1-1-1993

# Optimal Control and Differential Games with Measures

E. N. Barron
*Loyola University of Chicago*

R. Jensen
*Loyola University of Chicago*

J. L. Menaldi
*Wayne State University*, menaldi@wayne.edu

## Recommended Citation

E. N. Barron, R. Jensen, and J.-L. Menaldi, *Optimal control and differential games with measures*, Nonlinear Analysis: Theory, Methods, and Applications **21** (1993), 241-268. doi: 10.1016/0362-546X(93)90019-O
Available at: https://digitalcommons.wayne.edu/mathfrp/37

# Optimal Control and Differential Games with Measures

**E.N.Barron*, R.Jensen****

Loyola University of Chicago

Department of Mathematical Sciences

Chicago,Illinois,60626

**J.L. Menaldi*****

Wayne State University

Department of Mathematics

Detroit, Michigan,48202

December 1990

Keywords:optimal control and differential games, bounded variation trajectories, viscosity solutions, singular control

*Abstract.* We consider control problems with trajectories which involve ordinary measureable control functions and controls which are measures. The payoff involves a running cost in time and a running cost against the control measures. In the optimal control problem we are trying to minimize this payoff with both controls. In the differential game problem we are trying to minimize the cost with the ordinary controls assuming that the measure controls are chosen to maximize the cost. We will characterize the value functions in both cases using viscosity solution theory by deriving the Bellman and Isaacs equations.

## 0.Introduction and Summary

The problems of this paper are motivated by models of physical controlled systems in which the trajectory is a function of bounded variation. The time of the jumps, if any, and the new spatial positions are under the control of the designer. In the optimal control problem the objective is to minimize a cost involving a running cost and a cumulative cost against the control measure. When the measure involves only jumps this will be a standard impulse control problem. But in this paper we are not restricting the measures merely to jumps but are allowing general Radon measures.

In many problems of interest we can manipulate the system only with ordinary controls and we wish to do so to minimize a cost. But when the system is subject to disturbances one seeks to design the system so as to perform well under the worst possible circumstances. In this situation we assume that the disturbances are modelled by a measure term in the

*Supported in part by AFOSR-86-0202, NSF DMS-9102967, and a grant from Loyola University

**Supported in part by AFOSR-86-0202, an NSF grant, and a grant from Loyola University

***Supported in part by NSF DMS-9101360

dynamics with a cost incurred in the payoff. The worst case analysis assumes that the measures are chosen so as to maximize this payoff. Therefore, this model is a differential game in which the dynamics is a function of bounded variation, the payoff involves the measures, and we are choosing an ordinary control to minimize this payoff while the opponent (in some cases considered to be nature) is choosing the measures to maximize the payoff.

A result of this paper is that the order of play of the maximizer and minimizer makes a difference. That is, the differential game with a maximizing measure and a minimizing ordinary control does not, in general, have a value. The upper value, i.e., the case when the maximizer has knowledge of the minimizer, is then the central object of interest in a worst case analysis. We will derive the results for both the upper and lower value and give a sufficient condition for the game to have a value.

The approach throughout this paper is dynamic programming leading to the value functions and the associated Bellman and Isaacs equations. In the optimal control case the Bellman equation becomes a standard variational inequality with two first order operators. In the differential game case the Isaacs equation is a highly nonlinear, first order problem involving a minimization over a set which depends on the derivatives of the value function. Precisely, the Isaacs equation for the upper value is

$$V_t^+ + \min_{z \in \mathcal{Z}(0,t,V_m^+,V_x^+)} \{V_x^+ \cdot f_1(t,x,z) + h_1(t,x,z)\} = 0,$$

where

$$\mathcal{Z}(0,t,V_m^+,V_x^+) = \{z \in Z : V_m^+ + V_x^+ \cdot f_2(t,z) + h_2(t,z) \leq 0\}.$$

This equation has a discontinuous, generally nonconvex hamiltonian. The equation for lower value is even more complicated. A theory of first order partial differential equations encompassing such equations is viscosity solution theory initiated by Crandall and Lions [10].

The first example of a Bellman equation with control sets depending on the solution arose in the consideration of an optimal control problem with a minimax cost [7,8]. That is, the minimax problem consists of finding a control which minimizes the $L^\infty$ norm of a function of time, the state, and the control. Using the well known fact that the $L^\infty$ norm of a function is the maximum over subprobability measures of the function integrated against the measure, we see that the minimax problem is a special case of the subject of this paper. This example is included at the end of this paper.

Some justification for taking the dynamic programming approach to the problems of this paper may be necessary. Control problems involving measures are extremely difficult to solve via necessary conditions [21,25]. Such necessary conditions are not even known for the differential game. The Pontryagin conditions involve knowing *a priori* the support of the optimal measures, which in turn depends on the unknown adjoint variables. Further, one then must verify that one actually has an optimal control. The determination of the value function by solving the Bellman equation is not beyond the scope of numerical methods. Moreover, the Bellman equation leads to the candidate feedback optimal controls in the usual way. Finally, it is well known, and proved in [9], that for standard control problems there is an intimate connection between the adjoint variable in the Pontryagin conditions and the spatial gradient of the value function. In fact, the adjoint variable is the spatial gradient of the optimal cost evaluated along the optimal trajectory. Such a result is not so clear in problems involving measures.

Finally, we mention that previous work regarding problems with measures in one form or another appears in [2,6,7,10,15,19-26]. Necessary conditions are derived in [22] and [25]. Problems with measures are more commonly called singular control problems. See [18,24] for related examples.

## 1. The Optimal Control Problem

We consider the following model on the finite horizon $[0,T]$. The dynamics are:

$$d\xi = f_1(\tau,\xi(\tau),\zeta(\tau))d\tau + f_2(\tau,\zeta(\tau))d\mu(\tau) \qquad \text{if } t<\tau\leq T, \tag{1.1}$$

$$\xi(t) = x \in R^1 \qquad \mu(t) = m \in [0,1]. \tag{1.2}$$

The controls are $(\zeta,\mu)$, chosen from the class $(Z\times\mathcal{M}_m)[t,T]$ where

$$Z[t,T] \equiv \{\zeta \mid \zeta:[t,T]\to Z,\ \zeta \text{ is Borel measurable}\}$$

$$\mathcal{M}_m[t,T] \equiv \{\mu \mid \mu:[t,T]\to[0,1],\ \mu \text{ is non decreasing on } [t,T],$$

$$\mu(t)=m,\ \mu \text{ is right continuous on } [t,T]\}.$$

$Z$ is a compact subset of some $R^p$, $p\geq 1$, and $m$ is any point in $[0,1]$. Since $\zeta\in Z[t,T]$ is bounded and Borel measureable, $\tau\mapsto(h_2(\tau,\zeta(\tau)),\ f_2(\tau,\zeta(\tau))$ are $d\mu$ integrable for any $\mu\in\mathcal{M}_m[t,T]$. We use the convention that $\mu(t-)=m$, and so $d\mu$ may have a point mass at the initial point $t$. There is a one-one correspondence between Radon measures $d\mu$ and distribution functions $\mu$.

The objective in this section will be to *minimize* the following cost over the class $(Z\times\mathcal{M}_m)[t,T]$ :

$$P_{t,x,m}(\zeta,\mu) = \int_t^T h_1(r,\xi(r),\zeta(r))\,dr + \int_{[t,T]} h_2(r,\zeta(r))\,d\mu(r). \tag{1.3}$$

We define the value function $V:[0,T]\times R^1\times[0,1]\to R^1$ as follows:

$$V(t,x,m) = \inf_{(\zeta,\mu)\in(Z\times\mathcal{M}_m)[t,T]} P_{t,x,m}(\zeta,\mu)\,. \tag{1.4}$$

We will make the following assumptions regarding the given functions $f_i$ and $h_i, i=1,2$:

**(A).** *For* $\varphi_i = f_i, h_i$ *we assume* $\varphi_1 : [0,T] \times R^1 \times Z \to R^1$ *and* $\varphi_2 : [0,T] \times Z \to R^1$ *are continuous in all arguments and there is a constant* $K > 0$ *such that,*

$$|\varphi_1(t,x,z)| \leq K, \ |\varphi_2(t,z)| \leq K, \ \forall (t,x,z) \in [0,T] \times R^1 \times Z$$

*and*

$$|\varphi_1(t,x,z) - \varphi_1(t,y,z)| \leq K|x-y|, \quad \forall (t,z) \in [0,T] \times Z, \text{ and } x, y \in R^1;$$

$$|\varphi_2(t,z) - \varphi_2(t',z)| \leq K|t-t'|, \quad z \in Z.$$

The assumption (A) is more than sufficient to guarantee that for each pair of controls $(\zeta, \mu) \in (Z \times \mathcal{M}_m)[t,T]$ there will be a unique trajectory $\xi(\cdot)$ on the interval $[t,T]$. This trajectory is not necessarily absolutely continuous but it will be of bounded total variation. In general, a unique trajectory will not exist if we allow dependence of $f_2$ on $x$. For given admissible controls $(\zeta, \mu)$, the associated trajectory starting from $x \in R^1$ is, by definition, the solution of

$$\xi(\tau) = x + \int_t^\tau f_1(r, \xi(r), \zeta(r))\, dr + \int_{[t,\tau]} f_2(r, \zeta(r))\, d\mu(r).$$

¿From the fact that $\int_{[t,T]} d\mu \leq 1$, we easily verify that $\sup_{t \leq \tau \leq T} \|\xi(\tau)\| \leq K$ independent of controls. Furthermore, $\xi$ is right continuous.

*Remark.*We will be considering only the 1-dimensional case in this paper to simplify the presentation. The extension to the n- dimensional case involves interpreting appropriately the meaning of the expressions $f_2 \cdot d\mu$ and $h_2 \cdot d\mu$. This can be done in several ways, c.f. [25].

Our first theorem establishes the continuity of the value function

**Theorem 1.1.** *Under the condition (A),* $V$ *is a continuous function of* $(t,x,m) \in [0,T] \times R^1 \times [0,1]$. *In fact,*

*(1)* $|V(t,x,m) - V(t,y,m)| \leq K|x-y|$,

*(2)* $0 \leq V(t,x,m_2) - V(t,x,m_1) \leq K(m_2 - m_1)$ *if* $0 \leq m_1 \leq m_2 \leq 1$,

*(3)* $-\frac{K}{T-t_2}(t_2 - t_1) \leq V(t_1,x,m) - V(t_2,x,m) \leq K(t_2 - t_1)$ *if* $0 \leq t_1 \leq t_2 < T$.

*Proof.* The hard part is establishing continuity in $t$, so we will first prove continuity in $m$. Continuity in $x$ is easy and we will leave it for the reader.

Fix $m_1 < m_2 \in [0,1]$ and fix $t \in [0,T], x \in R^1$. For each $\epsilon > 0$ we find $(\zeta_\epsilon, \mu_1) \in (Z \times \mathcal{M}_{m_1})[t,T]$, such that

$$V(t,x,m_1) \geq P_{t,x,m_1}(\zeta_\epsilon, \mu_1) - \epsilon. \tag{1.5}$$

Define $\tau_0$ as the first time after $t$ for which $\mu_1(\tau) + m_2 - m_1 \geq 1$; if this condition never occurs then set $\tau_0 = T$. Let

$$\mu_2(\tau) = \begin{cases} \mu_1(\tau) + m_2 - m_1, & \text{if } \tau < \tau_0 \\ 1, & \text{if } \tau \geq \tau_0. \end{cases}$$

Let $\xi_1$ be the trajectory using the controls $(\zeta_\epsilon, \mu_1)$ and $\xi_2$ the trajectory for $(\zeta_\epsilon, \mu_2)$. Then, for $i = 1, 2$,

$$\xi_i(\tau) = x + \int_t^\tau f_1(r, \xi_i(r), \zeta_\epsilon(r))\, dr + \int_{[t,\tau]} f_2(r, \zeta_\epsilon(r))\, d\mu_i(r)$$

These trajectories will be identical if $\tau_0 = T$, so we assume that $\tau_0 < T$. On the time interval $[t, \tau_0]$ the trajectories are identical. Let $\tau_0 < \tau \leq T$. We have from (A), with $K$ denoting a generic constant, that

$$\begin{aligned} |\xi_1(\tau) - \xi_2(\tau)| \leq K \int_t^\tau |\xi_1(r) - \xi_2(r)|\, dr \\ + \left| \int_{[t,\tau]} f_2(r, \zeta_\epsilon(r))\, d\mu_1(r) - \int_{[t,\tau]} f_2(r, \zeta_\epsilon(r))\, d\mu_2(r) \right| \\ = K \int_t^\tau |\xi_1(r) - \xi_2(r)|\, dr + \left| \int_{[\tau_0,\tau]} f_2(r, \zeta_\epsilon(r))\, d\mu_1(r) \right| \\ \leq K \int_t^\tau |\xi_1(r) - \xi_2(r)|\, dr + K \int_{[\tau_0,T]} d\mu_1(r) \\ \leq K \int_t^\tau |\xi_1(r) - \xi_2(r)|\, dr + K(m_2 - m_1). \end{aligned}$$

Gronwall's inequality allows us to conclude that

$$\sup_{t \leq \tau \leq T} |\xi_1(\tau) - \xi_2(\tau)| \leq K(m_2 - m_1).$$

It then follows, by a similar calculation that

$$|P_{t,x,m_1}(\zeta_\epsilon, \mu_1) - P_{t,x,m_2}(\zeta_\epsilon, \mu_2)| \leq K(m_2 - m_1).$$

Consequently, using (1.5),

$$\begin{aligned} V(t, x, m_1) &\geq P_{t,x,m_2}(\zeta_\epsilon, \mu_2) - K(m_2 - m_1) - \epsilon \\ &\geq V(t, x, m_2) - K(m_2 - m_1) - \epsilon. \end{aligned}$$

So, we conclude that

$$V(t, x, m_2) - V(t, x, m_1) \leq K(m_2 - m_1). \tag{1.6}$$

For the other side we use the lemma.

