DigitalCommons
@WayneState

Wayne State University

Mathematics Faculty Research Publications

Mathematics

5-7-2007

Some Results of Backward Itô Formula

Guiseppe Da Prato
Scuola Normale Superiore

Jose-Luis Menaldi
Wayne State University, menaldi@wayne.edu

Luciano Tubaro
Università di Trento

Recommended Citation

G. Da Prato, J.-L. Menaldi and L. Tubaro. Some results of backward Itô formula, *Stochastic Anal. Appl.* **25** 679-703 (2007). doi: 10.1080/07362990701283045
Available at: https://digitalcommons.wayne.edu/mathfrp/46

This Article is brought to you for free and open access by the Mathematics at DigitalCommons@WayneState. It has been accepted for inclusion in Mathematics Faculty Research Publications by an authorized administrator of DigitalCommons@WayneState.

Some Results of Backward Itô Formula

Giuseppe Da Prato
Scuola Normale Superiore
piazza dei Cavalieri, 7
56126 Pisa, Italia

Jose-Luis Menaldi
Wayne State University
Department of Mathematics
Detroit, Michigan 48202, USA

Luciano Tubaro
Università di Trento
Dipartimento di Matematica
38050 Povo (Trento), Italia

Abstract

We use the notion of backward integration, with respect to a general Lévy process, to treat, in a simpler and unifying way, various classical topics as: Girsanov theorem, first order partial differential equations, the Liouville (or Lyapunov) equations and the stochastic characteristic method.

1 Introduction

In this paper we extend the results of [2] to the case of a Lévy process. In [2] the notion of backward integration is widely used to treat, in a simpler and unifying way various results from Kunita [8], Krylov and Rozovskii [12] and Pardoux [11]. The idea comes from the following consideration: let $X(t,s,x)$ be the solution of a ordinary stochastic differential equation (possibly with jumps), such that $X(s,s,x)=x$ and let us consider the evolution operators $U_{s,t}$ acting on bounded measurable functions $\varphi\colon \mathbb{R}^d \mapsto \mathbb{R}$ defined as

$$U_{s,t}\varphi(x) = \varphi(X(t,s,x)), \qquad 0 \le s \le t \le T.$$

It is easy to check that the following backward evolution property

$$U_{s,t} = U_{s,r}U_{r,t}, \qquad 0 \le s \le r \le t \le T,$$

holds thanks to the flow property of $X(t,s,x)$. Thus, it is important to study the behavior of $U_{s,t}$ in the variable s, because it is only in this variable that $U_{s,t}\varphi$ is the solution of a stochastic evolution equation: in this equation the backward integration is the "natural" tool, of which we recall the definition in the appendix. Theorem 2.1 is the main result, from which we deduce the various applications in §4. We observe also that we prove directly the backward Itô formula, because it seems to illustrate better the rule of backward integration.

For any nonnegative integer k and $0 \le \alpha \le 1$, we denote by $C^{k,\alpha}(\mathbb{R}^d)$ (resp. $C_b^{k,\alpha}(\mathbb{R}^d)$) the space of all functions from $\mathbb{R}^d$ into $\mathbb{R}$ which are continuous (resp.

continuous and bounded) together with their derivatives of order less or equal than k with the k-th derivatives α-Hölder continuous.

Suppose given functions $g\colon [0,T]\times\mathbb{R}^d \mapsto \mathbb{R}^d$, $\sigma\colon [0,T]\times\mathbb{R}^d \mapsto L(\mathbb{R}^d,\mathbb{R}^\ell)$ and $\gamma\colon \mathbb{R}^m_* \times [0,T]\times\mathbb{R}^d \mapsto \mathbb{R}^d$ satisfying the assumption:

Hypothesis 1.1. **(a)** The coefficients $g(t,x)$, $\sigma(t,x)$ and $\gamma(z,t,x)$ are always supposed Borel measurable, and because we are interested in global solutions defined on a prescribed bounded interval, say $[0,T]$, we impose a linear growth condition, namely, for every $p \geq 2$ there exists a constant $C_p > 0$ such that

$$|g(t,x)|^p + |\sigma(t,x)|^p + \int_{\mathbb{R}^m_*} |\gamma(z,t,x)|^p \pi(\mathrm{d}z) \leq C_p(1+|x|^p), \tag{1.1}$$

for every (t,x) in $[0,T]\times\mathbb{R}^d$. Thus, the initial condition x must be an $\mathcal{F}(t_0)$-measurable random variable (most of the time, a deterministic value).
(b) A clean existence and uniqueness theory is developed adding a uniform locally Lipschitz condition in the variable x, namely, for any $r > 0$ and $p \geq 2$ there exists a positive constant $M = M(r,p)$ such that

$$\begin{cases} |g(t,x) - g(t,x')|^p + |\sigma(t,x) - \sigma(t,x')|^p + \\ \qquad + \displaystyle\int_{\mathbb{R}^m_*} |\gamma(z,t,x) - \gamma(z,t,x')|^p \pi(\mathrm{d}z) \leq M|x-x'|^p, \end{cases} \tag{1.2}$$

for every (t,x), (t,x') in $[0,T]\times\mathbb{R}^d$ with $|x| \leq r$ and $|x'| \leq r$.
(c) The functions $g(t,x)$, $\sigma(t,x)$ and $\gamma(z,t,x)$ are twice continuously differentiable in x and locally bounded, i.e., if $\partial^\ell_x g(t,x)$, $\partial^\ell_x \sigma(t,x)$ and $\partial^\ell_x \gamma(z,t,x)$ denote any of the derivatives up to the order $\ell \leq 2$ then for any $r > 0$ and $p \geq 2$, we have

$$|\partial^\ell_x g(t,x)|^p + |\partial^\ell_x \sigma(t,x)|^p + \int_{\mathbb{R}^m_*} |\partial^\ell_x \gamma(z,t,x)|^p \pi(\mathrm{d}z) \leq K^\ell_{r,p}, \tag{1.3}$$

for any $0 \leq t \leq T$, $|x| \leq r$ and some constant $K^\ell_{r,p}$. □

Now, for a given $T > 0$, x in $\mathbb{R}^d$ and s in $[0,T]$, let us consider the stochastic differential equation

$$\begin{cases} X(t) = x + \displaystyle\int_s^t g(r,X(r))\mathrm{d}r + \int_s^t \sigma(r,X(r))\mathrm{d}w(r) + \\ \qquad\qquad + \displaystyle\int_{\mathbb{R}^m_*\times]s,t]} \gamma(z,r,X(r))\tilde{p}(\mathrm{d}z,\mathrm{d}r), \quad \forall t \in]s,T]. \end{cases} \tag{1.4}$$

Note that one may replace $\sigma(r,X(r))$ and $\gamma(z,r,X(r))$ with $\sigma(r,X(r-))$ and $\gamma(z,r,X(r-))$ in both stochastic integrals.

It is well known that, under Hypothesis 1.1, the stochastic differential equation (1.4) has a unique solution, that we denote by $X(t,s,x)$, t in $[s,T]$. Assumptions (1.2) and (1.3) are used with $p > 2d+4$ to show that the random field $X(t,s,x)$ has a version which is cad-lag in t, cag-lad in s and twice continuously differentiable in x. Clearly this is set on a complete filtered probability space $(\Omega,\mathcal{F},P)$ with a (standard) Wiener process w and a (standard) Poisson measure p with Lévy measure π. We are interested in the process (random field) $(s,x) \mapsto \varphi(X(t,s,x))$, which we'll denote by $u^t_\varphi(s,x)$.

2 Main result

We denote by $\mathcal{L}_s$, $\mathcal{M}_s$and $\mathcal{N}_s$ the linear operators defined for φ in $C_b^2(\mathbb{R}^d)$

$$\mathcal{L}_s\varphi(x) = \mathcal{L}_s^0\varphi(x) + \mathcal{L}_s^\gamma\varphi(x), \tag{2.1}$$

and

$$\mathcal{M}_s\varphi(x) = \sigma(s,x)^* D\varphi(x), \tag{2.2}$$

$$\mathcal{N}_s(z)\varphi(x) = \varphi(x+\gamma(z,s,x)) - \varphi(x), \tag{2.3}$$

where

$$\mathcal{L}_s^0\varphi(x) = \frac{1}{2}\mathrm{Tr}[D^2\varphi(x)\sigma(s,x)\sigma^*(s,x)] + \big(g(s,x), D\varphi(x)\big),$$
$$\mathcal{L}_s^\gamma\varphi(x) = \int_{\mathbb{R}_*^m} [\varphi(x+\gamma(z,s,x)) - \varphi(x) - (\gamma(z,s,x), D\varphi(x))]\pi(\mathrm{d}z),$$

To simplify the notation, it may be simpler to use

$$\big(\mathcal{M}_s\varphi(x), y\big) = \big(\sigma(s,x)y, D\varphi(x)\big), \quad \forall y \in \mathbb{R}^d,$$

where $(\cdot,\cdot)$ is the scalar (or dot) product in $\mathbb{R}^d$.

Let us introduce three mappings $F\colon [0,T]\times\mathbb{R}^d \mapsto \mathbb{R}$, $G\colon [0,T]\times\mathbb{R}^d \mapsto \mathbb{R}^\ell$ and $H\colon \mathbb{R}_*^m \times [0,T]\times\mathbb{R}^d \mapsto (-1,\infty)$, and let us assume that F is twice continuously differentiable in x, G and H have locally Hölder continuous second derivative in x, i.e., for some $0 < \theta < 1$, for any $r > 0$ and any order of derivative $\ell \le 2$, there exist a positive constants $M = M(r)$ and a measurable function $z \mapsto \beta(z) = \beta(z,r)$ such that

$$\begin{cases} |\partial_x^\ell G(t,x) - \partial_x^\ell G(t,x')| \le M|x-x'|^\theta, \\ |\partial_x^\ell H(z,t,x) - \partial_x^\ell H(z,t,x')| \le \beta(z)|x-x'|^\theta, \\ \displaystyle\int_{\mathbb{R}_*^m} |\beta(z)|^p\pi(\mathrm{d}z) < \infty, \quad \forall p \ge 2, \end{cases} \tag{2.4}$$

for every (t,x), (t,x') in $[0,T]\times\mathbb{R}^d$ with $|x| \le r$ and $|x'| \le r$, and there exist measurable functions $\alpha(z) \le 0 \le \beta(z)$ satisfying

$$\begin{cases} -1 < \alpha(z) \le H(z,t,x) \le \beta(z), \quad \forall z,t,x, \\ \displaystyle\int_{\mathbb{R}_*^d} |\beta(z)|^p\pi(\mathrm{d}z) + \int_{\mathbb{R}_*^d} \big[\alpha(z) - \ln\big(1+\alpha(z)\big)\big]^{p/2}\pi(\mathrm{d}z) < \infty, \end{cases} \tag{2.5}$$

for every $p \ge 2$. Note that $\alpha^2/2 \le \alpha - \ln\big(1+\alpha\big)$ for any $-1 < \alpha \le 0$.

