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REGULARIZING EFFECT FOR INTEGRO–DIFFERENTIAL PARABOLIC EQUATIONS

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1. INTRODUCTION.

We first give a general definition of an integro–differential operator and discuss an analytic approach to the study of second order parabolic integro–differential equations.

Consider the integro–differential operator related with a jump process, i.e.

$$I\varphi(x,t) = \int_{\mathcal{R}^d_\star} [\varphi(x+z,t) - \varphi(x,t) - z\cdot\nabla\varphi(x,t)] M(x,t,dz) \ , \qquad (1.1)$$

where the Levy kernel $M(x,t,dz)$ is a Radon measure on $\mathcal{R}^d_\star = \mathcal{R}^d \setminus \{0\}$ for any fixed $x \in \mathcal{R}^d,\ t \in [0,T]$, and such that at least

$$\left\{ \begin{array}{c} \displaystyle\int_{|z|\le 1} |z|^\gamma M(x,t,dz) + \int_{|z|>1} |z| M(z,t,dz) < +\infty \ , \\ \forall\, x,t,\ 0 \le \gamma \le 2 \, . \end{array} \right. \qquad (1.2)$$

The number γ will called the "order of the operator". From condition (1.2) follows that the measure M can be singular when $z = 0$, but it is regular when z goes to infinity. The exact control at the origen is given by the order γ.

The reader is referred to the books of Gikhman and Skorokhod [15, p. 245], Bensoussan and Lions [3, p. 178] Garroni and Menaldi [Chap. II] among others.

If the function φ is smooth then we can rewrite

$$I\varphi(x,t) = \int_0^1 (1-\theta)d\theta \int_{\mathcal{R}^d_\star} [z \cdot \nabla^2 \varphi(x+\theta z, t) z] M(x,t,dz) \ . \tag{1.3}$$

So, in view of (1.2) the expression $I\varphi$ makes sense at least when the second order derivatives of φ in x (i.e. $\nabla^2\varphi$) are continuous and bounded in $\mathcal{R}^d \times [0,T]$.

A priori the integro–differential operator (1.1) is defined only for functions $\varphi(x,t)$ with x in the whole space $\mathcal{R}^d$ and t in [0,T]. However, we want to consider equations on either a bounded or an unbounded region $\overline{\Omega}$ of $\mathcal{R}^d$, with either Dirichlet or Neumann boundary conditions and even with oblique boundary conditions. We therefore need to localize the operator into $\overline{\Omega}$, e.g. by extending the data φ outside of $\overline{\Omega}$. Thus (1.1) becomes

$$I\varphi(x,t) = \int_{\mathcal{R}^d_\star} [\tilde{\varphi}(x+z,t) - \varphi(x,t) - z \cdot \nabla\varphi(x,t)] M(x,t,dz) \ , \tag{1.4}$$

where φ is a function defined on $\overline{\Omega} \times [0,T]$ and $\tilde{\varphi}$ is an extension of φ to the whole space $\mathcal{R}^d \times [0,T]$.

If we are working with homogeneous Dirichlet boundary conditions, then it is natural to use the zero extension, i.e. $\tilde{\varphi}(x,t) = \varphi(x,t)$ if $x \in \overline{\Omega},\ t \in [0,T]$, and $\tilde{\varphi}(x,t) = 0$ otherwise. From the probabilistic viewpoint, this corresponds to the stopping of the random process at the first exit time of the domain $\overline{\Omega}$.

Assuming φ smooth in $\overline{\Omega} \times [0,T]$, we can have only a global Lipschitz continuous zero extension $\tilde{\varphi}$ because of the homogeneous Dirichlet boundary condition. However, we may need $\nabla^2\tilde{\varphi}$ in order to use expression (1.3) for giving sense to (1.4). This is a delicate point which is not very clear in the literature (cf. Gimbert and Lions [16]).

Under convenient hypotheses on Ω, one may use another extension, say a smooth extension to $\mathcal{R}^d \times [0,T]$, but this does not usually have a good probabilistic interpretation. We will make use of a condition under which the extension will not be necessary, see condition (2.17).

However, we will be more specific about the dependency on the variables x, t of kernel $M(x,t,dz)$ and will have enough flexibility to include the modulation of the jumps (well adapted for the stochastic differential equation theory, cf. Bensoussan and Lions [3, p. 244] and Gikhman and Skorokhod [15, p.

215]) and the density control (better adapted for the martingale problems theory, cf. Bensoussan and Lions [3, p. 251]). Then we will express the integro–differential operator (1.1) in the following form

$$\begin{cases} I\varphi(x,t) &= \displaystyle\int_F [\varphi(x+j(x,t,\zeta),t) - \varphi(x,t) - j)(x,t,\zeta)\cdot \\ & \quad \cdot\nabla\varphi(x,t)]m(x,t,\zeta)\pi(d\zeta)\ , \end{cases} \tag{1.5}$$

where $\pi(\cdot)$ is a Radon measure on the measurable space $(F, \mathcal{F})$.

Our hypotheses on the structure of the jumps cover the main cases, and at the same time they are sufficiently specific to allow us to carry over even the construction of the Green function. Notice that the so–called regularizing property in the parabolic problems depends on the specific "good" properties of the corresponding Green function or of the fundamental solution. From a probabilistic point of view this is related with the fact that the associate probability measure is absolutely continuous with respect to the Lebesgue measure on R^d.

2. INTEGRO–DIFFERENTIAL OPERATOR.

We consider first the data in the whole space (Section 2.1), and then we discuss the situation in a bounded region, (Section 2.2). In Section 2.3, we only state the existence and uniqueness results, which we will use in proving the regularizing effect. For the proofs and for general results in the Sobolev spaces and in the Hölder weighted spaces see Garroni and Menaldi [9].

2.1. In the Whole Space.

In the whole space $\mathcal{R}^d$ we give the following form to the integro–differential operator (1.1):

$$\begin{cases} I\varphi(x,t) &= \displaystyle\int_F [\varphi(x+j(x,t,\zeta),t) - \varphi(x,t) - j(x,t,\zeta)\cdot \\ & \quad \cdot\nabla\varphi(x,t)]m(x,t,\zeta)\pi(d\zeta), \end{cases} \tag{2.1}$$

acting on functions $\varphi(x,t)$ defined for x in $\mathcal{R}^d$, t in $[0,T]$, and where $\pi(\cdot)$ is a Radon measure on the measurable space $(F, \mathcal{F})$ and ∇ denotes the gradient operator in the first variable x.

The jumps coefficient $j(x,t,\zeta)$ and the density $m(x,t,\zeta)$ satisfy *for some* $0 \le \gamma \le 2$ the following condition:

$$
\begin{cases}
\text{the functions } j(x,t,\zeta),\ m(x,t,\zeta) \text{ are measurable in} \\
\mathcal{R}^d \times [0,T] \times F, \text{ and there exist a } \mathcal{F}\text{–measurable and positive} \\
\text{function } j_\gamma(\zeta) \text{ and a constant } C_0 \text{ such that for every } x,t,\zeta \\
\quad |j(x,t,\zeta)| \le j_\gamma(\zeta), \quad 0 \le m(x,t,\zeta) \le 1, \\
\quad \int_F [j_\gamma(\zeta)]^p (1 + j_\gamma(\zeta))^{-1} \pi(d\zeta) = C_0 < \infty\,,\ \forall\, p \in [\gamma, 2]\,.
\end{cases}
\tag{2.2}
$$

Remark 2.1 *It is clear that (2.2) is a condition of type (1.2), uniform in* x,t, *for the Levy kernel*

$$
M(x,t,A) = \int_{\{\zeta\,:\,j(x,t,\zeta)\in A\}} m(x,t,\zeta)\pi(d\zeta)\ , \tag{2.3}
$$

with $A \subset \mathcal{R}^d_\star$, *Borel measurable.* □

Definition 2.2 *The number* γ *in condition (2.2) is called the order of the operator.* □

Notice that condition (2.2) means that the measure $\pi(d\zeta)$ can be singular when $j_\gamma = 0$, but it is regular when j_γ goes to infinity.