**Lemma 1.2.** *$V$ is monotone nondecreasing in $m \in [0,1]$, i.e.,*

$$V(t,x,m_2) - V(t,x,m_1) \geq 0, \qquad 1 \geq m_2 \geq m_1 \geq 0. \tag{1.7}$$

*Proof.* For each $\epsilon > 0$ choose $(\zeta_\epsilon, \mu_2) \in (Z \times \mathcal{M}_{m_2})\,[t,T]$, such that

$$V(t,x,m_2) \geq P_{t,x,m_2}(\zeta_\epsilon, \mu_2) - \epsilon.$$

Let $\mu_1 \equiv \mu_2 - m_2 + m_1$. Then $\mu_1$ starts at $m_1$ and is simply $\mu_2$ shifted down by $m_2 - m_1$. Further, $d\mu_1 \equiv d\mu_2$ so that the associated trajectories are identical. Therefore,

$$\begin{aligned} V(t,x,m_2) &\geq P_{t,x,m_2}(\zeta_\epsilon, \mu_2) - \epsilon \\ &= P_{t,x,m_1}(\zeta_\epsilon, \mu_1) - \epsilon \\ &\geq V(t,x,m_1) - \epsilon, \end{aligned}$$

completing the proof of the lemma. □

Combining (1.6) and (1.7), continuity in $m$ and (2) is established.

Now we turn to continuity in $t \in [0,T)$. Fix $0 \leq t_1 < t_2 < T$ and fix $x \in R^1,\ m \in [0,1)$. For each $\epsilon > 0$ there exists $(\zeta_2, \mu_2) \in (Z \times \mathcal{M}_m)\,[t_2,T]$, such that

$$V(t_2,x,m) \geq P_{t_2,x,m}(\zeta_2, \mu_2) - \epsilon. \tag{1.8}$$

Set $\zeta_1(\tau) = \zeta_2(t_2)$, if $t_1 \leq \tau < t_2$; $\zeta_1(\tau) \equiv \zeta_2(\tau)$, if $t_2 \leq \tau \leq T$. Let $\mu_1(\tau) = m$ if $t_1 \leq \tau \leq t_2$, and $\mu_1(\tau) \equiv \mu_2(\tau)$ if $t_2 \leq \tau \leq T$. Then, $(\zeta_1, \mu_1) \in (Z \times \mathcal{M}_m)\,[t_1,T]$. Finally, let $\xi_2$ be the trajectory on $[t_2,T]$ for the controls $(\zeta_2, \mu_2)$ and let $\xi_1$ be the trajectory on $[t_1,T]$, also starting from $x$, for the controls $(\zeta_1, \mu_1)$.

Then, it follows from (A) and the fact that $d\mu_1(\tau) = 0$ if $t_1 \leq \tau \leq t_2$, and $d\mu_1(\tau) \equiv d\mu_2(\tau)$ if $t_2 \leq \tau \leq T$, that

$$|\xi_1(t_2) - x| \leq K(t_2 - t_1) \qquad \text{and} \qquad \sup_{t_2 \leq \tau \leq T} |\xi_1(\tau) - \xi_2(\tau)| \leq K(t_2 - t_1);$$

and, so

$$|P_{t_1,x,m}(\zeta_1,\mu_1) - P_{t_2,x,m}(\zeta_2,\mu_2)| \leq \int_{t_1}^{t_2} |h_1(r,\xi_1(r),z)|\,dr + K(t_2 - t_1) \leq K(t_2 - t_1).$$

Therefore, from (1.8),

$$V(t_2,x,m) \geq P_{t_1,x,m}(\zeta_1,\mu_1) - K(t_2 - t_1) - \epsilon \geq V(t_1,x,m) - K(t_2 - t_1) - \epsilon.$$

We conclude that

$$V(t_1,x,m) - V(t_2,x,m) \leq K(t_2 - t_1). \tag{1.9}$$

Next, we need to show that

$$V(t_2, x, m) - V(t_1, x, m) \le \frac{K}{T - t_2}(t_2 - t_1). \tag{1.10}$$

We begin again with an $\epsilon > 0$ and $(\zeta_1, \mu_1) \in (Z \times \mathcal{M}_m)\,[t_1, T]$, such that

$$V(t_1, x, m) \ge P_{t_1,x,m}(\zeta_1, \mu_1) - \epsilon.$$

Define the functions $s : [t_1, T] \to [t_2, T],\ \tau : [t_2, T] \to [t_1, T]$, by

$$s(\tau) = t_2 + \frac{T - t_2}{T - t_1}(\tau - t_1), \qquad \tau(s) = t_1 + \frac{T - t_1}{T - t_2}(s - t_2).$$

Denote the class of continuous functions on $[a, b]$ by $C[a, b]$. For the purpose of proving (1.13) below we define the mapping $\Theta$ taking $C[t_1, T]$ into $C[t_2, T]$ by

$$(\Theta f)\,(s) \equiv f(\tau(s)).$$

The map $\Theta$ is a linear isomorphism with norm 1. Now we consider the adjoint operator $\Theta^*$, which is also an isomorphism from Radon measures on $[t_2, T]$ to Radon measures on $[t_1, T]$. Therefore, there exists a Radon measure $\mu_2$ such that $\Theta^*(\mu_2) = \mu_1$, and for any $\varphi \in C[t_1, T]$,

$$\begin{aligned} < \varphi, \mu_1 > &= \int_{[t_1,T]} \varphi(r)\, d\mu_1(r) \\ &=< \varphi, \Theta^* \mu_2 >=< \Theta\varphi, \mu_2 >= \int_{[t_2,T]} (\Theta\varphi)\,(r)\, d\mu_2(r) \\ &= \int_{[t_2,T]} \varphi(\tau(r))\, d\mu_2(r). \end{aligned} \tag{1.11}$$

It is not hard to see, by suitably choosing $\varphi$, that $\mu_2 \in \mathcal{M}_m[t_2, T]$. We can extend $\Theta$ and the relation (1.11) to the space of bounded, Borel measureable functions since the Borel $\sigma-$field is contained in the $\mu_2-$ measureable algebra. Then, by approximating a Borel measureable function by a sequence of continuous functions and using the dominated convergence theorem, we see that (1.11) will hold for any $\varphi$ which is bounded and Borel measureable. Note that we are not saying that a continuous linear functional on the space of Borel functions is represented by a Radon measure. We are saying that the Radon measure representation of the continuous linear functional (with the sup norm) $\Theta$ can be extended to Borel functions using the $L^1$ norm with the $\mu$ measure.

Define $\zeta_2(s) = (\Theta\zeta_1)\,(s) = \zeta_1(\tau(s))$. Let $\xi_1$ be the trajectory on $[t_1, T]$ corresponding to $(\zeta_1, \mu_1)$ and let $\xi_2$ be the trajectory on $[t_2, T]$ starting from $x$ corresponding to $(\zeta_2, \mu_2)$.

**Lemma 1.3.** *There is a constant $K$, independent of controls and $\epsilon$, such that*

$$\sup_{t_1\le\tau\le T}|\xi_1(\tau)-\xi_2(s(\tau))|\le\frac{K}{T-t_2}(t_2-t_1) \tag{1.12}$$

$$\sup_{t_2\le s\le T}|\xi_1(\tau(s))-\xi_2(s)|\le\frac{K}{T-t_1}(t_2-t_1) \tag{1.13}$$

*Proof.* We will only prove (1.13) since the proof of (1.12) is similar. (See Theorem 2.1 below for the preliminaries for (1.12).)

We have that

$$\xi_2(s)=x+\int_{t_2}^{s}f_1(r,\xi_2(r),\zeta_2(r))\,dr+\int_{[t_2,s]}f_2(r,\zeta_2(r))\,d\mu_2(r)$$

and

$$\begin{aligned}\xi_1(\tau(s))&=x+\int_{t_1}^{\tau(s)}f_1(b,\xi_1(b),\zeta_1(b))\,db+\int_{[t_1,\tau(s)]}f_2(b,\zeta_1(b))\,d\mu_1(b)\\&=x+\int_{t_1}^{\tau(s)}f_1(b,\xi_1(b),\zeta_1(b))\,db+\int_{[t_1,T]}f_2(b,\zeta_1(b))\mathbf{1}_{[t_1,\tau(s)]}(b)\,d\mu_1(b).\end{aligned} \tag{1.14}$$

We use the notation that $\mathbf{1}_A$ is the characteristic function of the set $A$.

Make the substitution $b=\tau(r)$ in the first integral in (1.14) and use the definition of $\Theta$ given in (1.11) in the second integral to get

$$\begin{aligned}\xi_1(\tau(s))&=x+\frac{T-t_1}{T-t_2}\int_{t_2}^{s}f_1(\tau(r),\xi_1(\tau(r)),\zeta_1(\tau(r)))\,dr\\&\qquad+\int_{[t_2,T]}\Theta\left(f_2(b,\zeta_1(b))\mathbf{1}_{[t_1,\tau(s)]}(b)\right)\,d\mu_2(b)\\&=x+\frac{T-t_1}{T-t_2}\int_{t_2}^{s}f_1(\tau(r),\xi_1(\tau(r)),\zeta_2(r))\,dr\\&\qquad+\int_{[t_2,T]}f_2(\tau(r),\zeta_2(r))\mathbf{1}_{[t_1,\tau(s)]}(\tau(r))\,d\mu_2(r)\\&=x+\frac{T-t_1}{T-t_2}\int_{t_2}^{s}f_1(\tau(r),\xi_1(\tau(r)),\zeta_2(r))\,dr\\&\qquad+\int_{[t_2,T]}f_2(\tau(r),\zeta_2(r))\mathbf{1}_{[t_2,s]}(r)\,d\mu_2(r)\\&=x+\frac{T-t_1}{T-t_2}\int_{t_2}^{s}f_1(\tau(r),\xi_1(\tau(r)),\zeta_2(r))\,dr\\&\qquad+\int_{[t_2,s]}f_2(\tau(r),\zeta_2(r))\,d\mu_2(r)\end{aligned}$$

Now we use the following facts:

$$|\frac{T-t_1}{T-t_2}-1|=\frac{t_2-t_1}{T-t_2}, \qquad \text{and} \quad |s-\tau(s)|=(t_2-t_1)\frac{T-s}{T-t_2}\leq (t_2-t_1)\frac{T-t_1}{T-t_2}.$$

Then, using condition (A),

$$\begin{aligned}\xi_1(\tau(s)) &= x+\frac{T-t_1}{T-t_2}\int_{t_2}^{s} f_1(\tau(r),\xi_1(\tau(r)),\zeta_2(r))\,dr+\int_{[t_2,s]} f_2(\tau(r),\zeta_2(r))\,d\mu_2(r)\\ &= x+\int_{t_2}^{s} f_1(r,\xi_1(\tau(r)),\zeta_2(r))\,dr+\int_{[t_2,s]} f_2(r,\zeta_2(r))\,d\mu_2(r)+O(\frac{t_2-t_1}{T-t_2}).\end{aligned}$$

Combining these facts, again using condition (A), we get the estimate that

$$|\xi_1(\tau(s))-\xi_2(s)|\leq K\int_{t_2}^{s}|\xi_1(\tau(r))-\xi_2(r)|\,dr+O(\frac{t_2-t_1}{T-t_2}).$$

Gronwall's inequality then establishes that (1.13) holds. □

Now that we have an estimate on the trajectories it is easy to verify that

$$|P_{t_1,x,m}(\zeta_1,\mu_1)-P_{t_2,x,m}(\zeta_2,\mu_2)|\leq \frac{K}{T-t_2}(t_2-t_1).$$

We conclude that

$$V(t_1,x,m)\geq P_{t_1,x,m}(\zeta_1,\mu_1)-\epsilon\geq P_{t_2,x,m}(\zeta_2,\mu_2)-\frac{K}{T-t_2}(t_2-t_1)-\epsilon.$$

which gives the desired estimate (1.10).

