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Green and Poisson Functions with Wentzell Boundary Conditions

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Abstract: We discuss the construction and estimates of the Green and Poisson functions associated with a parabolic second order integro-differential operator with Wentzell boundary conditions.

Keywords: Green and Poisson functions, Wentzell boundary conditions, Levy processes, fundamental solutions, stochastic differential equations with jumps, integro-differential second-order operator, heat-kernel estimates.

AMS (MOS) Subject Classification: 60J60, 60J75, 35K20, 35K65

1 Introduction

Let A_2 be a second order (uniformly) elliptic differential operator with bounded and Hölder continuous coefficients in the open half-space $\mathbb{R}_+^d = \{x : x_d > 0\}$, i.e.,

$$\begin{cases} A_2(t)\varphi(x) := A_0(t)\varphi(x) + \sum_{i=1}^d a_i(t, x)\partial_i\varphi(x) - a_0(t, x)\varphi(x), \\ A_0(t)\varphi(x) := \frac{1}{2} \sum_{i,j=1}^d a_{ij}(t, x)\partial_{ij}\varphi(x), \end{cases} \quad (1.1)$$

where $a_0 \geq 0$, $a_{ij} = a_{ji}$ for any i, j and φ is any continuously differentiable function with a compact support in the closed half-space \mathbb{R}_+^d . Also, we suppose that for some positive constants c_0, C_0 and any $t > 0, x$ in \mathbb{R}_+^d we have

$$c_0|\xi|^2 \leq \sum_{i,j=1}^d a_{ij}(t, x)\xi_i\xi_j \leq C_0|\xi|^2, \quad \forall \xi \in \mathbb{R}^d, \quad (1.2)$$

and

$$a_{ij}, a_i, a_0 \in C_b^\alpha([0, \infty) \times \mathbb{R}_+^d), \quad \forall i, j. \quad (1.3)$$

Now, consider an integro-differential operator $I(t)$ with bounded and Hölder continuous coefficients, i.e.,

$$\begin{cases} I(t)\varphi(x) = \int_{\mathbb{R}_+^m} [\varphi(x + j(\zeta, t, x)) - \varphi(x) - j(\zeta, t, x) \cdot \nabla\varphi(x)] \times \\ \times m(\zeta, t, x)\pi(d\zeta), \end{cases} \quad (1.4)$$

where the coefficients satisfy

$$\begin{cases} |j(\zeta, t, x)| \mathbb{1}_{\{m(\zeta, t, x) > 0\}} \leq \bar{j}(\zeta), & 0 \leq m(\zeta, t, x) \leq 1, \\ \int_{\{\bar{j} < 1\}} [\bar{j}(\zeta)]^\gamma \pi(d\zeta) + \int_{\{\bar{j} \geq 1\}} \bar{j}(\zeta)\pi(d\zeta) \leq C_0, \end{cases} \quad (1.5)$$

for every ζ, t, x and for some positive constants $C_0, 0 \leq \gamma < 2$ and some positive measurable function $\bar{j}(\cdot)$. Also there exist another positive measurable function (again denoted by) $\bar{j}(\cdot)$ and some constant

$M_0 > 0$ such that for any t, t', x, x' and ζ we have

$$\begin{cases} |j(\zeta, t, x) - j(\zeta, t', x')| \leq \bar{j}(\zeta)[|t - t'|^{\alpha/2} + |x - x'|^\alpha], \\ |m(\zeta, t, x) - m(\zeta, t', x')| \leq M_0[|t - t'|^{\alpha/2} + |x - x'|^\alpha], \\ \int_{\{\bar{j} < 1\}} [\bar{j}(\zeta)]^\gamma \pi(d\zeta) + \int_{\{\bar{j} \geq 1\}} \bar{j}(\zeta) \pi(d\zeta) \leq M_0. \end{cases} \quad (1.6)$$

We also assume that $j(\zeta, t, x)$ is continuously differentiable in x for any fixed ζ , and that there exists a constant $c_0 > 0$ such that

$$c_0|x - x'| \leq |(x - x') + \theta[j(\zeta, t, x) - j(\zeta, t, x')]| \leq c_0^{-1}|x - x'|, \quad (1.7)$$

for any t, x, x' and $0 \leq \theta \leq 1$. Since $I(t)$ is a non-local operator, we need to assume

$$j_d(\zeta, t, x) \geq 0, \quad \forall \zeta, t, x \quad (1.8)$$

so that $I(t)$ acts on functions defined only on the half-space \mathbb{R}_+^d .

Next, for a complete second order integro-differential operator of the form $A = A_2 + I$, let B be a (uniform) *Wentzell type* boundary second order differential operator with bounded and Hölder continuous coefficients, i.e.,

$$\begin{cases} B(t)\varphi(x) := B_0(t)\varphi(x) + b_d(t, \tilde{x})\partial_d\varphi(x) - \rho(t, \tilde{x})A(t)\varphi(x), \\ B_0(t) := \frac{1}{2} \sum_{i,j=1}^{d-1} b_{ij}(t, \tilde{x})\partial_{ij} + \sum_{i=1}^{d-1} b_i(t, \tilde{x})\partial_i - b_0(t, \tilde{x}), \end{cases} \quad (1.9)$$

where $b_{ij} = b_{ji}$ is a symmetric non-negative definite matrix, $b_0, b_d, \rho \geq 0$, and for some positive constant c_0 and any $t > 0, \tilde{x}$ in \mathbb{R}^{d-1} we have

$$\rho(t, \tilde{x}) \geq c_0 \quad \text{or} \quad b_d(t, \tilde{x}) \geq c_0, \quad (1.10)$$

and

$$b_{ij}, b_i, b_0, \rho \in C_b^\alpha([0, \infty) \times \mathbb{R}^{d-1}), \quad \forall i, j. \quad (1.11)$$

Note that all coefficients are trivially extended to the whole half-space. Sometimes, we will use the notation $A\varphi(x) = A(t)\varphi(x) = A(t, x)\varphi(x)$ and $B\varphi(x) = B(t)\varphi(x) = B(t, \tilde{x})\varphi(x)$ to emphasize the (t, x) -dependency of the coefficients.

Consider the boundary value problem with a terminal condition (instead of an initial condition)

$$\begin{cases} \partial_t u(t, x) + A(t)u(t, x) = f(t, x), \quad \forall t < T, x \in \mathring{\mathbb{R}}_+^d, \\ B(t)u(t, \tilde{x}, 0) + g(t, \tilde{x}) = 0, \quad \forall t < T, x \in \mathbb{R}^{d-1}, \\ u(T, x) = \varphi(x), \quad \forall x \in \mathring{\mathbb{R}}_+^d, \end{cases} \quad (1.12)$$

and the representation formula

$$\begin{cases} u(t, x) = \int_t^T ds \int_{\mathbb{R}^{d-1}} P_{A,B}(t, x, s, \tilde{y})g(s, \tilde{y})d\tilde{y} + \\ + \int_t^T ds \int_{\mathbb{R}_+^d} G_{A,B}(t, x, s, y)f(s, y)dy + \int_{\mathbb{R}_+^d} G_{A,B}(t, x, T, y)\varphi(y)dy, \end{cases} \quad (1.13)$$

where $G_{A,B}$ and $P_{A,B}$ are the Green and the Poisson functions. If $\rho > 0$ then the Green function $G_{A,B}$ contains a Dirac delta measure on the boundary $\partial\mathbb{R}_+^d = \mathbb{R}^{d-1} \times \{0\}$. Certainly, the compatibility condition $B(T)\varphi(\tilde{x}, 0) + \psi(T, \tilde{x}) = 0$ for any \tilde{x} in \mathbb{R}^{d-1} is necessary.

We consider a terminal condition so that a clean and neat relation with a stochastic differential equation with boundary condition can be written. Namely, the measure $P = P_{t,x}$ generated by the pair $A(\cdot)$ and $B(\cdot)$ with initial condition $P\{x(t) = x\}$, $x(t) := \omega(t)$, is a probability measure on the canonical space $C([t, \infty), \mathbb{R}_+^d)$ such that there exists an adapted, non-decreasing and continuous process $\ell(t)$, so-called *local time*, satisfying the conditions $\ell(t) = 0$,

$$\begin{aligned} \int_t^s \mathbb{1}_{\partial\mathbb{R}_+^d}(x(r))d\ell(r) &= \ell(s), \quad \forall s \geq t, \\ \int_t^s \mathbb{1}_{\partial\mathbb{R}_+^d}(x(r))dr &= \int_t^s \rho(r, x(r))d\ell(r), \quad \forall s \geq t, \end{aligned}$$

and such that the process

$$M_\varphi(s) := \varphi(x(s)) - \varphi(x(t)) - \int_t^s A(r)\varphi(x(r))dr - \int_t^s B(s)\varphi(x(r))d\ell(r),$$

is a martingale for any smooth function φ . Hence, we have the representation

$$u(t, x) = \mathbb{E}^{tx} \left\{ e(t, T)\varphi(T, x(T)) + \int_t^T e(s, T)f(s, x(s))ds + \int_t^T e(s, T)g(s, x(s))d\ell(s) \right\},$$

where the exponential process is given by

$$e(t, T) := \exp \left[- \int_t^T a_0(s, x(s))ds \right],$$

and $\mathbb{E}^{tx}\{\cdot\}$ denotes the mathematical expectation with respect to the probability measure P_{tx} . In short, the Green function is the transition probability density of the Markov process $x(\cdot)$ and the Poisson function is the transition density of the local time $\ell(\cdot)$.

Notice that the boundary operator $B(\cdot)$ is determined by the diffusion measure up to a multiplicative constant, i.e., if $\bar{B}(s)\varphi(\tilde{y}) := \alpha(s, \tilde{y})B(s)\varphi(\tilde{y})$, with $\alpha(s, \tilde{y})$ being a positive and continuous functions on the boundary $\partial\mathbb{R}_+^d \sim \mathbb{R}^{d-1}$, then

$$\int_t^s B(r)\varphi(\tilde{x}(r))d\ell(r) = \int_t^s \bar{B}(r)\varphi(\tilde{x}(r))d\bar{\ell}(r),$$

where

$$\bar{\ell}(s) := \int_t^s \alpha(r, \tilde{x}(r))\varphi(\tilde{x}(r))d\ell(r), \quad \forall s \geq t.$$

On the other hand, the martingale process $M_\varphi(\cdot)$ can be re-written as

$$M_\varphi(s) = \varphi(x(s)) - \varphi(x(t)) - \int_t^s \mathbb{1}_{\mathbb{R}_+^d}(x(r)) A(r)\varphi(x(r))dr - \int_t^s \mathbb{1}_{\partial\mathbb{R}_+^d}(x(r)) B'(r)\varphi(x(r))d\ell(r), \quad \forall s \geq t,$$

where $B' = B_0 + b_d\partial_d$, i.e., $B = B' - \rho A$, and \mathbb{R}_+^d and $\partial\mathbb{R}_+^d$ are the interior and the boundary of the half-space \mathbb{R}_+^d .

The Dirichlet boundary conditions correspond to $b_0 \geq c_0$, $\rho = 0$ and $b_d = 0$, i.e., instead of

$$B(t)u(t, \tilde{x}, 0) + g(t, \tilde{x}) = 0, \quad \forall t < T, x \in \mathbb{R}^{d-1},$$

we have

$$u(t, \tilde{x}, 0) = g(t, \tilde{x}), \quad \forall t < T, x \in \mathbb{R}^{d-1},$$

with the compatibility condition $\varphi(\tilde{x}, 0) = \psi(T, \tilde{x})$ for any \tilde{x} in \mathbb{R}^{d-1} . The representation formula results

$$u(t, x) = \mathbb{E}^{tx} \left\{ e(t, \tau_{tx})\varphi(\tau_{tx}, x(\tau_{tx})) + \int_t^{\tau_{tx}} e(s, \tau_{tx})f(s, x(s))ds + \int_t^{\tau_{tx}} e(s, \tau_{tx})g(s, x(s))ds \right\},$$

where the functional

$$\tau_{tx} = \inf \{s \in [t, T] : x_d(s) = 0\},$$

with $\tau_{tx} = T$ if $x_d(s) \geq 0$ for any s in $[t, T]$.

Various particular cases of these parabolic boundary values problems are very well treated in the literature. For instance, the purely differential case has a classic treatment for Dirichlet and

Neumann boundary conditions (e.g. Friedman [?] and Ladyzhenskaya et al. [?]), while the oblique derivatives case and more general Wentzell boundary conditions are less typical. Essentially, better well known are results about existence and uniqueness of the solution (to the PDE, e.g., Lieberman [?], Lunardi [?]) than the actual construction and estimates of the Green and Poisson functions. Always for the purely differential case, general boundary conditions can be found in Eidelman [?], Solonnikov [?], Skubachevskii [?]. In particular, Ivasišen [?] studied the construction and estimates of the Green and Poisson functions for parabolic systems. Certainly, there are many more references (in the form of books, papers or memoirs, e.g., Taira [?]) that we can quote, which are found in the above references and in a search under the key word *parabolic equations*. As soon as we move to the complete second order integro-differential equations, the literature is scarce. For instance, one finds existence and uniqueness results for Dirichlet (and sometimes Neumann) boundary conditions in Bensoussan and Lions [?], while the construction and estimates for the Green function with oblique boundary conditions can be found in [?, ?]. Also recently, e.g., see Favini et al. [?], Taira [?].

On the other hand, Wentzell boundary conditions are well known in stochastic processes, e.g., Ikeda and Watanabe [?], Gihman and Skorohod [?]. However, one finds much less material for a complete second order integro-differential equations, e.g., Anulova [?, ?], Chaleyat-Maurel et al. [?], Komatsu [?, ?], and Menaldi and Robin [?], among others. Probabilistic arguments yields the construction of the fundamental solution and sometime the Green function, but heat kernel type estimates are always found by analytic means.

In this paper, we follow [?, ?] to construct and estimate the Green and Poisson functions corresponding to a complete second order integro-differential operator with Wentzell boundary conditions. In the process, we quickly review the parametrix method with indication on how to transport the arguments to the non-local case. Part of our arguments (and our calculations) to treat Wentzell boundary conditions seem to be new, even for the purely differential case. We begin with the constant coefficients, then we add a non-local term. Finally, we give detailed indication for the variable coefficient case, for which full calculations may require much more effort. Adding a non-local (tangential, i.e., acting on \mathbb{R}^{d-1}) to the Wentzell boundary operator B presents not difficulty, however, including the jumps in some oblique direction (with positive component with respect to the normal) requires another analysis, not treated in this draft.

Most of our analysis can be carry out in a smooth domain of \mathbb{R}^{d+1} , however we discuss only the cylindrical case in the half-space, i.e., in $[0, \infty) \times \mathbb{R}_+^d$, but estimates are given on $[0, T] \times \mathbb{R}_+^d$, for any given positive constant T . The general case is treated by means of local coordinates, although, much more detail is necessary.