If condition (2.2) is satisfied with γ in $[0,1]$, then the operator (2.1) can be split into an integral form and a first order differential operator. Indeed, in view of (2.2) with $p = 1$ and $p = 2$ we have

$$
\begin{cases}
\text{(i)} \quad \int_{|j_\gamma(\zeta)|\le 1} |j(x,t,\zeta)|\pi(d\zeta) < \infty \\
\text{(ii)} \quad \int_{|j_\gamma(\zeta)|> 1} |j(x,t,\zeta)|\pi(d\zeta) < \infty\ ,
\end{cases}
\tag{2.4}
$$

which allows us to write

$$
\begin{aligned}
I\varphi(x,t) \;&=\; \int_F [\varphi(x + j(x,t,\zeta),t) - \varphi(x,t)] m(x,t,\zeta)\pi(d\zeta) - \\
&\quad - \int_F j(x,t,\zeta) m(x,t,\zeta)\pi(d\zeta) \cdot \nabla\varphi(x,t)\ .
\end{aligned}
$$

In this case, we use the notation $I\varphi$ only for the integral part, i.e.

$$I\varphi(x,t) = \int_F [\varphi(x + j(x,t,\zeta),t) - \varphi(x,t)] m(x,t,\zeta)\pi(d\zeta)\ .$$

Condition (2.2) can be simplified for $0 \le \gamma \le 1$. Indeed for γ in $[0,1]$, condition (2.2) is equivalent to

$$\int_F [j_\gamma(\zeta)]^p \pi(d\zeta) \le C_\gamma\,, \quad \forall\, p \in [\gamma, 1]\ . \tag{2.5}$$

Now, it makes sense to rewrite the integro–differential operator (2.1) as

$$\begin{cases} I_\gamma\varphi(x,t) = \displaystyle\int_{F_\gamma} [\varphi(x + j(x,t,\zeta),t) - \varphi(x,t)] m(x,t,\zeta)\pi(d\zeta)\ , \\ \qquad\qquad\qquad\qquad \text{if } \ 0 \le \gamma \le 1, \\ I_\gamma\varphi(x,t) = \displaystyle\int_{F_\gamma} [\varphi(x + j(x,t,\zeta),t) - \varphi(x,t) - j(x,t,\zeta)\cdot \\ \qquad \cdot\nabla\varphi(x,t)] m(x,t,\zeta)\pi(d\zeta) \qquad \text{if } \ 1 < \gamma \le 2\ , \end{cases} \tag{2.6}$$

If φ is a smooth function then instead of (2.6), for $\gamma \in (1,2]$ we can rewrite

$$\begin{cases} I\varphi(x,t) = \displaystyle\int_0^1 (1-\theta)d\theta \int_F [j(x,t,\zeta)\cdot\nabla^2\varphi(x + \theta j(x,t,\zeta),t) \\ \qquad j(x,t,\zeta)] m(x,t,\zeta)\pi(d\zeta)\ . \end{cases} \tag{2.7}$$

In order to understand a little bit the integro–differential operator (2.1) we have

Lemma 2.3 *Under assumption (2.2), for every $\varepsilon > 0$ there exists a constant $C(\varepsilon)$ such that*

$$\begin{cases} \|I\varphi\|_{L^\infty} \le \varepsilon\|\nabla\varphi\|_{L^\infty} + C(\varepsilon)\|\varphi\|_{L^\infty}\ , \quad 0 \le \gamma \le 1\ , \\ \|I\varphi\|_{L^\infty} \le \varepsilon\|\nabla^2\varphi\|_{L^\infty} + C(\varepsilon)\big[\|\varphi\|_{L^\infty} + \|\nabla\varphi\|_{L^\infty}\big], \quad 1 < \gamma \le 2 \end{cases} \tag{2.8}$$

for every smooth function φ. □

We will need some other assumption on $j(x,t,\zeta)$ in order to have a property similar to (2.8) for the Lebesgue spaces L^p, $1 \le p < \infty$. We assume that

$$\begin{cases} \text{the function } \ j(x,t,\zeta) \ \text{ is continuously differentiable in } \ x \\ \text{for any fixed } t,\zeta, \ \text{ and there exists a constant } \ c_0 > 0 \\ \text{such that for every } x,x',t,\zeta \ \text{ and } \ 0 \le \theta \le 1 \ \text{ we have} \\ \\ c_0|x - x'| \le |(x - x') + \theta[j(x,t,\zeta) - j(x',t,\zeta)]| \le c_0^{-1}|x - x'|\ . \end{cases} \tag{2.9}$$

This condition (2.9) implies that the change of variable $X = x + \theta j(x,t,\zeta)$ is a diffeomorphism of class C^1 in $\mathcal{R}^d$, for any fixed t in $[0,T]$ and ζ in F. Moreover, the Jacobian of the transformation satisfies

$$0 < c_1 \le \det(I_d + \theta \nabla j(x,t,\zeta)) \le C_1, \;\; 0 \le \theta \le 1, \tag{2.10}$$

for some constants $C_1 \ge c_1 > 0$ and every x in $\mathcal{R}^d$, t in $[0,T], \zeta$ in F. Here I_d is the identity matrix in $\mathcal{R}^d$, $\nabla j(x,t,\zeta)$ is the matrix of the first partial derivative in x, and $\det(\cdot)$ denotes the determinant of a matrix.

Lemma 2.4 *Under assumptions (2.2) and (2.9), for every $\varepsilon > 0$ there exists a constant $C(\varepsilon)$ such that*

$$\begin{cases} \|I\varphi(\cdot,t)\|_{L^p} \le \varepsilon\|\nabla\varphi(\cdot,t)\|_{L^p} + C(\varepsilon)\|\varphi(\cdot,t)\|_{L^p}\,, \;\; 0 \le \gamma \le 1 \\ \|I\varphi(\cdot,t)\|_{L^p} \le \varepsilon\|\nabla^2\varphi(\cdot,t)\|_{L^p} + C(\varepsilon)\Big[\|\varphi(\cdot,t)\|_{L^p} + \\ \qquad + \|\nabla\varphi(\cdot,t)\|_{L^p}\Big]\,, \;\; 1 < \gamma \le 2\,, \end{cases} \tag{2.11}$$

for every smooth function φ, any t in $[0,T]$, and where $L^p = L^p(\mathcal{R}^d)$, $1 \le p < \infty$. □

In order to study the integro–differential operator (2.1) in the Hölder spaces $C^{\alpha,\frac{\alpha}{2}}$, we need Hölder continuity of the coefficients $j(x,t,\zeta)$ and $m(x,t,\zeta)$.

Specifying, for some exponent $0 < \alpha < 1$, we assume that there exist a constant $M_0 > 0$ such that for every x, x', t, t', ζ, and $0 \le \theta \le 1$

$$\begin{cases} |j(x,t,\zeta) - j(x',t',\zeta)| \le j_\gamma(\zeta)(|x-x'|^\alpha + |t-t'|^{\alpha/2}), \\ |[x + \theta j(x,t,\zeta)] - [x' + \theta j(x',t',\zeta)]| \le \\ \qquad \le M_0(|x-x'| + |t-t'|^{1/2}), \\ |m(x,t,\zeta) - m(x',t',\zeta)| \le (|x-x'|^\alpha + |t-t'|^{\alpha/2})\,. \end{cases} \tag{2.12}$$

Denote by $\|\cdot\|_{C^{\alpha,\frac{\alpha}{2}}}$ the norm in the space $C^{\alpha,\frac{\alpha}{2}}(\overline{\mathcal{R}}^d \times [0,T])$ of Hölder continuous and bounded functions. Taking into account of expression (2.6) we have

Lemma 2.5 *Under assumptions (2.2) and (2.12), for every $\varepsilon > 0$ there exists a constant $C(\varepsilon)$ such that*

$$\begin{cases} \|I\varphi\|_{C^{\alpha,\frac{\alpha}{2}}} \le \varepsilon \|\nabla\varphi\|_{C^{\alpha,\frac{\alpha}{2}}} + C(\varepsilon)\|\varphi\|_{C^{\alpha,\frac{\alpha}{2}}} \,, \ 0 \le \gamma \le 1 \,, \\ \|I\varphi\|_{C^{\alpha,\frac{\alpha}{2}}} \le \varepsilon \|\nabla^2\varphi\|_{C^{\alpha,\frac{\alpha}{2}}} + C(\varepsilon)\Big[\|\varphi\|_{C^{\alpha,\frac{\alpha}{2}}} + \|\nabla\varphi\|_{C^{\alpha,\frac{\alpha}{2}}}\Big] \,, \\ \qquad 1 < \gamma \le 2 \,. \end{cases} \tag{2.13}$$

for every smooth function φ, $\Box$.