The proof of theorem 1.1 is completed using the next proposition.This result gives us the terminal and boundary conditions and shows that $V$ is continuous on $[0,T]\times R^1\times[0,1]$.

**Proposition 1.4.** *$V$ satisfies the terminal condition*

$$\begin{aligned}\lim_{t\to T} V(t,x,m)=V(T,x,m) &= \min_{z\in Z, m\leq a\leq 1} h_2(T,z)(a-m)\\ &= \min\{(1-m)\min_{z\in Z} h_2(T,z),\ 0\}.\end{aligned} \tag{1.15}$$

*and boundary condition*

$$V(t,x,1)=\gamma(t,x), \tag{1.16}$$

*where*

$$\gamma(t,x)=\inf_{\zeta\in Z[t,T]}\int_t^T h_1(r,\xi(r),\zeta(r))\,dr, \quad \textit{with } d\xi/d\tau=f_1(\tau,\xi(\tau),\zeta(\tau)),\ \xi(t)=x,$$

*is the value function for the optimal control problem in which the measures do not appear.*

*Proof.* Fix $m \le a \le 1,\ z \in Z$ and choose $\mu(t-0) = m,\ \mu(\tau) = a$ if $t \le \tau \le T$. We have a point mass at $t$ if $m < a$. Then from (1.3)-(1.4),

$$V(t,x,m) \le \int_t^T h_1(r,\xi(r),z)\,dr + h_2(t,z)(a-m).$$

Let $t \uparrow T$ to get $\limsup_{t\to T} V(t,x,m) \le \min_{z\in Z, m\le a} h_2(T,z)(a-m)$.

For the other side, let $(\zeta,\mu) \in (Z \times \mathcal{M}_m)[t,T]$ be arbitrary. Then from (A),

$$\begin{aligned}
&\int_t^T h_1(r,\xi(r),\zeta(r))\,dr + \int_{[t,T]} h_2(r,\zeta(r))\,d\mu(r)\\
&\qquad \ge \int_t^T h_1(r,\xi(r),\zeta(r))\,dr + \int_{[t,T]} h_2(T,\zeta(r)) - K(T-r)\,d\mu(r)\\
&\qquad \ge \int_t^T h_1(r,\xi(r),\zeta(r))\,dr + \int_{[t,T]} \min_{z\in Z} h_2(T,z) - K(T-r)\,d\mu(r)\\
&\qquad \ge \int_t^T h_1(r,\xi(r),\zeta(r))\,dr + \left(\min_{z\in Z} h_2(T,z) - K(T-t)\right)(\mu(T)-m)\\
&\qquad \ge \int_t^T h_1(r,\xi(r),\zeta(r))\,dr + \min_{z\in Z, m\le a}\left(h_2(T,z) - K(T-t)\right)(a-m).
\end{aligned}$$

Consequently, letting $t \uparrow T$ we see that since $\zeta$ and $\mu$ were arbitrary, $\liminf_{t\to T} V(t,x,m) \ge \min_{z\in Z, m\le a} h_2(T,z)(a-m)$. and the terminal condition (1.15) is verified.

Finally, to see that the boundary condition (1.16) is satisfied we simply observe that if the controls $\mu$ must start at 1 and be nondecreasing then they must stay at 1. That is $\mathcal{M}_1[t,T] \equiv \{1\}$, and the result follows immediately from the proof of continuity of $V$ in $m$.

The proof of proposition 1.4 as well as theorem 1.1 is complete. □

*Remark:* Suppose that we had a terminal cost, say $g(\xi(T))$, as well as a running cost, i.e., the cost functional is

$$g(\xi(T)) + P_{t,x,m}(\zeta,\mu).$$

In this case, the terminal condition becomes

$$V(T,x,m) = \min_{z\in Z, m\le a\le 1}\left(h_2(T,z)(a-m) + g(x + f_2(T,z)(a-m))\right),$$

and the boundary condition becomes

$$V(t,x,1) = \gamma(t,x) = \inf_{\zeta\in Z[t,T]}\{g(\xi(T)) + \int_t^T h_1(r,\xi(r),\zeta(r))\,dr\},$$

where $d\xi/d\tau = f_1(\tau,\xi(\tau),\zeta(\tau)),\ \xi(t) = x$.

The next result contains the dynamic programming principle for the optimal control problem

**Proposition 1.5.** *Let (A) hold. Then for any $t < s \le T$ we have that*

$$V(t,x,m) = \inf_{(\zeta,\mu)\in(Z\times\mathcal{M}_m)[t,s]} \{ \int_t^s h_1(r,\xi(r),\zeta(r)\, dr$$

(DP1)
$$+ \int_{[t,s]} h_2(r,\zeta(r))\, d\mu(r) + V(s,\xi(s-),\mu(s-))\}.$$

*and*

(DP2)
$$V(t,x,m) = \min_{z\in Z, 1-m\ge\delta\ge 0} \{h_2(t,z)\delta + V(t,x+\delta f_2(t,z), m+\delta)\}.$$

*Proof.* We will prove (DP2); the the proof of (DP1) is standard and furthermore is very similar to [4,theorem 2.1,2.3].

Let $F(t,x,m)$ denote the right hand side of (DP2). Since we can choose $\delta = 0$ we see that $F(t,x,m) \le V(t,x,m)$.

For the other side, let $z \in Z$ be fixed and $\zeta(\tau) \equiv z, t \le \tau \le T$. Fix $0 \le \delta \le 1-m$. Let $\mu \in \mathcal{M}_m[t,T]$ be defined by $\mu(t-) = m,$ and $\mu(\tau) = \delta + m,$ if $t \le \tau \le T$. Let $\xi(\cdot)$ be the trajectory for the controls $\zeta, \mu$. Then, for any $\epsilon > 0,$ with $t+\epsilon \le T$, we have from (DP1) that

(1.17)
$$\begin{aligned} V(t,x,m) \le \int_t^{t+\epsilon} & h_1(r,\xi(r),\zeta(r))dr \\ & + \int_{[t,t+\epsilon]} h_2(r,\zeta(r))d\mu(r) + V(t+\epsilon,\xi(t+\epsilon-0),\mu(t+\epsilon-0)) \\ = \int_t^{t+\epsilon} & h_1(r,\xi(r),\zeta(r))dr + h_2(t,z)\delta + V(t+\epsilon,\xi(t+\epsilon-0),\mu(t+\epsilon-0)) \end{aligned}$$

Letting $\epsilon \to 0$, since $\xi(t+\epsilon) \to x + f_2(t,z)\delta$, and $\mu$ is right continuous, we conclude from (1.5) and the continuity of $V$ that

$$V(t,x,m) \le h_2(t,z)\delta + V(t,x+f_2(t,z)\delta, m+\delta), \qquad \forall\, z \in Z,\ \forall\, 1-m \ge \delta \ge 0.$$

Therefore, $V(t,x,m) \le F(t,x,m)$ and the result is proved. $\square$

Using the same method of proof we easily derive the following.

**Lemma 1.6.** *The map $\delta \mapsto \min_{z\in Z}\{h_2(t,z)\delta + V(t,x+f_2(t,z)\delta, m+\delta)\}$ is nondecreasing on $[0,1-m]$.*

*Remark:* We can combine (DP1) and (DP2) to get

$$\begin{aligned} V(t,x,m) = \inf_{(\zeta,\mu)\in(Z\times\mathcal{M}_m)[t,s]} \{ & \int_t^s h_1(r,\xi(r),\zeta(r))\, dr + \int_{[t,s)} h_2(r,\zeta(r))\, d\mu(r) \\ + & \min_{z\in Z, 1-\mu(s-)\ge\delta\ge 0} \{h_2(s,z)\delta + V(s,\xi(s-)+\delta f_2(s,z), \mu(s-)+\delta)\}\}. \end{aligned}$$

Next we will derive the Bellman equation for the problem and prove that $V$ is the viscosity solution of the equation. Define the hamiltonians $H_1 : [0,T] \times R^2 \to R^1$, and $H_2 : [0,T] \times R^1 \to R^1$ by

$$H_1(t,x,p_x) = \min_{z\in Z} \left(p_x f_1(t,x,z) + h_1(t,x,z)\right), \qquad H_2(t,p_x) = \min_{z\in Z} \left(p_x f_2(t,z) + h_2(t,z)\right).$$

**Theorem 1.7.** *Let (A) hold. The value function $V$ is the unique viscosity solution on the set $\Omega \equiv (0,T) \times R^1 \times (0,1)$ of*

$$\min\{V_t + H_1(t,x,V_x),\ V_m + H_2(t,V_x)\} = 0 \tag{1.18}$$

*and $V$ satisfies the terminal condition (1.15) and boundary condition (1.16).*

Before we give the proof of the theorem we recall from [16,17] the definition of a (possibly discontinuous) viscosity solution of a Hamilton-Jacobi equation.

**Definition 1.8.** *A function $u : R^n \to R^1$ is a viscosity subsolution (supersolution) of the equation*

$$G(x,u,D_x u) = 0, \text{ where } G : R^n \times R^1 \times R^n \to R^1,$$

*if for any $\varphi \in C^1(R^n)$ for which $u^* - \varphi$ has a maximum ($u_* - \varphi$ has a minimum) at the point $y$, we have*

$$G^*(y,u^*(y),D_x\varphi) \geq 0 \text{ (respectively } G_*(y,u_*(y),D_x\varphi) \leq 0 \text{ ) at } y.$$

*where $u^*, u_*$ denote the upper and lower semicontinuous envelopes of $u$, respectively. Similarly for $G^*, G_*$.*

In general, we see that a viscosity solution as well as the function $G$ may be discontinuous. In our problem we have already proved the continuity of the proposed solution and we have the continuous function $G$ given by

$$G(t,x,m,p_t,p_m,p_x) = \min\{p_t + H_1(t,x,p_x),\ p_m + H_2(t,p_x)\}.$$

We now turn to the proof of the theorem.

*Proof.* Let $\varphi$ be a smooth function on $\Omega$ and suppose that $V - \varphi$ achieves a strict zero maximum at the point $(t_0,x_0,m_0)$. We can always arrange, by modifying $\varphi$ if necessary (c.f. [11,12]), to have $(t_0,x_0,m_0) \in (0,T) \times R^1 \times (0,1)$. From (DP2) we have that

$$\begin{aligned} V(t_0,x_0,m_0) = \varphi(t_0,x_0,m_0) &= \min_{z\in Z, 1-m_0\geq\delta\geq 0}\{h_2(t_0,z)\delta + V(t_0,x_0+\delta f_2(t_0,z),m_0+\delta)\} \\ &\leq \min_{z\in Z, 1-m_0\geq\delta\geq 0}\{h_2(t_0,z)\delta + \varphi(t_0,x_0+\delta f_2(t_0,z),m_0+\delta)\}. \end{aligned}$$

Therefore, for every $\delta > 0$

$$0 \le \min_{z\in Z}\{h_2(t_0, z) + \delta^{-1}\left(\varphi(t_0, x_0 + \delta f_2(t_0, z), m_0 + \delta) - \varphi(t_0, x_0, m_0)\right)\}.$$

Let $\delta \to 0$ and use the differentiability of $\varphi$ to get that

$$0 \le \min_{z\in Z}\{h_2(t_0, z) + \varphi_x(t_0, x_0, m_0) \cdot f_2(t_0, z) + \varphi_m(t_0, x_0, m_0)\}. \tag{1.19}$$

Define the control $\mu \in \mathcal{M}_{m_0}[t_0, T]$ by $\mu(\tau) \equiv m_0, t_0 \le \tau \le T$. From (DP1) we get for any $t_0 \le s \le T$,

$$\begin{aligned}\varphi(t_0, x_0, m_0) &= V(t_0, x_0, m_0)\\ &\le \inf_{\zeta\in Z[t_0,s]}\{\int_{t_0}^{s} h_1(r, \xi(r), \zeta(r))\, dr + \int_{[t_0,s]} h_2(r, \zeta(r))\, d\mu(r) + \varphi(s, \xi(s), \mu(s))\}\\ &= \inf_{\zeta\in Z[t_0,s]}\{\int_{t_0}^{s} h_1(r, \xi(r), \zeta(r))\, dr + \varphi(s, \xi(s), m_0)\}\end{aligned}$$

Notice that for the control $\mu$ the trajectory for each $\zeta$ is given by $d\xi/d\tau = f_1(\tau, \xi, \zeta), T \ge \tau > t_0, \xi(t_0) = x_0$, and there are no jumps in either $\mu$ or $\xi$.

Set $s = t_0 + \epsilon$, in the preceding; divide by $\epsilon$ and let $\epsilon \to 0$ to obtain that

$$0 \le \min_{z\in Z}\{h_1(t_0, x_0, z) + \varphi_x(t_0, x_0, m_0) \cdot f_1(t_0, x_0, z) + \varphi_t(t_0, x_0, m_0)\}. \tag{1.20}$$

Combining (1.19) and (1.20) we see that $V$ is a subsolution of (1.18).

