \bar{D}, \ \ \forall x \in \Re^n. \tag{5.8}$$

Thus, we define for any x in $\Re^n$ the control

$$\nu_x(t) = p(x) - x - \int_0^t \nabla v(y_{p(x)}(s)) d\xi_{p(x)}(s), \quad \forall t \geq 0, \tag{5.9}$$

where the processes $(y_{p(x)}(t), \xi_{p(x)}, t \geq 0)$ are given by (4.24) with x replaced by $p(x)$. It is clear then that $(\nu_x(t), t \geq 0)$ as in (4.30) is an optimal ergodic control for initial state x in $\Re^n$. □

Final Comments

Once the convergence (2.14) of Theorem 2.1 has been established, it is clear that ε-optimal controls of the α-discounted problem produce ε-optimal controls for the ergodic problem, as α vanishes.

Usually, if we look for a pair (λ, v) as the solution of the Hamilton-Jacobi-Bellman then, the constant λ is unique and the potential value function v is unique up to an additive constant. However, we could not prove that fact completely, i.e. that the conditions (2.15) and (2.16) are enough to determine a unique solution.

Another hard question is the regularity of the free boundary ∂D. This is very related to the $W^{3,\infty}$-regularity of the value function v. Results in this direction can be found in Soner and Shreve [32], where a two-dimensional case with unidirectional control is studied, and in Williams, Chow and Menaldi [43], where local regularity (outside of some lower dimensional region) is obtained.

Notice that the potential value function v is in $W^{2,\infty}(\Re^n)$ if the matrix f has all eigenvalues strictly negative. This follows from the estimate (3.43) of Corollary 3.6. □

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