2.2. In a Bounded Region.

To study the integro–differential operator (2.1) in a bounded domain Ω of $\mathcal{R}^d$, we need to localize the jumps to $\overline{\Omega}$. Moreover, the expression (2.7) is not always valid, since the segment $[x,\ x + j(x,t,\zeta)]$ need not to lay inside the domain $\overline{\Omega}$.

We express the integro–differential operator as

$$\begin{cases} I_\gamma\varphi(x,t) = \displaystyle\int_{F_\gamma} [\varphi(x + j(x,t,\zeta),t) - \varphi(x,t)] m(x,t,\zeta)\pi(d\zeta) \,, \\ \qquad\qquad \text{if } \ 0 < \gamma \le 1, \\ I_\gamma\varphi(x,t) = \displaystyle\int_{F_\gamma} \varphi(x + j(x,t,\zeta),t) - \varphi(x,t) - j(x,t,\zeta)\cdot \\ \qquad \cdot\nabla\varphi(x,t)] m(x,t,\zeta)\pi(d\zeta) \,, \qquad \text{or} \\ \qquad = \displaystyle\int_0^1 d\theta \int_{F_\gamma} j'(x,t,\zeta,\theta)\cdot[\nabla\varphi(x + j(x,t,\zeta,\theta),t) - \\ \qquad -\nabla\varphi(x,t)] m(x,t,\zeta)\pi(d\zeta), \qquad \text{if } \ 1 < \gamma \le 2 \,. \end{cases} \tag{2.14}$$

Here we assume that $\pi(\cdot)$ is a Radon measure on $(F,\mathcal{F})$, and that $j(x,t,\zeta)$, $j(x,t,\zeta,\theta)$, $j'(x,t,\zeta,\theta)$, $m(x,t,\zeta)$ are measurable functions for (x,t,ζ,θ) in $\overline{\Omega} \times [0,T] \times F \times [0,1]$ satisfying:

$$\begin{cases} j(x,t,\zeta) = j(x,t,\zeta,1), \quad j(x,t,\zeta,\theta) = \displaystyle\int_0^\theta j'(x,t,\zeta,\tau)d\tau \,, \\ 0 \le m(x,t,\zeta) \le 1, \quad |j'(x,t,\zeta,\theta)| \le j_\gamma(\zeta) \,, \\ \displaystyle\int_{F_\gamma} [j_\gamma(\zeta)]^p \pi(d\zeta) \le C_0, \ \ \forall p \in [\gamma,1], \ \text{if } \ 0 < \gamma \le 1 \,, \\ \displaystyle\int_{F_\gamma} [j_\gamma(\zeta)]^p (1 + j_\gamma(\zeta))^{-1} \pi(d\zeta) \le C_0, \ \ \forall p \in [\gamma,2], \ \text{if } \ 1 < \gamma \le 2 \,, \end{cases} \tag{2.15}$$

$$\det(I_d + \nabla_x j(x,t,\zeta,\theta)) \ge c_1, \ \ 0 < c_1 \le 1 \tag{2.16}$$

$$\text{if } m(x,t,\zeta) \neq 0 \text{ and } \zeta \in F_\gamma \text{ then } x + j(x,t,\zeta,\theta) \in \overline{\Omega}\,,\ \forall \theta \in [0,1] \tag{2.17}$$

$$\begin{cases} |m(x,t,\zeta) - m(x',t',\zeta)| \le |x-x'|^\alpha + |t-t'|^{\alpha/2}\ , \\ |(x-x') + [j(x,t,\zeta,\theta) - j(x',t',\zeta,\theta)]| \le \\ \qquad \le M_0(|x-x'| + |t-t'|^{1/2})\ , \end{cases} \tag{2.18}$$

$$\begin{cases} |j'(x,t,\zeta,\theta) - j'(x',t',\zeta,\theta)| \le j_\gamma(\zeta)(|x-x'|^\alpha + |t-t'|^{\alpha/2}), \\ \text{with } j_\gamma(\zeta) \text{ satisfying (2.15)}, \end{cases} \tag{2.19}$$

Remark 2.6 *Notice that the expression (2.14) makes sense for every smooth function φ by virtue of assumption (2.17). For the particular homogeneous Dirichlet boundary conditions we can assume the integro differential operator in the form*

$$I = I_0 + I_\gamma, \quad 0 < \gamma \le 2, \tag{2.20}$$

where

$$I_0\varphi(x,t) = \int_{F_0} [\tilde\varphi(x + j(x,t,\zeta),t) - \varphi(x,t)] m(x,t,\zeta)\pi(d\zeta)\ , \tag{2.21}$$

$$\tilde\varphi(x,t) = \begin{cases} \varphi(x,t) & \text{if } x \in \overline{\Omega},\ t \in [0,T], \\ 0 & \text{otherwise}, \end{cases} \tag{2.22}$$

and I_γ, $0 < \gamma \le 2$, is given by (2.14).

Condition (2.17) is not necessarly satisfied for $\gamma = 0$. Conditions (2.15)–(2.19) still hold for $\gamma > 0$. □

Remark 2.7 *If Ω is a convex domain then we can take $j(x,t,\zeta,\theta) = \theta j(x,t,\zeta)$ and consequently $j'(x,t,\zeta,\theta) = j(x,t,\zeta)$.* □

Theorem 2.8 *Under assumptions (2.14),..., (2.19) the following estimates hold. For $1 < \gamma \le 2$ we have*

$$\begin{cases} \|I\varphi(\cdot,t)\|_{L^p(\Omega)} \le \varepsilon \|\nabla^2\varphi(\cdot,t)\|_{L^p(\Omega)} + C(\varepsilon)\Big[\|\varphi(\cdot,t)\|_{L^p(\Omega)} + \\ \qquad + \|\nabla\varphi(\cdot,t)\|_{L^p(\Omega)}\Big]\ , \end{cases} \tag{2.23}$$

$$\begin{cases} \|I\varphi\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{Q}_T)} & \leq & \varepsilon\|\nabla^2\varphi\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{Q}_T)} + C(\varepsilon)\Big[\|\varphi\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{Q}_T)} + \\ & & +\|\nabla\varphi\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{Q}_T)}\Big] \,, \end{cases} \tag{2.24}$$

and for $0 \leq \gamma \leq 1$ *we have*

$$\|I\varphi(\cdot,t)\|_{L^p(\Omega)} \leq \varepsilon\|\nabla\varphi(\cdot,t)\|_{L^p(\Omega)} + C(\varepsilon)\|\varphi(\cdot,t)\|_{L^p(\Omega)}, \tag{2.25}$$

$$\|I\varphi\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{Q}_T)} \leq \varepsilon\|\nabla\varphi\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{Q}_T)} + C(\varepsilon)\|\varphi\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{Q}_T)}, \tag{2.26}$$

for every smooth function φ *and any* t *in* $[0,T]$, $1 \leq p \leq \infty$, *and* $0 < \alpha < 1$, $\overline{Q}_T = \overline{\Omega} \times [0,T]$.

We refer to Garroni and Menaldi [9] for the proof of the above results. □

2.3. Existence and Uniqueness Results.

In the present section we recall the existence and uniqueness results for the boundary value problems to an integro–differential parabolic equation of second order in a cylindrical domain $Q_T = \Omega \times (0,T)$.

The proofs are based on the properties of operator I stated in Theorem 2.8. We make use of the fixed point arguments, starting from the existence and uniqueness results for the corresponding differential problems. Nonlinear results for $0 \leq \gamma \leq 1$ can be found in Garroni et al. [12,13], and in Garroni and Vivaldi [14]. We refer also to Menaldi [21], and Menaldi and Robin [22,23] for related studies.