We need to prove finally that $V$ is a supersolution of (1.18). Thus, suppose that $V - \varphi$ has a strict zero minimum at the point $(t_0, x_0, m_0) \in \Omega$ with $\varphi$ a smooth function. Assume to the contrary that there is a constant $C > 0$ such that

$$\varphi_t(t_0, x_0, m_0) + \min_{z\in Z}\{\varphi_x(t_0, x_0, m_0) \cdot f_1(t_0, x_0, z) + h_1(t_0, x_0, z)\} \ge C \tag{1.21}$$

and

$$\varphi_m(t_0, x_0, m_0) + \min_{z\in Z}\{\varphi_x(t_0, x_0, m_0) \cdot f_2(t_0, z) + h_2(t_0, z)\} \ge C \tag{1.22}$$

Fix $z \in Z$ and define $\xi(\cdot)$ by $d\xi(m)/dm = f_2(t_0, z),\ \xi(m_0) = x_0$. From (1.22), since $\varphi$ is smooth, we see that for all $m \in [m_0, m_0 + \delta]$ for small $e > \delta > 0$,

$$\varphi_m(t_0, \xi(m), m) + \varphi_x(t_0, \xi(m), m) \cdot f_2(t_0, z) + h_2(t_0, z) \ge C/2$$

Consequently,

$$\frac{d}{dm}\varphi(t_0, \xi(m), m) + h_2(t_0, z) \ge C/2.$$

Integrate this from $m_0$ to $m_0 + \delta$ to get

$$\varphi(t_0, x_0 + \delta f_2(t_0, z), m_0 + \delta) - \varphi(t_0, x_0, m_0) + \delta h_2(t_0, z) \ge \delta C/2. \tag{1.23}$$

Since $V - \varphi$ has a strict zero minimum at $(t_0, x_0, m_0)$ we obtain from (1.23)

$$\begin{aligned}&V(t_0, x_0 + \delta f_2(t_0, z), m_0 + \delta) + \delta h_2(t_0, z)\\ &\qquad \ge \varphi(t_0, x_0, m_0) + \delta C/2 = V(t_0, x_0, m_0) + \delta C/2,\end{aligned} \tag{1.24}$$

for all $z \in Z$ and sufficiently small $e > \delta > 0$. This inequality says that it is not optimal to jump to a better position at time $t_0$.

**Lemma 1.9.** *If (1.24) holds, then there exists an $\epsilon > 0$ such that for all $t_0 < s < t_0 + \epsilon$,*

$$V(t_0, x_0, m_0) = \inf_{\zeta,\mu}\{\int_{t_0}^{s} h_1(r, \xi(r), \zeta(r))\, dr + \int_{[t_0,s]} h_2(r, \zeta(r))\, dr + V(s, \xi(s), \mu(s))\},$$

*where the infimum on $\mu$ is taken on the class $\mathcal{M}_{m_0}[t_0, s] \cap C[t_0, s]$.*

*Proof.* For each integer $n = 1, 2, \ldots$, there exists $(\zeta_n, \mu_n) \in (Z \times \mathcal{M}_{m_0})\,[t_0, T]$ such that

$$V(t_0, x_0, m_0) + \frac{1}{n} \geq P_{t_0,x_0,m_0}\,(\zeta_n, \mu_n)\,.$$

Let $s_n \geq t_0$ be the first point of discontinuity of $\mu_n$. We have that, with $\lambda_n \equiv \mu_n(s_n) - \mu_n(s_n - 0)$,

$$\begin{aligned} V(t_0, x_0, m_0) + \frac{1}{n} &\geq P_{t_0,x_0,m_0}\,(\zeta_n, \mu_n) \\ &\geq \int_{t_0}^{s_n} h_1(r, \xi_n(r), \zeta_n(r))\, dr + \int_{[t_0,s_n)} h_2(r, \zeta_n(r))\, d\mu_n(r) \\ &+ \min_{z\in Z}\{V(s_n, \xi_n(s_n - 0) + \lambda_n f_2(s_n, z), \mu_n(s_n) + \lambda_n) + \lambda_n h_2(s_n, z)\}\}. \end{aligned}$$

If there is a subsequence such that $s_n \to t_0$ and $\delta' > 0$ with $\lambda_n \to \delta'$, then using the continuity of $V$ we obtain that if $n \to \infty$,

$$V(t_0, x_0, m_0) \geq \min_{z\in Z}\{V(t_0, x_0 + \delta' f_2(t_0, z), m_0 + \delta') + \delta' h_2(t_0, z)\}.$$

Using Lemma 1.6 we have reached a contradiction of (1.24). $\square$

Now fix $\epsilon$ given by lemma 1.9. Let $0 < \rho_0 < \epsilon$ and $\zeta \in Z[t_0, t_0 + \rho_0]$ and $\mu \in \mathcal{M}_{m_0} \cap C[t_0, t_0 + \rho_0]$ with (1.21) and (1.22) (with $C$ replaced by $C/2$) holding at $(r, \xi(r), \mu(r)),\ t_0 \leq r \leq t_0 + \rho_0$. Then compute

$$\begin{aligned} &\varphi(t_0 + \rho_0, \xi(t_0 + \rho_0), \mu(t_0 + \rho_0)) - \varphi(t_0, x_0, m_0) \\ &= \int_{t_0}^{t_0+\rho_0} \varphi_t(r, \xi(r), \mu(r)) + \varphi_x(r, \xi(r), \mu(r)) f_1(r, \xi(r), \zeta(r))\, dr \\ &\quad + \int_{[t_0,t_0+\rho_0]} \varphi_m(r, \xi(r), \mu(r)) + \varphi_x(r, \xi(r), \mu(r)) f_2(r, \zeta(r))\, d\mu(r) \\ &\geq \int_{t_0}^{t_0+\rho_0} -h_1(r, \xi(r), \zeta(r))\, dr + \int_{[t_0,t_0+\rho_0]} -h_2(r, \zeta(r))\, d\mu(r) + \rho_0 C. \end{aligned}$$

That is,

$$\begin{aligned} \varphi(t_0 + \rho_0, &\xi(t_0 + \rho_0), \mu(t_0 + \rho_0)) - \varphi(t_0, x_0, m_0) \\ &+ \int_{t_0}^{t_0+\rho_0} h_1(r, \xi(r), \zeta(r))\, dr + \int_{[t_0,t_0+\rho_0]} h_2(r, \zeta(r))\, d\mu(r) \geq \rho_0\, C. \end{aligned} \tag{1.25}$$

Since $V - \varphi$ has a strict zero minimum at $(t_0, x_0, m_0)$ we obtain from (1.25) that

$$
\begin{aligned}
&V(t_0+\rho_0, \xi(t_0+\rho_0), \mu(t_0+\rho_0)) - V(t_0, x_0, m_0) \\
(1.26) \qquad &+ \int_{t_0}^{t_0+\rho_0} h_1(r, \xi(r), \zeta(r))\, dr + \int_{[t_0, t_0+\rho_0]} h_2(r, \zeta(r))\, d\mu(r) \geq \rho_0\, C.
\end{aligned}
$$

This inequality is a contradiction. Therefore, $V$ is shown to be a viscosity supersolution of (1.18) as well.

Finally, the fact that $V$ is the only viscosity solution of (1.18) follows from more general uniqueness results for first order Hamilton-Jacobi equations (c.f.[1,3,13]). □

Now we introduce the following optimal control problem with unbounded controls:

$$
(1.27) \qquad \text{Minimize } P_{t,x,m}(\zeta, \alpha) = \int_t^T h_1(r, \xi(r), \zeta(r))\, dr + \int_t^T h_2(r, \zeta(r))\alpha(r)\, \mathbf{1}_{\{\mu<1\}} dr
$$

subject to

$$
\begin{aligned}
(1.28) \qquad & d\xi/d\tau = f_1(\tau, \xi(\tau), \zeta(\tau)) + f_2(\tau, \zeta(\tau))\alpha(\tau)\mathbf{1}_{\{\mu<1\}}, \qquad t < \tau \leq T \\
& d\mu/d\tau = \alpha(\tau)\mathbf{1}_{\{\mu<1\}}, \qquad t < \tau \leq T, \\
& \xi(t) = x \in R^1 \qquad \mu(t) = m \in [0,1],
\end{aligned}
$$

over the class of controls $(\zeta, \alpha) \in Z[t,T] \times L^1_+[t,T]$, where $L^1_+[t,T] = \{\alpha : [t,T] \to [0,\infty) \mid \int_t^T \alpha(r)\, dr < \infty\}$. The function $\mathbf{1}_{\{\mu<1\}}$ is the characteristic function of the set $\{\mu(\tau) < 1\}$. For any control $\alpha \in L^1_+[t,T]$ we see that $\mu(\tau) \in [0,1]$, for all $t \leq \tau \leq T$. Furthermore, since

$$
\left| \int_t^T f_2(r, \zeta(r))\alpha(r)\mathbf{1}_{\{\mu<1\}}(r)\, dr \right| \leq K \int_t^T \alpha(r)\mathbf{1}_{\{\mu<1\}}(r)\, dr \leq K\,[\mu(T) - \mu(t)] \leq K,
$$

we see that $\|\xi(\tau)\|_{L^\infty} \leq K$, independently of controls.

The value function for this problem is defined by

$$
W(t,x,m) = \inf_{(\zeta,\alpha)\in Z\times L^1_+} P_{t,x,m}(\zeta, \alpha).
$$

It is easily seen that $W$ is a bounded function under the assumption (A).

**Theorem 1.10.** *Let (A) hold.*

(1) *$W$ is a viscosity solution of (1.18) and satisfies the terminal condition (1.15) and boundary condition (1.16).*

(2) *The value function $W$ is also the unique continuous viscosity solution of*

$$
(1.29) \qquad W_t(t,x,m) + H(t,x,W_m,W_x) = 0
$$

*where*

$$
(1.30) \qquad H(t,x,p_m,p_x) \equiv \begin{cases} H_1(t,x,p_x), & \text{if } p_m + H_2(t,p_x) \geq 0 \\ -\infty, & \text{if } p_m + H_2(t,p_x) < 0. \end{cases}
$$

(3) *$W = V$ on $\overline{\Omega}$.*

*Proof.* The proof that $W$ satisfies the terminal and boundary conditions is similar to that in proposition 1.5 and is left to the reader.

We will prove that $W$ is a viscosity solution of (1.18). In fact, this follows immediately from Theorem I.1 of [2] but we will provide the details.

The idea of the proof is to bound the controls $\alpha$ which then results in a standard optimal control problem to which classical results apply. Therefore, we consider the control problem (1.27)-(1.28) but we must choose the controls $\alpha$ from the class

$$A_B[t,T] = \{\alpha : [t,T] \to [0,B] : \alpha \in L^1_+[t,T]\},$$

for each fixed $B > 0$. When we use this class we will denote the corresponding value function by $W^B$. Now, using standard theory, $W^B$ is the unique viscosity solution of

$$W_t + H^B(t,x,W_m,W_x) = 0 \qquad \text{on } \Omega,$$

where

$$H^B(t,x,p_m,p_x) \equiv \min_{z\in Z}\{p_x f_1(t,x,z) + h_1(t,x,z) - B\,(p_m + p_x f_2(t,z) + h_2(t,z))^-\}.$$

By considering classes of control functions it is clear that $B \geq B'$ implies that $W \leq W^B \leq W^{B'}$. We conclude that $W^B$ converges to some function $\Gamma \geq W$ which is upper semicontinuous. In fact it is not hard for the reader to verify that on $(0,T) \times R^1 \times [0,1]$, $\Gamma = W$. Therefore, $W$ is at least upper semicontinuous.

We will now use the fact that $W^B \searrow W$ to show that $W$ is a viscosity solution of (1.18).

Let $W - \varphi$ achieve a zero unique maximum at the point $(t_0, x_0, m_0)$ with $\varphi$ a smooth function. We arrange, if necessary, to have $t_0 > t$ and $0 < m_0 < 1$. Then, by Lemma A.2 in Barles and Perthame [5], for each $B > 0, W^B - \varphi$ achieves a maximum at $(t_B, x_B, m_B)$ and $(t_B, x_B, m_B) \to (t_0, x_0, m_0)$ as $B \to \infty$. Since $W^B$ is a subsolution, at $(t_B, x_B, m_B)$

$$\varphi_t + \min_{z\in Z}\{\varphi_x f_1(t_B,x_B,z) + h_1(t_B,x_B,z) - B\,(\varphi_m + \varphi_x f_2(t_B,z) + h_2(t_B,z))^-\} \geq 0. \tag{1.31}$$

Since the expression in parentheses is nonnegative we may drop it to get

$$\varphi_t + \min_{z\in Z}\{\varphi_x f_1(t_B,x_B,z) + h_1(t_B,x_B,z)\} \geq 0 \qquad \text{at } (t_B,x_B,m_B).$$

Let $B \to \infty$ to see that

$$\varphi_t + \min_{z\in Z}\{\varphi_x f_1(t_0,x_0,z) + h_1(t_0,x_0,z)\} \geq 0 \qquad \text{at } (t_0,x_0,m_0). \tag{1.32}$$

Also, divide through by $B$ in (1.31), let $B \to \infty$ and use condition (A) to obtain

$$\min_{z\in Z}\Big(-(\varphi_m + \varphi_x f_2(t_0,z) + h_2(t_0,z))^-\Big) \geq 0$$

which implies immediately that

$$\varphi_m + \min_{z\in Z} (\varphi_x f_2(t_0, z) + h_2(t_0, z)) \geq 0. \tag{1.33}$$

Combining (1.32) and (1.33) we conclude that $W$ is a viscosity subsolution of (1.18).

