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VARIATIONAL ANALYSIS IN SEMI-INFINITE AND INFINITE PROGRAMMING, II: NECESSARY OPTIMALITY CONDITIONS¹

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Abstract. This paper concerns applications of advanced techniques of variational analysis and generalized differentiation to problems of semi-infinite and infinite programming with feasible solution sets defined by parameterized systems of infinitely many linear inequalities of the type intensively studied in the preceding development [5] from our viewpoint of robust Lipschitzian stability. We present meaningful interpretations and practical examples of such models. The main results establish necessary optimality conditions for a broad class of semi-infinite and infinite programs, where objectives are generally described by nonsmooth and nonconvex functions on Banach spaces and where infinite constraint inequality systems are indexed by arbitrary sets. The results obtained are new in both smooth and nonsmooth settings of semi-infinite and infinite programming.

Key words. semi-infinite and infinite programming, parametric optimization, variational analysis, necessary optimality conditions, linear infinite inequality systems, generalized differentiation, coderivatives, lower and upper subdifferentials

AMS subject classification. 90C34, 90C05, 49J52, 49J53, 65F22

1 Introduction

The paper mainly deals with optimization problems formalized as

$$\text{minimize } \varphi(p, x) \text{ subject to } x \in \mathcal{F}(p), \quad (1.1)$$

where $\varphi: P \times X \rightarrow \overline{\mathbb{R}} := (-\infty, \infty]$ is an extended-real-valued *cost* function defined on the product of Banach spaces, and where $\mathcal{F}: P \rightrightarrows X$ is a set-valued mapping of *feasible solutions* given by

$$\mathcal{F}(p) := \{x \in X \mid \langle a_t^*, x \rangle \leq b_t + \langle c_t^*, p \rangle, \ t \in T\} \quad (1.2)$$

with an *arbitrary* (possibly infinite) *index set* T and with fixed elements $a_t^* \in X^*$, $c_t^* \in P^*$, and $b_t \in \mathbb{R}$ for all $t \in T$. Optimization problems of this type relate to *semi-infinite programming* provided that the space X is finite-dimensional and to *infinite programming* if X is infinite-dimensional; see, e.g., [1, 10].

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Note that a usual framework of sensitivity (or stability) analysis for semi-infinite/infinite programs corresponds to formalism (1.2) specified as

$$\mathcal{F}(p) := \{x \in X \mid \langle a_t^*, x \rangle \leq b_t + p_t, \quad t \in T\}, \quad p = (p_t)_{t \in T}, \quad (1.3)$$

where $p \in P$ is treated as a *perturbation parameter*. This is a particular case of (1.2) when $c_t^* = \delta_t$ is the classical *Dirac measure* at $t \in T$. A natural choice of P in the latter case is the space $l_\infty(T)$ of all bounded functions on T with the supremum norm

$$\|p\|_\infty := \sup_{t \in T} |p_t| = \sup \{|p(t)| \mid t \in T\}.$$

When the index set T is compact (which is not assumed in what follows) and the perturbations $p(\cdot)$ are restricted to be continuous on T , the supremum above is realized, and thus $l_\infty(T)$ reduces to the classical space $\mathcal{C}(T)$ of continuous functions over a compact set.

In the preceding paper [5] we developed and applied advanced tools of variational analysis and generalized differentiation to fully characterize *robust Lipschitzian stability* of the feasible solution map in (1.3) expressed entirely via its initial data. The main goal of the present paper is to employ these and related techniques and some results from [5] to derive verifiable *necessary conditions* for *optimal solutions* to semi-infinite and infinite programs (1.1) with feasible solution maps given by (1.2) in the case of arbitrary Banach spaces X and P and arbitrary index sets T . If (1.2) reduces to (1.3) in (1.1) with $P = l_\infty(T)$, the results obtained below recover those established in our preliminary Research Report [4].

Note that optimization in (1.1) is conducted with respect to both variables (p, x) , which are interconnected through the infinite inequality system (1.2). This means in fact that we have *two groups of decision variables* represented by x and p . One player specifies p and the other solves (1.1) in x subject to (1.2) with the specified p as a parameter. The first one, having the same objective, varies his/her parameter p to get the best outcome via the so-called *optimistic approach*. We could treat this as a *two-level design*: optimizing the basic parameter p at the upper level, while at the lower level the cost function is optimized with respect to x for the given p . The reader is referred to, e.g., [15] and the bibliography therein for various tuning and tolerancing problems of such types arising in engineering design.

Other classes of optimization models that could be described in the two-variable form (1.1) with semi-infinite/infinite constraints of type (1.2) appear in *optimal control* and *approximation theory*; see, e.g., [1, 10, 11] for more details. In Section 2 of this paper we discuss some new particular models related to *electricity markets*, *water resources*, and *multiobjective optimization* that can be naturally formalized as problem (1.1) involving two groups of decision variables and infinitely many linear inequality constraints of type (1.2). The first and the third of these models are given in fact in the more special form (1.3).

It is worth mentioning that our basic problem (1.1) is written in the format of the so-called *abstract MPECs* (mathematical programs with equilibrium constraints) [17, 22, 24], but the main emphasis there is the *generalized equation/variational condition* (in Robinson's sense [25]) structure of the set-valued mapping $\mathcal{F}$ in (1.1) given by

$$\mathcal{F}(p) := \{x \in X \mid 0 \in f(p, x) + Q(x)\}$$

with a single-valued mapping $f: P \times X \rightarrow Y$ and a set-valued mapping $Q: X \rightrightarrows Y$, which particularly encompasses solution maps to the classical variational inequalities and complementarity problems when $Q(x) = N(x; \Omega)$ is the normal cone mapping to a convex set $\Omega \subset X$. The underlying *infinite inequality* structure (1.2) of the mapping $\mathcal{F}$ in our framework is completely different from the MPEC case. The main goal and achievement of this paper is involving infinite inequality systems (1.2) and (1.3) into the general optimization framework (1.1) and deriving verifiable necessary optimality conditions for such problems entirely in terms of their initial data.

Note that the results obtained below are new not only in the general case of nonsmooth and nonconvex cost functions $\varphi(p, x)$ and arbitrary Banach spaces X and P as well as arbitrary index sets T in (1.1)–(1.3), but also for conventional classes of one-variable semi-infinite programs. We refer the reader to [1, 2, 3, 6, 8, 7, 10, 11, 13, 14, 16, 29, 30] and the bibliographies therein for a range of approaches and results concerning optimal solutions to various problems of semi-infinite and infinite programming; see more discussions in Sections 4 and 5.

The rest of the paper is organized as follows. In Section 2 we present the three particular optimization models commented above. Section 3 is devoted to overviewing the basic tools of generalized differentiation used subsequently for deriving the main results of the paper given in Sections 4 and 5. These results establish first-order necessary conditions for optimal solutions to infinite and semi-infinite programs (1.1) with general nonsmooth and nonconvex cost functions and feasible solution maps defined by (1.2) and (1.3).

In Section 4 we present necessary optimality conditions in the so-called *lower subdifferential* type, which are expressed via appropriate extensions of the subdifferential of convex analysis to the general class of lower semicontinuous cost functions. Besides well-developed calculus rules for the corresponding subdifferentials, a major role in deriving these conditions is played by the *coderivative* of the feasible solution maps under consideration, which is constructively computed in [5] entirely in terms of the initial data. The results obtained are generally given in a verifiable *qualified asymptotic* form introduced in this paper, while they are presented in an *extended Karush-Kuhn-Tucker* (KKT) form under some *closedness* (Farkas-Minkowski type) *constraint qualification*.

The final Section 5 contains necessary optimality conditions for nonsmooth problems (1.1) with infinitely many inequality constraints (1.2) and (1.3) established in a relatively new for minimization *upper subdifferential/superdifferential* form that has never been used before in semi-infinite and infinite programming. The upper subdifferential optimality conditions obtained in this section are generally independent of the lower subdifferential ones derived in Section 4. In fact, both agree for smooth (i.e., continuously differentiable) objectives, while the upper conditions may be strictly better in the case of cost functions that are merely Fréchet differentiable at optimal solutions. The main difference is as follows: the upper subdifferential conditions provide trivial information in the case of convex cost functions and the like when the lower subdifferential ones play a major role but, on the other hand, the upper conditions give significantly stronger results for broad classes of “upper regular” functions, e.g., for minimization problems involving concave and semiconcave objectives; see more details and discussions in Section 5. Similarly to the lower subdifferential conditions

of Section 4, the upper ones take advantages of precise computing the coderivative of the feasible solution maps in (1.2) and (1.3) and thus express necessary optimality conditions entirely in terms of the initial data of the basic model (1.1).

