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Impulse Control of Stochastic Navier-Stokes Equations

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Abstract. In this paper we study stopping time and impulse control problems for stochastic Navier-Stokes equation. Exploiting a local monotonicity property of the nonlinearity, we establish existence and uniqueness of strong solutions in two dimensions which gives a Markov-Feller process. The variational inequality associated with the stopping time problem and the quasi-variational inequality associated with the impulse control problem are resolved in a weak sense, using semigroup approach with a convergence uniform over path.

Key words. Impulse control, Stochastic Navier-Stokes equation, Dynamic programming, Variational inequality, Variational data assimilation, Stochastic optimal control.

AMS Classification. 35Q30, 49N25, 76D05.

1 Introduction

Optimal control theory of fluid dynamics has numerous applications such as aero/hydrodynamic control, combustion control, Tokomak magnetic fusion as well as ocean and atmospheric prediction. During the past decade several fundamental advances have been made by a number of researchers as documented in Sritharan [21]. In this paper we develop a new direction to this subject, namely we mathematically formulate and resolve impulse and stopping time problems. Impulse control of Navier-Stokes equations has significance beyond control theory. In fact, in optimal weather prediction the task of updating the initial data optimally at strategic times can be reformulated precisely as an impulse control problem for the primitive cloud equations (which consist of the Navier-Stokes equation coupled with temperature and species evolution equations, cf. Dymnikov and Filatov [11]), see Bennett [1], Daley [9], Monin [15].

For the study of optimal stopping problem alone, it is possible to impose regularity assumptions on the stopping cost. However, in our case, optimal stopping problems are used as intermediate steps to treat the impulse control problem through an iteration process. This dictates that we must work with stopping costs which have only continuity property.

Optimal stopping and impulse control problems are very well known, particularly for diffusion processes (e.g., see the books of Bensoussan and Lions [3, 4]), for degenerate diffusion with jumps (e.g., Menaldi [13]) and for general Markov process (e.g., Robin [18], Shiryayev [19], Stettner [20]). The main technical challenge is to give a characterization of the value function (or optimal cost) and to exhibit an optimal control. In these works certain conditions are imposed on the data which make the theory not applicable to fluid dynamics. Although the variational technique has been adapted to Gauss-Sobolev spaces (e.g., Chow and Menaldi [8], Zabczyk [26]) with partial results, but because of the technical difficulties associated with the domain of the generator, we prefer to follow the semigroup approach. Certainly, most of the effort is dedicated to give a suitable sense to the stochastic Navier-Stokes equation in a two-dimensional domain (cf. [14], among others) to produce a Markov-Feller process in a Hilbert space (non-locally compact) with a weakly continuous semigroup. Some related results can be found in Bensoussan [2] and Zabczyk [25], but they are not directly applicable to our model. To discuss only optimal stopping time, we may impose regularity on the stopping cost. However, due to the iteration procedure used to study impulse control problems, we need to be able to treat stopping costs which are only continuous. For numerical approximation, we can use the general arguments presented in Quadrat [17].

The organization of the paper is as follows. First in Section 2, we discuss the Markov-Feller process generated by stochastic Navier-Stokes equation in a two-dimensional domain. Next, in Section 3 we study stopping time problems and finally in Section 4, we consider switching and impulse control problems. Notice that Sections 3 and 4 are actually independent of the Navier-Stokes equation, only key conditions established in Section 2 are necessary to completely develop the theory of impulse control for Markov-Feller semigroup not necessarily strongly continuous.

2 Fluid Dynamics as Markov-Feller Process

Let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\mathcal{O}$. Denote by $\mathbf{u}$ and p the velocity and the pressure fields. The Navier-Stokes problem (with Newtonian constitutive) can be written in a compact form as follows:

$$\partial_t \mathbf{u} + A\mathbf{u} + B(\mathbf{u}) = \mathbf{f} \quad \text{in } \mathbb{L}^2(0,T;\mathbb{V}'), \tag{2.1}$$

with the initial condition

$$\mathbf{u}(0) = \mathbf{u_0} \quad \text{in } \mathbb{H}, \tag{2.2}$$

where now $\mathbf{u_0}$ belong to $\mathbb{H}$ and the field $\mathbf{f}$ is in $\mathbb{L}^2(0,T;\mathbb{H})$. The Sobolev spaces and operator used are as follows:

$$\mathbb{V} = \{\mathbf{v} \in \mathbb{H}_0^1(\mathcal{O},\mathbb{R}^2) : \nabla\cdot\mathbf{v} = 0 \text{ a.e. in } \mathcal{O}\}, \tag{2.3}$$

with the norm

$$\|\mathbf{v}\|_{\mathbb{V}} := \left(\int_{\mathcal{O}} |\nabla\mathbf{v}|^2 dx\right)^{1/2} = \|\mathbf{v}\|, \tag{2.4}$$

and $\mathbb{H}$ is the closure of $\mathbb{V}$ in the $\mathbb{L}^2$-norm

$$\|\mathbf{v}\|_{\mathbb{H}} := \left(\int_{\mathcal{O}} |\mathbf{v}|^2 dx\right)^{1/2} = |\mathbf{v}|. \tag{2.5}$$

The linear operators

$$\begin{cases} P_{\mathbb{H}} : \mathbb{L}^2(\mathcal{O},\mathbb{R}^2) \longrightarrow \mathbb{H}, \quad \text{orthogonal projection,} \\ A : \mathbb{H}^2(\mathcal{O},\mathbb{R}^2) \cap \mathbb{V} \longrightarrow \mathbb{H}, \quad A\mathbf{u} = -\nu P_{\mathbb{H}} \Delta\mathbf{u}, \quad \nu > 0 \end{cases} \tag{2.6}$$

and the nonlinear operator

$$B : \mathcal{D}_B \subset \mathbb{H}\times\mathbb{V} \longrightarrow \mathbb{H}, \quad B(\mathbf{u},\mathbf{v}) = P_{\mathbb{H}}(\mathbf{u}\cdot\nabla\mathbf{v}), \tag{2.7}$$

with the notation $B(\mathbf{u}) = B(\mathbf{u},\mathbf{u})$, and clearly, the domain of B requires that $(\mathbf{u}\cdot\nabla\mathbf{v})$ belongs to the Lebesgue space $\mathbb{L}^2(\mathcal{O},\mathbb{R}^2)$.

Using the triple duality $\mathbb{V} \subset \mathbb{H} = \mathbb{H}' \subset \mathbb{V}'$ we may consider A as mapping $\mathbb{V}$ into its dual $\mathbb{V}'$. The inner product in the Hilbert space $\mathbb{H}$ (i.e., $\mathbb{L}^2$-scalar product) is denoted by $(\cdot,\cdot)$ and the induced duality by $\langle\cdot,\cdot\rangle$.

Let us consider the Navier-Stokes equation (2.1) subject to a random (Gaussian) term i.e., the forcing field $\mathbf{f}$ has a mean value still denoted by $\mathbf{f}$ and a noise denoted by $\dot{\mathbf{G}}$. We can write[1] $\mathbf{f}(t) = \mathbf{f}(x,t)$ and the noise process $\dot{\mathbf{G}}(t) = \dot{\mathbf{G}}(x,t)$ as a series $d\mathbf{G}_k = \sum_k \mathbf{g}_k(x,t)dw_k(t)$, where $\mathbf{g} = (\mathbf{g}_1, \mathbf{g}_2, \cdots)$ and $w = (w_1, w_2, \ldots)$ are regarded as ℓ^2-valued functions. The stochastic noise process represented by $\mathbf{g}(t)dw(t) = \sum_k \mathbf{g}_k(x,t)dw_k(t,\omega)$ (notice that most of the time we omit the variable ω) is normal distributed in $\mathbb{H}$ with a trace-class co-variance operator denote by $\mathbf{g}^*\mathbf{g} = \mathbf{g}^*\mathbf{g}(t)$ and given by

$$\begin{cases} (\mathbf{g}^*\mathbf{g}(t)\mathbf{u},\mathbf{v}) := \displaystyle\sum_k (\mathbf{g}_k(t),\mathbf{u})\,(\mathbf{g}_k(t),\mathbf{v}) \\ \mathrm{Tr}(\mathbf{g}^*\mathbf{g}(t)) := \displaystyle\sum_k |\mathbf{g}_k(t)|^2 < \infty, \end{cases} \tag{2.8}$$

i.e., the mapping (stochastic integral) induced by the noise

$$\mathbf{v} \mapsto \int_0^T (\mathbf{g}(t)dw(t),\mathbf{v}) := \sum_k \int_0^T (\mathbf{g}_k(t),\mathbf{v})\,dw_k(t) \tag{2.9}$$

is a continuous linear functional on $\mathbb{H}$ with probability 1 and the noise is the formal time-derivative of the Gaussian process $\mathbf{G}(t) = \int_0^t \mathbf{g}(t)dw(t)$. A multiplicative noise of the form $\mathbf{g}(t,u)dw(t)$, where $g(t,u)$ is a continuous operator from $\mathbb{V}$ into $L^2(0,T;\ell_2(\mathbb{H}))$, can be also considered, however, for the sake of simplicity we adopt only additive noise, cf [14].

We interpret the stochastic Navier-Stokes equation as an Itô stochastic equation in variational form

$$\begin{cases} d(\mathbf{u}(t),\mathbf{v}) + \langle A\mathbf{u}(t) + B(\mathbf{u}(t)),\mathbf{v}\rangle\,dt = \\ \qquad\qquad = (\mathbf{f},\mathbf{v})\,dt + \displaystyle\sum_k (\mathbf{g}_k,\mathbf{v})\,dw_k(t), \end{cases} \tag{2.10}$$

in $(0,T)$, with the initial condition

$$(\mathbf{u}(0),\mathbf{v}) = (\mathbf{u}_0,\mathbf{v}), \tag{2.11}$$

[1] to simplify notation we use time-invariant forces

for any $\mathbf{v}$ in the space $\mathbb{V}$.

We may use as initial time a stopping time τ (random variable) with respect to the natural filtration $(\mathcal{F}_t,\, t \geq 0)$ (right-continuous and completed) associated with the Wiener process, and initial value $\mathbf{u_0} = \mathbf{u}_\tau(x,\omega)$ which is a $\mathcal{F}_\tau$–measurable random variable. Similarly, we may allow random forcing terms $\mathbf{f}(x,t,\omega)$ and $\mathbf{g}(x,t,\omega)$ or even having a nice dependency on the solution $\mathbf{u}$. For the random initial conditions we have to write the stochastic Navier-Stokes equation (2.10), (2.11) in its integral (variational) form, namely

$$\begin{cases} (\mathbf{u}(\theta),\mathbf{v}) + \displaystyle\int_\tau^\theta \langle A\mathbf{u}(t) + B(\mathbf{u}(t)),\mathbf{v}\rangle\, dt = (\mathbf{u}_\tau,\mathbf{v}) + \\ \qquad\qquad + \displaystyle\int_\tau^\theta (\mathbf{f}(t),\mathbf{v})\, dt + \sum_k \int_\tau^\theta (\mathbf{g}_k(t),\mathbf{v})\, dw_k(t), \end{cases} \tag{2.12}$$

for any stopping time $\tau \leq \theta \leq T$ and any $\mathbf{v}$ in the space $\mathbb{V}$. Actually, by a density argument we may allow any adapted process $\mathbf{v}(t)$ in $L^2(\Omega; L^2(0,T;\mathbb{V})) \cap \mathbb{L}^4(\mathcal{O}\times(0,T)\times\Omega)$. We now state the following result valid for smooth bounded and unbounded domains.

