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A Distributed Parabolic Control with Mixed Boundary Conditions

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Abstract

We study the asymptotic behavior of an optimal distributed control problem where the state is given by the heat equation with mixed boundary conditions. The parameter α intervenes in the Robin boundary condition and it represents the heat transfer coefficient on a portion Γ_1 of the boundary of a given regular n-dimensional domain. For each α, the distributed parabolic control problem optimizes the internal energy g. It is proven that the optimal control $\hat{g}_\alpha$ with optimal state $u_{\hat{g}_\alpha \alpha}$ and optimal adjoint state $p_{\hat{g}_\alpha \alpha}$ are convergent as $\alpha \to \infty$ (in norm of a suitable Sobolev parabolic space) to $\hat{g}$, $u_{\hat{g}}$ and $p_{\hat{g}}$, respectively, where the limit problem has Dirichlet (instead of Robin) boundary conditions on Γ_1. The main techniques used are derived from the parabolic variational inequality theory.

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1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ with a regular boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$, which is the union of two essentially disjoint (and regular) portions Γ_1 and Γ_2, where Γ_1 has a positive $(n-1)$-Hausdorff measure. Also suppose given a time interval $[0, T]$, for some $T > 0$. Consider the following two-state evolution heat conduction problems with mixed boundary conditions,

$$\partial_t u - \Delta u = g \text{ in } \Omega, \qquad u\big|_{\Gamma_1} = b, \qquad -\partial_n u\big|_{\Gamma_2} = q, \tag{1.1}$$

and, for a parameter $\alpha > 0$,

$$\partial_t u - \Delta u = g \text{ in } \Omega, \qquad -\partial_n u\big|_{\Gamma_1} = \alpha(u - b), \qquad -\partial_n u\big|_{\Gamma_2} = q, \tag{1.2}$$

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both with an initial condition

$$u(0) = v_b, \tag{1.3}$$

where g is the internal energy in Ω, b is the temperature (of the external neighborhood) on Γ_1 for (1.1) (for (1.2)), q is the heat flux on Γ_2 and α is the heat transfer coefficient of Γ_1 (Newton's law on Γ_1). All data, g, q, b, v_b and the domain Ω with the boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$ are assumed to be sufficiently smooth so that the problems (1.1) and (1.2) admit variational solutions in Sobolev spaces.

The data b, v_b and q are fixed, sufficiently smooth and satisfy the compatibility condition $v_b = b$ on Γ_1, while g is taken as a control variable in $L^2\big(0,T;L^2(\Omega)\big)$, and α as a (singular) parameter destined to approaches infinite. Thus, denote by u_g and $u_{g\alpha}$ the solution of (1.1) and (1.2), respectively, with the initial condition (1.3) in the following standard variational form

$$\begin{cases} u_g - v_b \in L^2(0,T;V_0), \quad u_g(0) = v_b \quad \text{and} \quad \dot{u}_g \in L^2(0,T;V_0') \\ \text{such that} \quad \langle \dot{u}_g(t), v\rangle + a(u_g(t), v) = L_g(t,v), \quad \forall v \in V_0, \end{cases} \tag{1.4}$$

and

$$\begin{cases} u_{g\alpha} \in L^2(0,T;V), \quad u_{g\alpha}(0) = v_b \quad \text{and} \quad \dot{u}_{g\alpha} \in L^2(0,T;V') \\ \text{such that} \quad \langle \dot{u}_{g\alpha}(t), v\rangle + a_\alpha(u_{g\alpha}(t), v) = L_{g\alpha}(t,v), \quad \forall v \in V, \end{cases} \tag{1.5}$$

where

$$\begin{aligned} & V_0 := \{v \in H^1(\Omega) : v\big|_{\Gamma_1} = 0\}, \\ & H := L^2(\Omega), \qquad (g,h)_H := \int_\Omega gh \, \mathrm{d}x, \\ & L_g(t,v) := (g(t), v)_H - \int_{\Gamma_2} q(t) v \, \mathrm{d}\gamma, \\ & a(u,v) := \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x, \\ & a_\alpha(u,v) := a(u,v) + \alpha \int_{\Gamma_1} uv \, \mathrm{d}\gamma, \\ & L_{g\alpha}(t,v) := L_g(t,v) + \alpha \int_{\Gamma_1} bv \, \mathrm{d}\gamma, \end{aligned} \tag{1.6}$$

and $\langle \cdot, \cdot \rangle$ denotes the duality bracket. Note that the dual space V_0' (and V') of V_0 (and V) is not an space of distributions, since $\mathcal{D}(\Omega)$ is not dense in $V_0 \subset V$, due to the non-zero boundary conditions on Γ_2. The norm in V_0 is given by $v \mapsto \|\nabla v\|_H$, while the norm in V is $(\|v\|_H^2 + \|\nabla v\|_H^2)^{1/2}$. Nevertheless, $v \mapsto L_g(t,v)$ and $v \mapsto L_{g\alpha}(t,v)$ are linear continuous functional satisfying

$$\begin{aligned} & \|L_g(t,\cdot)\|_{V_0'} \leq \|g(t)\|_{V_0'} + \|q(t)\|_{H^{-1/2}(\Gamma_2)}, \quad \forall v \in V_0, \\ & \|L_{g\alpha}(t,\cdot)\|_{V} \leq \|g(t)\|_{V'} + \|q(t)\|_{H^{-1/2}(\Gamma_2)} + \alpha \|b\|_{H^{1/2}(\Gamma_1)}, \quad \forall v \in V, \end{aligned}$$

and $a(\cdot,\cdot)$ and $a_\alpha(\cdot,\cdot)$ are bilinear symmetric continuous forms on V_0 and V, respectively. Also, it is clear the compatibility assumption $v_b = b$ on Γ_1 and that if $b = 0$ then $L_g(t,\cdot) = L_{g,\alpha}(t,\cdot)$.

One should remark that an element u of $L^2(0,T;V)$ such that $\dot{u}$ belongs to $L^2(0,T;V')$ then u can be regarded as a continuous function from $[0,T]$ into H. This makes clear the meaning of the initial condition at $t = 0$ (and idem with V_0 replacing V).

On the space $\mathcal{H} := L^2(\Omega\times]0,T[)$ with norm $\|\cdot\|_{\mathcal{H}}$ and inner product $(\cdot,\cdot)_{\mathcal{H}}$, i.e.,

$$(u,v)_{\mathcal{H}} = \int_0^T \big(u(t),v(t)\big)_H \,\mathrm{d}t, \quad \forall u,v \in \mathcal{H},$$

consider the nonnegative functional costs J and J_α, defined by the expressions

$$J(g) := \frac{1}{2}\|u_g - z_d\|^2_{\mathcal{H}} + \frac{m}{2}\|g\|^2_{\mathcal{H}}, \tag{1.7}$$

and

$$J_\alpha(g) := \frac{1}{2}\|u_{g\alpha} - z_d\|^2_{\mathcal{H}} + \frac{m}{2}\|g\|^2_{\mathcal{H}}, \tag{1.8}$$

where z_d is a given element in $\mathcal{H} = L^2(\Omega\times]0,T[)$ and m is a strictly positive constant.

