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Singular Ergodic Control for Multidimensional Gaussian-Poisson Processes

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Abstract

Singular control for multidimensional Gaussian-Poisson processes with a long-run (or ergodic) and a discounted criteria are discussed. The dynamic programming yields the corresponding Hamilton-Jacobi-Bellman equations, which are discussed. Full details on the proofs and further extensions are left for coming works.

Keywords: Singular control, ergodic control, Gaussian processes, Poisson processes.

AMS Subject Classification: Primary 49J40 and Secondary 60J60, 60J75.

1 Introduction

We use our joint paper with Michael Taksar [9] as our starting point to extend (most of the results there) to a class of Gaussian-Poisson processes, following most of the calculation developed in [6] and [10]. Checking recent references, the reader can find that the so-called singular control has interesting applications (including in finance), but the mathematical setting has been mainly on diffusion (or Gaussian) processes on finite or infinite time horizon, and sometimes with a long-run (ergodic) criteron, but not very often, the combination of diffusion with jumps and ergodic cost. A comprehensive work is the books by Arapostathis et al. [1], Bensoussan [2], Fleming and Soner [4] and Øksendal and Sulem [11], among many others. A possible explanation is the fact that suitable estimates for the transition density function related to jump-diffusion processes are more complicated and recently developed, e.g. see the book Garroni and Menaldi [5].

The "singular" control refers to the fact that the controls in the cost functional are not necessarily absolutely continuous with respect to time. Actually, we assume that the fluctuation of the stochastic system under control is described by an d-dimensional Gaussian-Poisson process $(y(t) : t \geq 0)$ with a time-varying drift, a constant diffusion matrix and jump term. The control is realized by a non-anticipating process of bounded variation $(\nu(t) : t \geq 0)$, i.e., the state equation is the following stochastic differential equation in the Itô's sense:

$$\begin{cases} \mathrm{d}y(t) = \big(g + fy(t)\big)\mathrm{d}t + \sigma \mathrm{d}w(t) + \displaystyle\int_{\mathbb{R}^d_*} z\tilde{p}(\mathrm{d}z, \mathrm{d}t) + \mathrm{d}\nu(t), \quad t > 0, \\ y(0) = x, \end{cases} \tag{1}$$

where $w(t) : t \geq 0)$ is a d-dimensional standard Wiener process, $p(\mathrm{d}z, \mathrm{d}t)$ is a Poisson measure in $\mathbb{R}^d_*$ with Lévy measure $\pi(\mathrm{d}z) = \mathbb{E}\{p(\mathrm{d}z,]0, t])\}/t$, and $\tilde{p}(\mathrm{d}z, \mathrm{d}t) = p(\mathrm{d}z, \mathrm{d}t) - \mathbb{E}\{p(\mathrm{d}z,]0, t])\}$ is a martingale or centered Poisson measure, and all this is realized in a filtered probability space $(\Omega, P, \mathcal{F}(t) : t \geq 0)$, where the filtration $(\mathcal{F}(t) : t \geq 0)$ satisfies the usual conditions. The coefficients g, f and σ are constant, i.e. g is a d-dimensional column vector, f and σ are $d \times d$ square matrices. Adjusting the dimensions of the coefficients involved, a n-dimensional standard Wiener process $(d \neq n)$ and a m-dimensional Poisson measure $(m \neq d)$ could be used, but it seems no essential to this model.

The cost associated with the position of the process is measured by a convex nonnegative function h, and the cost of controlling is proportional to the displacement (i.e., the variation of ν) induced by this control. We are interested in minimizing the limiting time average expected (i.e., ergodic or long-run) cost

$$\lambda = \inf_{\nu} \lim_{T\to\infty} \frac{1}{T} \mathbb{E}\Big\{ \int_0^T h(y(t))\mathrm{d}t + c|\nu|(T) \Big\}, \tag{2}$$

where c is a positive real number, and $|\nu|(T)$ denotes the total variation of ν on $[0, T)$. More precisely, if $(\nu(t) : t \geq 0)$, $\nu(t) = (\nu_1(t), \ldots, \nu_d(t))$ is an adapted process with bounded variation (on any bounded time-interval) then the one-dimensional variation process $t \mapsto |\nu|(t)$ is given by

$$\begin{cases} |\nu|(T) = \sup\big\{|\nu(t_0)| + \displaystyle\sum_{i=1}^{k} |\nu(t_i) - \nu(t_{i-1})| \ : \\ \qquad\qquad\qquad\qquad\qquad\qquad : \ 0 = t_0 < t_1 < \cdots < t_k = T\big\}, \end{cases} \tag{3}$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^d$. Note that under the action such a control ν, the initial condition in the stochastic differential equation (1) becomes $y(0) = x + \nu(0)$.