The notation of this paper is basically standard and conventional in the areas of variational analysis and semi-infinite/infinite programming; see, e.g., [10, 21]. Unless otherwise stated, all the spaces under consideration are *Banach* with the corresponding norm $\|\cdot\|$. Recall that w^* indicates the weak* topology of a dual space, and we use the symbol $w^*\text{-lim}$ for the weak* *topological* limit, which generally means the weak* convergence of *nets* denoted usually by $\{x_\nu^*\}_{\nu \in \mathcal{N}}$. In the case of sequences we use the standard notation $\mathbb{N} := \{1, 2, \dots\}$ for the collections of all natural numbers.

Given a subset $\Omega \subset Z$ of a Banach space, the symbols $\text{int } \Omega$, $\text{cl } \Omega$, $\text{co } \Omega$, and $\text{cone } \Omega$ stand, respectively, for the interior, closure, convex hull, and conic convex hull of Ω ; the notation $\text{cl}^* \Theta$ signifies the weak* closure of a subset $\Theta \subset Z^*$ in the dual space. Given a set-valued mapping $F: Z \rightrightarrows Y$, we denote its *domain* and *graph* by, respectively,

$$\text{dom } F = \{z \in Z \mid F(z) \neq \emptyset\} \quad \text{and} \quad \text{gph } F := \{(z, y) \in Z \times Y \mid y \in F(z)\}.$$

Considering finally an arbitrary index set T , let $\mathbb{R}^T$ be the *product space* of $\lambda = (\lambda_t \mid t \in T)$ with $\lambda_t \in \mathbb{R}$ for all $t \in T$, let $\mathbb{R}^{(T)}$ be the collection of $\lambda \in \mathbb{R}^T$ such that $\lambda_t \neq 0$ for *finitely many* $t \in T$, and let $\mathbb{R}_+^{(T)}$ be the *positive cone* in $\mathbb{R}^{(T)}$ defined by

$$\mathbb{R}_+^{(T)} := \{\lambda \in \mathbb{R}^{(T)} \mid \lambda_t \geq 0 \text{ for all } t \in T\}. \quad (1.4)$$

2 Two-Variable Models of (Semi-)Infinite Programming

In this section we present three examples of particular models arising in applications, which can be naturally described in the semi-infinite/infinite programming form (1.1) with two groups of decision variables interconnected via infinite linear inequality constraints (1.3).

Example 2.1 (electricity markets). Suppose that an electrical company has $n \in \mathbb{N}$ plants located in a certain geographical region Ω . The company must decide at what level of production these plans should be operated during a fixed period of time T (e.g., a season). In competitive market economy, suppose that the company can buy and sell electricity in the electricity markets and denote by $b_t \geq 0$ the demand of electricity in the region Ω at each particular time $t \in T$. These data are given in advance (e.g., predicted according to the past experience). Let $a_t = (a_{1t}, \dots, a_{nt}) \in \mathbb{R}^n$, where a_{it} is the rate of electricity production of the i th electrical plant $i \in \{1, \dots, n\}$ at time $t \in T$ that is assumed to be known from the past experience and technological assessments. Denote further by $x_i \in [0, 1]$ the production intensitivity level of the plant $i \in \{1, \dots, n\}$ along the whole time interval T assuming in this way that each plant operates at a fixed level along the time interval T to ensure regular and stable production plans. This decision variable relates to workforce contracted, equipment installed, investments made in the plant, etc. Finally, denote by $p_t \in \mathbb{R}$ the amount of electricity bought (if positive) or sold (if negative) at each time $t \in T$. The decision maker can decide to produce more in order to sell the excess or to

produce less if it is more profitable to buy the deficit. Thus the cost function $\varphi(p, x)$ in such a model depends on the aggregated operation level $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and on the imported/exported electricity $p = (p_t)_{t \in T}$. In particular, $\varphi(p, x)$ could be written in the form of $\text{cost}(x) - \text{benefit}(p)$.

As we see, our model can be formalized as a *semi-infinite program* of type (1.1) with two groups of decision variables and infinitely many linear inequality constraints given by

$$\langle a_t, x \rangle + p_t \geq b_t, \quad t \in T,$$

which reduces to the canonical form (1.3) by changing the sign, provided that the constraint $x \in [0, 1]^n$ is incorporated to the objective function by adding the corresponding indicator function; otherwise we would use model (1.2) to consider this constraint.

The next example concerns control of water resources via systems of reservoirs.

Example 2.2 (water resources). Consider a system of n reservoirs $R_1, R_2, \dots, R_n$ from which a time-varying water demand is required during a fixed period of time $T = [a, b]$. Let c_i be the capacity of the reservoir R_i , and let water flow into R_i at rate $r_i(t)$ for each $i = 1, \dots, n$ and $t \in T$. Denote by $D(t)$ the rate of water demand at t and suppose that all these nonnegative functions are piecewise continuous on T and are known in advance.

If it is enough water to fill all the reservoir capacity, then the rest can be sold to a neighboring dry area provided that the demand is satisfied. Conversely, if the inflows are short and the reservoirs have free capability for holding additional water, then some water can be bought from outside to meet the inner demand in the region.

Denote by $x_i(t)$ the rate at which water is fed from the reservoir R_i at time $t \in T$. The *feeder constraints* can be expressed by

$$0 \leq x_i(t) \leq \eta_i, \quad i = 1, \dots, n, \quad (2.1)$$

with fixed bounds $\eta_i \geq 0$. The *selling rate* of water from the reservoir R_i at time t is given by $dp_i(t)$, where $p_i(t) \in \mathbb{R}$ as $t \in T$ is a function of bounded variation on T , and hence it is bounded itself for each $i = 1, \dots, n$. Note that we are actually buying water at time $t \in T$ if the selling rate $dp_i(t)$ is negative. Denoting by $s_i \geq 0$ the amount of water initially stored in R_i , we formulate the *storage constraints*

$$\begin{aligned} 0 &\leq \int_a^t [r_i(\tau) - x_i(\tau)] d\tau - \int_a^t dp_i(\tau) + s_i \\ &= \int_a^t [r_i(\tau) - x_i(\tau)] d\tau - p_i(t) + s_i \\ &\leq c_i \quad \text{for all } t \in T \text{ and } i = 1, \dots, n \end{aligned} \quad (2.2)$$

and finally arrive at the following *constrained optimization problem*:

$$\left\{ \begin{array}{l} \text{minimize } \varphi(p, x) \text{ subject to} \\ (2.1), (2.2), \text{ and} \\ \sum_{i=1}^n x_i(t) \geq D(t), \quad t \in T, \end{array} \right. \quad (2.3)$$

where the cost/objective function $\varphi(p, x)$ is determined by the cost of water, environmental requirements in the region, and the technology of reservoir processes. It is clear that we should impose the relationship

$$D(t) \leq \sum_{i=1}^n \eta_i, \quad t \in T,$$

in order to ensure the consistency of the constraints in (2.3).

Let us show that optimization problem (2.3) can be reduced to the form of *infinite programming* (1.1) with two groups of variable $(p, x) \in P \times X$ and infinitely many linear inequality constraints of type (1.2). It is natural to assume in our basic model that $x_i \in \mathcal{C}(T)$ for all $i = 1, \dots, n$, which thus gives $x \in X := \mathcal{C}(T)^n$, and that $p \in P := BV(T)^n$, where $BV(T)$ denotes the space of functions of bounded variation on T .

Our next step consists of justifying the representation

$$\int_a^t x_i(\tau) d\tau = \langle \alpha_i, x_i \rangle, \quad t \in T, \quad (2.4)$$

with a certain function $\alpha_t \in \mathcal{C}(T)^*$. Observe that the Riesz representation theorem (see, e.g., [9, Proposition 2.19]) ensures that each element of $\mathcal{C}(T)$ may be identified with a function of bounded variation on T by means of the expression

$$z \mapsto \langle z, \gamma \rangle := \int_a^b z(\tau) d\gamma(\tau)$$

via the Stieltjes integral. In the particular case (2.4) we consider the family of functions $\alpha_t: T \rightarrow \mathbb{R}$ of bounded variation on T given by

$$\alpha_t(\tau) := \begin{cases} \tau & \text{if } a \leq \tau \leq t, \\ t & \text{otherwise.} \end{cases} \quad (2.5)$$

Note the fulfillment of the relationship

$$d\alpha_t(\tau) = \chi_{[a,t]}(\tau) d\tau, \quad t \in T,$$

where $\chi_{[a,t]}$ is the standard characteristic function of $[a, t]$. Define further

$$\beta_i(t) := \int_a^t r_i(\tau) d\tau \quad \text{for } i = 1, \dots, n, \quad t \in T$$

and observe that the constraints in (2.2) can be rewritten in the format

$$0 \leq \beta_i(t) - \langle \alpha_t, x_i \rangle - p_i(t) + s_i \leq c_i, \quad t \in T, \quad i = 1, \dots, n,$$

which is equivalent to the required representation for all $i = 1, \dots, n$ and $t \in T$:

$$\begin{cases} \langle \alpha_t, x_i \rangle + p_i(t) \leq \beta_i(t) + s_i, \\ \langle -\alpha_t, x_i \rangle - p_i(t) \leq c_i - s_i - \beta_i(t). \end{cases}$$

The last constraint in (2.3) can be written as

$$\sum_{i=1}^n \langle \delta_t, x_i \rangle \geq D(t), \quad t \in T,$$

via the classical *Dirac measure* δ_t at $t \in T$ satisfying

$$\langle \delta_t, x_i \rangle = x_i \quad \text{as } t \in T.$$

This completes the modeling of the water recourse problem under consideration in the two-variable format of infinite programming (1.1) with linear inequality constraints (1.2)

In the final example of this section we show that semi-infinite and infinite programming models of type (1.1) with two groups of variables and linear inequality constraints in form (1.3) naturally appear in reducing constrained problems of *vector/multiobjective optimization* to single-objective ones via the so-called *goal programming* approach.