Our interest is on the distributed parabolic (or evolution) optimal control problems

$$\text{Find} \quad \hat{g} \quad \text{such that} \quad J(\hat{g}) \le J(g), \quad \forall g \in \mathcal{H} \tag{1.9}$$

and

$$\text{Find} \quad \hat{g}_\alpha \quad \text{such that} \quad J_\alpha(\hat{g}_\alpha) \le J_\alpha(g), \quad \forall g \in \mathcal{H}, \tag{1.10}$$

as well as the asymptotic behavior as the parameter α approaches infinite.

This type of optimal distributed control problems have been extensively studied, e.g., see the book Lions [10] among others. As point out early, our interest is the convergence as $\alpha \to \infty$, a parabolic version of Gariboldi and Tarzia [8], which is related to Ben Belgacem et al. [4] and Tabacman and Tarzia [11].

2 Parabolic Equations with Mixed Conditions

Note that if via Riesz' representation $H = H'$ then one has $V \subset H \subset V'$ and $V_0 \subset H \subset V_0'$ with a continuous and dense inclusion.

As mentioned early the control parameter g belongs to $\mathcal{H}$, and the data for the optimal control problems are z_d and m satisfying

$$z_d \in \mathcal{H} = L^2\big(0,T;L^2(\Omega)\big), \quad \text{and} \quad m > 0. \tag{2.1}$$

The regularity of the domain Ω, the boundary $\Gamma_1 \cup \Gamma_2$ and the regularity of the boundary data v_b, b and q are summarized on the assumption

$$\begin{aligned} &\text{there exists} \quad \psi \in L^2\big(0,T;H^2(\Omega)\big) \quad \text{with} \quad \dot{\psi} \in L^2\big(0,T;L^2(\Omega)\big) \\ &\text{such that} \quad \psi(0) = v_b, \quad \psi\big|_{\Gamma_1} = b, \quad \partial_n \psi\big|_{\Gamma_1} = 0, \quad -\partial_n \psi\big|_{\Gamma_2} = q, \end{aligned} \tag{2.2}$$

with the standard notation of Sobolev and Lebesgue spaces and the compatibility assumption $v_b = b$ on Γ_1. Note the over conditioning for ψ on Γ_1, which is not necessary but convenient in some way (e.g., the adjoint state has a very similar equation with homogeneous boundary conditions).

Thus, the change of unknown function u into $u - \psi$ reduces to analysis the case where the boundary data v_b, b and q are all zero, and g is replaced by $g - (\partial_t - \Delta)\psi$. However, for $\alpha > 0$ a new term appears, namely,

$$\langle g_\psi(t), v\rangle = (g(t), v)_H + \int_{\Gamma_1} v \partial_n \psi(t)\, \mathrm{d}\gamma, \quad \forall v \in V, \tag{2.3}$$

i.e., the new Robin boundary condition is non-homogeneous and

$$\|g_\psi(t)\|_{V'} = \sup_{\|v\|_V \le 1} \big|\langle g_\psi(t), v\rangle\big| \le \|g(t)\|_{L^2(\Omega)} + \|\partial_n \psi(t)\|_{H^{-1/2}(\Gamma_1)}.$$

Thus, because of the over conditioning on Γ_1 one has $g_\psi = g$. Anyway, both problems, (1.4) and (1.5) become

$$\begin{cases} u_g \in L^2(0,T;V_0), \quad \text{with} \quad u_g(0) = 0 \quad \text{and} \quad \dot{u}_g \in L^2(0,T;V_0') \\ \text{such that} \quad \langle \dot{u}_g(t), v\rangle + a(u_g(t), v) = (g(t), v)_H, \quad \forall v \in V_0 \end{cases} \tag{2.4}$$

and

$$\begin{cases} u_{g\alpha} \in L^2(0,T;V), \quad \text{with} \quad u_{g\alpha}(0) = 0 \quad \text{and} \quad \dot{u}_{g\alpha} \in L^2(0,T;V') \\ \text{such that} \quad \langle \dot{u}_{g\alpha}(t), v\rangle + a_\alpha\big(u_{g\alpha}(t), v\big) = (g(t), v)_H, \quad \forall v \in V, \end{cases} \tag{2.5}$$

where $(\cdot,\cdot)_H$, $a(\cdot,\cdot)$ and $a_\alpha(\cdot,\cdot)$ are as in (1.6). Again $V_0 \subset V$ with inclusion continuous but not dense, so that V' is not identifiable with a subset of V_0'. However, by Hahn-Banach Theorem, any element in V_0' can be extended to an element in V' preserving its norm.

Recall that for any element u in $L^2(0,T;V)$ with $\dot{u}$ in $L^2(0,T;V')$ such that the distribution $(\partial_t - \Delta)u$ belongs to $L^2(\Omega\times]0,T[)$ one can integrate by parts to interpret $\partial_n u$ as an element in $L^2\big(0,T;H^{-1/2}(\partial\Omega)\big)$, where $H^{-1/2}(\partial\Omega)$ is the dual space of $H^{1/2}(\partial\Omega) = \gamma\big(H^1(\Omega)\big)$ and γ is the trace operator from $H^1(\Omega)$ onto $H^{1/2}(\partial\Omega)$. Again, to simplify the arguments, one may assume that $\partial\Omega = \Gamma_1 \cup \Gamma_2$ such that for any v_i in $H^{1/2}(\Gamma_i)$ there exists v in $H^1(\Omega)$ satisfying $v = v_i$ on Γ_i, for $i = 1, 2$, e.g., the two pieces of the boundary are strictly disjoint, $\Gamma_1 \cap \Gamma_2 = \emptyset$ (i.e., $\Gamma_i = \partial\Omega_i$ and $\overline{\Omega}_1 \subset \Omega_2$). Therefore, the parabolic equations (2.4) and (2.5) mean the following:

- *space of the solution*: u_g in $L^2(0,T;V_0)$ with $\dot{u}_g$ in $L^2(0,T;V_0')$, and $u_{g\alpha}$ in $L^2(0,T;V)$ with $\dot{u}_{g\alpha}$ in $L^2(0,T;V')$,
- *initial condition*: for either $u = u_g$ or $u = u_{g\alpha}$ the solution u belongs to $C^0\big(0,T;L^2(\Omega)\big)$ and so $u(0) = 0$ in $L^2(\Omega)$,
- *equation in* $\Omega\times]0,T[$: for either $u = u_g$ or $u = u_{g\alpha}$ the solution u is considered as a distribution so that $(\partial_t - \Delta)u = g$ in $\mathcal{D}'(\Omega\times]0,T[)$,
- *boundary condition on* Γ_2: for either $u = u_g$ or $u = u_{g\alpha}$ the trace of the solution u is defined and $\partial_n u = 0$ in $L^2\big(0,T;H^{-1/2}(\Gamma_2)\big)$,
- *boundary condition on* Γ_1: $u_g = 0$ in $L^2\big(0,T;H^{1/2}(\Gamma_1)\big)$ and $\partial_n u_{g\alpha} + \alpha u_{g\alpha} = 0$ in $L^2\big(0,T;H^{-1/2}(\Gamma_1)\big)$.