Proposition 2.1 (2-D). *Let τ and $\mathbf{u}_\tau$ be an stopping time with respect to $(\mathcal{F}_t,\, t \geq 0)$ and a $\mathcal{F}_\tau$–measurable random variable such that*

$$0 \leq \tau \leq T, \qquad \mathbf{u}_\tau \in L^p(\Omega;\mathbb{H}), \tag{2.13}$$

for some $p \geq 4$. Suppose $\mathbf{f}(x,t)$ and $\mathbf{g}(x,t)$ satisfy

$$\mathbf{f} \in L^p(0,T;\mathbb{V}'), \qquad \mathbf{g} \in L^p(0,T;\ell_2(\mathbb{H})). \tag{2.14}$$

Then there exists a unique adapted process $\mathbf{u}(t,x,\omega)$ with the regularity

$$\mathbf{u} \in L^p(\Omega; C^0(\tau,T;\mathbb{H})) \cap L^2(\Omega; L^2(\tau,T;\mathbb{V})) \tag{2.15}$$

and satisfying (2.12) *and the following a priori bound holds,*

$$\begin{cases} E\big\{ \displaystyle\sup_{\tau\leq t\leq T} |\mathbf{u}(t)|^p + \int_\tau^T |\nabla\mathbf{u}(t)|^2 |\mathbf{u}(t)|^{p-2} dt \big\} \leq \\ \quad \leq C_p E\big\{ |\mathbf{u}(\tau)|^p + \displaystyle\int_\tau^T \big[\|\mathbf{f}(t)\|_{\mathbb{V}'}^p + \mathrm{Tr}(\mathbf{g}^*\mathbf{g}(t))^{p/2} \big] dt \big\}, \end{cases} \tag{2.16}$$

for any $p \geq 2$ *and some constant* $C_p = C(T, \nu, p)$ *depending only on* $T > 0$, $\nu > 0$ *and* $p \geq 2$. *Moreover, if* $\bar{\mathbf{u}}(t, x, \omega)$ *is the solution with another initial data, we have*

$$|\mathbf{u}(\theta) - \bar{\mathbf{u}}(\theta)| \exp\Big[-\frac{32}{\nu^3} \int_\tau^\theta \|\mathbf{u}(t)\|^4_{\mathbb{L}^4(\mathcal{O})} dt\Big] \leq |\mathbf{u}_\tau - \bar{\mathbf{u}}(\tau)|, \tag{2.17}$$

with probability 1, *for any* $\tau \leq \theta \leq T$. □

The proof of this result can be found in [14]. The reader is referred to the books by Vishik and Fursikov [24] and Capinski and Cutland [6] for a comprehensive treatment on statistical and stochastic fluid dynamics. Our strong solution can be considered as a variational version of the result reported in Da Prato and Zabczyk [10, Chapter 15]. Using methods similar to the proof of estimate (2.17) we can get with probability 1,

$$\begin{cases} |\mathbf{u}(\theta) - \bar{\mathbf{u}}(\theta)|^2 e^{-r(\theta,\tau,\mathbf{u})} + \nu \displaystyle\int_\tau^\theta \|\mathbf{u}(t) - \bar{\mathbf{u}}(t)\|^2 e^{-r(t,\tau,\mathbf{u})} dt \leq \\ \quad \leq |\mathbf{u}(\tau) - \bar{\mathbf{u}}(\tau)|^2 + \displaystyle\int_\tau^\theta \langle \delta\mathbf{f}(t), \mathbf{u}(t) - \bar{\mathbf{u}}(t) \rangle e^{-r(t,\tau,\mathbf{u})} dt + \\ \quad + \displaystyle\sum_k \int_\tau^\theta \langle \delta\mathbf{g}_k(t), \mathbf{u}(t) - \bar{\mathbf{u}}(t) \rangle e^{-r(t,\tau,\mathbf{u})} dw_k(t), \end{cases} \tag{2.18}$$

where $r(t, \tau, \mathbf{u}) := \frac{32}{\nu^3} \int_\tau^\theta \|\mathbf{u}(s)\|^4_{\mathbb{L}^4(\mathcal{O})} ds$ and $\bar{\mathbf{u}}(\cdot)$ is the solution corresponding to the data $\mathbf{f} - \delta\mathbf{f}$ and $\mathbf{g} - \delta\mathbf{g}$.

As direct consequence of Proposition 2.1, we have a realization in the canonical space $C^0(0, T; \mathbb{H})$ of the Markov-Feller process associated with the (non linear) stochastic Navier-Stokes equation (2.10). We also have

Proposition 2.2 ($\mathbb{V}$–regularity). *Let the assumptions* (2.13) *and* (2.14) *hold as in* Proposition 2.1. *If*

$$\mathbf{u}_\tau \in L^p(\Omega; \mathbb{V}), \qquad \mathbf{f} \in L^p(0, T; \mathbb{H}), \qquad \mathbf{g} \in L^p(0, T; \ell_2(\mathbb{V})), \tag{2.19}$$

with $p \geq 2$, *then the solution* $\mathbf{u}(t)$ *of the stochastic Navier-Stokes equation* (2.10) *with initial condition* (2.12) *has the regularity*

$$\mathbf{u} \in C^0(\tau, T; \mathbb{V}) \cap L^2(\tau, T; \mathbb{H}^2(\mathcal{O}; \mathbb{R}^2)) \tag{2.20}$$

with probability 1, and the following estimate

$$
\begin{cases} E\Big\{\sup_{\tau\le t\le T}\big[|\nabla\mathbf{u}(t)|^p e^{-r(t,\tau,\mathbf{u})}\big]+ \\ \quad + \int_\tau^T |\Delta\mathbf{u}(t)|^2|\nabla\mathbf{u}(t)|^{p-2}e^{-r(t,\tau,\mathbf{u})}dt\Big\} \le C_p\big\{E\{|\nabla\mathbf{u}(\tau)|^p\}+ \\ \quad + \int_\tau^T \big[\|\mathbf{f}(t)\|_{\mathbb{H}}^p + (\mathrm{Tr}_{\mathbb{V}}(\mathbf{g}^*\mathbf{g}(t)))^{p/2}\big]e^{-r(t,\tau,\mathbf{u})}dt\big\}, \end{cases} \tag{2.21}
$$

where $r(t,\tau,\mathbf{u}) := c_\nu \int_\tau^t \|\mathbf{u}(s)\|^4_{\mathbb{L}^4(\mathcal{O})}ds$, *for some constants* C_p, c_ν *depending only on* p, T *and* $\nu > 0$.

Proof. In general, if $\mathbf{u}(t)$ belongs to $\mathbb{H}\cap\mathbb{H}^2(\mathcal{O},\mathbb{R}^2)$ then $\Delta\mathbf{u}(t)$ (respectively $\nabla\mathbf{u}(t)$ does not necessarily belong to $\mathbb{H}$ (respectively $\mathbb{V}$), however the norms $|\Delta\cdot|$ (respectively $|\nabla\cdot|$) and $|A\cdot|$ (respectively $|A^{1/2}\cdot|$) are equivalent, for instance we refer to Temam [23] for details and more comments. Let us assume that with probability 1, the solution $\mathbf{u}(t)$ of the stochastic Navier-Stokes equation (2.10) belongs to $L^2(0,T;\mathbb{H}^2(\mathcal{O},\mathbb{R}^2))$ and that $\partial_t\mathbf{u}(t)$ belongs to $L^2(0,T;\mathbb{H})$. Then multiplying equation (2.10) by $-P_{\mathbb{H}}\Delta\mathbf{u}(t)$ we have

$$
\begin{aligned} \frac{1}{2}d|\nabla\mathbf{u}(t)|^2 + \nu|P_{\mathbb{H}}\Delta\mathbf{u}(t)|^2dt = \big[\langle B(\mathbf{u}(t)),\Delta\mathbf{u}(t)\rangle + (\mathbf{f}(t),\Delta\mathbf{u}(t))+ \\ +\frac{1}{2}\sum_k |\nabla\mathbf{g}_k(t)|^2\big]dt + \sum_k(\nabla\mathbf{g}_k(t),\nabla\mathbf{u}(t))dw_k(t), \end{aligned}
$$

after recalling that $P_{\mathbb{H}}\mathbf{f}(t) = \mathbf{f}(t)$ and $P_{\mathbb{H}}\mathbf{g}_k(t) = \mathbf{g}_k(t)$. Next, using Hölder inequality and estimating the $\mathbb{L}^4$-norm we find a constant $C_0 > 0$ such that

$$
\begin{aligned} |\langle B(\mathbf{u}),\Delta\mathbf{u}\rangle| &\le 2|\Delta\mathbf{u}|\,|\nabla\mathbf{u}|_{\mathbb{L}^4}\,\|\mathbf{u}\|_{\mathbb{L}^4} \le \\ &\le C_0|\Delta\mathbf{u}|\,|\nabla\mathbf{u}|^{1/2}\,(|\nabla\mathbf{u}|^{1/2} + |\Delta\mathbf{u}|^{1/2})\,\|\mathbf{u}\|_{\mathbb{L}^4}. \end{aligned}
$$

Because $|P_{\mathbb{H}}\Delta\mathbf{u}|$ is equivalent to $|\Delta\mathbf{u}|$, there is a constant $c_0 > 0$ such that $c_0|\Delta\mathbf{u}| \le |P_{\mathbb{H}}\Delta\mathbf{u}|$ and then we obtain

$$
\begin{aligned} d\big[|\nabla\mathbf{u}(t)|^2e^{-r(t,\tau,\mathbf{u})}\big] + c_0\nu\,|\Delta\mathbf{u}(t)|^2e^{-r(t,\tau,\mathbf{u})}dt = \\ = F(t)e^{-r(t,\tau,\mathbf{u})}dt + 2\sum_k(\nabla\mathbf{g}_k(t),\nabla\mathbf{u}(t))e^{-r(t,\tau,\mathbf{u})}dw_k(t), \end{aligned}
$$

where

$$
\begin{aligned}
& r(t,\tau,\mathbf{u}) := c_\nu \int_\tau^t \big[\|\mathbf{u}(s)\|^4_{\mathbb{L}^4(\mathcal{O})} + 1\big] ds, \\
& F(t) \le C_1\big(|\mathbf{f}(t)|^2 + \sum_k |\nabla \mathbf{g}_k(t)|^2\big) + \frac{c_0\nu}{2} |\Delta \mathbf{u}|^2,
\end{aligned}
$$

for suitable constants C_1, $c_\nu > 0$. depending only on ν, c_0 and C_0. Since

$$
\begin{aligned}
& E\Big\{ \sup_{\tau\le t\le T} \Big| \sum_k \int_\tau^t (\nabla \mathbf{g}_k(t), \nabla \mathbf{u}(t)) e^{-r(t,\tau,\mathbf{u})} dw_k(t) \Big| \Big\} \le \\
& \quad \le 2\Big[\sum_k \int_\tau^T |\nabla \mathbf{g}_k(t)|^2 e^{-r(t,\tau,\mathbf{u})} dt\Big]^{1/2} \Big[E\Big\{ \sup_{\tau\le t\le T} |\nabla \mathbf{u}(t)|^2 e^{-r(t,\tau,\mathbf{u})} \Big\}\Big]^{1/2}
\end{aligned}
$$

we deduce the a priori estimate (2.21) for $p = 2$. Actually, we need to redo the above arguments on the finite-dimensional approximation of the solution and then pass to the limit to justify the result. For $p > 2$, we use Itô formula for the (real) process $|\nabla \mathbf{u}(t)|^2$ and the function $(\cdot)^{p/2}$ to get

$$
\begin{aligned}
d|\nabla \mathbf{u}(t)|^p = \frac{p}{2} |\nabla \mathbf{u}(t)|^{p-2} d|\nabla \mathbf{u}(t)|^2 + & \\
& + \frac{p(p-2)}{8} \sum_k |\nabla \mathbf{u}(t)|^{p-4} |(\nabla \mathbf{g}_k(t), \nabla \mathbf{u}(t))|^2 dt.
\end{aligned}
$$

By means of Hölder inequality, we can show (2.21) for $p > 2$ with arguments similar to the case $p = 2$. □

In what follows and for the sake of simplicity, we assume that the processes $\mathbf{f}(x,t,\omega)$ and $\mathbf{g}(x,t,\omega)$ are independent of t, i.e.,

$$
\mathbf{f} \in \mathbb{V}' \quad \text{and} \quad \mathbf{g} \in \ell_2(\mathbb{H}) \tag{2.22}
$$

and we denote by $\mathbf{u}(t;\mathbf{u_0})$ the random field, i.e., the solution of Navier-Stokes equation (2.10), (2.11), usually we substitute $\mathbf{u_0}$ with $\mathbf{v}$.