Denote by $\mathcal{A}(x,t,\partial_x,\partial_t)$ the linear parabolic differential operator with "regular" coefficients

$$\mathcal{A}(x,t,\partial_x,\partial_t)u = \partial_t u - L(x,t,\partial_x)u - Iu \,, \tag{2.27}$$

where

$$-L(x,t,\partial_x)u = -a_{ij}(x,t)\partial_{ij}u + a_i(x,t)\partial_i u + a_0(x,t)u \ . \tag{2.28}$$

We assume, that this operator is *uniformly parabolic*, with coefficients a_{ij} at least continuous, namely

$$\begin{cases} a_{ij}(x,t) \quad \text{are continuous and bounded,} \\ a_{ij}(x,t)\xi_i\xi_j \geq \mu|\xi|^2, \quad \forall\, \xi \in \mathcal{R}^d \,,\ \mu > 0 \ , \end{cases} \tag{2.29}$$

in the domain where the mentioned problems are studied. We assume that the integral operator has the form (2.14) for a bounded domain and satisfies at least (2.15) and (2.16), for $0 \le \gamma \le 2$. Also in a bounded region the boundary operator is *regular*, with coefficients at least Hölder continuous, namely

$$\begin{cases} \mathcal{B}(x,t,\partial_x) = b_i(x,t)\partial_i + b_0(x,t) \\ b_i(x,t),\ b_0(x,t) \quad \text{are of class } C^{\alpha,\frac{\alpha}{2}} \\ b_i(x,t)n_i(x) \ge c_0 > 0 \quad \forall\ (x,t) \in \Sigma_T\,. \end{cases} \tag{2.30}$$

These are the minimal assumptions used in this paper. Suppose that Ω is a bounded domain with the boundary $\partial\Omega$ sufficiently smooth, for instance of class C^2. In the cylindrical domain $Q_T = \Omega \times (0,T)$, with lateral surface $\Sigma_T = \partial\Omega \times [0,T]$, we will consider the Cauchy–Dirichlet problem

$$\begin{cases} \mathcal{A}(x,t,\partial_x,\partial_t)u(x,t) = f(x,t) & \text{in } Q_T\ , \\ u(x,0) = \varphi(x) & \text{in } \Omega\ , \\ u(x,t) = \psi(x,t) & \text{on } \Sigma_T\ , \end{cases} \tag{2.31}$$

and the problem with oblique derivative

$$\begin{cases} \mathcal{A}(x,t,\partial_x,\partial_t)u(x,t) = f(x,t) & \text{in } Q_T\ , \\ u(x,0) = \varphi(x) & \text{in } \Omega\ , \\ \mathcal{B}(x,t,\partial_x)u(x,t) = \psi(x,t) & \text{on } \Sigma_T\ . \end{cases} \tag{2.32}$$

Notice that for the homogeneous Dirichlet problem we can add a zero order term, this means that we can consider the integro–differential operator I given by (2.20), i.e. $Iu = I_0u + I_\gamma u$, with $0 < \gamma \le 2$, where in I_0 we use the zero extension (2.21).

We must also assume that the functions f, φ and ψ in (2.31) or (2.32) satisfy the "usual" compatibility conditions.

We can give the results on the solvability of these problems in the Hölder functions spaces $C^{k+\alpha,\frac{k+\alpha}{2}}(\overline{Q}_T)$, $k \in N$, $k \ge 2$, $0 < \alpha < 1$. We will state here only the main results for $k = 2$. The proofs can be found in Garroni and Menaldi [9].

Theorem 2.9 *Let a_{ij}, a_i, $a_0 \in C^{\alpha,\frac{\alpha}{2}}(\overline{Q}_T)$, $0 < \alpha < 1$. Assume that the boundary $\partial\Omega$ is of class $C^{2+\alpha}$ and that I satisfies (2.15),…,(2.19) with $0 \le \gamma \le 2$. Then for any $f \in C^{\alpha,\frac{\alpha}{2}}(\overline{Q}_T)$, $\varphi \in C^{2+\alpha}(\overline{\Omega})$, $\psi \in C^{2+\alpha,\frac{2+\alpha}{2}}(\Sigma_T)$ satisfying the compatibility condition*

$$\begin{cases} \varphi(x) = \psi(x,0) & \forall x \in \partial\Omega \\ A(x,t,\partial_x)\varphi(x) + f(x,t) = \partial_t\psi(x,t) & \forall x \in \partial\Omega,\ t = 0\,, \end{cases} \tag{2.33}$$

problem (2.31) has a unique solution from the class $C^{2+\alpha,\frac{2+\alpha}{2}}(\overline{Q}_T)$. *It satisfies the inequality*

$$\|u\|_{2+\alpha,\overline{Q}_T} \leq C\left(\|f\|_{\alpha,\overline{Q}_T} + \|\varphi\|_{2+\alpha,\overline{\Omega}} + \|\psi\|_{2+\alpha,\Sigma_T}\right), \tag{2.34}$$

with the constant C *not depending on* f, φ *and* ψ. □

Theorem 2.10 *Let* a_{ij}, a_i, $a_0 \in C^{\alpha,\frac{\alpha}{2}}(\overline{Q}_T)$, $0 < \alpha < 1$. *Assume that* $\partial\Omega$ *is of class* $C^{2+\alpha}$, *that* I *satisfies (2.15),...,(2.19) with* $0 \leq \gamma \leq 2$, *and that* b_i, $b_0 \in C^{1+\alpha,\frac{1+\alpha}{2}}(\Sigma_T)$. *Then for arbitrary* $f \in C^{\alpha,\frac{\alpha}{2}}(\overline{Q}_T)$, $\psi \in C^{1+\alpha,\frac{1+\alpha}{2}}(\Sigma_T)$, $\varphi \in C^{2+\alpha}(\overline{\Omega})$ *satisfying the compatibility condition*

$$b_i(x,0)\partial_i\varphi(x) + b_0(x,0)\varphi(x) = \psi(x,0) \quad \forall x \in \partial\Omega, \tag{2.35}$$

problem (2.32) has a unique solution $u \in C^{2+\alpha,\frac{2+\alpha}{2}}(\overline{Q}_T)$ *and*

$$\|u\|_{2+\alpha,\overline{Q}_T} \leq C\left(\|f\|_{\alpha,\overline{Q}_T} + \|\varphi\|_{2+\alpha,\overline{\Omega}} + \|\psi\|_{1+\alpha,\Sigma_T}\right) \tag{2.36}$$

with the constant C *not depending on* f, φ *and* ψ. □

3. ESTIMATE ON THE GREEN FUNCTION.

It is interesting to notice that we are expecting a *regularizing* property for the parabolic problems (2.31) and (2.32) for $f = 0$, $\psi = 0$, i.e., if we start with non–homogeneous initial data not necessarily smooth (say only continuous) at time $t = 0$ we expect to have a $C^{2+\alpha,\frac{2+\alpha}{2}}$ solution at a time $t > 0$. This property cannot be deduced by means of the tecnique used in proving the above results and it has not been considered in the standard references such as in Anulova [1,2], Bensoussan and Lions [3], Bony et al. [4], Chaleyat–Maurel et al. [5], Gikhman and Skorokhod [15], Gimbert and Lions [16], Komatsu [17], Lenhard [19], Lepeltier and Marchal [20], Menaldi and Robin [24], Protter [25], Stroock [26], Taira [27].

It is clear that the regularizing property for the parabolic second order differential operator depends on the specific well known properties for the corresponding Green function.

In this section we want to use the Green function constructed in Garroni and Menaldi [7,9] to generate a Markov–Feller process.

First we recall a series of norms and seminorms used to define the Green Function Spaces. These seminorms will replace most of the essential properties of heat kernel type functions. The $C(\cdot,\cdot)$ and $K(\cdot,\cdot)$ norms give control of the L^∞ and L^1 norms, the $M(\cdot,\cdot,\alpha)$ and $N(\cdot,\cdot,\alpha)$ are the equivalent of the $C^{\alpha,\frac{\alpha}{2}}$ seminorms, and the $R(\cdot,\cdot,\alpha)$ is viewed as a kind of diagonal seminorms mixing the independent variables and the "frozen" variables. We give the following