Firstly, note that $u_{g\alpha}\big|_{\Gamma_1}$ belongs to $L^2\big(0,T;H^{1/2}(\Gamma_1)\big)$ and

$$L^2\big(0,T;H^{1/2}(\Gamma_1)\big) \subset L^2\big(0,T;L^2(\Gamma_1)\big) \subset L^2\big(0,T;H^{-1/2}(\Gamma_1)\big),$$

with continuous and dense inclusion. Secondly, when comparing the solutions u_g and $u_{g\alpha}$ one has both in the larger space $L^2\big(0,T;V\big)$. However, the continuous inclusion $V_0 \subset V$ is not dense, and so the inclusion $V' \subset V_0'$ is not injective, one has $\dot{u}_g$ and $\dot{u}_{g\alpha}$ elements in $L^2\big(0,T;V_0'\big)$, which are not identifiable as distributions.

3 State and Adjoint State Equations

To study the optimal control problem (1.9), denote by u_0 the solution u_g of the parabolic variational equality either (1.4) or equivalently (2.4) corresponding to $g = 0$, and define the (linear) operator $C\colon \mathcal{H} \to L^2(0,T;V_0)$, given by $C(g) := u_g - u_0$. We have

Proposition 3.1. *With the previous notation and assumptions, the functional* (1.7) *can be expressed as*

$$J(g) = \frac{1}{2}\pi(g,g) - \ell(g) + \frac{1}{2}\|z_d - u_0\|^2_{\mathcal{H}}, \quad \forall g \in \mathcal{H},$$

where $\pi(g,h) := \big(C(g),C(h)\big)_{\mathcal{H}} + m(g,h)_{\mathcal{H}}$ *is a symmetric, continuous and coercive bilinear form on* $\mathcal{H}$ *and* $\ell(g) := \big(C(g), z_d - u_0\big)_{\mathcal{H}}$ *is a linear continuous functional on* $\mathcal{H}$*. Moreover,* J *is strictly convex and its Gateaux derivative is given by* $\langle J'(g),h\rangle = \big(u_g - z_d, C(g)\big)_{\mathcal{H}} + m(g,h)_{\mathcal{H}}$*. Furthermore, as a consequence, the optimal control problem* (1.9) *has a unique minimizer* $\hat{g}$ *in* $\mathcal{H}$*, i.e.,* $J(\hat{g}) \le J(g)$*, for every* g *in* $\mathcal{H}$*, any solution* $\bar{g}$ *of the equation* $J'(\bar{g}) = 0$ *is indeed a minimizer. Also, if* p_g *is the adjoint state defined by the parabolic variational equality with a terminal condition*

$$\begin{cases} p_g \in L^2(0,T;V_0), \quad \textit{with} \quad p_g(T) = 0 \quad \textit{and} \quad \dot{p}_g \in L^2(0,T;V_0') \\ \textit{such that} \quad -\langle \dot{p}_g(t), v\rangle + a(u_g(t),v) = (u_g - z_d, v)_H, \quad \forall v \in V_0, \end{cases} \tag{3.1}$$

then $J'(g) = mg + p_g$ *for every* g *in* $\mathcal{H}$ *and* $J'(\hat{g}) = m\hat{g} + p_{\hat{g}} = 0$.

Proof. Note the boundary conditions for the adjoint state p_g are

$$p_g(t) = 0 \quad \text{on} \quad \Gamma_1 \qquad \text{and} \qquad \partial_n p_g(t) = 0 \quad \text{on} \quad \Gamma_2.$$

for almost every t in $]0, T[$.

First, we check the expression of J, if $z'_d := z_d - u_0$ then

$$\begin{aligned} J(g) &= \frac{1}{2}\|C(g) - z'_d\|_{\mathcal{H}}^2 + \frac{m}{2}\|g\|_{\mathcal{H}}^2 = \\ &= \frac{1}{2}\Big[\|C(g)\|_{\mathcal{H}}^2 + \|z'_d\|_{\mathcal{H}}^2 - 2(C(g), z'_d)_{\mathcal{H}}\Big] + \frac{m}{2}\|g\|_{\mathcal{H}}^2 = \\ &= \frac{1}{2}\pi(g,g) - L(g) + \frac{1}{2}\|z_d - u_0\|_{\mathcal{H}}^2. \end{aligned}$$

To verify that $g \mapsto C(g)$ is a linear application, one checks that the function $r_1 u_{g_1} + r_2 u_{g_2} + (1 - r_1 - r_2)u_0$ is a solution of the parabolic variational equality (1.4) with $g = r_1 g_1 + r_2 g_2$, for every real numbers r_1, r_2; and by uniqueness one has

$$u_{r_1 g_1 + r_2 g_2} = r_1 u_{g_1} + r_2 u_{g_2} + (1 - r_1 - r_2)u_0, \tag{3.2}$$

for every r_i, r_2 in $\mathbb{R}$ and g_1, g_2 in $\mathcal{H}$. Hence,

$$\begin{aligned} C(r_1 g_1 + r_2 g_2) &= u_{r_1 g_1 + r_2 g_2} - u_0 = r_1 u_{g_1} + r_2 u_{g_2} + (1 - r_1 - r_2)u_0 - u_0 = \\ &= r_1(u_{g_1} - u_0) + r_2(u_{g_2} - u_0) = r_1 C(g_1) + r_2 C(g_2), \end{aligned}$$

i.e., the operator C is linear.

Now to check the continuity of C, we note that since Γ_1 has positive measure, Poincaré inequality implies that the bilinear form $a(\cdot,\cdot)$ is coercive on V_0, i.e., there exists $\lambda_0 > 0$ such that

$$a(v, v) \geq \lambda_0 \|\nabla v\|_H^2, \quad \forall v \in V_0. \tag{3.3}$$

We have

$$(\dot{u}_g(t) - \dot{u}_0(t), v)_H + a(u_g(t) - u_0(t), v) = (g(t), v)_H, \quad \forall v \in V_0,$$

and, in particular, for $v = u_g(t) - u_0(t)$,

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|u_g(t) - u_0(t)\|_H^2\Big) + \lambda_0\big\|\nabla\big(u_g(t) - u_0(t)\big)\big\|_H^2 \leq \\ &\qquad \leq \big(g(t), u_g(t) - u_0(t)\big)_H \leq \frac{1}{2\lambda_0}\|g(t)\|_{V'_0}^2 + \frac{\lambda_0}{2}\|\nabla\big(u_g(t) - u_0(t)\big)\|_{V_0}^2, \end{aligned}$$

where the dual norm is given by

$$\|v\|_{V'_0}^2 = \sup\big\{(v, \varphi)_H : \varphi \in V_0,\ \|\varphi\|_{V_0} \leq 1\big\}.$$

This yields

$$\|\nabla C(g)\|_{\mathcal{H}} \leq \frac{1}{\lambda_0}\Big[\int_0^T \|g(t)\|_{V_0'}^2 \,\mathrm{d}t\Big]^{1/2},$$

$$\sup_{0\leq t\leq T} \|C(g)(t)\|_H \leq \frac{1}{\sqrt{\lambda_0}}\Big[\int_0^T \|g(t)\|_{V_0'}^2 \,\mathrm{d}t\Big]^{1/2},$$

and going back to the equation, we get

$$\Big[\int_0^T \Big\|\frac{\mathrm{d}}{\mathrm{d}t}\big(C(g)(t)\big)\Big\|_{V_0'}^2 \,\mathrm{d}t\Big]^{1/2} \leq \frac{2}{\lambda_0}\Big[\int_0^T \|g(t)\|_{V_0'}^2 \,\mathrm{d}t\Big]^{1/2}$$

Hence the operator

$$C\colon L^2\big(0,T;V_0'\big) \to \big\{v \in L^2(0,T;V_0) \cap L^\infty(0,T;H) : \dot v \in L^2(0,T;V_0')\big\}$$

is actually continuous. As a consequence, the bilinear form $\pi(\cdot,\cdot)$ is symmetric, continuous and coercive on $\mathcal{H}\times\mathcal{H}$, since $\mathcal{H} \subset L^2\big(0,T;V_0'\big)$.