Example 2.3 (multiobjective optimization). Consider the following Pareto-type problem of constrained multiobjective optimization:

$$\text{Min } f(x) \text{ subject to } x \in \Omega, \quad (2.6)$$

where $\Omega \subset X$ is a bounded subset of a Banach space X , and where a cost vector-valued mapping $f: X \rightarrow \ell_\infty(T)$ is given by

$$f(x) := (\langle a_t^*, x \rangle)_{t \in T}, \quad x \in X,$$

with $a_t^* \in X^*$ for all t from an arbitrary index set T . We assume that the set $\{a_t^* \mid t \in T\}$ is bounded in X^* . Recall that $\bar{x} \in \Omega$ is a *Pareto optimal/efficient solution* to (2.6) if

$$[x \in \Omega, \langle a_t^*, x \rangle \leq \langle a_t^*, \bar{x} \rangle \text{ for all } t \in T] \implies [\langle a_t^*, x \rangle = \langle a_t^*, \bar{x} \rangle \text{ for all } t \in T].$$

To find Pareto optimal solutions to (2.6), we employ the goal programming approach that consists of the following:

- Observe that the boundedness assumptions imposed on Ω and $\{a_t^* \mid t \in T\}$ imply the existence of $b = (b_t)_{t \in T} \in \ell_\infty(T)$ satisfying

$$-\infty < b_t < \inf \{ \langle a_t^*, x \rangle \mid x \in \Omega \} \text{ for all } t \in T$$

and then pick any such an “utopian point” b .

- Consider the *goal programming problem* with respect to two variable $(p, x) \in \ell_\infty(T) \times X$:

$$\begin{cases} \text{minimize } \|p\| & \text{subject to} \\ x \in \Omega & \text{and } \langle a_t^*, x \rangle - p_t \leq b_t \text{ for all } t \in T. \end{cases} \quad (2.7)$$

The above choice of b_t ensures that $p_t > 0$ as $t \in T$ for any feasible solution to (2.7), which allows us to reduce the Pareto-type constrained multiobjective optimization problem (2.6) to the scalar minimization form (2.7).

• Replacing the geometric constraints in (2.7) by minimizing the *indicator function* $\delta(x; \Omega)$ of Ω equal to 0 for $x \in \Omega$ and ∞ for $x \notin \Omega$, we finally reduce (2.7) and the original multiobjective optimization problem (2.6) to the *semi-infinite/infinite programming* format (1.1) with constraints (1.3) in $l_\infty(T) \times X$ written as

$$\begin{cases} \text{minimize } \varphi(p, x) := \|p\| + \delta(x; \Omega) & \text{subject to} \\ \langle a_t^*, x \rangle - p_t \leq b_t & \text{for all } t \in T. \end{cases}$$

3 Generalized Differentiation

Our main goal in this paper is to derive necessary optimality conditions for semi-infinite and infinite programming problems of type (1.1) with infinitely many linear inequality constraints given in form (1.2) and by its specification (1.3). We consider general cases of (1.2) when the cost function $\varphi(p, x)$ is nonsmooth and nonconvex, which surely require the usage of appropriate tools of *generalized differentiation*. Furthermore, developing an approach of variational analysis and taking into account the specific structure of problem (1.1) with set-valued constraints $x \in \mathcal{F}(p)$, we use generalized differentiation even for the case of *smooth* objectives φ employing in this way the *coderivative* construction for set-valued mappings and its full computing in [5] for the feasible solution maps $\mathcal{F}$ under consideration.

In this section we briefly overview some tools of generalized differentiation needed in the subsequent sections; the reader can find more details and discussions in [12, 21, 27, 28] and the bibliographies therein. We consider first (*lower*) *subdifferentials*, or collections of subgradients, for extended-real-valued functions that reduce to the classical subdifferential of convex analysis in the case of convex functions and are conventionally employed in *minimization* problems with “*less or equal* ($\leq$)” inequality constraints. Note that the adjective “*lower*” is usually taken for granted and is dropped in subdifferential studies and applications; see, however, the discussions and results in Section 5.

Let Z be an arbitrary Banach space, let $\varphi: Z \rightarrow \overline{\mathbb{R}}$ be an extended-real-valued function finite at the reference point $\bar{z}$, and let $\varepsilon \geq 0$. The ε -*subdifferential* of φ at $\bar{z}$ is defined by

$$\widehat{\partial}_\varepsilon \varphi(\bar{z}) := \left\{ z^* \in Z^* \mid \liminf_{z \rightarrow \bar{z}} \frac{\varphi(z) - \varphi(\bar{z}) - \langle z^*, z - \bar{z} \rangle}{\|z - \bar{z}\|} \geq -\varepsilon \right\}. \quad (3.1)$$

For $\varepsilon = 0$ in (3.1), the construction $\widehat{\partial} \varphi(\bar{z}) := \widehat{\partial}_0 \varphi(\bar{z})$ is known as the *regular* (or *viscosity*, or *Fréchet*) *subdifferential* of φ at this point. It reduces to the classical subdifferential of convex analysis for convex functions φ while may be empty in the absence of convexity (as, e.g., for $\varphi(x) = -|x|$ at $\bar{x} = 0$) and does not generally satisfy required calculus rules. Employing the sequential limiting procedure

$$\begin{aligned} \partial \varphi(\bar{z}) := \left\{ z^* \in Z^* \mid \right. & \exists \text{ seqs. } (\varepsilon_k, z_k, z_k^*)_{k \in \mathbb{N}} \subset \mathbb{R}_+ \times Z \times Z^* \text{ with } z_k^* \in \widehat{\partial}_{\varepsilon_k} \varphi(z_k) \\ & \left. \text{and } \varepsilon_k \downarrow 0, z_k \rightarrow \bar{z}, \varphi(z_k) \rightarrow \varphi(\bar{z}), z_k^* \xrightarrow{w^*} z^* \text{ as } k \rightarrow \infty \right\}, \end{aligned} \quad (3.2)$$

we arrive at the robust subdifferential construction known as the *limiting* (or *basic*, or *Mordukhovich*) *subdifferential* of φ at $\bar{z}$. Note that the limiting operation in (3.2) can be

symbolically written in the *sequential* form of the Painlevé-Kuratowski outer/upper limit

$$\partial\varphi(\bar{z}) = \text{Lim sup}_{\substack{z \xrightarrow{\varphi} \bar{z} \\ \varepsilon \downarrow 0}} \widehat{\partial}_\varepsilon \varphi(z),$$

where the symbol $z \xrightarrow{\varphi} \bar{z}$ signifies that $z \rightarrow \bar{z}$ with $\varphi(z) \rightarrow \varphi(\bar{z})$. If the function φ is lower semicontinuous (l.s.c.) around $\bar{z}$ and the space Z is *Asplund* (i.e., each of its separable subspace has a separable dual; see [9, 21] for more details), then we can equivalently put $\varepsilon_k \equiv 0$ in (3.2) and get the representation

$$\partial\varphi(\bar{z}) = \text{Lim sup}_{z \xrightarrow{\varphi} \bar{z}} \widehat{\partial}\varphi(z). \quad (3.3)$$

It turns out furthermore that, in spite of (actually due to) nonconvexity of the subgradient sets $\partial\varphi(\bar{z})$, the limiting subdifferential (3.3) admits *full calculus* in the Asplund space setting that is mainly based on *variational/extremal principles* of variational analysis; see [21] for the comprehensive study and references. On the other hand, the enlarged subdifferential construction (3.2), having many useful properties and applications in arbitrary Banach spaces (see, in particular, [21, Chapters 1 and 4] and [22, Chapters 5 and 6]), may fail to satisfy important calculus rules in general nonsmooth settings of non-Asplund spaces. This is the case of the space $P = \ell_\infty(T)$ naturally appeared in modeling infinite inequality constraints of type (1.3); see [5, Proposition 2.5].