2 Constant Coefficients

For the sake of clarity, we re-write the operators in the case of constant coefficients. Suppose we are given a second order constant elliptic differential operator

$$\begin{cases} A\varphi(x) := A_0\varphi(x) + I\varphi(x) + \sum_{i=1}^d a_i \partial_i \varphi(x) - a_0 \varphi(x) = A_2\varphi(x) + I\varphi(x), \\ A_0\varphi(x) := \frac{1}{2} \sum_{i,j=1}^d a_{ij} \partial_{ij} \varphi(x), \\ I\varphi(x) := \int_{\mathbb{R}_*^m} [\varphi(x + \mathbf{j}(\zeta)) - \varphi(x) - \mathbf{j}(\zeta) \cdot \nabla \varphi(x)] \mathbf{m}(\zeta) \pi(d\zeta) \end{cases} \quad (2.1)$$

for every continuously differentiable functions φ with a compact support in the closed half-space \mathbb{R}_+^d , where (a_{ij}) is a symmetric positive definite matrix and $a_0 \geq 0$. The functions \mathbf{j} and \mathbf{m} are measurable with respect to the measure π in $\mathbb{R}_*^m = \mathbb{R}^m \setminus \{0\}$, and satisfy $0 \leq \mathbf{m} \leq m_0$,

$$\int_{\{\mathbf{j} < 1\}} |\mathbf{j}(\zeta)|^\gamma \pi(d\zeta) + \int_{\{\mathbf{j} \geq 1\}} |\mathbf{j}(\zeta)| \pi(d\zeta) \leq C_0, \quad (2.2)$$

for some positive constants m_0 , C_0 , and $0 \leq \gamma < 2$. If necessary, \underline{a} denotes the d -dimensional square matrix (a_{ij}) and also, \underline{a} (or \underline{a}) denotes the d -dimensional vector (\tilde{a}, a_d) , where \tilde{a} (or $\tilde{\underline{a}}$) is equal to (a_1, \dots, a_{d-1}) . Also, assume that B is a (constant) *Wentzell type* boundary differential operator of the

form

$$\begin{cases} B\varphi(x) := B_0\varphi(x) + b_d\partial_d\varphi(x) - \rho A\varphi(x), & \forall x \in \mathbb{R}_+^d, \\ B_0\varphi(x) := \frac{1}{2} \sum_{i,j=1}^{d-1} b_{ij}\partial_{ij}\varphi(x) + \sum_{i=1}^{d-1} b_i\partial_i\varphi(x) - b_0\varphi(x), \end{cases} \quad (2.3)$$

where (b_{ij}) is a symmetric non-negative definite matrix, $b_0, b_d, \rho \geq 0$, and $\rho > 0$ or $b_d > 0$. If necessary, $\underline{\underline{b}}$ denotes the $(d-1)$ -dimensional square matrix (b_{ij}) and also, \underline{b} (or $\underline{\underline{b}}$) denotes the d -dimensional vector (\tilde{b}, b_d) , where \tilde{b} (or $\underline{\underline{b}}$) is equal to (b_1, \dots, b_{d-1}) . Most of the time, the tilde $\tilde{}$ sign is used to emphasize the $(d-1)$ dimension of the given element, while the underline $\underline{}$ and the double underline $\underline{\underline{}}$ refer to a vector and a matrix, respectively. It may be convenient to write $b_{ij} = \sum_k \zeta_{ik}\zeta_{jk}$, where the matrix ζ is the product of an orthogonal matrix $\tilde{\varrho}$ and a diagonal matrix $(\tilde{\beta})^{1/2}$, i.e., $\underline{\underline{b}} = \tilde{\varrho}\tilde{\beta}\tilde{\varrho}^*$ and $\zeta_{ij} = \varrho_{ij}\sqrt{\beta_j}$, where $\beta_1, \dots, \beta_{d-1}$ are the eigenvalues of the matrix (b_{ij}) .

Denote by Γ_d the heat kernel (or Gaussian kernel) of dimension d , i.e.,

$$\Gamma_d(t, x) = \Gamma_d(t, \tilde{x}, x_d) := (2\pi t)^{-d/2} \exp\left(-\frac{1}{2t} \sum_{i=1}^d x_i^2\right), \quad (2.4)$$

with x in \mathbb{R}^d , which may be written as (\tilde{x}, x_d) or $(x_1, \dots, x_{d-1}, x_d)$. It is clear that $\Gamma_d(rt, r^2x) = r^{-d/2}\Gamma_d(t, x)$, for every $r > 0$. As long as confusion does not arrive, it may be convenient to use the notation

$$\Gamma_d(q, x) := (2\pi)^{-d/2} \det(q)^{-1/2} \exp\left(-\frac{1}{2} \sum_{i,j=1}^d x_i q^{ij} x_j\right), \quad (2.5)$$

for any symmetric (strictly) positive matrix $q = (q_{ij})$ with inverse matrix (q^{ij}) . Thus, if $\mathbb{1}$ denotes the identity matrix then $\Gamma_d(t\mathbb{1}, x) = \Gamma_d(t, \tilde{x}, x_d)$. Clearly, one has $\Gamma_d(q, x) = \det(q)^{-1/2}\Gamma(\mathbb{1}, q^{-1/2}x)$ and $x \mapsto \Gamma_d(q, x - m)$ is the normal or Gaussian distribution in \mathbb{R}^d , with mean vector m and co-variance matrix q . Since q is a symmetric positive matrix, it can be written as $q = \rho\lambda\rho^*$, where λ is a diagonal matrix (of eigenvalues) and ρ a orthogonal matrix, and then

$$\Gamma_d(q, x) = \det(\lambda)^{-1/2}\Gamma_d(\mathbb{1}, \lambda^{-1/2}\rho^*x) = \prod_{i=1}^d \Gamma_1(\lambda_i, (\rho^*x)_i),$$

where $\lambda_i > 0$ are the eigenvalues of the matrix q . Note that

$$-\partial_d\Gamma_d(t, x) = \frac{x_d}{t}\Gamma_d(t, x) = (2\pi t)^{-d/2} \frac{x_d}{t} \exp\left(-\frac{|x|^2}{2t}\right),$$

for any $t > 0$ and x in \mathbb{R}^d ,

The fundamental solution corresponding to the purely differential operator A_2 as in (??), is given by the expression

$$F_0(t, x) = e^{-a_0 t}\Gamma_d(\underline{a}, x - \underline{a}), \quad \forall t > 0, x \in \mathbb{R}^d. \quad (2.6)$$

Hence, proposing $F = F_0 + F_0 \star Q$ one finds the fundamental solution corresponding to the whole second order integro-differential operator A as in (??), where \star means the kernel convolution in $[0, \infty) \times \mathbb{R}^d$, i.e.,

$$[\varphi \star \phi](t, x) = \int_0^t ds \int_{\mathbb{R}^d} \varphi(t-s, x-y)\psi(s, y)dy, \quad \forall t \geq 0, x \in \mathbb{R}^d.$$

Indeed, one must solve the following Volterra equation for either F or Q ,

$$F = F_0 + F_0 \star IQ, \quad Q = IF_0 + IF_0 \star Q.$$

If $Q_0 = IF_0$ then the formal series

$$F = \sum_{k=0}^{\infty} F_k, \quad F_k = F_0 \star IF_{k-1} \quad \text{and} \quad Q = \sum_{k=0}^{\infty} Q_k, \quad Q_k = Q_0 \star Q_{k-1}, \quad (2.7)$$

represent the (unique) solutions. Clearly, the presence of the non-local operator I makes disappear the heat kernel type estimates and the difficulty is the convergence of these series.

For the particular case

$$I\varphi(x) = \mathfrak{m}[\varphi(x + \mathfrak{j}) - \varphi(x)], \quad (2.8)$$

one can calculate explicitly the solution. Indeed, by means of the identity

$$F_0(t + s, x) = \int_{\mathbb{R}^d} F_0(t, x - z) F_0(s, z) dz = [F_0(t, \cdot) * F_0(s, \cdot)](x)$$

we get

$$F_k(t, \cdot) = \frac{t^k}{k!} I^k F_0(t, \cdot) \quad \text{and} \quad Q_k(t, \cdot) = \frac{t^k}{k!} I^{k+1} F_0(t, \cdot), \quad (2.9)$$

where

$$I^k \varphi(x) = \sum_{i=0}^k \binom{k}{i} (-1)^{k-i} \varphi(x + i\mathfrak{j}) \mathfrak{m}^k.$$

Hence

$$\begin{aligned} F(t, x) &= \sum_{k=0}^{\infty} \sum_{i=0}^k \binom{k}{i} (-1)^{k-i} \frac{(\mathfrak{m}t)^k}{k!} F_0(t, x + i\mathfrak{j}) = \\ &= e^{-\mathfrak{m}t} \sum_{k=0}^{\infty} \frac{(\mathfrak{m}t)^k}{k!} F_0(t, x + k\mathfrak{j}), \quad \forall t > 0, x \in \mathbb{R}^d. \end{aligned}$$

A posteriori, we can check the convergence of the series (and all its derivatives), but the heat kernel estimates are lost, there are many singular points, not just the origin.

Now for a general I , we let $\varepsilon \rightarrow 0$ with

$$I_\varepsilon \varphi(x) = \int_{|\mathfrak{j}(\zeta)| \geq \varepsilon} [\varphi(x + \mathfrak{j}(\zeta)) - \varphi(x) - \mathfrak{j}(\zeta) \cdot \nabla \varphi(x)] \mathfrak{m}(\zeta) \pi(d\zeta),$$

to establish the relation (??). Nevertheless, the expression of the power of I is more complicate, and simple explicit calculations are not longer possible. As studied in [?], several semi-norm are introduced, but only two are necessary (with a slight change of notation) for the constant coefficient case, namely, for any kernel $\varphi(t, x)$ and k real (usually non-negative) we define

$$\left\{ \begin{array}{l} K_0(\varphi, k) = \inf \{ K_0 \geq 0 : |\varphi(t, x)| \leq K_0 t^{-1+(k-d)/2}, \forall t, x \}, \\ K_n(\varphi, k) = \inf \{ K_n \geq 0 : \int_{\mathbb{R}^n} |\varphi(t, \tilde{x}^{(n)}, x^{(n)})| d\tilde{x}^{(n)} \leq \\ \leq K_n t^{-1+(k-d+n)/2}, \forall t, x^{(n)} \}, \\ K_d(\varphi, k) = \inf \{ K_d \geq 0 : \int_{\mathbb{R}^d} |\varphi(t, x)| dx \leq K_d t^{-1+k/2}, \forall t \}, \\ K(\varphi, k) = \max \{ K_0(\varphi, k), \dots, K_d(\varphi, k) \}, \\ K^2(\varphi, k) = \max \{ K(\varphi, 2+k), K(\nabla \varphi, 1+k), K(\nabla^2 \varphi, k) \}, \end{array} \right. \quad (2.10)$$

where $x = (\tilde{x}^{(n)}, x^{(n)})$, $\tilde{x}^{(n)} = (x_1, \dots, x_n)$, $x^{(n)} = (x_{n+1}, \dots, x_d)$, $n = 1, \dots, d-1$. Actually, K_0 , K_{d-1} and K_d are the most relevant semi-norms. In view of the heat kernel estimates satisfied by the fundamental function F_0 , the semi-norm $K(\partial^\ell F_0, 2-\ell)$ is finite, for any $\ell = 0, 1, \dots$, where ∂^ℓ denotes any derivative of order ℓ_x in x and order ℓ_t in t with $\ell = 2\ell_t + \ell_x$.

Theorem 2.1. *Under the assumptions (??) and (??) the series (??) converges in the sense that for any parabolic order of derivative $\ell = 1, 2, \dots$ there exists a positive constant C such that for every k we have*

$$K^2(\partial^\ell F_k, k(2-\gamma) - \ell) \leq C^k (k!)^{-(2-\gamma)} K^2(\partial^\ell F_0, -\ell), \quad (2.11)$$

where F_k is as in (??) and the constant C depending only on ℓ , on the dimension d , the constants C_0 , m_0 , and the matrix-norms $\|\underline{a}\|$, $\|\underline{a}^{-1}\|$ and the vector-norm $\|\underline{a}\|$.

Proof. The constant $0 \leq \gamma < 2$ in assumption (??) plays an important role. Indeed, for $0 \leq \gamma \leq 1$, the operator I may be supposed of a simpler form (by putting together the parts of the operator with $\mathbf{j}\nabla$ and \underline{a}), i.e.,

$$I\varphi(x) = \int_{\mathbb{R}^m} [\varphi(x + \mathbf{j}(\zeta)) - \varphi(x)] \mathbf{m}(\zeta) \pi(d\zeta),$$

which yields

$$K(I\varphi, 1 + k - \gamma) \leq m_0 C_0 [K(\varphi, 1 + k) + K(\nabla\varphi, k)].$$

Similarly, for $1 \leq \gamma \leq 2$, by means of the expression

$$I\varphi(x) = \int_0^1 d\theta \int_{\mathbb{R}_*^m} \mathbf{j}(\zeta) \cdot [\nabla\varphi(x + \theta\mathbf{j}(\zeta)) - \nabla\varphi(x)] \mathbf{m}(\zeta) \pi(d\zeta),$$

we get

$$K(I\varphi, 2 + k - \gamma) \leq m_0 C_0 [K(\nabla\varphi, 1 + k) + K(\nabla^2\varphi, k)].$$

Hence $K(\partial^\ell F_k, 2 + k(2 - \gamma) - \ell)$ is finite. Moreover, from the identities

$$F_0(t + s, \cdot) = F_0(t, \cdot) * F_0(s, \cdot) \quad \text{and} \quad IF_0(t + s, \cdot) = IF_0(t, \cdot) * F_0(s, \cdot),$$

we have the desired estimate, as in [?, Chapter 3] □

Remark 2.2. The layer potentials or jump relations are satisfied by the fundamental solution F_0 , namely,

$$\lim_{x_d \rightarrow 0^+, x_d > 0} \sum_{i=1}^d \int_0^t ds \int_{\mathbb{R}^{d-1}} a_{di} \partial_i F_0(t - s, \tilde{x} - \tilde{y}, x_d) \psi(s, \tilde{y}) d\tilde{y} = -g(t, \tilde{x}),$$

for every $t > 0$, \tilde{x} in \mathbb{R}^{d-1} and any smooth function with compact support ψ . Note that $\frac{1}{2} \sum_{i=1}^d a_{di} \partial_i$ is the co-normal derivative, the kernel $(t, \tilde{x}) \mapsto \partial_i F_0(t, \tilde{x}, 0)$ is not integrable for any $i \neq d$, but the cancellation property

$$\int_{\mathbb{R}^{d-1}} \partial_i F_0(t, \tilde{x}, x_d) d\tilde{x} = 0, \quad \forall t > 0, x_d > 0, i \neq d,$$

and the normalization property

$$\int_0^\infty dt \int_{\mathbb{R}_+^d} \sqrt{a_{dd}} \partial_d F_0(t, \tilde{x}, x_d) d\tilde{x} = 1, \quad \forall x_d > 0,$$

hold true. It is relative simple to show the same jump relation for the fundamental solution F if $0 \leq \gamma < 1$, by means of the semi-norm $K_{d-1}(F_k, 2 + k(2 - \gamma))$. However, for $1 \leq \gamma < 2$ the situation is far more delicate. □

Remark 2.3. This same argument applies to a symmetric operator I of the form

$$\varphi(x) := \int_{\mathbb{R}_*^m} [\varphi(x + \mathbf{j}(\zeta)) + \varphi(x - \mathbf{j}(\zeta)) - 2\varphi(x)] \mathbf{m}(\zeta) \pi(d\zeta),$$

where the term in $\mathbf{m}(\zeta) \cdot \nabla\varphi(x)$ is not present, i.e., the case where the measure π and the coefficients are symmetric in \mathbb{R}_*^m . □