Beside the long-run cost (2), another class of infinite-horizon problems deals with the minimization of the total expected discounted cost

$$u_\alpha(x) = \inf_\nu \mathbb{E}\Big\{\int_0^\infty h(y(t))\mathrm{e}^{-\alpha t}\mathrm{d}t + c\int_0^\infty \mathrm{e}^{-\alpha t}\mathrm{d}|\nu|(t)\Big\}. \tag{4}$$

If the control $\nu = 0$ then the state equation (1) is related to the linear second-order integro-differential operator

$$\begin{cases} A\varphi(x) = \dfrac{1}{2}\displaystyle\sum_{i,j=1}^{d}\Big(\sum_{k=1}^{d}\sigma_{ik}\sigma_{jk}\Big)\partial_{ij}\varphi(x) + \sum_{i=1}^{d}\Big(g_i\sum_{j=1}^{d}f_{ij}x_j\Big)\partial_i\varphi(x) + \\ \qquad\qquad + \displaystyle\int_{\mathbb{R}^d}\big(\varphi(x+z) - \varphi(x) - z\cdot\nabla\varphi(x)\big)\pi(\mathrm{d}z), \end{cases} \tag{5}$$

with ∂_i, ∂_{ij} denoting the first and second partial derivatives, and $\nabla = (\partial_1, \cdots, \partial_d)$.

In the ergodic case, a formal application of the dynamic programming gives the following Hamilton-Jacobi-Bellman (HJB) equations

$$\min\big\{Av(x) - \lambda + h(x), c - |\nabla v(x)|\big\} = 0, \quad \text{in} \quad \mathbb{R}^d, \tag{6}$$

with two unknowns, the constant (optimal cost) λ and the function (potential) v.

In the discounted case, using again a formal of the dynamic programming, we obtain the following HJB equation

$$\min\big\{Au_\alpha(x) - \alpha u_\alpha(x) + h(x), c - |\nabla u_\alpha(x)|\big\} = 0, \quad \text{in} \quad \mathbb{R}^d, \tag{7}$$

with only one unknown, the optimal cost u_α.

Our interest is to discuss theses two problems and their relation, as in our previous work with Taksar [9], where the simple case of Gaussian processes were considered.

2 Main Results

We assume that the parameters of the model satisfy the conditions

$$\begin{cases} \alpha, c \quad \text{are positive constants}, & \sigma \quad \text{is an invertible matrix}, \\ f \quad \text{is a stable matrix, i.e.,} \quad \mathsf{e}^{tf} & \text{is bounded as} \quad t \to \infty, \end{cases} \tag{8}$$

and the Lévy measure satisfies

$$\int_{\mathbb{R}^d} |z|^p \pi(\mathrm{d}z) < \infty, \quad \forall p \geq 2, \tag{9}$$

i.e., the uncontrolled evolution has finite moments of any order, but the small jumps are of second order. The d-dimensional standard Wiener process and the centered Poisson measure $\tilde{p}(\mathrm{d}z, \mathrm{d}t)$ are both independent martingales with respect to the complete and right continuous filtration $(\mathcal{F}(t) : t \geq 0)$. Some variations of the above assumption could be used, but the non degeneracy of σ (i.e., invertible) seems necessary in various points of the discussion below.

The set of control functionals $\mathcal{V}$ consists of all right continuous with left-hand limits (cad-lag) processes $(\nu(t) : t \geq 0)$ taking values in R^d, progressively measurable with respect to the complete and right continuous filtration $(\mathcal{F}(t) : t \geq 0)$ and such that the variation process $t \mapsto |\nu|(t)$ defined by (3) satisfies

$$E\{|\nu|(t)\} < \infty, \quad \forall t > 0, \tag{10}$$

and for technical reason, we adopt the convention that $\nu(0-) = 0$ and $|\nu(0)| = |\nu|(0)$.