Then we consider the following linear stochastic differential equation, for any t in $]s,T]$,

$$\begin{cases} \eta_s(t) = 1 + \displaystyle\int_s^t \eta_s(r)\, F(r, X(r,s,x))\mathrm{d}r + \\ \qquad + \displaystyle\int_s^t \eta_s(r)\big(G(r,X(r,s,x)), \mathrm{d}w(r)\big) + \\ \qquad + \displaystyle\int_{\mathbb{R}_*^m\times]s,t]} \eta_s(r)\, H(z,r,X(r,s,x))\tilde{p}(\mathrm{d}z,\mathrm{d}r), \end{cases} \tag{2.6}$$

where $X(t,s,x)$ is the solution of (1.4).

From now on, we denote this process $\eta_s(t)$ by $\eta^t(s,x)$, because we want to point out the dependency on (s,x). Indeed, the assumptions in G and H are sufficient to obtain a version of $\eta^t(s,x)$ which cad-lag in t, cag-lad in s and twice continuously differentiable in x.

To state the theorem let us introduce the following operators, for ψ in $C_b^2(\mathbb{R}^d)$,

$$\begin{aligned}\mathcal{L}_s^{F,G,H}\psi(x) &= \mathcal{L}_s\psi(x) + \big(\sigma(s,x)G(s,x), D\psi(x)\big) + \\ &+ \int_{\mathbb{R}_*^m} H(z,s,x)[\psi(x+\gamma(z,s,x)) - \psi(x)]\pi(\mathrm{d}z) + F(s,x)\psi(x),\end{aligned} \tag{2.7}$$

$$\begin{cases} \mathcal{M}_s^G\psi(x) = \mathcal{M}_s\psi(x) + G(s,x)\psi(x), \\ \\ \mathcal{N}_s^H(z)\psi(x) = \mathcal{N}_s(z)\psi(x) + H(z,s,x)\psi(x+\gamma(z,s,x)). \end{cases} \tag{2.8}$$

with the notation (2.1), (2.2) and (2.3).

Using the notation for the operators $\mathcal{L}_s^{F,G,H}$, $\mathcal{M}_s^G$ and $\mathcal{N}_s^H(z)$, defined by (2.7) and (2.8), we have the following theorem, that we prove in the following section

Theorem 2.1. *Assume that* Hypotheses 1.1, (2.4) *and* (2.5) *hold. Let φ in $C_b^2(\mathbb{R}^d)$ and denote*

$$y^t(s,x) = \varphi(X(t,s,x))\eta^t(s,x).$$

Then $y^t(s,x)$ satisfies the backward stochastic partial differential equation

$$\begin{cases} y^t(s,x) = \varphi(x) + \displaystyle\int_s^t \mathcal{L}_r^{F,G,H} y^t(r,\cdot)(x)\mathrm{d}r + \\ \qquad\qquad + \displaystyle\int_s^t \big(\mathcal{M}_r^G y^t(r,\cdot)(x), \hat{\mathrm{d}}w_r\big) + \\ \qquad\qquad + \displaystyle\int_{\mathbb{R}_*^m\times[s,t[} \mathcal{N}_r^H(z) y^t(r,\cdot)(x)\tilde{p}(\mathrm{d}z,\hat{\mathrm{d}}r). \end{cases} \tag{2.9}$$

Here recall that $\hat{\mathrm{d}}w(t)$ and $\tilde{p}(\mathrm{d}z,\hat{\mathrm{d}}t)$ mean the backward stochastic integration.

Recall that the stochastic integral is θ'-Hölder continuous in a parameter, the variable x in our case, if the integrand is θ-Hölder continuous in the parameter with $\theta' < \theta$, see Kunita [8]. The differential operators $\mathcal{M}_r^G$ and $\mathcal{N}_r^H$ acting on the integrands of the stochastic integrals are first-order, so that the resulting process is Hölder continuous in x. Also, it is clear that $y^t(s,x)$ has a version which is cad-lag in t, cag-lad in s and twice continuously differentiable in x.

We point out that, when F and G are bounded, and $H = 0$, Krylov and Rozovskii [6] found the quoted results using a different procedure. It is clear that our technique not uses any change of probability space to obtain a Girsanov formula.

3 Proof of the main result

To prove the main result we proceed by various steps.

Backward Itô formula I

In this section we will prove a backward Itô formula for the process $\varphi(X(t,s,x))$, where φ in $C_b^2(\mathbb{R}^d)$ and $X(t,s,x)$ is the solution to (1.4). The proof is based on Taylor formula and explicitly exploits the fact that $X(t,s,x)$ is the solution to the differential stochastic equation (1.4).

We have

Proposition 3.1. *Assume* Hypothesis 1.1 *and let φ in $C_b^2(\mathbb{R}^d)$. Then the random field $u_\varphi^t(s,x) = \varphi(X(t,s,x))$ satisfies the stochastic partial differential equation*

$$\begin{cases} u_\varphi^t(s,x) = \varphi + \int_s^t \mathcal{L}_r u_\varphi^t(r,x)\mathrm{d}r + \int_s^t \big(\mathcal{M}_r u_\varphi^t(r,x), \hat{\mathrm{d}}w(r)\big) + \\ \qquad\qquad + \int_{[s,t[\times\mathbb{R}_*^m} \mathcal{N}_r(z) u_\varphi^t(r,x)\tilde{p}(\mathrm{d}z,\hat{\mathrm{d}}r), \end{cases} \tag{3.1}$$

where we recall that $\hat{\mathrm{d}}w(t)$ and $\tilde{p}(\mathrm{d}z,\hat{\mathrm{d}}r)$ stand for backward stochastic integration.

Proof. The arguments are similar to those in Ikeda and Watanabe [3, Theorem II.5.1, p. 66], see also Jacod and Shiryaev [5]. The first step is to prove the result without the small jumps, e.g., we approximate the function γ in (1.4) and the measure π. Indeed, without any lost of generality we assume $\gamma(z,t,x)$ continuous in z and we set $\pi_n(B) = \pi(\{z \in B \,:\, n|z| \geq 1,\ |z| \leq n\})$, $n = 1,2,\ldots$ and B any Borel subset of $\mathbb{R}_*^m$, so that if p_n is the corresponding Poisson measure then there exist an increasing sequence of stopping times $\{\tau_1,\tau_2,\ldots\}$ and an adapted sequence of random variables (jumps) $\{z_1,z_2,\ldots\}$ such that

$$p_n(]0,t]\times\mathbb{R}_*^m) = \sum_{i=1}^{\infty} z_i 1_{\{\tau_i\leq t\}}, \quad \forall t > 0.$$

Clearly, $0 < \tau_i < \tau_{i+1}$ if $\tau_i < \infty$ and τ_i is the time of the i jump given by z_i.

Consider the process $X_n(t) = X_n(t,s,x)$ solution to the following stochastic differential equation

$$\begin{cases} X_n(t) = x + \int_s^t g(r,X_n(r))\mathrm{d}r + \int_s^t \sigma(r,X_n(r))\mathrm{d}w(r) + \\ \qquad\qquad + \int_{\mathbb{R}_*^m\times]s,t]} \gamma(z,r,X_n(r))\tilde{p}_n(\mathrm{d}z,\mathrm{d}r). \end{cases} \tag{3.2}$$

Such solution $X_n(t)$ has properties similar to those of equation (1.4), and if $u_{\varphi,n}^t(s,x) = \varphi(X_n(t,s,x))$ then converges (in probability) to $u_\varphi^t(s,x)$, together with their first and second derivatives in x, uniformly for $0 \leq s \leq t \leq T$ and $|x| \leq r$, for any fixed $r > 0$. Moreover, we can write the stochastic differential equation (3.2) as

$$X_n(t) = x + \int_s^t g(r,X_n(r))\mathrm{d}r + \int_s^t \sigma(r,X_n(r))\mathrm{d}w(r) + \\ + \sum_i \gamma(z_i,\tau_i,X_n(\tau_i))1_{\{s<\tau_i\leq t\}} - \int_s^t \mathrm{d}r \int_{\mathbb{R}_*^m} \gamma(z,r,X_n(r))\pi_n(\mathrm{d}z),$$

for any n.

Hence we have

$$\begin{cases} \varphi(X_n(t,s,x)) - \varphi(x) = \\ \qquad = \sum_i \{\varphi(X_n(t, s\vee\tau_{i-1},x)) - \varphi(X_n(t,s\vee\tau_i-,x))\} + \\ \qquad\quad + \sum_i \{\varphi(X_n(t,s\vee\tau_i-,x)) - \varphi(X_n(t,s\vee\tau_i,x))\}. \end{cases} \tag{3.3}$$

and we can deal with first sum as a continuous process. Therefore, we assume temporarily that

$$X_n(t) = x + \int_s^t \tilde{g}_n(r, X_n(r))\mathrm{d}r + \int_s^t \sigma(r, X_n(r))\mathrm{d}w(r), \tag{3.4}$$

where

$$\tilde{g}_n(t,x) = g(t,x) - \int_{\mathbb{R}^m_*} \gamma(z,t,x)\pi_n(\mathrm{d}z),$$

i.e., without changing notation we are working between two consecutive jumps, namely, in $]\tau_i, \tau_{i+1}[$ or alternatively, we do consider the jumps.