In what follows we proceed with applications of the aforementioned sequential subdifferential constructions to deriving necessary optimality conditions for nonsmooth problems (1.1) with general infinite inequality constraints (1.2) in the case of Asplund spaces X and P . To cover simultaneously infinite programs with arbitrary Banach spaces of decision variables, we employ the so-called approximate G -subdifferential by Ioffe [12], labeled in [12] as “the nucleus of the G -subdifferential,” which provides another (more complicate, topological) infinite-dimensional extension of the original construction by Mordukhovich [18] while turns out to be the most appropriate to work in the general Banach space settings including the underlying case of $P = \ell_\infty(T)$ as in (1.3).

The approximate subdifferential constructions on arbitrary Banach spaces are defined by the following multistep procedure. Given a function $\varphi: Z \rightarrow \overline{\mathbb{R}}$ finite at $\bar{z}$, consider first its *lower Dini* (or *Dini-Hadamard*) *directional derivative*

$$d^-\varphi(\bar{z}; v) := \liminf_{\substack{u \rightarrow v \\ t \downarrow 0}} \frac{\varphi(\bar{z} + tu) - \varphi(\bar{z})}{t}, \quad v \in Z,$$

and then define the *Dini ε -subdifferential* of φ at $\bar{z}$ by

$$\partial_\varepsilon^-\varphi(\bar{z}) := \{z^* \in Z^* \mid \langle z^*, v \rangle \leq d^-\varphi(\bar{z}; v) + \varepsilon\|v\| \text{ for all } v \in Z\}, \quad \varepsilon \geq 0.$$

As usual, put $\partial_\varepsilon^-\varphi(\bar{z}) := \emptyset$ if $\varphi(\bar{z}) = \infty$. The *A-subdifferential* of φ at $\bar{z}$ is defined via topological limits involving finite-dimensional reductions of ε -subgradients by

$$\partial_A\varphi(\bar{z}) := \bigcap_{\substack{L \in \mathcal{L} \\ \varepsilon > 0}} \overline{\text{Lim sup}_{z \xrightarrow{\varphi} \bar{z}} \partial_\varepsilon^-(\varphi + \delta(\cdot; L))(z)},$$

where $\mathcal{L}$ is the collection of all finite-dimensional subspaces of Z , where $\delta(\cdot; L)$ is the indicator function of L , and where $\overline{\text{Limsup}}$ stands for the *topological* Painlevé-Kuratowski upper/outer limit of a mapping $F: Z \rightrightarrows Z^*$ as $z \rightarrow \bar{z}$ defined by

$$\overline{\text{Limsup}}_{z \rightarrow \bar{z}} F(z) := \left\{ z^* \in Z^* \mid \exists \text{ a net } (z_\nu, z_\nu^*)_{\nu \in \mathcal{N}} \subset Z \times Z^* \text{ with } z_\nu^* \in F(z_\nu) \text{ and } (z_\nu, z_\nu^*) \rightarrow (\bar{z}, z^*) \text{ in the } \|\cdot\| \times w^* \text{ topology of } Z \times Z^* \right\}.$$

Then the *approximate G -subdifferential* of φ at $\bar{z}$ (the main construction called the “nucleus of the G -subdifferential” in [12]) is defined by

$$\partial_G \varphi(\bar{z}) := \left\{ z^* \in X^* \mid (z^*, -1) \in \bigcup_{\lambda > 0} \lambda \partial_A \text{dist}((\bar{z}, \varphi(\bar{z})); \text{epi } \varphi) \right\}, \quad (3.4)$$

where $\text{epi } \varphi := \{(z, \mu) \in Z \times \mathbb{R} \mid \mu \geq \varphi(z)\}$, and where $\text{dist}(\cdot; \Omega)$ stands for the *distance function* associated with the set in question.

We have the following relationship between the constructions (3.2) and (3.4) for every l.s.c. function on a Banach space:

$$\partial \varphi(\bar{z}) \subset \partial_G \varphi(\bar{z}), \quad (3.5)$$

where the equality holds when φ is locally Lipschitzian around $\bar{z}$ and Z is Asplund and weakly compactly generated (WCG); see [23, Theorem 9.2] and [21, Theorem 3.59]. Note the inclusion in (3.5) may be proper for Lipschitz continuous functions on (nonseparable) Asplund spaces; see, e.g., [21, Example 3.61]. Both constructions (3.2) and (3.4) are always smaller than the Clarke subdifferential; they may be substantially smaller even for simple functions on $\mathbb{R}$. We refer the reader to [21, Subsection 3.2.3] and [23, Sections 8 and 9] for more results and discussions in this direction. Note that both constructions (3.2) and (3.4) reduce, in any Banach space, to the classical strict derivative in the case of smooth functions and to the classical subdifferential of convex analysis when φ is convex.

We also recall the *singular* counterparts of (3.2) and (3.4) defined, respectively, by

$$\partial^\infty \varphi(\bar{z}) := \text{Lim sup}_{\substack{z \xrightarrow{\varphi} \bar{z} \\ \varepsilon, \lambda \downarrow 0}} \lambda \widehat{\partial}_\varepsilon \varphi(z), \quad (3.6)$$

$$\partial_G^\infty \varphi(\bar{z}) := \left\{ z^* \in X^* \mid (z^*, 0) \in \bigcup_{\lambda > 0} \lambda \partial_A \text{dist}((\bar{z}, \varphi(\bar{z})); \text{epi } \varphi) \right\}. \quad (3.7)$$

Similarly to (3.3), the singular subdifferential (3.6) can equivalently be represented with $\varepsilon = 0$ on the right-hand side if Z is Asplund and if φ is l.s.c. around $\bar{z}$. Similarly to (3.5), we always have the inclusion $\partial^\infty \varphi(\bar{z}) \subset \partial_G^\infty \varphi(\bar{z})$, where furthermore $\partial_G^\infty \varphi(\bar{z}) = \{0\}$ if φ is locally Lipschitzian around $\bar{z}$ on an arbitrary Banach space Z .

To deal with the set-valued term $\mathcal{F}$ in deriving necessary optimality conditions for infinite and semi-infinite programs (1.1), we use a generalized differential construction for set-valued mappings known as coderivatives. Given a set-valued mapping $F: Z \rightarrow Y$ between Banach spaces and following the scheme of [19], define the *coderivative* of F at

$(\bar{z}, \bar{y}) \in \text{gph } F$ generated by the normal cone N to its graph as a positively homogeneous mapping $D^*F(\bar{z}, \bar{y}): Y^* \rightrightarrows Z^*$ with the values

$$D^*F(\bar{z}, \bar{y})(y^*) := \{z^* \in Z^* \mid (z^*, -y^*) \in N((\bar{z}, \bar{y}); \text{gph } F)\}, \quad (3.8)$$

where $N(\cdot; \Omega) := \partial\delta(\cdot; \Omega)$ is the normal cone corresponding to some subdifferential ∂ of extended-real-valued functions. In this way we get coderivatives corresponding to the subdifferentials (3.2) and (3.4). Since both these subdifferentials reduce to the subdifferential of convex analysis for convex functions and since they are used in Sections 4 and 5 for the convex-graph mappings $\mathcal{F}: P \rightrightarrows X$ given by (1.2) and (1.3), the coderivatives of these mappings are the same for (3.2) and (3.4).

4 Lower Subdifferential Optimality Conditions

This section is devoted to establishing necessary optimality conditions of the lower subdifferential type for semi-infinite and infinite programs (1.1) with generally nonsmooth and nonconvex cost functions and infinitely many linear inequality constraints from (1.2) and (1.3). For these purposes we use the subdifferential (3.4) in the general Banach space setting and the smaller subdifferential (3.3) in the Asplund space framework combining them with the precised computation of the coderivative (3.8) of the corresponding feasible solution maps $\mathcal{F}$. The general results obtained are given in the so-called *asymptotic form* involving the weak* closure of a set constructively built upon the initial data of the constraint systems (1.2) and (1.3). Furthermore, they are presented in the more conventional (while new) *KKT form* under additional constraint qualifications.