Also, the holding cost function h satisfies the polynomial growth conditions below, namely, there exist positive constants C_0, C_1, C_2 and $m > 1$ such that, for any x, χ in $\mathbb{R}^d$, $0 < |\chi| < 1$, we have

$$\begin{cases} 0 \leq h(x) \leq C_0(1 + |x|^m), \\ |h(x) - h(x + \chi)| \leq C_1 |\chi| \big(1 + h(x)\big), \\ 0 < h(x + \chi) + h(x - \chi) - 2h(x) \leq C_2 |\chi|^2 \big(1 + h(x)\big). \end{cases} \tag{11}$$

Moreover, we suppose that h is strictly convex and that

$$|x|^{-1} h(x) \to \infty \quad \text{as} \quad |x| \to \infty. \tag{12}$$

For instance, $h(x) = x^2$ has this properties.

The costs are defined as follows:

$$\begin{cases} \mathcal{J}(x,\nu,\alpha) = \int_0^\infty h(y(t))\mathrm{e}^{-\alpha t}\mathrm{d}t + c\int_0^\infty \mathrm{e}^{-\alpha t}\mathrm{d}|\nu|(t), \\ J(x,\nu,\alpha) = \mathbb{E}\big\{\mathcal{J}(x,\nu,\alpha)\big\}, \\ K(x,\nu) = \lim_{T\to\infty}\frac{1}{T}\mathbb{E}\Big\{\int_0^T h(y(t))\mathrm{d}t + c|\nu|(T)\Big\}, \end{cases} \tag{13}$$

and therefore

$$\begin{cases} u_\alpha(x) = \inf\big\{J(x,\nu,\alpha) : \nu \in \mathcal{V}\big\}, \\ \lambda = \inf\big\{K(x,\nu) : \nu \in \mathcal{V}\big\}, \end{cases} \tag{14}$$

where λ is expected to be independent of x. Note that if the assumption (11) is imposed then the condition (9), for $p = 2$ and $p = \max\{2, m\}$, implies that the costs $J(x,\nu,\alpha)$ and $K(x,\nu)$ are both finite, at least for $\nu = 0$.

Our main results

Theorem 2.1. *Under the assumptions* (8),...,(12), *the ergodic cost λ is independent of the initial state x and*

$$\alpha u_\alpha(x) \to \lambda \quad \textit{as} \quad \alpha \to 0, \tag{15}$$

where the convergence is locally uniform in x belonging to $\mathbb{R}^d$.

Theorem 2.2. *There exist a convex and Lipschitz continuous function v with $v(0) = 0$ and a bounded, open, and nonempty region D in $\mathbb{R}^d$ such that*

$$\begin{cases} Av + h \geq \lambda \quad \textit{in} \quad \mathcal{D}(\mathbb{R}^d), \\ |\nabla v| \leq c \quad \textit{a.e. in } \mathbb{R}^d, \end{cases} \tag{16}$$

and v belongs to $W^{2,\infty}(D)$, and

$$\begin{cases} Au + h = \lambda \quad \textit{a.e. in } D, \\ |\nabla v| = c \quad \textit{a.e. in } \partial D. \end{cases} \tag{17}$$

Moreover, if ∂D is of class C^3, u is three times continuously differentiable on $\overline{D} = D \cup \partial D$, and ∇v is never tangent to ∂D, then there exists ν_x^* in $\mathcal{V}$ such that

$$K(x, \nu_x^*) = \lambda, \tag{18}$$

i.e., ν_x^* is an optimal ergodic (or stationary) policy.

Remark that $\mathcal{D}'(\mathbb{R}^d)$ denotes the space of Schwartz distributions in $\mathbb{R}^d$ and $W^{2,\infty}(D)$ is the Sobolev space of bounded functions with Lipschitz first derivatives in D.