To complete the argument, we choose $v = C(h)$ in (3.1) and $v = p_g$ in (1.4) with $g = 0$ and $g = h$ to obtain, after integrating in t, the equalities

$$-\big(\dot p_g, C(h)\big)_{\mathcal{H}} + \int_0^T a\big(p_g(t), C(h)(t)\big)\,\mathrm{d}t = \big(u_g - z_d, C(h)\big)_{\mathcal{H}}$$

and

$$\big(\dot u_h - \dot u_0, p_g\big)_{\mathcal{H}} + \int_0^T a\big(u_h(t) - u_0(t), p_g(t)\big)\,\mathrm{d}t = (h, p_g)_{\mathcal{H}}.$$

Thus

$$-\int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\big(p_g(t), C(h)(t)\big)_H \,\mathrm{d}t + (h,p_g)_{\mathcal{H}} = \big(u_g - z_d, C(h)\big)_{\mathcal{H}},$$

and because $p_g(T) = 0$ and $C(h)(0) = 0$, we deduce $J'(g) = mg + p_g$.

To show that $g \mapsto J(g)$ is strictly convex, one makes use of (1.7) and (3.2) to check that

$$(1-\theta)J(g_2) + \theta J(g_1) - J\big((1-\theta)g_1 + \theta g_2\big) =$$
$$= \frac{1}{2}\theta(1-\theta)\big[\|u_{g_1} - u_{g_2}\|_{\mathcal{H}}^2 + m\|g_1 - g_2\|_{\mathcal{H}}^2\big],$$

for every θ in $[0,1]$ and any g_1, g_2 in $\mathcal{H}$. □

Similarly, to study the optimal control problem (1.10), denote by $u_{0\alpha}$ the solution $u_{g\alpha}$ of the parabolic variational equality either (1.5) or equivalently (2.5) corresponding to $g = 0$, and define the (linear) operator $C_\alpha\colon \mathcal{H} \to L^2(0,T;V)$, given by $C_\alpha(g) := u_{g\alpha} - u_{0\alpha}$. We have

Proposition 3.2. *With the previous notation and assumptions, the functional (1.8) can be expressed as*

$$J_\alpha(g) = \frac{1}{2}\pi_\alpha(g,g) - \ell_\alpha(g) + \frac{1}{2}\|z_d - u_{0\alpha}\|^2_{\mathcal{H}}, \quad \forall g \in \mathcal{H},$$

where $\pi_\alpha(g,h) := \big(C_\alpha(g), C_\alpha(h)\big)_{\mathcal{H}} + m(g,h)_{\mathcal{H}}$ *is a symmetric, continuous and coercive bilinear form on* $\mathcal{H}$ *and* $\ell_\alpha(g) := \big(C_\alpha(g), z_d - u_{0\alpha}\big)_{\mathcal{H}}$ *is a linear continuous functional on* $\mathcal{H}$. *Moreover,* J_α *is strictly convex and its Gateaux derivative of* J_α *is given by* $\langle J'_\alpha(g), h\rangle = \big(u_g - z_d, C_\alpha(g)\big)_{\mathcal{H}} + m(g,h)_{\mathcal{H}}$. *Furthermore, as a consequence, the optimal control problem (1.10) has a unique minimizer* $\hat{g}_\alpha$ *in* $\mathcal{H}$, *i.e.,* $J_\alpha(\hat{g}_\alpha) \leq J_\alpha(g)$, *for every* g *in* $\mathcal{H}$, *and any solution* $\bar{g}_\alpha$ *of the equation* $J'(\bar{g}_\alpha) = 0$ *is indeed a minimizer. Also if* $p_{g\alpha}$ *is the adjoint state defined by the parabolic variational equality with a terminal condition*

$$\begin{cases} p_{g\alpha} \in L^2(0,T;V), \quad \text{with} \quad p_{g\alpha}(T) = 0 \quad \text{and} \quad \dot{p}_{g\alpha} \in L^2(0,T;V') \\ \text{such that } -\langle \dot{p}_{g\alpha}(t), v\rangle + a_\alpha(p_{g\alpha}(t), v) = (u_{g\alpha} - z_d, v)_H, \ \forall v \in V, \end{cases} \tag{3.4}$$

then $J'_\alpha(g) = mg_\alpha + p_{g_\alpha}$ *for every* g *in* $\mathcal{H}$ *and* $J'_\alpha(\hat{g}_\alpha) = m\hat{g}_\alpha + p_{\hat{g}_\alpha} = 0$.

Proof. The calculations are similar to the previous proposition. We remark that the boundary conditions for the adjoint state $p_{g\alpha}$ are

$$-\partial_n p_{g\alpha}(t) = \alpha p_{g\alpha} \quad \text{on} \quad \Gamma_1 \qquad \text{and} \qquad \partial_n p_{g\alpha}(t) = 0 \quad \text{on} \quad \Gamma_2.$$

for almost every t in $]0,T[$. Moreover, we assume $\alpha > 0$ so that the coerciveness (3.3) becomes

$$a_\alpha(v,v) \geq \lambda_1 \min\{1,\alpha\}\big[\|\nabla v\|^2_H + \|v\|^2_H\big], \quad \forall v \in V, \tag{3.5}$$

Indeed, by contradiction one can show that $a_1(v,v) \geq c_1\|v\|^2_H$ for every v in V, which implies (3.5). The continuity of $a(\cdot,\cdot)$ in V uses the continuity of the trace in $H^1(\Omega)$, namely, for some $\Lambda_1 > 0$ one has

$$a_\alpha(u,v) \leq \Lambda_1 \max\{1,\alpha\}\|u\|_V\|v\|_V, \quad \forall v \in V, \tag{3.6}$$

which depends on $\alpha > 0$.