Now suppose that $W_* - \varphi$ achieves a zero unique minimum at the point $(t_0, x_0, m_0)$ with $\varphi$ a smooth function. Then, again by Lemma A.2 in Barles and Perthame [5], for each $B > 0, W^B - \varphi$ achieves a minimum at $(t_B, x_B, m_B)$ and $(t_B, x_B, m_B) \to (t_0, x_0, m_0)$ as $B \to \infty$. At $(t_B, x_B, m_B)$

$$\varphi_t + \min_{z\in Z}\{\varphi_x f_1(t_B, x_B, z) + h_1(t_B, x_B, z) - B\left(\varphi_m + \varphi_x f_2(t_B, z) + h_2(t_B, z)\right)^-\} \leq 0 \tag{1.34}$$

If

$$\varphi_m + \min_{z\in Z}\{\varphi_x f_2(t_0, z) + h_2(t_0, z)\} \geq C > 0$$

then, by continuity, at $(t_B, x_B, m_B)$ for $B$ sufficiently large

$$\varphi_m + \min_{z\in Z}\{\varphi_x f_2(t_B, z) + h_2(t_B, z)\} \geq C/2.$$

¿From (1.34) we see that

$$\varphi_t + \min_{z\in Z}\{\varphi_x f_1(t_B, x_B, z) + h_1(t_B, x_B, z)\} \leq 0 \tag{1.35}$$

Letting $B \to \infty$ we see that (1.35) holds at the point $(t_0, x_0, m_0)$. Consequently, $W$ is a supersolution of (1.18).

Since the viscosity solution of (1.18) is known to be at least continuous, we know also from this fact that W must be continuous. Part (1) is proved.

*Remark:* It is not hard to directly establish the continuity of *W*. The details of the proof are similar to that of theorem 1.1

We will prove part (2) of the theorem from the lemma:

**Lemma 1.11.** *A continuous function is a viscosity solution of (1.18) if and only if it is also a viscosity solution of (1.29).*

*Proof.* Let $\Gamma$ be a viscosity solution of (1.18). It is obvious that $\Gamma$ is then also a subsolution of (1.29) so we need only show it is a supersolution of (1.29). To this end, if $\Gamma - \varphi$ has a minimum at $(t_0, x_0, m_0)$ then

$$\min\{\varphi_t + H_1(t_0, x_0, \varphi_x),\ \varphi_m + H_2(t_0, \varphi_x)\} \leq 0.$$

If $\varphi_m + H_2(t_0, \varphi_x) > 0$ then $\varphi_t + H_1(t_0, x_0, \varphi_x) \leq 0$. So, by (1.30), $\Gamma$ is a supersolution of (1.29). On the other hand, if $\varphi_m + H_2(t_0, \varphi_x) \leq 0$ then, using again the definition of the hamiltonian $H$, $\varphi_t + H(t_0, x_0, m_0, \varphi_m, \varphi_x) = -\infty$. In either case we conclude that $\Gamma$ is a supersolution of (1.29). Hence, a viscosity solution of (1.18) is also a viscosity solution of (1.29)

The proof that $\Gamma$ is a viscosity solution of (1.18) if it is a solution of (1.29) is similar and so we omit it. We conclude that the equations (1.18) and (1.29) are equivalent in the viscosity sense. $\square$

Finally we will prove that $W = V$. We can appeal to uniqueness theorems (c.f. Barles [1]) for (1.18) to conclude that $W = V$ because we have shown that $W$ and $V$ satisfy the same equation and boundary conditions. We can also prove this directly, however, by using proposition 5.3 of [25].

Clearly, $V(t, x, m) \le W(t, x, m)$. For the other side, given $\epsilon > 0$ there exists a pair of controls $(\zeta_\epsilon, \mu_\epsilon)$ with associated trajectory $\xi_\epsilon(\cdot)$ which are $\epsilon-$optimal

$$V(t, x, m) \ge P_{t,x,m}(\zeta_\epsilon, \mu_\epsilon) - \epsilon.$$

According to [25,proposition 5.3] there exists a sequence $(\zeta_i, \alpha_i)$ and associated trajectories $\xi_i$ such that

$$\xi_i \to \xi_\epsilon,\ d\xi_i \xrightarrow{*} d\xi_\epsilon,\ \alpha_i\, d\tau \xrightarrow{*} d\mu_\epsilon(\tau),\ \text{meas}\{\tau : \zeta_i(\tau) \ne \zeta_\epsilon(\tau)\} \to 0 \text{ as } i \to \infty$$

Finally, $h_2(\tau, \zeta_i(\tau))\alpha_i(\tau)d\tau \xrightarrow{*} h_2(\tau, \zeta_\epsilon(\tau))d\mu_\epsilon(\tau)$. Therefore, for $i$ sufficiently large

$$\begin{aligned} V(t, x, m) &\ge P_{t,x,m}(\zeta_\epsilon, \mu_\epsilon) - \epsilon \\ &\ge \int_t^T h_1(r, \xi_i(r), \zeta_i(r))\, dr + \int_t^T h_2(r, \zeta_i(r))\alpha_i(r)\mathbf{1}_{\{\mu<1\}}(r)\, dr - 2\epsilon \\ &\ge W(t, x, m) - 2\epsilon. \end{aligned}$$

and the result follows. $\square$

*Remarks:* 1. It follows from this result that the model with measures is not more general than that with unbounded control functions.

2. The Bellman equation formally tells us what the optimal controls are. For example, when $V_m + H_2(t, V_x) > 0$ the optimal measure control consists of doing nothing, i.e. $d\mu \equiv 0$. The optimal $\zeta$ control will then provide the minimum of the hamiltonian $H_1$. The $d\mu$ measure, or equivalently, the $\alpha$ control will be non zero only on the set where $V_m + H_2(t, V_x) = 0$. On this set the optimal $\zeta$ control will minimize the hamiltonian $H_2$. The optimal measure could have an absolutely continuous as well as a singular component. We leave as an open problem the rigorous connection between the Bellman equation and the optimal control.

## 2. The differential game

In this section we will consider the differential game associated with the dynamics (1.1)-(1.2) and payoff (1.3). The players will be the controls $\zeta$ and $\mu$ with $\zeta$ the minimizer and $\mu$ the maximizer of $P$. We will work within the framework of Elliott and Kalton's definition of differential games and refer to Elliott [14] for a basic synopsis of results on differential games in the connection with viscosity solutions.

Many of the results for differential games are proved in a manner similar to that for the optimal control case. In the interest of brevity we will only provide the proofs which are substantially distinct from those of section 1.

In order to be precise about the differential game let us define the terms. A *strategy* for the maximizer is a map $\alpha : Z[t,T] \to \mathcal{M}_m[t,T]$, such that $\zeta_1(\tau) = \zeta_2(\tau),\ t \le \tau \le s$, for each $t \le s \le T$, implies that $\alpha[\zeta_1](\tau) = \alpha[\zeta_2](\tau),\ t \le \tau \le s$. This defines $\alpha$ as a *nonanticipating* map. Let $\Gamma(t)$ denote the class of strategies for $\mu$ on $[t,T]$.

Similarly, the class of nonanticipating strategies for $\zeta$ on $[t,T]$, is denoted by $\Delta(t)$. A strategy for the minimizer is a nonanticipating map $\beta : \mathcal{M}_m[t,T] \to Z[t,T]$. We will sometimes write $\alpha \in \mathcal{M}_m[t,T], \beta \in Z[t,T]$ to signify that the strategies map into a control function in the class. An outcome of $(\zeta, \alpha(\zeta))$ (respectively $(\beta(\mu), \mu)$ must be an element of $(Z \times \mathcal{M}_m)[t,T]$. Then

**Definition 2.1.** *The upper value function $V^+ : [0,T] \times R^1 \times [0,1] \to R^1$ is defined by*

$$V^+(t,x,m) = \sup_{\alpha \in \Gamma(t)} \inf_{\zeta \in Z[t,T]} P_{t,x,m}(\zeta, \alpha[\zeta]).$$

*The lower value function $V^- : [0,T] \times R^1 \times [0,1] \to R^1$ is defined by*

$$V^-(t,x,m) = \inf_{\beta \in \Delta(t)} \sup_{\mu \in \mathcal{M}_m[t,T]} P_{t,x,m}(\beta[\mu], \mu).$$

**Theorem 2.2.** *Under assumption (A),*

1. *$V^\pm : [0,T] \times R^1 \times [0,1] \to R^1$ are bounded and continuous and satisfy the terminal, boundary conditions*

$$V^\pm(T,x,m) = \min_{z \in Z} \left(h_2(T,z)\right)^+ (1-m), \tag{TC}$$

$$V^\pm(t,x,1) = \gamma(t,x), \tag{BC}$$

*where $\gamma$ is defined in (1.16).*

2. *$V^+$ satisfies the dynamic programming principles*

$$V^+(t,x,m) = \min_{z \in Z} \max_{1-m \ge \delta \ge 0} \{h_2(t,z)\delta + V^+(t, x + \delta f_2(t,z), m+\delta)\}, \tag{2.1}$$

(2.2)

$$V^+(t,x,m) = \sup_{\alpha\in\mathcal{M}_m[t,s]}\inf_{\zeta\in Z[t,s]}\{\int_t^s h_1(r,\xi(r),\zeta(r))dr + \int_{[t,s]} h_2(r,\zeta(r))dr + V^+(s,\xi(s-),\mu(s-))\}.$$

(3) $V^-$ *satisfies the dynamic programming principles*

(2.3) $$V^-(t,x,m) = \max_{1-m\geq\delta\geq 0}\min_{z\in Z}\{h_2(t,z)\delta + V^-(t,x+\delta f_2(t,z),m+\delta)\},$$

(2.4)

$$V^-(t,x,m) = \inf_{\beta\in Z[t,s]}\sup_{\mu\in\mathcal{M}_m[t,s]}\{\int_t^s h_1(r,\xi(r),\zeta(r))dr + \int_{[t,s]} h_2(r,\zeta(r))dr + V^-(s,\xi(s-),\mu(s-))\}.$$

*Remark:* If we add a terminal cost to the payoff, say $g(\xi(T))$, then the terminal condition becomes

(2.5) $$V^+(T,x,m) = \min_{z\in Z}\max_{m\leq a\leq 1}\{g(x+f_2(T,z)(a-m)) + h_2(T,z)(a-m)\},$$

for $V^+$ and

(2.6) $$V^-(T,x,m) = \max_{m\leq a\leq 1}\min_{z\in Z}\{g(x+f_2(T,z)(a-m)) + h_2(T,z)(a-m)\},$$

for $V^-$. Of course these terminal conditions will not be the same in general. One should not, therefore, expect the game with measures to always have value.

*Proof.* We will only prove some of the results stated and only for the upper value. The proofs for the lower value are similar.

We prove first that $V^+$ is continuous in $t$ in one direction.

Fix $(x,m)\in R^1\times(0,1)$. Let $0<t_1<t_2<T$ and let $\epsilon>0$ be given. Then, there is a strategy $\alpha_1\in\Gamma(t_1)$ such that

$$V^+(t_1,x,m)\leq P_{t_1,x,m}(\zeta_1,\alpha_1[\zeta_1])+\epsilon, \qquad \forall\,\zeta_1\in Z[t_1,T].$$

Define the maps $s:[t_1,T]\to[t_2,T]$, $\tau:[t_2,T]\to[t_1,T]$ as in §1. Define the map $\Theta: C[t_2,T]\to C[t_1,T]$ by $(\Theta f)(\tau)=f(s(\tau))$.