3 A Priori Estimates

Considering the Gaussian-Poisson process

$$\beta(t) = gt + \sigma w(t) + \int_{\mathbb{R}^d_* \times]0,t]} z\tilde{p}(\mathrm{d}z, \mathrm{d}s), \tag{19}$$

the state $(y(t) : t \geq 0)$ of the system (1) can be written as

$$y(t) = \mathsf{e}^{tf} x + \int_0^t \mathsf{e}^{(t-s)f} \mathrm{d}\beta(s) + \int_0^t \mathsf{e}^{(t-s)f} \mathrm{d}\nu(s), \tag{20}$$

and if each control ν is decomposed into its continuous part ν^c and its purely jump part ν^j, with $\nu = \nu^c + \nu^j$, $\nu^c(0) = 0$ and $\nu^j(0-) = 0$, all by components, then the cost of controlling becomes the constant c times the integral

$$\int_0^\infty \mathsf{e}^{-\alpha t} \mathrm{d}|\nu|(t) = \int_0^\infty \mathsf{e}^{-\alpha t} \mathrm{d}|\nu^c|(t) + \sum_{t \geq 0} \mathsf{e}^{-\alpha t} |\nu^j(t) - \nu^j(t-)|, \tag{21}$$

Note that ν^c and ν^j have locally bounded variation, ν^j is right continuous with left-hand limits and with countably many discontinuities. Essentially based on [6] and [10] we have

Proposition 3.1. *If* (8),...,(12) *are satisfied then exists a constant* $K_0 > 1$ *such that, for any* x, χ *in* $\mathbb{R}^d$, $0 \leq |\chi| \leq 1$, *and* $\alpha > 0$, *we have*

$$\begin{cases} 0 \leq u_\alpha(x) \leq c|x| + (K_0 - 1)\alpha^{-1}, \\ |u_\alpha(x) - u_\alpha(x + \chi)| \leq C_1 |\chi| \big(c|x| + K_0 \alpha^{-1}\big), \\ 0 \leq u_\alpha(x + \chi) + u_\alpha(x - \chi) - 2u_\alpha(x) \leq C_2 |\chi|^2 \big(c|x| + K_0 \alpha^{-1}\big). \end{cases} \tag{22}$$

where c *is the constant in* (8) *that appears in the cost* (13)*, and* C_1, C_2 *are the constants in assumptions* (11)*.*

Proof. We follows the arguments in the joint paper with Taksar [9]. For instance, to show the first part (22), consider the reflected diffusion processes with jumps

$$\begin{cases} \mathrm{d}y_0(t) = f y_0(t)\mathrm{d}t + \mathrm{d}\beta(t) - y_0(t)\mathrm{d}\xi_0(t), \\ y_0(0) = 0, \quad \xi_0(0) = 0, \\ |y_0(t)| \leq 1, \quad \forall t \geq 0, \\ \mathrm{d}\xi_0(t) \neq 0 \quad \text{only if} \quad |y_0(t)| = 1, \end{cases} \tag{23}$$

where β given by (19) and $(\xi_0(t) : t \geq 0)$ is a one-dimensional increasing cad-lag (continuous from the right with left-hand limits) process, e.g., see [8]. Then Itô formula applied to the function $(y,t) \mapsto |y|^2 \mathsf{e}^{-\alpha t}$ yields

$$\begin{cases} |y_0(T)|^2 \mathsf{e}^{-\alpha T} + \alpha \int_0^T |y_0(t)|^2 \mathsf{e}^{-\alpha t}\mathrm{d}t + 2\int_0^T \mathsf{e}^{-\alpha t}\mathrm{d}\xi_0(t) = \\ \qquad = 2\int_0^T y_0(t)\cdot [f y_0(t)\mathrm{d}t + \mathrm{d}\beta(t)] + \int_0^T \mathrm{Tr}(\sigma\sigma^*)\mathsf{e}^{-\alpha t}\mathrm{d}t + \\ \qquad\qquad + 2\int_0^T \mathsf{e}^{-\alpha t}\mathrm{d}t \int_{\mathbb{R}^d_*} |z|^2 \pi(\mathrm{d}z), \end{cases} \tag{24}$$

which implies the estimate

$$\alpha \mathbb{E}\Big\{ \int_0^T \mathsf{e}^{-\alpha t}\mathrm{d}\xi_0(t) \Big\} \leq |g| + |f| + \frac{1}{2}\mathrm{Tr}(\sigma\sigma^*) + \int_{\mathbb{R}^d_*} |z|^2 \pi(\mathrm{d}z). \tag{25}$$

The reflected diffusion processes with jumps shows that, under the action of the bounded variation control process $\mathrm{d}\nu(t) = y_0(t)\mathrm{d}\xi_0(t)$, the state of system (i.e., $y_0(t)$) remains within the unit ball if it starts in this ball. Thus, a recipe to keep the state of the system within the unit ball is to make an initial jump to the origin and then reflect the process on the boundary.