Proposition 2.3 (continuity). *Under the condition* (2.22) *the random field* $\mathbf{u}(t;\mathbf{v})$ *is locally uniformly continuous in* $\mathbf{v}$, *locally uniformly for* t *in* $[0,\infty)$. *Moreover, for any* $p > 0$ *and* $\alpha > 0$ *there is a positive constant* λ *sufficiently large such that the following estimate*

$$
E\{e^{-\alpha t}(\lambda + |\mathbf{u}(t;\mathbf{v})|^2)^{p/2}\} \le (\lambda + |\mathbf{v}|^2)^{p/2}, \quad \forall t \ge 0, \ \mathbf{v} \in \mathbb{H} \tag{2.23}
$$

holds, even for any stopping time $t = \tau$. Furthermore, if $\mathbf{f}$ *and* $\mathbf{g}$ *belong to* $\mathbb{H}$ *and* $\ell_2(\mathbb{V})$*, respectively, then the random field is also locally uniformly continuous in* t*, locally uniformly for* $\mathbf{v}$ *in* $\mathbb{V}$*.*

Proof. Let us re-phrase the fact that the random field $\mathbf{u}(t;\mathbf{v})$ is locally uniformly continuous in v, locally uniformly for t in $[0,\infty)$ as follows: for any $\varepsilon > 0$ there is a $\delta > 0$ such that for any T in $[0, 1/\varepsilon]$ and $\mathbf{v}$, $\bar{\mathbf{v}}$ in $\mathbb{H}$ satisfying $|\mathbf{v}-\bar{\mathbf{v}}| < \delta$, $|\mathbf{v}| \leq 1/\varepsilon$ and $|\bar{\mathbf{v}}| \leq 1/\varepsilon$ we have $P\big\{ \sup_{0\leq t\leq T} |\mathbf{u}(t;\mathbf{v}) - \mathbf{u}(t;\bar{\mathbf{v}})| \geq \varepsilon\big\} < \varepsilon$. To show this fact, we notice that if we set

$$\begin{cases} r(t;\mathbf{v}) := \displaystyle\int_0^t \|\mathbf{u}(s;\mathbf{v})\|^4_{\mathbb{L}^4(\mathcal{O})} ds, \qquad \text{and} \\ \tau_r(\mathbf{v}) := \inf\big\{ t \geq 0 \ : \ |\mathbf{u}(t;\mathbf{v})|^4 + r(t;\mathbf{v}) \geq r \big\}, \end{cases} \tag{2.24}$$

then from estimates (2.16) with p=4 and

$$\|\varphi\|^4_{L^4} \leq 2\|\varphi\|^2_{L^2}\|\nabla\varphi\|^2_{\mathbb{L}^2},$$

we deduce that for any t in the stochastic interval $[s, \tau_r(\mathbf{v})]$

$$|\mathbf{u}(t;\mathbf{v})|^4 + r(t;\mathbf{v}) \leq r, \tag{2.25}$$

and for any $T > s$ there is a constant $C_T > 0$, which depends only on T and ν, such that

$$r\, P\{\tau_r(\mathbf{v}) \leq T\} \leq C_T \,\big\{ |\mathbf{v}|^4 + \|\mathbf{f}\|^4_{\mathbb{V}'} + [\mathrm{Tr}(\mathbf{g}^*\mathbf{g})]^2 \big\}, \tag{2.26}$$

Thus, even though balls are not compact on $\mathbb{H}$, we can get uniform convergence on any ball. Indeed, from estimate (2.17) we have

$$|\mathbf{u}(t\wedge\tau_r;\mathbf{v}) - \mathbf{u}(t\wedge\tau_r;\bar{\mathbf{v}})| \leq C_r|\mathbf{v}-\bar{\mathbf{v}}|,$$

for some constant C_r, while (2.26) yields

$$P\{\mathbf{u}(t;\mathbf{v}) \neq \mathbf{u}(t\wedge\tau_r;\mathbf{v})\} \leq P\{\tau_r < T\} \leq C_T \frac{|\mathbf{v}|^4 + \|\mathbf{f}\|^4_{\mathbb{V}'} + [\mathrm{Tr}(\mathbf{g}^*\mathbf{g})]^2}{r},$$

for any $0 \leq t \leq T$ and some constants C_T depending only on T and ν. This establishes the continuity in $\mathbf{v}$.

The locally uniformly continuity of the random field in t can be re-phrased as follows: for any $\varepsilon > 0$ there is a $\delta > 0$ such that for any $\mathbf{v}$ in $\mathbb{V}$, with

$\|\mathbf{v}\| < 1/\varepsilon$ have $P\big\{\sup_{0\le t\le\delta}\sup_{0\le s\le 1/\varepsilon}|\mathbf{u}(t+s;\mathbf{v})-\mathbf{u}(s;\mathbf{v})|\ge\varepsilon\big\}<\varepsilon$. To estimate the modulus of continuity of the random field $\mathbf{u}(t,\mathbf{v})$ we make use of the following estimate. For any $\alpha\;\beta>0$ there exists a constant $C_0=C_0(\alpha,\beta)$ such that

$$\sup_{|t-s|<\delta}|\mathbf{v}(t)-\mathbf{v}(s)|^{\alpha}\le C_0\,\delta^{\beta}\int_0^T dt\int_0^T\frac{|\mathbf{v}(t)-\mathbf{v}(s)|^{\alpha}}{|t-s|^{2+\beta}}ds, \tag{2.27}$$

for any measurable function $\mathbf{v}$ on $[0,T]$ and any $\delta>0$, cf. Da Prato and Zabczyk [10, Theorem B.1.5, pp. 311-316]. Therefore, if for some constants $p,\;q,\;C>0$ a process $\mathbf{v}(t,\omega)$ satisfies

$$E\big\{|\mathbf{v}(t)-\mathbf{v}(s)|^{p}\big\}\le C|t-s|^{1+q},\quad\forall t,s\in[0,T], \tag{2.28}$$

then for any $0<r<q$ there is another constant $C_0=C_0(p,q,C,r)$ such that if $\rho_T(\delta;\mathbf{v}):=\sup\big\{|\mathbf{v}(t)-\mathbf{v}(s)|\;:\;t,s\in[0,T],\;|t-s|<\delta\big\}$, we have $E\big\{[\rho_T(\delta;\mathbf{v})]^p\big\}\le C_0\,\delta^r$, for any $\delta>0$. Therefore, in view of estimate (2.26), to show the continuity in t it suffices to prove an estimate of the form (2.28) for the stopped random field $\mathbf{u}_r(t;\mathbf{v}):=\mathbf{u}(t\wedge\tau_r;\mathbf{v})$, where τ_r is given by (2.24). To this purpose, from estimate (2.21) and definition (2.24) we obtain

$$\begin{cases} E\big\{\sup\limits_{0\le s\le T}\big[|\nabla\mathbf{u}_r(s;\mathbf{v})|^p+|\mathbf{u}_r(s;\mathbf{v})|^p_{\mathbb{L}^4(\mathcal{O})}\big]\big\}\le \\ \qquad\qquad\le C_{r,T}\big\{|\nabla\mathbf{v}|^p+|\mathbf{f}|^p+(\mathrm{Tr}_{\mathbb{V}}(\mathbf{g}^*\mathbf{g}))^{p/2}\big\}, \end{cases} \tag{2.29}$$

for any $r>0$, $T>0$ and $p\ge 4$, and some constant $C_{r,t}$ depending only on r, T, p, and ν. On the other hand, by means of (2.18) with $\bar{\mathbf{u}}(t):=\mathbf{v}$, $\delta\mathbf{f}(t):=\mathbf{f}-A\mathbf{v}-B(\mathbf{v})$, and $\delta\mathbf{g}(t):=\mathbf{g}$, and the bounds

$$\|A\mathbf{v}\|_{\mathbb{V}'}\le\nu\|\mathbf{v}\|_{\mathbb{V}},\qquad\|B(\mathbf{v})\|_{\mathbb{V}'}\le\|\mathbf{v}\|^2_{\mathbb{L}^4(\mathcal{O})},$$

we get

$$\begin{cases} E\big\{|\mathbf{u}_r(t;\mathbf{v})-\mathbf{v}|^{2p}\big\}\le C_{r,T}\big[|\nabla\mathbf{v}|^{2p}+|\mathbf{v}|^{4p}_{\mathbb{L}^4(\mathcal{O})}+\|\mathbf{f}\|^{2p}_{\mathbb{V}'}+ \\ \qquad\qquad+(\mathrm{Tr}(\mathbf{g}^*\mathbf{g}))^p\big]\,t^p,\quad\forall t\in[0,T]. \end{cases} \tag{2.30}$$

Next, the strong Markov property, estimates (2.29), (2.30), the fact that $\mathbb{H}\subset\mathbb{V}'$ and $\mathbb{V}\subset\mathbb{L}^4((O))\cap\mathbb{H}$ yield

$$\begin{cases} E\big\{|\mathbf{u}_r(t+s;\mathbf{v})-\mathbf{u}_r(s;\mathbf{v})|^p\big\}\le C_{r,T}\big\{1+\|\mathbf{v}\|^{2p}+|\mathbf{f}|^{2p}+ \\ \qquad\qquad+(\mathrm{Tr}_{\mathbb{V}}(\mathbf{g}^*\mathbf{g}))^p\big\}\,t^{p/2},\quad\forall t,s\in[0,T], \end{cases} \tag{2.31}$$

for any $p \geq 2$, $r > 0$, $T > 0$ and some other constant $C_{r,T}$ depending only on p, r, T, and ν. Hence, the desired estimate on the modulus of continuity of the form (2.28) follows.

To prove estimate (2.23), we notice that in view of the energy equation

$$\begin{cases} d|\mathbf{u}(t)|^2 + 2\nu\,|\nabla \mathbf{u}(t)|^2 dt = \\ \qquad\qquad = \mathrm{Tr}(\mathbf{g}^*\mathbf{g})dt + 2\,\langle \mathbf{f}, \mathbf{u}(t)\rangle\, dt + 2\sum_k (\mathbf{g}_k, \mathbf{u}(t))\, dw_k(t), \end{cases} \tag{2.32}$$

we can apply Itô's formula to the (real-valued) process $y(t) := |\mathbf{u}(t)|^2$ and the function $(\lambda + y)^{p/2} e^{-\alpha t}$, with positive constants λ, p and α, to get

$$\begin{aligned} d(\lambda + |\mathbf{u}(t)|^2)^{p/2} e^{-\alpha t} = \bigg[&\frac{p}{2}\, \frac{\mathrm{Tr}(\mathbf{g}^*\mathbf{g}) + 2\,|\langle \mathbf{f}, \mathbf{u}(t)\rangle| - 2\nu\,|\nabla \mathbf{u}(t)|^2}{\lambda + |\mathbf{u}(t)|^2} + \\ &+ \frac{p(p-2)}{4}\, \frac{\mathrm{Tr}(\mathbf{g}^*\mathbf{g})\,|\mathbf{u}(t)|^2}{(\lambda + |\mathbf{u}(t)|^2)^2} - \alpha \bigg] (\lambda + |\mathbf{u}(t)|^2)^{p/2} e^{-\alpha t}\, dt + \\ &+ p\sum_k (\mathbf{g}_k, \mathbf{u}(t))\,(\lambda + |\mathbf{u}(t)|^2)^{p/2-1} e^{-\alpha t}\, dw_k(t). \end{aligned}$$

Since

$$\begin{cases} \alpha_0(\lambda) := \sup \Big\{ 2p\, \dfrac{\mathrm{Tr}(\mathbf{g}^*\mathbf{g}) + 2\,|\langle \mathbf{f}, \mathbf{u}\rangle| - \nu\,|\nabla \mathbf{u}|^2}{\lambda + |\mathbf{u}|^2} + \\ \qquad\qquad + p(p-2)\, \dfrac{\mathrm{Tr}(\mathbf{g}^*\mathbf{g})\,|\mathbf{u}|^2}{(\lambda + |\mathbf{u}|^2)^2} \ : \ \mathbf{u} \in \mathbb{V} \Big\}, \end{cases} \tag{2.33}$$

is an infinitesimal $0(1/\sqrt{\lambda})$, for any fixed p. Thus for any $\alpha > 0$ and $p > 0$ there is a λ sufficiently large such that $4\alpha \geq \alpha_0(\lambda)$ and then estimate (2.23) holds. Actually, we also have

$$E\{\sup_{t \geq 0} e^{-\alpha t} (\lambda + |\mathbf{u}(t; \mathbf{v})|^2)^{p/2}\} \leq C_p (\lambda + |\mathbf{v}|^2)^{p/2}, \tag{2.34}$$

for any $\mathbf{v}$ in $\mathbb{H}$ and for some constant $C_p > 0$. □

Then the Navier-Stokes semigroup (*NS–Semigroup*) $(\Phi(t),\, t \geq 0)$ defined by $\Phi(t)h(\mathbf{v}) := E\{h(\mathbf{u}(t; \mathbf{v}))\}$, is indeed a Markov-Feller semigroup on the space $C_b(\mathbb{H})$ of continuous and bounded real function on $\mathbb{H}$ endowed with the sup-norm. Since the base space $\mathbb{H}$ is not locally compact, the NS–Semigroup is not *strongly continuous*. After establishing the *strong* Feller property of the NS–Semigroup, i.e., $(t, \mathbf{v}) \mapsto \Phi(t)h(\mathbf{v})$ is continuous for any $t > 0$, $\mathbf{v}$ in $\mathbb{H}$ and any Borel and bounded function, we can use the energy estimate to show

$$\lim_{r\to\infty} \sup \Big\{ \frac{1}{T} \int_0^T P\{|\nabla \mathbf{u}(t; \mathbf{v})| > r\}\, dt \Big\},$$

for some $T_0 > 0$ and $\mathbf{v}$ in $\mathbb{H}$. If the domain $\mathcal{O}$ is bounded, then we can use the results in Chow and Khasminskii [7] to obtain an invariant measure μ, i.e., $\langle h, \mu\rangle = \langle \Phi(t)h, \mu\rangle$, for any Borel and bounded function h. Details regarding the uniqueness of the invariant measure are reported in Flandoli and Maslowski [12]. This approach allows us to consider the NS–Semigroup in a Gauss-Sobolev space of the type $L^2(\mathbb{H}, \mu)$, similar to Chow and Menaldi [8], where $(\Phi(t),\, t \geq 0)$ becomes a strongly continuous semigroup. In our approach, it is convenient to work with unbounded functions. To that purpose, we proceed as follows.