2.1 Dirichlet Boundary Conditions

For a purely second order differential operator A_2 as in (??), the Green and the Poisson functions take the form

$$\begin{cases} G_{0,D}(t, x, y_d) := e^{-ta_0} [\Gamma_d(t\underline{a}, \tilde{x} - t\underline{a}, x_d - ta_d - y_d) - \\ \quad - e^{2a_d y_d / \sqrt{a_{dd}}} \Gamma_d(t\underline{a}, \tilde{x} - t\underline{a}, x_d - ta_d + y_d)], \\ P_{0,D}(t, x) := e^{-ta_0} \frac{x_d}{t\sqrt{a_{dd}}} [\Gamma_d(t\underline{a}, \tilde{x} - t\underline{a}, x_d - ta_d)], \end{cases}$$

where $\underline{\tilde{a}} = (a_1, \dots, a_{d-1})$, with underline $\underline{\quad}$ to emphasize the difference with the matrix double-underline notation. This is to say that for any smooth functions with compact support $f(t, x)$, $\varphi(x)$ and $g(t, \tilde{x})$, the expression

$$\left\{ \begin{aligned} u(t, x) &= \int_t^T ds \int_{\mathbb{R}_+^d} G_{0,D}(s-t, \tilde{x} - \tilde{y}, x_d, y_d) f(s, y) dy + \\ &\quad + \int_{\mathbb{R}_+^d} G_{0,D}(T-t, \tilde{x} - \tilde{y}, x_d, y_d) \varphi(y) dy + \\ &\quad + \int_t^T ds \int_{\mathbb{R}^{d-1}} P_{0,D}(s-t, \tilde{x} - \tilde{y}, x_d) g(s, \tilde{y}) d\tilde{y} \end{aligned} \right.$$

provides the solution to the Dirichlet problem with terminal condition, i.e.,

$$\left\{ \begin{aligned} \partial_t u(t, x) + A_2(t)u(t, x) &= f(t, x), \quad \forall t > 0, x \in \mathbb{R}_+^d, \\ u(t, \tilde{x}, 0) &= g(t, \tilde{x}), \quad \forall t > 0, x \in \mathbb{R}^{d-1}, \\ u(0, x) &= \varphi(x), \quad \forall x \in \mathbb{R}_+^d. \end{aligned} \right.$$

This fact is easy to prove once the above expression is known. Hence we can construct the Poisson function as follows:

Theorem 2.4. Assume (??), (??) and

$$j_d(\zeta) \geq 0, \quad \forall \zeta \in \mathbb{R}_*^m. \quad (2.12)$$

Then, with the notation (??), the function

$$P_{A,D}(t, x) = \frac{x_d}{t\sqrt{a_{dd}}} F(t, x) = \frac{x_d}{t\sqrt{a_{dd}}} \sum_{k=0}^{\infty} F_k(t, x) \quad (2.13)$$

is the Poisson function corresponding to the complete second order integro-differential operator A as in (??) with Dirichlet boundary conditions in the half-space \mathbb{R}_+^d .

Proof. First we need to check that, for any $k = 1, 2, \dots$ and any smooth function g with compact support, we have

$$\int_0^t \frac{x_d}{s} ds \int_{\mathbb{R}^{d-1}} F_k(s, \tilde{y}, x_d) g(t-s, \tilde{x} - \tilde{y}) d\tilde{y} \rightarrow 0 \quad \text{as } x_d \rightarrow 0, \quad (2.14)$$

where $F_k(t, \cdot) = t^k I^k F_0(t, \cdot)/k!$. Indeed, by means of the explicit expression of the heat kernel F_0 and the condition (??) we deduce

$$\int_{\mathbb{R}^{d-1}} |IF_0(t, \tilde{y}, x_d)| d\tilde{y} \leq C_1 t^{-1-\gamma/2} \exp\left(-\frac{x_d^2}{2t}\right),$$

which yields (??) for $k = 1$. Similarly, we have

$$\int_{\mathbb{R}^{d-1}} |F_k(t, \tilde{y}, x_d)| d\tilde{y} \leq C_k t^{-1+k(2-\gamma)/2} \exp\left(-\frac{x_d^2}{2t}\right),$$

for any $k = 2, \dots$, and then

$$\int_0^t \frac{x_d}{s\sqrt{a_{dd}}} ds \int_{\mathbb{R}^{d-1}} P_{A,D}(s, \tilde{y}, x_d) g(t-s, \tilde{x} - \tilde{y}) d\tilde{y} \rightarrow g(t, \tilde{x}),$$

as $x_d \rightarrow 0$, for any smooth function g with compact support.

Next, we have to check that $P_{A,D}$ solves the homogeneous integro-differential equation in \mathbb{R}_+^d , i.e.,

$$[\partial_t - A(t)]P_{A,D}(t, x) = 0, \quad \forall t > 0, x \in \mathbb{R}_+^d. \quad (2.15)$$

Indeed, because of the construction of the fundamental solution F which define $P_{A,D}$, it suffices to show the above equation for I_ε instead of I , see (??). Moreover, we can move the differential part to A_2 , i.e., change the first order coefficients \underline{a} into

$$\underline{a} + \int_{\{j(\zeta) \geq \varepsilon\}} j(\zeta) \mathfrak{m}(\zeta) \pi(d\zeta),$$

so that we are reduced to the case

$$I\varphi(x) = \int_{\mathbb{R}_*^m} [\varphi(x + \mathbf{j}(\zeta)) - \varphi(x)] \mathbf{m}(\zeta) \pi(d\zeta),$$

under the condition $0 \leq \mathbf{m}(\zeta) \leq m_0$ and

$$\int_{\mathbb{R}_*^m} |\mathbf{j}(\zeta)| \pi(d\zeta) \leq C_0,$$

actually, $\mathbf{j}(\zeta)$ is bounded and $\pi(\mathbb{R}_*^m) < \infty$. Thus, define $\varphi_d(x) = x_d$ and

$$I'_d \varphi(x) = \int_{\mathbb{R}_*^m} \varphi(x + \mathbf{j}(\zeta)) \mathbf{j}_d(\zeta) \mathbf{m}(\zeta) \pi(d\zeta)$$

to see that

$$[I(\varphi_d \varphi)](x) = \varphi_d(x) I\varphi(x) + I'_d \varphi(x).$$

Furthermore, we may rotate the coordinates so that the matrix of the second order coefficients \underline{a} is diagonal. Hence

$$\sqrt{a_{dd}} [a_d F_0(t, x) - \partial_d F_0(t, x)] = \varphi_d(x) F_0(t, x)$$

and therefore

$$\sqrt{a_{dd}} [a_d t I F_0(t, x) - t I \partial_d F_0(t, x)] = \varphi_d(x) I F_0(t, x) + I'_d F_0(t, x).$$

Again

$$I \{ \sqrt{a_{dd}} [a_d t I F_0(t, x) - t I \partial_d F_0(t, x)] \} = \varphi_d(x) I^2 F_0(t, x) + I I'_d F_0(t, x) + I'_d I F_0(t, x),$$

and because I and I'_d commute, we deduce by induction

$$I^{k-1} \{ \sqrt{a_{dd}} [a_d t I F_0(t, x) - t I \partial_d F_0(t, x)] \} = \varphi_d(x) I^k F_0(t, x) + k I'_d I^{k-1} F_0(t, x),$$

for any $k \geq 1$. This proves that

$$\sqrt{a_{dd}} [a_d F_k(t, x) - \partial_d F_k(x)] = \frac{x_d}{t} F_k(x) + I'_d F_{k-1}(x), \quad \forall k \geq 1.$$

Next, by means of the series (??) we obtain

$$\sqrt{a_{dd}} [a_d F(t, x) - \partial_d F(t, x)] = \frac{x_d}{t} F(t, x) + I'_d F(t, x), \quad \forall t > 0, x \in \mathbb{R}^d. \quad (2.16)$$

Since F , $\partial_d F$ and $I'_d F$ are solutions of the homogeneous equation then $x_d F(t, x)/t$ is also a solution, i.e., the equation (??) is satisfied. \square

Now, to construct the Green function corresponding to the second order integro-differential operator $A = A_2 + I$ with Dirichlet boundary conditions in \mathbb{R}_+^d , we may proceed as in the case of the fundamental solution, by solving the following Volterra equation for either G or Q , namely,

$$G = G_0 + G_0 \star IQ, \quad Q = IG_0 + IG_0 \star Q,$$

where G_0 is the Green function corresponding to the purely differential part A_2 , and now the (non-commutative) kernel convolution \star is in $[0, \infty] \times \mathbb{R}_+^d$, i.e.,

$$(\varphi \star \psi)(t, \tilde{x}, x_d, y_d) = \int_0^t ds \int_{\mathbb{R}_+^d} \varphi(t-s, \tilde{x} - \tilde{z}, x_d, z_d) \psi(s, \tilde{z}, z_d, y_d) dz,$$

with $z = (\tilde{z}, z_d)$. If $Q_0 = IG_0$ then the formal series

$$G = \sum_{k=0}^{\infty} G_k, \quad G_k = G_0 \star IG_{k-1} \quad \text{and} \quad Q = \sum_{k=0}^{\infty} Q_k, \quad Q_k = Q_0 \star Q_{k-1},$$

provide the (unique) solutions. Clearly, we do have the property

$$G_0(t+s, \tilde{x}, x_d, y_d) = \int_{\mathbb{R}_+^d} G_0(t, \tilde{x} - \tilde{z}, x_d, z_d) G_0(s, \tilde{z}, z_d, y_d) dz,$$

but we have no longer the equality $I(\varphi \star \psi) = \varphi \star (I\psi)$, which would yield $G_k(t, \cdot) = t^k I^k G_0(t, \cdot) / k!$. On the other hand, we need to modify the definition of the semi-norms, namely, for kernels $\varphi(t, x, y_d)$,

$$\left\{ \begin{array}{l} K_0(\varphi, k) = \inf \{ K_0 \geq 0 : |\varphi(t, x, y_d)| \leq K_0 t^{-1+(k-d)/2}, \forall t, x, y_d \}, \\ K_d(\varphi, k) = \inf \left\{ K_d \geq 0 : \int_{\mathbb{R}_+^d} |\varphi(t, \tilde{z}, z_d, y_d)| dz + \right. \\ \left. + \int_{\mathbb{R}_+^d} |\varphi(t, \tilde{z}, x_d, z_d)| dz \leq K_d t^{-1+k/2}, \forall t, x_d, y_d \right\}, \\ K(\varphi, k) = \max \{ K_0(\varphi, k), K_d(\varphi, k) \}, \end{array} \right. \quad (2.17)$$

where $x = (\tilde{x}, x_d)$. Also, we may use semi-norms of the type $K_n(\cdot, \cdot)$, for $n = 1, \dots, d-1$. Moreover, estimating the G is simple, but it may be complicate to handle $\partial^\ell G$ for $\ell = 1, 2, \dots$

An alternative way to construct the Green function is to propose

$$G_{A,D}(t, x, y_d) = F(t, \tilde{x}, x_d - y_d) - V(t, x, y_d)$$

and to calculate the integral

$$V(t, \tilde{x}, x_d, y_d) = \int_0^t \frac{x_d}{(t-s)\sqrt{a_{dd}}} ds \int_{\mathbb{R}^{d-1}} F(t-s, \tilde{x} - \tilde{y}, x_d) F(s, \tilde{y}, y_d) d\tilde{y},$$

where the most singular term, namely

$$\int_0^t \frac{x_d}{(t-s)\sqrt{a_{dd}}} ds \int_{\mathbb{R}^{d-1}} F_0(t-s, \tilde{x} - \tilde{y}, x_d) F_0(s, \tilde{y}, y_d) d\tilde{y} = F_0(t, \tilde{x}, x_d + y_d),$$

is computed exactly, and all other lower order terms can be estimated with the semi-norms

$$\left\{ \begin{array}{l} K_0(\varphi, k) = \inf \{ K_0 \geq 0 : |\varphi(t, \tilde{x}, x_d)| \leq \\ \leq K_0 t^{-1+\frac{k-d}{2}} \exp(-c_0 \frac{x_d^2}{t}), \forall t, \tilde{x}, x_d \}, \\ K_{d-1}(\varphi, k) = \inf \left\{ K_{d-1} \geq 0 : \int_{\mathbb{R}^{d-1}} |\varphi(t, \tilde{x}, x_d)| dx \leq \right. \\ \left. \leq K_{d-1} t^{-1+\frac{k-1}{2}} \exp(-c_0 \frac{x_d^2}{t}), \forall t, x_d \right\}, \\ K(\varphi, k) = \max \{ K_0(\varphi, k), K_{d-1}(\varphi, k) \}, \end{array} \right. \quad (2.18)$$

as in the case of the fundamental solution. The constant $c_0 > 0$ is taken so that $2c_0\sqrt{a_{dd}} < 1$. So, for future reference, we may state

Theorem 2.5. *Under the assumptions (??), (??) and (??) the Green function corresponding to the second order integro-differential operator A in the half-space with Dirichlet boundary conditions is given by*

$$G_{A,D}(t, x, y_d) = F_0(t, \tilde{x}, x_d - y_d) - F_0(t, \tilde{x}, x_d + y_d) + F_I(t, x, y_d),$$

where the kernel F_0 is given by (??) and the semi-norms $K(\partial^\ell F_I, 4 - \gamma - \ell)$ are finite, for any $\ell = 0, 1, \dots$ \square

Note that we can also write $G_{A,D} = G_{0,D} + G_I$ where the semi-norms $K(\partial^\ell G_I, 4 - \gamma - \ell)$ are finite, for any $\ell = 0, 1, \dots$

2.2 Degenerate Equations

To simplify the notation for the heat kernel (??), we use $\tilde{\Gamma}_0(t, \tilde{x}) = \Gamma_{d-1}(t, \tilde{x})$ and $\Gamma_0(t, x) = \Gamma_0(t, \tilde{x}, x_d) = \Gamma_d(t, x)$, for any $t > 0$ and $x = (\tilde{x}, x_d)$ in \mathbb{R}^d .