This means that if for each x in $\mathbb{R}^d$ we define

$$\begin{cases} \nu_x(t) = -x - \int_0^t y_0(s)\mathrm{d}\xi_0(s), \quad \forall t \geq 0, \\ y_x(t) = y_0(t), \quad \forall t \geq 0, \quad \text{as in (23)}, \end{cases} \tag{26}$$

then $(y_x(t) : t \geq 0)$ is the solution of the stochastic differential equation (20) for the initial condition x and the control ν_x, and

$$u_\alpha(x) \leq J(x, \nu_x, \alpha) = c|x| + J(x, \nu_0, \alpha). \tag{27}$$

Next, in view of the estimate (25), we deduce the first part of (22) with

$$K_0 = 1 + |g| + |f| + \frac{1}{2}\,\mathrm{Tr}(\sigma\sigma^*) + \int_{\mathbb{R}^d_*} |z|^2 \pi(\mathrm{d}z) + \sup_{|x|\le 1} |h(x)|. \tag{28}$$

To verify the second part of (22), use the inequality

$$|u_\alpha(x) - u_\alpha(x + \lambda\chi)| \le \sup_\nu |J(x,\nu,\alpha) - J(x + \lambda\chi, \nu, \alpha)|,$$

for $|\chi| = 1$ and λ in $(0,1]$, where the supremum is taken over all controls ν satisfying

$$J(x + \lambda\chi, \nu, \alpha) \le c(|x| + 1) + (K_0 - 1)\alpha^{-1}.$$

Next, if C_1 is the constant in assumption (11) then combine the above inequalities with

$$\big|h\big(y_x(t)\big) - h\big(y_{x+\lambda\chi}(t)\big)\big| \le C_1 \lambda |\mathrm{e}^{tf}\chi| \big[1 + h\big(y_x(t)\big)\big],$$

to deduce the second part of (22), after remarking that $|\mathrm{e}^{tf}\chi| \le 1$.

For the last part, as above begin with

$$\begin{aligned}|u_\alpha(x + \lambda\chi) + u_\alpha(x - \lambda\chi) - 2u_\alpha(x)| &\le \\ &\le \sup_\nu |J(x + \lambda\chi, \nu, \alpha) + J(x - \lambda\chi, \nu, \alpha) - 2J(x,\nu,\alpha)|,\end{aligned}$$

for $|\chi| = 1$ and λ in $(0,1]$, where again the supremum is taken over all controls ν satisfying

$$J(x \pm \lambda\chi, \nu, \alpha) \le c(|x| + 1) + (K_0 - 1)\alpha^{-1}.$$

Next, if C_2 is the constant in assumption (11) then combine the above inequalities with

$$\begin{aligned}\big|h\big(y_{x+\lambda\chi}(t)\big) + h\big(y_{x-\lambda\chi}(t)\big) - 2h\big(y_x(t)\big)\big| &\le \\ &\le C_2 \lambda^2 |\mathrm{e}^{tf}\chi|^2 \big[1 + h\big(y_x(t)\big)\big],\end{aligned}$$

to complete the proof. $\square$

A direct consequence of the above Proposition is the fact that if the second derivatives of h are bounded and all eigenvalues of f are strictly negative then the second derivatives of u_α are equi-bounded in $\alpha > 0$.

Following the arguments in [6], [10] and [9], the optimal cost u_α satisfies

$$\begin{cases} u_\alpha \in W^{2,\infty}_{loc}(\mathbb{R}^d), \quad \text{i.e., locally Lipschitz first detivatives,} \\ Au_\alpha - \alpha u_\alpha + h \geq 0, \quad \text{a.e. in } \mathbb{R}^d, \\ |\nabla u_\alpha| \leq c \quad \text{in } \mathbb{R}^d, \\ Au_\alpha - \alpha u + \alpha_h = 0, \quad \text{a.e. in} \quad [|\nabla u_\alpha| < c], \end{cases} \tag{29}$$

where $[|\nabla u_\alpha| < c]$ denotes the set of points x in $\mathbb{R}^d$ satisfying $|\nabla u_\alpha(x)| < c$.