Following Definition 2.2 in [5] given for the case of system (1.3), we say that the *strong Slater condition* (SSC) holds for system (1.2) if there is a pair $(\hat{p}, \hat{x}) \in P \times X$ such that

$$\sup_{t \in T} [\langle a_t^*, \hat{x} \rangle - \langle c_t^*, \hat{p} \rangle - b_t] < 0, \quad (4.1)$$

where $(\hat{p}, \hat{x})$ is called the *strong Slater point* for (1.2). The reader can easily check the fulfillment of the equivalent descriptions of the SSC in (4.1) similar to [5, Lemma 2.3].

We say also that system (1.2) has the *Farkas-Minkowski property* if the convex cone

$$\text{cone}\{(-c_t^*, a_t^*, b_t) \in P^* \times X^* \times \mathbb{R} \mid t \in T\} \quad (4.2)$$

is weak* closed in $P^* \times X^* \times \mathbb{R}$. We refer the reader to [2, 6, 7, 10, 16, 30] for sufficient conditions ensuring the validity of this property, its relationships with other constraint qualifications, and various applications to problems of semi-infinite and infinite programming.

Now we are ready to formulate and prove the main results of this section, which we present in two similar while independent theorems. The first theorem gives necessary optimality conditions for problem (1.1) with infinite constraints (1.2) in arbitrary Banach spaces X and P employing the G -subdifferential (3.4) and its singular counterpart (3.7). The second theorem holds for Asplund spaces X and P while using the smaller subdifferential constructions (3.3), (3.6) and thus providing more selective necessary optimality conditions in the latter framework.

Theorem 4.1 (lower subdifferential optimality conditions for nonsmooth infinite programs in arbitrary Banach spaces). *Let $(\bar{p}, \bar{x}) \in \text{gph } \mathcal{F}$ be a local minimizer for problem (1.1) with the general linear constraint system $\mathcal{F}$ given by infinite inequalities (1.2). Assume that both spaces X and P are Banach and that the cost function $\varphi: P \times X \rightarrow \overline{\mathbb{R}}$ is l.s.c. around $(\bar{p}, \bar{x})$ with $\varphi(\bar{p}, \bar{x}) < \infty$. Suppose also that:*

- (a) *either φ is locally Lipschitzian around $(\bar{p}, \bar{x})$;*
- (b) *or $\text{int}(\text{gph } \mathcal{F}) \neq \emptyset$ (which holds, in particular, when $\mathcal{F}$ satisfies the SSC in (4.1) and the set $\{(a_t^*, c_t^*) \mid t \in T\}$ is bounded in $X^* \times P^*$) and the system*

$$(p^*, x^*) \in \partial_G^\infty \varphi(\bar{p}, \bar{x}), \quad -(p^*, x^*, \langle (p^*, x^*), (\bar{p}, \bar{x}) \rangle) \in \text{cl}^* \text{cone} \{(-c_t^*, a_t^*, b_t) \mid t \in T\} \quad (4.3)$$

admits only the trivial solution $(p^, x^*) = (0, 0)$. Then there exists a G -subgradient pair $(p^*, x^*) \in \partial_G \varphi(\bar{p}, \bar{x})$ such that*

$$-(p^*, x^*, \langle p^*, \bar{p} \rangle + \langle x^*, \bar{x} \rangle) \in \text{cl}^* \text{cone} \{(-c_t^*, a_t^*, b_t) \mid t \in T\}. \quad (4.4)$$

If furthermore the constraint system (1.2) satisfies the Farkas-Minkowski property (4.2), then the asymptotic condition (4.4) can be equivalently written in the Karush-Kuhn-Tucker form: there are $(p^, x^*) \in \partial_G \varphi(\bar{p}, \bar{x})$ and $\lambda = (\lambda_t)_{t \in T} \in \mathbb{R}_+^{(T)}$ for which we have*

$$(p^*, x^*) + \sum_{t \in T(\bar{p}, \bar{x})} \lambda_t (-c_t^*, a_t^*) = 0, \quad (4.5)$$

where $T(\bar{p}, \bar{x}) := \{t \in T \mid \langle a_t^, \bar{x} \rangle - \langle c_t^*, \bar{p} \rangle = b_t\}$, and where $\mathbb{R}_+^{(T)}$ is defined in (1.4).*

Theorem 4.2 (lower subdifferential optimality conditions for nonsmooth infinite programs in Asplund spaces). *In the framework of Theorem 4.1, suppose that the spaces X and P are Asplund and that assumption (b) is replaced by the weaker one on the triviality of solutions to the system*

$$(p^*, x^*) \in \partial^\infty \varphi(\bar{p}, \bar{x}), \quad -(p^*, x^*, \langle (p^*, x^*), (\bar{p}, \bar{x}) \rangle) \in \text{cl}^* \text{cone} \{(-c_t^*, a_t^*, b_t) \mid t \in T\} \quad (4.6)$$

with the singular subdifferential of φ defined in (3.6). Then we have the stronger necessary optimality conditions for the given solution $(\bar{p}, \bar{x})$ with the replacement $(p^, x^*) \in \partial_G \varphi(\bar{p}, \bar{x})$ in (4.4) and (4.5) by $(p^*, x^*) \in \partial \varphi(\bar{p}, \bar{x})$ from the limiting subdifferential (3.3).*

We prove Theorems 4.1 and 4.2 simultaneously by using the corresponding calculus rules for the subdifferentials (3.4) and (3.3) in Banach and Asplund spaces, respectively.

Proofs of Theorem 4.1 and Theorem 4.2. The original infinite programming problem (1.1) can be obviously rewritten as a mathematical program with geometric constraints:

$$\text{minimize } \varphi(p, x) \text{ subject to } (p, x) \in \text{gph } \mathcal{F}, \quad (4.7)$$

which is equivalently described by unconstrained minimization with “infinite penalties”:

$$\text{minimize } \varphi(p, x) + \delta((p, x); \text{gph } \mathcal{F})$$

via the indicator function of the graph of the feasible map $\mathcal{F}$ given in (1.2). Considering the general Banach space setting of Theorem 4.1 and applying the generalized Fermat rule (see, e.g., [21, Proposition 1.14]) to the latter problem at its local minimizer $(\bar{p}, \bar{x})$, we have

$$(0, 0) \in \partial_G [\varphi + \delta(\cdot; \text{gph } \mathcal{F})](\bar{p}, \bar{x}). \quad (4.8)$$

Employing further the G -subdifferential sum rule to (4.8), formulated in [12, Theorem 7.4] for the “nuclei”, we obtained from (4.8) that

$$(0, 0) \in \partial_G \varphi(\bar{p}, \bar{x}) + N((\bar{p}, \bar{x}); \text{gph } \mathcal{F}) \quad (4.9)$$

provided that either φ is locally Lipschitzian around $(\bar{p}, \bar{x})$, or the interior of $\text{gph } \mathcal{F}$ is nonempty and the qualification condition

$$\partial_G^\infty \varphi(\bar{p}, \bar{x}) \cap [-N((\bar{p}, \bar{x}); \text{gph } \mathcal{F})] = \{(0, 0)\} \quad (4.10)$$

is satisfied. It is easy to check that the strong Slater condition (4.1) and the boundedness of $\{(a_t^*, c_t^*) \mid t \in T\}$ surely imply that the interior of $\text{gph } \mathcal{F}$ is nonempty; cf. [5, Remark 2.4].

Observe that by the coderivative definition (3.8) we get

$$(p^*, x^*) \in -N((\bar{p}, \bar{x}); \text{gph } \mathcal{F}) \text{ if and only if } -p^* \in D^* \mathcal{F}(\bar{p}, \bar{x})(x^*)$$

Following the proofs of [5, Proposition 3.1] and [5, Theorem 3.2] in the case of the feasible map $\mathcal{F}$ from (1.2), we see that the previous conditions are equivalent to

$$(p^*, -x^*, -(\langle p^*, \bar{p} \rangle + \langle x^*, \bar{x} \rangle)) \in \text{cl}^* \text{cone}\{(-c_t^*, a_t^*, b_t) \mid t \in T\}.$$

Employing the above characterization of $-N((\bar{p}, \bar{x}); \text{gph } \mathcal{F})$ in (4.9) and (4.10), we respectively arrive at the necessary optimality condition (4.4) under the qualification condition (4.3). If furthermore the Farkas-Minkowski property (4.2) is satisfied, then the operation cl^* in (4.4) can be omitted, and the latter qualification condition easily reduces to (4.5). This completes the proof of Theorem 4.1.