For a *degenerate* second order differential operator B_0 given by (??), we are interested in the following two problems in the (open) half-space $\mathbb{R}_+^d = \mathbb{R}^{d-1} \times (0, \infty)$, without any boundary condition at $\partial\mathbb{R}_+^d$, both with a representation formula, and where the maximum principle ensures the uniqueness, at least for smooth data rapidly decreasing as $|x| \rightarrow \infty$. For $\rho > 0$, a parabolic problem in $[0, \infty) \times \mathbb{R}_+^d$ with initial condition

$$\begin{cases} B_0 u(t, x) + b_d \partial_d u(t, x) - \rho \partial_t u(t, x) + v(t, x) = 0, & \forall t > 0, x \in \mathbb{R}_+^d, \\ u(0, x) = u_0(x), & \forall x \in \mathbb{R}_+^d, \end{cases} \quad (2.19)$$

with

$$\begin{cases} u(t, x) = e^{-b_0(t/\rho)} \int_{\mathbb{R}^{d-1}} \tilde{\Gamma}_0\left(\frac{t}{\rho}, \tilde{z}\right) u_0\left(\tilde{x} - \tilde{\zeta}\tilde{z} + \tilde{b}\frac{t}{\rho}, x_d + b_d\frac{t}{\rho}\right) d\tilde{z} + \\ + \int_0^{t/\rho} e^{-b_0 r} dr \int_{\mathbb{R}^{d-1}} \tilde{\Gamma}_0(r, \tilde{z}) v(t - \rho r, \tilde{x} - \tilde{\zeta}\tilde{z} + \tilde{b}r, x_d + b_d r) d\tilde{z}. \end{cases} \quad (2.20)$$

For $\rho = 0$ and $b_d > 0$, a parabolic problem in $\mathbb{R}^{d-1} \times [0, \infty)$ with terminal condition

$$\begin{cases} B_0 u(\tilde{x}, x_d) + b_d \partial_d u(\tilde{x}, x_d) + v(\tilde{x}, x_d) = 0, & \forall x_d > 0, \tilde{x} \in \mathbb{R}^{d-1}, \\ \lim_{x_d \rightarrow \infty} u(\tilde{x}, x_d) = 0, & \forall \tilde{x} \in \mathbb{R}^{d-1}, \end{cases} \quad (2.21)$$

with

$$u(\tilde{x}, x_d) = \int_0^\infty e^{-b_0 r} dr \int_{\mathbb{R}^{d-1}} \tilde{\Gamma}_0(r, \tilde{z}) v(\tilde{x} - \tilde{\zeta}\tilde{z} + \tilde{b}r, x_d + b_d r) d\tilde{z}. \quad (2.22)$$

Clearly, if $\tilde{\zeta}$ vanishes then the above representations are simplified, without the use of the fundamental solution $\tilde{\Gamma}_0 = \Gamma_{d-1}$ as in (??).

To verify the representation or inversion formula (??), we check that

$$\begin{aligned} \sum_{i=1}^d b_i \partial_i \left[\int_0^\infty e^{-b_0 r} dr \int_{\mathbb{R}^{d-1}} \tilde{\Gamma}_0(r, \tilde{z}) v(\tilde{x} - \tilde{\zeta}\tilde{z} + \tilde{b}r, x_d + b_d r) d\tilde{z} \right] = \\ = \int_0^\infty e^{-b_0 r} dr \int_{\mathbb{R}^{d-1}} \tilde{\Gamma}_0(r, \tilde{z}) \partial_r v(\tilde{x} - \tilde{\zeta}\tilde{z} + \tilde{b}r, x_d + b_d r) d\tilde{z} \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2} \sum_{i,j=1}^{d-1} b_{ij} \partial_{ij} \left[\int_0^\infty e^{-b_0 r} dr \int_{\mathbb{R}^{d-1}} \tilde{\Gamma}_0(r, \tilde{z}) v(\tilde{x} - \tilde{\zeta}\tilde{z} + \tilde{b}r, x_d + b_d r) d\tilde{z} \right] = \\ = \int_0^\infty e^{-b_0 r} dr \int_{\mathbb{R}^{d-1}} \tilde{\Gamma}_0(r, \tilde{z}) \frac{1}{2} \Delta_{\tilde{z}} v(\tilde{x} - \tilde{\zeta}\tilde{z} + \tilde{b}r, x_d + b_d r) d\tilde{z} = \\ = \lim_{\varepsilon \downarrow 0} \int_\varepsilon^\infty e^{-b_0 r} dr \int_{\mathbb{R}^{d-1}} [\partial_r \tilde{\Gamma}_0(r, \tilde{z})] v(\tilde{x} - \tilde{\zeta}\tilde{z} + \tilde{b}r, x_d + b_d r) d\tilde{z}. \end{aligned}$$

This yields

$$\begin{aligned} B_0 \left[\int_0^\infty e^{-b_0 r} dr \int_{\mathbb{R}^{d-1}} \tilde{\Gamma}_0(r, \tilde{z}) v(\tilde{x} - \tilde{\zeta}\tilde{z} + \tilde{b}r, x_d + b_d r) d\tilde{z} \right] = \\ = \lim_{\varepsilon \downarrow 0} \int_\varepsilon^\infty \partial_r \left[e^{-b_0 r} dr \int_{\mathbb{R}^{d-1}} \tilde{\Gamma}_0(r, \tilde{z}) v(\tilde{x} - \tilde{\zeta}\tilde{z} + \tilde{b}r, x_d + b_d r) d\tilde{z} \right], \end{aligned}$$

which prove (??). Similarly, we show the validity of the other representation or inversion formula (??).

2.3 Wentzell Boundary Conditions

First, to follow better the arguments, let us assume $A = \frac{1}{2}\Delta$ and let us use the general notation $P(t, x, \tilde{y})$ and $G(t, x, y)$ instead of the particular expressions $\tilde{P}(t, x)$ and $G(t, x, y_d)$ for the Green and

Poisson functions. Then, consider the (parabolic) Green function and the Poisson for the Dirichlet problem in the half-space $\dot{\mathbb{R}}_+^d$,

$$\begin{cases} G_D(t, x, y) = \Gamma_d(t, \tilde{x} - \tilde{y}, x_d - y_d) - \Gamma_d(t, \tilde{x} - \tilde{y}, x_d + y_d) = \\ \quad = \Gamma_{d-1}(t, \tilde{x} - \tilde{y}) [\Gamma_1(t, x_d - y_d) - \Gamma_1(t, x_d + y_d)], \end{cases} \quad (2.23)$$

and

$$P_D(t, x, \tilde{y}) = -\partial_d \Gamma_d(t, \tilde{x} - \tilde{y}, x_d), \quad (2.24)$$

where $\Gamma_d(t, \tilde{x}, x_d) = \Gamma_d(t, x)$ for any $x = (\tilde{x}, x_d)$ is given by (??). Recall that we write $\Gamma_0(t, x) = \Gamma_0(t, \tilde{x}, x_d) = \Gamma_d(t, \tilde{x}, x_d)$.

Without giving all details, let us mention that the solution of the heat equation in the half-space with a Dirichlet boundary condition

$$\begin{cases} \partial_t u(t, x) = \frac{1}{2} \Delta u(t, x) + f(t, x), & \forall t > 0, x \in \dot{\mathbb{R}}_+^d, \\ u(t, x) = g(t, x), & \forall t > 0, x \in \partial \dot{\mathbb{R}}_+^d, \\ u(0, x) = u_0(x), & \forall x \in \dot{\mathbb{R}}_+^d, \end{cases} \quad (2.25)$$

is given by the expression

$$\begin{cases} u(t, x) = \int_0^t ds \int_{\mathbb{R}^{d-1}} P_D(t-s, x, \tilde{y}) g(s, \tilde{y}) d\tilde{y} + \\ \quad + \int_{\mathbb{R}_+^d} G_D(t, x, y) u_0(y) dy + \int_0^t ds \int_{\mathbb{R}_+^d} G_D(t-s, x, y) f(s, y) dy, \end{cases}$$

for any sufficiently smooth data f , g and u_0 . Here, we identify the boundary $\partial \dot{\mathbb{R}}_+^d$ with the $(d-1)$ -dimensional space \mathbb{R}^{d-1} , so that $g(t, x)$ with x in $\partial \dot{\mathbb{R}}_+^d$ can be written as $g(t, \tilde{x})$ with \tilde{x} in \mathbb{R}^{d-1} .

The Green function G_D , considered as a distribution in $(0, \infty) \times \dot{\mathbb{R}}_+^d$, satisfies (??) with $f(t, x) = \delta(t, x)$, $g = 0$ and $u_0 = 0$, while as a distribution in $\dot{\mathbb{R}}_+^d$, it satisfies (??) with $f(t, x) = 0$, $g = 0$ and $u_0(x) = \delta(x)$. On the other hand, the Poisson function P_D satisfies (??) with $f(t, x) = 0$, $g(t, \tilde{x}) = \delta(t, \tilde{x})$ and $u_0 = 0$, considered as a distribution in $\dot{\mathbb{R}}_+^d$.

Remark 2.6. Another typical case is the Green G_N and the Poisson P_N functions with Neumann boundary conditions, i.e.,

$$\begin{cases} \partial_t u(t, x) = \frac{1}{2} \Delta u(t, x) + f(t, x), & \forall t > 0, x \in \dot{\mathbb{R}}_+^d, \\ \partial_d u(t, x) + g(t, x) = 0, & \forall t > 0, x \in \partial \dot{\mathbb{R}}_+^d, \\ u(0, x) = u_0(x), & \forall x \in \dot{\mathbb{R}}_+^d. \end{cases}$$

It is known that,

$$\begin{aligned} G_N(t, x, y) &= \Gamma_d(t, \tilde{x} - \tilde{y}, x_d - y_d) + \Gamma_d(t, \tilde{x} - \tilde{y}, x_d + y_d), \\ P_N(t, x, \tilde{y}) &= \Gamma_d(t, \tilde{x} - \tilde{y}, x_d), \end{aligned}$$

This is discussed as a particular case of what follows. Note the relations $P_D = -\partial_d P_N$ and

$$\begin{aligned} P_D(t, \tilde{x} - \tilde{y}, y_d) &= \partial_d^x G_D(t, \tilde{x}, 0, \tilde{y}, y_d), \\ P_D(t, \tilde{x} - \tilde{y}, x_d) &= -\partial_d^y G_D(t, x, \tilde{y}, 0), \end{aligned}$$

where ∂_d^x or ∂_d^y means partial derivatives with respect to the variable x_d or y_d , respectively. \square

To solve the heat equation in the half-space with a Wentzell type boundary condition, i.e.,

$$\begin{cases} \partial_t u(t, x) = \frac{1}{2} \Delta u(t, x) + f(t, x), & \forall t > 0, x \in \dot{\mathbb{R}}_+^d, \\ Bu(t, x) + g(t, x) = 0, & \forall t > 0, x \in \partial \dot{\mathbb{R}}_+^d, \\ u(0, x) = u_0(x), & \forall x \in \dot{\mathbb{R}}_+^d, \end{cases} \quad (2.26)$$

with B given by (??), we may proceed as follows. If $f = 0$ then $\partial_t u = \frac{1}{2} \Delta u$ and the boundary condition $Bu + g = 0$ is equivalent to the degenerate parabolic equations discussed in the previous subsection,

(??) and (??). Thus, the corresponding Poisson function P_B is obtained by using the representations or inversion formulae (??) and (??) with P_D , i.e., $P_B(t, x, \tilde{y}) = Q_B(t, \tilde{x} - \tilde{y}, x_d)$, where Q_B is given as follows, for $\rho > 0$,

$$\begin{cases} Q_B(t, x) = -\partial_d \int_0^{t/\rho} e^{-b_0 r} dr \int_{\mathbb{R}^{d-1}} \tilde{\Gamma}_0(r, \tilde{z}) \times \\ \quad \times \Gamma_0(t - \rho r, \tilde{x} - \tilde{\zeta} \tilde{z} + \tilde{b}r, x_d + b_d r) d\tilde{z}, \end{cases} \quad (2.27)$$

and for $\rho = 0$ and $b_d > 0$,

$$Q_B(t, x) = -\partial_d \int_0^\infty e^{-b_0 r} dr \int_{\mathbb{R}^{d-1}} \tilde{\Gamma}_0(r, \tilde{z}) \Gamma_0(t, \tilde{x} - \tilde{\zeta} \tilde{z} + \tilde{b}r, x_d + b_d r) d\tilde{z}. \quad (2.28)$$

Note that the variable t is a parameter in the expressions (??), and that the partial derivative ∂_d can be calculated inside or outside the integral signs.

Remark 2.7. It is clear that the above integrals defining Q_B are non-singular for $t > 0$ and $x_d > 0$, and that upper estimates of the heat kernel type (??) are necessary to make the above formula workable. Later, we are going to verify these estimates only in particular cases, with explicit calculations. \square

To find the expression of the Green function G_B , first we remark that if u is a solution of the Wentzell type boundary condition problem (??) with $f = 0$, $g = 0$ and u_0 , then Bu is a solution of the Dirichlet problem (??) with $f = 0$, $g = 0$ and u_0 replaced by Bu_0 . Also we note that

$$\begin{cases} \int_{\mathbb{R}_+^d} G_D(t, x, y) B_0^y u_0(y) dy = \int_{\mathbb{R}_+^d} B_0^x G_D(t, x, y) u_0(y) dy + \\ \quad + 2b_d \int_{\mathbb{R}_+^d} \partial_d \Gamma_0(t, \tilde{x} - \tilde{y}, x_d + y_d) u_0(y) dy. \end{cases}$$

and

$$\begin{cases} \int_{\mathbb{R}_+^d} G_D(t, x, y) \Delta^y u_0(y) dy = \int_{\mathbb{R}_+^d} \Delta^x G_D(t, x, y) u_0(y) dy - \\ \quad - 2 \int_{\mathbb{R}_+^d} \partial_d^x \Gamma_0(t, \tilde{x} - \tilde{y}, x_d) u_0(\tilde{y}, 0) d\tilde{y}, \end{cases}$$

i.e., we have

$$\begin{cases} \int_{\mathbb{R}_+^d} G_D(t, x, y) B^y u_0(y) dy = 2b_d \int_{\mathbb{R}_+^d} \partial_d \Gamma_d(t, \tilde{x} - \tilde{y}, x_d + y_d) u_0(y) dy + \\ \quad + \rho \int_{\mathbb{R}_+^d} \partial_d \Gamma_d(t, \tilde{x} - \tilde{y}, x_d) u_0(\tilde{y}, 0) d\tilde{y} - \int_{\mathbb{R}_+^d} B^x G_D(t, x, y) u_0(y) dy. \end{cases}$$

Now the inversion formulae and the uniqueness (e.g., for $v = B^x G_D$ we get $u = G_D$ from the representation) yield

$$G_B(t, x, y) = G_D(t, x, y) + 2b_d Q_B(t, \tilde{x} - \tilde{y}, x_d + y_d) + \rho \delta_0(y_d) Q_B(t, \tilde{x} - \tilde{y}, x_d),$$

where Q_B is given by (??) or (??) according to the various cases, and δ_0 is the delta measure in the variable y_d .