Now, let Σ_{st} be the set of all decompositions (or partitions) $\sigma = \{s = s_0 < s_1 < \ldots < s_N = t\}$ of the interval $[s,t]$, partially ordered in the usual way. For any σ in Σ_{st} we set

$$|\sigma| = \max\big\{s_k - s_{k-1} \,:\, k = 1,\ldots,N\big\}.$$

and we have

$$\begin{aligned} \varphi(X_n(t,s,x)) - \varphi(x) &= \sum_{k=1}^N \big[\varphi(X_n(t,s_{k-1},x)) - \varphi(X_n(t,s_k,x))\big] = \\ &= \sum_{k=1}^N \big[\varphi(X_n(t,s_k,X_n(s_k,s_{k-1},x))) - \varphi(X_n(t,s_k,x))\big]. \end{aligned}$$

To simplify the notation, we set

$$\begin{aligned} &\varphi_n(t,s,x) = \varphi\big(X_n(t,s,x)\big) = u^n_{t,\varphi}(s,x), \\ &\varphi_n'(t,s,x) = D_x\big[\varphi\big(X_n(t,s,\cdot)\big)\big](x), \\ &\varphi_n''(t,s,x) = D_x^2\big[\varphi\big(X_n(t,s,\cdot)\big)\big](x), \end{aligned}$$

where D_x means derivative in the variable x. It follows

$$\begin{aligned} \varphi(X_n(t,s,x)) - \varphi(x) &= \sum_{k=1}^N \big(\varphi_n(t,s_k,x), X_n(s_k,s_{k-1},x) - x\big) + \\ &= \tfrac{1}{2}\sum_{k=1}^N \big(\varphi_n''(t,s_k,x)(X_n(s_k,s_{k-1},x) - x), X_n(s_k,s_{k-1},x) - x\big) + \\ &\qquad + R_1(|\sigma|), \end{aligned}$$

Then

$$\sum_{k=1}^{N}\big(\varphi_n'(t,s_k,x),X_n(s_k,s_{k-1},x)-x\big)=$$
$$=\sum_{k=1}^{N}\big(\varphi_n'(t,s_k,x),\tilde g(s_k,x)\big)(s_k-s_{k-1})+$$
$$+\sum_{k=1}^{N}\big(\varphi_n'(t,s_k,x),\sigma(s_k,x)(w(s_k)-w(s_{k-1}))\big)+R_2(|\sigma|),$$

and

$$\tfrac12\sum_{k=1}^{N}\big(\varphi_n''(t,s_k,x)(X_n(s_k,s_{k-1},x)-x),X_n(s_k,s_{k-1},x)-x\big)=$$
$$=\tfrac12\sum_{k=1}^{N}\big(\varphi_n''(t,s_k,x)\sigma(s_k,x),\sigma(s_k,x)\big)(s_k-s_{k-1})+R_3(|\sigma|).$$

Gathering all, we find

$$\left\{\begin{aligned}&\varphi(X_n(t,s,x))-\varphi(x)=\sum_{k=1}^{N}\big(\varphi_n'(t,s_k,x),\tilde g(s_k,x)\big)(s_k-s_{k-1})+\\&\quad+\sum_{k=1}^{N}\big(\varphi_n'(t,s_k,x),\sigma(s_k,x)(w(s_k)-w(s_{k-1}))\big)+\\&\quad+\tfrac12\sum_{k=1}^{N}\big(\varphi_n''(t,s_k,x)\sigma(s_k,x),\sigma(s_k,x)\big)(s_k-s_{k-1})+R(|\sigma|),\end{aligned}\right.\tag{3.5}$$

where

$$R(|\sigma|)=R_1(|\sigma|)+R_2(|\sigma|)+R_3(|\sigma|).$$

Now, by proceeding as in the usual proof of Itô formula, taking into account that we have

$$X_n(s_k,s_{k-1},x)-x=\int_{s_{k-1}}^{s_k}\tilde g(r,X_n(r,s_{k-1},x))\mathrm{d}r+$$
$$+\int_{s_{k-1}}^{s_k}\sigma(r,X_n(r,s_{k-1},x))\mathrm{d}w(r),$$

we deduce $R(|\sigma|)\to 0$ in probability. So now, by setting $\varphi_n(s)=\varphi(t,s,x)=\varphi(X_n(t,s,x))$ with $X_n(t,s,x)$ given by (3.4), and letting $|\sigma|$ tend to 0 in (3.5), we obtain

$$\varphi_n(s)=\varphi+\int_s^t\mathcal{L}_r^0\varphi_n(r)\mathrm{d}r-\int_s^t\int_{\mathbb{R}_*^m}\big(\gamma(z,r,x),D\varphi_n(r)\big)\pi_n(\mathrm{d}z)\mathrm{d}r+$$
$$+\int_s^t\big(\mathcal{M}_r\varphi_n(r),\hat{\mathrm{d}}w(r)\big),$$

This means that for the first sum in (3.3) yields the contribution

$$\varphi(x)+\int_s^t\mathcal{L}_r^0u_{\varphi,n}^t(r,x)\mathrm{d}r-\int_s^t\int_{\mathbb{R}_*^m}\big(\gamma(z,r,x),Du_{\varphi,n}^t(r,x)\big)\pi_n(\mathrm{d}z)\mathrm{d}r+$$
$$+\int_s^t\big(\mathcal{M}_ru_{\varphi,n}^t(r),\hat{\mathrm{d}}w(r)\big),$$

where $u^t_{\varphi,n}(s,x) = \varphi(X_n(t,s,x))$ with $X_n(t,s,x)$ given by (3.2).

For the second sum in (3.3), i.e., the sum of jumps, we set

$$\gamma_n(z,s) = \gamma(z,s,X_n(t,s-,x))$$

and we have

$$\begin{aligned}
&\sum_i \{\varphi(X_n(t,s\vee\tau_i-,x)) - \varphi(X_n(t,s\vee\tau_i,x))\} = \\
&\quad = \sum_i \{\varphi(X_n(t,\tau_i-,x)) - \varphi(X_n(t,\tau_i,x))\}1_{\{s<\tau_i\leq t\}} = \\
&\quad = \int_{[s,t)\times\mathbb{R}^m_*} \big[\varphi(X_n(t,r-,x)+\gamma(z,r,x)) - \varphi(X_n(t,r-,x))\big] p_n(\mathrm{d}z,\hat{\mathrm{d}}r) = \\
&\quad = \int_{[s,t)\times\mathbb{R}^m_*} \big[\varphi(X_n(t,r-,x)+\gamma(z,r,x)) - \varphi(X_n(t,r-,x))\big] \tilde{p}_n(\mathrm{d}z,\hat{\mathrm{d}}r) + \\
&\qquad + \int_s^t \mathrm{d}r \int_{\mathbb{R}^m_*} \big[\varphi(X_n(t,r-,x)+\gamma(z,r,x)) - \varphi(X_n(t,r-,x))\big] \pi_n(\mathrm{d}z),
\end{aligned}$$

thus, establishing (3.1), replacing $\tilde{p}$ with $\tilde{p}_n$, for $u^t_{\varphi,n}(s,x)$.

Now, we can pass to the limit for $n\to\infty$. One have $X_n(t,s,x)\to X(t,s,x)$, $D_xX_n(t,s,x)\to D_xX(t,s,x)$ and $D^2_xX_n(t,s,x)\to D^2_xX(t,s,x)$, a.s. (also uniformly in finite intervals with respect to s and x). Consequently, $u^t_{\varphi,n}(s,x)\to u^t_\varphi(s,x)$ and $\mathcal{L}^0_r u^t_{\varphi,n}(s,x)\to \mathcal{L}^0_r u^t_\varphi(s,x)$ a.s. (also uniformly in finite intervals with respect to s and x).

Also, it is easy to see, by the dominated convergence theorem, that

$$\int_0^t \mathcal{L}^0_r u^t_{\varphi,n}(r,x)\mathrm{d}r \to \int_0^t \mathcal{L}^0_r u^t_\varphi(r,x)\mathrm{d}r, \quad \text{a.s.},$$

$$\int_0^t \big(\mathcal{M}_r u^t_{\varphi,n}(r), \hat{\mathrm{d}}w(r) \to \int_0^t \big(\mathcal{M}_r u^t_\varphi(r), \hat{\mathrm{d}}w(r) \ \text{ in } \ L^2([0,T]\times\Omega),$$

$$\begin{aligned}
&\int_s^t \mathrm{d}r \int_{\mathbb{R}^m_*} \big[\varphi(X_n(t,r-,x)+\gamma(z,r,x)) - \varphi(X_n(t,r-,x))\big]\pi_n(\mathrm{d}z) \\
&\quad \to \int_s^t \mathrm{d}r \int_{\mathbb{R}^m_*} \big[\varphi(X(t,r-,x)+\gamma(z,r,x)) - \varphi(X(t,r-,x))\big]\pi(\mathrm{d}z),
\end{aligned}$$

a.s., and

$$\begin{aligned}
&\int_{[s,t)\times\mathbb{R}^m_*} \big[\varphi(X_n(t,r-,x)+\gamma(z,r,x)) - \varphi(X_n(t,r-,x))\big]\tilde{p}_n(\mathrm{d}z,\hat{\mathrm{d}}r) \\
&\quad \to \int_{[s,t)\times\mathbb{R}^m_*} \big[\varphi(X(t,r-,x)+\gamma(z,r,x)) - \varphi(X(t,r-,x))\big]\tilde{p}(\mathrm{d}z,\hat{\mathrm{d}}r),
\end{aligned}$$

where the convergence is meant in $L^2([0,T]\times\Omega)$. Thus the proof is now complete. □

Remark 1. Here, we recall a remark in [2]. In the case of only diffusions, using Stratonovich integral, instead of backward Itô integral, the second order term in $\mathcal{L}_t$ disappears; consequently, equation (3.1) reduces to the first order stochastic partial differential equation studied by Kunita [9, 10]. We can say that equation (3.1) is essentially a *first order equation*. The same equation was found by Krylov and Rozovskii [12]. □