Definition 3.1 (Green Function Spaces) *Let us denote by* $\mathcal{G}_k^{\alpha,\frac{\alpha}{2}}$ *(or* $\mathcal{G}_k^{\alpha,\frac{\alpha}{2}}$ $(\overline{\Omega}\times[0,1],\mathcal{R}^n)$ *when necessary),* $k\in\mathcal{R}$, $n\in\mathcal{N}$ *and* $0<\alpha<1$, *the space of all continuous functions (or kernels)* $\varphi(x,t,y,s)$ *defined for* x,y *in* $\overline{\Omega}\subset\mathcal{R}^d$ *and* $0\le s<t\le 1$, *with values in* $\mathcal{R}^n$ *(usually* $n=1$ *and* $k\ge 0$*) and such that the following infima (3.1),...,(3.15) (of order k) are finite. Similarly, if* $\alpha=0$ *we denote by* $\mathcal{G}_k^0$ *the previous space if only the infima (3.1) and (3.2) are finite.*

$$\left\{\begin{aligned} C(\varphi,k) &= \inf\{C\ge 0 : |\varphi(x,t,y,s)|\le C(t-s)^{-1+(k-d)/2}\ , \\ &\qquad \forall x,t,y,s\}\ , \end{aligned}\right. \tag{3.1}$$

$$\left\{\begin{aligned} K(\varphi,k) &= \inf\{K\ge 0 : \int_\Omega [|\varphi(x,t,z,s)|+|\varphi(z,t,y,s)|]dz \le \\ &\qquad \le K(t-s)^{-1+k/2},\ \forall x,t,y,s\}, \end{aligned}\right. \tag{3.2}$$

$$\left\{\begin{aligned} M(\varphi,k,\alpha) &= M_1(\varphi,k,\alpha)+M_2(\varphi,k,\alpha)+M_3(\varphi,k,\alpha)+ \\ &\quad +M_4(\varphi,k,\alpha) \end{aligned}\right. \tag{3.3}$$

$$\left\{\begin{aligned} M_1(\varphi,k,\alpha) &= \inf\{M_1\ge 0 : |\varphi(x,t,y,s)-\varphi(x',t,y,s)|\le \\ &\qquad \le M_1|x-x'|^\alpha (t-s)^{-1+(k-d-\alpha)/2}, \\ &\qquad \forall x,x',t,y,s\} \end{aligned}\right. \tag{3.4}$$

$$\left\{\begin{aligned} M_2(\varphi,k,\alpha) &= \inf\{M_2\ge 0 : |\varphi(x,t,y,s)-\varphi(x,t',y,s)|\le \\ &\qquad \le M_2|t-t'|^{\alpha/2}[(t-s)^{-1+(k-d-\alpha)/2}\vee \\ &\qquad \vee(t'-s)^{-1+(k-d-\alpha)/2}],\ \forall x,t,t',y,s\}, \end{aligned}\right. \tag{3.5}$$

$$\left\{\begin{aligned} M_3(\varphi,k,\alpha) &= \inf\{M_3\ge 0 : |\varphi(x,t,y,s)-\varphi(x,t,y',s)|\le \\ &\qquad \le M_3|y-y'|^\alpha (t-s)^{-1+(k-d-\alpha)/2}, \\ &\qquad \forall x,t,y,y',s\}, \end{aligned}\right. \tag{3.6}$$

$$\left\{\begin{array}{lcl} M_4(\varphi,k,\alpha) & = & \inf\{M_4 \geq 0 : |\varphi(x,t,y,s) - \varphi(x,t,y,s')| \leq \\ & & \leq M_4|s-s'|^{\alpha/2}[(t-s)^{-1+(k-d-\alpha)/2} \vee \\ & & \vee (t-s')^{-1+(k-d-\alpha)/2},\ \forall x,t,y,s'\} \end{array}\right. \tag{3.7}$$

$$N(\varphi,k,\alpha) = N_1(\varphi,k,\alpha) + N_2(\varphi,k,\alpha) + N_3(\varphi,k,\alpha) + N_4(\varphi,k,\alpha) \tag{3.8}$$

$$\left\{\begin{array}{lcl} N_1(\varphi,k,\alpha) & = & \inf\{N_1 \geq 0 : \int_\Omega |\varphi(x,t,z,s) - \varphi(x',t,z,s)|dz \leq \\ & & \leq N_1|x-x'|^\alpha (t-s)^{-1+(k-\alpha)/2}, \\ & & \forall x,x',t,s\}\ , \end{array}\right. \tag{3.9}$$

$$\left\{\begin{array}{lcl} N_2(\varphi,k,\alpha) & = & \inf\{N_2 \geq 0 : \int_\Omega [|\varphi(x,t,z,s) - \varphi(x,t',z,s)| + \\ & & + |\varphi(z,t,y,s) - \varphi(z,t',y,s)]dz \leq \\ & & \leq N_2|t-t'|^{\alpha/2}[(t-s)^{-1+(k-\alpha)/2} \vee \\ & & \vee (t'-s)^{-1+(k-\alpha)/2}],\ \forall x,t,t',y,s\}\ , \end{array}\right. \tag{3.10}$$

$$\left\{\begin{array}{lcl} N_3(\varphi,k,\alpha) & = & \inf\{N_3 \geq 0 : \int_\Omega |\varphi(z,t,y,s) - \varphi(z,t,y',s)|dz \leq \\ & & \leq N_3|y-y'|^\alpha (t-s)^{-1+(k-\alpha)/2}, \\ & & \forall t,y,y',s\}, \end{array}\right. \tag{3.11}$$

$$\left\{\begin{array}{lcl} N_4(\varphi,k,\alpha) & = & \inf\{N_4 \geq 0 : \int_\Omega [|\varphi(x,t,z,s) - \varphi(x,t,z,s')| + \\ & & + |\varphi(z,t,y,s) - \varphi(z,t,y,s')]dz \leq \\ & & \leq N_4|s-s'|^{\alpha/2}[(t-s)^{-1+(k-\alpha)/2} \vee \\ & & \vee (t-s')^{-1+(k-\alpha/2}],\ \forall x,t,y,s'\}, \end{array}\right. \tag{3.12}$$

$$R(\varphi,k,\alpha) = R_1(\varphi,k,\alpha) + R_2(\varphi,k,\alpha), \tag{3.13}$$

$$\left\{\begin{array}{lcl} R_1(\varphi,k,\alpha) & = & \inf\{R_1 \geq 0 : \int_\Omega |\varphi(Z,t,y,s) - \varphi(Z',t,y,s)| \\ & & J_\eta(Z,Z')dz \leq R_1\eta^\alpha (t-s)^{-1+(k-\alpha)/2}, \\ & & \forall Z,Z',t,y,s\ \ and\ \ \eta > 0\}, \end{array}\right. \tag{3.14}$$

$$\left\{ \begin{aligned} R_2(\varphi,k,\alpha) \; &= \; \inf\{R_2 \geq 0 : \int_\Omega |\varphi(x,t,Z,s) - \varphi(x,t,Z',s)| \\ &\qquad J_\eta(Z,Z')dz \leq R_2 \eta^\alpha (t-s)^{-1+(k-\alpha)/2}, \\ &\qquad \forall x,t,Z,Z',s \;\; \textit{and} \;\; \eta > 0\}, \end{aligned} \right. \tag{3.15}$$

where the change of variables $Z(z)$ and $Z'(z)$ are diffeomorphisms of class C^1 in $\mathcal{R}^d$, and the Jacobian

$$J_\eta(Z,Z') = \begin{cases} |\det(\nabla Z)| \wedge |\det(\nabla Z')| & \textit{if } |Z - Z'| \leq \eta \\ & \textit{and } Z, Z' \in \overline{\Omega}, \\ 0 & \textit{otherwise}, \end{cases} \tag{3.16}$$

$\det(\cdot)$ *means the determinant of a $d \times d$ matrix, ∇Z, $\nabla Z'$ stand for the matrices of the first partial derivatives of $Z(z), Z'(z)$ with respect to the variable z, and $\wedge, \vee$ denote the minimum, maximum (resp.) between two real numbers.* □

Notice that we are considering the kernel φ of four variables x, t, y, s. The first two variables will be the actually independent variables x, t to which the subindices $1, 2$ in the α–Hölder type seminorms $M(\cdot,\cdot,\alpha)$ and $N(\cdot,\cdot,\alpha)$ refer. The second two variables y, s will play the role of frozen parameters; we use the subindices $3, 4$ for these variables. For the diagonal seminorms $R(\cdot,\cdot,\alpha)$ the variables t and s are parameters, so the subindices $1, 2$ refer to the variables x, y, respectively.

It is proved in Garroni and Menaldi [7,9] that under assumptions as in Theorems 2.9 or 2.10 and $0 \leq \gamma < 2$, there exists a Green function associated with the integro–differential operator $\mathcal{A}$ and Dirichlet or oblique derivative boundary conditions (relative to boundary operator $\mathcal{B}$) denoted by $G(x,t,y,s)$. Moreover, this Green function has the representation $G = G_L + G_I$, where G_L is the classic Green function associated with the differential operator L. If $\gamma < 2 - \alpha$, then G_I belongs to the Green space $\mathcal{G}_{4-\gamma}^{2+\alpha,\frac{2+\alpha}{2}}$, which is defined from $\mathcal{G}_k^{\alpha,\frac{\alpha}{2}}$ (as the classic spaces $C^{2+\alpha,\frac{2+\alpha}{2}}$ is defined from $C^{\alpha,\frac{\alpha}{2}}$). If $2 - \alpha \leq \gamma < 2$, then G_I belongs to the Green space $\mathcal{G}_{4-\gamma}^{2+\varepsilon,\frac{2+\varepsilon}{2}}$, for any $\varepsilon \in (0, 2-\gamma)$.