The expression for the kernel Q_B can be simplified as follows. Indeed, recall $b_{ij} = \sum_{k=1}^{d-1} \varrho_{ik} \beta_k \varrho_{jk}$ and $\varsigma_{ij} = \varrho_{ij} \sqrt{\beta_j}$, i.e., $\tilde{\zeta} = \tilde{\varrho}(\tilde{\beta})^{1/2}$, the diagonal matrix $\tilde{\beta}$, with entries $\beta_i \geq 0$, $i = 1, \dots, d-1$ (in the diagonal) are the eigenvalues of the matrix (b_{ij}) , and the orthogonal matrix $\tilde{\varrho}$ satisfied $|\det(\tilde{\varrho})| = 1$ and $|\tilde{\varrho}\tilde{x}| = |\tilde{x}|$, for every \tilde{x} in \mathbb{R}^{d-1} . Since $\tilde{\Gamma}_0(r, \tilde{\varrho}\tilde{z}) = \tilde{\Gamma}_0(r, \tilde{z})$, we deduce

$$\int_{\mathbb{R}^{d-1}} \tilde{\Gamma}_0(r, \tilde{z}) \tilde{\Gamma}_0(t, \tilde{x} - \tilde{\zeta}\tilde{z}) d\tilde{z} = \int_{\mathbb{R}^{d-1}} \tilde{\Gamma}_0(r, \tilde{z}) \tilde{\Gamma}_0(t, \tilde{\varrho}^* \tilde{x} - \tilde{\beta}^{1/2} \tilde{z}) d\tilde{z}.$$

Next, after the individual change of variables $y_i = z_i \sqrt{\beta_i}$ only if $\beta_i > 0$, remarking that $\Gamma_1(r, y_i \sqrt{\beta_i}) = \sqrt{\beta_i} \Gamma_1(\beta_i r, y_i)$ and

$$\int_{\mathbb{R}^d} \Gamma_d(s, y) \Gamma_d(t, x - y) dy = \Gamma_d(s + t, x), \quad \forall t, s > 0, x \in \mathbb{R}^d, d = 1, 2, \dots,$$

we get

$$\begin{aligned} \int_{\mathbb{R}^{d-1}} \tilde{\Gamma}_0(r, \tilde{z}) \tilde{\Gamma}_0(t, \tilde{x} - \tilde{c}\tilde{z}) d\tilde{z} &= \int_{\mathbb{R}^{d-1}} \left(\prod_{i=1}^{d-1} \Gamma_1(\beta_i r, y_i) \right) \tilde{\Gamma}_0(t, \tilde{\varrho}^* \tilde{x} - \tilde{y}) d\tilde{y} = \\ &= \prod_{i=1}^{d-1} \Gamma_1(t + \beta_i r, (\tilde{\varrho}^* x)_i) = \det(t\tilde{\mathbb{1}} + r\tilde{\beta})^{-1/2} \tilde{\Gamma}_0(1, (t\tilde{\mathbb{1}} + r\tilde{\beta})^{-1/2} \tilde{\varrho}^* x), \end{aligned}$$

where $\tilde{\mathbb{1}}$ denotes the identity matrix of dimension $d-1$. Alternatively, by means of the Fourier transform we can check that the convolution is indeed a centered normal distribution with (invertible) covariance matrix $(t\tilde{\mathbb{1}} + r\tilde{c}\tilde{c}^*)$, i.e.,

$$\begin{aligned} \int_{\mathbb{R}^{d-1}} \tilde{\Gamma}_0(r, \tilde{z}) \tilde{\Gamma}_0(t, \tilde{x} - \tilde{c}\tilde{z}) d\tilde{z} &= \Gamma_{d-1}(t\tilde{\mathbb{1}} + r\tilde{c}\tilde{c}^*, x) = \\ &= (2\pi)^{-d/2} [\det(t\tilde{\mathbb{1}} + r\tilde{c}\tilde{c}^*)]^{-1/2} \exp[-|(t\tilde{\mathbb{1}} + r\tilde{c}\tilde{c}^*)^{-1/2} x|^2], \end{aligned}$$

with the notation (??), which agrees with the previous expression.

This is to say that the formulae for Q_B can be reduced, without the integral in \mathbb{R}^{d-1} , i.e, for $\rho > 0$,

$$Q_B(t, x) = -\partial_d \int_0^{t/\rho} e^{-b_0 r} \Gamma_d((t - \rho r)\mathbb{1} + r\underline{b}, x + br) dr, \quad (2.29)$$

and for $\rho = 0$ and $b_d > 0$,

$$Q_B(t, x) = -\partial_d \int_0^\infty e^{-b_0 r} \Gamma_d(t\mathbb{1} + r\underline{b}, x + br) dr, \quad (2.30)$$

where \underline{b} is the matrix (b_{ij}) enlarged by zeros to be a square d -dimensional matrix, and the notation (??) is used. Clearly, the expression (??) becomes (??) as ρ approaches zero. Thus, formula (??) represents all cases, with the convention $t/\rho = \infty$ if $\rho = 0$.

Remark 2.8. To summarize, we have shown that

$$\begin{cases} P_B(t, x, \tilde{y}) = Q_B(t, \tilde{x} - \tilde{y}, x_d), \\ G_B(t, x, y) = [\Gamma_d(t, \tilde{x} - \tilde{y}, x_d - y_d) - \Gamma_d(t, \tilde{x} - \tilde{y}, x_d + y_d)] + \\ \quad + 2b_d Q_B(t, \tilde{x} - \tilde{y}, x_d + y_d) + \rho \delta_0(y_d) Q_B(t, \tilde{x} - \tilde{y}, x_d), \end{cases}$$

are the Poisson and Green functions corresponding to the heat-equation with a Wentzell type boundary condition (??), where the kernel Q_B is given by the formula (??) and satisfies

$$BQ_B(t, x) = \partial_d \Gamma_d(t, x), \quad \forall t > 0, x \in \mathbb{R}_+^d.$$

Clearly, these equalities prove the heat kernel type estimates P_B and G_B knowing the heat kernel estimates for the kernel Q_B , namely: Denote by D_B^k any derivative of order k with respect to some coefficients of the operator B , i.e., with respect to any ς_{ij} (or ϱ_{ij} or β_i), any b_0, b_1, \dots, b_d and ρ . Also, denote by D_{tx}^n any partial derivative in the variable t and $x = (x_1, \dots, x_d)$ of parabolic order n , i.e., $n = 2n_0 + n_1 + \dots + n_d$ with $D_{tx}^n = \partial_t^{n_0} \partial_1^{n_1} \dots \partial_d^{n_d}$. Then, for any n and k there exist positive constants $C_0 = C_{nk}$ and $c_0 = c_{nk}$ such that

$$|D_{tx}^n D_B^k Q_B(t, x)| \leq C_0 t^{-(d+n)/2} \exp\left(-c_0 \frac{|x|^2}{2t}\right), \quad (2.31)$$

for every $t > 0$ and x in \mathbb{R}_+^d . Moreover, even if B may contain second order derivative the expression BQ_B satisfies heat kernel type estimates as $\partial_d \Gamma_d$, i.e., a singularity comparable to first-order derivatives. \square

At this point, going back to the particular expressions $P(t, x)$ and $G(t, x, y_d)$ instead of the general $P(t, x, \tilde{y})$ and $G(t, x, y)$ for the Green and Poisson functions, and with the notation of the previous sections, we have

Theorem 2.9. *Let A and B be constant coefficients operators of the form (??) and (??), satisfying (??), (??) and (??) in the half-space \mathbb{R}_+^d . Denote by $G_{A,D}(t, x, y_d)$ and $P_{A,D}(t, x)$ the Green and*

Poisson functions with Dirichlet boundary conditions, and define the kernel $Q_{A,B}(t, x)$ by the formula

$$Q_{A,B}(t, x) = \int_0^{t/\rho} e^{-b_0 r} dr \int_{\mathbb{R}^{d-1}} \tilde{\Gamma}_0(r, \tilde{z}) P_{A,D}(t - \rho r, \tilde{x} - \tilde{\zeta} \tilde{z} + \tilde{b}r, x_d + b_d r) d\tilde{z}, \quad (2.32)$$

with the convention¹ that $t/\rho = \infty$ if $\rho = 0$. Then

$$[B_0 + b_d \partial_d - \rho \partial_t] Q_{A,B}(t, x) = P_{A,D}(t, x), \quad \forall t > 0, x \in \mathbb{R}_+^d$$

and

$$\begin{cases} P_{A,B}(t, x) = Q_{A,B}(t, x), \\ G_{A,B}(t, x, y_d) = G_{A,D}(t, x, y_d) + \frac{2b_d}{a_{dd}} Q_{A,B}(t, \tilde{x}, x_d + y_d) + \rho \delta_0(y_d) Q_{A,B}(t, x), \end{cases}$$

are the Poisson and Green functions with Wentzell boundary conditions. \square

Note that we do have an almost explicit expressions for $G_{A,D}$ and $P_{A,D}$ so that the first term in the series can be computed explicitly, and all other terms can be estimated with the K semi-norms.

2.4 Some Explicit Computations

We look here at two particular cases, elastic and sticky Brownian motions.

2.4.1 Elastic Case

In the elastic case, i.e., for $A = \frac{1}{2}\Delta$, $\rho = 0$, $\varsigma = 0$ and $b_d > 0$ we can compute the above integral, by first calculating

$$\int_0^\infty e^{-b_0 r} \Gamma_0(t, x + br) dr = (2\pi t)^{-d/2} \int_0^\infty e^{-b_0 r} e^{-\frac{|x+br|^2}{2t}} dr.$$

Indeed, by observing that

$$|x + br|^2 = |x|^2 + 2(x \cdot b)r + |b|^2 r^2 = \left[|b|r + \frac{(x \cdot b)}{|b|} \right]^2 + |x|^2 - \frac{(x \cdot b)^2}{|b|^2}$$

we have

$$\begin{aligned} \int_0^\infty e^{-b_0 r} \Gamma_0(t, x + br) dr &= \\ &= (2\pi t)^{-d/2} e^{\frac{(x \cdot b + b_0 t)^2 - |b|^2 |x|^2}{2|b|^2 t}} \int_0^\infty e^{-\frac{1}{2t} \left(|b|r + \frac{x \cdot b + b_0 t}{|b|} \right)^2} dr. \end{aligned}$$

Hence, the following natural change of variables

$$\rho = \frac{1}{\sqrt{2t}} \left(|b|r + \frac{x \cdot b + b_0 t}{|b|} \right),$$

yields

$$\begin{aligned} \int_0^\infty e^{-b_0 r} \Gamma_0(t, x + br) dr &= \\ &= (2\pi t)^{-d/2} e^{\frac{[x \cdot b + b_0 t]^2 - |b|^2 |x|^2}{2|b|^2 t}} \frac{\sqrt{2t}}{|b|} \int_{\frac{x \cdot b + b_0 t}{\sqrt{2t}|b|}}^\infty e^{-\rho^2} d\rho. \end{aligned}$$

Now, take the derivative of this expression with respect to x_d to get

$$\begin{aligned} Q_e(t, x) &= \frac{1}{|b|^2} \Gamma_0(t, x) \left\{ b_d + \right. \\ &\quad \left. + \sqrt{2} \frac{|b|^2 x_d - b_d [b \cdot x + b_0 t]}{|b| \sqrt{t}} e^{\frac{[b \cdot x + b_0 t]^2}{2t|b|^2}} \int_{\frac{b \cdot x + b_0 t}{\sqrt{2t}|b|}}^\infty e^{-z^2} dz \right\}, \end{aligned}$$

¹recall that either $\rho > 0$ or $b_d > 0$.

or

$$\left\{ \begin{aligned} Q_e(t, x) &= |b|^{-2} \Gamma_0(t, x) \left\{ b_d + \sqrt{\pi} \frac{|b|^2 x_d - b_d [b \cdot x + b_0 t]}{|b| \sqrt{2t}} \times \right. \\ &\quad \left. \times \exp\left(\frac{[b \cdot x + b_0 t]^2}{2t|b|^2}\right) \operatorname{Erfc}\left(\frac{b \cdot x + b_0 t}{\sqrt{2t}|b|}\right) \right\}. \end{aligned} \right. \quad (2.33)$$

using the complementary error function $\operatorname{Erfc}(\cdot)$. In particular, for the Neumann problem, i.e., $b_0 = 0$, $b_d = 1$ and $\tilde{b} = 0$, we found $Q_N = \Gamma_0$ which yields the well know formulae

$$\begin{aligned} P_N(t, x) &= \Gamma_d(t, x), \\ G_N(t, x, y_d) &= \Gamma_d(t, \tilde{x}, x_d - y_d) + \Gamma_d(t, \tilde{x}, x_d + y_d), \end{aligned}$$

as expected.

The above explicit formula allow a simple verification of the lower and upper heat kernel estimates for the elastic case, i.e., by means of the bounds

$$\left\{ \begin{aligned} \frac{2}{r + \sqrt{r^2 + 2}} &\leq \sqrt{\pi} e^{r^2} \operatorname{Erfc}(r) \leq \frac{2}{r + \sqrt{r^2 + 1}}, \quad \forall r \geq 0, \\ 2 = \operatorname{Erfc}(-\infty) &< \operatorname{Erfc}(r) \leq \operatorname{Erfc}(0) = 1, \quad \forall r \leq 0, \end{aligned} \right.$$

we can estimate the expression

$$R_e(t, x) = b_d + \sqrt{\pi} \frac{|b|^2 x_d - b_d [b \cdot x + b_0 t]}{|b| \sqrt{2t}} e^{\frac{[b \cdot x + b_0 t]^2}{2t|b|^2}} \operatorname{Erfc}\left(\frac{b \cdot x + b_0 t}{\sqrt{2t}|b|}\right)$$

appearing in the definition (??) of the kernel Q_e . Indeed, since

$$|b|^2 x_d - b_d [b \cdot x + b_0 t] = |\tilde{b}|^2 x_d - b_d (\tilde{b} \cdot \tilde{x} + b_0 t) \leq |b| |\tilde{b}| |x|, \quad \forall t \geq 0, x \in \mathbb{R}_+^d,$$

if $b \cdot x + b_0 t \leq 0$ we have $\tilde{b} \cdot \tilde{x} \leq -b_d x_d - b_0 t$,

$$[b \cdot x + b_0 t]^2 = (\tilde{b} \cdot \tilde{x})^2 + 2(\tilde{b} \cdot \tilde{x})(b_d x_d + b_0 t) + (b_d x_d + b_0 t)^2 \leq |\tilde{b}|^2 |\tilde{x}|^2 - (b_d x_d + b_0 t)^2$$

and

$$\frac{[b \cdot x + b_0 t]^2}{2|b|^2 t} \leq \frac{|\tilde{b}|^2 |\tilde{x}|^2 - (b_d x_d)^2}{2|b|^2 t} = \frac{|x|^2}{2t} - \frac{b_d^2 |x|^2}{2|b|^2 t},$$

so that

$$b_d \leq R_e(t, x) \leq b_d + \sqrt{2\pi} |\tilde{b}| \frac{|x|}{\sqrt{t}} \exp\left(\frac{|x|^2}{2t} - \frac{b_d^2 |x|^2}{2|b|^2 t}\right), \quad (2.34)$$

while if $b \cdot x + b_0 t \geq 0$ then the calculations are longer. Begin with

$$R_e(t, x) \leq \frac{2|b|^2 x_d + b_d [\sqrt{(b \cdot x + b_0 t)^2 + |b|^2 4t} - (b \cdot x + b_0 t)]}{(b \cdot x + b_0 t) + \sqrt{(b \cdot x + b_0 t)^2 + |b|^2 4t}},$$

and use

$$\begin{aligned} &[\sqrt{(b \cdot x + b_0 t)^2 + |b|^2 4t} - (b \cdot x + b_0 t)] = \\ &= \frac{[(b \cdot x + b_0 t)^2 + |b|^2 4t - (b \cdot x + b_0 t)^2]}{[\sqrt{(b \cdot x + b_0 t)^2 + |b|^2 4t} + (b \cdot x + b_0 t)]} = \\ &= \frac{|b|^2 4t}{\sqrt{(b \cdot x + b_0 t)^2 + |b|^2 4t} + (b \cdot x + b_0 t)}, \end{aligned}$$

to get

$$R_e(t, x) \leq \frac{2|b|^2 x_d [(b \cdot x + b_0 t) + \sqrt{(b \cdot x + b_0 t)^2 + |b|^2 4t}] + b_d |b|^2 4t}{[(b \cdot x + b_0 t) + \sqrt{(b \cdot x + b_0 t)^2 + |b|^2 4t}]^2}.$$