Let $C_p(\mathbb{H})$ be the space of real uniformly continuous functions on any ball and with a growth bounded by the norm to the $p \geq 0$ power, in another words, the space of real functions h on $\mathbb{H}$ such that $\mathbf{v} \mapsto h(\mathbf{v})(1 + |\mathbf{v}|^2)^{-p/2}$ is bounded and locally uniformly continuous, with the weighted sup-norm

$$\|h\| = \|h\|_{C_p} := \sup_{\mathbf{v}\in\mathbb{H}} \{|h(\mathbf{v})|(\lambda+|\mathbf{v}|^2)^{-p/2}\}, \tag{2.35}$$

where λ is a positive constant sufficiently large so that

$$\alpha \geq \alpha_0(p), \quad p \geq 0. \tag{2.36}$$

holds, where $\alpha_0(p)$ is given by (2.33). It is clear that $C_b(\mathbb{H}) \subset C_q(\mathbb{H}) \subset C_p(\mathbb{H})$ for any $0 \leq q < p$.

Then for any $\alpha \geq 0$, (linear) Navier-Stokes semigroup (NS–Semigroup) $(\Phi_\alpha(t),\, t \geq 0)$ with an α–exponential factor is defined as follows

$$\Phi_\alpha(t) : C_p(\mathbb{H}) \longrightarrow C_p(\mathbb{H}), \qquad \Phi_\alpha(t)h(\mathbf{v}) := E\{e^{-\alpha t} h[\mathbf{u}(t; \mathbf{v})]\}, \tag{2.37}$$

where $\mathbf{u}(t; \mathbf{v})$ denotes the solution $\mathbf{u}(x, t, \omega)$ of the stochastic Navier-Stokes equation (2.10) with initial (deterministic) value $\mathbf{u}(x, 0, \omega) = \mathbf{v}(x)$.

Proposition 2.4 (semigroup). *Under assumptions* (2.22) *and* (2.36) *the NS–Semigroup* $(\Phi_\alpha(t),\, t \geq 0)$ *is a weakly continuous Markov-Feller semigroup in the space* $C_p(\mathbb{H})$.

Proof. We the above notation, we have to show that

$$\begin{cases} \Phi_\alpha(t+s) = \Phi_\alpha(t)\Phi_\alpha(s), \quad \forall\, s,t \ge 0, \\ \|\Phi_\alpha(t)h\| \le \|h\|, \quad \forall\, h \in C_p(\mathbb{H}), \\ \Phi_\alpha(t)h(\mathbf{v}) \to h(\mathbf{v}) \ \text{ as } \ t \to 0, \quad \forall\, h \in C_p(\mathbb{H}), \\ \Phi_\alpha(t)h(\mathbf{v}) \ge 0, \quad \forall\, h \ge 0, \quad h \in C_p(\mathbb{H}). \end{cases} \tag{2.38}$$

The Feller character and the weak continuity follows from the locally uniform continuity of the random field with respect to time and initial data. Indeed, by means of Proposition 2.3, and the density of the space $\mathbb{V}$ into $\mathbb{H}$ we obtain

$$\lim_{\delta\to 0} E\Big\{ \sup_{0\le s\le T} \sup_{0\le t\le \delta} |\mathbf{u}(t+s;\mathbf{v}) - \mathbf{u}(s;\mathbf{v})| \Big\} = 0, \tag{2.39}$$

for any $T > 0$ and any $\mathbf{v}$ in $\mathbb{H}$.

To actually prove that $\Phi_\alpha(t)h$ is locally uniformly continuous we use the inequality

$$|\Phi_\alpha(t)h(\mathbf{v}) - \Phi_\alpha(t)h(\bar{\mathbf{v}})| \le |\Phi_\alpha(t)h_r(\mathbf{v})| + |\Phi_\alpha(t)h_r(\bar{\mathbf{v}})| + \\ + e^{-\alpha t} E\big\{ |h(\mathbf{u}(t;\mathbf{v})) - h(\mathbf{u}(t;\bar{\mathbf{v}}))| \mathbf{1}_{\mathbf{u}(t;\mathbf{v})<r,\ \mathbf{u}(t;\bar{\mathbf{v}})<r} \big\},$$

where $h_r(\mathbf{v}) := h(\mathbf{v})$ if $|\mathbf{v}| \ge r$ and $h_r(\mathbf{v}) := 0$ otherwise. Next, we use definition of the norm (2.35) to get

$$|\Phi_\alpha(t)h_r(\mathbf{v})| \le \|h\|_{C_p} E\{(\lambda + |\mathbf{u}(t;\mathbf{v})|^2)^{q/2} e^{-\alpha t}\}\, r^{p-q},$$

for any $q > p$, and in view of estimate (2.34), we deduce that $|\Phi_\alpha(t)h_r(\mathbf{v})|$ and $|\Phi_\alpha(t)h_r(\bar{\mathbf{v}})|$ approach zero as r goes to infinity, locally uniformly in $\mathbf{v}$ and $\bar{\mathbf{v}}$. Next, by means of the locally uniform continuity of the random field $\mathbf{u}(t;\mathbf{v})$, we conclude. □

Since the NS–Semigroup is not strongly continuous, we cannot consider the *strong* infinitesimal generator as acting on a dense domain in $C_p(\mathbb{H})$. However, this Markov-Feller semigroup $(\Phi_\alpha(t),\, t \ge 0)$ may be considered as acting on real Borel functions with p-polynomial growth, which is Banach space with the norm (2.35) and denoted by $B_p(\mathbb{H})$. It is convenient to define the family of semi-norms on $B_p(\mathbb{H})$

$$p_0(h,\mathbf{v}) := E\big\{ \sup_{s\ge 0} |h(\mathbf{u}(s;\mathbf{v}))|\, e^{-\alpha_0 s} \big\}, \quad \forall \mathbf{v} \in \mathbb{H}, \tag{2.40}$$

where $\alpha_0 = \alpha_0(\lambda)$ is given by (2.33) and λ is sufficiently large so that (2.36) holds. If a sequence $\{h_n\}$ of equi-bounded functions in $B_p(\mathbb{H})$ satisfies $p_0(h_n - h, \mathbf{v}) \to 0$ for any $\mathbf{v}$ in $\mathbb{H}$, we say that $h_n \to h$ boundedly pointwise convergence relative to the above family of semi-norms. In view of (2.39), it is clear that $p_0(\Phi_\alpha(t)h - h, \mathbf{v}) \to 0$ as $t \to 0$, for any function h in $C_p(\mathbb{H})$ and any $\mathbf{v}$ in $\mathbb{H}$.

Definition 2.5. Let $\bar{C}_p(\mathbb{H})$ be the subspace of functions $\bar{h}$ in $B_p(\mathbb{H})$ such that the mapping $t \mapsto \bar{h}[\mathbf{u}(t;\mathbf{v})]$ is almost surely continuous on $[0, +\infty)$ for any $\mathbf{v}$ in $\mathbb{H}$ and satisfies

$$\lim_{t\to 0} p_0(\Phi_\alpha(t)\bar{h} - \bar{h}, \mathbf{v}) = 0, \quad \forall\, \mathbf{v} \in \mathbb{H}. \tag{2.41}$$

where $p_0(\cdot,\cdot)$ is given by (2.40). □

This is the space of function (uniformly) continuous over the random field $\mathbf{u}(\cdot,\mathbf{v})$, relative to the family of semi-norms (2.40) and it is independent of α, as long as (2.36) holds. Hence, we may consider the NS–Semigroup on the Banach space $\bar{C}_p(\mathbb{H})$, endowed with the norm (2.35). The *weak* infinitesimal generator $-\bar{\mathcal{A}}_\alpha$ with domain $\mathcal{D}_p(\bar{\mathcal{A}}_\alpha)$ (as a subspace of $\bar{C}_p(\mathbb{H})$) is defined by boundedly pointwise limit $[h - \Phi_\alpha(t)h]/t \to \bar{\mathcal{A}}_\alpha h$ as $t \to 0$, relative to the family of semi-norms (2.40). By means of the finite-dimensional approximations, we can show that if h is *smooth cylindrical* function in $\mathbb{H}$ then the (weak) infinitesimal generator have the form

$$-\bar{\mathcal{A}}_\alpha h(\mathbf{u}) = \frac{1}{2}\mathrm{Tr}[\mathbf{g}^*\mathbf{g}D^2_{\mathbf{u}}h(\mathbf{u})] + \langle A\mathbf{u} + B(\mathbf{u}) - \alpha\,\mathbf{u}, D_{\mathbf{u}}h(\mathbf{u})\rangle,$$

when considering A and $B(\cdot)$ as mappings from $\mathbb{H}$ into the dual of $\mathbb{V} \cap \mathbb{H}^2(\mathcal{O},\mathbb{R}^2)$ and the dual $\mathbb{V} \cap \mathbb{W}^{1,\infty}(\mathcal{O},\mathbb{R}^2)$, respectively. Also, it is clear that $p_0(\Phi_\alpha(t)\bar{h}, \mathbf{v}) \le p_0(\bar{h}, \mathbf{v})$ for any $t \ge 0$, $\bar{h}$ in $\bar{C}_p(\mathbb{H})$ and $\mathbf{v}$ in $\mathbb{H}$.

Proposition 2.6 (density). *If assumptions* (2.22) *and* (2.36) *hold, then* $C_p(\mathbb{H}) \subset \bar{C}_p(\mathbb{H})$, *the NS–Semigroup leaves invariant the space* $\bar{C}_p(\mathbb{H})$ *and for any function* $\bar{h}$ *in* $\bar{C}_p(\mathbb{H})$, *there is a equi-bounded sequence* $\{\bar{h}_n\}$ *of functions in* $\mathcal{D}_p(\bar{\mathcal{A}}_\alpha)$ *satisfying* $p_0(\bar{h}_n - \bar{h}, \mathbf{v}) \to 0$ *for any* $\mathbf{v}$ *in* $\mathbb{H}$.

Proof. Indeed, since any function h in $C_p(\mathbb{H})$ is such that $\mathbf{v} \mapsto h(\mathbf{v})\,(\lambda + |\mathbf{v}|^2)^{-q/2}$, $q > p$ is uniformly continuous for $\mathbf{v}$ in $\mathbb{H}$, we may use estimate (2.34) to reduce the prove of property (2.41) to the following condition

$$\lim_{t\to 0} P\big\{\sup_{0\le s\le T} |\mathbf{u}(t+s;\mathbf{v}) - \mathbf{u}(s;\mathbf{v})|\big\} = 0, \quad \forall\, \mathbf{v} \in \mathbb{H},\ T > 0, \tag{2.42}$$

which follows from (2.39). This verifies the fact that $C_p(\mathbb{H}) \subset \bar{C}_p(\mathbb{H})$.