Given $\zeta_2\in Z[t_2,T]$ set $\zeta_1(\tau)\equiv\zeta_2(s(\tau))=(\Theta\zeta_2)(\tau)$. Also, set $\mu_1=\alpha_1[\zeta_1]$. Finally, define the strategy $\alpha_2\in\Gamma(t_2)$ by

$$\mu_2\equiv\alpha_2[\zeta_2]=\Theta^*(\alpha_1[\zeta_1])=\Theta^*(\mu_1).$$

As in section one, $\mu_2$ is a Radon measure with $\mu_2 \in \mathcal{M}_m[t_2, T]$. Furthermore, for any $\varphi \in C[t_2, T]$,

$$
\begin{aligned}
< \varphi, \mu_2 > &= \int_{[t_2,T]} \varphi(r)\, d\mu_2(r) \\
&=< \varphi, \Theta^* \mu_1 >=< \Theta\varphi, \mu_1 >= \int_{[t_1,T]} (\Theta\varphi)(r)\, d\mu_1(r) \\
&= \int_{[t_1,T]} \varphi(s(r))\, d\mu_1(r).
\end{aligned}
$$

In fact, by dominated convergence, this is valid for any bounded Borel measureable $\varphi$. Then, Lemma 1.3(1.12) holds and we conclude after some manipulation involving (A), that

$$
V^+(t_1, x, m) \leq P_{t_2,x,m}(\zeta_2, \alpha_2[\zeta_2]) + \frac{K}{T - t_2}(t_2 - t_1) + \epsilon, \qquad \forall\, \zeta_2 \in Z[t_2, T].
$$

This implies that

$$
V^+(t_1, x, m) \leq V^+(t_2, x, m) + \frac{K}{T - t_2}(t_2 - t_1).
$$

The remaining estimates for continuity are similar to that of Theorem 1.1 and are left to the reader.

Now we turn to the proof of (2.1). Let

$$
F(t, x, m) = \min_{z \in Z} \max_{1-m \geq \delta \geq 0} \{h_2(t, z)\delta + V^+(t, x + \delta f_2(t, z), m + \delta)\}.
$$

By setting $\delta = 0$ we see that $F(t, x, m) \geq V^+(t, x, m)$. Next, given any $\zeta \in Z[t, T]$ set $z = \zeta(t)$. We can find $1 - m \geq \delta' = \delta'(z) \geq 0$ so that

$$
\begin{aligned}
F(t, x, m) &\leq \max_{1-m \geq \delta \geq 0} \{h_2(t, z)\delta + V^+(t, x + \delta f_2(t, z), m + \delta)\} \\
&= h_2(t, z)\delta' + V^+(t, x + \delta' f_2(t, z), m + \delta')
\end{aligned}
$$

If $\delta' = 0$ we are done so we assume that $\delta' > 0$. Now, by definition of $V^+$, there exists a strategy $\alpha' \in \mathcal{M}_{m+\delta'}[t, T]$ such that

$$
V^+(t, x + \delta' f_2(t, z), m + \delta') \leq P_{t,x+\delta' f_2(t,z),m+\delta'}(\zeta, \alpha'[\zeta]) + \epsilon.
$$

Define the strategy $\alpha'' \in \mathcal{M}_m[t, T]$ by $\alpha''[\zeta](t-) = m$ and $\alpha''[\zeta](\tau) = \alpha'[\zeta](\tau)$ if $t \leq \tau \leq T$. Then, it is not hard to verify that

$$
P_{t,x+\delta' f_2(t,z),m+\delta'}(\zeta, \alpha'[\zeta]) + \delta' h_2(t, z) = P_{t,x,m}(\zeta, \alpha''[\zeta]),
$$

so that, combining the preceding, we get that

$$\begin{aligned} F(t,x,m) &\le h_2(t,z)\delta' + V^+(t, x+\delta' f_2(t,z), m+\delta') \\ &\le h_2(t,z)\delta' + P_{t,x+\delta' f_2(t,z),m+\delta'}(\zeta, \alpha'[\zeta]) + \epsilon \\ &= P_{t,x,m}(\zeta, \alpha''[\zeta]) + \epsilon. \end{aligned}$$

This evidently implies that $F(t,x,m) \le V^+(t,x,m)$, completing the proof. The remaining assertions of the theorem are left to the reader. □

We will now focus on the upper value, $V^+$, since we are taking the point of view that we are studying the differential game as a worst case analysis of a system subject to disturbances. Later we will state the results for the lower value, $V^-$.

Define the upper hamiltonian $H^+ : R^1 \times [0,T] \times R^3 \to R^1$ as

$$H^+(a,t,x,p_m,p_x) = \min_{z \in \mathcal{Z}(a,t,p_m,p_x)} \{p_x \cdot f_1(t,x,z) + h_1(t,x,z)\} \tag{2.7}$$

where

$$\mathcal{Z}(a,t,p_m,p_x) = \{z \in Z : p_m + p_x \cdot f_2(t,z) + h_2(t,z) \le a\}. \tag{2.8}$$

If $\mathcal{Z}(a,t,p_m,p_x) = \emptyset$ then we set $H^+ \equiv +\infty$.

In general, one cannot expect such hamiltonians to be continuous functions. In fact, this hamiltonian is not continuous. In view of the definition of viscosity solution with discontinuous hamiltonians, we have to calculate the upper and lower semicontinuous envelopes of $H^+$. We do so in the next lemma. The statement of the lemma is similar to that of [8,proposition 2.5] but the proof here is simpler.

**Lemma 2.3.** *The upper semicontinuous envelope, $(H^+)^*$ of $H^+$ is given by*

$$\left(H^+\right)^*(a,t,x,p_m,p_x) = H^+(a-0,t,x,p_m,p_x).$$

*The lower semicontinuous envelope is given by*

$$\left(H^+\right)_*(a,t,x,p_m.p_x) = H^+(a+0,t,x,p_m,p_x).$$

*Proof.* We will only prove the result for the upper semicontinuous envelope. By definition

$$\left(H^+\right)^*(a,t,x,p_m,p_x) = \limsup\{H^+(b,s,y,q_m,q_x);\ (b,s,y,q_m,q_x) \to (a,t,x,p_m,p_x)\}$$

Given $\epsilon > 0$, fix $(s,y) \in B_\epsilon(t,x)$, such that

$$\begin{aligned} &|\varphi(t,x,z) - \varphi(s,y,z)| \le K\epsilon, \qquad \varphi = f_1, h_1, \\ &|\varphi(t,z) - \varphi(s,z)| \le K\epsilon, \qquad \varphi = f_2, h_2. \end{aligned}$$

Also fix $(b, q_m, q_x) \in B_\epsilon(a, p_m, p_x)$. Now, by a standard result in finite dimensional penalization theory, we have that

$$
\begin{aligned}
H^+(b, s, y, q_m, q_x) &= \lim_{B\to\infty} \{\min_{z\in Z}\{q_x f_1(s, y, z) + h_1(s, y, z) \\
&\qquad + B\,(q_m + q_x f_2(s, z) + h_2(s, z) - b)^+\} \\
&\le \lim_{B\to\infty} \{\min_{z\in Z}\{p_x f_1(t, x, z) + h_1(t, x, z) + |p_x|\epsilon + K\epsilon \\
&\qquad + B\,(p_m + p_x f_2(t, z) + h_2(t, z) - a + K\epsilon)^+\} \\
&= \min_{z\in \mathcal{Z}(a-K\epsilon, t, p_m, p_x)} \{p_x f_1(t, x, z) + h_1(t, x, z)\} + |p_x|\epsilon + K\epsilon
\end{aligned}
$$

Consequently, since $\epsilon$ was arbitrary,

$$\left(H^+\right)^* (a, t, x, p_m, p_x) \le H^+(a - 0, t, x, p_m, p_x).$$

Since the reverse inequality follows from the definition of upper envelope, the proof is complete. □

The next lemma is the useful analogue of [8,prop.4.1] and Lemma 1.9 above.

**Lemma 2.4.** *A continuous function $u \in C(\Omega)$ is a viscosity solution of*

$$\max\{V_t + H^+(0, t, x, V_m, V_x), \quad V_m + H_2(t, V_x)\} = 0 \tag{2.9}$$

*if and only if $u$ is a viscosity solution of*

$$V_t + H^+(0, t, x, V_m, V_x) = 0. \tag{2.10}$$

The advantage of the formulation (2.9) is that the minimum in $H^+$ is always taken over a set which is nonempty.

With these preliminaries completed we can now state the main result of this section.

**Theorem 2.5.** *$V^+$ is a viscosity solution of (2.9) (or (2.10)) on $(0, T) \times R^1 \times (0, 1)$.*

*Proof.* We know that $V^+$ is continuous on $\overline{\Omega} = [0, T] \times R^1 \times [0, 1]$ so we need to verify the viscosity requirements.

Let $V^+ - \varphi$ achieve a strict maximum of zero at the point $(t_0, x_0, m_0)$. Without loss of generality we may assume that $(t_0, x_0, m_0) \in (0, T) \times R^1 \times (0, 1)$. We must show that

$$\varphi_t(t_0, x_0, m_0) + H^+(0 - 0, t_0, x_0, \varphi_m, \varphi_x) \ge 0 \text{ at } (t_0, x_0, m_0).$$

Suppose this is not true. Then there exists a $\beta > 0$ for which

$$\varphi_t(t_0, x_0, m_0) + H^+(-4\beta, t_0, x_0, \varphi_m, \varphi_x) \le -4\beta \text{ at } (t_0, x_0, m_0). \tag{2.11}$$

By definition of the hamiltonian, this implies that there exists $z^* \in \mathcal{Z}(-4\beta, t_0, \varphi_m, \varphi_x)$, such that

$$\varphi_t(t_0, x_0, m_0) + \varphi_x(t_0, x_0, m_0)\ f_1(t_0, x_0, z^*) + h_1(t_0, x_0, z^*) \leq -4\beta.$$

Consequently,

$$\varphi_t(s, y, \nu) + \varphi_x(s, y, \nu)\ f_1(s, y, z^*) + h_1(s, y, z^*) \leq -3\beta \tag{2.12}$$

and

$$\varphi_m(s, y, \nu) + \varphi_x(s, y, \nu)\ f_2(s, z^*) + h_2(s, z^*) \leq -3\beta \tag{2.13}$$

for every $(s, y, \nu) \in B_\delta(t_0, x_0, m_0)$ for some $\delta > 0$.

Set $\zeta^*(\tau) \equiv z^*$. Now, the fact that (2.13) holds at $(t_0, x_0, m_0)$ implies that there exists a $\delta' > 0$ such that

$$V^+(t_0, x_0 + \delta f_2(t_0, z^*), m_0 + \delta) + \delta' h_2(t_0, z^*) \leq V^+(t_0, x_0, m_0) + 3\beta\delta,\ \forall\, 0 < \delta < \delta'. \tag{2.14}$$

The proof of (2.14) is similar to that of (1.24). Next, (2.14) implies that there exists an $\epsilon > 0$ such that for any $t_0 < s < t_0 + \epsilon$ we have

$$V^+(t_0, x_0, m_0) \leq \sup_\alpha \{ \int_{t_0}^s h_1(r, \xi(r), \zeta^*(r))\, dr + \int_{[t_0, s]} h_2(r, \zeta^*(r))\, dr + V^+(s, \xi(s), \mu(s)) \}, \tag{2.15}$$

where the supremum is taken over strategies $\alpha$ which satisfy the property that the outcome $\mu$ of $(\zeta^*, \alpha[\zeta^*])$ is in $\mathcal{M}_{m_0} \cap C[t_0, s]$. Again, the proof of this is similar to that of Lemma 1.6 and uses the fact that

$$\delta \mapsto V^+(t_0, x_0 + \delta f_2(t_0, z^*), m_0 + \delta) + \delta h_2(t_0, z^*)$$

is nonincreasing on $[0, 1 - m_0]$.

Fix $t_0 < \rho < t_0 + \epsilon$ so that $(r, \xi(r), \mu(r)) \in B_\delta(t_0, x_0, m_0),\ t_0 \leq r \leq \rho$. Here, $\mu \in \mathcal{M}_{m_0} \cap C[t_0, t_0 + \epsilon]$ is arbitrary, and $\xi$ is the (continuous) trajectory corresponding to $(\mu, \zeta^*)$. Then, using (2.12) and (2.13) and the change of variable formula for Stieltjes integrals–noting that $\varphi$ is smooth–we get that

$$\begin{aligned}
\varphi(\rho, &\xi(\rho), \mu(\rho)) - \varphi(t_0, x_0, m_0) \\
&= \int_{t_0}^{\rho} \varphi_t(r, \xi(r), \mu(r)) + \varphi_x(r, \xi(r), \mu(r)) f_1(r, \xi(r), \zeta^*(r))\, dr \\
&\qquad + \int_{[t_0, \rho]} \varphi_m(r, \xi(r), \mu(r)) + \varphi_x f_2(r, \zeta^*(r))\, d\mu(r) \\
&\leq - \int_{t_0}^{\rho} h_1(r, \xi(r), \zeta^*(r))\, dr - \int_{[t_0, \rho]} h_2(r, \zeta^*(r))\, d\mu(r) \\
&\qquad\quad - 3\beta(\rho - t_0) - 3\beta[\mu(\rho) - m_0].
\end{aligned}$$

Consequently, since $V^+ - \varphi$ has a zero maximum at $(t_0, x_0, m_0)$ we get that

$$V^+(\rho, \xi(\rho), \mu(\rho)) + \int_{t_0}^{\rho} h_1(r, \xi(r), \zeta^*(r))\, dr + \int_{[t_0,\rho]} h_2(r, \zeta^*(r))\, d\mu(r)$$

$$\leq V^+(t_0, x_0, m_0) - 3\beta(\rho - t_0) - 3\beta[\mu(\rho) - m_0]. \tag{2.16}$$

This is true for every $\mu \in \mathcal{M}_{m_0} \cap C[t_0, \rho]$. Thus, using (2.15) we have arrived at a contradiction. Therefore, $V^+$ is a subsolution.