The operator C_α actually maps the space $L^2(0,T;V')$ into the space

$$\big\{v \in L^2(0,T;V) \cap L^\infty(0,T;H) : \dot{v} \in L^2(0,T;V')\big\}$$

and the estimates

$$\|\nabla C_\alpha(g)\|_{\mathcal{H}} \leq \frac{1}{\lambda_1}\Big[\int_0^T \|g(t)\|^2_{V'}\,\mathrm{d}t\Big]^{1/2},$$

$$\sup_{0\leq t\leq T}\|C_\alpha(g)(t)\|_H \leq \frac{1}{\sqrt{\lambda_1}}\Big[\int_0^T \|g(t)\|^2_{V'}\,\mathrm{d}t\Big]^{1/2},$$

$$\Big[\int_0^T \|\frac{\mathrm{d}}{\mathrm{d}t}(C_\alpha(g)(t))\|^2_{V'_0}\,\mathrm{d}t\Big]^{1/2} \leq \frac{2}{\lambda_1}\Big[\int_0^T \|g(t)\|^2_{V'}\,\mathrm{d}t\Big]^{1/2}$$

are independent of $\alpha > 1$, but

$$\Big[\int_0^T \|\frac{\mathrm{d}}{\mathrm{d}t}\big(C_\alpha(g)(t)\big)\|_{V'}^2\,\mathrm{d}t\Big]^{1/2} \le \frac{1+\alpha}{\lambda_1}\Big[\int_0^T \|g(t)\|_{V'}^2\,\mathrm{d}t\Big]^{1/2}$$

is depends on α. Certainly, also one deduces

$$\alpha\int_0^T |C_\alpha(g)(t)|_{L^2(\Gamma_1)}^2\,\mathrm{d}t \le \|g\|_{L^2(0,T;V')}\|C_\alpha(g)\|_{L^2(0,T;V)},$$

which is uniformly bounded in $\alpha > 1$. On the other hand, note that the functions b and q (or ψ) intervene to estimate $u_{0\alpha}$ and $\dot{u}_{0\alpha}$.

To show that $g \mapsto J_\alpha(g)$ is strictly convex, one show that

$$\begin{aligned}(1-\theta)J_\alpha(g_2) + \theta J_\alpha(g_1) - J_\alpha\big((1-\theta)g_1 + \theta g_2\big) &= \\ = \frac{1}{2}\theta(1-\theta)\big[\|u_{g_1\alpha} - u_{g_2\alpha}\|_{\mathcal{H}}^2 + m\|g_1 - g_2\|_{\mathcal{H}}^2\big],\end{aligned}$$

for every θ in $[0,1]$ and any g_1, g_2 in $\mathcal{H}$. □

Remark that one has nice estimates for the affine application $g \mapsto u_{g\alpha}$, namely

$$\|\nabla u_{g_1\alpha} - \nabla u_{g_2\alpha}\|_{\mathcal{H}} \le \frac{1}{\lambda_1}\|g_1 - g_2\|_{L^2(0,T;V')},$$
$$\sup_{0\le t\le T}\|u_{g_1\alpha}(t) - u_{g_2\alpha}(t)\|_H \le \frac{1}{\sqrt{\lambda_1}}\|g_1 - g_2\|_{L^2(0,T;V')},$$
$$\|\dot{u}_{g_1\alpha} - \dot{u}_{g_2\alpha}\|_{L^2(0,T;V_0')} \le \frac{2}{\lambda_1}\|g_1 - g_2\|_{L^2(0,T;V')},$$
$$\|\dot{u}_{g_1\alpha} - \dot{u}_{g_2\alpha}\|_{L^2(0,T;V')} \le \frac{1+\alpha}{\lambda_1}\|g_1 - g_2\|_{L^2(0,T;V')},$$
$$\|u_{g_1\alpha} - u_{g_2\alpha}\|_{L^2(0,T;L^2(\Gamma_1))} \le \frac{1}{\sqrt{\lambda_1\alpha}}\|g_1 - g_2\|_{L^2(0,T;V')},$$

and similarly, for the adjoint state mapping $g \mapsto p_{g\alpha}$, one obtain estimates as above replacing $u_{g_i\alpha}$ with $p_{g_i\alpha}$.

On the other hand, $u_{g_1\alpha} - u_{g_2\alpha}$ is the unique solution of a parabolic variational equality (1.5) with $q = 0$, $b = 0$ and $g = g_1 - g_2$, i.e., $(\partial_t - \Delta)(u_{g_1\alpha} - u_{g_2\alpha}) = g$ in $L^2(\Omega\times]0,T[)$ with homogeneous mixed (Robin on Γ_1 and Neumann on Γ_2) boundary conditions. Hence, regularity results implies that $u_{g_1\alpha} - u_{g_2\alpha}$ belongs to $L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega))$. Similar arguments apply to $u_{g_1} - u_{g_2}$, i.e., $(\partial_t - \Delta)(u_{g_1} - u_{g_2}) = g$ in $L^2(\Omega\times]0,T[)$ with homogeneous mixed (Dirichlet on Γ_1 and Neumann on Γ_2) boundary conditions. Note that some difficulties due to the mixed boundary conditions do arrives, e.g., see Grisvard [9], but our interest is on the asymptotic behavior as α becomes infinite.

4 Asymptotic Estimates

First one needs to obtain estimates on $u_{g\alpha}$ and $p_{g\alpha}$ uniformly in $\alpha > 1$ and any given g.

Proposition 4.1. *Under the previous assumptions one has the estimate*

$$\begin{aligned}\|u_{g\alpha}\|_{L^\infty(0,T;H)} + \|u_{g\alpha}\|_{L^2(0,T;V)} + \\ + \sqrt{(\alpha-1)}\|u_{g\alpha} - b\|_{L^2(\Gamma_1\times]0,T[)} \le C(1 + \|g_\psi\|_{L^2(0,T;V')}),\end{aligned} \tag{4.1}$$

for every $\alpha > 1$ and any g in $\mathcal{H}$, where the constant C depends only on the norms $\|\dot u_g\|_{L^2(0,T;V')}$, $\|\nabla u_g\|_{L^2(0,T;H)}$, and the coerciveness constant λ_1 in (3.5). Moreover, as $\alpha \to \infty$ one has $u_{g\alpha} \to u_g$ strongly in $L^2(0,T;V) \cap L^\infty(0,T;H)$ and $\dot u_{g\alpha} \to \dot u_g$ in norm $L^2(0,T;V_0')$.

Proof. First note that $V_0 \subset V$ is a continuous (non dense) inclusion and the norms $\|v\|_{V_0} = \|\nabla v\|_H$ is equivalently to $\|v\|_V = \sqrt{\|v\|_{V_0} + \|v\|_H}$ on V_0.

Let φ be a function in $L^2(0,T;V)$ such that $\dot\varphi$ belongs to $L^2(0,T;V')$, $\varphi(0) = v_b$ and $\varphi = b$ on Γ_1, e.g., an extension of b and v_b such as ψ in (2.2). Now, on the equality (1.5) defining $u_{g\alpha}$ take $v = u_{g\alpha}(t) - \varphi(t) := z_{g\alpha}(t)$ to get

$$\begin{aligned}\langle \dot u_{a\alpha}(t), z_{g\alpha}(t)\rangle + \big(\nabla u_{g\alpha}(t), \nabla z_{g\alpha}(t)\big)_H + \alpha\langle u_{g\alpha}(t), z_{g\alpha}(t)\rangle_{\Gamma_1} = \\ = (g(t), z_{g\alpha}(t))_H - \langle q(t), z_{g\alpha}(t)\rangle_{\Gamma_2} + \alpha\langle b, z_{g\alpha}(t)\rangle_{\Gamma_1},\end{aligned}$$

and because $\varphi = b$ on Γ_1 one deduces

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big\|z_{g\alpha}(t)\big\|_H^2 + \big\|\nabla z_{g\alpha}(t)\big\|_H^2 + \alpha\big\|z_{g\alpha}(t)\big\|_{L^2(\Gamma_1)}^2 = (g(t), z_{g\alpha}(t))_H - \\ - \langle q(t), z_{g\alpha}(t)\rangle_{L^2(\Gamma_2)} - \langle\dot\varphi(t), z_{g\alpha}(t)\rangle - (\nabla u_g, \nabla z_{g\alpha})_H,\end{aligned} \tag{4.2}$$

which together with coerciveness (3.5) and the condition $z_{g\alpha}(0) = 0$ yield the bound (4.1). By means of estimate (4.1), there exists a sequence $\alpha_n \to \infty$ and z in $L^2(0,T;V) \cap L^\infty(0,T;H)$ such that $z_{g\alpha_n} \to z$ weakly in $L^2(0,T;V)$ and weakly* in $L^\infty(0,T;H)$, and $z = 0$ on Γ_1, i.e., z belongs to $L^2(0,T;V_0)$.