Remark 2. If φ in $C_b^2(\mathbb{R}^d)$, then by taking expectation in (3.1) we readily see that function $v(s,x)$

$$v(s,x) = \mathbb{E}\big\{\varphi(X(t,s,x))\big\}, \quad 0 \le s \le t \le T,$$

gives a solution of backward Kolmogorov equation

$$\begin{cases} \partial_s v(s,x) + \mathcal{L}_s v(s,x) = 0, & \forall s \in]0,t], \\ v(t,x) = \varphi(x), & \forall x \in \mathbb{R}^d, \end{cases}$$

in the whole space $\mathbb{R}^d$. □

Remark 3. If the function φ also depends on s, i.e., $u_{t,\varphi}(s,x) = \varphi(s, X(t,s,x))$ then the backward Itô formula is modified by adding a term $\partial_s\varphi(s, X(t,s,x)))$ into the operator $\mathcal{L}$. □

Backward Itô formula II

Under Hypotheses 1.1, (2.4) and (2.5), the unique solution of problem (2.6) is given by

$$\begin{aligned} \eta^t(s,x) = \exp\Big\{ \int_s^t \big(G(r, X(r,s,x)), \mathrm{d}w(r)\big) + \\ + \int_s^t \big[F(r, X(r,s,x)) - \tfrac{1}{2}|G(r, X(r,s,x))|^2\big]\, \mathrm{d}r + \\ + \int_{\mathbb{R}^m_* \times]s,t]} \ln\big(1 + H(z,r,X(r,s,x))\big)\tilde{p}(\mathrm{d}z, \mathrm{d}r) + \\ + \int_s^t \mathrm{d}r \int_{\mathbb{R}^m_*} \big[\ln\big(1 + H(z,r,X(r,s,x))\big) - H(z,r,X(r,s,x))\big]\pi(\mathrm{d}z)\Big\}. \end{aligned} \tag{3.6}$$

We want to compute the backward Itô differential of $\eta^t(s,x)$ in s. In fact, by using the same technique as in the proof of Proposition 3.1, we can prove the following result:

Proposition 3.2. *Assume that* Hypotheses 1.1, (2.4) *and* (2.5) *hold. Then we have*

$$\begin{cases} \eta^t(s,x) = 1 + \displaystyle\int_s^t \mathcal{L}_r^{F,G,H} \eta^t(r,\cdot)(x)\mathrm{d}r + \\ \quad + \displaystyle\int_s^t \big(\mathcal{M}_r^G \eta^t(r,\cdot)(x), \hat{\mathrm{d}}w_r\big) + \int_{\mathbb{R}^m_* \times [s,t[} \mathcal{N}_r^H(z)\eta^t(r,\cdot)(x)\tilde{p}(\mathrm{d}z, \hat{\mathrm{d}}r), \end{cases} \tag{3.7}$$

for any $0 \le s < t \le T$, *and* x *in* $\mathbb{R}^d$.

Proof. For this end we note first that $\eta^t(s,x)$ verifies the following identity

$$\eta^t(s,x) = \eta^t(r, X(r,s,x))\, \eta^r(s,x), \tag{3.8}$$

where $s \le r \le t$.

As usual, the first step is to prove the result without the small jumps, actually we approximate π, p and $\tilde{p}$ by π_n, p_n and $\tilde{p}_n$ as before.

Considering $X_n(t)$ given by (3.2) and

$$\begin{aligned}\eta_n^t(s,x) = 1 &+ \int_s^t \eta_n^r(s,x)\, F(r, X_n(r,s,x))\mathrm{d}r + \\ &+ \int_s^t \eta_n^r(s,x)\big(G(r, X_n(r,s,x)), \mathrm{d}w(r)\big) + \\ &+ \int_{\mathbb{R}_*^m \times]s,t]} \eta_n^r(s,x)\, H_n(z,r,X_n(r,s,x))\tilde{p}_n(\mathrm{d}z,\mathrm{d}r),\end{aligned}$$

which can be rewrite as

$$\begin{aligned}\eta_n^t(s,x) = 1 &+ \int_s^t \eta_n^r(s,x)\, F(r, X_n(r,s,x))\mathrm{d}r + \\ &+ \int_s^t \eta_n^r(s,x)\big(G(r, X_n(r,s,x)), \mathrm{d}w(r)\big) + \\ &+ \sum_i \eta_n^{\tau_i}(s,x)\, H(z_i,\tau_i,X_n(\tau_i,s,x))1_{\{s<\tau_i\le t\}} - \\ &- \int_s^t \mathrm{d}r \int_{\mathbb{R}_*^m} \eta_n^r(s,x)\, H(z,r,X_n(r,s,x))\pi_n(\mathrm{d}z),\end{aligned}$$

then we have

$$\begin{aligned}\eta_n^t(s,x) - 1 = &\sum_i \{\eta_n^t(s\vee\tau_{i-1},x) - \eta_n^t(s\vee\tau_i -,x)\} + \\ &+ \sum_i \{\eta_n^t(s\vee\tau_i -,x) - \eta_n^t(s\vee\tau_i,x)\}.\end{aligned}$$

The first sum can be dealt as a continuous process, that is like (3.4) and

$$\begin{aligned}\eta_n^t(s,x) = 1 &+ \int_s^t \eta_n^r(s,x)\, \tilde{F}_n(r, X_n(r,s,x))\mathrm{d}r + \\ &+ \int_s^t \eta_n^r(s,x)\big(G(r, X_n(r,s,x)), \mathrm{d}w(r)\big)\end{aligned}$$

where

$$\tilde{F}_n(t,x) = F(t,x) - \int_{\mathbb{R}_*^m} H(z,t,x)\pi_n(\mathrm{d}z).$$

Then, let $\sigma = \{s = s_0 < s_1 < \ldots < s_N = t\}$ in Σ_{st}. Then we have

$$\eta_n^t(s,x) - 1 = \sum_{k=1}^N \left[\eta_n^t(s_{k-1},x) - \eta_n^t(s_k,x)\right].$$

Now we note that, in view of (3.8), we have

$$\eta_n^t(s_{k-1},x) - \eta_n^t(s_k,x) = \eta_n^t(s_k, X_n(s_k,s_{k-1},x))\, \eta_n^{s_k}(s_{k-1},x) - \eta_n^t(s_k,x),$$

and then we can write $\eta_n^t(s_{k-1},x) - \eta_n^t(s_k,x) = J_1^k + J_2^k + J_3^k$, where

$$\begin{aligned}J_1^k &= \eta_n^t(s_k, X_n(s_k,s_{k-1},x)) - \eta_n^t(s_k,x), \\ J_2^k &= \eta_n^t(s_k,x)\big(\eta_n^{s_k}(s_{k-1},x) - 1\big), \\ J_3^k &= \big[\eta_n^t(s_k, X_n(s_k,s_{k-1},x)) - \eta_n^t(s_k,x)\big]\big(\eta_n^{s_k}(s_{k-1},x) - 1\big).\end{aligned}$$

Considering $J_1 = \sum_{k=1}^N J_1^k$ and arguing as in preceding section, we can write

$$J_1 = \sum_{k=1}^N \big[\mathcal{L}^0_{s_k}\eta^t_n(s_k,\cdot)(x) - \int_{\mathbb{R}^m_*}(\gamma(z,s_k,x), D\eta^t_n(s_k,x))\pi_n(\mathrm{d}z)\big](s_k - s_{k-1}) + \sum_{k=1}^N \big(\mathcal{M}_{s_k}\eta^t_n(s_k,\cdot)(x), w(s_k) - w(s_{k-1})\big) + R'(|\sigma|).$$

Analogously, for $J_2 = \sum_{k=1}^N J_2^k$, we can write

$$J_2 = \sum_{k=1}^N \eta^t_n(s_k,x)\big[F(s_k,x) - \int_{\mathbb{R}^m_*} H(z,s_k,x)\pi_n(\mathrm{d}z)\big](s_k - s_{k-1}) + + \sum_{k=1}^N \eta^t_n(s_k,x)\big(G(s_k,x), w(s_k) - w(s_{k-1})\big) + R''(|\sigma|).$$

Finally, for $J_3 = \sum_{k=1}^N J_3^k$, we can write

$$J_3 = \sum_{k=1}^N \big(\mathcal{M}_{s_k}\eta^t_n(s_k,\cdot)(x), G(s_k,x)\big)(s_k - s_{k-1}) + R'''(|\sigma|).$$

Taking into account that $X_n(t,s,x)$ fulfills equation (1.4), and that $\eta^t_n(s,x)$ is the solution to (2.6), we have, arguing as in the proof of Itô formula,

$$R(|\sigma|) = R'(|\sigma|) + R''(|\sigma|) + R'''(|\sigma|) \to 0,$$

in probability as $|\sigma| \to 0$. Since $\eta^t_n(s,x) - 1 = J_1 + J_2 + J_3$, letting $|\sigma|$ tend to 0, we have the following contribution to $\eta^t_n(s,x)$ from the first sum

$$1 + \int_s^t \mathcal{L}^0_r\eta^t_n(r,x)\mathrm{d}r - \int_s^t \mathrm{d}r\int_{\mathbb{R}^m_*}\big(\gamma(z,r,x), D\eta^t_n(r,x)\big)\pi_n(\mathrm{d}z) + + \int_s^t \big(\mathcal{M}_r\eta^t_n(r,\cdot)(x), G(r,x)\big)\mathrm{d}r + + \int_s^t \eta^t_n(r,x)\big[F(r,x) - \int_{\mathbb{R}^m_*} H(z,r,x)\pi_n(\mathrm{d}z)\big]\mathrm{d}r + \int_s^t \big(\mathcal{M}_r\eta^t_n(r) + \eta^t_n(r,x)\,G(r,x), \hat{\mathrm{d}}w(r)\big).$$

For the second sum, i.e., the jumps, we have to use the analogous of identity (3.8), that is

$$\eta^t_n(\tau_i-,x) = \eta^t_n(\tau_i, X(\tau_i,\tau_i-,x))\eta^{\tau_i}_n(\tau_i-,x).$$