Next, from the strong Markov property we deduce

$$p_0(\Phi_\alpha(t)\bar{h}, \mathbf{v}) = E\big\{\sup_{s\geq 0} E\{|\bar{h}[\mathbf{u}(t+s;\mathbf{v})]|e^{-\alpha_0(t+s)} \ : \ \mathcal{F}_t\}e^{-(\alpha-\alpha_0)t}\big\} \leq$$
$$\leq E\big\{\sup_{s\geq 0} |\bar{h}[\mathbf{u}(t+s;\mathbf{v})]|e^{-\alpha_0(t+s)}\big\} = p_0(\bar{h}, \mathbf{v}),$$

for any $\mathbf{v}$ in $\mathbb{H}$ and $t \geq 0$. Therefore,

$$p_0(\Phi_\alpha(r+t)\bar{h} - \Phi_\alpha(t)\bar{h}, \mathbf{v}) = p_0(\Phi_\alpha(t)[\Phi_\alpha(r)\bar{h} - \bar{h}], \mathbf{v}) \leq$$
$$\leq p_0(\Phi_\alpha(r)\bar{h} - \bar{h}, \mathbf{v}),$$

which prove that the space $\bar{C}_p(\mathbb{H})$ is invariant under the NS–Semigroup.

Finally, to approximate any function $\bar{h}$ in $\bar{C}_p(\mathbb{H})$ by regular functions, we can define the sequence $\{\bar{h}_n \ n = 1, 2, \dots\}$ by

$$\bar{h}_n(\mathbf{v}) := n\int_0^\infty e^{-nt}\Phi_\alpha(t)\bar{h}(\mathbf{v})dt = \int_0^\infty e^{-t} E\big\{\bar{h}(\mathbf{u}(\frac{t}{n};\mathbf{v}))e^{-\alpha(\frac{t}{n})}\big\}dt,$$

and apply Markov property to get

$$\big|E\big\{\sup_{s\geq 0}[\bar{h}_n(\mathbf{u}(s;\mathbf{v})) - \bar{h}(\mathbf{u}(s;\mathbf{v}))]e^{-\alpha_0 s}\big\}\big| \leq$$
$$\leq \int_0^\infty e^{-t}\big[E\big\{\sup_{s\geq 0} |\bar{h}(\mathbf{u}(s+\frac{t}{n};\mathbf{v}))e^{-\alpha(\frac{t}{n})} - \bar{h}(\mathbf{u}(s;\mathbf{v}))|e^{-\alpha_0 s}\big\}\big]dt.$$

Thus, from estimates (2.34) and (2.39) we deduce

$$\lim_{n\to\infty} \big|E\big\{\sup_{s\geq 0}[\bar{h}_n(\mathbf{u}(s;\mathbf{v})) - \bar{h}(\mathbf{u}(s;\mathbf{v}))]e^{-\alpha_0 s}\big\}\big| = 0,$$

for any fixed $\mathbf{v}$ in $\mathbb{H}$. □

Under the assumption (2.22), a clear consequence of the above results is that given $\alpha > 0$, $p \geq 0$, λ sufficiently large to ensure (2.23) and a function $\bar{h}$ in $\bar{C}_p(\mathbb{H})$ there is another function $\bar{u}$ in $\mathcal{D}_p(\bar{\mathcal{A}}_\alpha)$ such that $\bar{\mathcal{A}}_\alpha\bar{u} = \bar{h}$, where the solution admits the explicitly representation

$$\bar{u} = \int_0^\infty \Phi_\alpha(t)\bar{h}\,dt. \tag{2.43}$$

The right-hand side is called the *weak* resolvent operator and denoted by either $\mathcal{R}_\alpha := \bar{\mathcal{A}}_\alpha^{-1}$ or $\mathcal{R}_\alpha := (\bar{\mathcal{A}}_0 + \alpha I)^{-1}$. Moreover, if $\alpha_0 = \alpha_0(\lambda)$ is the positive constant defined by (2.33) then for any $p > 0$ we have $\alpha_0(\lambda) \to 0$ as $\lambda \to \infty$, and for any stopping time τ,

$$\begin{cases} \frac{p\,\nu}{2} E\big\{ \int_0^\tau |\nabla \mathbf{u}(t;\mathbf{v})|^2 (\lambda + |\mathbf{u}(t;\mathbf{v})|^2)^{p/2-1} e^{-\alpha_0 t} dt \big\} + \\ \qquad + E\big\{ e^{-\alpha_0 \tau} (\lambda + |\mathbf{u}(\tau;\mathbf{v})|^2)^{p/2} \big\} \leq (\lambda + |\mathbf{v}|^2)^{p/2}, \quad \forall \mathbf{v} \in \mathbb{H}, \end{cases} \tag{2.44}$$

and then for any $\alpha > \alpha_0$ we obtain

$$\|\Phi_\alpha(t)\bar{h}\| \leq e^{-(\alpha-\alpha_0)t} \|\bar{h}\|, \qquad p_0(\Phi_\alpha(t)\bar{h}, \mathbf{v}) \leq e^{-(\alpha-\alpha_0)t} p_0(\bar{h}, \mathbf{v}), \tag{2.45}$$

for any $t \geq 0$, and

$$\|\mathcal{R}_\alpha \bar{h}\| \leq \frac{1}{\alpha - \alpha_0} \|\bar{h}\|, \qquad p_0(\mathcal{R}_\alpha \bar{h}, \mathbf{v}) \leq \frac{1}{\alpha - \alpha_0} p_0(\bar{h}, \mathbf{v}), \tag{2.46}$$

for any $\mathbf{v}$ in $\mathbb{H}$ and where the norm $\|\cdot\|$ and the semi-norms $p_0(\cdot, \mathbf{v})$ given by (2.35) and (2.40), respectively. Notice that $\alpha_0(\lambda) = 0$ for $p = 0$, and it is clear that for any $\bar{h} \leq h$ (pointwise) we have $\mathcal{R}_\alpha \bar{h} \leq \mathcal{R}_\alpha h$, which is a weak form of the maximum principle.

Notice that the weak infinitesimal used above is a variation of the one proposed in Priola [16].

3 Stopping Time Problem

For the sake of simplicity, we consider only the time-independent case, we assume (2.22), i.e. $\mathbf{f} \in \mathbb{V}'$ and $\mathbf{g} \in \ell_2(\mathbb{H})$. The time-evolution case can be studied with essentially the same techniques.

Recall that $\bar{C}_p(\mathbb{H})$ is as in Definition 2.5. Then, given two functions F and G in $\bar{C}_p(\mathbb{H})$ and $\alpha > 0$ we consider the *cost* functional

$$J(\mathbf{v}, \tau) := E\big\{ \int_0^\tau F(\mathbf{u}(t;\mathbf{v})) e^{-\alpha t} dt + \mathbf{1}_{\tau<\infty} G(\mathbf{u}(\tau;\mathbf{v})) e^{-\alpha\tau} \big\} \tag{3.1}$$

and the *optimal cost*

$$\hat{U}(\mathbf{v}) := \inf_\tau J(\mathbf{v}, \tau), \tag{3.2}$$

where the infimum is taken over all stopping times τ. Our purpose is to give a characterization of the optimal cost functional $\hat{U}$ and to exhibit an optimal stopping time $\hat{\tau}$.

This type of optimal stopping time problems is very well known, but only a few number of results are available for Markov processes on (not necessarily locally compact) polish spaces, cf. Bensoussan [2, Chapter 7, pp. 279–353] and Zabczyk [25], where some conditions are given under which the optimal cost (or value function) is continuous and the first moment of hitting the contact set is an optimal one. However, they cannot be used directly in this context.

A natural way of studying optimal stopping times is via the so-called penalized problems. Given $\alpha, \varepsilon > 0$ and F and G in $\bar{C}_p(\mathbb{H})$, we want to solve the nonlinear equation

$$U_\varepsilon \in \mathcal{D}_p(\bar{\mathcal{A}}_\alpha) \quad \text{such that} \quad \bar{\mathcal{A}}_\alpha U_\varepsilon + \frac{1}{\varepsilon}(U_\varepsilon - G)^+ = F, \tag{3.3}$$

where $(\cdot)^+$ denote the positive part and $\mathcal{D}_p(\bar{\mathcal{A}}_\alpha)$ is the domain of the weak infinitesimal generator $-\bar{\mathcal{A}}_\alpha$ of the NS–Semigroup $(\Phi_\alpha(t),\, t \geq 0)$. The solution U_ε of (3.3) can be interpreted as an optimal cost (or valued function) of a stochastic optimal control problem.

Proposition 3.1. *Let conditions* (2.22), (2.36) *and*

$$F,\ G \in \bar{C}_p(\mathbb{H}), \tag{3.4}$$

hold. Then, for any $\varepsilon > 0$, *there is one and only one solution of* (3.3). *Moreover, if F and G belong to $C_p(\mathbb{H})$ then U_ε also belongs to $C_p(\mathbb{H})$. Furthermore, if G belongs to $\mathcal{D}_p(\bar{\mathcal{A}}_\alpha)$ then the following estimate*

$$0 \leq U_\varepsilon(\mathbf{v}) - U_{\varepsilon'}(\mathbf{v}) \leq C_p\, \varepsilon\, \|(F - \bar{\mathcal{A}}_\alpha G)^+\|\, (\lambda + |\mathbf{v}|^2)^{p/2}, \tag{3.5}$$

is valid for any $0 < \varepsilon' < \varepsilon$, $\mathbf{v}$ *in* $\mathbb{H}$ *and* C_p *as in* (2.34).

Proof. First we notice that $(U_\varepsilon - G)^+ = U_\varepsilon - U_\varepsilon \wedge G$, where $\wedge$ denotes the minimum between two values. Thus

$$\bar{\mathcal{A}}_\alpha U_\varepsilon + \frac{1}{\varepsilon} U_\varepsilon = F + \frac{1}{\varepsilon}(U_\varepsilon \wedge G),$$

i.e., equation (3.3) is equivalent to a fixed point of the mapping T_ε from $\bar{C}_p(\mathbb{H})$ into itself defined by

$$T_\varepsilon(h) := (\bar{\mathcal{A}}_{\alpha+1/\varepsilon})^{-1}[F + \frac{1}{\varepsilon}(h \wedge G)].$$

Since

$$T_\varepsilon(h) - T_\varepsilon(\bar{h}) = \frac{1}{\varepsilon}(\bar{\mathcal{A}}_{\alpha+1/\varepsilon})^{-1}[h \wedge G - \bar{h} \wedge G],$$

we have

$$\|T_\varepsilon(h) - T_\varepsilon(\bar{h})\| \le \frac{1/\varepsilon}{\alpha - \alpha_0 + 1/\varepsilon} \|h \wedge G - \bar{h} \wedge G\|,$$

after using (2.46). Thus, T_ε is a contraction mapping on $\bar{C}_p(\mathbb{H})$, and there is a unique solution to equation (3.3), denoted by U_ε. Since, T_ε leave invariant the subspace $C_p(\mathbb{H})$, if F and G belong to $C_p(\mathbb{H})$ then U_ε belongs to $C_p(\mathbb{H})$.

Since

$$\bar{\mathcal{A}}_\alpha(U_\varepsilon - U_{\varepsilon'}) = -\big(\frac{1}{\varepsilon'} - \frac{1}{\varepsilon}\big)\big(U_{\varepsilon'} - G\big)^+ - \frac{1}{\varepsilon}\big[(U_{\varepsilon'} - G)^+ - (U_\varepsilon - G)^+\big],$$

we deduce that for $0 < \varepsilon' < \varepsilon$

$$\bar{\mathcal{A}}_\alpha(U_\varepsilon - U_{\varepsilon'}) \le 0 \quad \text{if} \quad U_{\varepsilon'} - U_\varepsilon > 0,$$

which yields $U_{\varepsilon'} \le U_\varepsilon$.

If G belongs to $\mathcal{D}_p(\bar{\mathcal{A}}_\alpha)$ then

$$\begin{aligned} U_\varepsilon - G = (\bar{\mathcal{A}}_{\alpha+1/\varepsilon})^{-1}[F - \frac{1}{\varepsilon}(G - U_\varepsilon)^+ - \bar{\mathcal{A}}_\alpha G] \le \\ \le (\bar{\mathcal{A}}_{\alpha+1/\varepsilon})^{-1}[F - \bar{\mathcal{A}}_\alpha G]. \end{aligned}$$

Hence

$$\begin{cases} \|(U_\varepsilon - G)^+\| \le \varepsilon \, \|(F - \bar{\mathcal{A}}_\alpha G)^+\|, \\ p_0([U_\varepsilon - G]^+, \mathbf{v}) \le \varepsilon \, p_0((F - \bar{\mathcal{A}}_\alpha G)^+, \mathbf{v}), \end{cases} \tag{3.6}$$

for any $\mathbf{v}$ in $\mathbb{H}$ and $\varepsilon > 0$.