Hence, note that $a_\alpha(u,v) = a(u,v)$ and $L_{g\alpha}(t,v) = L_g(t,v)$ if u belongs to V and v belongs to V_0, and take v in V_0 in the equations (1.4) and (1.5) defining u_g and $u_{g\alpha}$ to obtain $\langle \dot z_{g\alpha}, v\rangle + a(z_{g\alpha}, v) = 0$, for every $v \in V_0$. Therefore, $\dot z_{g\alpha_n} \to \dot z$ weakly in $L^2(0,T;V_0')$ and because $z_{g\alpha}(0) = 0$ and $z = 0$ on Γ_1, one deduces $z = 0$ in $L^2(0,T;V)$.

Thus, as $\alpha \to \infty$ one has $z_{g\alpha} \to 0$ weakly in $L^2(0,T;V)$ and weakly* in $L^\infty(0,T;H)$. It is clear that the inclusion $V_0 \subset V$ is continuous and because the norm of V restricted to V_0 is equivalent to the norm of V_0, Hahn-Banach Theorem implies that any element ϑ of V_0' can be extended to an element in V' preserving its norm, in particular $\dot u_g$ can be extended to be an element in $L^2(0,T;V')$. Then, take $\varphi = u_g$ in the equality (4.2) and considering $\dot u_g$ an element in $L^2(0,T;V')$, one deduces that the convergence of $u_{g\alpha}$ toward u_g is indeed strongly in $L^2(0,T;V) \cap L^\infty(0,T;H)$. Moreover, $z_{g\alpha} \to 0$ in norm $L^2(\Gamma\times]0,T[)$ and $\dot z_{g\alpha} \to 0$ in norm $L^2(0,T;V_0')$. □

Proposition 4.2. *Under the previous assumptions one has the estimate*

$$\begin{aligned}&\|p_{g\alpha}\|_{L^\infty(0,T;H)} + \|p_{g\alpha}\|_{L^2(0,T;V)} + \\ &\qquad + \sqrt{(\alpha-1)}\|p_{g\alpha}\|_{L^2(\Gamma_1\times]0,T[)} \leq C(1+\|u_{g\alpha}\|_{L^2(0,T;V')}),\end{aligned} \tag{4.3}$$

for every $\alpha > 1$ and any g in $\mathcal{H}$, where the constant C depends only on the norms $\|z_d\|_{\mathcal{H}}$, $\|\dot{p}_g\|_{L^2(0,T;V')}$, $\|\nabla p_g\|_{L^2(0,T;H)}$, and the coerciveness constant λ_1 in (3.5). Moreover, as $\alpha \to \infty$ one has $p_{g\alpha} \to p_g$ strongly in $L^2(0,T;V) \cap L^\infty(0,T;H)$ and $\dot{p}_{g\alpha} \to \dot{p}_g$ in norm $L^2(0,T;V_0')$.

Proof. Note that even when $b \neq 0$ the (Robin) boundary condition of p_g and $p_{g\alpha}$ on Γ_1 does not involve b directly. Certainly, the norm $\|u_{g\alpha}\|_{L^2(0,T;V')}$ is bounded by $\|u_{g\alpha}\|_{L^2(0,T;H)}$, which is uniformly bounded in α.

The technique used in Proposition 4.1 applies for the adjoint states $p_{g\alpha}$ and p_g. Perhaps the only point to remark is the convergence as $\alpha \to \infty$. Indeed, one needs to make use of the weak (and later strong) convergence $u_{g\alpha} \to u_g$ in $L^2(0,T;V')$, which is deduced for the convergence in $L^2(0,T;H)$. □

5 Optimal Control Problems

We are now ready to consider the distributed control problems (1.9) and (1.10). Our purpose is to establish

Theorem 5.1. *Let assumptions (2.1) and (2.2) be hold, and $\hat{g}$ and $\hat{g}_\alpha$ be the minimizers in $\mathcal{H}$ of problems (1.9) and (1.10), respectively. Then, as the parameter $\alpha \to \infty$, the minimizers $\hat{g}_\alpha \to \hat{g}$ strongly in $\mathcal{H}$. Moreover the corresponding optimal state and adjoint state satisfy $(u_{\hat{g}_\alpha\alpha}, \dot{u}_{\hat{g}_\alpha\alpha}) \to (u_{\hat{g}}, \dot{u}_{\hat{g}})$ and $(p_{\hat{g}_\alpha\alpha}, \dot{p}_{\hat{g}_\alpha\alpha}) \to (p_{\hat{g}}, \dot{p}_{\hat{g}})$ strongly in $L^2(0,T;V) \times L^2(0,T;V_0')$.*

Proof. We make several steps. First, be means of the estimate (4.1) in Proposition 4.1 one has

$$\|u_{0\alpha}\|_{\mathcal{H}} \leq C, \quad \forall \alpha > 1,$$

for some constant C. Now, from the inequality $J(\hat{g}_\alpha) \leq J(0)$ we deduce

$$\|\hat{g}_\alpha\|_{\mathcal{H}} + \|u_{\hat{g}_\alpha\alpha}\|_{\mathcal{H}} \leq C, \quad \forall \alpha > 1$$

for some constant independent of $\alpha > 1$.

Again, estimate (4.1) in Proposition 4.1 and estimate (4.2) in Proposition 4.2 yield

$$\begin{aligned}&\|u_{\hat{g}_\alpha\alpha}\|_{L^2(0,T;V)} + \|\dot{u}_{\hat{g}_\alpha\alpha}\|_{L^2(0,T;V_0')} + \\ &\qquad + \sqrt{(\alpha-1)}\|u_{\hat{g}_\alpha\alpha} - b\|_{L^2(0,T;L^2(\Gamma_1))} \leq C, \quad \forall \alpha > 1\end{aligned}$$

and

$$\begin{aligned}&\|p_{\hat{g}_\alpha\alpha}\|_{L^2(0,T;V)} + \|\dot{p}_{\hat{g}_\alpha\alpha}\|_{L^2(0,T;V_0')} + \\ &\qquad + \sqrt{(\alpha-1)}\|p_{\hat{g}_\alpha\alpha}\|_{L^2(0,T;L^2(\Gamma_1))} \leq C, \quad \forall \alpha > 1.\end{aligned}$$

Hence, there exist $\bar{g}$ in $\mathcal{H}$, $\hat{u}$ and $\hat{p}$ in $L^2(0,T;V_0)$ with $\dot{\hat{u}}$ and $\dot{\hat{p}}$ in $L^2(0,T;V_0')$ such that, for a convenient subsequence as $\alpha \to \infty$ we has $\hat{g}_\alpha \rightharpoonup \bar{g}$ weakly in $\mathcal{H}$, $u_{\hat{g}_\alpha\alpha} \rightharpoonup \hat{u}$ weakly in $L^2(0,T;V)$, $\dot{u}_{\hat{g}_\alpha\alpha} \rightharpoonup \dot{\hat{u}}$ weakly in $L^2(0,T;V_0')$, $p_{\hat{g}_\alpha\alpha} \rightharpoonup \hat{p}$ weakly in $L^2(0,T;V)$, $\dot{p}_{\hat{g}_\alpha\alpha} \rightharpoonup \dot{\hat{p}}$ weakly in $L^2(0,T;V_0')$.