It is clear that by following steps similar to those used to construct the Green function we can construct the fundamental solution for the integro–differential operator $\partial_t - L - I = \mathcal{A}$ under the assumptions

$$a_{ij},\, a_i,\, a_0 \in C^{\alpha,\frac{\alpha}{2}}(\overline{\mathcal{R}}^d \times [0,T]) \tag{3.17}$$

(2.2), (2.6), (2.9), (2.10) and (2.12). Let us denote by $\Gamma(x,t,y,s)$ the fundamental solution for the integro–differential operator $\mathcal{A}$. If we set

$$P(x,t,B,s) = \int_B \Gamma(x,t,y,s)dy, \quad \forall B \in \mathcal{B}(\mathcal{R}^d) \tag{3.18}$$

then $P(\cdot,\cdot,\cdot,\cdot)$ is a transition function, which defines the semigroup

$$\Phi(t,s)\varphi(x) = \int_{\mathcal{R}^d} \Gamma(x,t,y,s)\varphi(y)dy \tag{3.19}$$

satisfying the Feller property. Indeed, from the Weak Maximum Principle (cf. Garroni and Menaldi [10]) we deduce that $\varphi \geq 0$ implies $\Phi(t,s)\varphi \geq 0$. In turn, this shows that

$$\sup_{x\in\mathcal{R}^d} |\Phi(t,s)\varphi(x)| \leq \sup_{x\in\mathcal{R}^d} |\varphi(x)| \,. \tag{3.20}$$

The above a priori estimate, the fact that "*smooth* $\varphi(x)$ implies *smooth* $\Phi(t,s)$ $\varphi(x)$" and a classic *density* argument, prove that the semigroup given by (3.19) preserves continuity. At this point, we can ensure the existence of a Markov-Feller process $(X(t), t \geq 0)$ under a probability measure P on the sample space $D([0,\infty),\mathcal{R}^d)$ with transition function (3.18), e.g. Ethier and Kurtz [6, p. 169]. Due to the integro–differential operator I, the support of the probability measure P *is not* $C([0,\infty),\mathcal{R}^d)$. This Markov-Feller process is called a "Diffusion Process with Jumps". Except for some variations on the technical assumptions on the operator $\mathcal{A}$, the existence of such a diffusion process with jumps is well known (cf. Gikhman and Skorokhod [15], Komatsu [17], Lepeltier and Marchal [20], Stroock [26]). However, the specific description of the "density transition function", i.e. the fundamental solution Γ, is new to the best of our knowledge. Moreover, we expect to have a *regularizing* effect produced by the semigroup (3.19), i.e. for $t > s$ the function $\Phi(t,s)\varphi(x)$ is smooth (say $C^{2,1}$) even if $\varphi(x)$ is not so smooth. Nevertheless, this property requires some a priori regularities on the fundamental solution $\Gamma(x,t,y,s)$.

It is well know that the Markov-Feller process in the whole space can be used to represent the solution of Dirichlet boundary condition problem. Indeed, for a given smooth domain Ω in $\mathcal{R}^d$, we consider the stopping time

$$\tau = \inf\{t \geq s : X(t) \notin \Omega\}. \tag{3.21}$$

The function

$$u^D(x,t) = E\left\{\int_s^{\tau\wedge t} f(X(\lambda),\lambda)d\lambda \mid X(s) = x\right\} \tag{3.22}$$

is the solution to

$$\begin{cases} \mathcal{A}u^D &= f \ \text{ in } \ \Omega \times (s,T], \\ u^D &= 0 \ \text{ on } \ \bar{\Omega} \times \{s\} \cup \partial\Omega \times (s,T]. \end{cases} \tag{3.23}$$

Since the solution of (3.23) is unique (under the conditions of Section 2), this function u^D can be represented by means of the Green function relative to Ω with Dirichlet boundary condition, i.e.

$$u^D(x,t) = \int_s^t d\lambda \int_\Omega G^D(x,t,y,\lambda) f(y,\lambda) dy\,. \tag{3.24}$$

The conclusion is that the new Markov process $(X^D(t),\ t \geq 0)$ obtained by stopping $(X(t), t \geq 0)$ at the first exit time from Ω, i.e.

$$X^D(t) = \begin{cases} X(t) & \text{if } \ s \leq t \leq \tau, \\ X(\tau) & \text{if } \ t \geq \tau, \end{cases} \tag{3.25}$$

has

$$P^D(x,t,B,s) = \int_B G^D(x,t,y,s) dy \tag{3.26}$$

as the transition function.

To prove that the Markov process (3.25) is actually a Markov–Feller process and to construct the Markov–Feller process associated with the oblique derivative boundary condition, we need a complement to Theorems 2.9 and 2.10.

Theorem 3.2 *Suppose (2.14),...,(2.19) with* $0 \leq \gamma < 2$, *(2.27),...,(2.30), (3.17) and*

$$b_i, b_0 \in C^{1+\alpha, \frac{1+\alpha}{2}}(\overline{\mathcal{R}}^d \times [0,T])\,, \tag{3.27}$$

hold true. Then the Green function $G(x,t,y,s)$ *associated with the parabolic second order integro–differential operator* $\mathcal{A}$ *and one of the boundary conditions (Dirichlet, Neumann or oblique derivative) given by the operator* $\mathcal{B}$*, has the representation*

$$\begin{cases} G(x,t,y,s) = G_L(x,t,y,s) + \int_s^t d\tau \int_\Omega G_L(x,t,\xi,\tau) Q(\xi,\tau,y,s) d\xi\,, \\ \textit{for some } Q \textit{ in } \mathcal{G}_{2-\gamma}^{\alpha,\frac{\alpha}{2}} \textit{ if } \gamma < 2-\alpha\ , \\ [\textit{or in } \mathcal{G}_{2-\gamma}^{\varepsilon,\frac{\varepsilon}{2}}, \textit{ if } 2-\alpha \leq \gamma < 2\,,\ \varepsilon \in (0, 2-\gamma)]\ , \end{cases} \tag{3.28}$$

where $G_L(x,t,y,s)$ is the Green function associated with the differential operator L and the corresponding boundary condition. Moreover, for $\gamma < 2-\alpha$, G enjoys the following properties:
(i) *$G(x,t,y,s)$ is smooth, i.e.*

$$G(\cdot,\cdot,y,s) \in C^{2+\alpha,\frac{2+\alpha}{2}}(\overline{\Omega}\times(s,T]) \quad \forall\,(y,s)\in\Omega\times[0,T)\,, \tag{3.29}$$

(ii) *for any smooth function $\varphi(x)$ satisfying the compatibility conditions*

$$\begin{cases} \varphi(x) = A(x,s,\partial_x)\varphi(x) = 0 & \forall\, x\in\partial\Omega \quad \textit{(Dirichlet)},\\ \mathcal{B}(x,s,\partial_x)\varphi(x) = 0 & \forall\, x\in\partial\Omega \quad \textit{(oblique)}, \end{cases} \tag{3.30}$$

for a fixed $s\in[0,T)$, the function u_s given by

$$u_s(x,t) = \int_\Omega G(x,t,y,s)\varphi(y)dy, \quad \forall\,(x,t)\in\overline{\Omega}\times(s,T] \tag{3.31}$$

is the classic solution to the problem

$$\begin{cases} u_s \in C^{2+\alpha,\frac{2+\alpha}{2}}(\overline{\Omega}\times[s,T]) \quad \textit{and}\\ \mathcal{A}u_s = 0 \quad \textit{in}\ \Omega\times(s,T],\\ u_s = \varphi \quad \textit{in}\ \overline{\Omega}\times\{s\},\\ \textit{either} \quad u_s = 0 \quad \textit{or} \quad \mathcal{B}u_s = 0 \quad \textit{in}\ \partial\Omega\times[s,T], \end{cases} \tag{3.32}$$

and the following estimate holds for a constant C_T independent of φ

$$\|u_s\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(\overline{\Omega}\times[s,T])} \;\le\; C_T\|\varphi\|_{C^{2+\alpha}(\overline{\Omega})}\,, \tag{3.33}$$