Hence,

$$\begin{aligned}
&\sum_i \{\eta^t_n(s\vee\tau_i-,x)) - \eta^t_n(s\vee\tau_i,x))\} = \\
&\quad= \sum_i \{\eta^t_n(\tau_i-,x) - \eta^t_n(\tau_i,x)\}1_{\{s<\tau_i\le t\}} = \\
&\quad= \sum_i \{\eta^t_n(\tau_i, X(\tau_i,\tau_i-,x))\eta^{\tau_i}_n(\tau_i-,x) - \eta^t_n(\tau_i,x)\}1_{\{s<\tau_m\le t\}} = \\
&\quad= \sum_i \{\eta^t_n(\tau_i, X(\tau_i,\tau_i-,x)) - \eta^t_n(\tau_i,x) + \\
&\qquad+ \eta^t_n(\tau_i, X(\tau_i,\tau_i-,x))\,[\eta^{\tau_i}_n(\tau_i-,x) - 1]\}1_{\{s<\tau_m\le t\}}
\end{aligned}$$

Writing this sum as an integral, that is

$$\int_{[s,t)\times\mathbb{R}^m_*}\big[\eta^t_n(r-,x+\gamma(z,r,x))-\eta^t_n(r,x)\big]p_n(\mathrm{d}z,\hat{\mathrm{d}}r)+ \\ +\int_{[s,t)\times\mathbb{R}^m_*}\eta^t_n(r-,x+\gamma(z,r,x))\;H(z,r,x)\;p_n(\mathrm{d}z,\hat{\mathrm{d}}r),$$

we arrive to write it in the final form

$$\begin{aligned}
&\sum_i\{\eta^t_n(s\vee\tau_i-,x))-\eta^t_n(s\vee\tau_i,x))\}= \\
&=\int_{[s,t)\times\mathbb{R}^m_*}\big[\eta^t_n(r-,x+\gamma(z,r,x))-\eta^t_n(r,x)\big]\tilde{p}_n(\mathrm{d}z,\hat{\mathrm{d}}r)+ \\
&\quad+\int_s^t\int_{\mathbb{R}^m_*}\big[\eta^t_n(r-,x+\gamma(z,r,x))-\eta^t_n(r,x)\big]\pi_n(\mathrm{d}z)\,\mathrm{d}r+ \\
&\quad+\int_{[s,t)\times\mathbb{R}^m_*}\eta^t_n(r-,x+\gamma(z,r,x))\;H(z,r,x)\;\tilde{p}_n(\mathrm{d}z,\hat{\mathrm{d}}r)+ \\
&\quad+\int_s^t\int_{\mathbb{R}^m_*}\eta^t_n(r-,x+\gamma(z,r,x))\;H(z,r,x)\;\pi_n(\mathrm{d}z)\,\mathrm{d}r.
\end{aligned}$$

Thus, we have established (3.1) for $\eta^t_n(s,x)$. Now, we can pass to the limit for $n\to\infty$ in the same way as in the preceding section. □

To conclude we need the following lemma:

Lemma 3.3. *Let us suppose that $u(s)$, $v(s)$ be of the following form*

$$\begin{aligned}
u(s)-u(t)&=\int_s^t u_1(r)\mathrm{d}r+\int_s^t u_2(r)\,\hat{\mathrm{d}}w_r+\int_{[s,t)\times\mathbb{R}^m_*}u_3(z,r)\,\tilde{p}(\mathrm{d}z,\hat{\mathrm{d}}r),\\
v(s)-v(t)&=\int_s^t v_1(r)\mathrm{d}r+\int_s^t v_2(r)\,\hat{\mathrm{d}}w_r+\int_{[s,t)\times\mathbb{R}^m_*}v_3(z,r)\,\tilde{p}(\mathrm{d}z,\hat{\mathrm{d}}r).
\end{aligned}$$

Then we have

$$\left\{\begin{aligned}
&u(s)\,v(s)-u(t)\,v(t)= \\
&\quad=\int_s^t\Big(u_1(r)\,v(r)+u(r)\,v_1(r)+u_2(r)\,v_2(r)+ \\
&\qquad\qquad+\int_{\mathbb{R}^m_*}u_3(z,r)v_3(z,r)\pi(\mathrm{d}z)\Big)\mathrm{d}r+ \\
&\quad+\int_s^t\Big(u_2(r)\,v(r)+u(r)\,v_2(r)\Big)\hat{\mathrm{d}}w_r+ \\
&\quad+\int_{[s,t)\times\mathbb{R}^m_*}\Big(u_3(z,r)\,v(r)+u(r)\,v_3(z,r)+ \\
&\qquad\qquad+u_3(z,r)v_3(z,r)\Big)\tilde{p}(\mathrm{d}z,\hat{\mathrm{d}}r)
\end{aligned}\right. \tag{3.9}$$

Proof. The procedure is just the same as before. As usual, the first step is to prove the result without the small jumps, actually we approximate π, p and $\tilde{p}$ by π_n, p_n and $\tilde{p}_n$ as before, i.e., $\pi_n(B)=\pi(\{z\in B\;:\;n|z|\geq 1\})$, $n=1,2,\ldots$ and B any Borel subset of $\mathbb{R}^m_*$.

Thus, let us consider the following processes

$$
\begin{aligned}
u^{(n)}(s) - u^{(n)}(t) = \int_s^t u_1(r)\mathrm{d}r + \int_s^t u_2(r)\,\hat{\mathrm{d}}w_r + \\
+ \int_{[s,t)\times\mathbb{R}^m_*} u_3(z,r)\,\tilde{p}_n(\mathrm{d}z,\hat{\mathrm{d}}r) \\
v^{(n)}(s) - v^{(n)}(t) = \int_s^t v_1(r)\mathrm{d}r + \int_s^t v_2(r)\,\hat{\mathrm{d}}w_r + \\
+ \int_{[s,t)\times\mathbb{R}^m_*} v_3(z,r)\,\tilde{p}_n(\mathrm{d}z,\hat{\mathrm{d}}r),
\end{aligned}
$$

then, there exists a sequence of stopping times (where one jump occurs) $0 < \tau_1 < \tau_2 < \cdots$ and jumps $z_1, z_2, \ldots$, such that we can write the preceding equations as

$$
\begin{aligned}
u^{(n)}(s) - u^{(n)}(t) = \int_s^t u_1(r)\mathrm{d}r + \int_s^t u_2(r)\,\hat{\mathrm{d}}w_r + \\
+ \sum_i u_3(z_i,\tau_i)1_{\{s\le\tau_i<t\}} - \int_s^t \mathrm{d}r \int_{\mathbb{R}^m_*} u_3(z,r)\,\pi_n(\mathrm{d}z), \\
v^{(n)}(s) - v^{(n)}(t) = \int_s^t v_1(r)\mathrm{d}r + \int_s^t v_2(r)\,\hat{\mathrm{d}}w_r + \\
+ \sum_i v_3(z_i,\tau_i)1_{\{s\le\tau_i<t\}} - \int_s^t \mathrm{d}r \int_{\mathbb{R}^m_*} v_3(z,r)\,\pi_n(\mathrm{d}z),
\end{aligned}
$$

then we have

$$
\begin{aligned}
& u^{(n)}(s)v^{(n)}(s) - u^{(n)}(t)v^{(n)}(t) = \\
& = \sum_i \{u^{(n)}(s\vee\tau_{i-1})v^{(n)}(s\vee\tau_{i-1}) - u^{(n)}(s\vee\tau_i-)v^{(n)}(s\vee\tau_i-)\} + \\
& + \sum_i \{u^{(n)}(s\vee\tau_i-)v^{(n)}(s\vee\tau_i-) - u^{(n)}(s\vee\tau_i)v^{(n)}(s\vee\tau_i)\}.
\end{aligned}
$$

The first sum can be dealt as a continuous process, that is (we drop the index n to simplify notation)

$$
\begin{aligned}
u(s) - u(t) = \int_s^t \bar{u}_1(r)\mathrm{d}r + \int_s^t u_2(r)\,\hat{\mathrm{d}}w_r, \\
v(s) - v(t) = \int_s^t \bar{v}_1(r)\mathrm{d}r + \int_s^t v_2(r)\,\hat{\mathrm{d}}w_r
\end{aligned}
$$

with

$$
\begin{aligned}
\bar{u}_1(r) = u_1(r) - \int_{\mathbb{R}^m_*} u_3(z,r)\,\pi_n(\mathrm{d}z), \\
\bar{v}_1(r) = v_1(r) - \int_{\mathbb{R}^m_*} v_3(z,r)\,\pi(\mathrm{d}z).
\end{aligned}
$$

Hence, let us write

$$
\begin{aligned}
u(s)\,v(s) - u(t)\,v(t) = \sum_{k=1}^N \Big(u(s_{k-1})\,v(s_{k-1}) - u(s_k)\,v(s_k)\Big) \\
= \sum_{k=1}^N (u(s_{k-1}) - u(s_k))\,v(s_k) + \sum_{k=1}^N u(s_k)\,(v(s_{k-1}) - v(s_k)) + \\
+ \sum_{k=1}^N (u(s_{k-1}) - u(s_k))\,(v(s_{k-1}) - v(s_k)).
\end{aligned}
$$

Now we can write

$$u(s_{k-1}) - u(s_k) = \bar{u}_1(s_k)(s_k - s_{k-1}) + u_2(s_k)(w(s_k) - w(s_{k-1})) + L_k^1,$$

$$v(s_{k-1}) - v(s_k) = \bar{v}_1(s_k)(s_k - s_{k-1}) + v_2(s_k)(w(s_k) - w(s_{k-1})) + L_k^2,$$

and

$$(u(s_{k-1}) - u(s_k))\,(v(s_{k-1}) - v(s_k)) = u_2(s_k)v_2(s_k)(s_k - s_{k-1}) + L_k^3.$$

Setting

$$L(|\sigma|) = \sum_{k=1}^{n} \big\{ v(s_k)L_k^1 + u(s_k)L_k^2 + L_k^3 \big\},$$

it follows that $L(|\sigma|) \to 0$ in probability as $|\sigma| \to 0$. Hence

$$\begin{aligned}
&u^{(n)}(s)\,v^{(n)}(s) - u^{(n)}(t)\,v^{(n)}(t) = \\
&\qquad = \int_s^t \Big(u_1(r)\,v(r) + u(r)\,v_1(r) + u_2(r)\,v_2(r) - \\
&\qquad\quad - \int_{\mathbb{R}_*^m} u(r)v_3(z,r)\,\pi(\mathrm{d}z) - \int_{\mathbb{R}_*^m} u_3(z,r)v(r)\,\pi(\mathrm{d}z) \Big)\mathrm{d}r + \\
&\qquad\quad + \int_s^t \Big(u_2(r)\,v(r) + u(r)\,v_2(r) \Big)\,\hat{\mathrm{d}}w_r
\end{aligned}$$