By taking v in V_0 in the parabolic variational equality (2.5) and letting $\alpha \to \infty$ we deduce that $\hat{u}$ solves parabolic variational equality (2.4), and by uniqueness $\hat{u} = u_{\hat{g}}$. In particular $u_{\hat{g}_\alpha\alpha} \rightharpoonup u_{\hat{g}}$ weakly in $L^2(0,T;V_0')$. Thus, by taking v in V_0 in the parabolic variational equality defining the adjoint state $p_{\hat{g}_\alpha\alpha}$ in Proposition 3.2 and letting $\alpha \to \infty$ we deduce that $\hat{p} = p_{\bar{g}}$. On the other hand, taking limit in the equality $m\hat{g}_\alpha + p_{\hat{g}_\alpha\alpha} = 0$ we deduce $m\bar{g} + p_{\bar{g}} = 0$. Thus, by using Proposition 3.1, this proves that $\bar{g}$ is a minimizer for the control problem (1.9), and by uniqueness $\hat{g} = \bar{g}$.

At this point, we have

$$(\hat{g}_\alpha, u_{\hat{g}_\alpha\alpha}, \dot{u}_{\hat{g}_\alpha\alpha}, p_{\hat{g}_\alpha\alpha}, \dot{p}_{\hat{g}_\alpha\alpha}) \rightharpoonup (\hat{g}, u_{\hat{g}}, \dot{u}_{\hat{g}}, p_{\hat{g}}, \dot{p}_{\hat{g}})$$

weakly in the corresponding spaces, initially for a convenient subsequence as $\alpha \to \infty$, but in view of the uniqueness of the limit, the weak convergence whole as $\alpha \to \infty$.

To prove the strong convergence we use the weak semicontinuity of the norm and the optimality of $\hat{g}, \hat{g}_\alpha$, namely,

$$\begin{aligned} J(\hat{g}) = \frac{1}{2}\|u_{\hat{g}} - z_d\|^2_{\mathcal{H}} + \frac{m}{2}\|\hat{g}\|^2_{\mathcal{H}} \leq \liminf_{\alpha\to\infty} \big[\frac{1}{2}\|u_{\hat{g}_\alpha\alpha} - z_d\|^2_{\mathcal{H}} + \frac{m}{2}\|\hat{g}_\alpha\|^2_{\mathcal{H}}\big] \leq \\ \leq \limsup_{\alpha\to\infty} \big[\frac{1}{2}\|u_{\hat{g}_\alpha\alpha} - z_d\|^2_{\mathcal{H}} + \frac{m}{2}\|\hat{g}_\alpha\|^2_{\mathcal{H}}\big] \leq \limsup_{\alpha\to\infty} J_\alpha(g), \end{aligned}$$

for any g in $\mathcal{H}$. In view of Proposition 4.1, $u_{g\alpha} \to u_g$ strongly in $L^2(0,T;V)$ as $\alpha \to \infty$, which implies that

$$\limsup_{\alpha\to\infty} J_\alpha(g) = \lim_{\alpha\to\infty} \big[\frac{1}{2}\|u_{g\alpha} - z_d\|^2_{\mathcal{H}} + \frac{m}{2}\|g\|^2_{\mathcal{H}}\big] = J(g).$$

By taking infimum on g, all the above inequalities become equalities and therefore

$$\frac{1}{2}\|u_{\hat{g}_\alpha\alpha} - z_d\|^2_{\mathcal{H}} + \frac{m}{2}\|\hat{g}_\alpha\|^2_{\mathcal{H}} \to \frac{1}{2}\|u_{\hat{g}} - z_d\|^2_{\mathcal{H}} + \frac{m}{2}\|\hat{g}\|^2_{\mathcal{H}}.$$

This and the weak convergence imply that $(\hat{g}_\alpha, u_{\hat{g}_\alpha\alpha}) \to (\hat{g}, u_{\hat{g}})$ strongly in $\mathcal{H} \times \mathcal{H}$, as $\alpha \to \infty$.

Finally, if $z_\alpha = u_{\hat{g}_\alpha\alpha} - u_{\hat{g}}$ then we deduce

$$\begin{aligned} \int_0^T \big[\langle \dot{z}_\alpha(t), z_\alpha(t)\rangle + a_1(z_\alpha(t), z_\alpha(t)) + (\alpha - 1)\int_{\Gamma_1} |z_\alpha(x,t)|^2 \,\mathrm{d}x\big]\,\mathrm{d}t \leq \\ \leq \int_0^T \big[\langle \hat{g}_\alpha - \dot{u}_{\hat{g}}, z_\alpha\rangle - a(u_{\hat{g}}, z_\alpha) - \int_{\Gamma_2} q(x,t) z_\alpha(x,t)\,\mathrm{d}x\big]\,\mathrm{d}t. \end{aligned}$$

Since $z_\alpha \to 0$ weakly in $L^2(0,T;V)$ and $\hat{g}_\alpha \to \hat{g}$ strongly in $\mathcal{H}$, we obtain $u_{\hat{g}_\alpha \alpha} \to u_{\hat{g}}$ strongly in $L^2(0,T;V)$, as $\alpha \to \infty$. Now, going back to the equation one has

$$\langle \dot{z}_\alpha(t), v\rangle + a(z_\alpha(t), v) = \langle \hat{g}_\alpha - \hat{g}, v\rangle.$$

Now, taking sup for v in V_0 with $\|v_0\|_{V_0} \le 1$ and integrating in $]0,T[$ one obtains the strong convergence of the time derivative. Similarly, $(p_{\hat{g}_\alpha \alpha}, \dot{p}_{\hat{g}_\alpha \alpha}) \to (p_{\hat{g}}, \dot{p}_{\hat{g}})$ strongly in $L^2(0,T;V) \times L^2(0,T;V_0')$, as $\alpha \to \infty$. This completes the proof. $\square$

Also we have

Proposition 5.2. *If $\alpha_2 \ge \alpha_1 \ge \alpha_0 > 0$ then there exists a constant $C = C_{\alpha_0}$ such that for every g in $\mathcal{H}$ one has*

$$\|u_{g\alpha_1} - u_{g\alpha_2}\|_{L^2(0,T;V)} \le C_{\alpha_0}(\alpha_2 - \alpha_1)\|b - u_{g\alpha_2}\|_{L^2(0,T;H^{-1/2}(\Gamma_1))}, \tag{5.1}$$

and

$$\begin{aligned} \|p_{g\alpha_1} - p_{g\alpha_2}\|_{L^2(0,T;V)} \le C_{\alpha_0}(\alpha_2 - \alpha_1)\big(&\|p_{g\alpha_2}\|_{L^2(0,T;H^{-1/2}(\Gamma_1))} + \\ &+ \|b - u_{g\alpha_2}\|_{L^2(0,T;H^{-1/2}(\Gamma_1))}\big), \end{aligned} \tag{5.2}$$

i.e., the dependency in α is Lipschitz continuous.