Next, define the open set (the continuation region)

$$D_\alpha = \big\{ x \in \mathbb{R}^d \, : \, |\nabla u_\alpha(x)| < c \big\}, \tag{30}$$

and define the sets D and S as follows:

Definition 3.2. A point x in $\mathbb{R}^d$ belongs to the set D if there exits a constant $r = r(x) > 0$, and sequences $x_k \to x$ and $\alpha_k \to 0$ as $k \to \infty$ such that $\{|y - x_k| < r\} \subset D_{\alpha_k}$, for every k. While a point x in $\mathbb{R}^d$ belongs to S if there exit sequences $x_k \to x$ and $\alpha_k \to 0$ as $k \to \infty$ such that $x_k \notin D_{\alpha_k}$, for every k. □

As in [9], we can show that under the assumptions of Proposition 3.1 the set D is bounded and open, the set S is closed and $\mathbb{R}^d = D \cup S$. Actually, a key point is the use of assumption (12) to show that D_α is included in the ball $\{x \in \mathbb{R}^d : |x| \leq K_1\}$ with K_1 defined by

$$\begin{cases} K_1 = \sup \big\{ x \in \mathbb{R}^d \, : \, h(x) \leq a + b|x| \big\}, \\ \text{with } \; a = c|g| + K_0 \; \text{ and } \; b = c(1 + |f|), \end{cases} \tag{31}$$

where K_0 is the constant (28). Moreover

Theorem 3.3. *Under the assumptions* (8),…,(12)*, the set D given in Definition 3.2 is nonempty. Moreover, for every* $0 < \alpha < 1$*, we have*

$$|\nabla u_\alpha(x) - \nabla u_\alpha(x')| \leq K_2 |x - x'|, \quad \forall x, x' \in D_\alpha \tag{32}$$

for some constant K_2 independent of α, and outside of the continuation set D_α, the function u_α is linear along the gradient, i.e.,

$$u_\alpha\big(x + \theta \nabla u_\alpha(x)\big) = u_\alpha(x) + c^2\theta, \quad \forall x \notin D_\alpha \; \text{ and } \; \theta > 0, \tag{33}$$

where $u_\alpha(x)$ is the discounted optimal cost function (14)*, and c is the constant that appears in the cost* (13)*.*

Proof. Because u_α is convex we have

$$\int_{\mathbb{R}^d} \big(u_\alpha(x+z) - u_\alpha(x) - z \cdot \nabla u_\alpha\big)\pi(\mathrm{d}z) \geq 0, \tag{34}$$

which implies that

$$0 = Au_\alpha - \alpha u_\alpha + h \geq Lu_\alpha - \alpha u_\alpha + h \quad \text{in} \quad D_\alpha, \tag{35}$$

where L is the strict partial differential operator in A.

By means of the almost-local Schauder estimate for integro-differential operators (e.g., see [7] or Garroni and Menaldi [5]) the function u_α is smooth, and to show (32), we need to verify that for some set of d independent directions $(\chi_1, \ldots, \chi_d)$ in $\mathbb{R}^d$ we have

$$\sum_{k=1}^{d} \frac{\partial^2 u_\alpha(x)}{\partial \chi_k^2} \leq K_2, \quad \forall x \in D_\alpha. \tag{36}$$

Thus, combining (35), if σ_k denotes the k column of the matrix σ and $\chi_k = \sigma_k |\sigma_k|^{-1}$ then

$$\left\{ \begin{aligned} &\sum_{k=1}^{d} \frac{\partial^2 u_\alpha(x)}{\partial \chi_k^2} \leq \big(\min_k |\sigma_k|\big)^{-2} \big(\alpha u_\alpha(x) - \\ &\qquad\qquad - (g + fx) \cdot \nabla u_\alpha(x) - h(x)\big). \end{aligned} \right. \tag{37}$$

Hence, the constant K_2 is given by

$$K_2 = 2\big(\min_k |\sigma_k|\big)^{-2}\big(K_0 - 1 + c|g| + c(1 + |f|)K_1\big), \tag{38}$$

where K_0 and K_1 are the constants appearing in (28) and (31).