For the second sum we have simply

$$\begin{aligned}
&\sum_i \{ u^{(n)}(s \vee \tau_i -)v^{(n)}(s \vee \tau_i -) - u^{(n)}(s \vee \tau_i)v^{(n)}(s \vee \tau_i) \} = \\
&\quad = \sum_i \big\{ u^{(n)}(\tau_i -)v^{(n)}(\tau_i -) - u^{(n)}(\tau_i)v^{(n)}(\tau_i) \big\} 1_{\{s < \tau_i \le t\}} = \\
&\quad = \int_{[s,t)\times\mathbb{R}_*^m} \big[u^{(n)}(r)v_3(z,r) + u_3(z,r)v^{(n)}(r) + \\
&\qquad\qquad\qquad + u_3(z,r)v_3(z,r) \big] p_n(\mathrm{d}z, \hat{\mathrm{d}}r) = \\
&\quad = \int_{[s,t)\times\mathbb{R}_*^m} \big[u^{(n)}(r)v_3(z,r) + u_3(z,r)v^{(n)}(r) + \\
&\qquad\qquad\qquad + u_3(z,r)v_3(z,r) \big] \tilde{p}_n(\mathrm{d}z, \hat{\mathrm{d}}r) + \\
&\quad + \int_s^t \mathrm{d}r \int_{\mathbb{R}_*^m} \big[u^{(n)}(r)v_3(z,r) + u_3(z,r)v^{(n)}(r) + \\
&\qquad\qquad\qquad + u_3(z,r)v_3(z,r) \big] \pi_n(\mathrm{d}z).
\end{aligned}$$

Thus, we have established (3.9) for $u^{(n)}(s)v^{(n)}(s)$. Now, we can pass to the limit for $n \to \infty$ in the same way as in the preceding section. □

Proof of Theorem 2.1

Let us apply the previous lemma with $u(s) = \varphi(X(t,s,x))$ and $v(s) = \eta^t(s,x)$. Recalling (3.1) and (3.7), we easily arrive to the conclusion. □

Corollary 3.4. *Assume that* Hypotheses 1.1*,* (2.4) *and* (2.5) *hold, G is bounded and F is bounded above. If φ belongs to $C_b^2(\mathbb{R}^d)$, then the function*

$$u(s,x) = \mathbb{E}\big\{y^t(s,x)\big\}, \quad 0 < s < t,$$

is a solution to the (backward) Kolmogorov equation

$$\begin{cases} \partial_s u(s,x) + \mathcal{L}_s^{F,G,H} u(s,\cdot)(x) = 0, & \forall s < t, \\ u(t,x) = \varphi(x), & \forall x \in \mathbb{R}^d. \end{cases} \tag{3.10}$$

in the whole space $\mathbb{R}^d$. □

4 Some applications

In this section we point out several consequences of formula (2.9), obtaining simpler proofs of known results as: Feynman-Kac, Girsanov, and stochastic Feynman-Kac formula.

4.1 Feynman-Kac Formula

This result is taken from [2].

Theorem 4.1. *Assume that* Hypotheses 1.1 *hold, with $G = 0$ and $H = 0$, and that F is bounded above and twice continuously differentiable in x. If φ is in $C_b^2(\mathbb{R}^d)$, then the function*

$$u(s,x) = \mathbb{E}\big\{y^t(s,x)\big\}, \quad 0 < s < t,$$

where $y^t(s,x) = \varphi(X(t,s,x))\eta^t(s,x)$, is a solution to the (backward) Kolmogorov equation

$$\begin{cases} u_s(s,x) + \mathcal{L}_s u(s,\cdot)(x) + F(s,x)\, u(s,x) = 0, & \forall s < t,\ x \in \mathbb{R}^d, \\ u(t,x) = \varphi(x), & x \in \mathbb{R}^d \end{cases}$$

where $\eta^t(s,x)$ is given by (2.6) *or* (3.6)*.*

Proof. Since $G = H = 0$, we have from (3.6)

$$\eta^t(s,x) = \exp\Big\{ \int_s^t F(X(r,s,x))\mathrm{d}r \Big\}.$$

Moreover formula (2.9) becomes

$$y^t(s,x) = \varphi(x) + \int_s^t \big[\mathcal{L}_r y^t(r,\cdot)(x) + F(r,x) y^t(r,x)\big]\mathrm{d}r + \\ + \int_s^t \big(\mathcal{M}_r y^t(r,x), \hat{\mathrm{d}}w_r\big) + \int \mathcal{N}_r(z) y^t(r,x) \tilde{p}(\mathrm{d}z, \hat{\mathrm{d}}r).$$

Taking expectation, the conclusion follows. □

4.2 Girsanov Formula

First, let us consider the case of $F = H = 0$. Here we want to find an expression for the transition semigroup corresponding to the following the stochastic differential equation, with t in $[s, T]$,

$$\begin{cases} Y(t) = x + \displaystyle\int_s^t [g(r, Y(r)) + \sigma(r, Y(r))G(r, Y(r)) + \\ \qquad + \displaystyle\int_s^t \sigma(r, Y(r)) \mathrm{d}w(r) + \int_{\mathbb{R}^m_* \times]s,t]} \gamma(z, r, Y(r)) \, \tilde{p}(\mathrm{d}z, \mathrm{d}r), \end{cases} \tag{4.1}$$

in terms of $X(t)$, the solution to the "simpler" equation

$$\begin{cases} X(t) = x + \displaystyle\int_s^t g(r, X(r)) \, \mathrm{d}r + \int_s^t \sigma(r, X(r)) \mathrm{d}w(r) + \\ \qquad + \displaystyle\int_{\mathbb{R}^m_* \times]s,t]} \gamma(z, r, X(r)) \, \tilde{p}(\mathrm{d}z, \mathrm{d}r), \end{cases} \tag{4.2}$$

The following result can be considered as a formulation of Girsanov's theorem.

Theorem 4.2. *Assume that* Hypotheses 1.1, (2.4) *and* (2.5) *hold with* $F = 0$, $H = 0$, *with* G *bounded. Let* φ *be in* $C_b^2(\mathbb{R}^d)$, *and let* $Y(t, s, x)$ *and* $X(t, s, x)$ *be the solutions to* (4.1), *and* (4.2) *respectively. Then we have, for any* s *in* $[0, t]$ *and* x *in* $\mathbb{R}^d$

$$\mathbb{E}\{\varphi(Y(t, s, x))\} = \mathbb{E}\{\varphi(X(t, s, x))\eta^t(s, x)\}, \tag{4.3}$$

where $\eta^t(s, x)$ *is given by*

$$\eta^t(s, x) = \exp\Big\{ \int_s^t \big(G(X(r, s, x)), \mathrm{d}w(r)\big) - \frac{1}{2} \int_s^t |G(X(r, s, x))|^2 \mathrm{d}r.$$

Proof. We only need to prove that both sides of identity (4.3), considered as functions of (s, x) fulfill the Kolmogorov equation

$$\begin{cases} \partial_s u(s, x) + \mathcal{L}_s^{0,G,0} u(s, \cdot)(x) = 0, \quad \forall s < t, \; x \in \mathbb{R}^d, \\ u(t, x) = \varphi(x), \quad \forall x \in \mathbb{R}^d. \end{cases}$$

This is obviously true for the left hand side. For as the right hand side is concerned it is enough to apply formula (3.10), choosing $F = H = 0$. □

Now consider $0 \le s < T$ fixed and the stochastic differential equation associated with the integro-differential operator $\mathcal{L}_s^{0,G,H}$, see (2.1) and (2.7), considered in the variable r, i.e., the probability P_{sx} in the canonical space such that $P_{sx}(\{\omega \in D([s, T], \mathbb{R}^d) : \omega(s) = x\}) = 1$ and the process

$$M_\varphi(t) = \varphi\big(\omega(t)\big) + \int_s^t \mathcal{L}_r^{0,G,H} \varphi(\omega(r)) \mathrm{d}r, \quad \forall t \in [s, T],$$

is a P_{sx}-martingale for every smooth function φ.

Here we want to find an expression for the transition semigroup corresponding to the above probability P_{sx} in terms of the same $X(t)$ as in (4.2). The following result can be considered as a formulation of Girsanov's theorem.

Theorem 4.3. *Assume that* Hypotheses 1.1, (2.4) *and* (2.5) *hold with* $F = 0$, *with* G *and* H *bounded. Let* φ *be in* $C_b^2(\mathbb{R}^d)$, *and let* $X(t,s,x)$ *be the solution to* (4.2) *and* P_{sx} *be the unique martingale solution associated with the integro differential operator* $\mathcal{L}_r^{0,G,H}$. *Then we have, for any* t *in* $[s,T]$ *and* x *in* $\mathbb{R}^d$

$$\mathbb{E}_{sx}\big\{\varphi\big(\omega(t)\big)\big\} = \mathbb{E}\big\{\varphi(X(t,s,x))\eta^t(s,x)\big\}, \tag{4.4}$$

where $\mathbb{E}_{sx}$ *denotes the expectation with respect to* P_{sx} *and* $\eta^t(s,x)$ *is given now by*

$$\begin{aligned}\eta^t(s,x) = \exp\Big\{ \int_s^t \big(G(X(r,s,x)), \mathrm{d}w(r)\big) - \frac{1}{2}\int_s^t |G(X(r,s,x))|^2 \mathrm{d}r + \\ + \int_{\mathbb{R}_*^m \times]s,t]} \ln\big(1 + H(z,r,X(r,s,x))\big)\tilde{p}(\mathrm{d}z,\mathrm{d}r) + \\ + \int_s^t \mathrm{d}r \int_{\mathbb{R}_*^m} \big[\ln\big(1 + H(z,r,X(r,s,x))\big) - H(z,r,X(r,s,x))\big]\pi(\mathrm{d}z)\Big\}.\end{aligned}$$

Proof. This is essentially the same of the case $F = 0$ and $H = 0$, the only point to notice is that now, the right-hand side of (4.4) is used to define a probability which is the solution of the desired martingale problem. □