Since $0 < \varepsilon' < \varepsilon$ we have $U_{\varepsilon'} \le U_\varepsilon$, and then

$$\begin{aligned} &\bar{\mathcal{A}}_{\alpha+1/\varepsilon}(U_\varepsilon - U_{\varepsilon'}) \le \frac{1}{\varepsilon'}(U_{\varepsilon'} - G)^+, \quad \text{if } U_{\varepsilon'} \ge G, \\ &\bar{\mathcal{A}}_\alpha(U_\varepsilon - U_{\varepsilon'}) \le 0, \quad \text{if } U_{\varepsilon'} < G, \end{aligned}$$

yields

$$U_\varepsilon - U_{\varepsilon'} \le \frac{1}{\varepsilon'} E\Big\{ \int_0^\infty [U_{\varepsilon'}(\mathbf{u}(t;\mathbf{v})) - G(\mathbf{u}(t;\mathbf{v}))]^+ \, e^{-\alpha t} \, \chi_{\varepsilon,\varepsilon'}(t) dt \Big\},$$

with

$$\chi_{\varepsilon,\varepsilon'}(t) = \exp\big[-\frac{1}{\varepsilon}\int_0^t \mathbf{1}_{U_{\varepsilon'}(\mathbf{u}(s;\mathbf{v}))>G(\mathbf{u}(s;\mathbf{v}))} ds\big].$$

Hence

$$0 \le U_\varepsilon(\mathbf{v}) - U_{\varepsilon'}(\mathbf{v}) \le \frac{\|[u'_\varepsilon - \psi]^+\|}{\varepsilon'} \times \\ \times E\Big\{ \sup_{t\ge 0} e^{-\alpha t}\,(\lambda + |\mathbf{u}(t;\mathbf{v})|^2)^{p/2} \int_0^\infty \mathbf{1}_{U_{\varepsilon'}(\mathbf{u}(s;\mathbf{v}))>G(\mathbf{u}(s;\mathbf{v}))}\, \chi_{\varepsilon,\varepsilon'}(t)\, dt\Big\},$$

for any $\mathbf{v}$ in $\mathbb{H}$ and $\varepsilon > \varepsilon' > 0$. This yields estimate (3.5), after using estimates (2.34) and (3.6).

Notice that most of the above estimates can be obtained from the representation of u_ε as the following optimal cost

$$\begin{cases} U_\varepsilon(\mathbf{v}) := \inf\big\{J_0(\mathbf{v},\delta) \,:\, \delta \text{ adapted},\ 0 \le \varepsilon\delta \le 1\big\}, \quad \text{where} \\ J_0(\mathbf{v},\delta) = E\big\{ \int_0^\infty [F(\mathbf{u}(t;\mathbf{v})) + \delta(t) G(\mathbf{u}(t;\mathbf{v}))]\, e^{-\int_0^t(\alpha+\delta(s))ds}\, dt\big\}, \end{cases} \tag{3.7}$$

valid for any $\mathbf{v}$ in $\mathbb{H}$ and $\varepsilon > 0$. □

Let us consider the problem of finding

$$U \in \bar{C}_p(\mathbb{H}) \quad \text{such that} \quad U \le G, \quad \text{and} \quad \bar{\mathcal{A}}_\alpha U \le F, \tag{3.8}$$

usually referred to as a *sub-solution*. Notice that since U does not necessary belongs $\mathcal{D}_p(\bar{\mathcal{A}}_\alpha)$, the domain of the weak infinitesimal generator $-\bar{\mathcal{A}}_\alpha$ of the NS–Semigroup $(\Phi_\alpha(t),\, t \ge 0)$, the last inequality $\bar{\mathcal{A}}_\alpha U \le F$ is understood in the semigroup sense, i.e.,

$$U(\mathbf{v}) \le \Phi_\alpha(t) U(\mathbf{v}) + \int_0^t \Phi_\alpha(s) F(\mathbf{v}) ds, \quad \forall t \ge 0,\ \mathbf{v} \in \mathbb{H}. \tag{3.9}$$

We have

Theorem 3.2 (VI). *Under conditions* (2.22)*,* (2.36) *and* (3.4)*, the optimal cost* $\hat{U}$ *defined by* (3.2) *is the maximum sub-solution of problem* (3.9) *and it is given as the boundedly pointwise limit*[2] *of the penalized solutions* U_ε *of* (3.3)

[2] relative to the family of semi-norms (2.40)

as ε goes to zero. Moreover the exit time of the continuation region $\hat\tau = \hat\tau(\mathbf{v})$ defined by

$$\hat\tau(\mathbf{v}) := \inf \big\{ t \geq 0 \ : \ \hat U[\mathbf{u}(t;\mathbf{v})] = G[\mathbf{u}(t;\mathbf{v})] \big\}, \quad \forall \mathbf{v} \in \mathbb{H}, \tag{3.10}$$

is optimal, i.e., $\hat U(\mathbf{v}) = J(\mathbf{v}, \hat\tau)$. Furthermore, if G belongs to $\mathcal{D}_p(\bar{\mathcal{A}}_\alpha)$ then the Lewy-Stampacchia inequality

$$F \wedge \bar{\mathcal{A}}_\alpha G \leq \bar{\mathcal{A}}_\alpha \hat U \leq F \tag{3.11}$$

holds and U_ε converges to $\hat U$ in the sup–norm of $\hat C_p(\mathbb{H})$, therefore $\hat U$ belongs to $C_p(\mathbb{H})$, whenever F and G are in $C_p(\mathbb{H})$.

Proof. First, in view of (3.5) of Proposition 3.1, we can define

$$\bar U := \lim_{\varepsilon \to 0} U_\varepsilon, \tag{3.12}$$

as a monotone limit. If G belongs to $\mathcal{D}_p(\bar{\mathcal{A}}_\alpha)$ then the above convergence is also in norm, so that U belongs to $\hat C_p(\mathbb{H})$. Since u_ε can be re-written as an optimal cost in the form (3.7), from estimates (2.34) and (2.46) we obtain

$$\begin{cases} \|U_\varepsilon(G) - U_\varepsilon(\bar G)\| \leq C_p \, \|G - \bar G\|, \quad \text{and} \\ p_0(U_\varepsilon(G) - U_\varepsilon(\bar G), \mathbf{v}) \leq p_0(G - \bar G, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbb{H}, \end{cases} \tag{3.13}$$

where $U_\varepsilon(G)$ and $U_\varepsilon(\bar G)$ denote the penalized solutions corresponding to G and $\bar G$, respectively. Now, in view of the density of $\mathcal{D}_p(\bar{\mathcal{A}}_\alpha)$ in $\hat C_p(\mathbb{H})$ established in Proposition 2.6 and the above estimate (3.13), we deduce that the limit (3.12) used to define $\bar U$ holds true as a boundedly pointwise limit relative to the family of semi-norms (2.40).

Next, if U is a sub-solution, i.e., a solution of (3.9) then

$$U \leq (\bar{\mathcal{A}}_{\alpha+1/\varepsilon})^{-1}[F + \frac{1}{\varepsilon}(U \wedge G)] = T_\varepsilon(U),$$

and by iteration

$$U \leq T_\varepsilon U \leq T_\varepsilon^2 U \leq \cdots \leq T_\varepsilon^n U \to U_\varepsilon,$$

as n go to infinity. Therefore $U \leq U_\varepsilon$, which yields $U \leq \bar U$ proving that the function $\bar U$, given by the limit (3.12), is the maximum sub-solution of problem (3.8).

To establish the Lewy-Stampacchia inequality (3.11), which is interpreted in the semigroup sense (3.9), we consider the linear equation

$$V_\varepsilon \in \mathcal{D}_p(\bar{\mathcal{A}}_\alpha) \quad \text{such that} \quad \bar{\mathcal{A}}_{\alpha+1/\varepsilon} V_\varepsilon = \frac{1}{\varepsilon}(F - \bar{\mathcal{A}}_\alpha G)^+, \tag{3.14}$$

to notice that, as in the prove of estimate (3.7), we have

$$\frac{1}{\varepsilon}[U_\varepsilon(\mathbf{v}) - G(\mathbf{v})] \le V_\varepsilon(\mathbf{v}), \quad \forall \mathbf{v} \in \mathbb{H}. \tag{3.15}$$

Since

$$V_\varepsilon(\mathbf{v}) = \int_0^\infty e^{-t} \Phi_\alpha(\varepsilon t)[(F - \bar{\mathcal{A}}_\alpha G)^+](\mathbf{v}) dt, \quad \forall \mathbf{v} \in \mathbb{H},$$

we obtain $V_\varepsilon \to (F - \bar{\mathcal{A}}_\alpha G)^+$ boundedly pointwise and (3.11) follows.

It remains to prove that $\bar{U}$ is actually the optimal cost $\hat{U}$ given by (3.2). To that purpose, first we notice that for any stopping time τ we have

$$U_\varepsilon(\mathbf{v}) \le E\Big\{ \int_0^{\tau\wedge T} e^{-\alpha t} F[\mathbf{u}(t;\mathbf{v})] dt + e^{-\alpha\tau\wedge T} U_\varepsilon[\mathbf{u}(\tau\wedge T;\mathbf{v})] \Big\}, \tag{3.16}$$

and as $\varepsilon \to 0$ and $T \to \infty$ we get

$$\bar{U}(\mathbf{v}) \le E\Big\{ \int_0^{\tau} e^{-\alpha t} F[\mathbf{u}(t;\mathbf{v})] dt + e^{-\alpha\tau} G[\mathbf{u}(\tau;\mathbf{v})] \mathbf{1}_{\tau<\infty} \Big\}$$

after remarking that $\bar{U} \le G$. Thus $\bar{U}(\mathbf{v}) \le J(\mathbf{v}, \tau)$, for any $\mathbf{v}$ in $\mathbb{H}$ and any stopping time τ. On the other hand, take $\tau = \tau^\varepsilon$,

$$\tau^\varepsilon(\mathbf{v}) := \inf \big\{ t \ge 0 \ : \ U_\varepsilon[\mathbf{u}(t;\mathbf{v})] \ge G[\mathbf{u}(t;\mathbf{v})] \big\},$$

to have an equality in (3.16), i.e.,

$$U_\varepsilon(\mathbf{v}) = E\Big\{ \int_0^{\tau^\varepsilon\wedge T} e^{-\alpha t} F[\mathbf{u}(t;\mathbf{v})] dt + e^{-\alpha\tau^\varepsilon\wedge T} U_\varepsilon[\mathbf{u}(\tau^\varepsilon\wedge T;\mathbf{v})] \Big\}.$$

Hence, as $T \to \infty$ and $\varepsilon \to 0$ we obtain $\bar{U}(\mathbf{v}) = J(\mathbf{v}, \bar{\tau})$, where $\bar{\tau}$ is the (monotone increasing) limit of τ^ε. This proves that $\bar{U} = \hat{U}$.