Next we prove that $V^+$ is a supersolution of (2.9). Let $V^+ - \varphi$ achieve a strict minimum of zero at the point $(t_0, x_0, m_0)$. Again, without loss of generality we may assume that $(t_0, x_0, m_0) \in (0, T) \times R^1 \times (0, 1)$. We must show that

$$\varphi_t(t_0, x_0, m_0) + H^+(0 + 0, t_0, x_0, \varphi_m, \varphi_x) \leq 0 \text{ at } (t_0, x_0, m_0),$$

or, equivalently,

$$\max\{\varphi_t + H^+(0 + 0, t_0, x_0, \varphi_m, \varphi_x),\ \varphi_m + H_2^+(t_0, \varphi_x)\} \leq 0$$

at $(t_0, x_0, m_0)$.

Suppose to the contrary that there is a $\beta > 0$ such that

$$\varphi_t(t_0, x_0, m_0) + H^+(4\beta, t_0, x_0, \varphi_m, \varphi_x) \geq 4\beta \text{ at } (t_0, x_0, m_0).$$

Let $\zeta \in Z[t_0, T]$ be arbitrary. We claim that

$$\varphi_m(t_0, x_0, m_0) + \varphi_x(t_0, x_0, m_0)\ f_2(t_0, \zeta(t_0)) + h_2(t_0, \zeta(t_0)) \leq 4\beta. \tag{2.17}$$

Suppose instead that

$$\varphi_m(t_0, x_0, m_0) + \varphi_x(t_0, x_0, m_0)\ f_2(t_0, \zeta(t_0)) + h_2(t_0, \zeta(t_0)) > 4\beta. \tag{2.18}$$

In this case we let $\delta > 0$ be such that $m_0 + \delta \leq 1$ and

$$\varphi_m(t_0, x_0 + \delta f_2(t_0, \zeta(t_0)), m) + \varphi_x(t_0, x_0 + \delta f_2(t_0, \zeta(t_0), m)\ f_2(t_0, \zeta(t_0)) + h_2(t_0, \zeta(t_0)) > 3\beta$$

if $m_0 \leq m \leq m_0 + \delta$. Since $f_2$ and $h_2$ are bounded and $\varphi$ is smooth, we can choose $\delta$ to be independent of $\zeta$. Define on $[m_0, m_0 + \delta]$ the trajectory $d\xi(m)/dm = f_2(t_0, \zeta(t_0))$ with initial condition $\xi(m_0) = x_0$. We obtain

$$\frac{d}{dm}\,\varphi(t_0, \xi(m), m) + h_2(t_0, \zeta(t_0)) > 3\beta, \qquad m_0 \leq m \leq m_0 + \delta.$$

Integrating this from $m_0$ to $m_0 + \delta$

$$\varphi(t_0, x_0 + \delta f_2(t_0, \zeta(t_0)), m_0 + \delta) - \varphi(t_0, x_0, m_0) + \delta h_2(t_0, \zeta(t_0)) > 3\beta\delta.$$

Since $V^+ - \varphi$ has a minimum at $(t_0, x_0, m_0)$ we conclude that

$$V^+(t_0, x_0 + \delta f_2(t_0, \zeta(t_0)), m_0 + \delta) + \delta h_2(t_0, \zeta(t_0)) > V^+(t_0, x_0, m_0) + 3\beta\delta.$$

Since $\zeta$ was arbitrary, this is a contradiction of (2.1). Thus (2.17) must hold.

Define the strategy $\alpha[\zeta](\tau) \equiv \mu(\tau) \equiv m_0$ on $[t_0, T]$. Assume that we are given a control $\zeta \in Z \cap C[t_0, T]$. Then, from (2.17) there exists $t_0 < s \leq T$ such that

$$\varphi_m(\tau, \xi(\tau), \mu(\tau)) + \varphi_x(\tau, \xi(\tau), \mu(\tau))\ f_2(\tau, \zeta(\tau)) + h_2(\tau, \zeta(\tau)) \leq 3\beta, \quad t_0 \leq \tau \leq s,$$

where $\xi$ denotes the trajectory on associated with $(\zeta, \mu)$. We claim that there exists such an $s$ independent of $\zeta$. Indeed, if not then there would be a sequence $s_j \searrow t_0$ such that (2.18) would be true. But we have already seen that (2.18) leads to a contradiction. Consequently,we see that $\zeta \in \mathcal{Z}(3\beta, \tau, \varphi_m, \varphi_x)$ for all $t_0 \leq \tau \leq s$. Using the definition of $H^+$ we have,

$$\varphi_t(\tau, \xi(\tau), \mu(\tau)) + \varphi_x(\tau, \xi(\tau), \mu(\tau))\ f_1(\tau, \xi(\tau), \zeta(\tau)) + h_1(\tau, \xi(\tau), \zeta(\tau)) \geq 3\beta, \qquad t_0 \leq \rho \leq s.$$

This readily implies (we omit the frequently used details) that

$$\begin{aligned} V^+(s, \xi(s), \mu(s)) &+ \int_{t_0}^{s} h_1(r, \xi(r), \zeta(r))\ dr + \int_{[t_0, s]} h_2(r, \zeta(r))\ d\mu(r) \\ &\geq V^+(t_0, x_0, m_0) + 3\beta(s - t_0). \end{aligned}$$

Now, $\zeta$ was assumed continuous. But this inequality will hold for any $\zeta \in Z[t_0, t]$ since $s$ was independent of $\zeta$ and $\alpha$ is identically $m_0$ and so is also independent of $\zeta$. Consequently, we have found a strategy $\alpha \in \mathcal{M}_{m_0}[t_0, T]$ such that for all $\zeta \in Z[t_0, T]$ the previous inequality holds. But this contradicts (2.2). Thus, $V^+$ is also a supersolution. This completes the proof. $\square$

Next we prove that there is exactly one continuous viscosity solution of (2.10) satisfying the terminal condition (TC) and boundary condition (BC). We state the uniqueness result in the form of a comparison principle.

**Theorem 2.6.** *Let $u$ be a continuous viscosity subsolution and $v$ a continuous viscosity supersolution of (2.10), both satisfying the conditions (TC),(BC). Then $u \leq v$ on $\Omega$.*

*Proof.* Assume that $(u - v)(t_0, x_0, m_0) = \max(u - v) > 0$. Let $\beta > \gamma > 0$ satisfy $u(t_0, x_0, m_0) - \gamma > v(t_0, x_0, m_0) + \beta$.

Define $\tilde{u}(t, x, m) = u(t, x, m) - \frac{\gamma}{m} - \frac{\gamma}{t}$ and $\tilde{v}(t, x, m) = v(t, x, m) + \frac{\beta}{m} + \frac{\beta}{t}$. Then it is straightforward to check that $\tilde{u}$ is a subsolution of

$$\tilde{u}_t + \left(H^+\right)^* \left(\frac{\gamma}{m^2}, t, x, m, \tilde{u}_m, \tilde{u}_x\right) - \frac{\gamma}{t^2} = 0,$$

and $\tilde{v}$ is a supersolution of

$$\tilde{v}_t + \big(H^+\big)_* (-\frac{\beta}{m^2}, t, x, m, \tilde{v}_m, \tilde{v}_x) + \frac{\beta}{t^2} = 0.$$

Set $w(t,x,m,y,n) \equiv \tilde{u}(t,x,m) - \tilde{v}(t,y,n)$. Then, $w$ is a subsolution of

$$w_t + \big(H^+\big)^* (\frac{\gamma}{m^2}, t, x, m, w_m, w_x) - \big(H^+\big)_* (-\frac{\beta}{n^2}, t, y, n, -w_n, -w_y) - \frac{\beta+\gamma}{t^2} = 0. \tag{2.19}$$

Let $\epsilon > 0$ and consider the function

$$f_\epsilon(t,x,m,y,n) = w(t,x,m,y,n) - \frac{1}{2\epsilon}|x-y|^2 - \frac{1}{2\epsilon}|m-n|^2.$$

Assume that this function achieves its maximum at a point $(t_\epsilon, x_\epsilon, m_\epsilon, y_\epsilon, n_\epsilon)$. Then, it will follow from the continuity of $u$ and $v$, more generally from the upper (lower) semicontinuity of $u$ $(v)$, that

$$\frac{1}{2\epsilon}|x_\epsilon - y_\epsilon|^2 \to 0, \qquad \frac{1}{2\epsilon}|m_\epsilon - n_\epsilon|^2 \to 0 \text{ as } \epsilon \to 0, \tag{2.20}$$

and

$$w(t_\epsilon, x_\epsilon, m_\epsilon, y_\epsilon, n_\epsilon) \to \max w \qquad \text{as } \epsilon \to 0.$$

Now, we may assume that $\max(\tilde{u} - \tilde{v}) > 0$. It is clear that we will have $0 < t_\epsilon < T$ and $0 < m_\epsilon < 1$.

Since $w$ is a viscosity subsolution of (2.19), using the smooth test function $\varphi(x,m,y,n) = \frac{1}{2\epsilon}|x-y|^2 + \frac{1}{2\epsilon}|m-n|^2$, we have that

(2.21)
$$\begin{aligned} 0 &\le \big(H^+\big)^* (\frac{\gamma}{m_\epsilon^2}, t_\epsilon, x_\epsilon, m_\epsilon, \varphi_m, \varphi_x) - \big(H^+\big)_* (-\frac{\beta}{m^2}, t_\epsilon, y_\epsilon, n_\epsilon, -\varphi_n, -\varphi_y) - \frac{\beta+\gamma}{t_\epsilon^2} \\ &= H^+(\frac{\gamma}{m_\epsilon^2} - 0, t_\epsilon, x_\epsilon, m_\epsilon, \varphi_m, \varphi_x) - H^+(-\frac{\beta}{m^2} + 0, t_\epsilon, y_\epsilon, n_\epsilon, -\varphi_n, -\varphi_y) - \frac{\beta+\gamma}{t_\epsilon^2} \end{aligned}$$

Notice that $\varphi_m = -\varphi_n$ and $\varphi_x = -\varphi_y$ at $(t_\epsilon, x_\epsilon, m_\epsilon, y_\epsilon, n_\epsilon)$. Furthermore, since

$$\begin{aligned} \mathcal{Z}(\frac{\gamma}{m_\epsilon^2} - 0, t_\epsilon, q_m, q_x) &\supset \mathcal{Z}(0, t_\epsilon, q_m, q_x), \\ \mathcal{Z}(-\frac{\beta}{n_\epsilon^2} + 0, t_\epsilon, q_m, q_x) &\subset \mathcal{Z}(0, t_\epsilon, q_m, q_x), \end{aligned}$$

for every $(q_m, q_x) \in R^2$, we have that

$$H^+(\frac{\gamma}{m_\epsilon^2} - 0, t_\epsilon, x_\epsilon, m_\epsilon, \varphi_m, \varphi_x) \le H^+(0, t_\epsilon, x_\epsilon, m_\epsilon, \varphi_m, \varphi_x), \tag{2.22}$$

and

$$H^+(-\frac{\beta}{n_\epsilon^2} + 0, t_\epsilon, y_\epsilon, n_\epsilon, \varphi_m, \varphi_x) \geq H^+(0, t_\epsilon, y_\epsilon, n_\epsilon, \varphi_m, \varphi_x) \tag{2.23}$$

Combining (2.21)-(2.23) we see that

(2.24)

$$\begin{aligned} 0 &\leq H^+(0, t_\epsilon, x_\epsilon, m_\epsilon, \varphi_m, \varphi_x) - H^+(0, t_\epsilon, y_\epsilon, n_\epsilon, \varphi_m, \varphi_x) - \frac{\beta+\gamma}{t_\epsilon^2} \\ &\leq H^+(0, t_\epsilon, y_\epsilon, n_\epsilon, \varphi_m, \varphi_x) - H^+(0, t_\epsilon, y_\epsilon, n_\epsilon, \varphi_m, \varphi_x) + K|x_\epsilon - y_\epsilon| - \frac{\beta+\gamma}{t_\epsilon^2} \\ &= K|x_\epsilon - y_\epsilon| - \frac{\beta+\gamma}{t_\epsilon^2}. \end{aligned}$$

Since $\beta > \gamma > 0$ we can choose $\epsilon$ sufficiently small so that, using (2.20), the last part of (2.24) is nonpositive. This is a contradiction, so that we conclude that $u \leq v$.