To prove now Theorem 4.2, we observe first that the assumed Asplund property of the spaces X and P implies that their product $P \times X$ is also Asplund; see, e.g., [9]. Proceeding further as in the above proof of Theorem 4.1, we arrive at the generalized Fermat rule

$$(0, 0) \in \partial [\varphi + \delta(\cdot; \text{gph } \mathcal{F})](\bar{p}, \bar{x}) \quad (4.11)$$

in terms of the limiting subdifferential (3.3) in Asplund spaces and then apply to the sum in (4.11) the subdifferential sum rule from [21, Theorem 3.36] by taking into account the results of [21, Proposition 1.25 and Theorem 1.26] and recalling that the indicator function $\delta(\cdot; \text{gph } \mathcal{F})$ is l.s.c. on $P \times X$, since the graph $\text{gph } \mathcal{F}$ is a closed set. The aforementioned sum rule ensures the fulfillment of the inclusion

$$(0, 0) \in \partial \varphi(\bar{p}, \bar{x}) + N((\bar{p}, \bar{x}); \text{gph } \mathcal{F})$$

provided that either φ is locally Lipschitzian around $(\bar{p}, \bar{x})$, or the interior of $\text{gph } \mathcal{F}$ is nonempty and the qualification condition

$$\partial^\infty \varphi(\bar{p}, \bar{x}) \cap [-N((\bar{p}, \bar{x}); \text{gph } \mathcal{F})] = \{(0, 0)\}$$

is satisfied via the singular subdifferential (3.6) of φ at $(\bar{p}, \bar{x})$. The rest of the proof follows the lines in the above proof of Theorem 4.1. $\triangle$

Now we present several consequences of Theorem 4.1 and Theorem 4.2 all of which seem to be new in the settings under consideration. The first one concerns smooth semi-infinite and infinite programs with arbitrary index sets. Recall that a function $\varphi: Z \rightarrow \mathbb{R}$ is *strictly differentiable* at $\bar{z}$, with its gradient at this point denoted by $\nabla\varphi(\bar{z}) \in Z^*$, if we have

$$\lim_{z, u \rightarrow \bar{z}} \frac{\varphi(z) - \varphi(u) - \langle \nabla\varphi(\bar{z}), z - u \rangle}{\|z - u\|} = 0,$$

which surely holds if φ is continuously differentiable around $\bar{z}$.

Corollary 4.3 (necessary optimality conditions for smooth infinite programs in Banach spaces). *Let $(\bar{p}, \bar{x}) \in \text{gph } \mathcal{F}$ be a local minimizer for problem (1.1) in Banach spaces X and P with the constraint system $\mathcal{F}$ given in (1.2), and let the cost function $\varphi: P \times X \rightarrow \overline{\mathbb{R}}$ be strictly differentiable at $(\bar{p}, \bar{x})$. Then we have the inclusion*

$$\begin{aligned} & \left(\nabla_p \varphi(\bar{p}, \bar{x}), \nabla_x \varphi(\bar{p}, \bar{x}), \langle \nabla_p \varphi(\bar{p}, \bar{x}), \bar{p} \rangle + \langle \nabla_x \varphi(\bar{p}, \bar{x}), \bar{x} \rangle \right) \\ & \in \text{cl}^* \text{cone} \left\{ (c_t^*, -a_t^*, -b_t) \mid t \in T \right\}. \end{aligned} \quad (4.12)$$

If furthermore the constraint system (1.2) satisfies the Farkas-Minkowski property (4.2), then there exist multipliers $\lambda = (\lambda_t)_{t \in T} \in \mathbb{R}_+^{(T)}$ such that

$$\nabla_p \varphi(\bar{p}, \bar{x}) = \sum_{t \in T(\bar{p}, \bar{x})} \lambda_t c_t^*, \quad -\nabla_x \varphi(\bar{p}, \bar{x}) = \sum_{t \in T(\bar{p}, \bar{x})} \lambda_t a_t^*, \quad (4.13)$$

with $\mathbb{R}_+^{(T)}$ and $T(\bar{p}, \bar{x})$ described in Theorem 4.1.

Proof. We are based on Theorem 4.1, since Theorem 4.2 does not provide new information in comparison with Theorem 4.1 in this case. It is easy to check that

$$\partial_G \varphi(\bar{p}, \bar{x}) = \left\{ (\nabla_p \varphi(\bar{p}, \bar{x}), \nabla_x \varphi(\bar{p}, \bar{x})) \right\}$$

if φ is strictly differentiable at $(\bar{p}, \bar{x})$. Furthermore, a function strictly differentiable at some point is well known to be locally Lipschitzian around this point. Thus assumption (a) of Theorem 4.1 is satisfied; observe that the requirement on $\text{int}(\text{gph } \mathcal{F}) \neq \emptyset$ in (b) and hence the strong Slater condition for $\mathcal{F}$ with the boundedness of $\{a_t^* \mid t \in T\}$ are not needed. From (4.4) we arrive at (4.12), and (4.5) gives (4.13) in the case under consideration. $\triangle$

The next corollary specifies the necessary optimality conditions of Theorem 4.1 in the case of model (1.1) with infinite constraints in form (1.3) with $p \in l_\infty(T)$ considered in [4, 5]. For brevity and convenience we label such problems as l_∞ -programs.

Corollary 4.4 (lower subdifferential optimality conditions for l_∞ -programs). *Let $(0, \bar{x}) \in \text{gph } \mathcal{F}$ be a local minimizer for program (1.1) with the constraint system $\mathcal{F}$ given by (1.3), where $p = (p_t)_{t \in T} \in l_\infty(T)$. Assume that X is an arbitrary Banach space, that $\varphi: l_\infty(T) \times X \rightarrow \overline{\mathbb{R}}$ is l.s.c. around $(0, \bar{x})$ with $\varphi(0, \bar{x}) < \infty$, and that:*

- (a) either φ is locally Lipschitzian around $(0, \bar{x})$;
- (b) or $\text{int}(\text{gph } \mathcal{F}) \neq \emptyset$ (which holds, in particular, when the set $\{a_t^* \mid t \in T\}$ is bounded in X^*) and the system

$$(p^*, x^*) \in \partial_G^\infty \varphi(0, \bar{x}), \quad -\langle p^*, x^*, \langle x^*, \bar{x} \rangle \rangle \in \text{cl}^* \text{cone} \{(-\delta_t, a_t^*, b_t) \mid t \in T\} \quad (4.14)$$

has only the trivial solution $(p^*, x^*) = (0, 0)$. Then there exists a G -subgradient pair $(p^*, x^*) \in \partial_G \varphi(0, \bar{x})$ such that

$$-\langle p^*, x^*, \langle x^*, \bar{x} \rangle \rangle \in \text{cl}^* \text{cone} \{(-\delta_t, a_t^*, b_t) \mid t \in T\}. \quad (4.15)$$

If furthermore the constraint system (1.3) satisfies the Farkas-Minkowski property (4.2) with $c_t^* = \delta_t$, then condition (4.15) can be equivalently written in the Karush-Kuhn-Tucker form: there are $(p^*, x^*) \in \partial_G \varphi(0, \bar{x})$ and $\lambda = (\lambda_t)_{t \in T} \in \mathbb{R}_+^{(T)}$ from (1.4) for which

$$(p^*, x^*) + \sum_{t \in T(\bar{x})} \lambda_t (-\delta_t, a_t^*) = 0 \quad \text{with} \quad T(\bar{x}) := \{t \in T \mid \langle a_t^*, \bar{x} \rangle = b_t\}. \quad (4.16)$$

Proof. We can treat the constraint system (1.3) as a particular case of (1.2) with $c_t^* = \delta_t$, the Dirac measure at $t \in T$, which belongs to the dual space $P^* = l_\infty(T)^*$. It is easy to see then that conditions (4.3), (4.4), and (4.5) of Theorem 4.1 reduce to the corresponding conditions (4.14), (4.15), and (4.16) of the corollary, with the corresponding versions of the SSC, boundedness, and Farkas-Minkowski properties formulated in the corollary. Observe finally that for system (1.3) the presence of some feasible point yields the fulfillment of the SSC. Specifically, the inclusion $(0, \bar{x}) \in \text{gph } \mathcal{F}$ implies, e.g., that $(1_T, \bar{x})$ is a strong Slater point of (1.3), where the function $1_T \in l_\infty(T)$ is defined by $1_T(t) := 1$ for all $t \in T$. $\triangle$

As mentioned at the beginning of this section, there are various qualification conditions implying the fulfillment of the Farkas-Minkowski property for infinite inequality systems (1.2) and (1.3); see the references and discussions above. By Theorem 4.1 and Theorem 4.2, all such assumptions ensure the validity of necessary optimality conditions of the KKT type (4.5) for the nonsmooth problems of semi-infinite and infinite programming under consideration. The next corollary establishes one of the results of this type for semi-infinite programs with constraints (1.2) over compact index sets. For simplicity we present optimality conditions only for locally Lipschitzian cost functions while the reader can similarly extract from case (b) of the above theorems the corresponding result for l.s.c. objectives.