Let us show that $\bar{\tau}$ is actually the exit time of the continuation region to complete the proof. Indeed, since $U_\varepsilon \ge \hat{U}$, we get $\hat{\tau} \ge \tau^\varepsilon$, i.e., $\hat{\tau} \ge \bar{\tau}$. Now, if $\theta < \hat{\tau}$ then $\hat{U}[\mathbf{u}(t;\mathbf{v})] < G[\mathbf{u}(t;\mathbf{v})]$, for any $t \le \theta$. Using the fact that U_ε converges to $\hat{U}$ uniformly over trajectories of the random field $\mathbf{u}(t;\mathbf{v})$ for any fixed $\mathbf{v}$ in $\mathbb{H}$ and t in $[0, \theta]$, we must have $U_\varepsilon[\mathbf{u}(t;\mathbf{v})] < G[\mathbf{u}(t;\mathbf{v})]$, for any $t \le \theta$, provided ε is sufficiently small. This is $\bar{\tau} \ge \theta$, which yields $\bar{\tau} \ge \hat{\tau}$. □

Notice that by considering the semigroup formulation (3.9), the penalized problem (3.3) and the sub-solution problem (3.8) make sense for any function F and G in $B_p(\mathbb{H})$. Based on the monotonicity in F and G of the representation of U_ε, the main results of Proposition 3.1 and Theorem 3.2 are still valid if F and G are in the semi-space $\bar{C}_p^u(\mathbb{H})$ of upper-continuous functions over the random field, i.e. pointwise limits of decreasing equi-bounded sequences of functions in $\bar{C}_p(\mathbb{H})$. This means that for any data F and G in $\bar{C}_p^u(\mathbb{H})$, there is a maximum sub-solution $\hat{U}$ of (3.8) in $\bar{C}_p^u(\mathbb{H})$. Consider the following semi-space

Definition 3.3. Let $\mathcal{D}_p^u(\bar{\mathcal{A}}_\alpha)$ denote the semi-space of functions h such that there is a decreasing sequence $\{h_k\}$ in $\bar{C}_p(\mathbb{H})$ satisfying

$$\begin{cases} h_k(\mathbf{v}) \to h(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbb{H}, \\ \|h_k\|_p + \|\bar{\mathcal{A}}_\alpha h_k\| \le C, \quad \forall k = 1, 2, \dots, \end{cases} \tag{3.17}$$

for some constant C depending on h and where $\|\cdot\|$ is the norm (2.35). By convention we set $\bar{\mathcal{A}}_\alpha h := \wedge_k \bar{\mathcal{A}}_\alpha h_k$, meaning the pointwise infimum in k. □

Corollary 3.4. *Let us assume* (2.22), (2.36) *and* (3.4). *If G belongs to the semi-space $\mathcal{D}_p^u(\bar{\mathcal{A}}_\alpha)$ then the optimal cost $\hat{U}$ belongs to the semi-space $\mathcal{D}_p^u(\bar{\mathcal{A}}_\alpha)$, and estimates* (3.5) *and* (3.11) *hold. Moreover the penalized solution U_ε converges $\hat{U}$ in the norm* (2.35)*, in particular $\hat{U}$ belongs to $C_p(\mathbb{H})$ (is continuous) if F and G are in $C_p(\mathbb{H})$.*

Proof. Repeating the arguments of Proposition 3.1 and Theorem 3.2 we obtain the a priori estimates (3.5) and (3.11), after observing the monotonicity of $U\varepsilon$ with respect to G and the fact that $\wedge_{k\le n}\bar{\mathcal{A}}_\alpha G_k$ belongs to $\bar{C}_p(\mathbb{H})$, for any $n = 1, 2, \dots$, if G_k are also in $\mathcal{D}_p(\bar{\mathcal{A}}_\alpha)$. □

In general, if F and G are only Borel measurable functions in $B_p(\mathbb{H})$ then we cannot ensure neither that the maximum sub-solution exists nor that the penalized solutions U_ε converges to a sub-solution.

Another way to extend the meaning of the weak infinitesimal generator is to set up the sub-solution problem (3.8) as the following complementary problem. Find U in $\bar{C}_p(\mathbb{H})$ such that

$$U \le G, \quad \bar{\mathcal{A}}_\alpha U \le F, \quad \text{and} \quad \bar{\mathcal{A}}_\alpha U = F \quad \text{in} \quad [U < G], \tag{3.18}$$

where $\bar{\mathcal{A}}_\alpha$ is interpreted in the *martingale sense*, i.e., $\bar{\mathcal{A}}_\alpha U \leq F$ means that the process

$$U[\mathbf{u}(t;\mathbf{v})] + \int_0^t F[\mathbf{u}(s;\mathbf{v})]\,ds, \quad \forall t \geq 0,\ \mathbf{v} \in \mathbb{H}, \tag{3.19}$$

is a (continuous) sub-martingale, which is equivalent to the semigroup sense (3.9). The key point is the meaning given to $\bar{\mathcal{A}}_\alpha U = F$ in $[U < G]$ as

$$\begin{cases} \tau := \inf\{\, t \geq 0 \ : \ U[\mathbf{u}(t;\mathbf{v})] \geq G[\mathbf{u}(t;\mathbf{v})] \,\}, \\ U[\mathbf{u}(t \wedge \tau;\mathbf{v})] + \displaystyle\int_0^{t\wedge\tau} F[\mathbf{u}(s;\mathbf{v})]\,ds, \quad \forall t \geq 0,\ \mathbf{v} \in \mathbb{H} \end{cases} \tag{3.20}$$

is a (continuous) martingale. Thus, under the assumptions of Theorem 3.2, the maximum sub-solution of problem (3.8) or the optimal cost (3.2) is the unique solution of the so-called *variational inequality* (3.18).

4 Impulse Control Problem

In the previous section, our only action on the stochastic dynamic systems $\mathbf{u}(t;\mathbf{v})$ is to stop (or to continue) the evolution. Continue the evolution involves a *running cost* represented by the functional F and a decision to stop at the random time τ incurs in a *terminal cost* given by G. If the costs are reduced to money, then α is interpreted as the discount factor.

Now, we would like to sequentially control the evolution of the stochastic dynamic systems $\mathbf{u}(t;\mathbf{v})$ by changing the initial conditions $\mathbf{v}$. To that purpose, we are given a controlled Markov chain $\mathbf{q}_k(i)$ in $\mathbb{H}$ with transition operator $Q(k)$ where the control parameter k belongs to a compact metric space K. This is, for a sequence of independent identically distributed $\mathbb{H}$–valued random variables (ζ_i, $i = 1, 2, \ldots$) we have

$$\begin{cases} \mathbf{q}_k(i+1) = \mathbf{q}(\mathbf{q}_k(i), \zeta_i \,|\, k), \quad \forall i = 1, 2, \ldots, \\ E\big\{h(\mathbf{q}(\mathbf{v}, \zeta_1 \,|\, k))\big\} = Q(k)h(\mathbf{v}), \quad \forall \mathbf{v} \in \mathbb{H}, \end{cases} \tag{4.1}$$

for any initial value $\mathbf{q}(1)$, any bounded and measurable real-valued function h on $\mathbb{H}$ and any k in K. For the sake of simplicity, this Markov chain (i.e., each random variable ζ_i) is assumed to be independent of the Wiener process $w = (w_1, w_2, \ldots)$ used to model the disturbances in dynamic equation (2.10).

An impulse control is a sequence $\{\tau_i, k_i;\ i = 1, 2, \ldots\}$ of stopping times τ_i and decisions k_i such that τ_i approaches infinity. At time $t = \tau_i$ the system has an impulse described by the (controlled) Markov chain $\mathbf{q}_k(i)$ with $k = k_i$. Between two consecutive times $\tau_i \leq t < \tau_{i+1}$, the evolution follows the Navier-Stokes equation (2.10). This is

$$\begin{cases} \mathbf{u}(t) = \mathbf{u}(t, \tau_i; \mathbf{u}(\tau_i)), \quad \text{if} \quad \tau_i \leq t < \tau_{i+1}, \\ \text{and} \quad \mathbf{u}(\tau_i) = \mathbf{q}(\mathbf{u}(\tau_i-), \zeta_i \,|\, k_i), \end{cases} \tag{4.2}$$

where $\mathbf{u}(t, s; \mathbf{v})$ is the NS–random field with initial conditions $\mathbf{v}$ at the time s, and the τ_i- means the left-hand limit at τ_i. Since $\tau_i \to \infty$, we can construct the process $\mathbf{u}(t)$ by iteration of (4.2), for any impulse control $\{\tau_i, k_i;\ i = 1, 2, \ldots\}$ and initial condition $\mathbf{v}$ in $\mathbb{H}$. Therefore, the dynamic evolution is a stochastic process no continuous (even in probability), it is only right-continuous with left-hand limits. Notice that the control where all the stopping time $\tau_i = \infty$, is valid and means that we are keeping the same initial conditions, i.e., no-intervention decision. It is clear that τ_i is an stopping time with respect to the Wiener process enlarged by the σ–algebras generated by the random variables $\zeta_1, \zeta_2, \ldots, \zeta_{i-1}$. Also, the decision random variables k_i are measurable with respect to the σ–algebra generated by τ_i.

To each impulse we associate a strictly positive cost, referred to *cost-per-impulse* and given by the functional $L(\mathbf{v}, k)$. The total cost for an impulse control $\{\tau_i, k_i;\ i = 1, 2, \ldots\}$ and initial condition $\mathbf{v}$ is given by

$$J(\mathbf{v}, \{\tau_i, k_i\}) := E\Big\{ \int_0^\infty F(\mathbf{u}(t)) e^{-\alpha t} dt + \sum_i L(\mathbf{u}(\tau_i-), k_i) e^{-\alpha \tau_i} \Big\} \tag{4.3}$$

and the optimal cost

$$\hat{U}(\mathbf{v}) := \inf_{\{\tau_i, k_i\}} J(\mathbf{v}, \{\tau_i, k_i\}), \tag{4.4}$$

where the infimum is taken over all impulse controls, and $\mathbf{u}(t)$ is the evolution constructed by means of (4.2) with initial condition $\mathbf{v}$. Specific forms of F and L in fluid mechanics can be found in [21].

Although impulse control problems are very well known in finite dimensional setting, only a few results are available for Markov processes on general polish spaces (which are not necessarily locally compact, cf. Bensoussan [2, Chapter 8, pp. 355–397]). Hence as noted earlier we adapt the semigroup approach to follow the *hybrid control setting* described in Bensoussan and Menaldi [5]. The dynamic programming principle yields to the problem: Find U in $\bar{C}_p(\mathbb{H})$ such that

$$U \le MU, \quad \bar{\mathcal{A}}_\alpha U \le F, \quad \text{and} \quad \bar{\mathcal{A}}_\alpha U = F \quad \text{in} \quad [U < MU], \tag{4.5}$$

where $\bar{\mathcal{A}}_\alpha$ is interpreted in the *martingale sense* and M is the following non-linear operator on $\bar{C}(\mathbb{H})$ given by

$$Mh(\mathbf{v}) := \inf_k \big\{ L(\mathbf{v}, k) + Q(k)h(\mathbf{v}) \big\}, \quad \forall \mathbf{v} \in \mathbb{H}, \tag{4.6}$$

where the transition operator $Q(k)h(\mathbf{v}) = E\{h(\mathbf{q}(\mathbf{v}, \zeta_1 \,|\, k))\}$ is as in (4.1). Problem (4.5) is called a *quasi-variational inequality.*

To solve (4.5) we define by induction the sequence of variational inequalities

$$\begin{cases} \hat{U}^{n+1} \in \bar{C}_p(\mathbb{H}) \quad \text{such that} \quad \hat{U}^{n+1} \le M\hat{U}^n, \\ \bar{\mathcal{A}}_\alpha \hat{U}^{n+1} \le F \quad \text{and} \quad \bar{\mathcal{A}}_\alpha \hat{U}^{n+1} = F \quad \text{in} \quad [\hat{U}^{n+1} < M\hat{U}^n], \end{cases} \tag{4.7}$$

where $\hat{U}^0 = U^0$ solves the equation $\bar{\mathcal{A}}_\alpha U^0 = F$. Notice that (4.7) can be formulated as a maximum sub-solution problem

$$U^{n+1} \in \bar{C}_p(\mathbb{H}) \quad \text{such that} \quad U^{n+1} \le MU^n, \qquad \bar{\mathcal{A}}_\alpha U^{n+1} \le F, \tag{4.8}$$

for any $n \ge 0$. In view of Theorem 3.2 in the previous section, we need only to assume that M operates on the space $\bar{C}_p(\mathbb{H})$ to define the above sequence $\hat{U}^n$ of functions. This means that first, we impose the condition

$$\begin{cases} \|L(\cdot, k)\| \le C, \quad \forall\, k \in K, \\ \lim_{t \to 0} \sup_k \{p_0(\Phi_\alpha(t)L(\cdot, k) - L(\cdot, k), \mathbf{v})\} = 0, \quad \forall\, \mathbf{v} \in \mathbb{H}, \end{cases} \tag{4.9}$$

and next

$$\begin{cases} E\big\{|\mathbf{q}(\mathbf{v}, \zeta_1 \,|\, k)|^m\big\} \le C_m(1 + |\mathbf{v}|^m), \quad \forall\, k \in K, \ \mathbf{v} \in \mathbb{H}, \\ \lim_{t \to 0} \sup_k \{p_0(\Phi_\alpha(t)Q(k)h - Q(k)h, \mathbf{v})\} = 0, \quad \forall\, h \in \bar{C}_p(\mathbb{H}), \end{cases} \tag{4.10}$$

for any $m \ge 0$, some positive constant C_m and where the norm $\|\cdot\|$ and the semi-norms $p_0(\cdot, \mathbf{v})$ given by (2.35) and (2.40), respectively. Since the space K is compact, assumption (4.9) is not a strong restriction. However, condition (4.10) is essentially a smoothness property on the transition kernel of $\mathbf{q}(\mathbf{v}, \cdot \mid k))$ as well as on the distribution of the perturbation ζ_1 used in (4.1). One of the main differences between impulse and continuous type control is the positive cost-per-impulse, i.e., the requirement

$$L(\mathbf{v}, k) \geq \ell_0 > 0, \quad \forall \mathbf{v} \in \mathbb{H},\ k \in K, \tag{4.11}$$

which forbids the accumulation of impulses. We also need

$$F \in \bar{C}_p(\mathbb{H}), \qquad F(\mathbf{v}) \geq 0, \quad \forall \mathbf{v} \in \mathbb{H}, \tag{4.12}$$

to set up the sequence (4.7).