The only gap we need to close is the fact that the maxima in the proof may not be achieved due to the fact that $x$ is not known to be in a bounded set. We can fix this in the following way. Let $R > 0$ and $\kappa \in C^1(R^1)$ be a function with $0 \leq \kappa'(r) \leq 1$, $\kappa(r) = 0$ if $r \leq R$, and $\kappa(r) \to \infty$ as $r \to \infty$. Then, we modify the definition of $\tilde{u}$ and $\tilde{v}$ as follows:

$$\begin{aligned} \tilde{u}(t,x,m) &= u(t,x,m) - \gamma\kappa(|x|) - K\gamma(T-t) - \frac{\gamma}{m} - \frac{\gamma}{t} \\ \tilde{v}(t,x,m) &= v(t,x,m) + \beta\kappa(|x|) + K\beta(T-t) + \frac{\beta}{m} + \frac{\beta}{t}. \end{aligned}$$

The proof continues as before with minor modifications. This completes the proof of the theorem. □

*Remark:* It is not hard to show that

$$V^+(t,x,m) = \lim_{B\to\infty} W^B(t,x,m),$$

where $W^B(t,x,m)$ is the viscosity solution of

$$\begin{aligned} W_t^B(t,x,m) &+ \min_{z\in Z}\{W_x^B(t,x,m)\, f_1(t,x,z) + h_1(t,x,z) \\ &+ B[W_m^B(t,x,m) + W_x^B(t,x,m)\, f_2(t,z) + h_2(t,z)]^+\} = 0. \end{aligned}$$

In fact, the proof follows from [5] by establishing that the corresponding hamiltonian for $W^B(t,x,m)$ converges appropriately to the hamiltonian for $V^+$. Also, $W(t,x,m)$ satisfies the same terminal and boundary conditions as does $V^+(t,x,m)$. Notice that $W^B(t,x,m)$ is the upper value of the differential game in which the functions $\mu$ are absolutely continuous and $0 \leq d\mu/d\tau \leq B$.

Now we will state the results for the lower value. To do so we need the definition of the lower hamiltonian.

Let

$$\mathcal{F}(t,x) = \overline{co}\{(f_1(t,x,z), h_1(t,x,z), f_2(t,z), h_2(t,z)) : z \in Z\}$$

and

$$\mathcal{A}(a,t,x,p_m,p_x) = \{(\xi_1,\eta_1,\xi_2,\eta_2) \in \mathcal{F}(t,x) : p_m + p_x\xi_2 + \eta_2 \le a\}.$$

The notation $\overline{co}(A)$ denotes the closed convex hull of the set $A$. $H^-$ is defined by

$$H^-(a,t,x,p_m,p_x) = \min\{p_x\xi_1 + \eta_1 : (\xi_1,\eta_1,\xi_2,\eta_2) \in \mathcal{A}(a,t,x,p_m,p_x)\}. \tag{2.25}$$

and $H^-(a,t,x,p_m,p_x) \equiv +\infty$ if $\mathcal{A}(a,t,x,p_m,p_x) = \emptyset$.

**Theorem 2.7.** *The lower value function $V^-(t,x,m)$ is the unique continuous viscosity solution of*

$$V_t^- + H^-(0,t,x,V_m^-,V_x^-) = 0 \qquad (t,x,m) \in \Omega, \tag{2.26}$$

*and $V^-$ satisfies the terminal condition (TC) and boundary condition (BC).*

We will leave the proof of this theorem for the reader. We note however the following lemma which will explain the origin of the lower hamiltonian.

**Lemma 2.8.** *Fix $(t,x,b,p_m,p_x) \in (0,T) \times R^4$. Then*

$$\begin{aligned} \max_{\lambda \ge 0} \min_{z \in Z} \{p_x f_1(t,x,z) + h_1(t,x,z) + \lambda(p_m + p_x f_2(t,z) + h_2(t,z) - b)\} \\ = H^-(b,t,x,p_m,p_x). \end{aligned} \tag{2.27}$$

*Remark.* The left side of (2.27) arises from considering the lower differential game with unbounded maximizing control $\alpha$ as in (1.27)-(1.28).

*Proof of 2.8.* If $\mathcal{A}(b,t,x,p_m,p_x) = \emptyset$ it is clear that (2.27) trivially holds, so we assume this set is not empty.

Since $\min_{\rho \in \Lambda} \rho = \min_{\rho \in \overline{co}\Lambda} \rho$ we know that

$$\begin{aligned} \min_{z \in Z} \{p_x f_1(t,x,z) + h_1(t,x,z) + \lambda(p_m + p_x f_2(t,z) + h_2(t,z) - b)\} \\ = \min_{\mathcal{F}(t,x)} \{p_x\xi_1 + \eta_1 + a(p_m + p_x\xi_2 + \eta_2 - b)\} \end{aligned} \tag{2.28}$$

The function

$$(\lambda,\xi_1,\eta_1,\xi_2,\eta_2) \mapsto p_x\xi_1 + \eta_1 + \lambda(p_m + p_x\xi_2 + \eta_2 - b)$$

is certainly concave-convex since it is linear. Therefore, we may apply the minimax theorem (for example, [26, theorem 49.A]) to see that

$$\begin{aligned}
&\max_{\lambda\geq 0}\min_{z\in Z}\{p_x f_1(t,x,z)+h_1(t,x,z)+\lambda(p_m+p_x f_2(t,z)+h_2(t,z)-b)\}\\
&\quad=\max_{\lambda\geq 0}\min_{\mathcal{F}(t,x)}\{p_x\xi_1+\eta_1+\lambda(p_m+p_x\xi_2+\eta_2-b)\}\\
&\quad=\min_{\mathcal{F}(t,x)}\max_{\lambda\geq 0}\{p_x\xi_1+\eta_1+\lambda(p_m+p_x\xi_2+\eta_2-b)\}\\
&\quad=\min\{p_x\xi_1+\eta_1:(\xi_1,\eta_1,\xi_2,\eta_2)\in\mathcal{A}(b,t,x,p_m,p_x)\}\\
&\quad=H^-(b,t,x,p_m,p_x).
\end{aligned}$$

□

The next result follows from the uniqueness property.

**Corollary 2.9.** *If*

$$H^+(0,t,x,p_m,p_x)=H^-(0,t,x,p_m,p_x), \tag{2.30}$$

*then the differential game associated with (1.1)-(1.4) has value; i.e.,* $V^+=V^-$. *If the payoff has a terminal cost,* $g(\xi(T))$ *as well, with* $g$ *Lipschitz continuous and bounded, and if, in addition to (2.30)*

$$\begin{aligned}
&\max_{m\leq a\leq 1}\min_{z\in Z}\{g(x+f_2(T,z)(a-m))+h_2(T,z)(a-m)\}\\
&\qquad=\min_{z\in Z}\max_{m\leq a\leq 1}\{g(x+f_2(T,z)(a-m))+h_2(T,z)(a-m)\}
\end{aligned}$$

*then this differential game has value.*

*Remark:* When the differential game has a terminal cost and the maximizing player can jump it makes a difference which player has the last move.

We conclude this paper with the following special case of the results of this section.

We begin by noting that all of the results can be extended to the case $h_2=h_2(t,x,z)$ if $f_2=0$. Take $f_2=h_1=0$, and assume that $h_2\geq 0$. Thus the trajectory is continuous and we only have a cost against the measures. Then, the problem (2.10) for the upper value, $V^+(t,x,m)$, becomes

$$V_t^+ +\min\{V_x^+ f_1(t,x,z)\mid z\in Z \text{ such that } V_m^+ + h_2(t,x,z)\leq 0\}=0,\quad (t,x,m)\in\Omega,$$

with

$$V^+(T,x,m)=(1-m)\min_{z\in Z}h_2(T,x,z),\qquad V^+(t,x,1)=\gamma(t,x).$$

It is straightforward to verify that $V^+(t,x,m)=(1-m)W(t,x)$ if $(t,x,m)\in(0,T]\times R^1\times(0,1)$, where $W$ is the unique viscosity solution of

$$W_t+\min\{W_x f_1(t,x,z)\mid z\in Z \text{ such that } h_2(t,x,z)\leq W(t,x)\}=0,\quad (t,x)\in(0,T)\times R^1,$$

with

$$W(T,x) = \min_{z \in Z} h_2(T,x,z).$$

But, it was established in [8] that

$$W(t,x) = \inf_{\zeta \in Z[t,T]} ||h_2(r,\xi(r),\zeta(r))||_{L^\infty[t,T]} = \inf_{\zeta \in Z[t,T]} \sup_{t \le r \le T} h_2(r,\xi(r),\zeta(r)).$$

To understand the connection between this minimax problem and the problem with measures simply recall the basic fact that the $L^\infty$ norm of a function $f(x)$ is the norm of the functional on $L^1$, $\Gamma(g) = \int f(x) \cdot g(x)\ dx$. The results of this paper therefore generalize the main result of [8]. Furthermore, we have shown that $V_m^+ = -W$. This is connected to a result of Karatzas [18].

## Bibliography


1. G.Barles, *Uniqueness and regularity results for first order Hamilton -Jacobi equations*, Indiana Univ. Math. J. **39** (1990), 443–466.
2. G.Barles, *An approach of deterministic unbounded control problems and of first order Hamilton- Jacobi equations with gradient constraints*, Ph.D. Thesis Universite de Paris IX-Dauphine (1989), 2-31.
3. G.Barles, *Existence results for first order Hamilton - Jacobi equations*, Ann. Inst. Henri Poincare **1** (1984), 325–340.
4. G.Barles, *Deterministic impulse control problems*, SIAM J. Control and Optimization **23** (1985), 419–432.
5. G.Barles and B.Perthame, *Discontinuous solutions of deterministic optimal stopping time problems*, Mathematical Modelling and Numerical Analysis **21** (1987), 557–579.
6. E.N.Barron, *Viscosity solutions for the monotone control problem*, SIAM J. Control and Optimization **23** (1985), 161-171.
7. E.N.Barron, *Differential games with maximum cost*, Nonlinear Analysis, Theory, Methods, and Applications **14** (1990), 971–989.
8. E.N.Barron and H.Ishii, *The Bellman equation for minimizing the maximum cost*, Nonlinear Analysis, Theory, Methods, and Applications **13** (1989), 1067-1090.
9. E.N.Barron and R.Jensen, *The Pontryagin maximum principle from dynamic programming and viscosity solutions to first order partial differential equations*, Transactions A.M.S. **2** (1986), 635–641.
10. P.L.Chow, J.L.Menaldi, and M.Robin, *Additive control of stochastic linear systems with finite horizon*, SIAM J. Control and Optimization **23** (1985), 858–899.
11. M.G.Crandall and P.-L.Lions, *Viscosity solutions of Hamilton-Jacobi equations*, Transactions Amer. Math. Soc. **282** (1984), 1–42.
12. M.G.Crandall, L.C.Evans, and P.-L.Lions, *Some properties of viscosity solutions of Hamilton- Jacobi equations*, Trans. Amer. Math. Soc. **282** (1984), 487–502.
13. M.G.Crandall, H.Ishii, and P.-L.Lions, *Uniqueness of viscosity solutions revisited*, J.Math. Soc. Japan **39** (1987), 581–596.
14. R.J.Elliott, *Viscosity solutions and optimal control*, Pitman, London, 1986.
15. I. Capuzzo Dolcetta and L.C.Evans, *Optimal switching for ordinary differential equations*, SIAM J. Control and Optimization **22** (1984), 143–161.
16. H.Ishii, *Perron's method for Hamilton - Jacobi equations*, Duke Math. J. **55** (1987), 369–384.
17. H.Ishii, *Hamilton-Jacobi equations with discontinuous Hamiltonians on arbitrary open sets*, Bull. Fac. Sci.Eng., Chuo Univ. **28** (1985), 33– 77.
18. I. Karatzas, *Probabilistic aspects of finite-fuel stochastic control*, Proc. Natl. Acad. Sci. USA **82** (1985), 5579–5581.
19. J.L.Menaldi and M. Robin, *On some cheap control problems for diffusion processes*, Trans. Amer. Math. Soc. **278** (1983), 771–802.

20. J.L.Menaldi and M. Robin, *On singular control problems for diffusions with jumps*, IEEE Trans. Automatic Control **AC-29** (1984), 991–1004.
21. J.M.Murray, *Existence theorems for optimal control and calculus of variations problems where the states can jump*, SIAM J. Control and Optimization **24** (1986), 412–438.
22. R.Rishel, *An extended Pontryagin principle for control systems whose control laws contain measures*, SIAM J. Control **3** (1965), 191–205.
23. W.W.Schmaedke, *Optimal control theory for nonlinear vector differential equations containing measures*, SIAM J. Control **3** (1965), 231–280.
24. M. Sun, *Monotone control of a damped oscillator under random perturbations*, IMA J. Math. Control and Inf. **5** (1988), 169–186.
25. R.Vinter and F.Pereira, *A maximum principle for optimal processes with discontinuous trajectories*, SIAM J. Control and Optimization **26** (1988), 205–229.
26. E.Zeidler, *Nonlinear Functional Analysis and its Applications. III: Variational Methods and Optimization*, Springer Verlag, New York, 1985.