Corollary 4.5 (necessary optimality conditions of the KKT type for nonsmooth semi-infinite programs). *Let $(\bar{p}, \bar{x}) \in \text{gph } \mathcal{F}$ be a local minimizer for program (1.1) with the constraint system (1.2), where $X = \mathbb{R}^n$, $P = \mathbb{R}^m$, T is a compact Hausdorff space, and $\varphi: \mathbb{R}^m \times \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ is locally Lipschitzian around $(\bar{p}, \bar{x})$. Assume in addition that the mappings $t \in T \mapsto a_t^* \in \mathbb{R}^n$, $t \in T \mapsto c_t^* \in \mathbb{R}^m$, and $t \in T \mapsto b_t \in \mathbb{R}$ are continuous on T and that the SSC (4.1) holds for (1.2). Then there are subgradients $(p^*, x^*) \in \partial \varphi(\bar{p}, \bar{x})$ and multipliers $\lambda = (\lambda_t)_{t \in T} \in \mathbb{R}_+^{(T)}$ such that the KKT condition (4.5) is satisfied.*

Proof. Applying either Theorem 4.1 or Theorem 4.2 in this framework, it remains to check that the Farkas-Minkowski property (4.2) holds for (1.2) under the assumptions made.

Indeed, we directly get the boundedness of the set $\{(c_t^*, a_t^*, b_t) \mid t \in T\}$ in $\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}$ due to the assumed continuity of (c_t^*, a_t^*, b_t) and compactness of T . Further, the equivalent description of the strong Slater condition from [5, Lemma 2.3(ii)] ensures that

$$(0, 0, 0) \notin \text{co}\{(-c_t^*, a_t^*, b_t) \mid t \in T\},$$

which implies by [26, Corollary 9.6.1] that the conic hull $\text{cone}\{(-c_t^*, a_t^*, b_t) \mid t \in T\}$ is closed in $\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}$. The latter signifies the fulfillment of the Farkas-Minkowski property (4.2) for system (1.2) and thus completes the proof of the corollary. $\triangle$

Let us finally discuss some remarkable features of the necessary optimality conditions obtained in this section and compare them with known results in this direction.

Remark 4.6 (discussions on lower subdifferential optimality conditions). Observe first that the general necessary optimality conditions obtained in Theorem 4.1, Theorem 4.2, and their corollaries are given in the *normal form* involving nonzero multipliers for the cost function. Regarding constraints, these conditions are generally expressed in the *asymptotic form* that involves the weak* closure of the sets on the right-hand sides in (4.4) and (4.12). The latter new feature mainly relates to arbitrary index sets in the semi-infinite and infinite models considered in the paper. Under the Farkas-Minkowski property/constraint qualification (4.2), we get necessary optimality conditions in the non-asymptotic *KKT form* (4.5), which reduce to those recently obtained in [7, 8] for one-variable convex and DC (difference of convex) objectives in semi-infinite and infinite programs with arbitrary index sets. Results of such a KKT type have been mainly developed for smooth and convex semi-infinite and infinite programs with compact index sets; cf. Corollary 4.5 for a rather broad nonsmooth extension. We refer the reader to the recent paper [30], probably the first one on nonsmooth and nonconvex semi-infinite optimization, containing necessary optimality conditions of a Lagrangian type for nonsmooth and nonconvex semi-infinite programs with compact index sets and $\mathcal{C}(T)$ data. The necessary conditions obtained in [30] are expressed in terms of Clarke's generalized gradient, which can be significantly larger than the subdifferentials used in the corresponding results derived in this section.

5 Upper Subdifferential Optimality Conditions

The last section of the paper is devoted to deriving a new type of *upper subdifferential* necessary optimality conditions for the class of semi-infinite/infinite programs (1.1) with infinitely many linear inequality constraints (1.2) and (1.3). Optimality conditions of this type were initiated in [20] for other classes of nonsmooth minimization problems with finitely many constraints, while in fact they have their roots in the study of maximization (vs. minimization) problems for concave functions over convex sets; see, e.g., [26].

The main difference of the results derived in this section from those in Section 4 is the usage of upper subgradients (or supergradients) of *minimizing* cost functions instead of the conventional use of (lower) subgradients in minimization. In this way we obtain independent sets of necessary optimality conditions for the problems under consideration

in general Banach spaces; see Remark 5.5 for more details and discussions.

To proceed, we recall the notion of the *Fréchet upper subdifferential* (known also as the Fréchet or viscosity superdifferential) of $\varphi: Z \rightarrow \overline{\mathbb{R}}$ at $\bar{z}$ defined by

$$\widehat{\partial}^+ \varphi(\bar{z}) := \left\{ z^* \in Z^* \mid \limsup_{z \rightarrow \bar{z}} \frac{\varphi(z) - \varphi(\bar{z}) - \langle z^*, z - \bar{z} \rangle}{\|z - \bar{z}\|} \leq 0 \right\}, \quad (5.1)$$

which reduces to the classical gradient $\nabla \varphi(\bar{z})$ if φ is Fréchet differentiable at $\bar{z}$ and to the (upper) subdifferential of *concave* functions in the framework of convex analysis. Note that we always have the relationship $\widehat{\partial}^+ \varphi(\bar{z}) = -\widehat{\partial}(-\varphi)(\bar{z})$ between the upper subdifferential (5.1) and its *lower* Fréchet counterpart defined in (3.1) with $\varepsilon = 0$.

We have the following upper subdifferential necessary optimality conditions for the infinite and semi-infinite programs (1.1) with constraints (1.2) under consideration.

Theorem 5.1 (upper subdifferential optimality conditions for nonsmooth infinite programming in Banach spaces). *Let $(\bar{p}, \bar{x}) \in \text{gph } \mathcal{F}$ be a local minimizer for problem (1.1) with the infinite inequality constraints (1.2) in Banach spaces X and P . Then EVERY upper subgradient $(p^*, x^*) \in \widehat{\partial}^+ \varphi(\bar{p}, \bar{x})$ satisfies inclusion (4.4) of Theorem 4.1. If furthermore the constraint system (1.2) has the Farkas-Minkowski property (4.2), then the asymptotic condition (4.4) can be equivalently written in the upper subdifferential KKT form: for EVERY $(p^*, x^*) \in \widehat{\partial}^+ \varphi(\bar{p}, \bar{x})$ there are multipliers $\lambda = (\lambda_t)_{t \in T} \in \mathbb{R}_+^{(T)}$ such that the optimality condition (4.5) is satisfied.*

Proof. Pick any $(p^*, x^*) \in \widehat{\partial}^+ \varphi(\bar{p}, \bar{x})$ and, employing [21, Theorem 1.88(i)] held in arbitrary Banach spaces, construct a function $s: P \times X \rightarrow \overline{\mathbb{R}}$ such that

$$s(\bar{p}, \bar{x}) = \varphi(\bar{p}, \bar{x}), \quad \varphi(p, x) \leq s(p, x) \quad \text{for all } (p, x) \in P \times X, \quad (5.2)$$

and $s(\cdot)$ is Fréchet differentiable at $(\bar{p}, \bar{x})$ with the gradient $\nabla s(\bar{p}, \bar{x}) = (p^*, x^*)$. Taking into account that $(\bar{p}, \bar{x})$ is a local minimizer for (1.1) with constraints (1.2) and that

$$s(\bar{p}, \bar{x}) = \varphi(\bar{p}, \bar{x}) \leq \varphi(p, x) \leq s(p, x) \quad \text{for all } (p, x) \in \text{gph } \mathcal{F} \text{ near } (\bar{p}, \bar{x})$$

by (5.2), we conclude that $(\bar{p}, \bar{x})$ is a *local minimizer* for the auxiliary problem

$$\text{minimize } s(p, x) \text{ subject to } (p, x) \in \text{gph } \mathcal{F} \quad (5.3)$$

with the objective $s(\cdot)$ that is Fréchet differentiable at $(\bar{p}, \bar{x})$. Rewriting (5.3) in the infinite-penalty unconstrained form

$$\text{minimize } s(p, x) + \delta((p, x); \text{gph } \mathcal{F})$$

via the indicator function of $\text{gph } \mathcal{F}$, observe directly from definition (3.1) of the Fréchet subdifferential at a local minimizer that

$$(0, 0) \in \widehat{\partial}[s + \delta(\cdot; \text{gph } \mathcal{F})](\bar{p}, \bar{x}). \quad (5.4)$$

Since $s(\cdot)$ is Fréchet differentiable at $(\bar{p}, \bar{x})$, we easily get from (5.4) that

$$(0, 0) \in \nabla s(\bar{p}, \bar{x}) + N((\bar{p}, \bar{x}); \text{gph } \mathcal{F}),$$

which implies by $\nabla s(\bar{p}, \bar{x}) = (p^*, x^*)$ and the coderivative definition (3.8) that

$$-p^* \in D^*\mathcal{F}(\bar{p}, \bar{x})(x^*). \quad (5.5)$$

It follows from the proof of [5, Theorem 3.2] in the case of the feasible map $\mathcal{F}$ given in (1.2), that $p^* \in D^*\mathcal{F}(\bar{p}, \bar{x})(x^*)$ if and only if inclusion (??) holds. By (5.5) the latter justifies (4.4) for the given upper subgradient $(p^*, x^*) \in \hat{\partial}^+\varphi(\bar{p}, \bar{x})$. The KKT conclusion of the theorem is proved similarly to Theorem 4.1. $\triangle$

As a simple consequence of Theorem 5.1, we get an improvement of Corollary 4.3, where the cost function φ is assumed to be merely *Fréchet differentiable* at the optimal point $(\bar{p}, \bar{x})$ instead of the more restrictive assumption on its strict differentiability at this point.