Note that in the previous case, the stochastic equations for Y and X could be set (but non necessarily) in the same probability space with the same Wiener process and Poisson measure. However, when the jumps are involved, one may have the stochastic equations for X set in an arbitrary probability space, and by taken the image, one may suppose that (4.2) is really set on the canonical probability space $D([0,\infty),\mathbb{R}^d)$ where the canonical process $\omega(t) = X(t,\omega)$ solves the stochastic equation under the initial probability measure. Then, under the new probability measure P_{sx}, the same canonical process $\omega(t)$ solves the equivalent of equation (4.1), which is given as a martingale problem or by specifying the characteristics of the canonical process under P_{sx}, namely, the drift and diffusion terms are as in (4.1), but the integer measure ν (associated with the jumps of ω) has

$$\nu^p(B \times]a,b]) = \int_{]a,b]} \mathrm{d}r \int_{\big\{z \in \mathbb{R}_*^m : \gamma(z,r,\omega(r-)) \in B\big\}} (1 + H(z,r,\omega(r-)))\pi(\mathrm{d}z),$$

for every B in $\mathcal{B}(\mathbb{R}_*^d)$, $0 \le a < b$, as its predictable compensator, i.e., the Lévy measure or kernel has changed from $\pi\{z \in \mathbb{R}_*^m : \gamma(z,r,\cdot) \in B\}\,\mathrm{d}s$ into

$$\mathtt{M}(B,s,\cdot)\,\mathrm{d}s = \Big(\int_{\{z \in \mathbb{R}_*^m : \gamma(z,r,\cdot) \in B\}} (1 + H(z,r,\cdot))\pi(\mathrm{d}z)\Big)\mathrm{d}s,$$

as expected.

Remark 4. It is clear that the regularity assumption (1.3) is not necessary for Theorems 4.2 and 4.3, it suffices to approximate the coefficients. □

4.3 Stochastic Feynman-Kac Formula

Assume that we are given two probability filtered spaces $(\Omega_1, \mathcal{F}^{(1)}, P_1)$ and $(\Omega_2, \mathcal{F}^{(2)}, P_2)$, with respectively a pair of a k_1-dimensional Wiener process w_1 and a (Poisson) random measure $p_1(\mathrm{d}z_1, \mathrm{d}t)$ (with compensator $\pi_1(\mathrm{d}z_1)$), with jump values in $\mathbb{R}_*^{m_1}$, both adapted to a filtration $\{\mathcal{F}_t^{(1)}\}_{t\geq 0}$, and a pair of a k_2-dimensional Wiener process w_2 and a (Poisson) random measure $p_2(\mathrm{d}z_2, \mathrm{d}t)$ (with compensator $\pi_2(\mathrm{d}z_2)$), with jump values in $\mathbb{R}_*^{m_2}$, both adapted to a filtration $\{\mathcal{F}_t^{(2)}\}_{t\geq 0}$.

Let us consider $\Omega = \Omega_1 \times \Omega_2$, $\mathcal{F} = \mathcal{F}^{(1)} \otimes \mathcal{F}^{(2)}$, $w = (w_1, w_2)$ and that $G = (0, G_2)$ where G_2 is k_2-dimensional and the matrix $\sigma = (\sigma_1, \sigma_2)$ where σ_1 is $n \times k_1$ matrix and σ_2 is $n \times k_2$ matrix. Moreover, we can define on $\mathbb{R}_*^m$, where $m = m_1 + m_2$, the random measure

$$p(\mathrm{d}z, \mathrm{d}t) = p_1(\mathrm{d}z_1, \mathrm{d}t)\, \delta(\mathrm{d}z_2) + \delta(\mathrm{d}z_1)\, p_2(\mathrm{d}z_2, \mathrm{d}t)$$

with associated compensator

$$\pi(\mathrm{d}z) = \pi_1(\mathrm{d}z_1)\, \delta(\mathrm{d}z_2) + \delta(\mathrm{d}z_1)\, \pi_2(\mathrm{d}z_2),$$

and $\gamma(z, t, x) = \gamma_1(z_1, t, x) + \gamma_2(z_2, t, x)$; we can suppose that $\gamma_i(0, t, x) = 0$. hence, we consider the solution $X(t) = X(t, s, x)$ to the equation (1.4), that can be written as

$$\begin{aligned} X(t) = x &+ \int_s^t g(r, X(r))\, \mathrm{d}r + \\ &+ \int_s^t \sigma_1(r, X(r)) \mathrm{d}w_1(r) + \int_s^t \sigma_2(r, X(r)) \mathrm{d}w_2(r) + \\ &+ \int_{\mathbb{R}_*^{m_1} \times]s,t]} \gamma_1(z_1, r, X(r)) \tilde{p_1}(\mathrm{d}z_1, \mathrm{d}r) + \int_{\mathbb{R}_*^{m_2} \times]s,t]} \gamma_2(z_2, r, X(r)) \tilde{p_2}(\mathrm{d}z_2, \mathrm{d}r). \end{aligned}$$

We are here concerned with the following stochastic partial differential equation

$$\begin{aligned} u(s, x) = \varphi(x) &+ \int_s^t \Big\{ \mathcal{L}_r u(r, \cdot)(x) + \big(\sigma_2(r, x) G_2(r, x), Du(r, \cdot)(x)\big) + \\ &+ \int_{\mathbb{R}_*^{m_2}} H_2(z_2, r, x)[u(r, x + \gamma_2(z, r, x)) - u(r, x)] \pi_2(\mathrm{d}z_2) \Big\} \mathrm{d}r + \\ &+ \int_s^t \big(\sigma_2^*(r, x) Du(\cdot)(x) + G_2(r, x) u(r, x), \hat{\mathrm{d}}w_2(r)\big) + \\ &+ \int_{\mathbb{R}_*^{m_2} \times [s,t)} [u(r, x + \gamma_2(z_2, r, x)) - u(r, x) + H_2(z_2, r, x) u(r, x)]\, \tilde{p}_2(\mathrm{d}z_2, \hat{\mathrm{d}}r). \end{aligned} \tag{4.5}$$

In our case, we can write $\mathcal{L}_r = \mathcal{L}_r^0 + \mathcal{L}_r^{\gamma_1} + \mathcal{L}_r^{\gamma_2}$ where for $i = 1, 2$

$$\mathcal{L}_s^{\gamma_i} \varphi(x) = \int_{\mathbb{R}_*^{m_i}} [\varphi(x + \gamma_i(z, s, x)) - \varphi(x) - (\gamma_i(z, s, x), D\varphi(x))] \pi_i(\mathrm{d}z).$$

This problem arises in studying Filtering Theory, e.g., see Pardoux [11] or Rozovskii [12].

Let us consider

$$\begin{aligned} \eta^t(s, x) = \exp \Big\{ &\int_s^t \big(G_2(X(r, s, x)), dw_2(r)\big) - \frac{1}{2} \int_s^t |G_2(X(r, s, x))|^2) \mathrm{d}r + \\ &+ \int_{\mathbb{R}_*^{m_2} \times]s,t]} \ln\big(1 + H_2(z_2, r, X(r, s, x))\big) \tilde{p}_2(\mathrm{d}z_2, \mathrm{d}r) + \\ &+ \int_s^t \mathrm{d}r \int_{\mathbb{R}_*^m} \big[\ln\big(1 + H_2(z_2, r, X(r, s, x))\big) - H_2(z_2, r, X(r, s, x))\big] \pi_2(\mathrm{d}z_2) \Big\}. \end{aligned}$$

where $X(t,s,x)$ is the process indicated above.

The following result gives a representation formula for the solution to (4.5) proved, by a different method in Krylov and Rozovskii [6].

Theorem 4.4. *Assume that* Hypotheses 1.1, (2.4) *and* (2.5) *hold with* $F=0$, *and let* φ *be in* $C_b^2(\mathbb{R}^d)$. *Then function (here we denote by* $\mathbb{E}_1$ *the expectation value with respect to first variable* ω_1,

$$u(s,x) = \mathbb{E}_1\big\{\varphi(X(t,s,x))\eta^t(s,x)\big\}, \quad \forall s \in [0,t],$$

is a solution to the stochastic Kolmogorov equation (4.5).

Proof. Consider $y^t(s,x) = \varphi(X(t,s,x))\eta^t(s,x)$; this process satisfies the equation

$$\begin{aligned} y^t(s,x) = \varphi(x) &+ \int_s^t \big[\mathcal{L}_r y^t(r,\cdot)(x) + \big(\sigma_2(r,x)G_2(r,x), Dy^t(r,\cdot)(x)\big)\big]\mathrm{d}r + \\ &+ \int_s^t \mathrm{d}r \int_{\mathbb{R}_*^{m_2}} H_2(z_2,r,x)[y^t(r,x+\gamma_2(z_2,r,x)) - y^t(r,x)]\pi_2(\mathrm{d}z_2) + \\ &+ \int_s^t \big(Dy^t(r,\cdot)(x), \sigma_1(r,x)\hat{\mathrm{d}}w_1(r)\big) + \\ &+ \int_s^t \big(\sigma_2^*(r,x)Dy^t(r,\cdot)(x) + G_2(r,x)y^t(r,x), \hat{\mathrm{d}}w_2(r)\big) + \\ &+ \int_{\mathbb{R}_*^{m_1}\times[s,t)} [y^t(r,x+\gamma_1(z_1,r,x)) - y^t(r,x)]\,\tilde{p}_1(\mathrm{d}z_1,\hat{\mathrm{d}}r) + \\ &+ \int_{\mathbb{R}_*^{m_2}\times[s,t)} [y^t(r,x+\gamma_2(z_2,r,x)) - y^t(r,x) + \\ &\qquad\qquad + H_2(z_2,r,x)y^t(r,x)]\,\tilde{p}_2(\mathrm{d}z_2,\hat{\mathrm{d}}r). \end{aligned}$$

Taking conditional expectation $\mathbb{E}_1$, the conclusion follows. □

Appendix: backward integration

Let us recall briefly the backward integration for a standard Wiener process and a Poisson measure. In a given probability space $(\Omega,\mathcal{F},P)$ let $(w(t),\ t\geq 0)$ be a $\mathbb{R}^\ell$-valued Wiener process and let $\{p(\cdot,t) : t \geq 0\}$ be a standard Poisson measure with Lévy (characteristic or intensity) measure $\pi(\cdot)$ in $\mathbb{R}_*^m = \mathbb{R}^m \smallsetminus \{0\}$, and (local) martingale measure $\{\tilde{p}(\cdot,t) : t \geq 0\}$, $\tilde{p}(\cdot,t) = p(\cdot,t) - t\pi(\cdot)$. Given a fixed $T > 0$, define the standard Wiener process $\hat{w}_T(t) = w(T) - w(T-t)$ and the standard Poisson measure $\hat{p}_T(\cdot,t) := p(\cdot,T) - p(\cdot,T-t)$, for t in $[0,T)$, with Lévy (characteristic or intensity) measure $\hat{\pi}_T(\cdot) = \pi(T) - \pi(T-t)$, and (local) martingale measure $\tilde{\hat{p}}_T(\cdot,t) = \hat{p}_T(\cdot,t) - t\hat{\pi}_T(\cdot)$. Then the backward integral is the (forward) integral with respect to $\hat{w}_T$ and $\tilde{\hat{p}}_T$.