An important role is played by the function $\hat{U}^0 = U^0$, which solves $\bar{\mathcal{A}}_\alpha U^0 = F$, and by the function $\hat{U}_0 = U_0$, which are defined as the solution of the following variational inequality

$$\begin{cases} \hat{U}_0 \in \bar{C}_p(\mathbb{H}) \quad \text{such that} \quad \hat{U}_0 \leq \inf_k L(\cdot, k), \\ \bar{\mathcal{A}}_\alpha \hat{U}_0 \leq F \quad \text{and} \quad \bar{\mathcal{A}}_\alpha \hat{U}_0 = F \quad \text{in} \quad [\hat{U}_0 < \inf_k L(\cdot, k)], \end{cases} \tag{4.13}$$

or as the maximum sub-solution of the problem

$$U_0 \in \bar{C}_p(\mathbb{H}) \quad \text{such that} \quad U_0 \leq \inf_k L(\cdot, k), \qquad \bar{\mathcal{A}}_\alpha U_0 \leq F. \tag{4.14}$$

In view of estimate (3.5) of Proposition 3.1, we deduce that if the semi-norm $p_0([F - \bar{\mathcal{A}}_\alpha L(\cdot, k)]^+, \mathbf{v})$ vanishes[3] for any k and some $\mathbf{v}$ in $\mathbb{H}$ then $U^0(\mathbf{v}) = U_0(\mathbf{v})$ and indeed all the $\hat{U}^n(\mathbf{v})$ are equals to $\hat{U}^0(\mathbf{v})$.

Consider the following quasi-variational inequality (QVI)

$$\begin{cases} \hat{U} \in \bar{C}_p(\mathbb{H}) \quad \text{such that} \quad \hat{U} \leq M\hat{U}, \\ \bar{\mathcal{A}}_\alpha \hat{U} \leq F \quad \text{and} \quad \bar{\mathcal{A}}_\alpha \hat{U} = F \quad \text{in} \quad [\hat{U} < M\hat{U}], \end{cases} \tag{4.15}$$

or the maximum sub-solution of the problem

$$U \in \bar{C}_p(\mathbb{H}) \quad \text{such that} \quad U \leq MU, \qquad \bar{\mathcal{A}}_\alpha U \leq F, \tag{4.16}$$

under the condition

$$\text{there exist} \quad r \in (0, 1] \quad \text{such that} \quad r\, U^0(\mathbf{v}) \leq U_0(\mathbf{v}), \quad \forall \mathbf{v} \in \mathbb{H} \tag{4.17}$$

on which we will comments later.

[3] Recall that $[\cdot]^+$ denotes the positive part

Theorem 4.1 (QVI). *Let the assumptions* (2.22), (2.36), (4.9),..., (4.12) *hold. Then the* VI (4.7) *defines a (pointwise) decreasing sequence of functions* $\hat{U}^n(\mathbf{v})$ *which converges to the optimal cost* $\hat{U}(\mathbf{v})$, *given by* (4.4), *for any* $\mathbf{v}$ *in* $\mathbb{H}$. *Moreover, if* (4.17) *is satisfied then we have the estimate*

$$0 \le \hat{U}^n - \hat{U}^{n+1} \le (1-r)^n\, \hat{U}^0, \quad \forall n = 0, 1, \ldots, \tag{4.18}$$

the automaton impulse control $\{\hat{\tau}_i, \hat{k}_i\}$, *generated by the continuation region* $[\hat{U} < M\hat{U}]$ *and defined by* $\hat{\tau}_0 := 0$,

$$\begin{cases} \hat{\tau}_i := \inf\big\{\, t \ge \hat{\tau}_{i-1} \;:\; \hat{U}[\mathbf{u}(t;\mathbf{u}(\tau_{i-1}))] = M\hat{U}[\mathbf{u}(t;\mathbf{u}(\tau_{i-1}))]\,\big\}, \\ \hat{k}_i := \arg\min\big\{\, L(\mathbf{u}(\tau_i), k) + Q(k)\hat{U}(\mathbf{u}(\tau_i)) \;:\; k \in K\,\big\} \end{cases} \tag{4.19}$$

is optimal, i.e., $\hat{U}(\mathbf{v}) = J(\mathbf{v}, \{\hat{\tau}_i, \hat{k}_i\})$, *and the optimal cost* $\hat{U}$ *is the unique solution of the* QVI (4.15) *or the maximum sub-solution of problem* (4.16).

Proof. In view of the assumptions (4.9) and (4.10), the operator M defined by (4.6) maps the space $\bar{C}(\mathbb{H})$ into itself. Next, Theorem 3.2 ensures that the sequence of VI (4.7) is decreasing and well defined. Moreover, a simple application of the strong Markov property shows that $\hat{U}^n$ can be interpreted as the optimal cost of an impulse control problem where a maximum of n impulses are only allowed, i,e.,

$$\hat{U}^n(\mathbf{v}) := \inf\big\{\, J(\mathbf{v}, \{\tau_i, k_i\}) \;:\; \{\tau_i, k_i\},\ \text{with } \tau_i = \infty, \forall i > n\,\big\}. \tag{4.20}$$

Hence the sequence $\hat{U}^n(\mathbf{v})$ converges to the optimal cost $\hat{U}(\mathbf{v})$, for any $\mathbf{v}$.

Now, we need to establish estimate (4.18) to conclude. So first, define the nonlinear operator $V \mapsto \hat{U} := T(V)$ as the solution of the VI

$$\begin{cases} \hat{U} \in \bar{C}_p(\mathbb{H}) \quad \text{such that} \quad \hat{U} \le MV, \\ \bar{\mathcal{A}}_\alpha \hat{U} \le F \quad \text{and} \quad \bar{\mathcal{A}}_\alpha \hat{U} = F \quad \text{in} \quad [\hat{U} < MV]. \end{cases} \tag{4.21}$$

It is clear that T is a monotone (increasing) and concave operator on $\bar{C}(\mathbb{H})$, i.e.,

$$\begin{cases} U \le V \quad \text{implies} \quad T(U) \le T(V), \\ \theta T(U) + (1-\theta) T(V) \le T(\theta U + (1-\theta)V), \quad \forall\, \theta \in [0,1]. \end{cases} \tag{4.22}$$

Condition (4.17) means $T(0) = U_0 \geq rU^0$, which together with (4.22) yields

$$\begin{cases} \text{if} \quad T(V) \leq U_0 \quad \text{and} \quad V - U \leq \theta V, \quad \text{for some} \quad r \in (0,1] \\ \text{then} \quad T(U) - T(V) \leq \theta(1-r)T(U). \end{cases} \tag{4.23}$$

Indeed, $\theta 0 + (1-\theta)V \leq U$ implies

$$T(U) \geq T(\theta 0 + (1-\theta)V) \geq \theta T(0) + (1-\theta)T(V)$$

and so

$$T(V) - T(U) \leq \theta T(V) - \theta T(0) \leq \theta T(V) - \theta r U^0 \leq \theta(1-r)T(V).$$

Therefore, in view of $\hat{U}^{n+1} = T(\hat{U}^n)$ we can iterate (4.23) as follows. Since $T(\hat{U}^0) \leq \hat{U}^0$ and $\hat{U}^n \geq 0$ we have $0 \leq \hat{U}^0 - \hat{U}^1 \leq \hat{U}^0$. By means of (4.23) with $\theta = 1$ we get $0 \leq \hat{U}^1 - \hat{U}^2 \leq (1-r)\hat{U}^1$ and then (4.23) with $\theta = (1-r)^{n-1}$ yields

$$0 \leq \hat{U}^n - \hat{U}^{n+1} \leq (1-r)^n \hat{U}^n,$$

which provides estimate (4.18) after noticing that $\hat{U}^n \leq \hat{U}^0$. □

The argument above also shows that if instead of (4.17) we only know that

$$\begin{cases} \text{there exist} \quad r \in (0,1] \quad \text{and} \quad \mathbf{v} \in \mathbb{H} \\ \text{such that} \quad p_0((r\, U^0 - U_0)^+, \mathbf{v}) = 0, \end{cases} \tag{4.24}$$

then we have the estimate

$$p_0([\hat{U}^n - \hat{U}^{n+1} - (1-r)^n\, \hat{U}^0]^+, \mathbf{v}) = 0, \quad \forall n = 0, 1, \dots \tag{4.25}$$

and the automaton impulse control $\{\hat{\tau}_i, \hat{k}_i\}$, generated by the the continuation region $[\hat{U} < M\hat{U}]$, is optimal for that particular $\mathbf{v}$ in $\mathbb{H}$. It is also clear that $\hat{U}^0 \leq \|F\|/(\alpha - \alpha_0)$ and so (4.18) provides a uniform convergence of the sequence of VI (4.7).

If we impose

$$L(\mathbf{v}, k) \geq \ell_0(1 + |\mathbf{v}|^2)^{p/2} > 0, \quad \forall \mathbf{v} \in \mathbb{H},\ k \in K, \tag{4.26}$$

instead of (4.11), then assumption (4.17) holds for any $0 < r < 1$ such that $r \|F\| \leq \ell_0(\alpha - \alpha_0)$. In particular, we notice that condition (4.11) suffices to ensure (4.17), when $p = 0$ (i.e., when F and L are bounded).

It is clear that with the above technique we may consider to control also the coefficients ν, $\mathbf{f}$ and $\mathbf{g}$ of the Navier-Stokes equation (2.10), e.g, the transition operator $Q(k)$ is now acting on $\mathbb{H} \times \mathbb{V}'$ instead of only $\mathbb{H}$. In this case, the state of the dynamic system is the continuous evolution $\mathbf{u}(t)$ and the Markov chains with transition operators $Q(k)$. The QVI (4.15) becomes a system of QVI (e.g., indexed by the $\mathbf{f}$) of the type

$$\begin{cases} \hat{U}(\cdot, \mathbf{f}) \in \bar{C}_p(\mathbb{H}) \quad \text{such that} \quad \hat{U}(\cdot, \mathbf{f}) \leq M(\hat{U}, \cdot, \mathbf{f}), \\ \bar{\mathcal{A}}^{\mathbf{f}}_\alpha \hat{U}(\cdot, \mathbf{f}) \leq F(\cdot, \mathbf{f}), \quad \text{in} \quad \mathbb{H} \quad \text{and} \\ \bar{\mathcal{A}}^{\mathbf{f}}_\alpha \hat{U}(\cdot, \mathbf{f}) = F(\cdot, \mathbf{f}) \quad \text{in} \quad [\hat{U}(\cdot, \mathbf{f}) < M(\hat{U}, \cdot, \mathbf{f})], \end{cases} \tag{4.27}$$

where $\bar{\mathcal{A}}^{\mathbf{f}}_\alpha$ is the weak infinitesimal generator associated with $\mathbf{f}$ and the nonlinear operator $M(U) = M(U, \mathbf{v}, k)$ is now given by

$$M(U, \mathbf{v}, \mathbf{f}) := \inf_k \big\{ L(\mathbf{v}, \mathbf{f}, k) + Q(k) U(\mathbf{v}, \mathbf{f}) \big\}, \quad \forall \mathbf{v} \in \mathbb{H},\ \mathbf{f} \in \mathbb{V}', \tag{4.28}$$

where $L(\mathbf{v}, \mathbf{f}, k)$ and $F(\mathbf{v}, \mathbf{f})$ are the impulse and running costs, respectively. Despite the large indexing in $\mathbf{f}$ belonging to the dual space $\mathbb{V}'$, only a countable number is used (the state of the Markov chain) and it should be adapted to each particular application, i.e., if only a finite number of possible $\mathbf{f}$ are available then only those $\mathbf{f}$ are used and (4.27) is a finite system of QVI. Notice that the only *coupling* in the system of QVI (4.27) is produced by the nonlinear operator M, and therefore, the sequence of VI, similar to (4.7), corresponding to (4.27) is actually a system of *independent* (not coupled) VI to which Theorem 3.2 still applies. Hence, we may solve (4.27) in a way very similar to the one presented for (4.15)[4], but the notation and the numerical resolution is far more complicated.

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