(iii) *for any continuous and bounded function $\varphi(x)$ and for a fixed $s\in[0,T)$, the function u_s given by (3.31) is smooth, say $\in C^{2+\alpha,\frac{2+\alpha}{2}}(\overline{\Omega}\times(s,T])$, and the following estimate holds for a constant C_T independent of φ*

$$\begin{cases} \delta^{1+\frac{\alpha'}{2}}\langle u_s\rangle_{C^{2+\alpha',\frac{2+\alpha'}{2}}(\overline{\Omega}\times[s+\delta,T])} + \delta^{\frac{1+\alpha'}{2}}\langle u_s\rangle_{C^{1+\alpha',\frac{1+\alpha'}{2}}(\overline{\Omega}\times[s+\delta,T])} + \\ \qquad + \delta^{\frac{\alpha'}{2}}\langle u_s\rangle_{C^{\alpha',\frac{\alpha'}{2}}(\overline{\Omega}\times[s+\delta,T])} \le C_T\|\varphi\|_{C^0(\overline{\Omega})}\,, \quad 0\le\alpha'\le\alpha\,, \end{cases} \tag{3.34}$$

(iv) *for any uniformly continuous and bounded function $\varphi(x)$ in $\overline{\Omega}$ the function u_s defined by (3.31) satisfies the limit condition*

$$\lim_{(t-s)\to 0} u_s(\cdot,t) = \varphi \quad \textit{uniformly in}\ \overline{\Omega}\,. \tag{3.35}$$

Properties (i),...,(iv) still hold if $2-\alpha \le \gamma < 2$, replacing everywhere α with any $\varepsilon \in (0, 2-\gamma)$.

Proof. Representation (3.28) is the crucial fact. It is proved in Garroni and Menaldi [7] for $0 \leq \gamma \leq 1$ and can be extended to this case (cf. Garroni and Menaldi [9]).

First let us show that the Green function $G(x,t,y,s)$ is smooth for $x \in \overline{\Omega}$, $0 \leq s < t \leq T$, $y \in \Omega$. Indeed, let $0 < 3\delta \leq t-s$ and consider the last term in (3.28) denoted by $(G_L \bullet Q)(x,t,y,s)$. We have

$$(G_L \bullet Q)(x,t,y,s) = v^\delta_{ys}(x,t) + w^\delta_{ys}(x,t)\,,$$

where

$$\begin{aligned} v^\delta_{ys}(x,t) &= \int_s^t \rho_\delta(\tau - s)d\tau \int_\Omega G_L(x,t,\xi,\tau)Q(\xi,\tau,y,s)d\xi\,, \\ w^\delta_{ys}(x,t) &= \int_s^t [1-\rho_\delta(\tau - s)]d\tau \int_\Omega G_L(x,t,\xi,\tau)Q(\xi,\tau,y,s)d\xi\,, \end{aligned}$$

and $\rho_\delta(\theta)$ is a smooth function which vanishes for $\theta \leq \delta$ and equals 1 for $\theta \geq 2\delta$. Since the function $v^\delta_{ys}(x,t)$ is the solution of the following parabolic second order differential equation (with either Dirichlet or oblique boundary condition)

$$\begin{cases} \mathcal{L}v^\delta_{ys}(x,t) = f^\delta_{ys}(x,t) \quad \forall\,(x,t) \in \overline{\Omega} \times (s+\delta, T]\,, \\ v^\delta_{ys}(x,s) = 0 \quad \forall\, x \in \overline{\Omega}\,, \\ \mathcal{B}v^\delta_{ys}(x,t) = 0 \quad \forall\, x \in \partial\Omega \times [s+\delta, T]\,, \end{cases}$$

where $f^\delta_{ys}(x,t) = Q(x,t,y,s)\rho_\delta(t-s)$ and $\mathcal{L} = \partial_t - L$, we deduce that $v^\delta_{ys}(x,t)$ is smooth, i.e. $v^\delta_{ys} \in C^{2+\alpha,\frac{2+\alpha}{2}}(\overline{\Omega}\times[s+\delta,T])$. On the other hand, since $1-\rho_\delta(\theta)$ vanishes for $\theta \geq 2\delta$ we can write

$$w^\delta_{ys}(x,t) = \int_s^{t-\delta} [1-\rho_\delta(\tau - s)]d\tau \int_\Omega G_L(x,t,\xi,\tau)Q(\xi,\tau,y,s)d\xi\,,$$

which implies that $w^\delta_{ys}(x,t)$ is smooth, i.e. $w^\delta_{ys} \in C^{2+\alpha,\frac{2+\alpha}{2}}(\overline{\Omega} \times [s+3\delta, T])$. By virtue of the expression $G = G_L + G_L \bullet Q$ we get the first property (i) of the Green function.

To establish the second property (ii) we proceed as follows. Since $\varphi(x)$ is a smooth function satisfying the compatibility conditions (3.30), we can define a function $\Phi_s(x,t)$ as the unique solution in $C^{2+\alpha,\frac{2+\alpha}{2}}(\overline{\Omega}\times[s,T])$ of the parabolic second order differential equation (with either Dirichlet or oblique boundary condition)

$$\begin{cases} (\partial_t - L)\Phi_s(x,t) = -L\varphi(x) \quad \forall\,(x,t) \in \overline{\Omega} \times (s,T]\,, \\ \Phi_s(x,s) = \varphi(x) \quad \forall\, x \in \overline{\Omega}\,, \\ \mathcal{B}\Phi_s(x,t) = 0 \quad \forall\, x \in \partial\Omega \times [s,T]\,, \end{cases}$$

which satisfies estimate (3.33) with Φ_s in lieu of u_s. Now, set $f_s(x,t) = L\varphi(x) + I\Phi_s(x,t)$ and let $v_s(x,t)$ be the solution in $C^{2+\alpha,\frac{2+\alpha}{2}}(\overline{\Omega} \times [s,T])$ of the parabolic second order integro–differential equation (with either Dirichlet or oblique boundary condition)

$$\begin{cases} (\partial_t - L - I)v_s(x,t) = f_s(x,t) \quad \forall\,(x,t) \in \overline{\Omega} \times (s,T]\,, \\ v_s(x,s) = 0 \quad \forall\, x \in \overline{\Omega}\,, \\ \mathcal{B}v_s(x,t) = 0 \quad \forall\, x \in \partial\Omega \times [s,T]\,, \end{cases}$$

which can be solved by virtue of the compatibility condition (3.30). The following estimate holds (cf. Theorems 2.9 and 2.10)

$$\|v_s\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(\overline{\Omega}\times[s,T])} \le C_T \|L\varphi + I\Phi_s\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{\Omega}\times[s,T])}\,.$$

Since $\Phi_s(x,t) + v_s(x,t)$ solves equation (3.32), we obtain estimate (3.33). The fact that the function $u_s(x,t)$ given by (3.31) is a classic solution to problem (3.32) follows from the regularity and properties of the Green function. Thus, the uniqueness provided by the Maximum Principle (cf. Garroni and Menaldi [10]) completes the argument.

In order to prove the third property (iii) we need to show only the estimate (3.34). To that purpose, we notice that the function $u_s^L(x,t)$ defined by

$$u_s^L(x,t) = \int_\Omega G_L(x,t,y,s)\varphi(y)dy \tag{3.36}$$

enjoys the estimate (3.34) (with u_s^L in lieu of u_s). Thus we consider the function

$$u_s^Q(x,t) = \int_s^t d\tau \int_\Omega G_L(x,t,\xi,\tau)d\xi \int_\Omega Q(\xi,\tau,y,s)\varphi(y)dy\,. \tag{3.37}$$

Since

$$\partial_{xt}^l u_s^Q(x,t) = \int_s^t d\tau \int_\Omega \partial_{xt}^l G_L(x,t,\xi,\tau)d\xi \int_\Omega Q(\xi,\tau,y,s)\varphi(y)dy\,.$$

for $l = 0, 1$ and $Q \in \mathcal{G}_{2-\gamma}^{\alpha,\frac{\alpha}{2}}$ we get estimate (3.34), with the exception of the second order parts. For these second order part estimates we proceed as in the prove of (i). We express the function $u_s^Q(x,t)$ in the form

$$u_s^Q(x,t) = v_s^\delta(x,t) + w_s^\delta(x,t)\,,$$

where

$$\begin{aligned} v_s^\delta(x,t) &= \int_s^t \rho(\frac{\tau - s}{\delta})d\tau \int_\Omega G_L(x,t,\xi,\tau)d\xi \int_\Omega Q(\xi,\tau,y,s)\varphi(y)dy\,, \\ w_s^\delta(x,t) &= \int_s^t [1 - \rho(\frac{\tau - s}{\delta})]d\tau \int_\Omega G_L(x,t,\xi,\tau)d\xi \int_\Omega Q(\xi,\tau,y,s)\varphi(y)dy\,, \end{aligned}$$

and $\rho(\theta)$ is a smooth function satisfying

$$\rho(\theta) = \begin{cases} 0 & \forall\,\theta \in [0, \frac{1}{3}]\,, \\ (3\theta - 1) & \forall\,\theta \in [\frac{1}{3}, \frac{2}{3}]\,, \\ 1 & \forall\,\theta \in [\frac{2}{3}, 1]\,. \end{cases}$$