Define the two-index family of sub σ-algebras $\{\mathcal{F}_a^b : t > s \geq 0\}$ or $\{\mathcal{F}(b,a) : b \geq a \geq 0\}$ generated by the increments $w(t) - w(s)$ and $p(B,t) - p(B,s)$, $b \geq t > s \geq a$, $B \in \mathbb{R}_*^m$. Now, for a given $T > 0$, define $\{\hat{\mathcal{F}}_t^T : t \leq T\}$ also denoted by $\{\hat{\mathcal{F}}_T(t) : t \leq T\}$, where $\bar{\mathcal{F}}_T(t) = \cap_{\varepsilon>0}\mathcal{F}(T,t+\varepsilon)$, clearly the single-index family of sub σ-algebras $\{\bar{\mathcal{F}}_T(t) : t \leq T\}$ is decreasing and left-continuous, called *backward filtration*. An elementary or simple process (backward-predictable) has the form

(1)-*either* $f(t,\omega) = f_i(\omega)$ if $t_{i-1} \leq t < t_i$ with some $i = 1,\ldots,n$, where $0 \leq t_0 <$

$t_1 < \cdots < t_n = T$ are real numbers and $f_{i,j}$ is a $\bar{\mathcal{F}}_T(t_i)$ measurable bounded random variable for any i, and $f(t,\omega) = 0$ otherwise;
(2)-*or* $g(z,t,\omega) = g_{i,j}(\omega)$ if $t_{i-1} \le t < t_i$ and z belongs to K_j with some $i = 1,\ldots,n$, and $j = 1,\ldots,m$, where $0 \le t_0 < t_1 < \cdots < t_n = T$ are real numbers, K_j are disjoint sets with compact closure on $\mathbb{R}^m_*$ and $g_{i,j}$ is a $\bar{\mathcal{F}}_T(t_i)$ measurable bounded random variable for any i, and $g(z,t,\omega) = 0$ otherwise.

It is clear what the backward integral should be for any backward-predictable processes $f(t)$ and $g(z,t)$, namely

$$
\begin{aligned}
&\int_0^T f(s)\,\hat{\mathrm{d}}w(s) := \sum_{i=1}^n f_i\,[w(t_i) - w(t_{i-1})], \\
&\int_a^b f(s)\,\hat{\mathrm{d}}w(s) := \int_0^T f(s)\,1_{[a,b)}(s)\,\hat{\mathrm{d}}w(s),
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}^m_*\times[0,T)} g(z,s)\,\tilde{p}(\mathrm{d}z,\hat{\mathrm{d}}s) := \sum_{i=1}^n \sum_{j=1}^m g_{i,j}\,\tilde{p}(K_j\times]t_{i-1},t_i]), \\
&\int_{\mathbb{R}^m_*\times[a,b)} g(z,s)\,\tilde{p}(\mathrm{d}z,\hat{\mathrm{d}}s) := \int_{\mathbb{R}^m_*\times[0,T)} g(z,s)\,1_{[a,b)}(s)\,\tilde{p}(\mathrm{d}z,\hat{\mathrm{d}}s),
\end{aligned}
$$

for every $b > a \ge 0$.

The Poisson measure $p(\mathrm{d}z,\mathrm{d}s)$ with Lévy measure π satisfies $p(\mathbb{R}^m_*,\{0\}) = 0$ and can be approximated by another Poisson measure $p_\varepsilon(\mathrm{d}z,\mathrm{d}s)$ with Lévy measure $\pi_\varepsilon = 1_{K_\varepsilon}\pi$, where the support $K_\varepsilon = \{0 < \varepsilon \le |z| \le 1/\varepsilon\}$ of π_ε is a compact on $\mathbb{R}^m_*$, i.e., all jumps smaller than ε or larger than $1/\varepsilon$ have been eliminated. The integer measure p_ε is associated with a compound Poisson process and has a finite (random) number of jumps, i.e., for any $T > 0$ there is an integer $N = N(T,\omega)$, points $z_i = z_i(T,\omega)$ in K_ε for $i = 1,\ldots,N$ and positive reals $\theta_i = \theta_i(T,\omega)$, $i = 1,\ldots,N$ such that $p_\varepsilon(B,]a,b],\omega) = \sum_{n=1}^N 1_{z_i\in B}1_{a<\theta_i\le b}$, for every $B \in \mathcal{B}(\mathbb{R}^m_*)$, $0 \le a < b \le T$. In this case, the forward stochastic integral can be written as

$$
\int_{\mathbb{R}^m_*\times(0,T]} f(z,s)\,\tilde{p}_\varepsilon(\mathrm{d}z,\mathrm{d}s) = \sum_{i=1}^N f(z_i,\theta_i-) - \int_0^T \mathrm{d}s \int_{K_\varepsilon} f(z,s)\pi(\mathrm{d}z),
$$

for any adapted (forward, i.e. to $\mathcal{F}^s_0$, $s \ge 0$) cad-lag process $f(z,s)$, continuous in z. On the other hand, the backward stochastic integral is written as

$$
\int_{\mathbb{R}^m_*\times[0,T)} g(z,s)\,\tilde{p}_\varepsilon(\mathrm{d}z,\hat{\mathrm{d}}s) = \sum_{i=1}^N g(z_i,\theta_i) - \int_0^T \mathrm{d}s \int_{K_\varepsilon} g(z,s)\pi(\mathrm{d}z),
$$

for any adapted (backward, i.e. to $\mathcal{F}^T_s$, $s \le T$) cad-lag process $g(z,s)$, continuous in z. Recall that elementary forward processes are left-hand continuous while elementary backward processes are right-continuous. However, after taking limits for elementary processes, both, cad-lag and cag-lad adapted (either forward or backward) processes are integrable, but one takes the cag-lad version for the forward integral and the cad-lag version for the backward integral.

Finally, the backward stochastic integral is extended to all backward predictable processes, including all cag-lad (i.e., left-hand continuous having right-hand limit) processes, satisfying

$$\int_0^T |f(t)|^2 \mathrm{d}t < \infty \quad \text{and} \quad \int_0^T \mathrm{d}t \int_{\mathbb{R}^m_*} |g(z,t)|^2 \pi(\mathrm{d}z) < \infty$$

almost surely. It is then clear that

$$\int_a^b f(t)\, \hat{\mathrm{d}}w(t) = \int_{T-b}^{T-a} \hat{f}_T(t)\, \mathrm{d}\hat{w}_T(t),$$
$$\int_{\mathbb{R}^m_* \times [a,b)} g(z,t)\, \tilde{p}(\mathrm{d}z, \hat{\mathrm{d}}t) = \int_{\mathbb{R}^m_* \times (T-b, T-a]} \hat{g}_T(z,t)\, \tilde{\hat{p}}_T(\mathrm{d}z, \mathrm{d}t),$$

for any $0 \leq a < b \leq T$, where $\hat{f}_T(t) := f(T-t)$ and $\hat{g}_T(z,t) := g(z, T-t)$.

Remark 5. Note that by using backward integration with respect to a Wiener and a Poisson measure we avoid possible difficulties with time reversal, e.g. see Jacod and Protter [4]. □

References

[1] A. Bensoussan, J.L. Lions. Impulse Control and Quasi-Variational Inequalities, Gauthier-Villars, 1984

[2] G. Da Prato, L. Tubaro. Some remarks about backward Itô formula and applications, *Stochastic Anal. Appl.* **16**, no. 6, 993–1003 (1998)

[3] N. Ikeda, S. Watanabe. Stochastic Differential Equations and Diffusion Processes, North Holland, Amsterdam, 2nd edition, 1989

[4] J. Jacod and P. Protter. Time Reversal on Lévy Processes, *Ann. Probab.* **16**, 2, 620–641 (1988)

[5] J. Jacod and A.N. Shiryaev. Limit Theorems for Stochastic Processes, Springer-Verlag, New York, 1987

[6] N. V. Krylov, B. L. Rozovskii. On the first integrals and Liouville equations for diffusion processes, *Stochastic Differential systems,* Proc. 3rd SFSP-WG 7/1, Visegrád, Hungary 1980, Lect. Notes in Control and Information Sci., **36**, 117-125 (1981)

[7] N. V. Krylov, B. L. Rozovskii. Stochastic partial differential equations and diffusion processes, *Russian Math. Surveys* **37**, 81-105 (1982)

[8] H. Kunita. Stochastic differential equations and stochastic flows of diffeomorphisms, *Ecole d'été de St-Flour XII*, Lect. Notes Math. **1097**, 143-303 (1982)

[9] H. Kunita. First order stochastic partial differential equations, in *Stochastic Analysis* (K. Itô ed.), Kinokuniya, Tokyo, 249-269 (1984)

[10] H. Kunita. Stochastic flows and stochastic differential equations. Cambridge University Press 1990

[11] E. Pardoux. Stochastic partial differential equations and filtering of diffusion processes *Stochastics* **3**, no. 2, 127-167 (1979)

[12] B. L. Rozovskii. Stochastic Evolution Systems. Kluwer Academic Publishers 1990