Now, to show that D is nonempty, let x_α be a point in $\mathbb{R}^d$, where $u_\alpha(\cdot)$ attains its absolute minimum. Then $\nabla u_\alpha(x_\alpha) = 0$ and x_α belongs to D_α. Next, use estimate (32) to deduce that

$$\{x \in \mathbb{R}^d : |x - x_\alpha| \leq \varepsilon\} \subset D_\alpha \quad \text{for} \quad 0 < \alpha < 1,\ 0 < \varepsilon \leq cK_2^{-1}, \tag{39}$$

where K_2 is the constant given by (38), which appears in estimate (32). Therefore, any limit point of the family $\{x_\alpha : 0 < \alpha < 1\}$ belongs to D. Note that at least one limit point exists in view of the bound $D_\alpha \subset \{x \in \mathbb{R}^d : |x| \leq K_1\}$ with K_1 given by (31). □

As a Corollary of the above Theorem we can mention that if the second derivatives of h are bounded and all eigenvalues of f are strictly negative then the estimate (32) holds true for any x in $\mathbb{R}^d$. There are other interesting questions that we leave for a more detailed paper.

4 Potential Function and Optimal Control

In the previous section we studied most of the preliminary steps to deduce the main results. By no means, this presentation was comprehensive (actually, full details would need much more space), our intention was to convince the reader of the validity of our assertions. In this section, even with less details, we discuss briefly the proofs of Theorems 2.1 and 2.2. As mentioned earlier, full details (and extensions) of all these arguments will appear in a later work.

There are two points to cover, the passage to the limit as $\alpha \to 0$ and the validity of the Hamilton-Jacobi-Bellman equations, both points are obtained almost simultaneously. Indeed, essentially due to the a priori estimates on u_α, specially the fact that the gradients ∇u_α are equi-bounded and u_α are convex, we deduce that $\alpha u_\alpha(x) \to \lambda$ and $v_\alpha(x) = u_\alpha(x) - u_\alpha(0) \to v(x)$, locally uniformly in x, for some sequence $\alpha = \alpha_k \to 0$, where $\lambda \geq 0$ is a constant and v is a Lipschitz convex function. This implies the relation (16) and (17) as $\alpha \to 0$, actually, Av is a Radon measure. Again, the almost-local Schauder estimate for integro-differential operators implies that v is smooth on D and the estimates (32) and (33) of Theorem 3.3 hold true for v instead of u_α. This also implies that $|\nabla v(x)| = c$ for almost every x in $S = \mathbb{R}^d \smallsetminus D$.

If the continuation region D is smooth then there exists a smooth function ρ such that

$$\begin{cases} D = \big\{x \in \mathbb{R}^d \,:\, \rho(x) < 0\big\}, \\ \partial D = \big\{x \in \mathbb{R}^d \,:\, \rho(x) = 0\big\}, \\ |\nabla \rho(x)| \geq 1 \quad \text{on} \quad \partial D, \end{cases} \tag{40}$$

and there exists a function $M(x)$ from a neighborhood of ∂D into the set of $d \times d$ symmetric matrices, which is twice-continuously differentiable and satisfies

$$\begin{cases} z \cdot M(x) z > 0, \quad \forall z \in \mathbb{R}^d,\ z \neq 0,\ \forall x, \\ -\nabla v(x) = M(x) \nabla \rho(x), \quad \forall x \in \partial D, \end{cases} \tag{41}$$

i.e., the free boundary ∂D and the potential v are smooth, and ∇v is never tangent to ∂D. Under these assumptions we can build the reflected diffusion process on D (see e.g., [8]), which provides an optimal stationary control policy.

Another hard question is the regularity of the free boundary ∂D. This is very related to the $W^{3,\infty}$-regularity of the value function v. Results in this direction can be found in Soner and Shreve [12], where a two-dimensional case with unidirectional control is studied, and in Williams et al [13], where local regularity (outside of some lower-dimensional region) is obtained. Certainly, there are several recent papers on this type of singular control, e.g. Bensoussan et al [3], among many others.

Remark that if the matrix f is not assumed to be stable then the cost of the control (due to the rate function h) may force the system to be stable. Moreover, the bounded variation controls may cause the system to remain within a bounded region, see (23). Thus, a little more analysis is necessary to deal with the case when the matrix f is not necessarily stable, this (and other related questions) will be discussed in a coming paper.

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