Corollary 5.2 (necessary optimality conditions for infinite programs with Fréchet differentiable objectives). *Let $(\bar{p}, \bar{x}) \in \text{gph } \mathcal{F}$ be a local minimizer for problem (1.1) and constraints (1.2) in Banach spaces X and P . Assume that the cost function φ is Fréchet differentiable at $(\bar{p}, \bar{x})$ with the gradient $\nabla\varphi(\bar{p}, \bar{x})$. Then the necessary optimality condition (4.12) is satisfied. If furthermore the constraint system (1.2) has the Farkas-Minkowski property (4.2), then there exist multipliers $\lambda = (\lambda_t)_{t \in T} \in \mathbb{R}_+^{(T)}$ such that (4.13) holds.*

Proof. It follows directly from Theorem 5.1 due to the fact that $\hat{\partial}^+\varphi(\bar{p}, \bar{x}) = \{\nabla\varphi(\bar{p}, \bar{x})\}$ when φ is Fréchet differentiable at $(\bar{p}, \bar{x})$. $\triangle$

Let us further present upper subdifferential counterparts of Corollary 4.4 and Corollary 4.5 from the previous section.

Corollary 5.3 (upper subdifferential optimality conditions for l_∞ -programs). *Let $(0, \bar{x}) \in \text{gph } \mathcal{F}$ be a local minimizer for problem (1.1) with the infinite constraint system (1.3) in Banach spaces. Then EVERY upper subgradient $(p^*, x^*) \in \hat{\partial}^+\varphi(0, \bar{x})$ satisfies inclusion (4.15) in Corollary 4.4. If furthermore the constraint system (1.3) satisfies the Farkas-Minkowski property (4.2) with $c_t^* = \delta_t$, then condition (4.15) can be equivalently written in the upper subdifferential Karush-Kuhn-Tucker form: for EVERY $(p^*, x^*) \in \hat{\partial}^+\varphi(0, \bar{x})$ there are multipliers $\lambda = (\lambda_t)_{t \in T} \in \mathbb{R}_+^{(T)}$ from (1.4) such that (4.16) holds.*

Proof. Follows from Theorem 5.1 similarly to the proof of Corollary 4.4. $\triangle$

Corollary 5.4 (upper subdifferential optimality conditions of the KKT type for nonsmooth semi-infinite programs). *Let $(\bar{p}, \bar{x}) \in \text{gph } \mathcal{F}$ be a local minimizer for program (1.1) with the constraint system (1.2), where $X = \mathbb{R}^n$, $P = \mathbb{R}^m$, T is a compact Hausdorff space, and $\varphi: \mathbb{R}^m \times \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ is finite at $(\bar{p}, \bar{x})$. Assume in addition that the mappings $t \in T \mapsto a_t^* \in \mathbb{R}^n$, $t \in T \mapsto c_t^* \in \mathbb{R}^m$, and $t \in T \mapsto b_t \in \mathbb{R}$ are continuous on T and that the SSC (4.1) holds for (1.2). Then for EVERY upper subgradient $(p^*, x^*) \in \hat{\partial}^+\varphi(\bar{p}, \bar{x})$ there are multipliers $\lambda = (\lambda_t)_{t \in T} \in \mathbb{R}_+^{(T)}$ from (1.4) such that the KKT condition (4.5) is satisfied.*

Proof. Follows from Theorem 5.1 similarly to the proof of Corollary 4.5. $\triangle$

Finally, we discuss the major relationships between the lower and upper subdifferential optimality conditions obtained in this paper focusing mainly on comparison between the corresponding conditions of Theorems 4.1, 4.2 and Theorem 5.1. Note that there is no particular counterpart of Theorem 5.1 in the Asplund space setting.

Remark 5.5 (comparison between lower and upper subdifferential optimality conditions for infinite and semi-infinite programs). We can see that the necessary optimality conditions in Theorems 4.1, 4.2 and Theorem 5.1 are formulated in the similar formats with two visible distinctions:

- (i) The upper subdifferential conditions in the asymptotic form hold in Theorem 5.1 with *no assumptions* imposed on φ and $\mathcal{F}$ in contrast to those in Theorems 4.1 and 4.2.
- (ii) The resulting inclusion (4.4) is proved to hold for *every* Fréchet upper subgradient $(p^*, x^*) \in \widehat{\partial}^+ \varphi(\bar{p}, \bar{x})$ in Theorem 5.1 in comparison with just *some* (lower) subgradients $(p^*, x^*) \in \partial_G \varphi(\bar{p}, \bar{x})$ and $(p^*, x^*) \in \partial \varphi(\bar{p}, \bar{x})$ in the lower subdifferential result of Theorem 4.1 and Theorem 4.2, respectively.

The underlying issue to draw the reader's attention is that the Fréchet upper subdifferential $\widehat{\partial}^+ \varphi(\bar{p}, \bar{x})$ may be *empty* in many important situations (e.g., for convex cost functions) while the lower subdifferentials $\partial_G \varphi(\bar{p}, \bar{x})$ and $\partial \varphi(\bar{p}, \bar{x})$ are surely *nonempty* at least for any locally Lipschitzian functions on Banach and Asplund spaces, respectively. Note that the optimality condition of Theorem 5.1 holds trivially if $\widehat{\partial}^+ \varphi(\bar{p}, \bar{x}) = \emptyset$, while even in this case it provides some easily checkable information on optimality without taking constraints into account. Of course, a real strength of upper subdifferential optimality conditions as in Theorem 5.1 should be exhibited for *nonsmooth* cost functions admitting Fréchet upper subgradients at the point in question.

There are remarkable classes of nonsmooth functions enjoying the latter property. First we mention *concave* continuous functions on arbitrary Banach spaces and also *DC* (difference of convex) functions whose minimization can be reduced to minimizing concave functions subject to convex constraints. Another important class of functions admitting a nonempty set of Fréchet upper subgradients consists of the so-called *semiconcave* functions, known also under various other names (e.g., upper subsmooth, paraconcave, approximately concave, etc.) and being particularly important for applications to optimization, viscosity solutions of the Hamilton-Jacobi partial differential equations, optimal control, and differential games; see more discussions and references in [22, Commentary 5.5.4, pp. 135–136].

Since the subdifferentials $\partial_G \varphi(\bar{p}, \bar{x})$ and $\partial \varphi(\bar{p}, \bar{x})$ used above are smaller than the Clarke generalized gradient $\partial_C \varphi(\bar{p}, \bar{x})$ for every l.s.c. function in any Banach space, Theorem 4.1 and Theorem 4.2 immediately imply their counterparts with some *C*-subgradient $(p^*, x^*) \in \partial_C \varphi(\bar{p}, \bar{x})$ therein. It is worth emphasizing that the latter lower subdifferential optimality condition is *significantly weaker* than the upper subdifferential one in Theorem 5.1 for concave and other “upper regular” functions (see [21]) including those mentioned above. Considering for simplicity the case of concave continuous functions, we have

$$\widehat{\partial}^+ \varphi(\bar{z}) = -\widehat{\partial}(-\varphi)(\bar{z}) = -\partial_C(-\varphi)(\bar{z}) = \partial_C \varphi(\bar{z}) \neq \emptyset$$

due to the plus-minus symmetry of the generalized gradient for locally Lipschitzian functions. Thus Theorem 5.1 dramatically strengthens the C -counterpart of Theorem 4.1 and Theorem 4.2 in such cases justifying the necessary optimality condition held for *every* $(p^*, x^*) \in \partial_C \varphi(\bar{p}, \bar{x})$ instead of just *one* element from this set.

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