Similarly, we obtain

$$R_e(t, x) \geq \frac{2|b|^2 x_d [(b \cdot x + b_0 t) + \sqrt{(b \cdot x + b_0 t)^2 + |b|^2 2t}] + b_d |b|^2 2t}{[(b \cdot x + b_0 t) + \sqrt{(b \cdot x + b_0 t)^2 + |b|^2 2t}]^2}.$$

This shows that for

$$r = \frac{(b \cdot x + b_0 t) + \sqrt{(b \cdot x + b_0 t)^2 + |b|^2 2t}}{\sqrt{t}} \quad \text{and} \quad \rho = \frac{x_d}{\sqrt{t}}$$

we have

$$R_e(t, x) \leq \max \{b_d, 2|b|^2(\rho r^{-1} + 2b_d r^{-2})\},$$

$$2|b| \leq r \leq 2|b| + 2\frac{|b|^2}{b_d} \rho, \quad \rho \geq 0,$$

i.e., if $b \cdot x + b_0 t \geq 0$ then

$$R_e(t, x) \leq b_d + \frac{x_d}{\sqrt{t}}, \quad \forall t > 0, x \in \mathbb{R}_+^d. \quad (2.35)$$

A lower bound is found similarly, namely, for

$$r = \frac{(b \cdot x + b_0 t) + \sqrt{(b \cdot x + b_0 t)^2 + |b|^2 2t}}{\sqrt{t}} \quad \text{and} \quad \rho = \frac{x_d}{\sqrt{t}}$$

we have

$$R_e(t, x) \geq \min \{b_d, 2|b|^2(\rho r^{-1} + b_d r^{-2})\},$$

$$2|b| \leq r, \quad r^2 \leq 4\left(\frac{[b \cdot x + b_0 t]^2}{t} + |b|^2\right), \quad \rho \geq 0,$$

i.e., if $b \cdot x + b_0 t \geq 0$ then

$$R_e(t, x) \geq \frac{b_d}{2} \left(\frac{[b \cdot x + b_0 t]^2}{|b|^2 t} + 1\right)^{-1}, \quad \forall t > 0, x \in \mathbb{R}_+^d. \quad (2.36)$$

These estimates (??), (??) and (??) yield upper and lower estimates on Q_e , i.e., for every $c_0 > 0$ such that $b_d^2 c_0 < |b|$ there exists a constant $C_0 > 0$ (depending only on c_0 , $|b|$ and $b_d > 0$) satisfying

$$Q_e(t, x) \leq C_0 t^{-d/2} \exp\left(-c_0 \frac{|x|^2}{2t}\right), \quad \forall t > 0, x \in \mathbb{R}_+^d, \quad (2.37)$$

and for any $c_1 > 1$ there exist a constant $C_1 > 0$ (depending only on c_1 , $|b|$ and $b_d > 0$) satisfying

$$Q_e(t, x) \geq C_1 (1 + b_0 t)^{-1} t^{-d/2} \exp\left(-c_1 \frac{|x|^2}{2t}\right), \quad \forall t > 0, x \in \mathbb{R}_+^d. \quad (2.38)$$

Moreover, upper bound estimate can also be found for all derivatives of Q_e as in (??).

The elastic case for $b_b \rightarrow 0$ and $\zeta = 0$ yields

$$Q_e(t, x) = \Gamma_0(t, x) \sqrt{\pi} \frac{x_d}{\sqrt{2t|\tilde{b}|}} \exp\left(\frac{[\tilde{b} \cdot \tilde{x} + b_0 t]^2}{2t|\tilde{b}|^2}\right) \operatorname{Erfc}\left(\frac{\tilde{b} \cdot \tilde{x} + b_0 t}{\sqrt{2t|\tilde{b}|}}\right).$$

However, we see that as $\tilde{b} \cdot \tilde{x} \rightarrow -\infty$, the heat kernel type estimate is lost.

2.4.2 Sticky Case

In the sticky case, i.e., if $A = \frac{1}{2}\Delta$, $\rho > 0$, but $b_0 = 0$ and $\zeta = 0$ then

$$Q_s(t, x) = -\partial_d \int_0^{t/\rho} \Gamma_0(t - \rho r, x + br) dr =$$

$$= -(2\pi)^{-d/2} \partial_d \int_0^{t/\rho} (t - \rho r)^{-d/2} e^{-\frac{|x-br|^2}{2(t-\rho r)}} dr.$$

This integral can be expressed in terms of the function $\Phi_\nu(t, x)$ defined by the integral

$$\Phi_\nu(t, x) = \frac{1}{2^\nu} \int_0^t \frac{1}{s^{\nu+1}} e^{-(s+\frac{x^2}{4s})} ds, \quad \forall t, x, \nu > 0.$$

Observing that, for any odd dimension d , this function can be further simplified using the complementary error function $\text{Erfc}(\cdot)$. Notice that

$$\Phi_{\nu+1}(t, x) = -\frac{1}{x} \frac{\partial}{\partial x} \Phi_\nu(t, x), \quad \forall t, x, \nu > 0.$$

and that for $t \rightarrow \infty$ the function reduces to the so-called modified Bessel functions of second kind (also called Kelvin or MacDonald functions) defined by

$$K_\nu(x) = \frac{1}{2} \left(\frac{x}{2}\right)^\nu \int_0^\infty \frac{1}{s^{\nu+1}} e^{-(s+\frac{x^2}{4s})} ds, \quad x, \nu > 0.$$

We have

$$\begin{aligned} Q_s(t, x) = & -\frac{1}{\rho} \frac{1}{(2\pi)^{d/2}} \exp\left[(x + b\frac{t}{\rho}) \cdot (b\frac{1}{\rho})\right] \times \\ & \times \left[b_d |b|^{d-2} \frac{1}{\rho^{d-1}} \Phi_{\frac{d}{2}-1}\left(\frac{|b|^2 t}{2\rho^2}, \frac{1}{\rho} |b| |x + b\frac{t}{\rho}|\right) - \right. \\ & \left. - (x_d + b_d \frac{t}{\rho}) |b|^d \frac{1}{\rho^d} \Phi_{\frac{d}{2}}\left(\frac{|b|^2 t}{2\rho^2}, \frac{1}{\rho} |b| |x + b\frac{t}{\rho}|\right) \right], \end{aligned} \quad (2.39)$$

where we remark the homogeneity in b/ρ as expected.

Clearly, dimension $d = 1$ corresponds to $\nu = -1/2$. In this case, we can calculate

$$\begin{cases} \int_0^t e^{-(s+\frac{x^2}{4s})} \frac{ds}{\sqrt{s}} = e^{-x} \int_{\frac{x-2t}{2\sqrt{t}}}^\infty e^{-z^2} dz - e^x \int_{\frac{x+2t}{2\sqrt{t}}}^\infty e^{-z^2} dz = \\ \\ = \frac{\sqrt{\pi}}{2} e^{-x} \text{Erfc}\left(\frac{x-2t}{2\sqrt{t}}\right) - \frac{\sqrt{\pi}}{2} e^x \text{Erfc}\left(\frac{x+2t}{2\sqrt{t}}\right). \end{cases} \quad (2.40)$$

In particular, for $t \rightarrow \infty$, one gets

$$\int_0^\infty e^{-(s+\frac{x^2}{4s})} \frac{ds}{\sqrt{s}} = e^{-x} \sqrt{\pi}.$$

Indeed, performing the substitution $s = r^2$ one get

$$\int_0^t e^{-(s+\frac{x^2}{4s})} \frac{ds}{\sqrt{s}} = 2 \int_0^{\sqrt{t}} e^{-(r^2+\frac{x^2}{4r^2})} dr.$$

Now observe that

$$r^2 + \frac{x^2}{4r^2} = \left(r - \frac{x}{2r}\right)^2 + x,$$

and that the invertible substitution $\rho = r - \frac{x}{2r}$ yields

$$r = \frac{1}{2} (\rho + \sqrt{\rho^2 + 2x}) \quad \text{and} \quad 2dr = \left(1 + \frac{\rho}{\sqrt{\rho^2 + 2x}}\right) d\rho.$$

Hence

$$\int_0^t e^{-(s+\frac{x^2}{4s})} \frac{ds}{\sqrt{s}} = e^{-x} \int_{-\infty}^{\frac{2t-x}{2\sqrt{t}}} e^{-\rho^2} \left(1 + \frac{\rho}{\sqrt{\rho^2 + 2x}}\right) d\rho,$$

which can be written as (??), after remarking that on any symmetric intervals about zero, the integration with respect the measure $(\rho/\sqrt{\rho^2 + 2x})d\rho$ is zero.

Thus, for example we have

$$\Phi_{-\frac{1}{2}}(t, x) = \sqrt{2} \left(e^{-x} \int_{\frac{x-2t}{2\sqrt{t}}}^\infty e^{-z^2} dz - e^x \int_{\frac{x+2t}{2\sqrt{t}}}^\infty e^{-z^2} dz \right)$$

and

$$\Phi_{\frac{1}{2}}(t, x) = -\frac{1}{x} \frac{\partial}{\partial x} \Phi_{-\frac{1}{2}}(t, x) = \frac{\sqrt{2}}{x} \left(e^{-x} \int_{\frac{x-2t}{2\sqrt{t}}}^{\infty} e^{-z^2} dz + e^x \int_{\frac{x+2t}{2\sqrt{t}}}^{\infty} e^{-z^2} dz \right),$$

which gives

$$Q_s(t, x) = \frac{1}{\sqrt{2\pi}} e^{\frac{t+2cx}{4c^2}} \left[-\frac{t+2cx}{4c^2} \Phi_{\frac{1}{2}}\left(\frac{t}{8c^2}, \frac{t+2cx}{4c^2}\right) + \Phi_{-\frac{1}{2}}\left(\frac{t}{8c^2}, \frac{t+2cx}{4c^2}\right) \right],$$

and using the complementary error function $\text{Erfc}(\cdot)$,

$$Q_s(t, x) = -\frac{2}{\sqrt{\pi}} e^{\frac{t+cx}{2c^2}} \int_{\frac{t+cx}{c\sqrt{2t}}}^{\infty} e^{-z^2} dz = \exp\left(\frac{t+cx}{2c^2}\right) \text{Erfc}\left(\frac{t+cx}{c\sqrt{2t}}\right),$$

for $d = 1$, $b = b_d = 1$ and $\rho/2 = c$.

3 Variable Coefficients

Here we are under the conditions of the introduction, the operators A and B have now variable coefficients. By means of a simple application of Green's Theorem (i.e., integration by parts) in \mathbb{R}_+^d and assuming that all coefficients are sufficiently smooth, we can show the following relations among the various Green and Poisson functions.

First, as in Garroni and Menaldi [?, Section 2.4] we discuss the adjoint operator for the properly integro-differential part I given by (??) under the assumptions (??), (??) and (??). We need to treat two cases, first when $0 \leq \gamma \leq 1$, we have

$$I(t)\varphi(x) = \int_{\mathbb{R}_*^m} [\varphi(x + \mathbf{j}(\zeta, t, x)) - \varphi(x)] \mathbf{m}(\zeta, t, x) \pi(d\zeta). \quad (3.1)$$

Then, assuming for some constant M_γ

$$\begin{cases} |\mathbf{m}(\zeta, t, x) - \mathbf{m}(\zeta, t, x')| \leq M_\gamma |x - x'|^\gamma, \\ |\nabla \mathbf{j}(\zeta, t, x)| \mathbb{1}_{\{\bar{j}(\zeta) < 1\}} \leq M_\gamma \bar{j}^\gamma(\zeta), \end{cases} \quad (3.2)$$

for every $t > 0$, x, x' in \mathbb{R}^d , ζ in \mathbb{R}_*^m and the same $\bar{j}(\zeta)$ of (??), we deduce

$$\int_{\mathbb{R}^d} I\varphi(x) \psi(x) dx = \int_{\mathbb{R}^d} \varphi(x) I^* \psi(x) dx, \quad \forall \varphi, \psi \in \mathcal{D}(\mathbb{R}^d),$$

where the adjoint operator I^* is defined by

$$\begin{cases} I^* \psi = \int_{\mathbb{R}_*^m} [\psi(\cdot - \mathbf{j}^*(\zeta, \cdot)) - \psi] \mathbf{m}^*(\zeta, \cdot) \pi(d\zeta) + \\ \quad + \psi \int_{\mathbb{R}_*^m} [\mathbf{m}^*(\zeta, \cdot) - \mathbf{m}(\zeta, \cdot)] \pi(d\zeta), \end{cases} \quad (3.3)$$

and

$$\begin{aligned} \mathbf{j}^*(\zeta, t, X) &= \mathbf{j}(\zeta, t, x(t, X, \zeta)), \\ \mathbf{m}^*(\zeta, t, X) &= \mathbf{m}(\zeta, t, x(t, X, \zeta)) \det(\partial x(t, X, \zeta) / \partial X) \end{aligned}$$

and with the change of variable $X = x + \mathbf{j}(\zeta, t, x)$.

Now for $1 < \gamma \leq 2$, we keep the expression (??) and we assume

$$\begin{cases} |\nabla \mathbf{m}(\zeta, t, x)| \leq M_\gamma, \\ |\nabla \mathbf{m}(\zeta, t, x) - \nabla \mathbf{m}(\zeta, t, x')| \leq M_\gamma |x - x'|^{\gamma-1}, \\ |\nabla \mathbf{j}(\zeta, t, x)| \mathbb{1}_{\{\bar{j}(\zeta) < 1\}} \leq M_\gamma \bar{j}^{\gamma-1}(\zeta), \\ |\nabla \cdot \mathbf{j}(\zeta, t, x) - \nabla \cdot \mathbf{j}(\zeta, t, x + \mathbf{j}(\zeta, t, x))| \mathbb{1}_{\{\bar{j}(\zeta) < 1\}} \leq M_\gamma \bar{j}^\gamma(\zeta), \end{cases} \quad (3.4)$$

for every $t > 0$, x, x' in \mathbb{R}^d , ζ in \mathbb{R}_*^m and the same $\bar{j}(\zeta)$ of (??). Note that $\nabla \cdot j(\zeta, \cdot, t, x)$ means the divergence of the function $x \mapsto j(\zeta, t, x)$ for any fixed ζ . Then, the adjoint operator I^* is written as $I_0^* + D_1$, where

$$I_0^* \psi = \int_{\mathbb{R}_*^m} [\psi(\cdot - j^*(\zeta, \cdot)) - \psi - j^*(\zeta, \cdot) \cdot \nabla \psi] \mathfrak{m}^*(\zeta, \cdot) \pi(d\zeta) \quad (3.5)$$

and

$$\left\{ \begin{aligned} D_1 \psi &= \left(\int_{\mathbb{R}_*^m} [j(\zeta, \cdot) \mathfrak{m}(\zeta, \cdot) - j^*(\zeta, \cdot) \mathfrak{m}^*(\zeta, \cdot)] \pi(d\zeta) \right) \cdot \nabla \psi + \\ &+ \int_{\mathbb{R}_*^m} [\mathfrak{m}^*(\zeta, \cdot) + \mathfrak{m}(\zeta, \cdot) \nabla \cdot j(\zeta, \cdot) + j(\zeta, \cdot) \cdot \nabla \mathfrak{m}(\zeta, \cdot) - \mathfrak{m}(\zeta, \cdot)] \pi(d\zeta) \end{aligned} \right\} \psi, \quad (3.6)$$

which is a first-order differential operator.