Proof. For a fixed g and $\alpha_2 \ge \alpha_1 \ge \alpha_0 > 0$ set $z = u_{g\alpha_2} - u_{g\alpha_1}$ to obtain from the equation (1.5) with α_i the identity

$$\langle \dot{z}(t), v\rangle + a_{\alpha_1}(z(t), v) = (\alpha_2 - \alpha_1)\int_{\Gamma_1}(b - u_{g\alpha_2})v\,\mathrm{d}\gamma, \quad \forall v \in V.$$

By taking $v = z(t)$ and by means of the inequalities

$$\Big|\int_0^T \mathrm{d}t \int_{\Gamma_1}(b - u_{g\alpha_2})z\,\mathrm{d}\gamma\Big| \le C_0\|b - u_{g\alpha_2}\|_{L^2(0,T;H^{-1/2}(\Gamma_1))}\|z\|_{L^2(0,T;V)}$$

and

$$a_\alpha(v,v) \ge \lambda(\alpha_0)\|v\|_V^2, \quad \forall v \in V,\ \alpha \ge \alpha_0,$$

we deduce the desired estimate with $C_{\alpha_0} = C_0/\lambda(\alpha_0)$.

Similarly, for a fixed g and $\alpha_2 \ge \alpha_1 \ge \alpha_0 > 0$ set $w = p_{g\alpha_2} - p_{g\alpha_1}$ to obtain from the equation (3.5) with α_i the identity

$$\langle \dot{w}(t), v\rangle + a_{\alpha_1}(w(t), v) = (\alpha_1 - \alpha_2)\int_{\Gamma_1} p_{g\alpha_2} v\,\mathrm{d}\gamma + (u_{g\alpha_2} - u_{g\alpha_1}, v)_H,$$

for every v in V. By taking $v = w(t)$ and in view of the estimate (5.1), we conclude. $\square$

Under some more restrict assumption we have monotonicity on α

Proposition 5.3. *Let us assume the data b constant on Γ_1, $v_b \leq b$ on Ω, $g \leq 0$ in $\Omega\times]0,T[$ and $q \geq 0$ on $\Gamma_2\times]0,T[$. Then $u_{g\alpha} \leq u_g \leq b$ for every $\alpha > 0$. Moreover, if $0 < \alpha_1 \leq \alpha_2$ then $u_{g\alpha_1} \leq u_{g\alpha_2} \leq u_g \leq b$ in $\Omega\times]0,T[$. Furthermore, if $b \leq z_d$ in $\Omega\times]0,T[$ then $p_{g\alpha_1} \leq p_{g\alpha_2} \leq p_g \leq 0$ in $\Omega\times]0,T[$, for every $\alpha_2 \geq \alpha_1 > 0$.*

Proof. First, the maximum principle implies that $u_{g\alpha} \leq b$. Indeed, if $z = (u_{g\alpha} - b)$ then we have

$$\langle \dot{z}(t), z^+(t)\rangle + a\big(z(t), z^+(t)\big) + \alpha \int_{\Gamma_1} z(t)z^+(t)\,\mathrm{d}\gamma = \\ = \big(g(t), z^+(t)\big) - \int_{\Gamma_2} q(t)z^+(t)\,\mathrm{d}\gamma$$

after using the fact that b is constant, which implies $z^+ = 0$.

Similarly, if $w = u_{g\alpha_2} - u_{g\alpha_1}$ with $\alpha_2 > \alpha_1$ then we get

$$\langle \dot{w}(t), w^+(t)\rangle + a_{\alpha_1}\big(w(t), w^+(t)\big) + (\alpha_2 - \alpha_1) \int_{\Gamma_1} (b - u_{g\alpha_2}(t) z^+(t)\,\mathrm{d}\gamma = 0,$$

which yields $w \leq 0$, i.e., $u_{g\alpha_2} \leq u_{g\alpha_1}$.

Finally, if $y = u_{g\alpha} - u_g$ then we obtain

$$\langle \dot{y}(t), y^+(t)\rangle + a\big(y(t), y^+(t)\big) + \alpha \int_{\Gamma_1} (b - u_{g\alpha}(t) y^+(t)\,\mathrm{d}\gamma = 0,$$

which yields $y \leq 0$, i.e., $u_{g\alpha} \leq u_g$.

The estimate on the adjoint state follows from a comparison with the solution r of the parabolic variational equality with terminal condition

$$\begin{cases} r \in L^2(0,T;V), \quad r(T) = 0 \quad \text{and} \quad \dot{r} \in L^2(0,T;V') \\ \text{such that} \quad -\langle \dot{r}(t), v\rangle + a(r(t), v) = (b - z_d, v)_H, \quad \forall v \in V. \end{cases} \tag{5.3}$$

Indeed, if $b \leq z_d$ in $\Omega\times]0,T[$ then the maximum principle (as above) yields $p_g \leq r \leq 0$. Next, similarly to the state u with $b = 0$, one deduces that $p_{g\alpha_1} \leq p_{g\alpha_2} \leq p_g \leq r \leq 0$ in $\Omega\times]0,T[$, for every $\alpha_2 \geq \alpha_1 > 0$. $\square$

Certainly, the maximum principle yields $u_{g_1} \leq u_{g_2}$ and $u_{g_1\alpha} \leq u_{g_2\alpha}$ if $g_1 \leq g_2$, but a priori, it is not clear when the minimizers satisfy $\hat{g} \geq \hat{g}_\alpha$ to deduce the monotonicity $u_{g_{\alpha_1}\alpha_1} \leq u_{g_{\alpha_2}\alpha_2} \leq u_{g_\alpha} \leq b$.

6 Final Comments

Variational inequalities was popular in the 70's, most of the main techniques for parabolic variational inequalities can be found in various classic books, e.g., Bensoussan and Lions [5], among other.

It is well known that the regularity of the mixed problem is problematic when both portions of the boundary Γ_1 and Γ_2 have a nonempty intersection, e.g. see the book Grisvard [9]. Recently, sufficient conditions (on the data) to obtain a H^2 regularity for a (elliptic) mixed boundary conditions are given in Bacuta et al. [3], see also Azzam and Kreyszig [1], among others.

Numerical analysis of a parabolic PDE with mixed boundary conditions (Dirichlet and Neumann) is studied in Babuska and Ohnimus [2], while a parabolic control problem with Robin boundary conditions is considered in Chrysafinos et al. [7] and Bergounioux and Troltzsch [6].

The state equation, i.e., a parabolic PDE with mixed boundary conditions (Robin and Neumann) has been discussed in Ben Belgacem et al. [4] and Tarzia [12].

Certainly, there are several possible extensions, e.g., a state equation of the form

$$\partial_t u - \operatorname{div}\big(A(x,t)\nabla u\big) + b(t,x)u = f \quad \text{in} \quad \Omega \times]0,T[,$$

with mixed boundary conditions. A carefully analysis is necessary, but the main techniques used to let $\alpha \to \infty$ in the parabolic variational inequality seems to be very well adaptable to more general situations.

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