Since the function $\rho(\frac{\tau - s}{\delta})$ vanishes for $\tau \le s + \delta/3$ we deduce that the function v_s^δ is the solution of a parabolic second order differential equation, namely

$$\begin{cases} v_s^\delta \in C^{2+\alpha,\frac{2+\alpha}{2}}(\overline{\Omega} \times [s,T]) \ \text{ and} \\ \mathcal{L}v_s^\delta = f_s^\delta \quad \text{in } \Omega \times (s,T], \\ v_s^\delta(\cdot,s) = 0 \quad \text{in } \overline{\Omega}, \\ \mathcal{B}v_s^\delta = 0 \quad \text{in } \partial\Omega \times [s,T]\,, \end{cases}$$

where

$$f_s^\delta(x,t) = \rho(\frac{t - s}{\delta}) \int_\Omega Q(x,t,y,s)\varphi(y)dy\,.$$

Thus, assuming $\gamma \ge 1$ without loss of generality, we have for $t - s \ge \delta$

$$\begin{aligned} |f_s^\delta(x,t)| &\le \rho(\frac{t - s}{\delta})K(Q, 2 - \gamma)(t - s)^{-1+(2-\gamma)/2}\|\varphi\|_{C^0(\overline{\Omega})} \le \\ &\le C\delta^{-1}\|\varphi\|_{C^0(\overline{\Omega})} \end{aligned}$$

and

$$\begin{aligned} |f_s^\delta(x,t) - f_s^\delta(x',t')| \le \rho(\frac{t - s}{\delta})N(Q, 2 - \gamma, \alpha)(t \wedge t' - s)^{-1+(2-\gamma-\alpha)/2} \times \\ \times\|\varphi\|_{C^0(\overline{\Omega})}(|x - x'|^\alpha + |t - t'|^{\alpha/2}) + |\rho(\frac{t - s}{\delta}) - \rho(\frac{t' - s}{\delta})| \times \\ \times K(Q, 2 - \gamma)(t' - s)^{-1+(2-\gamma)/2}\|\varphi\|_{C^0(\overline{\Omega})} \le \\ \le C\delta^{-1-\alpha/2}\|\varphi\|_{C^0(\overline{\Omega})}(|x - x'|^\alpha + |t - t'|^{\alpha/2})\,, \end{aligned}$$

for any $x, x' \in \overline{\Omega}$, $t, t' \in [s+\delta, T]$, and some constant C independent of φ. Next, in view of global Schauder's estimates, we obtain the estimate (3.34) corresponding to the second order for the function v_s^δ instead of u_s. Now, for the function $w_s^\delta(x,t)$ we remark that the integral is not singular (in view of the function ρ), and that for $t - s \geq \delta$

$$\begin{aligned} \partial^2_{xt} w_s^\delta(x,t) &= \int_s^{t-3\delta} [1 - \rho(\frac{\tau - s}{\delta})] d\tau \int_\Omega \partial^2_{xt} G_L(x,t,\xi,\tau) d\xi \\ & \int_\Omega Q(\xi,\tau,y,s)\varphi(y) dy \ . \end{aligned}$$

Hence, the pointwise estimates of the heat-hernel type on the Green function G_L (cf. Ladyzenskaja et al. [18]) complete the proof.

The last property (iv) can be obtained from the representation $G = G_L + G_L \bullet Q$. Indeed, the function u_s^L given by (3.36) enjoys the limit condition (3.35). On the other hand, the function u_s^Q given by (3.37) satisfies for $\gamma \geq 1$

$$\begin{aligned} |u_s^Q(x,t)| &\leq K(G_0,2)K(Q,2-\gamma)\|\varphi\|_{C^0(\overline{\Omega})} \int_s^t (\tau - s)^{-1+(2-\gamma)/2} \leq \\ &\leq C\|\varphi\|_{C^0(\overline{\Omega})}(t-s)^{(2-\gamma)/2} \,, \end{aligned}$$

for some constant C independent of φ. Thus the limit condition (3.35) follows.

The last statement, regarding the case $2 - \alpha \leq \gamma < 2$, is immediately obtained by running trough of the different stages of dimonstration and replacing everywhere α with $\varepsilon \in (0, 2-\gamma)$. □

Now, we can define the semigroup

$$\Phi^D(t,s)\varphi(x) = \int_\Omega G^D(x,t,y,s)\varphi(y) dy. \tag{3.38}$$

Theorem 3.2 proves that $\Phi^D(t,s)$ is indeed a continuous semigroup, on $C_0(\Omega)$ (i.e. the space of continuous functions on $\overline{\Omega}$ vanishing at infinity and on $\partial\Omega$). Then, there exists a unique Markov–Feller process $(X^D(t), t \geq 0)$ under a probability measure P^D on the sample space $D([0,\infty),\Omega)$ with transition probability function (3.26).

The semigroup (3.38) admits the following representations:

$$\Phi^D(t,s)\varphi(x) = E^D\{\varphi(X^D(t)) \mid X^D(s) = x\}$$

and

$$\Phi^D(t,s)\varphi(x) = E\{\varphi(X(t \wedge \tau)) \mid X(s) = x\} \tag{3.39}$$

where τ is given by (3.21).

Thus, the stopped Markov process defined by (3.25) is a Markov–Feller process. Notice that the Feller character of $(X^D(t),\ t \geq 0)$ follows from Theorem 3.2, but this property can be proved independently by studying the functional τ defined by (3.21) and by proving that τ is continuous $P-$ a.s. This proof involves the use of barrier functions (cf. Bensoussan and Lions [3]).

Regarding the construction of the Markov–Feller process associated with the oblique derivatives boundary conditions, the references are scarce.

As a direct consequence of Theorem 3.2, we have

Theorem 3.3 *Let us assume (2.14),…,(2.19) with* $0 \leq \gamma < 2$, *(2.27),…, (2.32), (3.17), (3.27), and*

$$\begin{cases} a_0(x,t) &= 0, \quad \forall (x,t) \in \overline{\Omega} \times [0,T], \\ b_0(x,t) &= 0, \quad \forall (x,t) \in \partial\Omega \times [0,T] \end{cases} \tag{3.40}$$

hold true. Then there exists a Markov-Feller process $(X^b(t), t \geq 0)$ *under a probability measure* P^b *on the sample space* $D([0,\infty), \overline{\Omega})$ *with transition density function* $G^b(x,t,y,s)$. *The process is unique if the initial distribution is prescribed.* □

This theorem can be regarded as a generalization of the construction of reflected diffusion processes with jumps reported on Anulova [1,2], Chaleyat–Maurel et al. [5], Menaldi and Robin [24].

Remark 3.4 *If we drop the conditions on the Hölder continuity of zero and first order coefficients of* L *and on the Hölder continuous properties on the coefficients of the proper integral operator* I, *then the above theorems remain true. The only difference is that in Theorem 3.2 the properties of the Green function should be understood in a "weak" sense, i.e. the spaces* $C^{2+\alpha,\frac{2+\alpha}{2}}$ *and* C^0 *should be replaced by* $W_p^{2,1}$ *and* L^p *respectively. For instance, if* φ *belongs to* L^p *then* $u_s(\cdot,t)$ *belongs to* W_p^2 *for* $t > s$. *Consequently the transition functions are now the weak Green functions. This holds for* $0 \leq \gamma < 2$. *Moreover, we can drop the regularity assumption on the coefficients* b_i, b_0 *[only Hölder continuous coefficients for the boundary operator* $\mathcal{B}$*] and we still obtain the Theorem 3.3 for* $\gamma \leq 1$. *The transition function* $G^b(x,t,y,s)$, *however, has some explosion near the boundary for the second order derivatives with respect to* x *and the first order derivative with respect to* t *(cf. Garroni and Menaldi [7]).* □

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