Next, if the coefficients are smooth, then it is convenient to define the formal adjoint operators

$$\left\{ \begin{aligned} A^*(t) \varphi(x) &:= A_0^*(t) \varphi(x) + I^*(t) \varphi(x) + \sum_{i=1}^d \partial_i (a_i^*(t, x) \varphi(x)) - a_0^*(t, x) \varphi(x), \\ A_0^*(t) \varphi(x) &:= \frac{1}{2} \sum_{i,j=1}^d \partial_j (a_{ij}(t, x) \partial_i \varphi(x)), \end{aligned} \right. \quad (3.7)$$

and

$$\left\{ \begin{aligned} B^*(t) \varphi(x) &:= B_0^*(t) \varphi(x) + b_d(t, \tilde{x}) \partial_d \varphi(x) - \rho A^* \varphi(x), \\ B_0^*(t) &:= \frac{1}{2} \sum_{i,j=1}^{d-1} \partial_j (b_{ij}(t, \tilde{x}) \partial_i \cdot) + \sum_{i=1}^{d-1} \partial_i (b_i^*(t, \tilde{x}) \cdot) - b_0^*(t, \tilde{x}), \end{aligned} \right. \quad (3.8)$$

where the adjoint coefficients may be computed as follows

$$a_{ij}(t, x) = a_{ij}^*(t, x), \quad a_0^*(t, x) = a_0(t, x) + \sum_{i=1}^d \partial_i a_i(t, x),$$

$$a_i^*(t, x) = -a_i(t, x) - \sum_{j=1}^d \partial_j a_{ij}(t, x),$$

and

$$b_{ij}^*(t, \tilde{x}) = b_{ij}(t, \tilde{x}), \quad b_0^*(t, \tilde{x}) = b_0(t, \tilde{x}) + \sum_{i=1}^{d-1} \partial_i b_i(t, \tilde{x}),$$

$$b_i^*(t, \tilde{x}) = -b_i(t, \tilde{x}) - \sum_{j=1}^{d-1} \partial_j b_{ij}(t, \tilde{x}),$$

Remark that in the construction of the Green and Poisson functions we require $a_0(t, x) \geq 0$ and $b_0(t, \tilde{x}) \geq 0$ (among other assumptions) but not necessarily $a_0^*(t, x) \geq 0$ and $b_0^*(t, \tilde{x}) \geq 0$. Thus, the adjoint problem does not always satisfy the conditions for the direct construction.

Because of the assumption (??) all jumps are interior in \mathbb{R}_+^d and so there is not contribution from the jumps on the boundary. Thus, define the co-normal differential operators ∂_A and ∂_A^* ,

$$\left\{ \begin{aligned} \partial_A \varphi(x) &:= \frac{1}{2} \sum_{i=1}^d (a_{id}(t, \tilde{x}, 0) \partial_i \varphi(x)), \\ \partial_A^* \varphi(x) &:= \frac{1}{2} \sum_{j=1}^d \partial_j (a_{dj}(t, \tilde{x}, 0) \varphi(x)) - a_d(t, \tilde{x}, 0) \varphi(x) \end{aligned} \right. \quad (3.9)$$

on the boundary $\partial \mathbb{R}_+^d \simeq \mathbb{R}^{d-1}$. Note that for $A = \frac{1}{2} \Delta$ we have $\partial_A = \partial_A^* = \frac{1}{2} \partial_d$. We have the relations

$$\left\{ \begin{aligned} P_{A,D}(t, x, s, \tilde{y}) &= -\partial_A^* G_{A,D}(t, x, s, \tilde{y}, 0), \\ G_{A,D}(t, x, s, y) &= G_{A,D}^*(s, y, t, x), \\ P_{A,N}(t, x, s, \tilde{y}) &= G_{A,N}(t, x, s, \tilde{y}, 0), \end{aligned} \right. \quad (3.10)$$

for Dirichlet and co-normal (Neumann) boundary conditions, where $G_{A,D}^*$ means the Green function associated with the adjoint operator A^* with Dirichlet boundary conditions. Note that the co-normal first order differential operator $\partial_{\tilde{\lambda}}^*$, defined by (??), is acting on the variable y . Moreover, the Green function $G_{A,B}$ may be found as solving the adjoint problem in the variable (s, y) , i.e., $(s, y) \mapsto G(t, x, s, y)$ satisfies

$$[-\partial_s - A^*(s)]G_{A,B}(t, x, \cdot, \cdot) = 0, \quad G(t, x, t, \cdot) = \delta_x,$$

for $t > s \geq 0$ and x, y in \mathbb{R}_+^d , where A^* is as above, plus a suitable complementary boundary condition.

If $\rho = 0$ and one normalize B with respect to A , i.e., assuming

$$a_{dd}(t, \tilde{x}, 0) = 2b_d(t, \tilde{x}), \quad \forall t \geq 0, \tilde{x} \in \mathbb{R}^{d-1}, \quad (3.11)$$

then the function $(s, y) \mapsto G_{A,B}(t, x, s, y)$ satisfies the boundary condition

$$B_A^* G_{A,B}(t, x, s, \tilde{y}, 0) = 0, \quad \forall s < t, \tilde{y} \in \mathbb{R}^{d-1},$$

where

$$B_A^* := B_0^* + \tilde{\partial}_A^* + \partial_A^* = B_0^* + 2\tilde{\partial}_A^* + \partial_d(a_{dd} \cdot) - a_d,$$

i.e.,

$$P_{A,B}(t, x, s, \tilde{y}) 2b_d(s, \tilde{y}) = G_{A,B}(t, x, s, \tilde{y}, 0) a_{dd}(s, \tilde{y}, 0),$$

for the case $\rho = 0$. However, in general we have

$$\begin{cases} P_{A,B}(t, x, s, \tilde{y}) = Q_{A,B}(t, x, s, \tilde{y}, 0), \\ G_{A,B}(t, x, s, y) = G_{A,D}(t, x, s, y) + \frac{2b_d(s, \tilde{y})}{a_{dd}(s, \tilde{y}, 0)} Q_{A,B}(t, x, s, y) + \\ \quad + Q_{A,B}(t, x, s, \tilde{y}, 0) \rho(s, \tilde{y}) \delta(y_d), \end{cases} \quad (3.12)$$

for a suitable kernel $Q_{A,B}$. Note that $G_{A,D}(t, x, s, y) = 0$ for either $x_d = 0$ or $y_d = 0$.

If $b_d = 0$ then we can calculate the Poisson function as

$$P_{A,B}(t, x, s, \tilde{y}) := \int_s^t d\tau \int_{\mathbb{R}^{d-1}} (-\partial_{\tilde{\lambda}}^* G_{A,D}(t, x, \tau, \tilde{\xi}, 0)) P_B(\tau, \tilde{\xi}, s, \tilde{y}) d\tilde{\xi},$$

where $P_B(t, \tilde{x}, s, \tilde{y})$ is the fundamental solution corresponding to $B_\rho^* = B_0^* - (\partial_t \rho)$ with initial condition $P_B(t, \tilde{x}, t, \cdot) \rho(t, \cdot) = \delta_{\tilde{x}}$. Also, the Green function $G_{A,B} = G_{A,D}$. Under the normalization condition (??) we have $P_{A,B}(t, x, s, \tilde{y}) = G_{A,B}(t, x, s, \tilde{y}, 0)$ when $2b_d(s, \tilde{y}) = a_{dd}(s, \tilde{y}, 0)$, ignoring the $\delta(y_d) \rho(s, \tilde{y})$.

3.1 Successive Approximations

First we consider the purely differential case, and then we give some indication of how to extend the method to the nonlocal case as in Garroni and Menaldi [?]. One of the arguments used in the construction of the fundamental functions for variables coefficients, the parametrix method of Levi, is essentially based on the study of a Volterra equation for heat-type kernels $Q(t, x, \tau, \xi)$, namely,

$$\begin{cases} Q(t, x, \tau, \xi) = Q_0(t, x, \tau, \xi) + (Q_0 \star Q)(t, x, \tau, \xi), \\ (Q_0 \star Q)(t, x, \tau, \xi) := \int_\tau^t ds \int_{\mathbb{R}^d} Q_0(t, x, s, y) Q(s, y, \tau, \xi) dy, \end{cases} \quad (3.13)$$

where the given kernel Q_0 satisfies the estimates

$$|Q_0(t, x, \tau, \xi)| \leq C_0 (t - \tau)^{-\frac{d+2-\alpha}{2}} \exp\left(-c_0 \frac{|x - \xi|^2}{t - \tau}\right), \quad (3.14)$$

for any $t > \tau$, and x, ξ in \mathbb{R}^d , and some $\alpha > 0$. For a given $c_0 > 0$, it is convenient to denote by $[[Q_0]]_{(\alpha)}$ the smallest constant C_0 for which the bound (??) is satisfied.

Mainly using the Beta function and the equality

$$\begin{cases} \int_{\mathbb{R}^d} \exp\left(-c_0 \frac{|x - y|^2}{t - s}\right) \exp\left(-c_0 \frac{|y - \xi|^2}{s - \tau}\right) dy = \\ = \left[\left(\frac{\pi}{c_0}\right) \left(\frac{(t - s)(s - \tau)}{t - \tau}\right)\right]^{\frac{d}{2}} \exp\left(-c_0 \frac{|x - \xi|^2}{t - \tau}\right), \end{cases}$$

one can prove that the sequence of kernels Q_n defined by recurrence as

$$Q_{n+1}(t, x, \tau, \xi) = (Q_0 \star Q_n)(t, x, \tau, \xi)$$

satisfies

$$\llbracket Q_n \rrbracket_{(n\alpha)} \leq \frac{q_\alpha}{(n!)^{\frac{\alpha}{2}}}, \quad \forall n = 1, 2, \dots,$$

where the constant q_α depends only on c_0, C_0, α and d . Hence, the Volterra equation (??) has a (unique) solution given by the series

$$Q(t, x, \tau, \xi) = \sum_{n=0}^{\infty} Q_n(t, x, \tau, \xi),$$

where the limit is done in the uniform convergence on compact sets in $\{(t, x, \tau, \xi) : t > \tau, x, \xi \in \mathbb{R}^d\}$.

This same argument can be used with the Green function in half-space \mathbb{R}_+^d , where the Volterra equation has the form

$$\begin{cases} Q(t, x, \tau, \xi) = Q_0(t, x, \tau, \xi) + (Q_0 \star Q)(t, x, \tau, \xi), \\ (Q_0 \star Q)(t, x, \tau, \xi) := \int_{\tau}^t ds \int_{\mathbb{R}_+^d} Q_0(t, x, s, y) Q(s, y, \tau, \xi) dy, \end{cases} \quad (3.15)$$

with a given kernel Q_0 defined within any \mathbb{R}_+^d instead \mathbb{R}^d , and also, with the Poisson function in half-space \mathbb{R}_+^d , where now the Volterra equation has the form

$$\begin{cases} R(t, x, \tau, \tilde{\xi}) = R_0(t, x, \tau, \tilde{\xi}) + (R_0 \tilde{\star} R)(t, x, \tau, \tilde{\xi}), \\ (R_0 \tilde{\star} R)(t, x, \tau, \tilde{\xi}) := \int_{\tau}^t ds \int_{\mathbb{R}^{d-1}} R_0(t, x, s, \tilde{y}) R(s, \tilde{y}, 0, \tau, \tilde{\xi}) d\tilde{y}, \end{cases} \quad (3.16)$$

with $y = (\tilde{y}, y_d)$ and $R(s, y, \tau, \tilde{\xi}) = R(s, \tilde{y}, y_d, \tau, \tilde{\xi})$. The Volterra equation (??) works very similar to the initial equation (??) in \mathbb{R}^d , and its (unique) solution is expressed as (convergent) series of kernels $R_{n+1} = R_0 \tilde{\star} R_n$. However, to study the Volterra equation (??) we need to have a kernel satisfying

$$|R_0(t, x, \tau, \xi)| \leq C_0 (t - \tau)^{-\frac{d+1-\alpha}{2}} \exp\left(-c_0 \frac{|x - \xi|^2}{t - \tau}\right), \quad (3.17)$$

for any $t > \tau$, and x, ξ in \mathbb{R}^d , i.e., the heat kernel type estimates in \mathbb{R}_+^d like (??) with $d - 1$ instead of d . If we just keep heat kernel type estimates like (??) in \mathbb{R}_+^d , then, because the kernel convolution $\tilde{\star}$ is only in dimension $(d - 1)$ the second integral in s involves a factor of the form $(t - s)^{(\alpha-3)/2}$, which is not integrable if $\alpha \leq 1$. Alternatively, we may assume that the kernel R_0 satisfies a variation of (??), namely

$$\begin{cases} |R_0(t, x, \tau, \tilde{\xi})| \leq C_0 \left(\frac{x_d}{\sqrt{t - \tau}} + 1\right) (t - \tau)^{-\frac{d+1-\alpha}{2}} \times \\ \times \exp\left(-c_0 \frac{|x - \tilde{\xi}|^2}{t - \tau}\right), \end{cases} \quad (3.18)$$

for any $t > \tau$, and x, ξ in \mathbb{R}_+^d , and some $\alpha > 0$. Thus, if $\tilde{\llbracket} R_0 \rrbracket_{(\alpha)}$ denotes the smallest constant C_0 for which the bound (??) is satisfied, then we have

$$\tilde{\llbracket} R_n \rrbracket_{(n\alpha)} \leq \frac{r_\alpha}{(n!)^{\frac{\alpha}{2}}}, \quad \forall n = 1, 2, \dots,$$

where the constant r_α depends only on c_0, C_0, α and d .

Next, to obtain a Hölder estimate of the k -type, namely,

$$\begin{cases} |Q(t, x, \tau, \xi) - Q(t', x', \tau', \xi')| \leq C_0 [|t - t'|^{\alpha/2} + |x - x'|^\alpha + \\ + |\tau - \tau'|^{\alpha/2} + |\xi - \xi'|^\alpha] (t - \tau)^{-\frac{d+2+k}{2}} \exp\left(-c_0 \frac{|x - \xi|^2}{t - \tau}\right), \end{cases} \quad (3.19)$$

for any $t > \tau, t' > \tau'$, and x, ξ, x', ξ' in \mathbb{R}^d , with $(t - \tau)|x' - \xi'|^2 \leq (t' - \tau')|x - \xi|^2$, for the same $\alpha > 0$ and $c_0 > 0$, is harder. A more complicate argument (essentially based on some cancellation property

of Q_0) is used to show the validity of (??) for the kernel Q , solution of the Volterra equation (??) with $k = 0$. Similarly for the kernel R .

Now, to extend this method to the nonlocal one uses semi-norms of the type (??), (??), (??), (??) as in the previous section, plus some variation regarding the Hölder character of the kernel, the semi-norms M , N and R as developed in the books [?, ?], where the case of oblique derivative is fully discussed, where to simplify the arguments, it is assumed that $\gamma + \alpha < 2$, so that no cancellation property for a kernel IQ is necessary. Certainly, some adaptation of the technique is necessary for the various type of boundary conditions. In what follows, we give some indication confined to the purely differential case.

3.1.1 Fundamental Solution

The problem is set in the whole space, and boundary conditions are replaced by growth conditions on the functions and their derivatives. For instance, comprehensive details on this classic case can be found in the books Friedman [?] or Ladyzhenskaya et al. [?].

The fundamental solution $G(t, x, s, y)$ defined for $t > s \geq 0$ and x, y in \mathbb{R}^d is expressed as

$$F(t, x, s, y) = F_0(t - s, x - y; s, y) + F_0 \star Q(t, x, s, y),$$

where $F_0(t, x; s, y)$ is the fundamental solution with *frozen* coefficients and Q is a kernel to be determined, This is usually referred to as the *parametrix method*. Clearly, constant or parameterized by (s, y) means

$$F_0(t, x; s, y) = e^{-ta_0(s, y)} \Gamma_d(\underline{ta}(s, y), x - \underline{ta}(s, y)),$$

with the notation (??), only the part with the matrix \underline{a} is most relevant, the terms with a_0 and the vector \underline{a} may be omitted, i.e., they can be part of the kernel Q .

If $A(s, y)$ denotes the second order differential operator (??) with parameterized coefficients (but acting on the variable x) and set

$$Q_0(t, x, s, y) := [A(s, y) - A(t, x)]F_0(t - s, x - y; s, y),$$

then the kernel Q is found as the solution of the Volterra equation

$$Q = Q_0 + Q_0 \star Q,$$

which can be solved by the method of successive approximations in view of the non-degeneracy and bounded Hölder continuity assumptions (??), (??) on the coefficients, and the heat kernel type estimates proved on the explicit expression of F_0 .

The next step is to establish the validity heat kernel estimates for the fundamental solution F , based on the above expression.

3.1.2 Dirichlet Conditions

Essentially, the Green function with Dirichlet boundary conditions is constructed with the same arguments used to build the fundamental solution, but the initial G_0 is the Green function with constant (or parameterized) coefficients corresponding to Dirichlet boundary conditions. Again, G_0 has an explicit expression as discussed in previous sections. In this case, the Volterra equation is solved by the method of successive approximations in the half-space \mathbb{R}_+^d and the Green function G_D is obtained as a series.

However, the arguments to construct the Poisson function are more delicate since the heat kernel type estimates have a stronger singularity, $(t - s)^{-1/2}$ higher than the Green function. If the coefficient were smooth, then the Poisson function could be calculated as normal derivative of the adjoint Green function with Dirichlet boundary conditions, via Green identity. For bounded Hölder continuous coefficients, the expression

$$P_D := P_0 + G_D \star [A_0 - A]P_0,$$

provided the Poisson function, where $P_0(t - s, x - y; s, y)$ is the Poisson function corresponding to constant (or parameterized) coefficients and

$$[A_0 - A]P_0(t - s, x - y; s, y) := [A(s, y) - A(t, x)]P_0(t - s, x - y; s, y),$$

with both differential operators A_0 and A acting on the variable x . The hard point is to establish the heat kernel type estimates for $P_D - P_0$ given by the above relation. Essentially, some kind of integration by parts is used to relate the singular integral $G_D \star [A_0 - A]P_0$ with the non-singular integral $[A_0 - A]G_D \star P_0$.

Alternatively, one may begin with the fundamental solution for variable coefficients denoted by $F(t, x, s, y)$ and then one solves the Dirichlet problem in the variables t and x ,

$$AF_1 = 0 \text{ in } \mathbb{R}_+^d \quad \text{and} \quad F_1 = F \text{ on } \partial\mathbb{R}_+^d,$$

with vanishing initial condition, and finally setting $G = F - F_1$ as the Green function with Dirichlet boundary conditions. Here, the point is to show the estimates necessary to allow the construction of the kernel F_1 . At the same time, this procedure produces estimates for the Poisson kernel P_D .

For instance, details can be found in Ivashišen [?] and Solonnikov [?] for parabolic systems. Also, in the books Eidelman [?] and Friedman [?] the interested reader will find some useful discussion.

3.1.3 Oblique Derivative

This is the case where the assumptions $\rho = 0$, $b_0 \geq 0$, normalization $2b_d = a_{dd}$, no second order derivatives, non-degeneracy (??), (??) and bounded Hölder continuous coefficients (??), (??) are imposed.

The arguments are similar to those of the fundamental solution, but a two-step method is necessary, one step to make variable the coefficients of the interior differential operator A and another step for the boundary operator B . Indeed, first set

$$G_1 = G_0 + G_0 \star Q,$$

and determine the kernel Q by the Volterra equation

$$Q = Q_0 + Q_0 \star Q, \quad Q_0 := [A_0 - A]G_0,$$

where $G_0(t - s, x - y; s, y)$ is the Green function corresponding to constant (or parameterized) coefficients, and again both differential operators $A_0 = A(s, y)$ and $A = A(t, x)$ act on the x variable. In view of the heat kernel type estimates on G_0 , this Volterra equation is solved by the method of successive approximations in the half-space \mathbb{R}_+^d . The Green function G_1 and the Poisson function (as mentioned in the previous section) are related by the equality

$$P_1(t, x, s, \tilde{y}) := G_1(t, x, s, \tilde{y}, 0) \frac{a_{dd}(s, \tilde{y}, 0)}{2b_d(s, \tilde{y})},$$

which corresponds to interior variable coefficients and constant (or parameterized) coefficients on the boundary, i.e., satisfying on the boundary

$$B_0G_1 = 0 \quad \text{and} \quad B_0P_1 = \tilde{\delta},$$

where the boundary differential operator $B_0 = B(s, y)$ is acting on the variable x , and $\tilde{\delta}$ is the delta measure on (t, \tilde{x}) concentrated at (s, \tilde{y}) .

The next step is to set

$$P = P_1 + P_1 \tilde{\star} R, \quad \text{and} \quad G = G_1 + P \tilde{\star} [B_0 - B]G_1,$$

and to determine the kernel R by solving

$$R = R_0 + R_0 \tilde{\star} R, \quad R_0 := [B_0 - B]P_1,$$

where G_1 and P_1 are as above, and both boundary differential operators $B_0 = B(s, y)$ and $B = B(t, x)$ are acting on the variable x . Note that because P_1 is the Poisson function one has

$$BP = \tilde{\delta} + [B - B_0]P_1 + R - R_0 \tilde{\star} R = \tilde{\delta},$$

and

$$BG = \tilde{\delta} + [B - B_0]G_1 + [B_0 - B]G_1 = 0,$$

which reproduces the desired equations.

Since the boundary operator B does not contain second order derivatives, the kernel R_0 has a weak (integrable) singularity $(t-s)^{(-d-1+\alpha)/2}$, so that the Volterra equation for R is solvable and heat type estimates are possible for the (surface) kernel convolution.

Alternatively, first we may set

$$P_1 = P_0 + P_0 \tilde{\star} R,$$

and determining R by solving the Volterra equation

$$R = R_0 + R_0 \tilde{\star} R, \quad R_0 := [B_0 - B]P_0,$$

where $P_0(t-s, \tilde{x} - \tilde{y}, x_d)$ is the Poisson function corresponding to constant (or parameterized) coefficients. Thus, once P_1 has been found, the expression

$$G_1 = G_0 + P_1 \tilde{\star} [B_0 - B]G_0$$

gives the Green function, i.e.,

$$A_0 G_1 = \delta, \quad \text{and} \quad B G_1 = 0.$$

Next, we have to solve the Volterra equation

$$Q = Q_0 + Q_0 \star Q, \quad Q_0 := [A_0 - A]G_1$$

and then $G = G_1 + G_1 \star Q$ results the expression of the Green function corresponding to A and B .

Certainly, a great effort is needed to establish the heat kernel type estimates for the Green functions $G = G_{A,B}$ and its derivatives. Note that in this case, the Poisson function $P_{A,B}$ is equal to the Green function on the boundary, i.e., $P_{A,B}(t, x, s, \tilde{y}) := G_{A,B}(t, x, s, \tilde{y}, 0)$, provided the normalization conditions $a_{dd}(t, \tilde{x}, 0) = 2b_d(t, \tilde{x})$ holds. Full details can be found in Garroni and Solonnikov [?], Ivashišen [?] and Solonnikov [?].

3.1.4 Sticky Boundary

This is the case $\rho > 0$, $b_0 \geq 0$ and $b_d > 0$. Hence, we normalize by setting $\rho = 1$, i.e., defining a new boundary operator \hat{B} by the relation $\rho \hat{B} = B$. Then, we proceed as in the previous case of oblique derivative or alternatively, one may begin setting

$$\hat{P}_1 = \hat{P}_0 + \hat{P}_0 \tilde{\star} R,$$

and determining R by solving the Volterra equation

$$R = \hat{R}_0 + \hat{R}_0 \tilde{\star} R, \quad \hat{R}_0 := [\hat{B}_0 - \hat{B}]P_0,$$

where $\hat{P}_0(t-s, \tilde{x} - \tilde{y}, x_d)$ is the Poisson function corresponding to constant (or parameterized) coefficients. Remark that $[\hat{B}_0 - \hat{B}]$ contains only derivatives up to the first order (even if $\rho > 0$, the term with A_0 is unchanged), and so the previous Volterra equation can be solved. Thus, once \hat{P}_1 has been found, the expression

$$\hat{G}_1 = \hat{G}_0 + \hat{P}_1 \tilde{\star} [\hat{B}_0 - \hat{B}]\hat{G}_0$$

gives associated Green function, i.e.,

$$A_0 \hat{G}_1 = \delta, \quad \text{and} \quad \hat{B} \hat{G}_1 = 0.$$

Note that $\hat{B} - A_0$ is a first order differential boundary operator. Next, we have to solve the Volterra equation

$$\hat{Q} = Q_0 + Q_0 \star \hat{Q}, \quad Q_0 := [A_0 - A]G_1$$

and then

$$Q_{A,\hat{B}} = \hat{G}_1 + \hat{G}_1 \star \hat{Q},$$

$$P_{A,B}(t, x, s, \tilde{y}) := Q_{A,\hat{B}}(t, x, s, \tilde{y}, 0) \frac{a_{dd}(s, \tilde{y}, 0)}{2b_d(s, \tilde{y})}$$

is the expression of the Poisson function corresponding to A and B , and

$$G_{A,B}(t, x, s, y) = Q_{A,\hat{B}}(t, x, s, y) + Q_{A,\hat{B}}(t, x, s, \tilde{y}) \rho(s, \tilde{y}) \delta(y_d)$$

is the Green function, see previous section.

3.1.5 Independent Conditions

This is the case $\rho > 0$, $b_0 \geq 0$ and $b_d = 0$. Clearly, this reduces to the Dirichlet boundary condition, and really independent conditions when $b_0 = 0$, the Poisson function $P_{A,B}$ and the Green functions are found independently.

3.1.6 Second order derivatives

When the boundary differential operator contains second order (tangential) derivative in x_i , $i = 1, \dots, d-1$, the calculations are more delicate, but essentially the same arguments are valid. In particular, as with Dirichlet boundary conditions, one may begin with the fundamental solution for variable coefficients denoted by $F(t, x, s, y)$ and then one solves the boundary value problem in the variables t and x ,

$$AF_1 = 0 \text{ in } \mathbb{R}_+^d \quad \text{and} \quad BF_1 = BF \text{ on } \partial\mathbb{R}_+^d,$$

with vanishing initial condition, and finally setting $G = F - F_1$ as the Green function with the boundary conditions given by the operator B . The point here is that BF is a smooth (Hölder continuous) function for x on the boundary $\partial\mathbb{R}_+^d$, as long as y is in the interior of \mathbb{R}_+^d . Thus, general Theorem can be used to find a unique solution F_1 , but a lot of effort should be done to produce sharp estimates leading to the mentioned heat kernel estimates, e.g., see Eidelman [?] and Solonnikov [?].

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References

- [1] S.V. Anulova. On process with Lévy generating operators in a half-space. *Math. USSR-Izv.*, 13:9–51, 1979.
- [2] S.V. Anulova. On stochastic differential equations with boundary conditions in a half-space. *Math. USSR-Izv.*, 18:423–437, 1982.
- [3] A. Bensoussan and J.L. Lions. *Impulse Control and Quasi-Variational Inequalities*. Gauthier-Villars, Paris, 1984. (pages are referred to the French edition).
- [4] S.D. Eidelman. *Parabolic Systems*. North Holland, Amsterdam, 1969.
- [5] A. Favini, G.R. Goldstein, J.A. Goldstein and S. Romanelli General Wentzell boundary conditions, differential operators and analytic semigroups in $C[0, 1]$. *Bol. Soc. Paran. Mat.*(3s.) v. 20 1/2: 93104, 2002.
- [6] A. Friedman. *Partial Differential Equations of Parabolic Type*. Prentice-Hall, Englewood Cliff, 1964.
- [7] M.G. Garroni and J.L. Menaldi. *Green Functions for Second Order Integral-Differential Problems*. Pitman Research Notes in Mathematics Series No 275. Longman, Essex, 1992.
- [8] M.G. Garroni and J.L. Menaldi. *Second Order Elliptic Integro-Differential Problems*. Research Notes in Mathematics Series No 430. Chapman & Hall / CRC Press, Boca Raton (FL), 2002.
- [9] M.G. Garroni and V.A. Solonnikov. On parabolic oblique derivative problem with Hölder continuous coefficients. *Comm. Partial Diff. Eq.*, 9:1323–1372, 1984.
- [10] I.I. Gihman and A.V. Skorohod. *The Theory of Stochastic Processes*, volume I, II and III. Springer-Verlag, New York, 1974, 1975 and 1979.
- [11] N. Ikeda and S. Watanabe. *Stochastic Differential Equations and Diffusion Processes*. North Holland, Amsterdam, 2nd edition, 1989.

- [12] S.D. Ivasišen. Green's matrices of boundary value problems for Petrovskii parabolic systems of general form, I and II. *Math. USSR Sbornik*, 42:93–144, and 461–489, 1982.
- [13] T. Komatsu. Markov processes associated with certain integro-differential operator. *Osaka J. Math.*, 10:271–303, 1973.
- [14] T. Komatsu. Uniform estimates for fundamental solutions associated with non-local Dirichlet form. *Osaka J. Math.*, 32:833–860, 1995.
- [15] O.A. Ladyzhenskaya, V.A Solonnikov, and N.N. Uraltseva. *Linear and Quasilinear Equations of Parabolic Type*. Am. Math. Soc., Providence, Rhode Island, 1968.
- [16] G.M. Lieberman. *Second Order Parabolic Differential Equations*. World Scientific Publishing, Singapur, 1997.
- [17] A. Lunardi. *Analytic Semigroups and Optimal Regularity in Parabolic Problems*. Birkhäuser, Boston, 1995.
- [18] N. El Karoui M. Chaleyat-Maurel and B. Marchal. Reflexion discontinue et systemes stochastiques. *Ann. Probab.*, 8:1049–1067, 1980.
- [19] J.L. Menaldi and M. Robin. Invariant measure for diffusions with jumps. *Appl. Math. Optim.*, 40:105–140, 1999.
- [20] A.L. Skubachevskii. *Elliptic Functional Differential Equations and Applications*. Birkhäuser, Boston, 1997.
- [21] V.A. Solonnikov. *On A Boundary Value Problems for Linear General Parabolic Systems of Differential Equations*. Trudy Math. Inst. Steklov. English transl., Amer. Math. Soc., Providence, Rhode Island, 1967.
- [22] K. Taira. *Diffusion processes and partial differential equations*, Academic Press Inc., Boston, MA, 1988.
- [23] K. Taira. *Semigroups, boundary value problems and Markov processes*, Springer-Verlag, 2004.